

TURÁN'S THEOREM FOR THE FANO PLANE

LOUIS BELLMANN AND CHRISTIAN REIHER

ABSTRACT. Confirming a conjecture of Vera T. Sós in a very strong sense, we give a complete solution to Turán's hypergraph problem for the Fano plane. That is we prove for $n \geq 8$ that among all 3-uniform hypergraphs on n vertices not containing the Fano plane there is indeed exactly one whose number of edges is maximal, namely the balanced, complete, bipartite hypergraph. Moreover, for $n = 7$ there is exactly one other extremal configuration with the same number of edges: the hypergraph arising from a clique of order 7 by removing all five edges containing a fixed pair of vertices.

For sufficiently large values n this was proved earlier by Füredi and Simonovits, and by Keevash and Sudakov, who utilised the stability method.

§1. INTRODUCTION

With his seminal work [19], Turán initiated extremal graph theory as a separate subarea of combinatorics. After proving his well known extremal result concerning graphs not containing a clique of fixed order, he proposed to study similar problems for graphs arising from platonic solids and for hypergraphs. For instance, for a given 3-uniform hypergraph F and a given natural number n , there arises the question to determine the largest number $\text{ex}(n, F)$ of edges that a 3-uniform hypergraph H can have without containing F as a subhypergraph.

Here a *3-uniform hypergraph* $H = (V, E)$ consists of a set V of *vertices* and a collection $E \subseteq V^{(3)}$ of 3-element subsets of V , that are called the *edges* of H . Since all hypergraphs occurring in this article are 3-uniform, we will henceforth abbreviate the terminology and just say “hypergraph” when we mean “3-uniform hypergraph.”

Despite tremendous efforts over the last 70 years, our knowledge about these Turán functions $n \mapsto \text{ex}(n, F)$ is very limited, even for very innocent looking hypergraphs F such as the tetrahedron $F = K_4^{(3)}$. It is thus customary to focus on the *Turán densities*

$$\pi(F) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, F)}{\binom{n}{3}},$$

the existence of which follows from the fact that the sequences $n \mapsto \text{ex}(n, F)/\binom{n}{3}$ are, by a result of Katona, Nemetz, and Simonovits [7], monotonically decreasing. These Turán

2010 *Mathematics Subject Classification.* 05C65, 05D05.

Key words and phrases. Turán's hypergraph problem, Fano plane.

The second author was supported by the European Research Council (ERC grant PEPCo 724903).

densities are not understood very well either and all one knows in this regard about the tetrahedron are the estimates

$$\frac{5}{9} \leq \pi(K_4^{(3)}) \leq 0.5616. \quad (1.1)$$

The lower bound follows from an explicit construction due to Turán himself (see e.g. [2]), which is widely believed to be optimal. The upper bound was established by Razborov [15] by means of his flag algebraic approach introduced in [14].

Vera T. Sós proposed to study Turán’s hypergraph problem in the special case where $F = \mathcal{F}$ is the *Fano plane*, i.e., the projective plane over the field with two elements. More precisely, one takes \mathcal{F} to be the hypergraph with 7 vertices, which are the *points* of the Fano plane, and whose 7 edges correspond to the *lines* of the Fano plane (see Fig. 1.1).

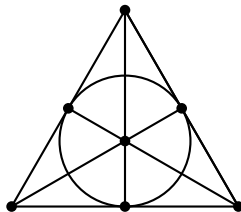


FIGURE 1.1. Fano plane

One verifies easily that no matter how the vertices of the Fano plane get coloured with two colours, there will always be a monochromatic edge, which suggests that bipartite hypergraphs could be relevant to the problem under discussion. Given a natural number n , we denote the *balanced, complete, bipartite* hypergraph on n vertices by B_n . This hypergraph is defined so as to have a partition $V(B_n) = X \cup Y$ of its n -element vertex set with $||X| - |Y|| \leq 1$ such that a triple $e \subseteq V(B_n)$ is an edge of B_n if and only if it intersects both X and Y . The above observation on vertex colourings implies $\mathcal{F} \not\subseteq B_n$ and, hence, that $\text{ex}(n, \mathcal{F}) \geq b(n)$, where

$$b(n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lfloor (n+1)/2 \rfloor}{3}$$

denotes the number of edges of B_n . This number rewrites more conveniently as

$$b(n) = \begin{cases} \frac{1}{8}n^2(n-2) & \text{if } n \text{ is even,} \\ \frac{1}{8}(n^2-1)(n-2) & \text{if } n \text{ is odd.} \end{cases} \quad (1.2)$$

Sós conjectured this construction to be optimal, i.e., that

$$\text{ex}(n, \mathcal{F}) = b(n) \quad (1.3)$$

and that, moreover, B_n is the unique n -vertex hypergraph with $b(n)$ edges not containing a Fano plane. According to Füredi [4], this conjecture of Sós was widely known since

the 1970's. In her problem and survey article [18], which often serves as a reference for this problem, she discusses several connections between design theory and extremal hypergraph theory, even though (1.3) does not seem to be mentioned there.

The first result in this direction is due to de Caen and Füredi [1], who proved that $\pi(\mathcal{F}) = \frac{3}{4}$ holds for the Fano plane \mathcal{F} . Their article introduced the so-called *link multigraph method* on which all further progress on Sós's conjecture is based, and which has since then found many further applications (see e.g. [8, 12]). A few years later it turned out that by combining the work in [1] with Simonovits' *stability method* [17] one can prove (1.3) for all sufficiently large n . This was done by Füredi and Simonovits in [6] and, independently, by Keevash and Sudakov in [9]. It is not straightforward to extract optimal quantitative information from either of those articles but it seems safe to say that following [6] closely (1.3) would be hard to show for all $n \geq 10^{100}$ and easy for $n \geq 10^{300}$, while the arguments in [9] would probably require n to be larger than 10^{900} .

The main result of the present work proves (1.3) for all $n \geq 7$. Furthermore, we show that for $n \geq 8$ the balanced, complete, bipartite hypergraph is indeed the only extremal configuration. For $n = 7$, however, there is a second extremal example, which is the hypergraph J_7 remaining from the complete hypergraph $K_7^{(3)}$ when one deletes all five edges involving a fixed pair of vertices. Plainly J_7 has $\binom{7}{3} - 5 = 30 = b(7)$ edges and $\mathcal{F} \not\subseteq J_7$ follows from the fact that in the Fano plane every pair of points determines a line.

Theorem 1.1. *For every integer $n \geq 7$ we have*

$$\text{ex}(n, \mathcal{F}) = b(n) = \frac{n-2}{2} \cdot \left\lfloor \frac{n^2}{4} \right\rfloor,$$

where \mathcal{F} denotes the Fano plane. Moreover, for $n \geq 8$ the only extremal hypergraph is the balanced, complete, bipartite hypergraph B_n , while for $n = 7$ there are exactly two extremal hypergraphs, namely B_7 and J_7 .

The proof of Theorem 1.1 proceeds by induction on n and uses the link multigraph method. Let us mention for completeness that for $n \leq 6$ one trivially has $\text{ex}(n, \mathcal{F}) = \binom{n}{3}$, the unique extremal configuration being the complete hypergraph $K_n^{(3)}$.

Organisation. We prove Theorem 1.1 in Section 4. Some auxiliary considerations dealing with small hypergraphs and inductive characterisations of balanced, complete, bipartite hypergraphs are gathered in Section 2. The results on multigraphs we shall require are developed in Section 3.

§2. PRELIMINARIES

2.1. Tetrahedra. The n -vertex hypergraphs we need to deal with in the proof of our main result will have $b(n)$ edges and, hence, an edge density of $\frac{3}{4} + o(1)$. In view of (1.1) such

hypergraphs contain tetrahedra provided that n is sufficiently large. Later on it will be important to know that this actually holds for small values of n as well, which can be seen by means of the following well-known, elementary argument.

Starting from the obvious fact $\text{ex}(4, K_4^{(3)}) = 3$ one uses the monotonicity of the sequence

$$n \mapsto \frac{\text{ex}(n, K_4^{(3)})}{\binom{n}{3}}$$

in order to obtain

$$\text{ex}(n, K_4^{(3)}) \leq \frac{3}{4} \binom{n}{3}$$

for every $n \geq 4$. Together with the estimate

$$\frac{3}{4} \binom{n}{3} = \frac{n(n-1)(n-2)}{8} < \frac{(n+1)(n-1)(n-2)}{8} \stackrel{(1.2)}{\leq} b(n),$$

which holds for all $n \geq 3$, this leads to the following statement.

Fact 2.1. *For $n \geq 4$ every hypergraph on n vertices with $b(n)$ edges contains a tetrahedron.*

2.2. Finding Fano planes. This subsection discusses two ways of looking at the Fano plane \mathcal{F} that turn out to be helpful for realising that a given hypergraph H contains a copy of \mathcal{F} .

The first of them goes back to the work of de Caen and Füredi [1] and reappeared in all subsequent articles addressing the Turán problem for the Fano plane. Given a vertex x of an arbitrary hypergraph H one may form its so-called *link graph* with vertex set $V(H)$ in which two vertices u and v are declared to be adjacent if and only if the triple uvx is an edge of H . Now the simple yet important observation one frequently uses is that if xyz denotes an arbitrary edge of the Fano plane \mathcal{F} , then the six further edges of \mathcal{F} correspond to certain edges of the link graphs of x , y , and z . Moreover, these edges in the link graphs use four vertices only and they form a configuration which has, for obvious reasons, been called “*three crossing pairs*” in [6] (see Fig. 2.1).

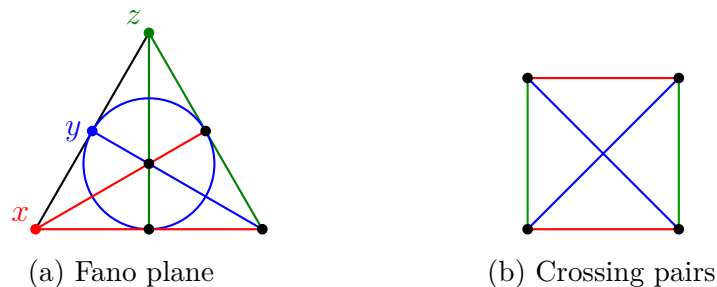


FIGURE 2.1. The edge xyz and the link graphs of x , y , and z .

Another way of locating Fano planes in dense hypergraphs focuses on the link graph of a single vertex. Plainly, every vertex x of the Fano plane \mathcal{F} belongs to three edges of \mathcal{F} , which correspond to a perfect matching M in the link graph of x restricted to the six remaining vertices of \mathcal{F} . There are four further edges in \mathcal{F} forming a certain tripartite hypergraph \mathcal{P} , whose partition classes are given by M . Owing to its connection with Pasch's axiom in the axiomatic approach to planar Euclidean geometry (see [13, §2, Grundsatz IV]), \mathcal{P} is often called the *Pasch hypergraph* (see Fig. 2.2).

This perspective on the Fano plane is especially useful when combined with the stability method, for the Pasch hypergraph is known to have vanishing Turán density—a fact exploited both in [6] and in [9]. In the present work, the Pasch hypergraph plays a much less prominent rôle and it will only be mentioned in the proof of Lemma 2.3 below.

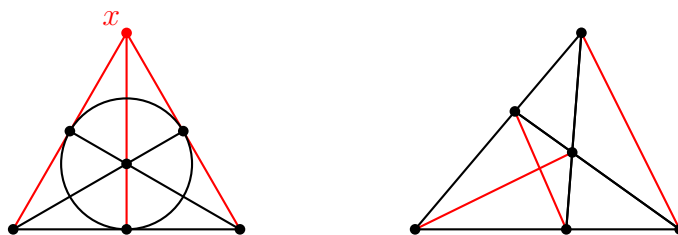


FIGURE 2.2. The Pasch hypergraph contained in the Fano plane.

2.3. Small hypergraphs. In this subsection we gather several auxiliary statements addressing hypergraphs on 7 or 8 vertices. We begin with the case $n = 7$ of Theorem 1.1, which will later constitute the start of an induction.

Lemma 2.2. *Every hypergraph with 7 vertices and 30 edges not containing a Fano plane is isomorphic to either B_7 or J_7 .*

Proof. Let H be such a hypergraph with vertex set $[7]$ and write \overline{H} for its complement, which has 5 edges.

For every permutation $\pi \in S_7$ we denote the number of triples among

$$\begin{aligned} &\pi(1)\pi(2)\pi(3), \quad \pi(3)\pi(4)\pi(5), \quad \pi(1)\pi(5)\pi(6), \quad \pi(1)\pi(4)\pi(7), \\ &\pi(3)\pi(6)\pi(7), \quad \pi(2)\pi(5)\pi(7), \quad \text{and} \quad \pi(2)\pi(4)\pi(6), \end{aligned}$$

which are edges of \overline{H} , by $A(\pi)$. As these seven triples form a Fano plane, the number $A(\pi)$ cannot vanish for any $\pi \in S_7$, wherefore

$$\sum_{\pi \in S_7} A(\pi) \geq |S_7| = 7!.$$

On the other hand, every edge of \overline{H} appears in the above list for precisely $7 \cdot 3! \cdot 4! = 7!/5$ permutations π and a double-counting argument yields

$$\sum_{\pi \in S_7} A(\pi) = \frac{7!}{5} \cdot e(\overline{H}) = 7!.$$

For these reasons, we have $A(\pi) = 1$ for every $\pi \in S_7$. If \overline{H} would have two edges intersecting in a single vertex, then an appropriate permutation $\pi \in S_7$ would satisfy $A(\pi) \geq 2$, which has just been proved to be false. Therefore, any two distinct edges of \overline{H} are either disjoint or they intersect in a pair.



FIGURE 2.3. Possibilities for \overline{H} .

A quick case analysis discloses that there are only two hypergraphs on 7 vertices with 5 edges having this property, namely the disjoint union of a tetrahedron and a single edge (see Fig. 2.3a), and the hypergraph whose edges are the five triples containing a fixed pair of vertices (see Fig. 2.3b). In the former case H is isomorphic to B_7 and in the latter case one has $H \cong J_7$. \square

The next lemma analyses certain Fano-free hypergraphs on 7 vertices with possibly only 29 edges. It will allow us later to exclude several configurations on six vertices in a hypothetical minimal counterexample to Theorem 1.1.

Lemma 2.3. *Let H be a hypergraph on 7 vertices not containing a Fano plane. If some vertex v of H satisfies $d(v) \geq 11$ and $e(H \setminus v) \geq 18$, then $H \setminus v$ is isomorphic to B_6 .*

Proof. Set $K = V(H) \setminus \{v\}$ and let L denote the link graph of v restricted to K . It has 6 vertices and at least 11 edges, and thus it contains a perfect matching M , say with edges x_1x_2, x_3x_4, x_5x_6 . Assuming indirectly that $H \setminus v$ is not isomorphic to B_6 one checks easily that this hypergraph contains a copy of $K_5^{(3)}$ and we may henceforth suppose that $\{x_1, \dots, x_5\}$ is a clique of order 5 in H . In other words, x_6 belongs to all edges of the complement H_* of $H \setminus v$.

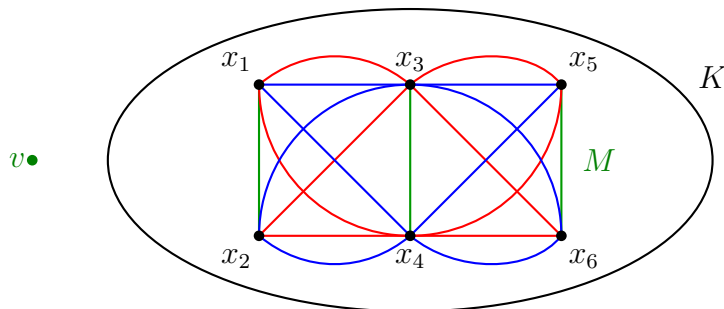


FIGURE 2.4. The matching M in the link of v and two Pasch hypergraphs (drawn red and blue).

Now both

$$x_1x_3x_5, x_1x_4x_6, x_2x_3x_6, x_2x_4x_5$$

and

$$x_2x_4x_6, x_2x_3x_5, x_1x_4x_5, x_1x_3x_6$$

are edge configurations forming Pasch hypergraphs that together with the matching M in the link of v would yield a Fano plane (see Figure 2.4). Thus both of the above disjoint rows contain a triple which fails to be an edge of H . On the other hand, by $e(H \setminus v) \geq 18$ the complement H_* can have at most two edges.

So without loss of generality we may suppose that the edges of H_* are $x_1x_4x_6$ and $x_2x_4x_6$. Now x_1 and x_2 are the only vertices of H_* having degree 1. If there were a different perfect matching M' in L not pairing these two vertices with each other, we could repeat the entire argument with M' in place of M and would thus find a Fano plane in H .

This shows that all perfect matchings of L use the edge x_1x_2 . Hence the graph $L \setminus x_1x_2$ with 6 vertices and at least 10 edges has no perfect matchings at all, which is only possible if this graph consists of a K_5 and an isolated vertex. As the edge x_1x_2 cannot belong to this K_5 , the isolated vertex must be either x_1 or x_2 . In both cases

$$x_1x_2, x_3x_5, x_4x_6$$

is a perfect matching in L . Together with the Pasch hypergraph

$$x_1x_3x_4, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5$$

it leads to a Fano plane in H , which is absurd. Thus we have indeed $(H \setminus v) \cong B_6$. \square

Finally, the last statement of this subsection will allow us later to eliminate a somewhat annoying case that arises in the induction step from 7 to 8 due to the non-uniqueness of the extremal hypergraph on 7 vertices.

Fact 2.4. *Let H be a hypergraph on 8 vertices not containing a Fano plane. If $K_6^{(3)} \subseteq H$, then $e(H) \leq 46 < b(8)$.*

Proof. Write $V(H) = K \cup \{x, y\}$, where K induces a $K_6^{(3)}$ in H . By Lemma 2.3 applied to $H \setminus y$ there are at most 10 edges containing x but not y . Similarly, there are at most 10 edges containing y but not x . Finally, H can have at most $|K| = 6$ edges containing both x and y . So altogether we have indeed

$$e(H) \leq 20 + 10 + 10 + 6 = 46 < 48 = b(8). \quad \square$$

2.4. Characterisations of B_n . In our inductive proof of Theorem 1.1 we will consider a hypergraph H on some number $n \geq 8$ of vertices with $b(n)$ edges and $\mathcal{F} \not\subseteq H$. These assumptions will be shown to entail some strong structural properties of H and the purpose of this subsection is to check that we can actually conclude $H \cong B_n$ from those properties.

This is much easier when the number of vertices is even.

Lemma 2.5. *Suppose that $n \geq 6$ is even and that H is a hypergraph on n vertices. If for every vertex v of H the hypergraph $H \setminus v$ is isomorphic to B_{n-1} , then $H \cong B_n$.*

Proof. Let $y \in V(H)$ be arbitrary. Since $H \setminus y$ is isomorphic to B_{n-1} , there exists a partition $V(H) \setminus \{y\} = X \cup Y$ with $|X| = \frac{n}{2}$ and $|Y| = \frac{n}{2} - 1$ such that X and Y are independent sets in H . The same argument applies to every $y' \in Y$. Since B_{n-1} has a unique independent set of size $\frac{n}{2}$, the outcome must be the partition

$$V(H) \setminus \{y'\} = X \cup (Y \cup \{y\} \setminus \{y'\}) \quad \text{for each } y' \in Y.$$

This proves that H is isomorphic to B_n with vertex classes X and $Y \cup \{y\}$. \square

To handle the case where the number of vertices is odd we shall require the following lemma. Its initial assumption concerning the case $n = 8$ will turn out to be harmless, as we will already know its truth when using the lemma for the first time.

Lemma 2.6. *Assume that Theorem 1.1 holds for $n = 8$. Now let $n \geq 9$ be odd and let H be a hypergraph on n vertices with $b(n)$ edges which does not contain a Fano plane. Suppose that whenever a four-element set $K \subseteq V(H)$ induces a tetrahedron in H*

- (i) *we have $(H \setminus K) \cong B_{n-4}$*
- (ii) *and for every $v \in V(H \setminus K)$ there are exactly five pairs in $K^{(2)}$ forming together with v an edge of H .*

Then H is isomorphic to B_n .

Proof. Recall that by Fact 2.1 there is a tetrahedron contained in H , say with vertex set $K \subseteq V(H)$. Owing to condition (i) there is a partition $V \setminus K = X \cup Y$ witnessing

that $H \setminus K$ is indeed isomorphic to B_{n-4} . Notice that due to $n \geq 9$ we may suppose that $|X| \geq 2$ and $|Y| \geq 3$.

Now consider any four distinct vertices $x, x' \in X$ and $y, y' \in Y$. By clause (ii) applied to the tetrahedra K and $K' = \{x, x', y, y'\}$ we obtain

$$e(K \cup K') = e(K) + e(K') + 5(|K| + |K'|) = 2 \cdot 4 + 5 \cdot 8 = 48 = b(8)$$

and by the hypothesised validity of Theorem 1.1 for hypergraphs on 8 vertices it follows that $K \cup K'$ induces a copy of B_8 in H . As K induces a tetrahedron, there exists an enumeration $K = \{v_1, v_2, v_3, v_4\}$ such that the two independent 4-sets of this B_8 are, possibly after relabelling y and y' ,

- (a) either $\{v_1, v_2, x, x'\}$ and $\{v_3, v_4, y, y'\}$
- (b) or $\{v_1, v_2, x, y\}$ and $\{v_3, v_4, x', y'\}$.

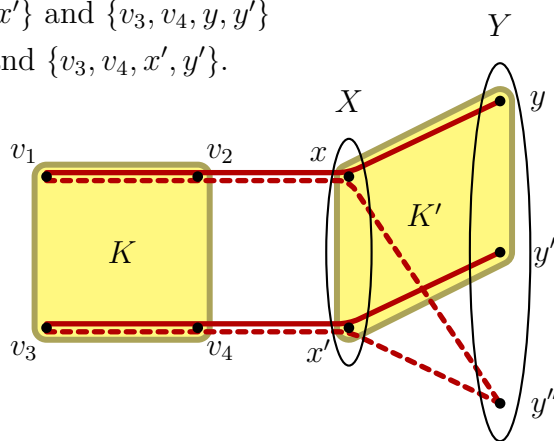


FIGURE 2.5. The impossible case (b). Tetrahedra are drawn as yellow quadruples and independent sets as red lines.

Now assume for the sake of contradiction that the latter possibility occurs (see Fig. 2.5). Let $y'' \in Y$ be an arbitrary vertex distinct from y and y' . When applying the argument of the foregoing paragraph to $\{x, x', y, y''\}$ instead of K' we still have the independent set $\{v_1, v_2, x, y\}$ and, consequently $\{v_3, v_4, x', y''\}$ is independent as well. Proceeding similarly with the tetrahedron $\{x, x', y', y''\}$ we learn that the set $\{v_1, v_2, x, y''\}$ is likewise independent. In particular, neither $v_1 v_2 y''$ nor $v_3 v_4 y''$ belongs to $E(H)$, meaning that K and y'' violate condition (ii). This proves that alternative (b) is indeed impossible.

Summarising the discussion so far, we know that depending on any four distinct vertices $x, x' \in X$ and $y, y' \in Y$ there is an enumeration $K = \{v_1, v_2, v_3, v_4\}$ such that the two independent 4-sets of the copy of B_8 induced by $K \cup \{x, x', y, y'\}$ are as mentioned in (a).

Now if we keep y and y' fixed and let the pair x, x' vary through X we will always get the same independent set $\{v_3, v_4, y, y'\}$ and thus the entire set $X \cup \{v_1, v_2\}$ is independent in H . Similarly, $Y \cup \{v_3, v_4\}$ is independent as well. Consequently H is indeed isomorphic to B_n with partition classes $X \cup \{v_1, v_2\}$ and $Y \cup \{v_3, v_4\}$. \square

§3. MULTIGRAPHS

This section builds upon [6, Section 2-4] and collects several extremal results on multigraphs that will be applied at a later occasion to certain link multigraphs arising in hypergraphs not containing Fano planes.

Definition 3.1. For a positive integer p , a p -tuple $\vec{G} = (G_1, \dots, G_p)$ of graphs on the same vertex set $V(\vec{G})$ will be referred to as a p -multigraph.

Extending some pieces of graph theoretic notation to the context of multigraphs, we will write $e(\vec{G}) = \sum_{i=1}^p e(G_i)$ for the *total number of edges* of a p -multigraph $\vec{G} = (G_1, \dots, G_p)$. Similarly, for every $X \subseteq V(\vec{G})$ we put $e(X) = \sum_{i=1}^p e_{G_i}(X)$ and if the members of X are enumerated explicitly we will omit a pair of curly braces and write, e.g., $e(x, y, z)$ instead of the more baroque $e(\{x, y, z\})$. In the special case of two-element sets, the number $e(x, y)$ will be called the *multiplicity* of the pair xy .

With each p -multigraph one can associate a corresponding *weighted graph* (V, e) given by the set of vertices $V = V(\vec{G})$ and the multiplicity function $(x, y) \mapsto e(x, y)$. There is a rich literature on extremal problems in weighted graphs and the topic is studied both for its own sake (see e.g. [5, 16]) and due to its applicability to other parts of extremal combinatorics, such as Turán's hypergraph problem and the Ramsey-Turán theory of graphs (see e.g. [3, 10, 11]).

The main difference between multigraphs and weighted graphs is that the former do also keep track of the sets $M(x, y) \subseteq [p]$ containing for every pair xy of vertices those indices $i \in [p]$ for which xy is an edge of G_i . Therefore, there is a richer variety of extremal questions that can be asked in the setting of multigraphs. The following such problem is closely tied to the Turán number of the Fano plane.

Definition 3.2. For $p \geq 3$ a p -multigraph $\vec{G} = (G_1, \dots, G_p)$ is said to contain *three crossing pairs* (see Figure 3.1) if there are three distinct indices $i, j, k \in [p]$ and four distinct vertices $w, x, y, z \in V(\vec{G})$ such that

- $wx, yz \in E(G_i)$;
- $wy, xz \in E(G_j)$;
- and $wz, xy \in E(G_k)$.

The maximum total number of edges that a p -multigraph on n vertices can have without containing three crossing pairs is denoted by $f_p(n)$.

The function $f_4(\cdot)$ was determined by Füredi and Simonovits in [6, Theorem 2.2]. Their result plays an important rôle in the proof of our main result and reads as follows.

Theorem 3.3. *For every $n \geq 4$ one has*

$$f_4(n) = 2 \binom{n}{2} + 2 \left\lfloor \frac{n^2}{4} \right\rfloor.$$

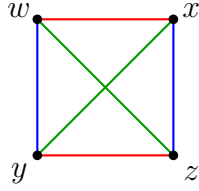


FIGURE 3.1. Three crossing pairs in (G_i, G_j, G_k) .

We would like to mention that Füredi and Simonovits also obtained a characterisation of the extremal configurations on $n \geq 8$ vertices (see Figure 3.2). Namely, if $\vec{G} = (G_1, G_2, G_3, G_4)$ denotes a 4-multigraph on at least 8 vertices with $f_4(n)$ edges that does not contain three crossing pairs, then there are a partition $V(\vec{G}) = X \cup Y$ and a permutation $\pi \in S_4$ such that

- $|X| = \lfloor \frac{n}{2} \rfloor$, $|Y| = \lfloor \frac{n+1}{2} \rfloor$,
- $E(G_{\pi(1)}) = E(G_{\pi(2)}) = X^{(2)} \cup K(X, Y)$,
- and $E(G_{\pi(3)}) = E(G_{\pi(4)}) = Y^{(2)} \cup K(X, Y)$,

where $K(X, Y)$ denotes the collection of all pairs xy with $x \in X$ and $y \in Y$.

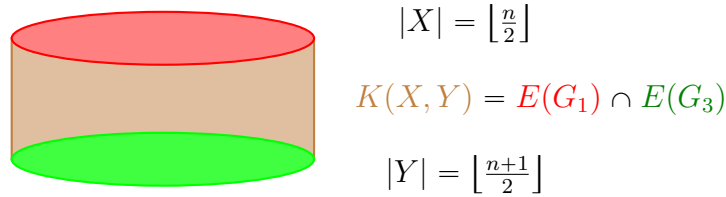


FIGURE 3.2. An extremal 4-multigraph (G_1, G_2, G_3, G_4) with $\pi = \text{id}$, $G_1 = G_2$, and $G_3 = G_4$.

It can be shown that this characterisation of the extremal configurations extends to the case $n = 7$ as well, but for $n \in \{4, 5, 6\}$ further extremal multigraphs are mentioned in [6].

* * *

The remainder of this section deals with the function $f_5(\cdot)$. Two instructive examples of 5-multigraphs without three crossing pairs are the following.

- Let $G_1 = G_2 = G_3 = G_4 = G_5$ be a K_4 -free Turán graph on n vertices. Notice that this 5-multigraph has $5 \lfloor \frac{n^2}{3} \rfloor$ edges.
- Let $\vec{G}_* = (G_1, G_2, G_3, G_4)$ be an extremal 4-multigraph without three crossing pairs with vertex partition $V(\vec{G}_*) = X \cup Y$ as described earlier, take G_5 to be the complete bipartite graph between X and Y and consider $\vec{G} = (G_1, \dots, G_5)$. Clearly the 5-multigraph \vec{G} does not contain three crossing pairs either and its number of edges is $2 \binom{n}{2} + 3 \lfloor \frac{n^2}{4} \rfloor$.

These examples demonstrate

$$f_5(n) \geq \max \left(5 \left\lfloor \frac{n^2}{3} \right\rfloor, 2 \binom{n}{2} + 3 \left\lfloor \frac{n^2}{4} \right\rfloor \right) \quad (3.1)$$

and, as a matter of fact, one can show that equality holds for every n . The only proof of this statement, which is known to us, is, however, quite laborious and relies on extensive case distinctions. For this reason we will state and prove below a weaker result on $f_5(\cdot)$ which still suffices for the application we have in mind.

Proposition 3.4. *We have $f_5(n) \leq \frac{1}{4}(7n^2 - n)$ for every natural number $n \geq 3$.*

Before we turn to the proof of this fact we take a closer look at the case $n = 4$.

Lemma 3.5. *Let $\vec{G} = (G_1, \dots, G_5)$ be a 5-multigraph on four vertices not containing three crossing pairs and set $e = e(\vec{G})$.*

(i) *If $e \geq 23$, then there exists an enumeration $V(\vec{G}) = \{w, x, y, z\}$ such that*

$$e(w, x) + e(y, z) \leq 5.$$

(ii) *If $e \geq 22$, then there exist two distinct vertices u and v with $e(u, v) = 5$.*

Proof. Write $V(\vec{G}) = \{w, x, y, z\}$ and define $a = e(w, x) + e(y, z)$, $b = e(w, y) + e(x, z)$, as well as $c = e(w, z) + e(x, y)$ to be the sums of the multiplicities of the three pairs of disjoint edges. By symmetry we may suppose that the enumeration of $V(\vec{G})$ we started with has been chosen in such a way that $a \leq b \leq c$ holds.

Now suppose for the sake of contradiction that

$$a \geq 6, \quad b \geq 7, \quad \text{and} \quad c \geq 8. \quad (\star)$$

Due to $a \geq 6$ there is an index $i \in [5]$ such that wx and yz are edges of G_i . Similarly, $b \geq 7$ implies that there are at least two indices $j \in [5]$ with the property that wy and xz are edges of G_j and, hence, at least one of them is distinct from i . Proceeding in the same way with $c \geq 8$ one finds an index $k \neq i, j$ for which wz and xy are edges of G_k . We have thereby found three crossing pairs in (G_i, G_j, G_k) and this contradiction proves that (\star) is indeed false.

Now part (i) of the lemma follows from the observation that $a + b + c = e \geq 23$ and $10 \geq c \geq b \geq a$ entail $c \geq 8$ and $b \geq 7$. So the failure of (\star) yields $a \leq 5$, as desired.

For the proof of part (ii) we notice that $a + b + c = e \geq 22$ and $c \geq b \geq a$ still imply $c \geq 8$. The falsity of (\star) shows that at least one of the estimates $a \leq 5$ or $b \leq 6$ holds. In both cases we obtain $c \geq 9$, meaning that at least one of the two pairs wz or xy has multiplicity 5. \square

Proof of Proposition 3.4. The trivial bound $f_5(3) \leq 5 \binom{3}{2} = 15$ shows that our claim holds for $n = 3$. Next, an easy averaging argument yields $f_5(n) \leq \frac{5}{4} f_4(n)$ for every natural

number n . Due to Theorem 3.3 and (3.1) this gives the exact values

$$f_5(4) = 25 \quad \text{and} \quad f_5(5) = 40, \quad (3.2)$$

which establish the desired estimate for $n \in \{4, 5\}$. Arguing indirectly we now let $n \geq 6$ denote the least integer for which there exists a 5-multigraph $\vec{G} = (G_1, \dots, G_5)$ on n vertices with more than $\frac{1}{4}(7n^2 - n)$ edges that does not contain three crossing pairs. For every set $X \subseteq V = V(\vec{G})$ of vertices we shall write $e^+(X) = e(\vec{G}) - e(V \setminus X)$ for the total number of edges having at least one endvertex in X . As long as $0 < |X| \leq n - 3$ the minimality of n yields

$$e(V \setminus X) \leq \frac{1}{4}(7(n - |X|)^2 - (n - |X|)),$$

whence

$$e^+(X) > \frac{1}{4}(14n|X| - 7|X|^2 - |X|).$$

In particular, we obtain

$$e^+(X) \geq \begin{cases} \frac{1}{2}(7n - 3) & \text{if } |X| = 1, \\ 7n - 7 & \text{if } |X| = 2, \\ \frac{21}{2}n - 16 & \text{if } |X| = 3. \end{cases} \quad (3.3)$$

Owing to $e(\vec{G}) > \frac{7}{2}\binom{n}{2}$ the average edge multiplicity in \vec{G} is greater than $\frac{7}{2}$. Therefore, there exist a set $Q \subseteq V$ consisting of four vertices with $e(Q) > 6 \cdot \frac{7}{2} = 21$, and by Lemma 3.5(ii) it follows that there are two distinct vertices x and y with $e(x, y) = 5$.

According to (3.3) we have

$$\sum_{z \in V \setminus \{x, y\}} (e(x, z) + e(y, z)) = e^+(x, y) - 5 \geq 7n - 12 > 7(n - 2).$$

Consequently, there exists a vertex z distinct from x and y with $e(x, z) + e(y, z) \geq 8$.

Altogether we have thereby shown that there exist triples (x_*, y_*, z_*) of distinct vertices with

$$e(x_*, y_*) = 5 \quad \text{and} \quad e(x_*, z_*) + e(y_*, z_*) \geq 8$$

and for the remainder of the proof we fix one such triple with the additional property that $e(x_*, z_*) + e(y_*, z_*) \geq 8$ is maximal. Set $\alpha = e(x_*, z_*)$ as well as $\beta = e(y_*, z_*)$ and observe that we may suppose $\alpha \geq \beta$ for reasons of symmetry. Clearly (α, β) is one of the four ordered pairs $(5, 5)$, $(5, 4)$, $(5, 3)$, or $(4, 4)$.

Because of

$$\begin{aligned} \sum_{v \in V \setminus \{x_*, y_*, z_*\}} (e(v, x_*) + e(v, y_*) + e(v, z_*)) &= e^+(x_*, y_*, z_*) - (5 + \alpha + \beta) \\ &\stackrel{(3.3)}{\geq} \left(\frac{21}{2}n - 16\right) - 15 > 10(n - 3) \end{aligned} \quad (3.4)$$

there exists a vertex $v_* \neq x_*, y_*, z_*$ satisfying

$$e(v_*, x_*) + e(v_*, y_*) + e(v_*, z_*) \geq 11. \quad (3.5)$$

By applying the left part of (3.2) to the quadruple $\{v_*, x_*, y_*, z_*\}$ we learn

$$\alpha + \beta \leq 25 - 11 - 5 = 9,$$

meaning that the pair (α, β) cannot be $(5, 5)$.

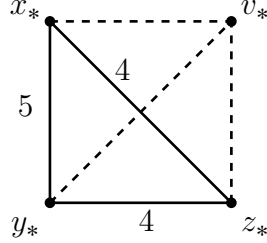


FIGURE 3.3. The case $\alpha = \beta = 4$.

Assume next that $\alpha = \beta = 4$ (see Figure 3.3), which yields $e(v_*, x_*, y_*, z_*) \geq 13 + 11 = 24$. Due to Lemma 3.5(i) it follows that either $e(v_*, x_*) \leq 1$, $e(v_*, y_*) \leq 1$, or $e(v_*, z_*) = 0$. The last alternative would contradict (3.5), so by symmetry we may suppose that $e(v_*, y_*) \leq 1$. Invoking (3.5) once more we infer that $e(v_*, x_*) = e(v_*, z_*) = 5$. But now the edges of the triangle (v_*, x_*, z_*) have multiplicities 5, 5, and 4, contrary to the maximal choice of $\alpha + \beta$. We have thereby proved that $(\alpha, \beta) \neq (4, 4)$.

For these reasons, it must be the case $\alpha = 5$ and $\beta \in \{3, 4\}$ (see Figure 3.4). Adding

$$\sum_{v \in V \setminus \{x_*, y_*, z_*\}} e(v, x_*) = e^+(x_*) - 10 \stackrel{(3.3)}{\geq} \frac{7}{2}n - \frac{23}{2}$$

to (3.4) we infer

$$\begin{aligned} \sum_{v \in V \setminus \{x_*, y_*, z_*\}} (2e(v, x_*) + e(v, y_*) + e(v, z_*)) &\geq \left(\frac{21}{2}n - 16 - 14\right) + \left(\frac{7}{2}n - \frac{23}{2}\right) \\ &> 14(n - 3), \end{aligned}$$

which shows that there exists a vertex $w_* \neq x_*, y_*, z_*$ such that

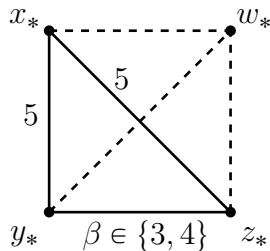
$$2e(w_*, x_*) + e(w_*, y_*) + e(w_*, z_*) \geq 15. \quad (3.6)$$

In particular, we have $e(w_*, x_*) + e(w_*, y_*) + e(w_*, z_*) \geq 10$ and, consequently,

$$e(w_*, x_*, y_*, z_*) \geq 13 + 10 = 23.$$

Appealing to Lemma 3.5(i) again we deduce that at least one of the three cases

$$e(w_*, x_*) \leq 2, \quad e(w_*, y_*) = 0, \quad \text{or} \quad e(w_*, z_*) = 0$$

FIGURE 3.4. The case $\alpha = 5$ and $\beta < 5$.

occurs. The first of them is incompatible with (3.6), so by symmetry we may suppose that $e(w_*, y_*) = 0$. In combination with (3.6) this yields $e(w_*, x_*) = e(w_*, z_*) = 5$. But now the triangle (w_*, x_*, z_*) contradicts the supposed maximality of $\alpha + \beta$. \square

Let us finally summarise the properties of f_5 that shall be utilised in the next section.

Conclusion 3.6. *If $n \geq 9$ is odd, then*

- (a) $b(n - 5) + f_5(n - 5) + 7(n - 5) + 10 < b(n)$ and
- (b) $(b(n - 6) + \frac{1}{2}(n - 9)) + f_5(n - 6) + \binom{n-6}{2} + 10(n - 6) + 20 < b(n)$.

Proof. Due to the explicit formula (1.2) for $b(n)$ and the estimate on $f_5(n - 5)$ provided by Proposition 3.4, part (a) is a consequence of $1 < \frac{1}{8}(n - 5)^2$, which is trivially valid. Similarly, part (b) reduces to $0 < \frac{1}{4}(3n - 23)$, which is likewise obvious. \square

§4. PROOF OF THE MAIN THEOREM

This entire section is dedicated to the proof of Theorem 1.1, which proceeds by induction on n . Since the base case $n = 7$ was already treated in Lemma 2.2, we may suppose that $n \geq 8$ and that (1.1) as well as our statement addressing the extremal hypergraphs hold for every $n' \in [7, n)$ in place of n . Now let $H = (V, E)$ be a hypergraph on $|V| = n$ vertices with $|E| = b(n)$ edges that does not contain a Fano plane. We are to prove that $H \cong B_n$. Let us distinguish two cases according to the parity of n .

First Case: $n \geq 8$ is even.

For every vertex $v \in V$ we have $e(H \setminus v) \leq b(n - 1)$, since otherwise the induction hypothesis would yield a Fano plane in $H \setminus v$. This shows that

$$d(v) \geq b(n) - b(n - 1) = 3 \binom{n/2}{2} = \frac{3|E|}{n}$$

holds for every $v \in V$. Due to $\sum_{v \in V} d(v) = 3|E|$ this is only possible if every vertex has degree $3 \binom{n/2}{2}$. But now it follows that for every $v \in V$ the hypergraph $H \setminus v$ has exactly $b(n - 1)$ edges. So if $n \geq 10$ the induction hypothesis informs us that the assumption of Lemma 2.5 is satisfied, meaning that H is indeed isomorphic to B_n . In the remaining

case $n = 8$ the same conclusion can still be drawn unless there is a vertex $v \in V$ with $(H \setminus v) \cong J_7$. But this would entail $K_6^{(3)} \subseteq J_7 \subseteq H$ and Fact 2.4 would show $|E| < b(8)$, which is absurd.

Second Case: $n \geq 9$ is odd.

For $K \subseteq V$ and $i \in \{0, 1, 2, 3\}$ let $e_i(K)$ denote the number of edges of H with exactly i vertices in K . Clearly, we have

$$e_0(K) + e_1(K) + e_2(K) + e_3(K) = |E| = b(n) \quad (4.1)$$

for every $K \subseteq V$.

We need to know later that H cannot contain a clique on five vertices and the claim that follows prepares the proof of this fact.

Claim 4.1. *If some six vertices of H span at least 18 edges, then they induce a copy of B_6 .*

Proof. Let $K = \{v_1, \dots, v_6\} \subseteq V$ span at least 18 edges of H and suppose for the sake of contradiction that the subhypergraph of H induced by K is not isomorphic to B_6 . This hypergraph must then contain a clique on five vertices and there arises no loss of generality by assuming that $\{v_1, \dots, v_5\}$ induces a $K_5^{(3)}$ in H .

For $n \geq 13$ we have $n - 6 \geq 7$ and the induction hypothesis yields, in particular,

$$e_0(K) \leq b(n - 6) + \frac{1}{2}(n - 9), \quad (4.2)$$

where the additional term $\frac{1}{2}(n - 9)$ is actually not needed. The reason for including it here is that for $n \in \{9, 11\}$ it makes the right side equal to the trivial upper bound $\binom{n-6}{3}$. Therefore (4.2) holds in all possible cases.

Now consider the 6-multigraph $\vec{G} = (G_1, \dots, G_6)$ with vertex set $V \setminus K$, where for $j \in [6]$ the edges of G_j are inherited from the link graph of v_j . Since $\{v_1, \dots, v_5\}$ is a clique in H , three crossing pairs in (G_1, \dots, G_5) would give rise to a Fano plane in H . Hence $e(G_1) + \dots + e(G_5) \leq f_5(n - 6)$ and together with the trivial bound $e(G_6) \leq \binom{n-6}{2}$ we obtain

$$e_1(K) = e(\vec{G}) \leq f_5(n - 6) + \binom{n - 6}{2}. \quad (4.3)$$

Moreover, Lemma 2.3 shows that every vertex $v \in V \setminus K$ can contribute at most 10 edges to $e_2(K)$, wherefore

$$e_2(K) \leq 10(n - 6). \quad (4.4)$$

By plugging (4.2), (4.3), (4.4), and the trivial upper bound $e_3(K) \leq \binom{6}{3} = 20$ into (4.1) we arrive at the estimate

$$b(n) \leq (b(n - 6) + \frac{1}{2}(n - 9)) + f_5(n - 6) + \binom{n - 6}{2} + 10(n - 6) + 20,$$

which contradicts Conclusion 3.6(b). This proves Claim 4.1. \square

Claim 4.2. $K_5^{(3)} \not\subseteq H$

Proof. Assume contrariwise that $K = \{v_1, \dots, v_5\} \subseteq V$ induces a $K_5^{(3)}$ in H . We contend that $e_0(K) \leq b(n-5)$. For $n \geq 13$ this follows indeed from the induction hypothesis, for $n = 9$ we just need to appeal to the trivial bound $e_0(K) \leq \binom{n-5}{3} = 4 = b(n-5)$ and for $n = 11$ the desired estimate holds in view of Claim 4.1.

A link multigraph argument similar to the one encountered in the foregoing proof of (4.3) shows that $e_1(K) \leq f_5(n-5)$. Owing to Claim 4.1 every vertex in $V \setminus K$ belongs to at most 7 edges contributing to $e_2(K)$, whence $e_2(K) \leq 7(n-5)$. Combining all these estimates and $e_3(K) = 10$ with (4.1) we learn

$$b(n) \leq b(n-5) + f_5(n-5) + 7(n-5) + 10,$$

which, however, contradicts Conclusion 3.6(a). Thereby Claim 4.2 is proved. \square

In order to conclude the proof of our main result we will now show that H satisfies the assumptions of Lemma 2.6. Suppose to this end that $K \subseteq V$ induces a tetrahedron in H . For $n \geq 11$ the induction hypothesis gives $e_0(K) \leq b(n-4)$ and for $n = 9$ this estimate could only fail if $V \setminus K$ induces a $K_5^{(3)}$ in H , which would contradict Claim 4.2. Thus we obtain

$$b(n) \leq b(n-4) + f_4(n-4) + 5(n-4) + 4$$

in the usual manner, where the factor 5 in front of $(n-4)$ comes from the absence of 5-cliques in H . A short calculation exploiting Theorem 3.3 reveals that this estimate does actually hold with equality. In particular, this yields

$$e_0(K) = b(n-4) \tag{4.5}$$

and

$$e_2(K) = 5(n-4).$$

The latter equation proves immediately that K obeys clause (ii) from Lemma 2.6. It remains to check that, similarly, (4.5) leads to (i), i.e., to $(H \setminus K) \cong B_{n-4}$. For $n \geq 13$ this is indeed true due to the induction hypotheses. For $n = 11$ we need to point out additionally that $H \setminus K$ cannot be isomorphic to J_7 , as this hypergraph contains a copy of $K_5^{(3)}$, whilst H does not. Finally, for $n = 9$ the desired statement is a simple consequence of the fact that $B_5 \cong K_5^{(3)-}$ is the only hypergraph on 5 vertices with $b(5) = 9$ edges. This concludes the proof of our main result.

ACKNOWLEDGEMENT

We would like to thank Miklós Simonovits for sending us a copy of [18] and Zoltán Füredi [4] for further information regarding the history of the problem.

REFERENCES

- [1] D. De Caen and Z. Füredi, *The maximum size of 3-uniform hypergraphs not containing a Fano plane*, J. Combin. Theory Ser. B **78** (2000), no. 2, 274–276, DOI [10.1006/jctb.1999.1938](https://doi.org/10.1006/jctb.1999.1938). MR1750899 ↑1, 2.2
- [2] P. Erdős, *Paul Turán, 1910–1976: his work in graph theory*, J. Graph Theory **1** (1977), no. 2, 97–101. MR0441657 (56 #61) ↑1
- [3] P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi, *More results on Ramsey-Turán type problems*, Combinatorica **3** (1983), no. 1, 69–81, DOI [10.1007/BF02579342](https://doi.org/10.1007/BF02579342). MR716422 ↑3
- [4] Z. Füredi. Personal communication. ↑1, 4
- [5] Z. Füredi and A. Kündgen, *Turán problems for integer-weighted graphs*, J. Graph Theory **40** (2002), no. 4, 195–225, DOI [10.1002/jgt.10012](https://doi.org/10.1002/jgt.10012). MR1913847 ↑3
- [6] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Probab. Comput. **14** (2005), no. 4, 467–484, DOI [10.1017/S0963548305006784](https://doi.org/10.1017/S0963548305006784). MR2160414 ↑1, 2.2, 2.2, 3, 3, 3
- [7] G. Katona, T. Nemetz, and M. Simonovits, *On a problem of Turán in the theory of graphs*, Mat. Lapok **15** (1964), 228–238 (Hungarian, with Russian and English summaries). MR0172263 ↑1
- [8] P. Keevash and D. Mubayi, *The Turán number of $F_{3,3}$* , Combin. Probab. Comput. **21** (2012), no. 3, 451–456, DOI [10.1017/S0963548311000678](https://doi.org/10.1017/S0963548311000678). MR2912791 ↑1
- [9] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica **25** (2005), no. 5, 561–574, DOI [10.1007/s00493-005-0034-2](https://doi.org/10.1007/s00493-005-0034-2). MR2176425 ↑1, 2.2
- [10] C. M. Lüders and Chr. Reiher, *The Ramsey-Turán problem for cliques*, available at [arXiv:1709.03352](https://arxiv.org/abs/1709.03352). Submitted. ↑3
- [11] ———, *Weighted variants of the Andrásfai-Erdős-Sós Theorem*, available at [arXiv:1710.09652](https://arxiv.org/abs/1710.09652). Submitted. ↑3
- [12] D. Mubayi and V. Rödl, *On the Turán number of triple systems*, J. Combin. Theory Ser. A **100** (2002), no. 1, 136–152, DOI [10.1006/jcta.2002.3284](https://doi.org/10.1006/jcta.2002.3284). MR1932073 ↑1
- [13] M. Pasch, *Vorlesungen über neuere Geometrie*, Second edition, B. G. Teubner, Leipzig und Berlin, 1912 (German). ↑2.2
- [14] A. A. Razborov, *Flag algebras*, J. Symbolic Logic **72** (2007), no. 4, 1239–1282, DOI [10.2178/jsl/1203350785](https://doi.org/10.2178/jsl/1203350785). MR2371204 (2008j:03040) ↑1
- [15] ———, *On 3-hypergraphs with forbidden 4-vertex configurations*, SIAM J. Discrete Math. **24** (2010), no. 3, 946–963, DOI [10.1137/090747476](https://doi.org/10.1137/090747476). MR2680226 (2011k:05171) ↑1
- [16] V. Rödl and A. Sidorenko, *On the jumping constant conjecture for multigraphs*, J. Combin. Theory Ser. A **69** (1995), no. 2, 347–357, DOI [10.1016/0097-3165\(95\)90057-8](https://doi.org/10.1016/0097-3165(95)90057-8). MR1313901 ↑3
- [17] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319. MR0233735 ↑1
- [18] V. T. Sós, *Remarks on the connection of graph theory, finite geometry and block designs*, Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Accad. Naz. Lincei, Rome, 1976, pp. 223–233. Atti dei Convegni Lincei, No. 17 (English, with Italian summary). MR0543051 ↑1, 4
- [19] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452 (Hungarian, with German summary). MR0018405 ↑1

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, HAMBURG, GERMANY

E-mail address: Christian.Reiher@uni-hamburg.de

E-mail address: louisnbell@aol.com