SQUARES OF HAMILTONIAN CYCLES IN 3-UNIFORM HYPERGRAPHS

WIEBKE BEDENKNECHT, CHRISTIAN REIHER

ABSTRACT. We show that every 3-uniform hypergraph $H = (V, E)$ with $|V(H)| = n$ and minimum pair degree at least $(4/5 + o(1))n$ contains a squared Hamiltonian cycle. This may be regarded as a first step towards a hypergraph version of the Pósa-Seymour conjecture.

§1. INTRODUCTION

G. A. Dirac [3] proved in 1952 that every graph $G = (V, E)$ with $|V| \geqslant 3$ and minimum vertex degree $\delta(G) \geqslant |V|/2$ contains a Hamiltonian cycle. Since on any set V of at least three vertices there are graphs G with minimum degree $\delta(G) = \lceil |V|/2 \rceil - 1$, which do not contain a Hamiltonian cycle, this is an optimal result. Moreover, in 1962 Pósa conjectured that every graph $G = (V, E)$ with $|V| \geqslant 5$ and minimum degree $\delta(G) \geqslant 2|V|/3$ contains the square of a Hamiltonian cycle. This conjecture was generalised further by Seymour [14] to the so-called *Pósa-Seymour conjecture*, asking for the k-th power of a Hamiltonian cycle in graphs G with $\delta(G) \geqslant \frac{k}{k+1}|V|$.

A proof of this generalised conjecture for large graphs was obtained by Komlós, Sárközy, and Szemerédi [7]. Their proof is based on the regularity method for graphs and uses the so-called blow-up lemma [6] that was developed by the same authors shortly before. We study an analogous Pósa-type problem for 3-uniform hypergraphs, i.e., what minimum pair-degree condition guarantees the existence of a squared Hamiltonian cycle?

A 3-*uniform hypergraph* $H = (V, E)$ consists of a finite set $V = V(H)$ of vertices and a family $E = E(H)$ of 3-element subsets of V, which are called *(hyper)edges*. Throughout this article if we talk about hypergraphs we will always mean 3-uniform hypergraphs. We will write xy and xyz instead of $\{x, y\}$ and $\{x, y, z\}$ for edges and hyperedges. Similarly, we shall say that $wxyz$ is a *tetrahedron* or a $K_4^{(3)}$ in a hypergraph H if the triples wxy, wxz, wyz, and xyz are edges of H.

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There are at least two concepts of minimum degree and several notions of cycles like tight, loose and Berge cycles [1] (see also [2]). Here we will only introduce some of these notions.

If $H = (V, E)$ is a hypergraph and $v \in V$ is a vertex of H, then we denote by

$$d_H(v) = |\{e \in E : v \in e\}|$$

the *degree of* v and by

$$\delta_1(H) = \min\{d_H(v) : v \in V\}$$

the *minimum vertex degree* of H taken over all $v \in V$.

Similarly, for two vertices $u, v \in V$ we denote by

$$d_H(u, v) = |N_H(u, v)| = |\{e \in E : u, v \in e\}|$$

the *pair-degree of* u *and* v and by

$$\delta_2(H) = \min\{d_H(u, v) : uv \in V^{(2)}\}$$

the *minimum pair-degree* of H taken over all pairs of vertices of H.

We call a hypergraph P a *tight path of length* ℓ, if $|V(P)| = \ell + 2$ and there exists an ordering of the vertices $V(P) = \{v_1, \ldots, v_{\ell+2}\}$ such that a triple e forms a hyperedge of P iff $e = \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in [\ell]$. A *tight cycle* C *of length* $\ell \geqslant 4$ consists of a path $v_1 \ldots v_\ell$ of length $\ell - 2$ and the additional hyperedges $\{v_{\ell-1}, v_\ell, v_1\}$ and $\{v_\ell, v_1, v_2\}$.

Moreover, we call a hypergraph P' a *squared path of length* $\ell \geqslant 2$, if $|V(P')| = \ell + 2$ and there exists an ordering of the vertices $V(P') = \{v_1, \ldots, v_{\ell+2}\}$ such that a triple e forms a hyperedge iff $e \subseteq \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for some $i \in [\ell - 1]$. Similarly, a *squared cycle* C' *of length* $\ell \geqslant 5$ consists of a squared path $v_1 \ldots v_\ell$ of length $\ell - 2$ and the additional hyperedges e, which are 3-subsets of at least one of the sets $\{v_{\ell-2}, v_{\ell-1}, v_\ell, v_1\}$, $\{v_{\ell-1}, v_\ell, v_1, v_2\}$ or $\{v_\ell, v_1, v_2, v_3\}$.

Thus an n-vertex hypergraph H contains a *spanning* squared cycle if its vertices can be arranged on a circle in such a way that every triple of vertices contained in an interval of length 4 is an edge of H. Such spanning squared cycles will be called *squared Hamiltonian cycles* in this article. Clearly this is a natural analogue of the concept of squared Hamiltonian cycles in graphs, where any pair contained in an interval of length 3 is required to be an edge.

The first asymptotically optimal Dirac-type result for 3-uniform hypergraphs was obtained by Rödl, Ruciński, and Szemerédi, who proved in [11] that every n-vertex hypergraph H with $\delta_2(H) \geqslant \left(\frac{1}{2} + o(1)\right)n$ contains a Hamiltonian cycle. In [12] they showed this for large n under the optimal assumption $\delta_2(H) \geqslant \lfloor n/2 \rfloor$. Moreover, it was proved in [10] that a minimum vertex degree condition of $\delta_1(H) \geqslant \left(\frac{5}{9} + o(1)\right)\frac{n^2}{2}$ guarantees the existence of a

Hamiltonian cycle as well, where the constant 5/9 is again best possible. We will study which pair-degree condition implies a squared Hamiltonian cycle in 3-uniform hypergraphs and we will prove the following theorem.

Theorem 1.1. *For every $\alpha > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H with $n \geqslant n_0$ vertices and with minimum pair-degree $\delta_2(H) \geqslant (\frac{4}{5} + \alpha)n$ contains a squared Hamiltonian cycle.*

We will denote by $K_4^{(3)}$ the complete 3-uniform hypergraph on 4 vertices. Note that any four consecutive vertices in a squared Hamiltonian cycle span a copy of $K_4^{(3)}$. Therefore, if n is divisible by 4, a squared Hamiltonian cycle contains a $K_4^{(3)}$*-tiling*, i.e., $\frac{n}{4}$ vertex disjoint copies of $K_4^{(3)}$. The problem to enforce $K_4^{(3)}$-tilings by an appropriate pair-degree condition was studied by Pikhurko [9], who exhibited for every n divisible by 4 a hypergraph H on n vertices with $\delta_2(H) = \frac{3}{4}n - 3$ not containing a $K_4^{(3)}$-tiling. Moreover, he proved that every n-vertex hypergraph H with $\delta_2(H) \geqslant \big(\frac{3}{4} + o(1)\big)n$ contains vertex-disjoint copies of $K_4^{(3)}$ covering all but at most 14 vertices. We remark that based on Pikhurko's work [9] the pair-degree problem for $K_4^{(3)}$-tilings was solved by Keevash and Mycroft in [5]. They showed that all 3-uniform hypergraphs H of sufficiently large order n with $4 \mid n$ and minimum pair-degree

$$\delta_2(H) \geqslant \begin{cases} 3n/4 - 2 & \text{if } 8 \mid n, \\ 3n/4 - 1 & \text{otherwise} \end{cases}$$

contain a perfect $K_4^{(3)}$-tiling.

Notice that in view of Pikhurko's example the constant $\frac{4}{5}$ occurring in Theorem 1.1 cannot be replaced by anything below $\frac{3}{4}$ in case $4 \mid n$. In order to extend this observation to all congruence classes modulo 4 we take a closer look at the construction from [9]. Partition the vertex set $V = A_0 \dot\cup A_1 \dot\cup A_2 \dot\cup A_3$ such that $\big||A_i| - |A_j|\big| \leqslant 1$ for $0 \leqslant i < j \leqslant 3$. Let H be the hypergraph consisting of all the triples e that satisfy one of the following properties (see Fig. 1.1):

- $|A_0 \cap e| = 2$;
- e intersects each of A_0, A_i, A_j for some $1 \leqslant i < j \leqslant 3$;
- $e \subseteq A_i$ for some $i \in [3]$;
- $|e \cap A_i| = 1$ and $|e \cap A_j| = 2$ for some pair $ij \in [3]^{(2)}$.

Every $K_4^{(3)}$ intersecting A_0 has exactly 2 vertices in A_0, since A_0 spans no edge and if a $K_4^{(3)}$ would intersect A_0 in only one vertex, then its remaining three vertices must come from A_1, A_2, A_3 (one from each set), but three such vertices do not form an edge in H. A squared Hamiltonian cycle $C \subseteq H$ needs to contain at least one $K_4^{(3)}$ that intersects A_0,

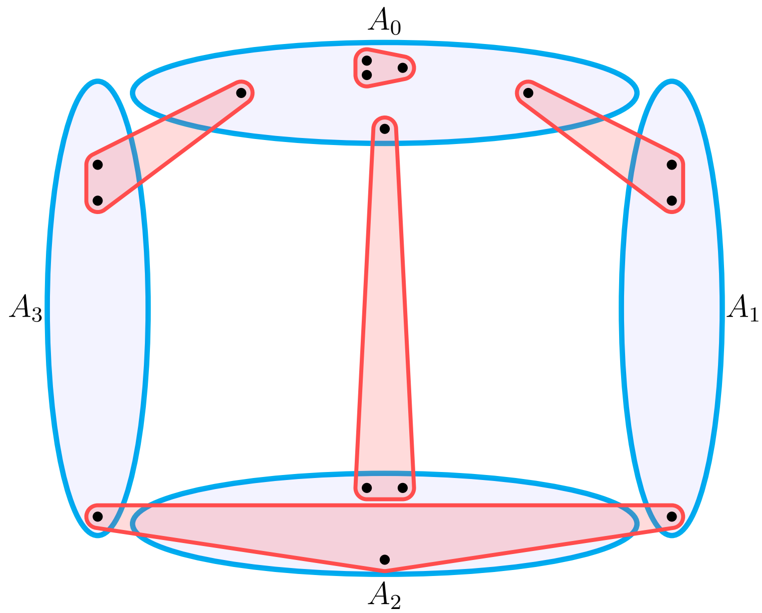

FIGURE 1.1. Complement of the hypergraph H, where the existing kinds of edges are indicated in red, e.g. all triples with 3 vertices in A_0 form an edge in the complement of H.

but then each $K_4^{(3)} \subseteq C$ needs to intersect A_0 in two vertices. This implies $|A_0| \geqslant n/2$, which contradicts our assumption and shows that H is indeed not containing a squared Hamiltonian cycle.

The proof of Theorem 1.1 is based on the *absorption method* developed by Rödl, Ruciński, and Szemerédi in [12]. In Section 2 we will discuss the general structure of the proof.

§2. Building squared Hamiltonian Cycles in Hypergraphs

In this section we will show the outline of the proof of Theorem 1.1. We start by presenting the dependencies of the auxiliary constants we use in the propositions required for the proof of Theorem 1.1. We write $a \gg b$ to indicate that b will be chosen sufficiently small depending on a and all other constants appearing on the left of b. In Theorem 1.1 some α with $1 \gg \alpha > 0$ is given. We fix the auxiliary constants ϑ_* and an integer $M \in \mathbb{N}$, such that

$$1 \gg \alpha \gg 1/M \gg \vartheta_* \gg 1/n\,.$$

The connecting lemma stated below plays a crucial rôle in the proof of Theorem 1.1. It asserts that any two disjoint triples of vertices can be connected by many "short" squared paths.

Proposition 2.1 (Connecting Lemma)**.** *There are an integer M and $\vartheta_* > 0$, such that for all sufficiently large hypergraphs $H = (V, E)$ with $\delta_2(H) \geqslant (4/5 + \alpha)|V|$ and all disjoint triples (a, b, c) and (x, y, z) with $abc, xyz \in E$ there exists some $m < M$ for which there are at least $\vartheta_* n^m$ squared paths from abc to xyz with m internal vertices.*

The proof of the connecting lemma forms the content of Section 3. We can connect any two squared paths by the connecting lemma using their start or endtriples, but for our constructions it will be important that we do not interfere with any already constructed subpath. Therefore we put a small *reservoir* of vertices aside, such that if we do not connect too many times it is possible to use vertices of the reservoir set only. The following lemma, which we prove in Section 4, shows the existence of such a set.

Proposition 2.2 (Reservoir Lemma)**.** *Suppose that for a given $\alpha > 0$ the constants $1/M \gg \vartheta_*$ are as provided by the connecting lemma and that $H = (V, E)$ is a sufficiently large hypergraph with $|V| = n$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$. Then there exists a reservoir set $\mathcal{R} \subseteq V$ of size $|\mathcal{R}| \leqslant \vartheta_*^2 n$ such that for all $\mathcal{R}' \subseteq \mathcal{R}$ with $|\mathcal{R}'| \leqslant \vartheta_*^4 n$ and for all disjoint triples (a, b, c) and (x, y, z) with $abc, xyz \in E$ there exists a connecting squared path in H with less than M internal vertices all of which belong to $\mathcal{R} \smallsetminus \mathcal{R}'$.*

Moreover, we put aside an absorbing path P_A, which will absorb an arbitrary but not too large set X of leftover vertices at the end of the proof, such that we get a squared Hamiltonian cycle.

Proposition 2.3 (Absorbing path)**.** *Let $\alpha \gg 1/M \gg \vartheta_*$ be as usual and let $H = (V, E)$ be a sufficiently large hypergraph with $|V| = n$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$. There exists an (absorbing) squared path $P_A \subseteq H - \mathcal{R}$ such that*

(1) $|V(P_A)| \leqslant \vartheta_* n$,
(2) for every set $X \subseteq V \smallsetminus V(P_A)$ with $|X| \leqslant 2\vartheta_*^2 n$ there is a squared path in H whose set of vertices is $V(P_A) \cup X$ and whose end-triples are the same as those of P_A.

In Section 5 we prove Proposition 2.3 and in Section 6 we will show the following theorem.

Theorem 2.4. *Given $\alpha, \mu > 0$ and $Q \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ all but at most μn vertices of H can be covered by vertex-disjoint squared paths with Q vertices.*

Also in Section 6 we use this theorem to prove the existence of an almost spanning squared cycle that covers all but at most $2\vartheta_*^2 n$ vertices.

Proposition 2.5. *Given $\alpha > 0$ let $\vartheta_* > 0$ and $M \in \mathbb{N}$ be the constants from the connecting lemma. There exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$ all but at most $2\vartheta_*^2 n$ vertices of H can be covered by a squared cycle such that some absorbing squared path P_A is an induced subgraph of this cycle.*

Combining Proposition 2.3 and Proposition 2.5 implies the existence of a squared Hamiltonian cycle and therefore proves Theorem 1.1.

§3. Connecting Lemma

We will show some of our results with the constant $\frac{3}{4}$ and others for $\frac{4}{5}$. Moreover we fix the auxiliary constants $\beta, \gamma, \vartheta_*$ and integers $K, \ell, M \in \mathbb{N}$ obeying the hierarchy

$$1 \gg \alpha \gg \beta, \gamma, 1/\ell \gg 1/K \gg 1/M \gg \vartheta_* \gg 1/n\,.$$

3.1. **Connecting properties.** We prove that the graph properties stated in the following lemma imply a connecting property and use this lemma later to show that some auxiliary graphs G_3 and G_v have this connecting property.

Lemma 3.1. *Let $\gamma \leqslant 1/16$ and let $G = (V, E)$ with $|V| = n$ be a graph with $\delta(G) \geqslant \sqrt{\gamma}n$ such that for every partition $X \cup Y = V$ of the vertex set with $|X|, |Y| \geqslant \sqrt{\gamma}n$ we have $e_G(X, Y) \geqslant \gamma n^2$.*

Then for every pair of distinct vertices $x, y \in V(G)$ there exists some $s = s(x, y) \leqslant 4/\gamma$ for which there are at least $\Omega(n^{s-1})$ many x-y-walks of length s.

Proof. For an arbitrary vertex $x \in V$ and an integer $i \geqslant 1$ we define

$$Z_x^i = \{z \in V\colon \text{ there are at least } (\gamma^2/4)^s n^{s-1} \ x\text{-}z\text{-walks of length } s \text{ in } G \text{ for some } s \leqslant i\}\,.$$

For $i \geqslant 2$ we have $Z_x^i \supseteq Z_x^{i-1}$ and therefore

$$|Z_x^i| \geqslant |Z_x^1| = |N_G(x)| \geqslant \delta(G) \geqslant \sqrt{\gamma}n\,.$$

Now we show that for every integer i with $1 \leqslant i \leqslant 2/\gamma$ at least one of the following holds:

$$|V \smallsetminus Z_x^i| < \sqrt{\gamma}n \qquad \text{or} \qquad |Z_x^{i+1} \smallsetminus Z_x^i| \geqslant \frac{\gamma n}{2}\,. \tag{3.1}$$

If $|V \smallsetminus Z_x^i| \geqslant \sqrt{\gamma}n$, then the assumption yields that

$$e_G(Z_x^i, V \smallsetminus Z_x^i) \geqslant \gamma n^2\,.$$

This implies that at least $\gamma n/2$ vertices in $V \smallsetminus Z_x^i$ have at least $\gamma n/2$ neighbours in Z_x^i. For such a vertex $u \in V \smallsetminus Z_x^i$ at least a proportion of $1/i \geqslant \gamma/2$ of its neighbours in Z_x^i is connected to x by walks of the same length, which implies $u \in Z_x^{i+1}$. As this argument applies to $\gamma n/2$ vertices outside Z_x^i we thus obtain $|Z_x^{i+1} \smallsetminus Z_x^i| \geqslant \gamma n/2$, which concludes the proof of (3.1).

It is not possible that the right outcome of (3.1) holds for each positive $i \leqslant 2/\gamma$. Therefore we have $|V \smallsetminus Z_x^j| < \sqrt{\gamma} n$ for $j = \lfloor 2/\gamma \rfloor$. So for $x, y \in V$ at least $n - 2\sqrt{\gamma} n \geqslant n/2$ vertices z are contained in the intersection $Z_x^j \cap Z_y^j$. For each $z \in Z_x^j \cap Z_y^j$ we get constants $s_1, s_2 \leqslant j \leqslant 2/\gamma$ such that there are at least $(\gamma^2/4)^{s_1} n^{s_1 - 1}$ x-z-walks of length s_1 and there are at least $(\gamma^2/4)^{s_2} n^{s_2-1}$ z-y-walks of length s_2. Therefore, for $s_z = s_1 + s_2 \geqslant 2$ there are at least $(\gamma^2/4)^{s_z} n^{s_z - 2}$ x-y-walks of length s_z passing through z.

There are at least $n/2$ vertices this argument applies to and by the box principle at least $\frac{n}{2}/\frac{4}{\gamma^2}$ of them give rise to the same pair (s_1, s_2) and, consequently, the same value of s_z. Moreover, the walks obtained for those vertices are distinct and hence for some $s(x, y) \in [2, 4/\gamma]$ there are at least

$$(\gamma^2 n/8) \cdot (\gamma^2/4)^{s(x,y)} n^{s(x,y)-2} \geqslant \tfrac{1}{2}(\gamma^2/4)^{4/\gamma+1} n^{s(x,y)-1}$$

x-y-walks of length $s(x, y)$. □

3.2. **The auxiliary graph G_3.** The first auxiliary graph we will study is the following.

Definition 3.2. *For a 3-uniform hypergraph $H = (V, E)$ we define the auxiliary graph G_3 (see Fig. 3.1) as the graph with vertex set $V(G_3) = V$ and*

$$xy \in E(G_3) \iff x \neq y \text{ and } \#\{(a, b, c) \in V^3 \colon abcx \text{ and } abcy \text{ are } K_4^{(3)}\} \geqslant \beta n^3 .$$

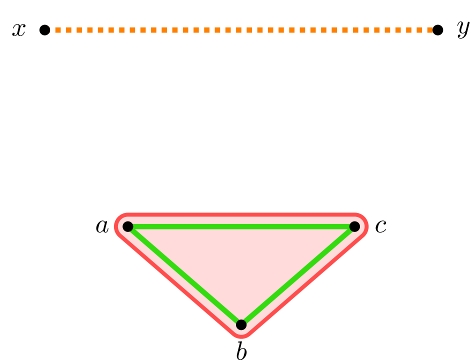

FIGURE 3.1. We have an edge $xy \in E(G_3)$ iff there are "many" edges $abc \in E(H)$ for which $ab, ac, bc \in L(x) \cap L(y)$.

The main result of this subsection is the following proposition.

Proposition 3.3. *Given $\alpha > 0$ there exist $n_0, \ell \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ for every pair of distinct vertices $x, y \in V(G)$ there exists some $t = t(x, y) \leqslant \ell$ for which there are at least $\Omega(n^{t-1})$ x-y-walks of length t in G_3.*

The next lemma gives us a lower bound on the minimum degree of G_3.

Lemma 3.4. *If $n \gg \alpha^{-1}$ and H is a hypergraph on n vertices with $\delta_2(H) \geqslant (3/4+\alpha)n$, then $\delta(G_3) \geqslant (1/4+\alpha)n$.*

Proof. Let $x \in V$ and $\beta < \alpha/8$. We count the ordered quadruples $(a,b,c,y) \in V^4$, such that $\{a,b,c,y\}$ and $\{x,a,b,c\}$ induce distinct tetrahedra in H. That is, we estimate the size of the set

$$A_x = \{(a,b,c,y) \in V^4 \colon x \neq y \text{ and } xabc \text{ and } abcy \text{ are } K_4^{(3)}\}\,.$$

Due to our assumption about $\delta_2(H)$ the number A of triples $(a,b,c) \in V^3$, which form a $K_4^{(3)}$ with x, can be estimated by

$$\begin{aligned} A &= \#\{(a,b,c) \in V^3 \colon abcx \text{ is a } K_4^{(3)}\} \\ &\geqslant (n-1)\Big(\frac{3n}{4}+\alpha n\Big)\Big(\frac{n}{4}+3\alpha n\Big) \\ &\geqslant \frac{n^3}{8} \end{aligned} \tag{3.2}$$

for n sufficiently large. Using the minimum pair-degree condition again we obtain

$$|A_x| \geqslant A\Big(\frac{n}{4}+3\alpha n-1\Big) \geqslant \Big(\frac{1}{4}+2\alpha\Big)An\,. \tag{3.3}$$

On the other hand, the assumption $d_{G_3}(x) \leqslant n/4+\alpha n$ would imply that

$$|A_x| = \sum_{y \in V \smallsetminus \{x\}} \#\{(a,b,c) \in V^3 \colon abcy \text{ and } abcx \text{ are } K_4^{(3)}\} \leqslant n \cdot \beta n^3 + (n/4+\alpha n)A\,.$$

Together with (3.3) this yields that

$$\Big(\frac{1}{4}+2\alpha\Big)An \leqslant \beta n^4 + \Big(\frac{1}{4}+\alpha\Big)An\,,$$

i.e., $\beta n^3 \geqslant \alpha A \overset{(3.2)}{\geqslant} \alpha n^3/8$. Since $\beta < \alpha/8$ this is a contradiction and shows that the minimum degree of G_3 is at least $(1/4+\alpha)n$. □

Lemma 3.5. *If $\beta, \gamma \ll \alpha$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4+\alpha)n$, then for every partition $X \,\dot\cup\, Y = V$ of the vertex set with $|X|, |Y| \geqslant (1/4+\alpha/2)n$ we have $e_{G_3}(X,Y) \geqslant \gamma n^2$.*

Proof. W.l.o.g. we can assume that $|X| \leqslant |Y|$. Since $|X| \geqslant (1/4+\alpha/2)n$, we know that $|Y| \leqslant (3/4-\alpha/2)n$. Counting the ordered triples with two vertices in X and one in Y

which induce an edge in H, we get

$$\begin{aligned}
&\#\{(x,x',y)\in X^2\times Y\colon xx'y\in E(H)\}\\
&=\sum_{(x,y)\in X\times Y}|N(x,y)\cap X|\\
&\geqslant |X||Y|\cdot(\delta_2(H)-|Y|)\\
&\geqslant \frac{3}{16}n^2\cdot\frac{3\alpha n}{2}=\frac{9\alpha}{32}n^3\,.
\end{aligned}$$

The number of $K_4^{(3)}$ including such a triple (x,x',y) can thus be estimated by

$$\begin{aligned}
&\big|\{(x,x',y,y')\in X^2\times Y^2\colon xx'yy' \text{ is a } K_4^{(3)}\}\big|+\big|\{(x,x',x'',y)\in X^3\times Y\colon xx'x''y \text{ is a } K_4^{(3)}\}\big|\\
&\qquad\geqslant \frac{9\alpha n^3}{32}\cdot\frac{n}{4}=\frac{9\alpha}{128}n^4\,.
\end{aligned}$$

Now we will distinguish two cases depending on whether the number of $K_4^{(3)}$ with exactly two or exactly three vertices in X is bigger than $\frac{9\alpha}{256}n^4$.

Case 1. $\#\{(x,x',y,y')\in X^2\times Y^2\colon xx'yy' \text{ is a } K_4^{(3)}\}\geqslant \dfrac{9\alpha}{256}n^4$

Define $A\subseteq X^2\times Y^2\times V$ to be the set of all quintuples (x,x',y,y',z) satisfying

(*i*) $xx'yy'$ is a $K_4^{(3)}$;

(*ii*) $zxx', zyy'\in E(H)$;

(*iii*) and at least three of $zxy, zx'y', zxy', zx'y$ are edges in H.

We claim that the size of A can be bounded from below by

$$|A|\geqslant\frac{9\alpha^2}{64}n^5\,. \tag{3.4}$$

Since we are in Case 1, it suffices to prove that every tetrahedron $(x,x',y,y')\in X^2\times Y^2$ extends to at least $4\alpha n$ members of A.

Writing

$$f(z)=|\{xy,xy',x'y,x'y'\}\cap E(L_z)|+2|\{xx',yy'\}\cap E(L_z)|$$

for every $z\in V$ we get

$$\begin{aligned}
\sum_{z\in V}f(z)&=d_H(x,y)+d_H(x,y')+d_H(x',y)+d_H(x',y')+2d_H(x,x')+2d_H(y,y')\\
&\geqslant 8\delta_2(H)\geqslant(6+8\alpha)n\,.
\end{aligned}$$

As $f(z)\leqslant 8$ holds for each $z\in V$ it follows that there are at least $4\alpha n$ vertices with $f(z)\geqslant 7$. For each of them we have $(x,x',y,y',z)\in A$. Thereby (3.4) is proved.

To derive an upper bound on $|A|$, we break the symmetry in (*iii*). Denoting by A' the set of quintuples $(x,x',y,y',z)\in X^2\times Y^2\times V$ satisfying (*i*), (*ii*), and (*iv*) $xy'z, x'yz, x'y'z \in E(H)$

we have

$$|A| \leqslant 4|A'|\,. \tag{3.5}$$

Moreover

$$\begin{aligned}|A'| &\leqslant \sum_{(x,y)\in X\times Y} \#\{(x',y',z)\in X\times Y\times V\colon xx'y'z \text{ and } x'yy'z \text{ are } K_4^{(3)}\}\\ &\leqslant e_{G_3}(X,Y)\cdot|X||Y||V| + |X||Y|\cdot\beta n^3\\ &\leqslant \frac{1}{4}e_{G_3}(X,Y)n^3 + \frac{1}{4}\beta n^5\,.\end{aligned}$$

Therefore with (3.4) and (3.5) it follows that

$$e_{G_3}(X,Y) \geqslant \Big(\frac{9\alpha^2}{64} - \beta\Big)n^2\,.$$

Case 2. $\#\{(x,x',x'',y)\in X^3\times Y\colon xx'x''y \text{ is a } K_4^{(3)}\} \geqslant \frac{9\alpha}{256}n^4$

Define $A \subseteq X^3\times Y\times V$ to be the set of all quintuples (x,x',x'',y,z) satisfying

(*i*) $xx'x''y$ is a $K_4^{(3)}$;
(*ii*) if $z\in Y$ at least one of the vertex sets $\{x,x'',y\},\{x,x',y\},\{x',x'',y\}$ induces a triangle in L_z;
(*iii*) if $z\in X$ the vertex set $\{x,x',x''\}$ induces a triangle in L_z.

We claim that the size of A can be bounded from below by

$$|A| \geqslant \frac{27\alpha^2}{256}n^5\,. \tag{3.6}$$

Since we are in Case 2, it suffices to prove that every tetrahedron $(x,x',x'',y)\in X^3\times Y$ extends to at least $3\alpha n$ members of A.

Writing

$$f(z) = |\{xy, xx', xx'', x'x'', x'y, x''y\}\cap E(L_z)|$$

for every $z\in V$ we get

$$\begin{aligned}\sum_{z\in V} f(z) &= d_H(x,y)+d_H(x,x')+d_H(x,x'')+d_H(x',x'')+d_H(x',y)+d_H(x'',y)\\ &\geqslant 6\delta_2(H) \geqslant (9/2+6\alpha)n\,.\end{aligned}$$

If $z\in Y$ is a vertex with $(x,x',x'',y,z)\notin A$ then $f(z)\leqslant 4$ and if $z\in X$ is a vertex with $(x,x',x'',y,z)\notin A$ then $f(z)\leqslant 5$. Hence we have

$$(9/2+6\alpha)n \leqslant 5|X|+4|Y|+\big|\{z\in X\colon (x,x',x'',y,z)\in A\}\big| + 2\big|\{z\in Y\colon (x,x',x'',y,z)\in A\}\big|\,.$$

Since $5|X| + 4|Y| = 4n + |X| \leqslant 9/2n$, it follows that

$$3\alpha n \leqslant \big|\{z \in X : (x, x', x'', y, z) \in A\}\big| + \big|\{z \in Y : (x, x', x'', y, z) \in A\}\big|,$$

as claimed.

Like before in Case 1 we obtain the upper bound

$$|A| \leqslant \beta n^5 + e_{G_3}(X, Y)n^3 .$$

Therefore with (3.6) it follows that

$$e_{G_3}(X, Y) \geqslant \Big(\frac{27\alpha^2}{256} - \beta\Big)n^2 . \qquad \square$$

Proof of Proposition 3.3. Because of Lemma 3.1, Lemma 3.4, and Lemma 3.5 it remains to check that for every partition $V = X \cup Y$ with $\sqrt{\gamma}n \leqslant |X| \leqslant (1/4 + \alpha/2)n$ we have $e_{G_3}(X, Y) \geqslant \gamma n^2$. This follows easily from

$$e_{G_3}(X, Y) = \sum_{x \in X} d_Y^{G_3}(x) \geqslant \delta(G_3) \cdot |X| - |X|^2$$

and Lemma 3.4. $\square$

3.3. **The auxiliary graphs G_v.** The second kind of auxiliary graphs we will study is the following.

Definition 3.6. *For a 3-uniform hypergraph $H = (V, E)$ and a vertex $v \in V$ we define the auxiliary graph G_v as the graph with vertex set $V(G_v) = V \setminus \{v\}$ and*

$$xy \in E(G_v) \iff x \neq y \text{ and } \#\{(a, b) \in V^2 : xabv \text{ and } yabv \text{ are } K_4^{(3)}\} \geqslant \beta n^2 .$$

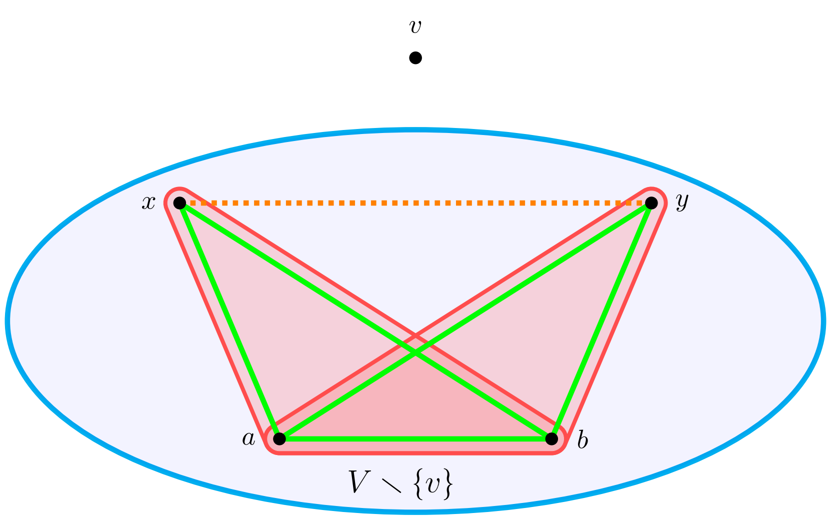

FIGURE 3.2. We have $xy \in E(G_v)$ iff there are "many" pairs $(a, b) \in V^2$ for which $abx, aby \in E(H)$ and abx, aby span triangles in L_v.

The main result of this subsection is the following proposition.

Proposition 3.7. *Given $\alpha > 0$ there exist $n_0, \ell \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4+\alpha)n$ for every pair of distinct vertices $x, y \in V(G)$ there exists some $t = t(x,y) \leqslant \ell$ for which there are at least $\Omega(n^{t-1})$ x-y-walks of length t in G_v.*

The next lemma gives us a lower bound on the minimum degree of G_v.

Lemma 3.8. *If $n \gg \alpha^{-1}$ and H is a hypergraph on n vertices with $\delta_2(H) \geqslant (3/4+\alpha)n$, then $\delta(G_v) \geqslant (1/4+\alpha)n$.*

Proof. Let $x \in V \smallsetminus \{v\}$. We count the triples $(a,b,y) \in V^3$, such that $\{y,a,b,v\}$ and $\{x,a,b,v\}$ induce distinct tetrahedra in H. That is, we estimate the size of the set

$$A_x = \{(a,b,y) \in V^3 \colon x \neq y \neq v \text{ and } xabv \text{ and } yabv \text{ are } K_4^{(3)}\}\,.$$

Due to our assumption about $\delta_2(H)$ the number A of pairs $(a,b) \in V^2$, which form a $K_4^{(3)}$ with x and v, can be estimated by

$$\begin{aligned} A &= \#\{(a,b) \in V^2 \colon abxv \text{ is a } K_4^{(3)}\} \\ &\geqslant \Big(\frac{3n}{4} + \alpha n\Big)\Big(\frac{n}{4} + 3\alpha n\Big) \\ &\geqslant \frac{n^2}{8}\,. \end{aligned} \tag{3.7}$$

Moreover we have

$$|A_x| \geqslant A\Big(\frac{n}{4} + 3\alpha n - 1\Big) \geqslant \Big(\frac{1}{4} + 2\alpha\Big)An\,. \tag{3.8}$$

On the other hand, the assumption $d_{G_v}(x) \leqslant n/4 + \alpha n$ would imply that

$$|A_x| = \sum_{y \in V \smallsetminus \{v,x\}} \#\{(a,b) \in V^2 \colon abvy \text{ and } abvx \text{ are } K_4^{(3)}\} \leqslant n \cdot \beta n^2 + (n/4 + \alpha n)A\,.$$

Together with (3.8) this yields that

$$\Big(\frac{1}{4} + 2\alpha\Big)An \leqslant \beta n^3 + \Big(\frac{1}{4} + \alpha\Big)An\,,$$

i.e., $\beta n^2 \geqslant \alpha A \overset{(3.7)}{\geqslant} \alpha n^2/8$. Since $\beta < \alpha/8$ this is a contradiction and shows that the minimum degree of G_v is at least $(1/4+\alpha)n$. □

Lemma 3.9. *If $n \gg \beta^{-1}, \gamma^{-1} \gg \alpha^{-1}$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4+\alpha)n$, then for every partition $X \cup Y = V \smallsetminus \{v\}$ of the vertex set with $|X|, |Y| \geqslant (1/4+\alpha/2)n$ we have $e_{G_v}(X,Y) \geqslant \gamma n^2$.*

Proof. We begin by showing that the set

$$A_\star = \{(x,y,z) \in X \times Y \times (V \smallsetminus \{v\}) \colon vxyz \text{ is a } K_4^{(3)} \text{ in } H\}\,,$$

satisfies

$$|A_\star| \geqslant \frac{n^3}{32}\,. \tag{3.9}$$

For the proof of this fact we may assume that $|X| \leqslant |Y|$. Thus $|X| \in [\frac{n}{4}, \frac{n}{2}]$ and hence

$$\begin{aligned}|A_\star| &\geqslant |X| \cdot (\delta_2(H) - |X|) \cdot (3\delta_2(H) - 2n)\\ &\geqslant |X| \cdot \Big(\frac{3}{4}n - |X|\Big) \cdot \frac{n}{4}\\ &\geqslant \frac{n^2}{8} \cdot \frac{n}{4} = \frac{n^3}{32}\,,\end{aligned}$$

as desired.

It follows that

$$|A_\star \cap (X \times Y \times X)| + |A_\star \cap (X \times Y^2)| = |A_\star| \geqslant \frac{n^3}{32}$$

and w.l.o.g. we can assume that $|A_\star \cap (X \times Y \times X)| \geqslant n^3/64$. Now we study the set

$$A_{\star\star} = \{(a,b,y,z) \in X^2 \times Y \times (V \smallsetminus \{v\}) \colon abvy, abvz \text{ are } K_4^{(3)} \text{ and } yz \in E(L_v)\}\,.$$

Given any triple $(a,y,b) \in A_\star \cap (X \times Y \times X)$ the quadruple $abvy$ forms a tetrahedron, there are at least $3\delta_2(H) - 2n$ vertices z for which $abvz$ forms a tetrahedron as well, and for at most $n - \delta_2(H)$ of those the condition $yz \in E(L_v)$ fails. Hence

$$\begin{aligned}|A_{\star\star}| &\geqslant |A_\star \cap (X \times Y \times X)| \cdot [(3\delta_2(H) - 2n) - (n - \delta_2(H))]\\ &\geqslant 4\alpha n \cdot |A_\star \cap (X \times Y \times X)| \geqslant \frac{\alpha}{16} n^4\,.\end{aligned}$$

Case 1. $|A_{\star\star} \cap (X^2 \times Y \times X)| \geqslant \alpha n^4/32$.

Owing to

$$\begin{aligned}\frac{\alpha n^4}{32} &\leqslant |A_{\star\star} \cap (X^2 \times Y \times X)|\\ &\leqslant \sum_{(z,y) \in X \times Y} \#\{(a,b) \in X^2 \colon abzv \text{ and } abvy \text{ are } K_4^{(3)}\}\\ &\leqslant \beta n^2 |X||Y| + e_{G_v}(X,Y) \cdot n^2\\ &\leqslant \beta n^2 \cdot n^2/4 + e_{G_v}(X,Y) \cdot n^2\end{aligned}$$

we have

$$e_{G_v}(X,Y) \geqslant \Big(\frac{\alpha}{32} - \frac{\beta}{4}\Big) n^2\,,$$

as desired.

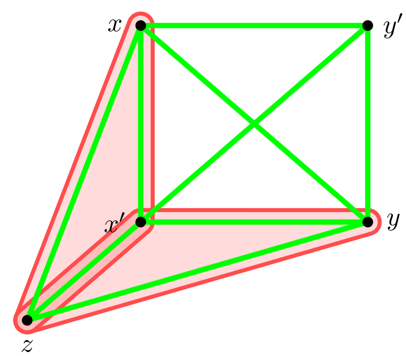

Figure 3.3. Example of a quintuple in A, where the link graph of v is indicated in green and hyperedges of H in red.

Case 2. $|A_{\star\star} \cap (X^2 \times Y^2)| \geqslant \alpha n^4/32$

Define $A \subseteq X^2 \times Y^2 \times (V \smallsetminus \{v\})$ to be the set of all quintuples (x, x', y, y', z) satisfying

(*i*) $xx'yy'$ is a K_4 in L_v

(*ii*) at least one of xx', yy' forms a $K_4^{(3)}$ with v and z

(*iii*) at least one of $xy, xy', x'y, x'y'$ forms a $K_4^{(3)}$ with v and z.

Notice that condition (*i*) holds for every $(x, x', y, y') \in A_{\star\star} \cap (X^2 \times Y^2)$. Let us now fix some such quadruple (x, x', y, y'). Due to our assumption about $\delta_2(H)$ we have

$$\begin{aligned} &d_H(x,y) + d_H(x,y') + d_H(x',y) + d_H(x',y') + 2\big(d_H(x,x') + d_H(y,y')\big) \\ &\quad +2\big(d_{L_v}(x) + d_{L_v}(x') + d_{L_v}(y) + d_{L_v}(y')\big) \geqslant 16\delta_2(H) \geqslant (12+16\alpha)n\,. \end{aligned}$$

So writing

$$f(z) = |\{xy, xy', x'y, x'y'\} \cap E(L_z)| + 2|\{xx', yy'\} \cap E(L_z)| + 2|\{vx, vx', vy, vy'\} \cap E(L_z)|$$

for every $z \in V$ we get

$$\sum_{z \in V} f(z) \geqslant (12+16\alpha)n\,.$$

If z is a vertex with $(x, x', y, y', z) \notin A$, then $f(z) \leqslant 12$, and hence we have

$$\#\{z \in V : (x, x', y, y', z) \in A\} \geqslant 16\alpha n/4 = 4\alpha n\,.$$

Applying this argument to every $(x, x', y, y') \in A_{\star\star} \cap (X^2 \times Y^2)$ we obtain, since we are in Case 2, that

$$|A| \geqslant \frac{\alpha}{32} n^4 \cdot 4\alpha n = \frac{\alpha^2}{8} n^5\,. \tag{3.10}$$

Now let A_x (resp. A_y) be the number of quintuples $(x, x', y, y', z) \in X^2 \times Y^2 \times (V \smallsetminus \{v\})$ such that

- $xx'vz$ (resp. $yy'vz$) and $x'yvz$ are $K_4^{(3)}$.

By symmetry we have

$$A_x + A_y \geqslant \frac{1}{4}|A| \overset{(3.10)}{\geqslant} \frac{\alpha^2}{32}n^5\,.$$

Consequently at least one of A_x, A_y is at least $\frac{\alpha^2}{64}n^5$. In either case one can prove that $e_{G_v}(X,Y) \geqslant \gamma n^2$ and below we display the argument assuming $A_x \geqslant \frac{\alpha^2}{64}n^5$. In this case

$$\begin{aligned}
A_x &\leqslant \sum_{(x,y)\in X\times Y} \#\{(x',y',z)\in V^3 \colon xx'zv \text{ and } yx'zv \text{ are } K_4^{(3)}\}\\
&\leqslant n \sum_{(x,y)\in X\times Y} \#\{(x',z)\in V^2 \colon xx'zv \text{ and } yx'zv \text{ are } K_4^{(3)}\}\\
&\leqslant |X||Y|\beta n^3 + e_{G_v}(X,Y)n^3
\end{aligned}$$

yields

$$e_{G_v}(X,Y) \geqslant \Big(\frac{\alpha^2}{64} - \frac{\beta}{4}\Big)n^2\,,$$

as desired. The case $A_y \geqslant \frac{\alpha^2}{64}n^5$ is similar. □

Proof of Proposition 3.7. Because of Lemma 3.8 and the fact that

$$e_{G_v}(X,Y) = \sum_{x\in X} d_Y^{G_v}(x) \geqslant \delta(G_v)\cdot|X| - |X|^2\,,$$

Lemma 3.9 is already true if $|X|,|Y| \geqslant \sqrt{\gamma}n$. Therefore the assumptions of Lemma 3.1 hold for the graph G_v, which implies Proposition 3.7. □

3.4. **Connecting Lemma.** For the rest of this section we will use the constant $\frac{4}{5}$, i.e., the minimum pair-degree hypothesis $\delta_2(H) \geqslant (4/5+\alpha)n$.

Definition 3.10. *For a 3-uniform hypergraph $H=(V,E)$ and vertices $v,r,s\in V$ we write*

$$N_v(r,s) = N(r,s,v) = N(r,v)\cap N(s,v)\cap N(r,s)\,.$$

Notice that our minimum pair-degree condition entails

$$|N_v(r,s)| \geqslant n/4 \tag{3.11}$$

for all $v,r,s\in V$.

Definition 3.11. *Given $n \gg \alpha^{-1}$, a hypergraph H on n vertices with minimum pair-degree $\delta_2(H) \geqslant (4/5+\alpha)n$ and two distinct vertices $v,w\in V(H)$ we define the auxiliary graph G_{vw} by $V(G_{vw}) = N(v,w)$ and*

$$uu'\in E(G_{vw}) \Longleftrightarrow uu'vw \text{ is a } K_4^{(3)}\,.$$

Due to our assumption about the minimum pair-degree we know that the size n' of the vertex set satisfies $n' = |V(G_{vw})| \geqslant (4/5+\alpha)n$.

Lemma 3.12. *Let $v, w \in V$ and $b, x \in V(G_{vw})$. There are at least $\Omega(n^2)$ walks of length 3 from b to x in G_{vw}.*

Proof. For a vertex $r \in V(G_{vw})$ we have

$$\begin{aligned} d_{G_{vw}}(r) &\geqslant |V(G_{vw})| - 2(n - \delta_2(H)) \\ &\geqslant \frac{|V(G_{vw})|}{2} + \frac{\delta_2(H)}{2} - 2(n - \delta_2(H)) \\ &= \frac{|V(G_{vw})|}{2} + \frac{5\delta_2(H)}{2} - 2n \geqslant \frac{n'}{2} + \frac{5\alpha n}{2} \geqslant \Big(\frac{1}{2} + \alpha\Big)n' \,. \end{aligned}$$

Thus the minimum degree of G_{vw} can be bounded from below by $\delta(G_{vw}) \geqslant (1/2 + \alpha)n'$ and any two vertices of G_{vw} have at least $2\alpha n'$ common neighbours in G_{vw}. Due to this and the minimum vertex degree condition in G_{vw} we can therefore find at least

$$\frac{n'}{2} \cdot 2\alpha n' = \alpha(n')^2 \geqslant \frac{\alpha}{2} n^2$$

walks of length 3 from b to x in G_{vw}. This shows Lemma 3.12. □

Lemma 3.13. *If $vbc, vxy \in E$ and $|N_v(b,c) \cap N_v(x,y)| = m$, then there are at least $\Omega(m^2 n^2)$ quadruples (w_0, b_1, c_1, w_1) such that $bcw_0b_1c_1w_1xy$ is*

- *a walk in H and*
- *a squared walk in L_v.*

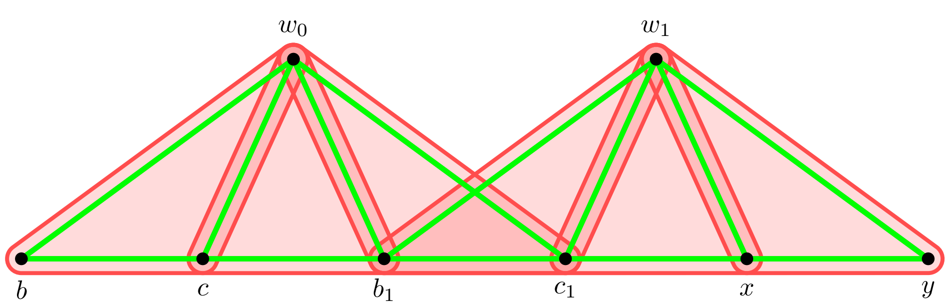

FIGURE 3.4. Quadruple (w_0, b_1, c_1, w_1) that fulfills the conditions of Lemma 3.13, where the link graph of v is indicated in green and hyperedges of H in red.

Proof. For every $w \in N_v(b,c) \cap N_v(x,y)$ Lemma 3.12 states that there are at least $\Omega(n^2)$ walks in G_{vw} from c to x of length 3. Let

$$X_{b_1c_1} = \{w \in N_v(b,c) \cap N_v(x,y) \colon cb_1c_1x \text{ is a walk in } G_{vw}\}$$

for $b_1, c_1 \in V$. Thus

$$\sum_{(b_1,c_1)\in V^2} |X_{b_1c_1}| \geqslant \Omega(mn^2)$$

and therefore the Cauchy-Schwarz inequality yields that

$$\sum_{(b_1,c_1)\in V^2} |X_{b_1c_1}|^2 \geqslant \Omega(m^2n^2)\,.$$

If $b_1, c_1 \in V$ and $w_0, w_1 \in X_{b_1c_1}$, then $bcw_0b_1c_1w_1xy$ has the desired properties. □

Proposition 3.14. *There is an integer K, such that for all edges $abc, xyz \in E$ and vertices $v \in N(a,b,c) \cap N(x,y,z)$ there are for some $k = k(abc, xyz) \leqslant K$ with $k \equiv 1 \pmod 3$ at least $\Omega(n^k)$ many $(u_1, \dots, u_k) \in V^k$ for which $abcu_1 \dots u_kxyz$ is*

- *a walk in H*
- *a squared walk in L_v.*

Proof. Recall that in Proposition 3.7 we found an integer ℓ and a function $t\colon V^{(2)} \to [\ell]$ such that for all distinct $r, s \in V$ there are $\Omega(n^{t(r,s)-1})$ walks of length $t(r,s)$ from r to s in G_v. By the box principle there exists an integer $t \leqslant \ell$ such that the set $\mathcal{Q} \subseteq N_v(b,c) \times N_v(x,y)$ of all pairs $(u,u') \in N_v(b,c) \times N_v(x,y)$ with $t(u,u') = t$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N_v(b,c)| \cdot |N_v(x,y)|}{\ell} \overset{(3.11)}{\geqslant} \frac{n^2}{16\ell}\,.$$

For each walk $v_0v_1 \dots v_t$ in G_v there are by Definition 3.6 at least $(\beta n^2)^t$ many $(2t)$-tuples $(b_1, c_1, \dots, b_t, c_t)$ such that

(*i*) $b_ic_iv \in E$ for $i = 1, \dots, t$,
(*ii*) $v_0 \in N_v(b_1, c_1)$ and $v_t \in N_v(b_t, c_t)$,
(*iii*) $v_i \in N_v(b_i, c_i) \cap N_v(b_{i+1}, c_{i+1})$ for $i = 1, \dots, t-1$.

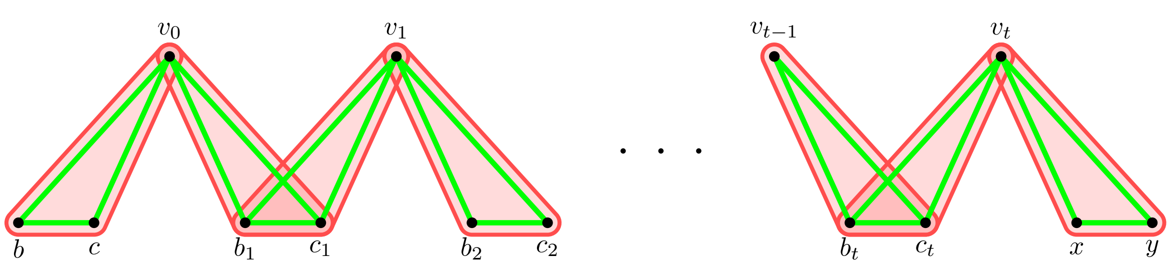

Figure 3.5. A $(3t+1)$-tuple $(v_0, v_1, \dots, v_t, b_1, c_1, \dots, b_t, c_t) \in V^{3t+1}$ satisfying (*i*), (*ii*), (*iii*), and (*iv*), where the link graph of v is indicated in green and hyperedges of H in red.

Consequently, there are at least

$$\frac{n^2}{16\ell}\cdot\Omega(n^{t-1})\cdot(\beta n^2)^t=\Omega(n^{3t+1})$$

$(3t+1)$-tuples $(v_0, v_1, \dots, v_t, b_1, c_1, \dots, b_t, c_t) \in V^{3t+1}$ satisfying (*i*), (*ii*), (*iii*) as well as

(*iv*) $v_0 \in N_v(b,c)$ and $v_t \in N_v(x,y)$.

On the other hand, we can also write the number of these $(3t+1)$-tuples as

$$\sum_{\vec{v}\in\Psi}|I_0(\vec{v})|\cdot|I_1(\vec{v})|\cdot\ldots\cdot|I_t(\vec{v})|\,,$$

where

$$\Psi=\{(b_1,c_1,\dots,b_t,c_t)\in V^{2t}\colon b_ic_iv\in E \text{ for } i=1,\dots,t\}$$

and for fixed $\vec{v}=(b_1,c_1,\dots,b_t,c_t)\in\Psi$

- $I_0(\vec{v})=N_v(b,c)\cap N_v(b_1,c_1)$
- $I_i(\vec{v})=N_v(b_i,c_i)\cap N_v(b_{i+1},c_{i+1})$ for $i=1,\dots,t-1$
- $I_t(\vec{v})=N_v(b_t,c_t)\cap N_v(x,y)$.

Altogether we have thereby shown that

$$\sum_{\vec{v}\in\Psi}|I_0(\vec{v})|\cdot|I_1(\vec{v})|\cdot\ldots\cdot|I_t(\vec{v})|\geqslant\Omega(n^{3t+1})\,. \tag{3.12}$$

Due to (3.12) and Lemma 3.13 there are at least

$$\begin{aligned}&\sum_{\vec{v}\in\Psi}\Omega(|I_0(\vec{v})|^2n^2)\cdot\ldots\cdot\Omega(|I_t(\vec{v})|^2n^2)\\ &\geqslant\Omega(n^{2t+2})\sum_{\vec{v}\in\Psi}(|I_0(\vec{v})|\cdot\ldots\cdot|I_t(\vec{v})|)^2\\ &\geqslant\Omega(n^{2t+2})\frac{\Big(\sum_{\vec{v}\in\Psi}|I_0(\vec{v})|\cdot\ldots\cdot|I_t(\vec{v})|\Big)^2}{|\Psi|}\\ &\geqslant\Omega(n^{2t+2})\Big(\frac{\Omega(n^{3t+1})}{n^t}\Big)^2=\Omega(n^{6t+4})\end{aligned}$$

$(6t+4)$-tuples, which fulfill the conditions of Proposition 3.14. Since $6t+4\equiv 1 \pmod 3$ this concludes the proof. □

Definition 3.15. *We call a sequence of vertices $v_1\dots v_h$ a squared v-walk from abc to xyz with h interior vertices if $abcv_1\dots v_hxyz$ is a walk in H and a squared walk in L_v.*

Proposition 3.16. *For all $abc, xyz\in E$ and $v\in N(a,b,c)\cap N(x,y,z)$ there are for some $k'=k'(abc,xyz,v)\leqslant K+2$ with $k'\equiv 0 \pmod 3$ at least $\Omega(n^{k'})$ many squared v-walks with k' interior vertices from abc to xyz.*

Proof. We choose vertices $d \in N_v(b,c)$ and $e \in N_v(c,d)$, and with Proposition 3.14 we find at least $\Omega(n^k)$ many squared v-walks from cde to xyz, where $k = k(cde,xyz) \leqslant K$ and $k \equiv 1 \pmod 3$. Notice that if $u_1 \dots u_k$ is such a walk, then $deu_1 \dots u_k$ is a squared v-walk from abc to xyz. Since $|N_v(b,c)|, |N_v(c,d)| \geqslant n/4$ holds by (3.11), there are for some $k \leqslant K$ with $k \equiv 1 \pmod 3$ at least $\frac{n^2/16}{K} = \Omega(n^2)$ pairs (d,e) with $k(cde,xyz) = k$. Now altogether there are $\Omega(n^{k+2})$ squared v-walks from abc to xyz with $k+2$ interior vertices. This implies Proposition 3.16, since $k+2 \equiv 0 \pmod 3$. $\square$

Lemma 3.17. *If $abc, xyz \in E$ and $|N(a,b,c) \cap N(x,y,z)| = m$, then there is an integer $t = t(abc,xyz) \leqslant (K+2)/3$ such that at least $\Omega(m^{t+1}n^{3t})$ squared walks from abc to xyz with $4t+1$ interior vertices exist.*

Proof. For every $w \in N(a,b,c) \cap N(x,y,z)$ Proposition 3.16 states that for some integer $k' = k'(w) \leqslant K+2$ with $k' \equiv 0 \pmod 3$ there are at least $\Omega(n^{k'})$ many squared w-walks from abc to xyz with k' interior vertices. By the box principle there exists an integer $k'' \leqslant K+2$ with $k'' \equiv 0 \pmod 3$ such that the set $\mathcal{Q} \subseteq N(a,b,c) \cap N(x,y,z)$ of all vertices $w' \in N(a,b,c) \cap N(x,y,z)$ with $k'(w) = k''$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N(a,b,c) \cap N(x,y,z)|}{K+2} = \frac{m}{K+2}\,.$$

For $P = (u_1, \dots, u_{k''}) \in V^{k''}$ let $X_P \subseteq \mathcal{Q}$ be the set of vertices $u \in \mathcal{Q}$ such that P is a squared u-walk from abc to xyz. Since $|\mathcal{Q}| \geqslant m/(K+2)$, the average size of X_P is at least $\Omega(m/(K+2)) = \Omega(m)$ by Proposition 3.16 and double counting. Since

$$\frac{\sum_{P \in V^{k''}} X_P^{k''/3+1}}{n^{k''}} \geqslant \Big(\frac{\sum_{P \in V^{k''}} X_P}{n^{k''}}\Big)^{k''/3+1} \geqslant \Omega(m^{k''/3+1})\,,$$

we get

$$\sum_{P \in V^{k''}} X_P^{k''/3+1} \geqslant \Omega(m^{k''/3+1} n^{k''})\,.$$

Since $k'' \equiv 0 \pmod 3$ and every ordered k''-tuple P of vertices gives rise to $X_P^{k''/3+1}$ squared walks from abc to xyz with $4k''/3+1$ interior vertices, this implies Lemma 3.17 with $t = k''/3$. $\square$

Finally we come to the main result of this section stated earlier as Proposition 2.1.

Proposition 3.18 (Connecting Lemma)**.** *There are an integer M and $\vartheta_* > 0$, such that for all disjoint triples (a,b,c) and (x,y,z) with $abc, xyz \in E$ there exists $m < M$ for which there are at least $\vartheta_* n^m$ squared paths from abc to xyz with m internal vertices.*

Proof. Recall that in Proposition 3.3 we found an integer ℓ and a function $t\colon V^{(2)} \to [\ell]$ such that for all distinct $r, s \in V$ there are $\Omega(n^{t(r,s)-1})$ walks of length $t(r,s)$ from r to s in G_3. By the box principle there exists an integer $t \leqslant \ell$ such that the set $\mathcal{Q} \subseteq N(a,b,c) \times N(x,y,z)$ of pairs $(u,u') \in N(a,b,c) \times N(x,y,z)$ with $t(u,u') = t$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N(a,b,c)| \cdot |N(x,y,z)|}{\ell} \geqslant \frac{n^2}{16\ell}\,.$$

For each path $v_0 v_1 \dots v_t$ in G_3 there are by Definition 3.2 at least $(\beta n^3)^t$ many $(3t)$-tuples $(a_1, b_1, c_1, \dots, a_t, b_t, c_t)$ such that

(i) $a_i b_i c_i \in E$ for $i = 1, \dots, t$

(ii) $v_0 \in N(a_1, b_1, c_1)$ and $v_t \in N(a_t, b_t, c_t)$

(iii) $v_i \in N(a_i, b_i, c_i) \cap N(a_{i+1}, b_{i+1}, c_{i+1})$ for $i = 1, \dots, t-1$.

Consequently, there are at least

$$\frac{n^2}{16\ell} \cdot \Omega(n^{t-1}) \cdot (\beta n^3)^t = \Omega(n^{4t+1})$$

$(4t+1)$-tuples $(v_0, \dots, v_t, a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in V^{4t+1}$ satisfying (i), (ii), (iii) as well as

(iv) $v_0 \in N(a,b,c)$ and $v_t \in N(x,y,z)$.

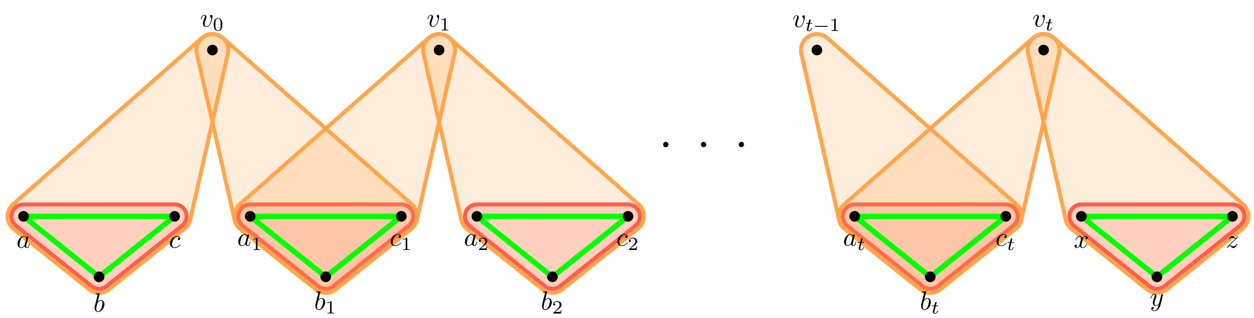

FIGURE 3.6. A $(4t+1)$-tuple $(v_0, \dots, v_t, a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in V^{4t+1}$ satisfying (i), (ii), (iii), and (iv), where orange quadruples indicate a copy of $K_4^{(3)}$, hyperedges of H are indicated in red, and green pairs are in the link graph of the corresponding v_i.

On the other hand, we can also write the number of these $(4t+1)$-tuples as

$$\sum_{\vec{v} \in \Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|\,,$$

where

$$\Psi = \{(a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in V^{3t} : a_i b_i c_i \in E \text{ for } i = 1, \dots, t\}$$

and for fixed $\vec{v} = (a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in \Psi$

- $I_0(\vec{v}) = N(a,b,c) \cap N(a_1, b_1, c_1)$
- $I_i(\vec{v}) = N(a_i, b_i, c_i) \cap N(a_{i+1}, b_{i+1}, c_{i+1})$ for $i = 1, \dots, t-1$
- $I_t(\vec{v}) = N(a_t, b_t, c_t) \cap N(x,y,z)$

Altogether we have thereby shown that

$$\sum_{\vec{v}\in\Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \geqslant \Omega(n^{4t+1})\,.$$

Lemma 3.17 gives us for every $\vec{v} \in \Psi$ some integers

- $t_0(\vec{v}) = t(abc, a_1b_1c_1)$
- $t_i(\vec{v}) = t(a_ib_ic_i, a_{i+1}b_{i+1}c_{i+1})$ for $i = 1, 2, \ldots, t-1$
- and $t_t(\vec{v}) = t(a_tb_tc_t, xyz)$.

By the box principle there are $\Psi_\star \subseteq \Psi$ and a $(t+1)$-tuple $(t_0, \ldots, t_t) \in [1, (K+2)/3]^{t+1}$ such that

$$\sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \geqslant \Omega(n^{4t+1}) \tag{3.13}$$

and $t_i(\vec{v}) = t_i$ for all $i \in \{0, \ldots, t\}$ and $\vec{v} \in \Psi_*$. Set $m = 4t + 4\sum_{i=0}^{t} t_i + 1$. Due to Lemma 3.17 there are at least

$$\begin{aligned}
&\sum_{\vec{v}\in\Psi_\star} \Omega(|I_0(\vec{v})|^{t_0+1} n^{3t_0}) \cdot \ldots \cdot \Omega(|I_t(\vec{v})|^{t_t+1} n^{3t_t}) \\
&= \Omega(n^{3\sum_{i=0}^{t} t_i}) \sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})|^{t_0+1} \cdot \ldots \cdot |I_t(\vec{v})|^{t_t+1}
\end{aligned}$$

m-tuples, which up to repeated vertices fulfill the conditions of Proposition 3.18. Let $T = \max(t_0, \ldots, t_t)$. Since

$$|I_i(\vec{v})|^{T+1} = |I_i(\vec{v})|^{t_i+1} \cdot |I_i(\vec{v})|^{T-t_i} \leqslant |I_i(\vec{v})|^{t_i+1} \cdot n^{T-t_i}\,,$$

we get

$$\begin{aligned}
n^{T(t+1)-\sum_{i=0}^{t} t_i} \sum_{\vec{v}\in\Psi_\star} \prod_{i=0}^{t} |I_i(\vec{v})|^{t_i+1} &= \sum_{\vec{v}\in\Psi_\star} \prod_{i=0}^{t} n^{T-t_i} |I_i(\vec{v})|^{t_i+1} \geqslant \sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})|^{T+1} \cdot \ldots \cdot |I_t(\vec{v})|^{T+1} \\
&= \sum_{\vec{v}\in\Psi_\star} (|I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|)^{T+1} \geqslant \left(\frac{\sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|}{|\Psi_\star|}\right)^{T+1} \cdot |\Psi_\star| \\
&\overset{(3.13)}{\geqslant} \left(\frac{\Omega(n^{4t+1})}{n^{3t}}\right)^{T+1} \cdot n^{3t} \geqslant \Omega(n^{3t+(t+1)(T+1)})\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Omega(n^{3\sum_{i=0}^{t} t_i}) \sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})|^{t_0+1} \cdot \ldots \cdot |I_t(\vec{v})|^{t_t+1} \\
&\geqslant \Omega(n^{3t+(t+1)+\sum_{i=0}^{t} t_i + 3\sum_{i=0}^{t} t_i}) = \Omega(n^m)\,.
\end{aligned}$$

At most $O(n^{m-1})$ tuples can fail being paths due to repeated vertices, thus there are $\Omega(n^m)$ squared paths from abc to xyz. This proves Proposition 3.18 with $M = \lceil 4\ell + 4(\ell+1) \cdot \frac{K+2}{3} + 2 \rceil$, since $m = 4t + 4\sum_{i=0}^{t} t_i + 1 \leqslant 4\ell + 4(\ell+1) \cdot \frac{K+2}{3} + 1$. □

§4. Reservoir Set

In all proofs using a reservoir lemma the reservoir set $\mathcal{R}$ is obtained by taking a random subset of V. On a technical level there are several possibilities which properties of $\mathcal{R}$ one actually requires and below we follow closely the approach of [10].

Proposition 4.1. *Let ϑ_* and M be the constants given by the Connecting Lemma. Then there exists a reservoir set $\mathcal{R} \subseteq V$ with $\frac{\vartheta_*^2 n}{2} \leqslant |\mathcal{R}| \leqslant \vartheta_*^2 n$, such that for all disjoint triples (a,b,c) and (x,y,z) with $abc, xyz \in E$ there are at least $\vartheta_* |\mathcal{R}|^{m(abc,xyz)}/2$ connecting squared paths in H all of whose $m(abc,xyz) < M$ internal vertices belong to $\mathcal{R}$.*

Proof. Consider a random subset $\mathcal{R} \subseteq V$ with elements included independently with probability

$$p = \Big(1 - \frac{3}{10M}\Big)\vartheta_*^2 .$$

Therefore $|\mathcal{R}|$ is binomially distributed and Chernoff's inequality yields

$$\mathbb{P}(|\mathcal{R}| < \vartheta_*^2 n/2) = o(1) . \tag{4.1}$$

Since

$$\vartheta_*^2 n \geqslant (4/3)^{1/M} pn \geqslant (1+c)\mathbb{E}[|\mathcal{R}|]$$

for some sufficiently small $c = c(M) > 0$, we have

$$\mathbb{P}(|\mathcal{R}| > \vartheta_*^2 n) \leqslant \mathbb{P}\big(|\mathcal{R}| > (4/3)^{1/M} pn\big) = o(1) . \tag{4.2}$$

The Connecting Lemma ensures that for all triples (a,b,c) and (x,y,z) there are at least $\vartheta_* n^m$ squared paths connecting them with $m = m(abc,xyz) < M$ internal vertices.

Let $X = X((a,b,c),(x,y,z))$ be the random variable counting the number of squared paths from (a,b,c) to (x,y,z) with m internal vertices in $\mathcal{R}$. We get

$$\mathbb{E}[X] \geqslant p^m \vartheta_* n^m . \tag{4.3}$$

Including or not including a particular vertex into $\mathcal{R}$ affects the random variable X by at most mn^{m-1}, wherefore the Azuma-Hoeffding inequality (see, e.g., [4, Corollary 2.27]) implies

$$\begin{aligned}\mathbb{P}\big(X \leqslant \tfrac{2}{3}\vartheta_*(pn)^m\big) &\overset{(4.3)}{\leqslant} \mathbb{P}\big(X \leqslant \tfrac{2}{3}\mathbb{E}[X]\big) \\ &\leqslant \exp\left(-\frac{2\mathbb{E}[X]^2}{9n(mn^{m-1})^2}\right) = \exp(-\Omega(n)) .\end{aligned} \tag{4.4}$$

Since there are at most n^6 pairs of triples that we have to consider, the union bound and (4.1), (4.2) tell us that asymptotically almost surely the reservoir $\mathcal{R}$ satisfies

$$\frac{\vartheta_*^2 n}{2} \leqslant |\mathcal{R}| \leqslant (4/3)^{1/M} pn \leqslant \vartheta_*^2 n \tag{4.5}$$

and

$$X((a,b,c),(x,y,z)) \geqslant \frac{2}{3}\vartheta_*(pn)^m \tag{4.6}$$

for all pairs of disjoint edges $abc, xyz \in E$. In particular, there is some $\mathcal{R} \subseteq V$ satisfying (4.5) and (4.6). Now it follows that

$$X((a,b,c),(x,y,z)) \geqslant \vartheta_*|\mathcal{R}|^m/2$$

holds for all $abc, xyz \in E$ as well, meaning that $\mathcal{R}$ has the desired properties. □

Lemma 4.2. *Let $\mathcal{R} \subseteq V$ be a reservoir set, ϑ_* the constant given by the Connecting Lemma and $\mathcal{R}' \subseteq \mathcal{R}$ an arbitrary subset of size at most $\vartheta_*^4 n$. Then for all triples (a,b,c) and (x,y,z) there exist a connecting squared path with $m(abc,xyz) < M$ internal vertices in H whose internal vertices belong to $\mathcal{R} \smallsetminus \mathcal{R}'$.*

Proof. Let $m = m(abc,xyz)$. Since $|\mathcal{R}| \geqslant \dfrac{\vartheta_*^2 n}{2}$ and $\vartheta_* \ll M^{-1}$, we can arrange that

$$|\mathcal{R}'| \leqslant \vartheta_*^4 n \leqslant \frac{\vartheta_*}{4m}|\mathcal{R}|\,.$$

Every vertex in $\mathcal{R}'$ is a member of at most $m|\mathcal{R}|^{m-1}$ squared paths with internal vertices in $\mathcal{R}$. Consequently, there are at least

$$\frac{\vartheta_*}{2}|\mathcal{R}|^m - |\mathcal{R}'|m|\mathcal{R}|^{m-1} \geqslant \frac{\vartheta_*}{2}|\mathcal{R}|^m - \frac{\vartheta_*}{4m}m|\mathcal{R}|^m > 0$$

such paths with all internal vertices in $\mathcal{R} \smallsetminus \mathcal{R}'$. □

To conclude this section we remark that taken together Proposition 4.1 and Lemma 4.2 entail Proposition 2.2.

§5. Absorbing Path

The goal of this section is to establish Proposition 2.3 which, let us recall, requires the minimum degree condition $\delta_2(H) \geqslant (4/5+\alpha)|V(H)|$. The common assumptions of all statements of this section are that we have

- $1 \gg \alpha \gg M^{-1} \gg \vartheta_* \gg n^{-1}$ such that the connecting lemma holds,
- a hypergraph $H = (V,E)$ with $|V| = n$ and $\delta_2(H) \geqslant (4/5+\alpha)n$,
- and a reservoir set $\mathcal{R} \subseteq V$ satisfying, in particular, that $|\mathcal{R}| \leqslant \vartheta_*^2 n$.

Definition 5.1. *Given a vertex $v \in V$ and a 6-tuple $(a,b,c,d,e,f) \in (V \smallsetminus \{v\})^6$ of distinct vertices, we call such a 6-tuple v-absorber if $abcdef$ and $abcvdef$ are squared paths in H.*

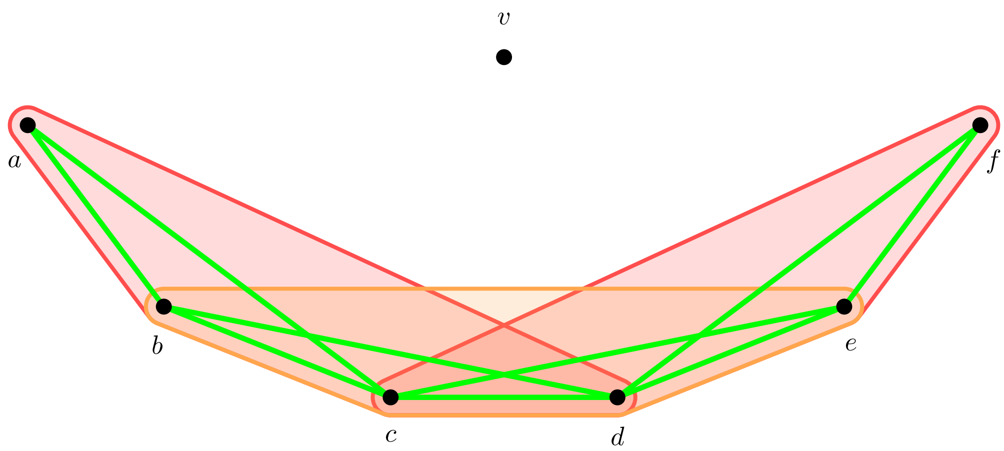

FIGURE 5.1. Example of a v-absorber, where the link graph of v is indicated in green and orange or red 4-edges indicate a copy of $K_4^{(3)}$.

Lemma 5.2. *For every $v \in V$ there are at least $\alpha^3 n^6$ many v-absorbers in $(V \smallsetminus \mathcal{R})^6$.*

Proof. Given $v \in V$ we choose the vertices of the 6-tuple in alphabetic order. For the first vertex we have n possible choices and for the second we still have more than $4n/5$ possibilities, since we only have the condition that $vab \in E$. For the third vertex we already have 3 conditions, namely $abc, vbc, vac \in E$. Consequently, we have more than $2n/5$ choices for c. For the vertices d, e, f we always have 5 conditions, so we have for each of them at least $5\alpha n$ possible choices. This implies that for given $v \in V$ we find more than

$$n \cdot 4n/5 \cdot 2n/5 \cdot (5\alpha n)^3 = 40\alpha^3 n^6$$

6-tuples meeting all the requirements from the v-absorber definition except that some of the 7 vertices $v, a, \ldots, f$ might coincide. There are at most $\binom{7}{2} n^5 = 21n^5$ such bad 6-tuples and at most $6\vartheta_*^2 n^6$ members of V^6 can use a vertex from the reservoir. Consequently, the number of v-absorbers in $(V \smallsetminus \mathcal{R})^6$ is at least $\big(40\alpha^3 - \frac{21}{n} - 6\vartheta_*^2\big)n^6 \geqslant \alpha^3 n^6$. $\square$

Lemma 5.3. *There is a set $\mathcal{F} \subseteq (V \smallsetminus \mathcal{R})^6$ with the following properties:*

(1) $|\mathcal{F}| \leqslant 8\alpha^{-3}\vartheta_*^2 n$,
(2) all vertices of every 6-tuple in $\mathcal{F}$ are distinct and the 6-tuples in $\mathcal{F}$ are pairwise disjoint,
(3) if $(a, b, c, d, e, f) \in \mathcal{F}$, then $abcdef$ is a squared path in H
(4) and for every $v \in V$ there are at least $2\vartheta_*^2 n$ many v-absorbers in $\mathcal{F}$.

Proof. Consider a random selection $\mathcal{X} \subseteq (V \smallsetminus \mathcal{R})^6$ containing each 6-tuple independently with probability $p = \gamma n^{-5}$, where $\gamma = 4\vartheta_*^2/\alpha^3$. Since $\mathbb{E}[|\mathcal{X}|] \leqslant pn^6 = \gamma n$, Markov's inequality yields

$$\mathbb{P}(|\mathcal{X}| > 2\gamma n) \leqslant 1/2\,. \tag{5.1}$$

We call two distinct 6-tuples from V^6 *overlapping* if there is a vertex occurring in both. There are at most $36n^{11}$ ordered pairs of overlapping 6-tuples. Let P be the random variable giving the number of such pairs both of whose components are in $\mathcal{X}$. Since $\mathbb{E}[P] \leqslant 36n^{11}p^2 = 36\gamma^2 n$ and $12\gamma \leqslant \vartheta_*$, Markov's inequality yields

$$\mathbb{P}(P > \vartheta_*^2 n) \leqslant \mathbb{P}(P > 144\gamma^2 n) \leqslant \frac{1}{4}\,. \tag{5.2}$$

In view of Lemma 5.2 for each vertex $v \in V$ the set A_v containing all v-absorbers in $(V \smallsetminus \mathcal{R})^6$ has the property $\mathbb{E}[|A_v \cap \mathcal{X}|] \geqslant \alpha^3 n^6 p = \alpha^3 \gamma n = 4\vartheta_*^2 n$. Since $|A_v \cap \mathcal{X}|$ is binomially distributed, Chernoff's inequality gives for every $v \in V$

$$\mathbb{P}(|A_v \cap \mathcal{X}| \leqslant 3\vartheta_*^2 n) \leqslant \exp(-\Omega(n)) < \frac{1}{5n}\,. \tag{5.3}$$

Owing to (5.1), (5.2), and (5.3) there is an "instance" $\mathcal{F}_\star$ of $\mathcal{X}$ satisfying the following:

- $|\mathcal{F}_\star| \leqslant 2\gamma n$,
- $\mathcal{F}_\star$ contains at most $\vartheta_*^2 n$ ordered pairs of overlapping 6-tuples,
- and for every $v \in V$ the number of v-absorbers in $\mathcal{F}_\star$ is at least $3\vartheta_*^2 n$.

If we delete from $\mathcal{F}_\star$ all the 6-tuples containing some vertex more than once, all that belong to an overlapping pair, and all violating (3), we get a set $\mathcal{F}$ which fulfills (1), since $|\mathcal{F}| \leqslant |\mathcal{F}_\star|$. The properties (2) and (3) hold by construction and for (4) we recall that v-absorbers satisfy (3) by definition. Therefore the set $\mathcal{F}$ has all the desired properties. $\square$

We are now ready to prove Proposition 2.3, which we restate for the reader's convenience.

Proposition 5.4 (Absorbing path)**.** *There exists an (absorbing) squared path $P_A \subseteq H - \mathcal{R}$ such that*

(1) $|V(P_A)| \leqslant \vartheta_* n$,

(2) for every set $X \subseteq V \smallsetminus V(P_A)$ with $|X| \leqslant 2\vartheta_*^2 n$ there is a squared path in H whose set of vertices is $V(P_A) \cup X$ and whose end-triples are the same as those of P_A.

Proof. Let $\mathcal{F} \subseteq (V \smallsetminus \mathcal{R})^6$ be as obtained in Lemma 5.3. Recall that $\mathcal{F}$ is a family of at most $8\alpha^{-3}\vartheta_*^2 n$ vertex-disjoint squared paths with six vertices.

We will prove that there is a path $P_A \subseteq H - \mathcal{R}$ with the following properties:

(a) P_A contains all members of $\mathcal{F}$ as subpaths,

(b) $|V(P_A)| \leqslant (M + 6)|\mathcal{F}|$.

Basically we will construct such a path P_A starting with any member of $\mathcal{F}$ by applying the connecting lemma $|\mathcal{F}| - 1$ times, attaching on further part from $\mathcal{F}$ each time.

Let $\mathcal{F}_* \subseteq \mathcal{F}$ be a maximal subset such that some path $P_A^* \subseteq H - \mathcal{R}$ has the properties (a) and (b) with $\mathcal{F}$ replaced by $\mathcal{F}_*$. Obviously $P_A^* \neq \varnothing$. From (b) and $1 \gg \alpha, M^{-1} \gg \vartheta_*$ we infer

$$|V(P_A^*)| \leqslant (M+6)|\mathcal{F}_*| \leqslant 2M|\mathcal{F}| \leqslant 16M\alpha^{-3}\vartheta_*^2 n \leqslant \vartheta_*^{3/2} n \tag{5.4}$$

and thus our upper bound on the size of the reservoir leads to

$$|V(P_A^*)| + |\mathcal{R}| \leqslant 2\vartheta_*^{3/2} n \leqslant \frac{\vartheta_* n}{2M}\,. \tag{5.5}$$

Assume for the sake of contradiction that $\mathcal{F}_* \neq \mathcal{F}$. Let (x, y, z) be the ending triple of P_A^* and let P be an arbitrary path in $\mathcal{F} \smallsetminus \mathcal{F}_*$ with starting triple (u, v, w). Then the connecting lemma tells us that there are at least $\vartheta_* n^m$ connecting squared paths with m interior vertices, where $m = m(xyz, uvw) < M$. By (5.5) at least half of them are disjoint to $V(P_A^*) \cup \mathcal{R}$. At least one such connection gives us a path $P_A^{**} \subseteq H - \mathcal{R}$ starting with P_A^*, ending with P and satisfying

$$|V(P_A^{**})| = |V(P_A^*)| + m + |V(P)| \leqslant |V(P_A^*)| + m + 6 \leqslant (M+6)(|\mathcal{F}_*| + 1)\,.$$

So $\mathcal{F}_* \cup \{P\}$ contradicts the maximality of $\mathcal{F}_*$ and proves that we have indeed $\mathcal{F}_* = \mathcal{F}$. Therefore there exists a path P_A with the properties (a) and (b).

As proved in (5.4) this path satisfies condition (1) of Proposition 5.4. To establish (2) one absorbs the up to at most $2\vartheta_*^2 n$ vertices in X one by one into P_A. This is possible due to (a) combined with (4) from Lemma 5.3. □

§6. Almost spanning cycle

The main work of this section goes into the proof of Theorem 2.4, which will occupy the Subsections 6.1–6.4. Having obtained this result we will deduce Proposition 2.5 in Subsection 6.5.

The proof of Theorem 2.4 itself is structured as follows. In Subsection 6.1 we derive an "approximate version" of Pikhurko's $K_4^{(3)}$-factor theorem (see Lemma 6.1) by imitating his proof from [9]. This lemma leads to Theorem 2.4 in the light of the hypergraph regularity method, which we recall in Subsection 6.2. In Subsection 6.3 we explain why "tetrahedra in the reduced hypergraph" correspond to regular "tetrads" large fractions of which can be covered by long squared paths. Finally in Subsection 6.4 we put everything together and complete the proof of Theorem 2.4.

6.1. **$K_4^{(3)}$-tilings.** The subsequent lemma will later be applied to a hypergraph obtained by means of the regularity lemma.

Lemma 6.1. *Let $t \geqslant 36$, $0 < \alpha < 1/4$ and $\tau \ll \alpha$. Given a hypergraph G on t vertices such that all but at most τt^2 unordered pairs $xy \in V^{(2)}$ of distinct vertices satisfy $d(x, y) \geqslant (3/4 + \alpha)t$, it is possible to delete at most $2\sqrt{\tau}t + 13$ vertices and find a $K_4^{(3)}$-factor afterwards.*

The following proof is similar to Pikhurko's argument establishing [9, Theorem 1].

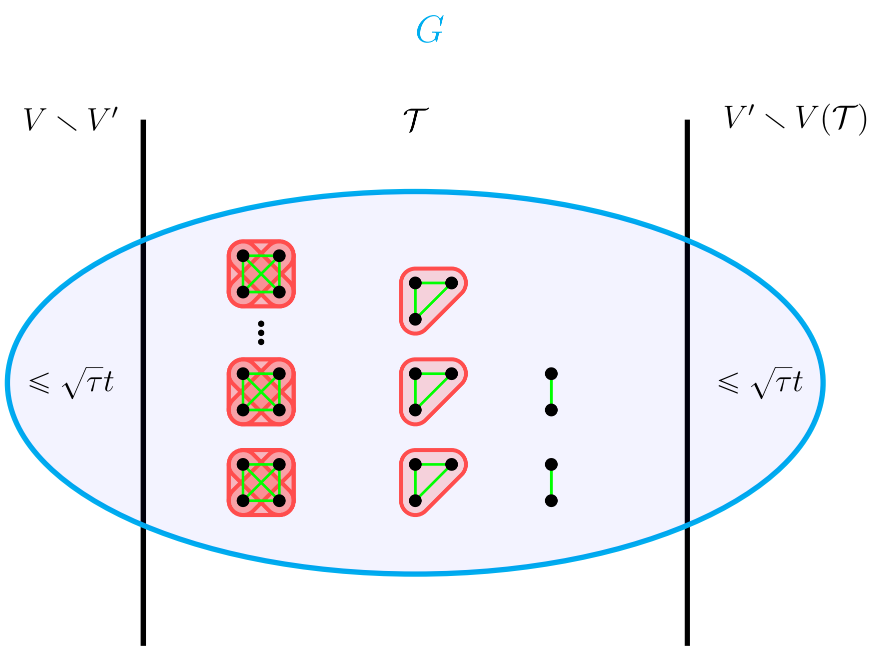

FIGURE 6.1. Example of a tiling $\mathcal{T}$ with maximal weight, where good pairs are indicated by green edges.

Proof. Let us call a pair of vertices *bad* if its pair-degree is smaller than $(3/4+\alpha)t$. Moreover we will call a subhypergraph of G *good* if it does not contain any bad pair of vertices.

First of all we will delete vertices which are in many bad pairs. More precisely we will successively delete vertices if such a vertex is in at least $\sqrt{\tau}t$ bad pairs. Since there are at most τt^2 bad pairs, we are deleting at most $\sqrt{\tau}t$ vertices and in the remaining hypergraph $G' = (V', E')$ every vertex is in at most $\sqrt{\tau}t$ bad pairs.

Let $\mathcal{F}$ be a set of hypergraphs. By an *$\mathcal{F}$-tiling* in G we mean a collection of vertex-disjoint good subgraphs, each of which is isomorphic to some member of $\mathcal{F}$. Moreover let $w_2 = 2$, $w_3 = 6$, and $w_4 = 11$ be weight factors.

In the following we will consider a $\{K_2^{(3)}, K_3^{(3)}, K_4^{(3)}\}$-tiling $\mathcal{T}$ in G' that maximises the weight function $w(\mathcal{T}) = w_2\ell_2 + w_3\ell_3 + w_4\ell_4$, where ℓ_i denote the number of copies of $K_i^{(3)}$ in $\mathcal{T}$.

At most $\sqrt{\tau}t$ vertices of V' are missed by the tiling $\mathcal{T}$. Indeed, otherwise we find a good subgraph isomorphic to $K_2^{(3)}$ not in the tiling, since every vertex in V' is in at most $\sqrt{\tau}t$ bad pairs. Because $w_2 > 0$ this is a contradiction to the maximality of $\mathcal{T}$.

We say a hypergraph $F \in \mathcal{T}$ makes a *connection* with the vertex $x \in V' \smallsetminus V(F)$ (denoted by $(F, x) \in \mathcal{C}$) if $|V(F)| \leqslant 3$ and $V(F) \cup \{x\}$ spans a complete good hypergraph. Examining the properties of connections, we get the following results.

- A $K_i^{(3)}$-subgraph $F \in \mathcal{T}$ with $i \leqslant 3$ can only make a connection to a vertex x that belongs to a $K_j^{(3)}$-subgraph of $\mathcal{T}$ with $j > i$.

Otherwise moving x to F would increase the weight of $\mathcal{T}$, since $w_4 + w_2 - 2w_3 = 1$, $w_4 - w_2 - w_3 = 3$, $w_3 - 2w_2 = 2$, and all other possible weight changes are positive as well.

- Each $K_2^{(3)}$-subgraph F in $\mathcal{T}$ makes at least $(\frac{3}{4} + \frac{\alpha}{2})t$ connections.

Let $\{a, b\}$ be the vertex set of $K_2^{(3)}$-subgraph F of $\mathcal{T}$. The subgraph F makes a connection with a vertex $x \in V' \smallsetminus V(F)$ if $abx \in E(G)$ and ab, ax, bx are good pairs. Recalling that ab is a good pair due to the definition of tiling, we can relax the second condition to ax, bx being good pairs. There are at least $(\frac{3}{4} + \alpha - \sqrt{\tau})t$ vertices in $V' \smallsetminus V(F)$ that form an edge with ab in G. Since every vertex in V' is in at most $\sqrt{\tau}t$ bad pairs, at most $2\sqrt{\tau}t$ vertices, which form an edge with ab in G, can fail the second condition. Thus, every $K_2^{(3)}$-subgraph F of $\mathcal{T}$ makes at least $(\frac{3}{4} + \alpha - 3\sqrt{\tau})t$ connections, which due to $\tau < \frac{\alpha^2}{36}$ is more than $(\frac{3}{4} + \frac{\alpha}{2})t$.

- Every $K_3^{(3)}$-subgraph F in $\mathcal{T}$ makes at least $(\frac{1}{4} + \alpha)t$ connections.

For each $K_3^{(3)}$-subgraph F of $\mathcal{T}$ there are at least $(\frac{9}{4} + \alpha)t$ edges that intersect it in exactly two vertices and consists of no bad pairs. Let c denote the number of connections made by a $K_3^{(3)}$-subgraph of $\mathcal{T}$. Thus, we get

$$\Big(\frac{9}{4} + \alpha\Big)t \leqslant 3c + 2(t - 3 - c)\,,$$

i.e.,

$$\Big(\frac{9}{4} + \alpha\Big)t - 2t + 6 \leqslant c\,.$$

- $\ell_3 \leqslant 3$.

Otherwise let F_1, F_2, F_3, F_4 be $K_3^{(3)}$-subgraphs in $\mathcal{T}$. All connections made by a F_i belong to a $K_4^{(3)}$-subgraph of $\mathcal{T}$ by the first bullet above. An upper bound for the number of $K_4^{(3)}$ in $\mathcal{T}$ is $\lfloor t/4 \rfloor$. Since

$$4\Big(\frac{1}{4} + \alpha\Big)t > 4\lfloor t/4 \rfloor\,,$$

the vertices of some $K_4^{(3)}$-subgraph F of $\mathcal{T}$ make at least 5 connections with F_1, F_2, F_3, F_4. Therefore we find two distinct vertices $x, y \in V(F)$ and $i, j \in [4]$ with $i \neq j$, such that

$(F_i, x), (F_j, y) \in \mathcal{C}$. Moving x to F_i and y to F_j and thereby reducing F to a $K_2^{(3)}$ would increase the weight of $\mathcal{T}$, since $2(w_4 - w_3) + (w_2 - w_4) = 1$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

Case 1. $\ell_2 \geqslant 3$

Let F_1, F_2, F_3 be $K_2^{(3)}$-subgraphs in $\mathcal{T}$.

- There is no $K_3^{(3)}$-subgraph $F \in \mathcal{T}$ with the property that F_1, F_2, F_3 make more than 3 connections to F.

Otherwise we could find distinct vertices $x, y \in V(F)$ and $i, j \in [3]$ with $i \neq j$, such that $(F_i, x), (F_j, y) \in \mathcal{C}$. Moving x to F_i and y to F_j and thereby eliminating F would increase the weight of $\mathcal{T}$, since $2(w_3 - w_2) - w_3 = 2$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

- There is no $K_4^{(3)}$-subgraph $F \in \mathcal{T}$ with the property that F_1, F_2, F_3 make more than 8 connections to F.

Otherwise we could find distinct vertices $x_1, x_2, x_3 \in V(F)$, such that $(F_i, x_i) \in \mathcal{C}$ for every $i \in [3]$. This is because every bipartite graph with nine edges and partition classes of size 3 and 4 contains a matching of size 3. Moving each x_i to F_i and thereby eliminating F would increase the weight of $\mathcal{T}$, since $3(w_3 - w_2) - w_4 = 1$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

Finally, by estimating the number of connections created by F_1, F_2, F_3 we obtain

$$3\Big(\frac{3}{4} + \frac{\alpha}{2}\Big)t \leqslant 3\ell_3 + 8\ell_4\,.$$

Since $\ell_3 \leqslant 3$ and $\ell_4 \leqslant \lfloor t/4 \rfloor$, we have

$$\Big(\frac{9}{4} + \frac{3}{2}\alpha\Big)t \leqslant 9 + 8\lfloor t/4 \rfloor\,,$$

which contradicts $t \geqslant 36$.

Case 2. $\ell_2 \leqslant 2$

We have deleted $\sqrt{\tau}t$ vertices from G to obtain the graph G', another $\sqrt{\tau}t$ vertices can be missed by the tiling $\mathcal{T}$, and at most $2\ell_2 + 3\ell_3 \leqslant 13$ vertices of $V(\mathcal{T})$ are not covered by $K_4^{(3)}$ subgraphs. Therefore it is possible to delete at most $2\sqrt{\tau}t + 13$ vertices and find a $K_4^{(3)}$-factor afterwards. □

6.2. **Hypergraph regularity method.** We denote by $K(X, Y)$ the complete bipartite graph with vertex partition $X \cup Y$. For a bipartite graph $P = (X \cup Y, E)$ we say it is (δ_2, d_2)-*quasirandom* if

$$\big|e(X', Y') - d_2|X'||Y'|\big| \leqslant \delta_2|X||Y|$$

holds for all subsets $X' \subseteq X$ and $Y' \subseteq Y$, where $e(X', Y')$ denotes the number of edges in P with one vertex in X' and one in Y'. Given a k-partite graph $P = (X_1 \cupdot \ldots \cupdot X_k, E)$ with $k \geqslant 2$ we say P is *(δ_2, d_2)-quasirandom*, if all naturally induced bipartite subgraphs $P[X_i, X_j]$ are (δ_2, d_2)-quasirandom. Moreover, for a tripartite graph $P = (X \cupdot Y \cupdot Z, E)$ we denote by

$$\mathcal{K}_3(P) = \big\{\{x, y, z\} \subseteq X \cup Y \cup Z \colon xy, xz, yz \in E\big\}$$

the triples of vertices in P spanning a triangle. For a (δ_2, d_2)-quasirandom tripartite graph $P = (X \cupdot Y \cupdot Z, E)$ the so-called *triangle counting lemma* implies that

$$d_2^3|X||Y||Z| - 3\delta_2|X||Y||Z| \leqslant |\mathcal{K}_3(P)| \leqslant d_2^3|X||Y||Z| + 3\delta_2|X||Y||Z|\,. \tag{6.1}$$

Definition 6.2. *Given a 3-uniform hypergraph $H = (V, E_H)$ and a tripartite graph $P = (X \cupdot Y \cupdot Z, E)$ with $X \cup Y \cup Z \subseteq V$ we say H is (δ_3, d_3)-quasirandom with respect to P if for every tripartite subgraph $Q \subseteq P$ we have*

$$\big||E_H \cap \mathcal{K}_3(Q)| - d_3|\mathcal{K}_3(Q)|\big| \leqslant \delta_3|\mathcal{K}_3(P)|\,.$$

Furthermore, we say H is δ_3-quasirandom with respect to P, if it is (δ_3, d_3)-quasirandom *for some $d_3 \geqslant 0$.*

We define the *relative density* of H with respect to P by

$$d(H|P) = \frac{|E_H \cap \mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|}\,,$$

where $d(H|P) = 0$ if $\mathcal{K}_3(P) = \varnothing$.

A refined version of the regularity lemma (see [13, Theorem 2.3]) states the following.

Lemma 6.3 (Regularity Lemma)**.** *For every $\delta_3 > 0$, every $\delta_2 \colon \mathbb{N} \to (0, 1]$, and every $t_0 \in \mathbb{N}$ there exists an integer T_0 such that for every $n \geqslant t_0$ and every n-vertex 3-uniform hypergraph $H = (V, E_H)$ the following holds.*

There are integers t and ℓ with $t_0 \leqslant t \leqslant T_0$ and $\ell \leqslant T_0$ and there exists a vertex partition $V_0 \cupdot V_1 \cupdot \ldots \cupdot V_t = V$ and for all $1 \leqslant i < j \leqslant t$ there exists a partition

$$\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \cupdot V_j, E^{ij}_\alpha) \colon 1 \leqslant \alpha \leqslant \ell\}$$

of the edge set of the complete bipartite graph $K(V_i, V_j)$ satisfying the following properties

(1) $|V_0| \leqslant \delta_3 n$ and $|V_1| = \ldots = |V_t|$,

(2) for every $1 \leqslant i < j \leqslant t$ and $\alpha \in [\ell]$ the bipartite graph P^{ij}_α is $(\delta_2(\ell), 1/\ell)$-quasirandom, and

(3) H is δ_3-quasirandom w.r.t $P^{ijk}_{\alpha\beta\gamma}$ for all but at most $\delta_3 t^3 \ell^3$ tripartite graphs

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_\alpha \cupdot P^{ik}_\beta \cupdot P^{jk}_\gamma = (V_i \cupdot V_j \cupdot V_k, E^{ij}_\alpha \cupdot E^{ik}_\beta \cupdot E^{jk}_\gamma)\,,$$

with $1 \leqslant i < j < k \leqslant t$ and $\alpha, \beta, \gamma \in [\ell]$.

The tripartite graphs $P^{ijk}_{\alpha\beta\gamma}$ appearing in (3) are usually called *triads*. Furthermore we will use the following version of the embedding lemma from [8].

Lemma 6.4. *For every $p \in \mathbb{N}$ and $\xi, d_3 > 0$ there exist $\delta_3 > 0$ and functions $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ such that the following holds.*

Let $\ell \in \mathbb{N}$ and let $G = \dot\bigcup_{1 \leqslant i < j \leqslant p} G^{ij}$ be a p-partite graph with vertex partition $V_1 \dot\cup \ldots \dot\cup V_p$, where $|V_1| = \ldots = |V_p| = n \geqslant N(\ell)$, such that each $G^{ij} = G[V_i, V_j]$ is $(\delta_2(\ell), 1/\ell)$-quasirandom. Moreover, let H be a 3-uniform hypergraph that is (δ_3, d_{ijk})-quasirandom with respect to G^{ijk} for all $1 \leqslant i < j < k \leqslant p$, where $G^{ijk} = G[V_i, V_j, V_k]$ and $d_{ijk} \geqslant d_3$. Then the number $|\mathcal{K}_p(H)|$ of complete, 3-uniform hypergraphs on p vertices in H with one vertex from each V_i satisfies

$$|\mathcal{K}_p(H)| \geqslant (1 - \xi) d_3^{\binom{p}{3}} (1/\ell)^{\binom{p}{2}} n^p .$$

6.3. **Squared paths in quasirandom tetrads.** The Embedding Lemma 6.4 can be utilised to find squared path in appropriate 4-partite environments.

Lemma 6.5. *Given $Q \in \mathbb{N}$ and $d_3 > 0$, there exist $\delta_3 > 0$, and functions $\delta_2 \colon \mathbb{N} \to (0,1]$ and $N \colon \mathbb{N} \to \mathbb{N}$, such that that the following holds for every $\ell \in \mathbb{N}$.*

Let $P = (V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4, E_P)$ be a 4-partite graph with $|V_1| = \ldots = |V_4| = n \geqslant N(\ell)$ such that $P^{ij} = (V_i \dot\cup V_j, E^{ij})$ is $(\delta_2(\ell), 1/\ell)$-quasirandom for every pair $ij \in [4]^{(2)}$. Suppose H is a 4-partite, 3-uniform hypergraph with vertex classes $V_1, \ldots, V_4$, which satisfies for every $ijk \in [4]^{(3)}$ that H is (δ_3, d_{ijk})-quasirandom w.r.t. the tripartite graphs $P^{ijk} = P^{ij} \dot\cup P^{ik} \dot\cup P^{jk}$ for some $d_{ijk} \geqslant d_3$. Then there exists a squared path with Q vertices in H.

Proof. For $p = Q$, $\xi = 1/2$ and the current d_3 let $\delta_3 > 0$ and functions $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ be given by Lemma 6.4. Moreover, let $W_1, \ldots, W_Q$ be disjoint vertex sets of size n. Choose for every $j \in [Q]$ and $i \in [4]$ with $i \equiv j \pmod 4$ a bijective function $\varphi_j \colon V_i \to W_j$. We copy E_P and $E(H)$ onto $W_1 \dot\cup \ldots \dot\cup W_Q$ in the following way.

- If for $1 \leqslant i < j \leqslant Q$ the integers $i', j' \in [4]$ satisfying $i \equiv i' \pmod 4$ and $j \equiv j' \pmod 4$ are distinct, let E^{ij}_W be the bipartite graph on $W_i \dot\cup W_j$ defined by

$$xy \in E^{i'j'} \iff \varphi_i(x)\varphi_j(y) \in E^{ij}_W$$

for all $x \in V_{i'}$ and $y \in V_{j'}$.

- If for $1 \leqslant i < j < k \leqslant Q$ the integers $i', j', k' \in [4]$ satisfying $i \equiv i' \pmod 4$, $j \equiv j' \pmod 4$, and $k \equiv k' \pmod 4$ are distinct, let H^{ijk}_W be the tripartite hypergraph on $W_i \dot\cup W_j \dot\cup W_k$ defined by

$$xyz \in E(H) \iff \varphi_i(x)\varphi_j(y)\varphi_k(z) \in H^{ijk}_W$$

for all $x \in V_{i'}$, $y \in V_{j'}$, and $z \in V_{k'}$.

For technical reasons we also need to specify bipartite graphs E_W^{ij} for distinct $i, j \in [Q]$ that are congruent modulo 4 in order to make Lemma 6.4 applicable. The precise choice of these graphs is immaterial in the following and we just take arbitrary $(\delta_2(\ell), 1/\ell)$-quasirandom bipartite graphs. E.g., we could declare all theses graphs to be isomorphic to P^{12}. Similarly, we need to define 3-partite hypergraphs H_W^{ijk} for distinct $i, j, k \in [Q]$ at least two of which are congruent modulo 4. This time we may just take the complete 3-partite hypergraphs between W_i, W_j, W_k, which are certainly $(\delta_3, 1)$-quasirandom with respect to $(W_i \cupdot W_j \cupdot W_k, E_W^{ij} \cupdot E_W^{ik} \cupdot E_W^{jk})$.

By Lemma 6.4 applied to $G_W = (W_1 \cupdot \ldots \cupdot W_Q, E_W)$, where $E_W = \dot\bigcup_{1 \leqslant i < j \leqslant Q} E_W^{ij}$ and the hypergraph $H_W = \dot\bigcup_{1 \leqslant i < j < k \leqslant Q} H_W^{ijk}$ we find at least $(1/2) n^Q (1/\ell)^{\binom{Q}{2}} d_3^{\binom{Q}{3}}$ squared paths $v_1 \ldots v_Q$ in H_W with $v_i \in W_i$ for every $i \in [Q]$. Notice that every squared such path in H_W corresponds to a squared walk in H via the inverses of the maps φ_i. It may happen that vertices get identified under this correspondence and therefore there might be squared paths in H_W not yielding squared paths in H. However $\binom{Q}{2} n^{Q-1}$ is a straightforward upper bound on the number of times this can occur and since for n sufficiently large we have

$$\frac{1}{2} n^Q (1/\ell)^{\binom{Q}{2}} d_3^{\binom{Q}{3}} > \binom{Q}{2} n^{Q-1},$$

we find at least one squared path in H. □

Lemma 6.6. *Given $Q \in \mathbb{N}$ with $Q \equiv 0 \pmod 4$, $d_3 > 0$, and $\nu > 0$. There exist $\delta_3 > 0$, $\delta_2 \colon \mathbb{N} \to (0,1)$, and $N \colon \mathbb{N} \to \mathbb{N}$, such that the following holds for every $\ell \in \mathbb{N}$. Let $P = (V_1 \cupdot V_2 \cupdot V_3 \cupdot V_4, E_P)$ be a 4-partite graph with $|V_1| = \ldots = |V_4| = n \geqslant N(\ell)$ and let $P^{ij} = (V_i \cupdot V_j, E^{ij})$ be $(\delta_2(\ell), 1/\ell)$-quasirandom for every $ij \in [4]^{(2)}$. Suppose that H is a 3-uniform hypergraph, which satisfies for every $ijk \in [4]^{(3)}$ that H is (δ_3, d_{ijk})-quasirandom with respect to the tripartite graph $P^{ijk} = P^{ij} \cupdot P^{ik} \cupdot P^{jk}$ for some $d_{ijk} \geqslant d_3$. Then all but at most νn vertices of $V_1 \cupdot \ldots \cupdot V_4$ can be covered by vertex-disjoint squared paths with Q vertices each.*

Proof. Let $\delta_3^* > 0$, $\delta_2^* \colon \mathbb{N} \to (0,1]$, $N^* \colon \mathbb{N} \to \mathbb{N}$ be the number and functions obtained by applying Lemma 6.5 to Q and $d_3/2$. Define

$$\delta_3 = \frac{\delta_3^* \nu^3}{128}, \quad \delta_2(\ell) = \min\left(\frac{\delta_2^*(\ell)\nu^2}{16}, \frac{\nu^2}{144\ell^3}\right), \quad N(\ell) = \left\lceil \frac{4N^*(\ell)}{\nu} \right\rceil$$

for each $\ell \in \mathbb{N}$. Let $P = (V_1 \cup V_2 \cup V_3 \cup V_4, E_P)$ and H be as described above for some $\ell \in \mathbb{N}$. Consider a maximal collection $S_1, \ldots, S_m$ of vertex-disjoint squared paths on Q vertices in H. For $i \in [4]$ let $V_i' \subseteq V_i$ denote the set of vertices not belonging to any of these paths. Due to $4 \mid Q$ the sets $V_1', \ldots, V_4'$ have the same size, say n^*. If $n^* < \nu n/4$ we are done, so assume from now on that $n^* \geqslant \nu n/4$. Then our choice of $\delta_2(\ell)$ implies that the bipartite graphs $P^{ij}[V_i' \cupdot V_j']$ are $(\delta_2^{**}(\ell), 1/\ell)$-quasirandom, where $\delta_2^{**}(\ell) = \min(\delta_2^*(\ell), \frac{1}{9\ell^3})$. So by Lemma 6.5 we get a contradiction to the maximality of m provided we can show that H is (δ_3^*, d_{ijk})-quasirandom w.r.t. the subtriads P_*^{ijk} of P^{ijk} induced by $V_i' \cup V_j' \cup V_k'$. This is indeed the case, since the triangle counting lemma yields that

$$\begin{aligned}
|\mathcal{K}_3(P^{123})| &\leqslant |V_1||V_2||V_3|\big(1/\ell^3 + 3\delta_2(\ell)\big) \\
&\overset{n^* \geqslant \nu n/4}{\leqslant} \frac{4^3|V_1'||V_2'||V_3'|}{\nu^3}(1/\ell^3 + 3\delta_2(\ell)) \\
&\leqslant \frac{64 \cdot \mathcal{K}_3(P[V_1', V_2', V_3'])}{\nu^3} \cdot \frac{(1/\ell^3 + 3\delta_2(\ell))}{(1/\ell^3 - 3\delta_2^{**}(\ell))} \\
&\leqslant 128 \cdot \frac{\mathcal{K}_3(P_*^{123})}{\nu^3}\,,
\end{aligned}$$

i.e.,

$$\delta_3|\mathcal{K}_3(P^{123})| \leqslant \delta_3^*|\mathcal{K}_3(P_*^{123})|\,,$$

and the same argument applies to every other triple $ijk \in [4]^{(3)}$. $\square$

6.4. **Vertex-disjoint squared paths with Q vertices.** Next we restate and prove Theorem 2.4.

Theorem 6.7. *Given $\alpha, \mu > 0$ and $Q \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ all but at most μn vertices of H can be covered by vertex-disjoint squared paths with Q vertices.*

Proof. As we could replace Q by $4Q$ if necessary we may suppose that Q is a multiple of 4. Pick sufficiently small $d_3, \nu, \tau \ll \alpha, \mu$ and let $\delta_3 > 0$, $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ be the number and functions obtained by applying Lemma 6.6 to Q, ν, and d_3. W.l.o.g. δ_3, $\delta_2(\cdot)$ are sufficiently small, such that $\delta_3 \ll \alpha, \tau$, and $\delta_2(\ell) \ll \alpha, \ell^{-1}, \tau$. For t_0 sufficiently large we can use Lemma 6.3 with δ_3, δ_2, t_0 and get an integer T_0. Finally we let n_0 be sufficiently large.

Now let H be a 3-uniform hypergraph with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (\frac{3}{4} + \alpha)n$. Due to Lemma 6.3 there exists a vertex partition $V_0 \cupdot V_1 \cupdot \ldots \cupdot V_t = V$ and pair partitions

$$\mathcal{P}^{ij} = \{P_\alpha^{ij} = (V_i \cupdot V_j, E_\alpha^{ij}) \colon 1 \leqslant \alpha \leqslant \ell\}$$

of the complete bipartite graphs $K(V_i, V_j)$ for $1 \leqslant i < j \leqslant t$ satisfying (1)-(3).

We call a triad $P_{\alpha\beta\gamma}^{ijk}$ *dense* if $d(H|P_{\alpha\beta\gamma}^{ijk}) \geqslant \alpha/10$. For every pair $i_*j_* \in [t]^{(2)}$ and every $\lambda \in [\ell]$ we denote the set of dense triads involving V_{i_*}, V_{j_*}, and $P_\lambda^{i_*j_*}$ by $\mathcal{D}_\lambda(i_*, j_*)$.

Claim 6.8. *For every $i_*j_* \in [t]^{(2)}$ we have $|\mathcal{D}_\lambda(i_*, j_*)| \geqslant (\frac{3}{4} + \frac{\alpha}{2})\ell^2 t$.*

Proof. Notice that Lemma 6.3(1) yields

$$\frac{n(1-\delta_3)}{t} \leqslant |V_k| \leqslant \frac{n}{t} \tag{6.2}$$

for every $k \in [t]$. Appealing to the $(\delta_2(\ell), 1/\ell)$-quasirandomness of $P^{i_* j_*}_\lambda$ we infer

$$\begin{aligned} |E^{i_* j_*}_\lambda| &\geqslant \Big(\frac{1}{\ell} - \delta_2(\ell)\Big)|V_{i_*}||V_{j_*}| \\ &\geqslant \Big(\frac{1}{\ell} - \delta_2(\ell)\Big)\Big(\frac{(1-\delta_3)n}{t}\Big)^2 . \end{aligned}$$

Together with the lower bound on $\delta_2(H)$ and $|V_0| \leqslant \delta_3 n$ it follows that

$$\Big(\frac{1}{\ell} - \delta_2(\ell)\Big)\Big(\frac{(1-\delta_3)n}{t}\Big)^2\Big(\frac{3}{4} + \alpha - \delta_3\Big)n \leqslant \sum_{xy \in E^{i_* j_*}_\lambda} |N(x,y) \smallsetminus V_0| . \tag{6.3}$$

On the other hand we can derive an upper bound on the right side by counting the edges in each triad using $E^{i_* j_*}_\lambda$ separately. Due to the triangle counting lemma and (6.2) each such triad contains at most

$$\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big)\Big(\frac{n}{t}\Big)^3$$

triangles. Therefore we have

$$\sum_{xy \in E^{i_* j_*}_\lambda} |N(x,y) \smallsetminus V_0| \leqslant t\ell^2 \frac{\alpha}{10}\Big(\frac{n}{t}\Big)^3\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big) + |\mathcal{D}_\lambda(i_*, j_*)|\Big(\frac{n}{t}\Big)^3\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big) .$$

Combined with (6.3) this leads because of $\delta_3 \ll \alpha$ and $\delta_2 \ll \alpha/\ell^3$ to

$$|\mathcal{D}_\lambda(i_*, j_*)| \geqslant (3/4 + \alpha/2)\ell^2 t . \qquad \square$$

For every $f\colon [t]^2 \to [\ell]$ we define a hypergraph J_f on the vertex set $[t]$ such that a 3-element set $\{i, j, k\}$ is an edge of J_f if the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$ is dense and H is δ_3-quasirandom w.r.t. this triad.

Claim 6.9. *There is $f\colon [t]^{(2)} \to [\ell]$ such that all but at most τt^2 pairs $ij \in [t]^{(2)}$ have at least pair-degree $(\frac{3}{4} + \frac{\alpha}{8})t$ in J_f.*

Proof. Let D_f be the hypergraph on $[t]$ whose edges are the triples ijk such that the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$ is dense, and let R_f be the hypergraph consisting of all sets $\{i, j, k\}$ such that H is δ_3-quasirandom with respect to the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$. Clearly, $J_f = D_f \cap R_f$. We will show that if we choose f uniformly at random, then with positive probability $E(\overline{R_f}) \leqslant 2\delta_3 t^3$ and $\delta_2(D_f) \geqslant (3/4 + \alpha/4)t$ hold.

The expected value of the number of missing edges in R_f is

$$\mathbb{E}(E(\overline{R_f})) \leqslant \frac{1}{\ell^3} \cdot \delta_3 t^3 \ell^3 = \delta_3 t^3 ,$$

since by Lemma 6.3(3) there are at most $\delta_3 t^3 \ell^3$ irregular triads. Thus, due to Markov's inequality

$$\mathbb{P}(E(\overline{R_f}) > 2\delta_3 t^3) < \frac{\delta_3 t^3}{2\delta_3 t^3} = \frac{1}{2}\,. \tag{6.4}$$

Now fix a pair $i_* j_* \in [t]^{(2)}$. Estimating the expected value of $d_{D_f}(i_*, j_*)$, we get for every $\lambda \in [\ell]$ that

$$\begin{aligned}
&\mathbb{E}\big(d_{D_f}(i_*, j_*) | f(i_*, j_*) = \lambda\big) \\
&= \frac{1}{\ell^{\binom{t}{2}-1}} \sum_{f\colon [t]^2 \to [\ell], f(i_*, j_*) = \lambda} d_{D_f}(i_*, j_*) \\
&= \frac{|\mathcal{D}_\lambda(i_*, j_*)|}{\ell^2}\,.
\end{aligned}$$

By Claim 6.8 it follows that

$$\mathbb{E}\big(d_{D_f}(i_*, j_*) | f(i_*, j_*) = \lambda\big) \geqslant (3/4 + \alpha/2)t\,.$$

Moreover, for $f\colon [t]^2 \to [\ell]$ with $f(i_*, j_*) = \lambda$ the value of $d_{D_f}(i_*, j_*)$ is completely determined by the $2(t-2)$ numbers $f(i,j)$ with $|\{i,j\} \cap \{i_*, j_*\}| = 1$ and if one changes one of these $2(t-2)$ values of f, then $d_{D_f}(i_*, j_*)$ can change by at most 1. Thus, the Azuma-Hoeffding inequality (see, e.g., [4, Corollary 2.27]) leads to

$$\mathbb{P}\big(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t \big| f(i_*, j_*) = \lambda\big) < \exp\Big(-\frac{2(\alpha t/4)^2}{2(t-2)}\Big)\,.$$

Therefore,

$$\mathbb{P}\big(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t \big| f(i_*, j_*) = \lambda\big) < e^{-\Omega(t)}$$

for each $\lambda \in [\ell]$ and hence

$$\mathbb{P}(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t) < e^{-\Omega(t)}\,. \tag{6.5}$$

Therefore the probability that some pair has a pair-degree less than $(3/4 + \alpha/4)t$ is less than $t^2/e^{\Omega(t)}$, which proves that with probability greater then $1/2$ the minimum pair-degree of D_f is at least $(3/4 + \alpha/4)t$. Together with (6.4) this shows that the probability that a function f fulfills $E(\overline{R_f}) \leqslant 2\delta_3 t^3$ and $\delta_2(D_f) \geqslant (3/4 + \alpha/4)t$ is greater than zero.

From now on let $f\colon [t]^2 \to [\ell]$ be a fixed function having these two properties. Notice that $D_f \cap R_f$ arise from D_f by deleting at most $2\delta_3 t^3$ edges. We can estimate the number $\overline{\tau} t^2$ of pairs, which have afterwards a pair-degree smaller than $(3/4 + \alpha/8)t$, by

$$\overline{\tau} t^2 \alpha t/8 \leqslant 6\delta_3 t^3\,.$$

Thus $\overline{\tau} \leqslant \frac{48\delta_3}{\alpha}$ and by our choice of $\delta_3 \ll \alpha, \tau$ it follows that $\overline{\tau} \leqslant \tau$. In other words, there are indeed at most τt^2 pairs $ij \in [t]^{(2)}$ whose pair-degree in J_f is smaller than $(\frac{3}{4} + \frac{\alpha}{8})t$. □

From now on we will denote the bipartite graph $P^{ij}_{f(i,j)}$ simply by P^{ij}, where f is the function obtained in Claim 6.9. Due to Claim 6.9 we can apply Lemma 6.1 to J_f with $\alpha' = \alpha/8$ instead of α and find a $K_4^{(3)}$-factor missing at most $2\sqrt{\tau}t + 13$ vertices with $\tau \ll \alpha'$. Since $Q \equiv 0 \pmod 4$, we can apply Lemma 6.6 to the "tetrads" corresponding to these $K_4^{(3)}$ in the reduced hypergraph. Therefore all but at most

$$\frac{n}{t}(2\sqrt{\tau}t + 13) + \frac{t}{4} \cdot \nu \cdot \frac{n}{t} + \delta_3 n \leqslant \mu n$$

vertices can be covered by vertex-disjoint squared paths with Q vertices. □

6.5. **Almost squared cycle.** Finally we establish Proposition 2.5 by connecting the absorbing path and a collection of many long squared paths provided by the foregoing theorem, which yields an almost spanning squared cycle.

Proposition 6.10. *Given $\alpha > 0$, let $\vartheta_* > 0$, $M \in \mathbb{N}$ be the constants from the connecting lemma and let P_A be an absorbing squared path. There exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$ all but at most $2\vartheta_*^2 n$ vertices of H can be covered by a squared cycle and P_A is an induced subhypergraph of this cycle.*

Proof. Applying Theorem 6.7 to the hypergraph $H \smallsetminus (P_A \cup \mathcal{R})$, where $\mathcal{R}$ is the reservoir set, with $\alpha' = \alpha/2$ instead of α, with some $Q \geqslant 2M\vartheta_*^{-4}$ divisible by 4, and $\mu = \vartheta_*^2$. We get less than n/Q squared paths with Q vertices and miss at most μn vertices. We will connect these paths and the absorbing path P_A to a squared cycle by using Lemma 4.2, which is applicable each time, since $M(\frac{n}{Q} + 1) \leqslant \vartheta_*^4 n$ for $Q \geqslant 2M\vartheta_*^{-4}$ and n sufficiently large. Therefore we just used vertices of the reservoir set. Because $\mu \leqslant \vartheta_*^2$ and $|R| \leqslant \vartheta_*^2 n$ we miss at most $\mu n + |R| \leqslant 2\vartheta_*^2 n$ vertices. □

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Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany

E-mail address: `Wiebke.Bedenknecht@uni-hamburg.de`

E-mail address: `Christian.Reiher@uni-hamburg.de`