# Topological field theory on $r$-spin surfaces and the Arf invariant 

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#### Abstract

We give a combinatorial model for $r$-spin surfaces with parametrised boundary based on [Nov]. The $r$-spin structure is encoded in terms of $\mathbb{Z}_{r}$-valued indices assigned to the edges of a polygonal decomposition. With the help of this model we count the number of mapping class group orbits on $r$-spin surfaces with parametrised boundary and fixed $r$-spin structure on each boundary component, extending (and giving a different proof of) results in [Ran, GG].

We use the combinatorial model to give a state sum construction of two-dimensional topological field theories on $r$-spin surfaces. We show that an example of such a topological field theory computes the Arf-invariant of an $r$-spin surface as introduced in [Ran, GG]. This implies in particular that the $r$-spin Arf-invariant is constant on orbits of the mapping class group, providing an alternative proof of that fact.


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## 1 Introduction

The rotation group in two dimensions is fundamentally different from the higher dimensional rotation groups. Namely, $S O(n)$ for $n \geq 3$ has universal cover $\operatorname{Spin}(n)$ which has a finite fibre (namely $\mathbb{Z}_{2}$ ), while the universal cover of $S O(2)$ is $\mathbb{R}$ which has an infinite fibre given by $\mathbb{Z}$. Accordingly, in two dimensions one can speak of $r$-spin structures, for $r \in \mathbb{Z}_{\geq 0}$, where one considers the connected cover of $S O(2)$ with fibre $\mathbb{Z}_{r}$. The special case $r=0$ is the universal cover. We review $r$-spin surfaces in detail in Section 2.1. Here we only mention that a 1 -spin surface is just an oriented surface, a 2 -spin surface is what is usually referred to as a surface with spin structure, and giving a 0 -spin structure on a surface is equivalent to giving a framing. We stress that the case $r=0$ is included in all of the following discussion.

In [Nov] a combinatorial description of $r$-spin surfaces is given based on the choice of a triangulation. For our applications, triangulations are cumbersome due to the large number of triangles required even for simple surfaces. We give a more convenient combinatorial model based on decompositions into polygons called PLCW-decompositions [Kir] (see Section 2.2). For example, this allows one to describe a genus $g$-surface with $b$ boundary components with $g+b \geq 1$ in terms of a single $(4 g+3 b)$-gon with appropriately identified edges.

We consider $r$-spin surfaces whose boundary components are parametrised by annuli with $r$-spin structure. The $r$-spin structures on these annuli are in bijection with $\mathbb{Z}_{r}$. The combinatorial representation of an $r$-spin structure on a surface $\Sigma$ is in terms of a marked PLCW-decomposition, that is:

- a PLCW decomposition of $\Sigma$ such that each boundary component contains a single edge and a single vertex,
- a choice of a marked edge for each face (before identification of the edges),
- an orientation of each edge,
- an edge index $s_{e} \in \mathbb{Z}_{r}$ for each edge $e$,
and where the edge indices need to satisfy a consistency condition around each vertex, see Section 2.3. To obtain an $r$-spin structure from the above data, one endows each face with its unique (up to isomorphism) $r$-spin structure and then uses the edge indices to define transition functions between the faces. Finally, one extends the $r$-spin structure to the vertices, which is possible due to the above consistency condition. Different sets of combinatorial data can describe isomorphic $r$-spin structures on a given surface, and we give an equivalence relation which precisely encodes that redundancy (Theorem 2.13).

The $r$-spin surfaces with parametrised boundary form a symmetric monoidal category $\mathcal{B}$ ord $d_{2}^{r}$, whose objects are "circles with $r$-spin structures", which we describe as finite lists of elements of $\mathbb{Z}_{r}$, and which dictate the restriction of the $r$-spin structure of a bordism to the in- and outgoing boundary components.

One defines a two-dimensional $r$-spin topological field theory (TFT) to be a symmetric monoidal functor

$$
\begin{equation*}
Z: \mathcal{B} \operatorname{ord}_{2}^{r} \rightarrow \mathcal{S} \tag{1.1}
\end{equation*}
$$

for a symmetric monoidal target category $\mathcal{S}$, which we will assume to be additive and idempotent-complete (and have countable direct sums in case $r=0$ ). We use the combinatorial model to give a state-sum construction of such TFTs which is easier to work with than the one given in [Nov] in terms of triangulations (Theorem 3.8). The input data for the state-sum construction is a Frobenius algebra $A \in \mathcal{S}$ whose Nakayama automorphism $N$ satisfies $N^{r}=\mathrm{id}_{A}$, and whose window element $\mu \circ \Delta \circ \eta: \mathbb{I} \rightarrow A$ is invertible (here $\mu$, $\Delta, \eta$ are the product, coproduct, and unit of $A$, respectively). State-sum constructions in the case of 2 -spin were considered previously in [BT, NR, GK].

Write $Z_{A}$ for the functor (1.1) obtained in this way. We show that

$$
\begin{equation*}
Z^{r}(A):=\bigoplus_{\lambda \in \mathbb{Z}_{r}} Z_{\lambda} \tag{1.2}
\end{equation*}
$$

where $Z_{\lambda}$ is the value of the functor $Z_{A}$ on the $r$-spin circle $\lambda$, gets equipped by $Z_{A}$ with a unital associative $\mathbb{Z}_{r}$-graded algebra structure which can be understood as a $\mathbb{Z}_{r}$-graded version of the centre of an algebra (Proposition 3.10). For $r=2$, this algebraic structure on state spaces has also been found in [MS].

In [DK] Frobenius algebras with $N^{r}=$ id appear under the name of $\Lambda_{r}$-Frobenius algebras in relation to $r$-spin surfaces. In [Ster] $\Lambda_{r}$-Frobenius algebras have been used to describe $r$-spin TFTs defined on "open bordisms", meaning that the objects in the bordism category are disjoint unions of intervals. Our $r$-spin TFTs are defined on "closed bordisms", meaning that objects are disjoint unions of circles.

As an example, let $\mathcal{S}$ be the category of super vector spaces over some field $k$ not of characteristic 2 and $A$ the Clifford algebra $C \ell(1)=k \oplus k \theta$ in one odd generator $\theta$. Assume that $r$ is even. One finds that $Z_{\lambda}=k \theta^{\lambda}$ for $\lambda \in \mathbb{Z}_{r}$ and that the following holds (Section 5.1 and Theorem 5.8):
Theorem 1.1. Let $\Sigma$ be an $r$-spin surface of genus $g$ with $b$ ingoing boundary components of $r$-spin structures $\lambda_{1}, \ldots, \lambda_{b} \in \mathbb{Z}_{r}$ and no outgoing boundary components. Then

$$
\begin{equation*}
Z_{C \ell(1)}(\Sigma)\left(\theta^{\lambda_{1}} \otimes \cdots \otimes \theta^{\lambda_{b}}\right)=2^{1-g}(-1)^{\operatorname{Arf}(\Sigma)} \tag{1.3}
\end{equation*}
$$

where $\operatorname{Arf}(\Sigma) \in \mathbb{Z}_{2}$ is the Arf-invariant of the $r$-spin structure of $\Sigma$ as defined in [Ran, GG].
By construction, $Z_{C \ell(1)}(\Sigma)$ is invariant under the action of the mapping class group of $\Sigma$. Thus the above theorem also proves that the $r$-spin Arf-invariant is constant on mapping class group orbits, a fact already shown in [Ran, GG] by different means. For usual spin structures, so $r=2$, the fact that a spin-TFT can compute the Arf-invariant (incidentally, for the same algebra) was already noticed in [MS, Gun, BT, GK]. From this point of view Theorem 1.1 is not surprising as an $r$-spin structure for even $r$ also defines a 2 -spin structure, and this correspondence is compatible with the Arf-invariant.

In [Ran, Thm. 2.9] mapping class group orbits of $r$-spin structures on a connected surface $\Sigma_{g, b}$ of genus $g$ with $b$ boundary components have been calculated for $g, b \geq 1$ when $r=2$, and for $g \geq 2, b \geq 1$ when $r>0$; in [GG, Prop. 5] the orbits are given for $g \geq 0$, $b=0$ in case $r>0$. The $r$-spin Arf invariant has been used to distinguish two orbits for $r$ even and $g \geq 2$. We extended these results for arbitrary $g$ and $b$ and give an alternative proof using the combinatorial formalism of Section 2. In order to state our theorem we need to fix some conventions.

We call an integer $d \in \mathbb{Z}_{\geq 0}$ a divisor of $r$ if there exists an integer $n$ such that $d \cdot n=r$. In particular, every non-negative integer is a divisor of 0 . Let us denote by $\operatorname{gcd}(a, b) \in \mathbb{Z}_{\geq 0}$ the non-negative generator of the ideal generated by $a$ and $b$ in $\mathbb{Z}$. Similarly one can define $\operatorname{gcd}(a, b, c) \in \mathbb{Z}_{\geq 0}$, etc. With this definition, $\operatorname{gcd}(a, 0)=a$ for all $a \in \mathbb{Z}_{\geq 0}$.

Let $\Sigma_{g, b}$ be a closed connected oriented surface of genus $g \geq 0$ with $b \geq 0$ ingoing boundary components and no outgoing boundary components. For $\lambda_{1}, \ldots, \lambda_{b} \in \mathbb{Z}_{r}$ denote by $\mathcal{R}^{r}(\Sigma)_{\lambda_{1}, \ldots, \lambda_{b}}$ the set of isomorphism classes of $r$-spin structures on $\Sigma_{g, b}$ which near the boundary circles restrict to the annulus $r$-spin structures given by $\lambda_{1}, \ldots, \lambda_{b}$ (see Section 2.1 for details).

We will also need the abelian group $O_{0}(r)$ defined as the quotient:

$$
\begin{equation*}
O_{0}(r):=\left(\mathbb{Z}_{r}\right)^{b} /\left\langle\hat{R}_{i}, \hat{H}_{i j}, G \mid i, j=1 \ldots b, i \neq j\right\rangle \tag{1.4}
\end{equation*}
$$

The generators $G, \hat{R}_{i}, \hat{H}_{i j} \in \prod_{i=1}^{b} \mathbb{Z}_{r}$ of the subgroup have components $(G)_{i}=1,\left(\hat{R}_{i}\right)_{k}=$ $\delta_{i, k}\left(\lambda_{i}-1\right),\left(\hat{H}_{i j}\right)_{i}=\left(\hat{H}_{i j}\right)_{j}=\lambda_{i}+\lambda_{j}-1$ and $\left(\hat{H}_{i j}\right)_{k}=0$ for $k \neq i, j$.

Our second main result is:
Theorem 1.2. Let $r \geq 0$ and let $\Sigma_{g, b}$ and $\lambda_{1}, \ldots, \lambda_{b}$ be as above.

1. The set of isomorphism classes of $r$-spin structures $\mathcal{R}^{r}(\Sigma)_{\lambda_{1}, \ldots, \lambda_{b}}$ is non-empty if and only if

$$
\begin{equation*}
2-2 g \equiv \sum_{i=1}^{b} \lambda_{i}(\bmod r) \tag{1.5}
\end{equation*}
$$

2. If the condition in Part 1 is satisfied, then the number of isomorphism classes is:

| $r$ | $b, g$ | $\left\|\mathcal{R}^{r}\left(\sum_{g, b}\right)_{\lambda_{1}, \ldots, \lambda_{b}}\right\|$ |
| :---: | :---: | :---: |
| 0 | $g=0$ and $b \in\{0,1\}$ | 1 |
|  | else | infinite |
| $>0$ | $b=0$ | $r^{2 g}$ |
|  | $b \geq 1$ | $r^{2 g+b-1}$ |

3. Suppose the condition in Part 1 is satisfied. Consider the action of the mapping class group of $\Sigma_{g, b}$ (which fixes the boundary pointwise) on $\mathcal{R}^{r}(\Sigma)_{\lambda_{1}, \ldots, \lambda_{b}}$ by pullback. The number of orbits is

| $g$ | conditions | number of orbits |
| :---: | :---: | :--- |
| 0 | (none) | $\left\|O_{0}(r)\right\|$ |
| 1 | $r$ even and at least one $\lambda_{i}$ odd | $2 \cdot \#\left(\right.$ divisors of $\left.\operatorname{gcd}\left(r, \lambda_{1}, \ldots, \lambda_{b}\right)\right)$ |
|  | else | $\#\left(\right.$ divisors of $\left.\operatorname{gcd}\left(r, \lambda_{1}, \ldots, \lambda_{b}\right)\right)$ |
| $\geq 2$ | $r$ even | 2 |
|  | $r$ odd | 1 |

Parts 1 and 2 of the theorem are proved in Proposition 2.19, Part 3 is proved in Section 6. The existence condition in Part 1 and the counting for $r>0$ in Part 2 is well-known for closed surfaces from complex geometry, where it relates to roots of the canonical bundle. The counting in Parts 2 and 3 extends results obtained in [Ran, GG] using different methods.

Remark 1.3. 1. We formulated Theorem 1.2 for ingoing boundary components to avoid notational complications. However, in the bordism category $\mathcal{B}$ ord $d_{2}^{r}$ one naturally has ingoing and outgoing boundary components. To incorporate these, define $R_{i}=\lambda_{i}-1$ for an ingoing boundary and $R_{i}=1-\lambda_{i}$ for an outgoing boundary. If one expresses Theorem 1.2 in terms of the $R_{i}$ by replacing $\lambda_{i}$ with $R_{i}+1$ everywhere, the result applies to connected bordisms with both ingoing and outgoing boundary components. The proof in Proposition 2.19 and in Section 6 is given in terms of the $R_{i}$.
2. Let $\Sigma: X \rightarrow Y$ be a connected morphism in $\mathcal{B o r d} d_{2}^{r}$. Let $B \subset \mathcal{B}$ ord $d_{2}^{r}(X, Y)$ be the subset of all morphisms which have the same underlying surface as $\Sigma$. Since morphisms in $\mathcal{B o r d} d_{2}^{r}$ are diffeomorphism classes of $r$-spin bordisms, Part 3 of Theorem 1.2 precisely computes the number of elements in $B$.
3. We will see in Section 6 that $O_{0}(r)$ as defined in (1.4), and which appears in Part 3 of Theorem 1.2, is naturally in bijection with orbits of the mapping class group for $g=0$ and $b \geq 0$. An explicit expression for the number of elements in $O_{0}(r)$ can be found in Lemma $2.18(b=0)$, Corollary $2.20(b=1)$, Equation $6.4(b=2)$ and Proposition $6.1(b \geq 2)$, but the general result is somewhat cumbersome. Here we just list the answer for $b=0,1,2$ :

| $b$ | condition | $\left\|O_{0}(r)\right\|$ |
| :---: | :---: | :--- |
| 0,1 | $($ none $)$ | 1 |
| 2 | $r=0$ and $\lambda_{1}=\lambda_{2}=1$ | infinite |
|  | else | $\operatorname{gcd}\left(r, \lambda_{1}-1\right)$ |

Recall that we assume the condition in Part 1 of Theorem 1.2 to hold. In particular, for $g=0, b=2$ we have $\lambda_{1}+\lambda_{2} \equiv 2(\bmod r)$.

This paper is organised as follows. In Section 2 we describe the combinatorial model for $r$-spin structures and state its main properties. In Section 3 we use this model to give a state-sum construction of $r$-spin TFTs, and we compute the value of these TFTs on several bordisms as an example. In Section 4, the action of a set of generators of the mapping class group on $r$-spin structures is expressed in terms of the data of the combinatorial model. In Section 5 we show that for $r$ even, the $r$-spin state-sum TFT for the twodimensional Clifford algebra computes the $r$-spin Arf-invariant. Section 6 contains the proof of Theorem 1.2 and also an explicit count of the mapping class group orbits in the genus 0 case. Finally, in Appendix A we relate the description of $r$-spin structures in terms of PLCW-decompositions that we use here to the triangulation-based model of [Nov]. We furthermore give the proofs of those properties of the combinatorial model and of $r$-spin state-sum TFTs which require the triangulation-based model and have been omitted in the main text.

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## 2 Combinatorial description of r-spin surfaces

In this section we present the combinatorial model for of $r$-spin structures and state its properties. We start by reviewing the definition of an $r$-spin structure (Section 2.1) and of the decomposition of surfaces we will use (Section 2.2). The main results in this section are the bijection of the combinatorial data modulo an appropriate equivalence relation and isomorphism classes of $r$-spin structures (Theorem 2.13 in Section 2.3) and the counting of these isomorphism classes for compact connected surfaces (Proposition 2.19 in Section 2.5).

## $2.1 r$-spin surfaces

Here we recall the definition of $r$-spin structures and of related notions, following [Nov]. Denote by $G L_{2}^{+}(\mathbb{R})$ the set of real $2 \times 2$ matrices of positive determinant, and let $p_{G L}^{r}$ : $\widetilde{G L}_{2}^{r} \rightarrow G L_{2}^{+}(\mathbb{R})$ be the $r$-fold connected cover for $r \in \mathbb{Z}_{>0}$ and the universal cover for $r=0$. Note that in both cases the fibres are isomorphic to $\mathbb{Z}_{r}=\mathbb{Z} / r \mathbb{Z}$. By a surface we mean an oriented two-dimensional smooth manifold. For a surface $\Sigma$ we denote by $F_{G L^{+}} \Sigma \rightarrow \Sigma$ the oriented frame bundle over $\Sigma$ ("oriented" means that orientation on the tangent space induced by the frame agrees with that of $\Sigma$ ).
Definition 2.1. 1. An $r$-spin structure on a surface $\Sigma$ is a pair $(\eta, p)$, where $\eta: P_{\widetilde{G L}} \Sigma \rightarrow$ $\Sigma$ is a principal $\widetilde{G L}{ }_{2}^{r}$-bundle and $p: P_{\widetilde{G L}} \Sigma \rightarrow F_{G L^{+}} \Sigma$ is a bundle map intertwining the $\widetilde{G L}_{2^{-}}^{r}$ and $G L_{2}^{+}$-actions on $P_{\widetilde{G L}} \Sigma$ and $F_{G L^{+}} \Sigma$ respectively.
2. An $r$-spin surface is a surface together with an $r$-spin structure.
3. A morphism of $r$-spin surfaces $\tilde{f}: \Sigma \rightarrow \Sigma^{\prime}$ is a bundle map between the $r$-spin surfaces, such that the underlying map of surfaces $f$ is a local diffeomorphism, and such that the diagram

commutes, where $d f_{*}$ denotes the induced map from the derivative of $f$.
4. A morphism of $r$-spin structures over $\Sigma$ is a morphism of $r$-spin surfaces whose underlying map of surfaces is the identity on $\Sigma$. We write

$$
\begin{equation*}
\mathcal{R}^{r}(\Sigma) \tag{2.2}
\end{equation*}
$$

for the set of isomorphism classes of $r$-spin structures on $\Sigma$.
Note that $p: P_{\widetilde{G L}} \Sigma \rightarrow F_{G L^{+}} \Sigma$ is a $\mathbb{Z}_{r}$-principal bundle $\left(r \in \mathbb{Z}_{\geq 0}\right)$. Also, morphisms of $r$-spin structures are always isomorphisms as they are maps of principal bundles. A diffeomorphism of $r$-spin surfaces is a morphism of $r$-spin surfaces with a diffeomorphism as underlying map of surfaces. Let us denote by

$$
\begin{equation*}
\mathcal{D}^{r}(\Sigma) \tag{2.3}
\end{equation*}
$$

the diffeomorphism classes of $r$-spin surfaces with underlying surface $\Sigma$. Note that by construction we have a surjection

$$
\begin{equation*}
\mathcal{R}^{r}(\Sigma) \rightarrow \mathcal{D}^{r}(\Sigma) \tag{2.4}
\end{equation*}
$$

given by passing to orbits under the action of the mapping class group of $\Sigma$ acting on $\mathcal{R}^{r}(\Sigma)$. As we shall see, this surjection is almost never injective.

Even though we do not need it in the rest of the paper, let us mention that a 0 -spin structure is the same as a framing. A framing of $\Sigma$ is a homotopy class of A framing of $\Sigma$ is a homotopy class of trivialisations of the oriented frame bundle over $\Sigma$. of the oriented frame bundle over $\Sigma$. Let $T(\Sigma)$ denote the set of framings of $\Sigma$. We have:
Proposition 2.2. There is a bijection $T(\Sigma) \xrightarrow{\sim} \mathcal{R}^{0}(\Sigma)$.

Proof. Take a framing and pick a representative trivialisation, i.e. an isomorphism of $G L_{2}^{+}$ principal bundles $\varphi: F_{G L} \Sigma \xrightarrow{\sim} G L_{2}^{+} \times \Sigma$. Define

$$
\begin{align*}
& p_{\varphi}:=[\widetilde{G L} 0 \\
& 0  \tag{2.5}\\
& \pi_{\varphi}:=\left[\widetilde{G L} \xrightarrow{p_{G L}^{0} \times \mathrm{id}_{\Sigma}} \times \Sigma \xrightarrow{p_{\varphi}} F_{G L} \Sigma \rightarrow \Sigma\right]
\end{align*}
$$

Then $\rho_{\varphi}:=\left(\pi_{\varphi}, p_{\varphi}\right)$ is a 0 -spin structure. Changing $\varphi$ by a homotopy gives an isomorphic 0 -spin structure [Hus, Ch. 4, Thm. 9.9]. This defines a map $F: T(\Sigma) \rightarrow \mathcal{R}^{0}(\Sigma)$.

Next we define a map in the opposite direction. Since $\widetilde{G L}_{2}^{0}$ is contractible, for any 0 -spin structure $\zeta=\left(\pi: P_{\widetilde{G L}} \Sigma \rightarrow \Sigma, p\right), \pi$ is a trivialisable $\widetilde{G L}_{2}^{0}$ principal bundle [Stee, Thm. 12.2]. Let $\tilde{\phi}_{\zeta}: P_{\widetilde{G L}} \Sigma \rightarrow \widetilde{G L}{ }_{2}^{0} \times \Sigma$ denote such a trivialisation. Then there exists a unique morphism of principal $G L_{2}^{+}$bundles $\phi_{\zeta}: F_{G L} \Sigma \rightarrow G L_{2}^{+} \times \Sigma$ such that

commutes. Again by contractability, any two choices of trivialisations $\tilde{\phi}_{\zeta}$ are homotopic and so the corresponding $\phi_{\zeta}$ are homotopic, too. By the same argument, different choices of representatives of isomorphism classes of 0 -spin structures give homotopic $\phi_{\zeta}$ 's. This defines a $\operatorname{map} G: \mathcal{R}^{0}(\Sigma) \xrightarrow{\sim} T(\Sigma)$.

The two maps $F$ and $G$ are inverse to each other. Indeed, for $[\zeta] \in \mathcal{R}_{\tilde{\sim}}^{0}(\Sigma)$, the 0 -spin structure one obtains after constructing $F(G([\zeta]))$ is isomorphic to $\zeta$ via $\tilde{\phi}_{\zeta}$ as in (2.6), so that indeed $F(G([\zeta]))=[\zeta]$. Conversely, starting from a homotopy class of trivialisations $[\varphi] \in T(\Sigma)$, in computing $G(F([\varphi]))$ we see that in (2.6) we can take $\tilde{\phi}_{\zeta}=\mathrm{id}$ and $\phi_{\zeta}=\varphi$, so that $G(F([\varphi]))=[\varphi]$.

After this aside on framings, let us return to $r$-spin surfaces and give a basic example which will later serve to parametrise the boundary components of $r$-spin bordisms.

Example 2.3. For $\kappa \in \mathbb{Z}$ let $\mathbb{C}^{\kappa}$ denote the $r$-spin structure on $\mathbb{C}^{\times}$given by the trivial principal $\widetilde{G L}_{2}^{r}$-bundle $\widetilde{G L}_{2}^{r} \times \mathbb{C}^{\times}$and the map

$$
\begin{align*}
p^{\kappa}: \widetilde{G L}_{2}^{r} \times \mathbb{C}^{\times} & \rightarrow G L_{2}^{+} \times \mathbb{C}^{\times} \\
(g, z) & \mapsto\left(z^{\kappa} \cdot p_{G L}^{r}(g), z\right), \tag{2.7}
\end{align*}
$$

where $z \in \mathbb{C}^{\times}$acts on $M \in G L_{2}^{+}$by

$$
z . M=\left(\begin{array}{cc}
\operatorname{Re} z & -\operatorname{Im} z  \tag{2.8}\\
\operatorname{Im} z & \operatorname{Re} z
\end{array}\right) M
$$

Since the $\widetilde{G L}{ }_{2}^{r}$-action is from the right and $p_{G L}^{r}$ is a group homomorphism, $p^{\kappa}$ indeed intertwines the $\widetilde{G L_{2}}{ }^{r}$ and $G L_{2}^{+}$-actions.
Lemma 2.4 ([Nov, Sec.3.4]). $\mathbb{C}^{\kappa}$ and $\mathbb{C}^{\kappa^{\prime}}$ are isomorphic $r$-spin structures if and only if $\kappa \equiv \kappa^{\prime}(\bmod r)$. The map $\mathbb{Z}_{r} \rightarrow \mathcal{R}^{r}\left(\mathbb{C}^{\times}\right), \kappa \mapsto\left[\mathbb{C}^{\kappa}\right]$ is a bijection.

In the case that $r>0$, it will be convenient to fix once and for all a set of representatives of $\mathbb{Z}_{r}$ in $\mathbb{Z}$, say $\{0,1, \ldots, r-1\}$, and to agree that for $\lambda \in \mathbb{Z}_{r}, \mathbb{C}^{\lambda}$ stands for $\mathbb{C}^{\kappa}$, with $\kappa \in \mathbb{Z}$ the chosen representative for $\lambda$.

Notations 2.5. For an $r$-spin surface $\Sigma$, by abuse of notation we will often use the same symbol $\Sigma$ to denote its underlying surface. That is, $\Sigma$ stands for the triple $\Sigma, \eta, p$ from Definition 2.1 (1).

A collar is an open neighbourhood of $\mathbb{S}_{1}$ in $\mathbb{C}^{\times}$. An ingoing (resp. outgoing) collar is the intersection of a collar with the set $\left\{z \in \mathbb{C}^{\times}| | z \mid \geq 1\right\}$ (resp. $\left\{z \in \mathbb{C}^{\times}| | z \mid \leq 1\right\}$ ). A boundary parametrisation of a surface $\Sigma$ is:

1. A disjoint decomposition $B_{\text {in }} \sqcup B_{\text {out }}=\pi_{0}(\partial \Sigma)$ (the in- and outgoing boundary components). $B_{\text {in }}$ and/or $B_{\text {out }}$ are allowed to be empty.
2. A collection of ingoing collars $U_{b}, b \in B_{\text {in }}$, and outgoing collars $V_{c}, c \in B_{\text {out }}$, together with a pair of orientation preserving embeddings

$$
\begin{equation*}
\phi_{\text {in }}: \bigsqcup_{b \in B_{\text {in }}} U_{b} \hookrightarrow \Sigma \hookleftarrow \bigsqcup_{c \in B_{\text {out }}} V_{c}: \phi_{\text {out }} . \tag{2.9}
\end{equation*}
$$

We require that for each $b$, the restriction $\left.\phi_{\text {in }}\right|_{U_{b}}$ maps $\mathbb{S}^{1}$ diffeomorphically to the connected component $b$ of $\partial \Sigma$, and analogously for $\left.\phi_{\text {out }}\right|_{V_{c}}$.
An $r$-spin boundary parametrisation of an $r$-spin surface $\Sigma$ is:

1. A boundary parametrisation of the underlying surface $\Sigma$ as above; we use the same notation notation as in (2.9).
2. A pair of maps fixing the restriction of the $r$-spin structure to the in- and outgoing boundary components

$$
\begin{align*}
\lambda: B_{\text {in }} & \rightarrow \mathbb{Z}_{r} & \text { and } & \mu: B_{\text {out }} \tag{2.10}
\end{align*}{\rightarrow \mathbb{Z}_{r} .}_{b} \mapsto \lambda_{b} \quad c \mapsto \mu_{c} .
$$

3. A pair of morphisms of $r$-spin surfaces which parametrise the in- and outgoing boundary components by collars with $r$-spin structure,

$$
\begin{equation*}
\varphi_{\text {in }}: \bigsqcup_{b \in B_{\text {in }}} U_{b}^{\lambda_{b}} \hookrightarrow \Sigma \hookleftarrow \bigsqcup_{c \in B_{\text {out }}} V_{c}^{\mu_{c}}: \varphi_{\text {out }} . \tag{2.11}
\end{equation*}
$$

Here, $U_{b}^{\lambda_{b}}$ is the restriction of $\mathbb{C}^{\lambda_{b}}$ to the ingoing collar $U_{b}$, and analogously $V_{c}^{\mu_{c}}:=$ $\left.\mathbb{C}^{\mu_{c}}\right|_{V_{c}}$. The maps of surfaces underlying $\varphi_{\text {in/out }}$ are required to be the maps $\phi_{\text {in }} /$ out in (2.9) from Part 1.

Note that by Lemma 2.4, the maps $\lambda, \mu$ in part 2 are not extra data, but are uniquely determined by the $r$-spin surface $\Sigma$ and the boundary parametrisation.

For diffeomorphisms between $r$-spin surfaces with parametrised boundary we only require that they respect germs of the boundary parametrisation. In more detail, let $\Sigma$ be as in (2.11) and let

$$
\begin{equation*}
\psi_{i n}: \bigsqcup_{d \in B_{\text {in }}^{\prime}} P_{d}^{\rho_{d}} \hookrightarrow \Xi \hookleftarrow \bigsqcup_{e \in B_{\text {out }}^{\prime}} Q_{e}^{\sigma_{e}}: \psi_{\text {out }} \tag{2.12}
\end{equation*}
$$

be another $r$-spin surface with boundary parametrisation. A diffeomorphism of $r$-spin surfaces with boundary parametrisation $\Sigma \rightarrow \Xi$ is an $r$-spin diffeomorphism $f: \Sigma \rightarrow \Xi$ subject to the following compatibility condition. Let $b \in B_{\mathrm{in}}^{\Sigma}$ be an ingoing boundary component of $\Sigma$ and let $f_{*}(b) \in \pi_{0}(\partial \Xi)$ be its image under $f$. We require that $f_{*}(b) \in B_{\mathrm{in}}^{\Xi}$ and that $\lambda_{b}=\rho_{f_{*}(b)}$. Furthermore, there has to exist an ingoing collar $C$ contained in both $U_{b}$ and $P_{f_{*}(b)}$ such that the diagram

of $r$-spin morphisms commutes. An analogous condition has to hold for each outgoing boundary component $c \in B_{\text {out }}$.

By an $r$-spin object we mean a pair $(X, \rho)$ consisting of a finite set $X$ and a map $\rho: X \rightarrow \mathbb{Z}_{r}, x \mapsto \rho_{x}$. Below we will construct a category whose objects are $r$-spin objects, and whose morphisms are certain equivalence classes of $r$-spin surfaces, which we turn to now.

Definition 2.6. Let $(X, \rho)$ and $(Y, \sigma)$ be two $r$-spin objects. An $r$-spin bordism from $(X, \rho)$ to $(Y, \sigma)$ is a compact $r$-spin surface $\Sigma$ with boundary parametrisation as in (2.11) together with bijections $\beta_{\text {in }}: X \xrightarrow{\sim} B_{\text {in }}$ and $\beta_{\text {out }}: Y \xrightarrow{\sim} B_{\text {out }}$ such that

and

commute. We will often abbreviate an $r$-spin bordism $\Sigma$ from $(X, \rho)$ to $(Y, \sigma)$ as $\Sigma: \rho \rightarrow \sigma$.
Given $r$-spin bordisms $\Sigma: \rho \rightarrow \sigma$ and $\Xi: \sigma \rightarrow \tau$, the glued $r$-spin bordism $\Xi \circ \Sigma: \rho \rightarrow \tau$ is defined as follows. Denote by $Y$ the source of $\sigma$, i.e. $\sigma: Y \rightarrow \mathbb{Z}_{r}$. For every $y \in Y$, the boundary component $\beta_{\text {out }}^{\Sigma}(y)$ of $\Sigma$ is glued to the boundary component $\beta_{\mathrm{in}}^{\Xi}(y)$ of $\Xi$ using
the $r$-spin boundary parametrisations $\varphi_{\text {out }}^{\Sigma}$ and $\varphi_{\text {in }}^{\Xi}$. The diagrams in (2.14) ensure that the $r$-spin structures on the corresponding collars are restrictions of the same $r$-spin structure on $\mathbb{C}^{\times}$.

Two $r$-spin bordisms between the same $r$-spin objects, $\Sigma, \Sigma^{\prime}:(X, \rho) \rightarrow(Y, \sigma)$ are called equivalent if there is a diffeomorphism $f: \Sigma \rightarrow \Sigma^{\prime}$ of $r$-spin surfaces with boundary parametrisation such that with $f_{*}: \pi_{0}(\partial \Sigma) \rightarrow \pi_{0}\left(\partial \Sigma^{\prime}\right)$,

commute. Let $[\Xi]: \sigma \rightarrow \tau$ and $[\Sigma]: \rho \rightarrow \sigma$ be equivalence classes of $r$-spin bordisms. The composition $[\Xi] \circ[\Sigma]:=[\Xi \circ \Sigma]: \rho \rightarrow \tau$ is well defined, that is independent of the choice of representatives $\Xi, \Sigma$ of the classes to be glued. In the following we will by abuse of notation write the same symbol $\Sigma$ for an $r$-spin bordism $\Sigma$ and its equivalence class $[\Sigma]$.

Definition 2.7. The category of $r$-spin bordisms $\mathcal{B}$ ord $d_{2}^{r}$ has $r$-spin objects as objects and equivalence classes of $r$-spin bordisms as morphisms.
$\mathcal{B}$ ord $d_{2}^{r}$ is a symmetric monoidal category with tensor product on objects and morphisms given by disjoint union. The identities and the symmetric structure are given by $r$-spin cylinders with appropriately parametrised boundary.

### 2.2 PLCW decompositions

In Section 2.3 we will use a cell decomposition to combinatorially encode $r$-spin structures on surfaces, and in Section 3.3 we will use this description to build an $r$-spin TFT. For explicit calculations it is helpful to keep the number of faces and edges to a minimum. The notion of a PLCW decomposition from [Kir], and which we review in this section, is


Figure 1: Glueing a torus from a rectangle. Each step is a regular cell map and each generalised cell decomposition is a PLCW decomposition.
well suited for such calculations. For example, there is a PLCW decomposition of a torus consisting of 1 face, 2 edges and 1 vertex, see Figure 1. For comparison, using simplicial sets would require at least 2 faces, 3 edges and 1 vertex; using simplicial complexes (i.e. triangulations, as in $[\mathrm{Nov}]$ ) would require at least 14 faces, 21 edges and 7 vertices (see e.g. [Lut]).

Now we turn to the definitions following [Kir]. Let $C \subset \mathbb{R}^{N}$ be a compact set, let $\dot{C}$ denote its interior and let $\dot{C}:=C \backslash \dot{C}$ denote its boundary. Let $B^{N}=[-1,1]^{N} \in \mathbb{R}^{N}$ denote the closed $N$-ball, or rather a piece-wise linear (PL for short) version thereof. Then $\dot{B}^{N}=S^{N-1}$ is the (PL-version of the) $(N-1)$-sphere. A PL map $\varphi: C \rightarrow \mathbb{R}^{M}$ is called a regular map if $\left.\varphi\right|_{\operatorname{Int}(C)}$ is injective. A compact subset $C \subset \mathbb{R}^{N}$ is a generalised $n$-cell (or simply cell), if $\dot{C}=\varphi\left(\dot{B}^{n}\right)$ and $\dot{C}=\varphi\left(\dot{B}^{n}\right)$ for a regular map $\varphi: B^{n} \rightarrow C$, which we call a characteristic map of $C$. A generalised cell decomposition is a finite collection of cells such that the interiors of cells do not intersect and the boundary of any cell is a union of cells. Examples are shown in Figure 1 and in Figure 2. We denote the $n$-skeleton of $K$ by $K^{n}$, which is the union of the set of $k$-cells $K_{k}$ with $k \leq n$, and we define the dimension $\operatorname{dim} K$ of $K$ to be the highest integer $n$ for which the set of $n$-cells is nonempty. We denote the set of boundaries of an $n$-cell $C \in K_{n}$ by $\partial(C) \subset K_{n-1}$. A regular cell map $f: L \rightarrow K$ between generalised cell decompositions $L$ and $K$ is a piecewise linear map $f: \bigcup_{C \in L} C \rightarrow \bigcup_{D \in K} D$ such that for every $C \in L$ with characteristic map $\varphi$ there is a cell $D=f(C) \in K$ for which $f \circ \varphi$ is a characteristic map. An example of a regular cell map is shown in Figure 1, a non-example is shown in Figure $2 b$ ).

Definition 2.8. A PLCW decomposition $K$ is a generalised cell decomposition of dimension $n$ such that if $n>0$

- $K^{n-1}$ is a PLCW decomposition and
- for any $n$-cell $A \in K_{n}$ with characteristic map $\varphi$ there is a PLCW decomposition $L$ of $S^{n-1}$, such that $\left.\varphi\right|_{S^{n-1}}: L \rightarrow K^{n-1}$ is a regular cell map.

Examples of PLCW decompositions are shown in Figure 1, Figure 2 b) and $c$ ). A generalised cell decomposition which is not a PLCW decomposition is shown in Figure 2 $a)$. Each PLCW decomposition can be related by a series of local elementary moves (cf. Section 2.4 below), and each PLCW decomposition can be refined to a simplicial complex [Kir, Thm. 6.3]. For more details see [Kir, Sec. 6-8].

From now on we specialise to 2 dimensional PLCW decompositions. Let $\Sigma$ be a compact surface with a PLCW decomposition $\Sigma=\Sigma_{2} \cup \Sigma_{1} \cup \Sigma_{0}$. We call these sets faces, edges and vertices respectively; one can think of faces as $n$-gons with $n \geq 1$. For $g+b \geq 1$, PLCW decompositions also allow for a decomposition of any compact connected surface $\Sigma_{g, b}$ of genus $g$ and with $b$ boundary components into a single face which is a ( $4 g+3 b$ )-gon, see Section 2.5.

To apply PLCW decompositions to smooth manifolds, we can use that a PLCW decomposition can be refined to a simplicial complex, and that PL cell maps for a simplicial complex can be approximated by smooth maps, giving smooth manifolds [Mun, Sec. 10].


Figure 2: a) A generalised cell decomposition which is not a PLCW decomposition. There are one 2-cell, four 1-cells and four 0-cells. One can visualise it by folding a paper and glueing it only along the bottom edge. b) A triangle with two sides identified and a 1-gon, both PLCW decompositions. The map between them is not a regular cell map as the edge in the middle has no image. c) A PLCW decomposition of a sphere into two faces, one edge (red line) and one vertex.

### 2.3 Combinatorial description of $r$-spin structures

In this section we extend the combinatorial description of $r$-spin structures in [Nov], which uses a triangulation of the underlying surface, to PLCW decompositions. We will only consider PLCW decompositions where the boundary components consist of a single vertex and a single edge.

Let $\Sigma$ be a surface with parametrised boundary, with a PLCW decomposition, with a marking of one edge of each face and an orientation of each edge. We do not require that the orientation of the boundary edges corresponds to the orientation of the boundary components, but we orient the faces according to the orientation of the surface. This induces an ordering of the edges of each face, the starting edge being the marked one, see Figure 3. By an edge index assignment we mean a map $s: \Sigma_{1} \rightarrow \mathbb{Z}_{r}, e \mapsto s_{e}$.

Definition 2.9. We call an assignment of edge markings, edge orientations and edge indices a marking of a PLCW decomposition and a PLCW decomposition together with a marking a marked PLCW decomposition.

For a vertex $v \in \Sigma_{0}$ let $D_{v}$ be the number of faces whose marked edge has $v$ as its boundary vertex in clockwise direction (with respect to the orientation of the face), as shown in Figure 3. Let $\partial^{-1}(v) \subset \Sigma_{1}$ denote the edges whose boundary contain $v$ :

$$
\begin{equation*}
\partial^{-1}(v):=\left\{e \in \Sigma_{1} \mid v \in \partial(e)\right\} \tag{2.16}
\end{equation*}
$$

The orientation of an edge gives a starting and an ending vertex, which might be the same. Let $N_{v}^{\text {start }}\left(\right.$ resp. $\left.N_{v}^{\text {end }}\right)$ be the number of edges starting (resp. ending) at the vertex $v$ and let

$$
\begin{equation*}
N_{v}=N_{v}^{\text {start }}+N_{v}^{\text {end }} \tag{2.17}
\end{equation*}
$$



Figure 3: Figure of a face with adjacent edges and vertices in a marked PLCW decomposition. The orientation of the face is that of the paper plane, the orientation of the edges is indicated by an arrow on them. The half-dot indicates the marked edge of the face the half-dot lies in. The arrow in the middle shows the clockwise direction along the marked edge $e$ and $v$ is the vertex sitting on the boundary of $e$ in clockwise direction. Note that the clockwise vertex $v$ of the edge $e$ is determined by the orientation of the face and not by the orientation of the edge $e$.

We note that an edge which starts and ends at $v$ contributes 1 to both $N_{v}^{\text {start }}$ and to $N_{v}^{\text {end }}$. For every edge $e \in \partial^{-1}(v)$ let

$$
\hat{s}_{e}= \begin{cases}-1 & \text { if } e \text { starts and ends at } v  \tag{2.18}\\ s_{e} & \text { if } e \text { is pointing out of } v \\ -1-s_{e} & \text { if } e \text { is pointing into } v\end{cases}
$$

Recall the maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}$ and $\mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$ from (2.10), as well as our convention that we only consider PLCW decompositions with exactly one vertex and one edge on each boundary component. For a vertex $u$ on a boundary component let us write by slight abuse of notation $u$ for this boundary component and let

$$
R_{u}:= \begin{cases}\lambda_{u}-1 & \text { if } u \in B_{\text {in }}  \tag{2.19}\\ 1-\mu_{u} & \text { if } u \in B_{\text {out }}\end{cases}
$$

We call a marking admissible with given maps $\lambda$ and $\mu$, if for every vertex $v \in \Sigma_{0}$ which is not on the boundary and for every vertex $u \in \Sigma_{0}$ on the boundary vertex and one edge on each boundary component) the following conditions are satisfied:

$$
\begin{array}{ll}
\sum_{e \in \partial^{-1}(v)} \hat{s}_{e} \equiv D_{v}-N_{v}+1 \\
\sum_{e \in \partial^{-1}(u)} \hat{s}_{e} \equiv D_{u}-N_{u}+1-R_{u} & (\bmod r) \tag{2.21}
\end{array}
$$

For an arbitrary marking of a PLCW decomposition of $\Sigma$ one can define an $r$-spin structure with $r$-spin boundary parametrisation on $\Sigma$ minus its vertices by taking the trivial $r$-spin structure on faces and fixing the transition functions using the marking. The above $r$-spin structure extends uniquely to the vertices of $\Sigma$, if and only if the marking is


Figure 4: Moves of Lemma 2.11 for a face of $\Sigma$. All edge orientations and markings are arbitrary unless shown explicitly. (1) Flipping the edge orientation of $e$. (2a), (2b) Moving the edge marking for a face. (3) Shifting the edge indices for a face. The dotted edges $e_{3}$ and $e_{5}$ are identified, hence the edge index remains unchanged. The edges $e_{1}$ and $e_{2}$ are counterclockwise oriented, hence the $+k$ shift of the corresponding edge indices $s_{1}$ and $s_{2}$, the edge $e_{4}$ is clockwise oriented, hence the $-k$ shift of $s_{4}$.
admissible for $\lambda$ and $\mu$. The $r$-spin boundary parametrisations are given by the inclusion of $r$-spin collars (as prescribed by $\lambda$ and $\mu$ ) over the collars of the boundary parametrisation of $\Sigma$. For more details on this construction we refer the reader to Appendix A.3.

Definition 2.10. Denote the $r$-spin structure with $r$-spin boundary parametrisation defined above by $\Sigma(s, \lambda, \mu)$.

There is some redundancy in the description of an $r$-spin structure via a marking. A one-to-one correspondence between certain equivalence classes of markings and isomorphism classes of $r$-spin structures will be given in Theorem 2.13 below. As preparation we first give a list of local modifications of the marking which lead to isomorphic $r$-spin structures.
Lemma 2.11. The following changes of the marking of the PLCW decomposition of $\Sigma$ (but keeping the PLCW decomposition fixed) give isomorphic $r$-spin structures:

1. Flip the orientation of an edge $e$ and change its edge index $s_{e} \mapsto-1-s_{e}$ (see Figure 4 (1)).
2. Move the marking on an edge $e$ of a polygon to the following edge counterclockwise and change the edge index of the previously marked edge $s_{e} \mapsto s_{e}-1$, if this edge is oriented counterclockwise, $s_{e} \mapsto s_{e}+1$ otherwise (see Figure 4 (2a) and (2b)).
3. Let $k \in \mathbb{Z}$. Shift the edge index of each edge of a polygon by $+k$, if the edge is oriented counterclockwise with respect to the orientation of the polygon, and by $-k$
otherwise. If two edges of a polygon are identified (i.e. are given by the same $e \in \Sigma_{1}$ ), do not change its edge index. For an illustration, see Figure 4 Part 3. We call this a deck transformation.

These operations on the marking commute with each other in the sense that the final edge indices do not depend on the order in which a given set of operations 1-3 is applied.

Note that the operation in 3 is the same as moving around the marking of a face completely by applying operation 2. This lemma is proved in Appendix A.4.

Let $\Sigma$ be a surface with a fixed PLCW decomposition. Write ( $m, o, s$ ) for a given marking of $\Sigma$, where $m$ denotes the edge markings of the faces, $o$ the edge orientations and $s$ the edge indices (cf. Definition 2.9). Let $\mathcal{M}(\Sigma)_{\lambda, \mu}^{P L C W}$ denote the set of all admissible markings for the maps $\lambda$ and $\mu$ on $\Sigma$. The operations in Lemma 2.11 generate an equivalence relation $\sim_{\text {fix }}$ on $\mathcal{M}(\Sigma)_{\lambda, \mu}^{P L C W}$. Let us denote equivalence classes by $[m, o, s]$. The following lemma gives a more concrete description of the equivalence classes.
Lemma 2.12. Let $(m, o, s) \in \mathcal{M}(\Sigma)_{\lambda, \mu}^{P L C W}$. We have:

1. For every choice $m^{\prime}, o^{\prime}$ there is some $s^{\prime}$ such that $[m, o, s] \sim_{f i x}\left[m^{\prime}, o^{\prime}, s^{\prime}\right]$.
2. For a given choice of edge indices $\tilde{s}$ we have $[m, o, s] \sim_{\text {fix }}[m, o, \tilde{s}]$ if and only if $s$ and $\tilde{s}$ are related by a sequence of deck transformations (operation 3) in Lemma 2.11.

Proof. The first statement is immediate from operations 1 and 2 in Lemma 2.11. For the second statement recall that operations 1-3 commute, and operation 3 is redundant. Any sequence of operations can thus be written as $M=\prod_{e}$ (op. 1 for edge $\left.e\right) \prod_{f}$ (op. 2 for face $f$ ). Since $m$ and $o$ do not change, operation 1 for an edge $e$ must occur in pairs, leaving $s_{e}$ unchanged, and operation 2 for a face $f$ must occur in multiples of the number of edges of that face, so that the total change is expressible in terms of operation 3, $M=\prod_{f}($ op. 3 for face $f)$.

Let $\mathcal{R}^{r}(\Sigma)_{\lambda, \mu}$ denote the isomorphism classes of $r$-spin structures with $r$-spin boundary parametrisation for the maps $\lambda$ and $\mu$. The following theorem is proved in Appendix A.4. Theorem 2.13. Let $\Sigma$ be a surface with PLCW decomposition. The map

$$
\begin{align*}
\mathcal{M}(\Sigma)_{\lambda, \mu}^{P L C W} / \sim_{\mathrm{fix}} & \longrightarrow \mathcal{R}^{r}(\Sigma)_{\lambda, \mu} \\
{[m, o, s] } & \longmapsto[\Sigma(s, \lambda, \mu)] \tag{2.22}
\end{align*}
$$

is a bijection. On the right hand side it is understood that the edge markings and orientations of $\Sigma$ are given by $m, o$.

Remark 2.14. When combined with Lemma 2.12, this shows that for a fixed edge marking and orientation the admissible edge index assignments up to deck transformations are in bijection with the isomorphism classes of $r$-spin structures with $r$-spin boundary parametrisation for the maps $\lambda$ and $\mu$.


Figure 5: Elementary moves of a marked PLCW decomposition. Figure a) shows edges between faces $f$ and $f^{\prime}$ (which are allowed to be the same). The edges are marked so that the vertex $w$ is the clockwise vertex for the face $f$ (cf. Figure 3). This convention is not restrictive as one can change the orientation of the edges and the markings using Lemma 2.11. In Figure b), on the left hand side the horizontal edge between the vertices $v$ and $v^{\prime}$ (which are allowed to be the same) is marked for the top polygon, but not for the bottom polygon, and it has edge index 0 . For the joint polygon on the right hand side, the marked edge is taken to be that from the bottom polygon on the left. Note that this latter convention for the markings is not restrictive, as using Lemma 2.11 one can move the markings around.

### 2.4 Elementary moves on marked PLCW decompositions

In the previous section we defined the $r$-spin structure $\Sigma(s, \lambda, \mu)$ in terms of a marked PLCW decomposition, and we explained how to change the marking while staying within a given isomorphism class of $r$-spin structures. In this section we state how the marking needs to change when modifying the underlying PLCW decomposition by elementary moves in order to produce isomorphic $r$-spin structures.
Definition 2.15. An elementary move on a PLCW decomposition of a surface is either

- removing or adding a bivalent vertex as shown in Figure 5 a), or
- removing or adding an edge as shown in Figure 5 b).

By [Kir, Thm. 7.4], any two PLCW decompositions can be related by elementary moves. We prove the following proposition in Appendix A.4.
Proposition 2.16. The elementary moves in Figure 5 induce isomorphisms of $r$-spin structures.

The edge index of an edge with a univalent vertex is fixed by the orientation and the marking of the edge, in particular it is independent of the rest of the edge indices. In Lemma A. 6 we will show that removing univalent vertices induces an isomorphism of $r$-spin structures. For an illustration, see Figure 6.

### 2.5 Example: Connected $r$-spin surfaces

In this section we illustrate how one can use the combinatorial formalism to count isomorphism classes of $r$-spin structures. This recovers results obtained in [Ran, GG] using a different formalism.


Figure 6: The edge index of an univalent vertex is fixed by only the marking and orientation of edges. Removing a univalent vertex induces an isomorphism of $r$-spin structures.

Notations 2.17. Whenever it does not cause confusion we will use the same symbols for edge labels and for edge indices. For example for $e \in \Sigma_{1}$ we will simply write $e \in \mathbb{Z}_{r}$ instead of $s_{e} \in \mathbb{Z}_{r}$.

Lemma 2.18. There exists $r$-spin structures on the sphere if and only if $r=1$ or $r=2$. If there exists an $r$-spin structure on the sphere then it is unique up to isomorphism.

Proof. Let us consider the sphere decomposed into two 1-gons, one edge $u$ and one vertex $v$ as in Figure $2 c$ ), with edge index $u$ (cf. Notations 2.17). Let us collect the ingredients for the vertex condition (2.20). The edge $u$ starts and ends at the vertex, therefore $\hat{u}=-1$ from (2.18).

The number $N_{v}$ of in- and outgoing edges for $v$ is $N_{v}=1+1=2$, cf. (2.17). The number of faces with $v$ in clockwise direction from their marked edge is $D_{v}=2$, since the edge is marked for both faces. The vertex condition (2.20) then reads

$$
-1 \equiv 2-2+1 \quad(\bmod r)
$$

which holds if and only if $r=1$ or $r=2$. The edge index $u$ can be set arbitrarily by operation 3 in Lemma 2.11, and together with Remark 2.14 we see that for any two values of $u$ the $r$-spin structures on the sphere are isomorphic.

Proposition 2.19. Let $\Sigma_{g, b}$ be a connected surface of genus $g$ with $b$ boundary components and with maps $\lambda$ and $\mu$. There exists an $r$-spin structure on $\Sigma_{g, b}$ if and only if

$$
\begin{equation*}
\chi\left(\Sigma_{g, b}\right) \equiv \sum_{u \in \pi_{0}(\partial \Sigma)} R_{u} \quad(\bmod r) \tag{2.23}
\end{equation*}
$$

where $\chi\left(\Sigma_{g, b}\right)=2-2 g-b$ denotes the Euler characteristic and $R_{u}$ was defined in (2.19). If (2.23) holds, the number $\left|\mathcal{R}^{r}\left(\Sigma_{g, b}\right)_{\lambda, \mu}\right|$ of isomorphism classes of $r$-spin structures on $\Sigma_{g, b}$ is given by:

| $r$ | $b, g$ | $\left\|\mathcal{R}^{r}\left(\Sigma_{g, b}\right)_{\lambda, \mu}\right\|$ |
| :---: | :---: | :---: |
| 0 | $g=0$ and $b \in\{0,1\}$ | 1 |
|  | else | infinite |
| $>0$ | $b=0$ | $r^{2 g}$ |
|  | $b \geq 1$ | $r^{2 g+b-1}$ |



Figure 7: PLCW decomposition of $\Sigma_{g, b}$ for $g+b \geq 1$ using only one face, shown after glueing (Fig.a) and before glueing (Fig. b) - the edges labeled with the same symbols are identified. In Fig. b, the bigger arrows indicate the marked edge, namely edge $r_{1}$ in case $b>0$ and edge $s_{1}$ in case $b=0$.

A similar result has been obtained for the existence of $r$-spin structures on closed hyperbolic orbifolds for $r>0$ in [GG, Thm. 3]. Note that in complex geometry, (2.23) (for $r>0$ and $b=0$ ) is just the condition for the existence of an $r$-th root of the canonical line bundle (see e.g. [Wit]).

Proof. The case $g=b=0$ has been discussed in Lemma 2.18, so we can assume $g+b \geq 1$. Decompose $\Sigma_{g, b}$ into a $(4 g+3 b)$-gon consisting of $2 g+2 b$ inner edges, $b$ boundary edges, one inner vertex $v_{0}$ and $b$ boundary vertices $v_{j}, j=1, \ldots, b$, as shown in Figure $7 a$ ) and $b$ ). Assign the edge indices $s_{i}, t_{i}, r_{j}$ and $u_{j}$, where $i=1, \ldots, g$ and $j=1, \ldots, b$. Mark the edge $s_{1}$ if $g \neq 0$ or the edge $r_{1}$ if $g=0$, see Figure $7 b$ ).

We now evaluate the admissibility condition at each vertex. For the boundary vertex $v_{j}$ there is the incoming inner edge $r_{j}$ and the boundary edge $u_{j}$ which starts and ends at the same vertex $v_{j}$. Therefore by (2.18), relative to $v_{j}$ one has $\hat{r}_{j}=-r_{j}-1$ and $\hat{u}_{j}=-1$. For either of the two markings (for $g \neq 0$ and for $g=0$ ) $D_{v_{j}}=0$ and $N_{v_{j}}=3$, therefore we have

$$
\begin{equation*}
-r_{j}-1-1 \equiv 0-3+1-R_{u_{j}} \quad(\bmod r) \quad \text { for } j=1, \ldots, b \tag{2.24}
\end{equation*}
$$

Thus the $r_{j}$ are uniquely fixed by the boundary parametrisation $\lambda, \mu$ to be $r_{j} \equiv R_{u_{j}}(\bmod r)$ for all $j$.

For the inner vertex $v_{0}$ there are $b$ edges leaving the vertex and $2 g$ edges which start and end there. Therefore by (2.18), relative to $v_{0}$ one has $\hat{r}_{j}=r_{j}$ and $\hat{s}_{i}=\hat{t}_{i}=-1 . D_{v_{0}}=1$
and $N_{v_{0}}=4 g+b$, and so

$$
\begin{equation*}
\sum_{j=1}^{b} r_{j}-2 g \equiv 1-(4 g+b)+1 \quad(\bmod r) \tag{2.25}
\end{equation*}
$$

Combining (2.24) and (2.25) one obtains (2.23).
By Remark 2.14, for a fixed marking and orientation, edge index assignments up to deck transformations are in bijection with $r$-spin structures. From (2.24) and (2.25) every $\left(s_{i}, t_{i}, u_{j}\right) \in\left(\mathbb{Z}_{r}\right)^{2 g+b}$ gives an admissible edge index assignment. A deck transformation on the face shifts the $u_{j}$ parameters simultaneously and leaves the $s_{i}$ and $t_{i}$ parameters fixed. By a simple counting we get the number of isomorphism classes of $r$-spin structures.

Corollary 2.20. There is a unique $r$-spin structure on the disk with boundary condition $\lambda=2$ (ingoing boundary) or $\lambda=0$ (outgoing boundary), and no $r$-spin structure else.

Let $R_{j}:=R_{u_{j}}$ from (2.19) and let us denote the $r$-spin structure on $\Sigma_{g, b}$ given by the parameters $s_{i}, t_{i}, u_{j} \in \mathbb{Z}_{r}$ for $i=1, \ldots, g$ and $j=1, \ldots, b$ from Figure 7 by

$$
\begin{equation*}
\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, R_{j}\right) \tag{2.26}
\end{equation*}
$$

(and recall from Notation 2.17 that the same symbols denote edges and the assigned edge indices).

## 3 State-sum construction of $r$-spin TFTs

Our first application of the combinatorial description of $r$-spin structures is a state-sum construction of $r$-spin TFTs, see [BT, NR, GK] for the 2-spin case and [Nov] for general $r$. We generalise the construction in [Nov] from triangulations to PLCW-decompositions, which are much more convenient for explicit computations. In Sections 3.1 and 3.2 we present some algebraic preliminaries, and in Section 3.3 we explain how suitable Frobenius algebras produce an $r$-spin TFT via a state-sum construction (Theorem 3.8). In Section 3.4 we compute the value of the state-sum TFT on connected $r$-spin bordisms.

### 3.1 Algebraic notions

Let $\mathcal{S}$ denote a strict symmetric monoidal category with tensor product $\otimes$, tensor unit $\mathbb{I}$ and braiding $c$. We use the graphical calculus as shown in Figure 8, and we will omit the labels for objects if they are understood, as e.g. in Figure 9.

An object $A \in \mathcal{S}$ together with morphisms $\mu \in \mathcal{S}(A \otimes A, A)$ (multiplication), $\eta \in \mathcal{S}(\mathbb{I}, A)$ (unit), $\Delta \in \mathcal{S}(A, A \otimes A)$ (comultiplication) and $\varepsilon \in \mathcal{S}(A, \mathbb{I})$ (counit), see Figure 9 , is a


Figure 8: String diagram notion of a morphism $f \in \mathcal{S}(A \otimes B \otimes C, D \otimes E)$, the symmetric braiding $c_{A, B}$ and the identity $\operatorname{id}_{A}$.


Figure 9: String diagrams we will use for the structure morphisms of a Frobenius algebra.

Frobenius algebra if the following relations hold:


These relations imply that a Frobenius algebra is in particular an associative algebra and coassociative coalgebra, see [Koc, Prop. 2.3.24]. For more details on the definition of algebras, coalgebras and Frobenius algebras in monoidal categories we refer to e.g. [FS].

For a Frobenius algebra $A$ we define the window element $\tau=\mu \circ \Delta \circ \eta$ [LP] and the Nakayama automorphism $N=\left(\mathrm{id}_{A} \otimes(\varepsilon \circ \mu)\right) \circ\left(c_{A, A} \otimes \mathrm{id}_{A}\right) \circ\left(\mathrm{id}_{A} \otimes(\Delta \circ \eta)\right)$, see Figure 10. Then $\tau$ is central (as follows from an easy calculation) and $N$ is a morphism of Frobenius


Figure 10: The window element $\tau$, the Nakayama automorphism $N$, its inverse $N^{-1}$ [Nov, Sec.5.3] and our string diagram abbreviation for the $k$ 'th power of $N$.
algebras (see [FS] and [NR, Prop. 4.5]). $A$ is called symmetric if $\varepsilon \circ \mu \circ c_{A, A}=\varepsilon \circ \mu$. It can be shown from a straightforward calculation that $A$ is symmetric if and only if $N=\mathrm{id}_{A}$.
Lemma 3.1. Let $A \in \mathcal{S}$ Frobenius algebra with Nakayama automorphism $N$. Then for every $n \in \mathbb{Z}$


Proof. The first and second equations are proven in [Nov, Lem. 5.12]. The third equation follows from a direct calculation.

A morphism $\kappa: \mathbb{I} \rightarrow A$ is called invertible if there is a morphism $\kappa^{\prime}: \mathbb{I} \rightarrow A$ such that $\mu \circ\left(\kappa \otimes \kappa^{\prime}\right)=\eta=\mu \circ\left(\kappa^{\prime} \otimes \kappa\right)$. In this case we write $\kappa^{-1}$ instead of $\kappa^{\prime}$ for the unique inverse.

Let $r \in \mathbb{Z}_{\geq 0},(A, \mu, \eta, \Delta, \varepsilon)$ be a Frobenius algebra in $\mathcal{S}$ with invertible window element $\tau$ and with Nakayama automorphism $N$, such that $N^{r}=\operatorname{id}_{A}$ (for $r=0$ this last condition is empty). A Frobenius algebra with $N^{r}=\mathrm{id}_{A}$ is called a $\Lambda_{r}$-Frobenius algebra in [DK, Prop. I.41]. Define

$$
\begin{equation*}
P_{\lambda}:=\left(\tau^{-1} \cdot(-)\right) \circ \mu \circ c_{A, A} \circ\left(\mathrm{id} \otimes N^{1-\lambda}\right) \circ \Delta \quad \in \operatorname{End}(A) . \tag{3.4}
\end{equation*}
$$

We collect some properties of $P_{\lambda}$ in the following lemma.
Lemma 3.2. $P_{\lambda}$ is an idempotent, and for any $\lambda_{1}, \lambda_{2} \in \mathbb{Z}_{r}$ one has that:

$$
\begin{align*}
N \circ P_{\lambda_{1}} & =P_{\lambda_{1}} \circ N & &  \tag{3.5}\\
\mu \circ\left(P_{\lambda_{1}} \otimes P_{\lambda_{2}}\right) & =P_{\lambda_{1}+\lambda_{2}} \circ \mu \circ\left(P_{\lambda_{1}} \otimes P_{\lambda_{2}}\right), & & \eta=P_{0} \circ \eta,  \tag{3.6}\\
\left(P_{\lambda_{1}} \otimes P_{\lambda_{2}}\right) \circ \Delta & =\left(P_{\lambda_{1}} \otimes P_{\lambda_{2}}\right) \circ \Delta \circ P_{\lambda_{1}+\lambda_{2}-2}, & & \varepsilon=\varepsilon \circ P_{2} . \tag{3.7}
\end{align*}
$$

Proof. That $P_{\lambda}$ is an idempotent is a direct generalisation of [Nov, Lem. 5.12(1)]. The additional $\left(\tau^{-1} \cdot(-)\right)$ removes the "bubble" $\mu \circ \Delta$. The identity (3.5) is immediate from the definition of $P_{\lambda}$ in (3.4) and the fact that $N$ is an automorphism of Frobenius algebras. The first identity in (3.6) is a more general version of [Nov, Lem. 6.8] and the proof works along the same lines. To show the second identity in each of (3.6) and (3.7) just write out the definition of $P_{\lambda}, N$ and $N^{-1}$. For the first identity in (3.7) use (3.6) together with

$$
\begin{equation*}
\left((\varepsilon \circ \mu) \otimes \operatorname{id}_{A}\right) \circ\left(\operatorname{id}_{A} \otimes P_{\lambda} \otimes \operatorname{id}_{A}\right) \circ\left(\operatorname{id}_{A} \otimes(\Delta \circ \eta)\right)=P_{2-\lambda}, \tag{3.8}
\end{equation*}
$$

which follows from a direct calculation.

### 3.2 The $\mathbb{Z}_{r}$-graded center

Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element. Let $\mathcal{S}$ be furthermore additive (in particular, finite direct sums distribute over tensor products) and assume that the idempotents $P_{\lambda}$ split in $\mathcal{S}$, i.e.

$$
\begin{equation*}
P_{\lambda}=\left[A \xrightarrow{\pi_{\lambda}} Z_{\lambda} \xrightarrow{\iota_{\lambda}} A\right], \quad\left[Z_{\lambda} \xrightarrow{\iota_{\lambda}} A \xrightarrow{\pi_{\lambda}} Z_{\lambda}\right]=\mathrm{id}_{Z_{\lambda}} \tag{3.9}
\end{equation*}
$$

for some object $Z_{\lambda} \in \mathcal{S}$. For $r=0$ assume furthermore that $\mathcal{S}$ has countably infinite direct sums which distribute over the tensor product. We can now define:

Definition 3.3. Let $r \in \mathbb{Z}_{\geq 0}$. The $\mathbb{Z}_{r}$-graded center of a Frobenius algebra $A$ with invertible window element and which satisfies $N^{r}=$ id is the direct sum

$$
\begin{equation*}
Z^{r}(A):=\bigoplus_{\lambda \in \mathbb{Z}_{r}} Z_{\lambda} \tag{3.10}
\end{equation*}
$$

This is a $\mathbb{Z}_{r}$-graded object and we call $\lambda \in \mathbb{Z}_{r}$ the degree of $Z_{\lambda}$.
Next we will endow $Z^{r}(A)$ with an algebra structure induced by $A$. Write

$$
\begin{equation*}
e_{\lambda}: Z_{\lambda} \rightarrow Z^{r}(A) \tag{3.11}
\end{equation*}
$$

for the embeddings of the summands in (3.11) and $p_{\lambda}$ for the induced projections which satisfy

$$
\begin{equation*}
\left[Z_{\lambda_{1}} \xrightarrow{e_{\lambda_{1}}} Z^{r}(A) \xrightarrow{p_{\lambda_{2}}} Z_{\lambda_{2}}\right]=\delta_{\lambda_{1}, \lambda_{2}} \operatorname{id}_{Z_{\lambda_{1}}} . \tag{3.12}
\end{equation*}
$$

Lemma 3.2 suggests to define, for $\lambda_{1}, \lambda_{2} \in \mathbb{Z}_{r}$,

$$
\begin{equation*}
\mu_{\lambda_{1}, \lambda_{2}}:=\left[Z_{\lambda_{1}} \otimes Z_{\lambda_{2}} \xrightarrow{\iota_{\lambda_{1}} \otimes \lambda_{2}} A \otimes A \xrightarrow{\mu} A \xrightarrow{\pi_{\lambda_{1}+\lambda_{2}}} Z_{\lambda_{1}+\lambda_{2}}\right] . \tag{3.13}
\end{equation*}
$$

By the universal property of direct sums (which in the countably infinite case for $r=0$ still distribute over $\otimes$ by our assumptions) there is a unique map

$$
\begin{equation*}
\bar{\mu}: Z^{r}(A) \otimes Z^{r}(A) \longrightarrow Z^{r}(A) \tag{3.14}
\end{equation*}
$$

which satisfies $\bar{\mu} \circ\left(e_{\lambda_{1}} \otimes e_{\lambda_{2}}\right)=e_{\lambda_{1}+\lambda_{2}} \circ \mu_{\lambda_{1}, \lambda_{2}}$. Let us furthermore define

$$
\begin{equation*}
\bar{\eta}:=\left[\mathbb{I} \xrightarrow{\eta} A \xrightarrow{\pi_{0}} Z_{0} \xrightarrow{e_{0}} Z^{r}(A)\right], \tag{3.15}
\end{equation*}
$$

The morphisms $\bar{\mu}$ and $\bar{\eta}$ are degree preserving. It is straightforward to verify that $Z^{r}(A)$ together with $\bar{\mu}$ and $\bar{\eta}$ becomes an associative unital $\mathbb{Z}_{r}$-graded algebra in $\mathcal{S}$.

One can restrict the Nakayama automorphism of $A$ on the $Z_{\lambda}$ 's by

$$
\begin{equation*}
N_{Z_{\lambda}}:=\left[Z_{\lambda} \xrightarrow{\iota_{\lambda}} A \xrightarrow{N} A \xrightarrow{\pi_{\lambda}} Z_{\lambda}\right] \tag{3.16}
\end{equation*}
$$

As in [Nov, Lem. 5.12/3] on verifies that

$$
\begin{equation*}
N_{Z_{\lambda}}^{\operatorname{gcd}(1-\lambda, r)}=\operatorname{id}_{Z_{\lambda}} \tag{3.17}
\end{equation*}
$$

Recall from the introduction that $\operatorname{gcd}(a, b)$ denotes the non-negative generator of the ideal $\langle a, b\rangle \subset \mathbb{Z}$. In particular, for $r=0$ we have $\operatorname{gcd}(1-\lambda, r)=|1-\lambda|$. The product $\bar{\mu}$ is in general not commutative, but a simple computation shows that its components satisfy:

$$
\begin{align*}
\mu_{\lambda_{1}, \lambda_{2}} \circ c_{Z_{\lambda_{2}}, Z_{\lambda_{1}}} & =\mu_{\lambda_{2}, \lambda_{1}} \circ\left(N_{\lambda_{2}}^{-\lambda_{1}} \otimes \operatorname{id}_{Z_{\lambda_{1}}}\right) \\
& =\mu_{\lambda_{2}, \lambda_{1}} \circ\left(\operatorname{id}_{Z_{\lambda_{2}}} \otimes N_{\lambda_{1}}^{+\lambda_{2}}\right) . \tag{3.18}
\end{align*}
$$

Let $\bar{N}: Z^{r}(A) \rightarrow Z^{r}(A)$ be the unique morphism such that $\bar{N} \circ e_{\lambda}=e_{\lambda} \circ N_{\lambda}$ for all $\lambda$. Combining the fact that $N$ is an automorphism of Frobenius algebras with the definition of $\bar{\mu}$ and $\bar{\eta}$ and using (3.5) shows that $\bar{N}$ is an algebra automorphism. By (3.17) we have $\bar{N}^{r}=$ id. We collect the above results in the following proposition.
Proposition 3.4. Let $A$ be as in Definition 3.3. The $\mathbb{Z}_{r}$-graded center $Z^{r}(A)$ of $A$ is an associative unital algebra via $\bar{\mu}, \bar{\eta}$ and is equipped with the algebra automorphism $\bar{N}$ satisfying $\bar{N}^{r}=\mathrm{id}$. The algebra $Z^{r}(A)$ satisfies the commutativity conditions, for $\lambda \in \mathbb{Z}_{r}$,

$$
\begin{align*}
& \bar{\mu} \circ c_{Z^{r}(A), Z^{r}(A)} \circ\left(\mathrm{id} \otimes e_{\lambda}\right)=\bar{\mu} \circ\left(\bar{N}^{-\lambda} \otimes e_{\lambda}\right), \\
& \bar{\mu} \circ c_{Z^{r}(A), Z^{r}(A)} \circ\left(e_{\lambda} \otimes \mathrm{id}\right)=\bar{\mu} \circ\left(e_{\lambda} \otimes \bar{N}^{\lambda}\right) . \tag{3.19}
\end{align*}
$$

Corollary 3.5. The component $Z_{0}$ of $Z^{r}(A)$ is a subalgebra and is the centre of $A$.

## Frobenius algebra structure for $r>0$

For the rest of this section let us assume that $r>0$. Since now $\mathbb{Z}_{r}$ is finite, we can define the coproduct as the sum

$$
\begin{equation*}
\bar{\Delta}:=\sum_{\lambda_{1}, \lambda_{2} \in \mathbb{Z}_{r}}\left[Z^{r}(A) \xrightarrow{p_{\lambda_{1}+\lambda_{2}-2}} Z_{\lambda_{1}+\lambda_{2}-2} \xrightarrow{\Delta_{\lambda_{1}, \lambda_{2}}} Z_{\lambda_{1}} \otimes Z_{\lambda_{2}} \xrightarrow{e_{\lambda_{1}} \otimes e_{\lambda_{2}}} Z^{r}(A) \otimes Z^{r}(A)\right] \tag{3.20}
\end{equation*}
$$

with component maps

$$
\begin{equation*}
\Delta_{\lambda_{1}, \lambda_{2}}:=\left[Z_{\lambda_{1}+\lambda_{2}-2} \xrightarrow{\iota_{\lambda_{1}+\lambda_{2}-2}} A \xrightarrow{\Delta \circ(\tau \cdot(-))} A \otimes A \xrightarrow{\pi_{\lambda_{1}} \otimes \pi_{\lambda_{2}}} Z_{\lambda_{2}} \otimes Z_{\lambda_{1}}\right] . \tag{3.21}
\end{equation*}
$$

We define the counit

$$
\begin{equation*}
\bar{\varepsilon}:=\left[Z^{r}(A) \xrightarrow{p_{2}} Z_{2} \xrightarrow{\iota_{2}} A \xrightarrow{\left(\tau^{-1} \cdot(-)\right)} A \xrightarrow{\varepsilon} \mathbb{I}\right] . \tag{3.22}
\end{equation*}
$$

The morphisms $\bar{\Delta}$ and $\bar{\varepsilon}$ have degree +2 and -2 respectively. Note that we inserted a multiplication with $\tau$ and its inverse in the definition of $\bar{\varepsilon}$ and $\bar{\Delta}$. The reason for this is
that we want these maps to match the structure maps calculated in Section 3.4 from the state-sum construction.

It is straightforward to see that altogether $Z^{r}(A)$ becomes a Frobenius algebra, just verify (3.1) and (3.2) restricted to individual summands of $Z^{r}(A)$ by using Lemma 3.2 to move projectors past structure maps of $A$ and by the properties of $A$ itself. Altogether we have:
Proposition 3.6. Let $A$ be as in Definition 3.3. For $r>0$, the $\mathbb{Z}_{r}$ graded center of $A$ together with $\bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\varepsilon}$ is a $\mathbb{Z}_{r}$-graded Frobenius algebra. The morphisms $\bar{\mu}$ and $\bar{\eta}$ have degree 0 , while $\bar{\Delta}$ has degree 2 and $\bar{\varepsilon}$ has degree -2 .

Remark 3.7. 1. The condition $N^{r}=\mathrm{id}_{A}$ amounts to $A$ being a representation of the group $\mathbb{Z}_{r}$. Instead of defining this in a general category, let $k$ be a field and let us assume that $A \in \operatorname{Rep}_{k}\left(\mathbb{Z}_{r}\right)$, the category of $k$-linear representations of $\mathbb{Z}_{r}$. Then the algebra $Z^{r}(A)$ is the full center of $A$ as defined in [Dav], and is in particular a commutative algebra in $\mathcal{Z}\left(\operatorname{Rep}_{k}\left(\mathbb{Z}_{r}\right)\right)$, the monoidal center of $\operatorname{Rep}_{k}\left(\mathbb{Z}_{r}\right)$. To see this one needs to check that $Z^{r}(A)$ has the form of the full center as given in [Dav, Prop. 9.6], which has been done in (3.18).
Note, however, that unless $r=1$ or $r=2$, the counit $\bar{\varepsilon}$ and the comultiplication $\bar{\Delta}$ are not degree preserving, i.e. $Z^{r}(A)$ is not a Frobenius algebra in $\mathcal{Z}\left(\mathbb{Z}_{r}\right)$ with these structure maps.
2. For $r=0$ one still obtains for every $\lambda_{1}, \lambda_{2} \in \mathbb{Z}$ a non-degeneracy condition, which we do not explain in detail.

### 3.3 State-sum construction

Let again $r \geq 0$ and $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element $\tau$ and with $N^{r}=\operatorname{id}_{A}$. In this section we define a symmetric monoidal functor $Z_{A}: \mathcal{B}_{\text {ord }}^{r} \boldsymbol{r} \rightarrow \mathcal{S}$, that is, a TFT on two-dimensional $r$-spin bordisms.

Recall the direct sum decomposition $Z^{r}(A)=\bigoplus_{\lambda \in \mathbb{Z}_{r}} Z_{\lambda}$ of the $\mathbb{Z}_{r}$-graded centre from Definition 3.3. We define the TFT $Z_{A}$ on objects as follows: Let $\rho: X \rightarrow \mathbb{Z}_{r}$ be an $r$-spin object. Then

$$
\begin{equation*}
Z_{A}(\rho):=\bigotimes_{x \in X} Z_{\rho_{x}} \tag{3.23}
\end{equation*}
$$

To define $Z_{A}$ on morphisms is more involved and will take up the remainder of this section. Let $(X, \rho)$ and $(Y, \sigma)$ be two $r$-spin objects. Let $\Sigma: \rho \rightarrow \sigma$ be an $r$-spin bordism with maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}, \mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$. Choose a decorated PLCW decomposition $\Sigma=\Sigma_{2} \cup \Sigma_{1} \cup \Sigma_{0}$ of the surface $\Sigma$ with admissible edge index assignment $s$ such that the $r$-spin structure with parametrised boundary $\Sigma(s, \lambda, \mu)$ from Definition 2.10 is isomorphic to the $r$-spin structure of the $r$-spin bordism $\Sigma: \rho \rightarrow \sigma$. Recall that $B_{\text {in }}$ and $B_{\text {out }}$ denote the in- and outgoing boundary components respectively and that by our conventions they are in bijection with edges on the boundary.


Figure 11: $a)$ Left and right sides $(e, l)$ and $(e, r)$ of an inner edge $e$, determined by the orientation of $\Sigma$ (paper orientation) and of $e$ (arrow). The edge index of $e$ is $s_{e}$. b) Convention for connecting tensor factors belonging to edge sides $(e, l)$ and $(e, r)$ of an inner edge $e$ with the tensor factors belonging to the morphism $g_{e}$.

For a face $f \in \Sigma_{2}$ which is an $n_{f}$-gon let us write $(f, k), k=1, \ldots, n_{f}$ for the sides of $f$, where $(f, 1)$ denotes the marked edge of $f$, and the labeling proceeds counter-clockwise with respect to the orientation of $f$. We collect the sides of all faces into a set:

$$
\begin{equation*}
S:=\left\{(f, k) \mid f \in \Sigma_{2}, k=1, \ldots, n_{f}\right\} . \tag{3.24}
\end{equation*}
$$

We double the set of edges by considering $\Sigma_{1} \times\{l, r\}$, where " $l$ " and " $r$ " stand for left and right, respectively. Let $E \subset \Sigma_{1} \times\{l, r\}$ be the subset of all $(e, l)$ (resp. $(e, r)$ ), which have a face attached on the left (resp. right) side, cf. Figure $11 a)$. Thus for an inner edge $e \in \Sigma_{1}$ the set $E$ contains both $(e, l)$ and $(e, r)$, but for a boundary edge $e^{\prime} \in \Sigma_{1}$ the set $E$ contains either $\left(e^{\prime}, l\right)$ or $\left(e^{\prime}, r\right)$. By construction of $S$ and $E$ we obtain a bijection

$$
\begin{equation*}
\Phi: E \xrightarrow{\sim} S \quad, \quad(e, x) \mapsto(f, k), \tag{3.25}
\end{equation*}
$$

where $e$ is the $k$ 'th edge on the boundary of the face $f$ lying on the side $x$ of $e$, counted counter-clockwise from the marked edge of $f$.

For every vertex $v \in \Sigma_{0}$ in the interior of $\Sigma$ or on an ingoing boundary component of $\Sigma$ choose a side of an edge $(e, x) \in E$ for which $v \in \partial(e)$. Let

$$
\begin{equation*}
V: \Sigma_{0} \backslash B_{\text {out }} \rightarrow E \tag{3.26}
\end{equation*}
$$

be the resulting function.
To define $Z_{A}(\Sigma)$ we proceed with the following steps.

1. Let us introduce the tensor products

$$
\begin{align*}
& \mathcal{A}_{S}:=\bigotimes_{(f, k) \in S} A^{(f, k)}, \quad \mathcal{A}_{E}:=\bigotimes_{(e, x) \in E} A^{(e, x)}, \\
& \mathcal{A}_{\text {in }}:=\bigotimes_{b \in B_{\text {in }}} A^{(b, i n)}, \quad \mathcal{A}_{\text {out }}:=\bigotimes_{c \in B_{\text {out }}} A^{(c, o u t)} \tag{3.27}
\end{align*}
$$

Every tensor factor is equal to $A$, but the various superscripts will help us distinguish tensor factors in the source and target objects of the morphisms we define in the remaining steps.
2. For an edge $e \in \Sigma_{1}$ we set

$$
g_{e}:= \begin{cases}A^{(e, i n)} \xrightarrow{N^{-s_{e}-1}} A^{(e, i n)} & ; e \in B_{\text {in }}  \tag{3.28}\\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text { id }_{A} \otimes N^{s_{e}+1}} A^{(e, l)} \otimes A^{(e, o u t)} & ; e \in B_{\text {out }}, \text { surface is left of } e \\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_{A} \otimes N^{s_{e}+1}} A^{(e, o u t)} \otimes A^{(e, r)} & ; e \in B_{\text {out }}, \text { surface is right of } e \\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_{A} \otimes N^{s_{e}+1}} A^{(e, l)} \otimes A^{(e, r)} & ; e \text { inner edge }\end{cases}
$$

cf. Figure 11. Define the linear map

$$
\begin{equation*}
\mathcal{C}:=\bigotimes_{e \in \Sigma_{1}} g_{e}: \mathcal{A}_{\text {in }} \rightarrow \mathcal{A}_{E} \otimes \mathcal{A}_{\text {out }} \tag{3.29}
\end{equation*}
$$

where it is understood that the tensor factors in $\mathcal{A}_{E} \otimes \mathcal{A}_{\text {out }}$ are assigned as indicated in (3.28).
3. Note that since all tensor factors in $\mathcal{A}_{E}$ are algebras, so is $\mathcal{A}_{E}$ itself. For $a: \mathbb{I} \rightarrow A$ and $(e, x) \in E$ write

$$
\begin{equation*}
a^{(e, x)}=\eta \otimes \cdots \otimes a \otimes \cdots \otimes \eta: \mathbb{I} \rightarrow \mathcal{A}_{E} \tag{3.30}
\end{equation*}
$$

where $a$ maps to the tensor factor $A^{(e, x)}$. Define $z: \mathbb{I} \rightarrow \mathcal{A}_{E}$ as the following product in the $k$-algebra $\mathcal{S}\left(\mathbb{I}, \mathcal{A}_{E}\right)$ :

$$
\begin{equation*}
z=\prod_{v \in \Sigma_{0} \backslash B_{\text {out }}}\left(\tau^{-1}\right)^{V(v)} \tag{3.31}
\end{equation*}
$$

Finally, we let $\mathcal{Z}$ be the endomorphism of $\mathcal{A}_{E}$ obtained by multiplying with $z$,

$$
\begin{equation*}
\mathcal{Z}:=\left[\mathcal{A}_{E} \xrightarrow{z \cdot(-)} \mathcal{A}_{E}\right] \tag{3.32}
\end{equation*}
$$

4. Let $\mu^{(1)}:=\operatorname{id}_{A}$ and let $\mu^{(n)}$ denote the $n$-fold product for $n \geq 2$. Assign to every face $f \in \Sigma_{2}$ obtained from an $n_{f}$-gon the morphism $\varepsilon \circ \mu^{\left(n_{f}\right)}: A_{(f, 1)} \otimes \cdots \otimes A_{\left(f, n_{f}\right)} \rightarrow \mathbb{I}$ and take their tensor product:

$$
\begin{equation*}
\mathcal{F}:=\bigotimes_{f \in \Sigma_{2}}\left(\varepsilon \circ \mu^{\left(n_{f}\right)}\right): \mathcal{A}_{S} \rightarrow \mathbb{I} \tag{3.33}
\end{equation*}
$$

5. We will now put the above morphisms together to obtain a morphism $\mathcal{L}: \mathcal{A}_{\text {in }} \rightarrow \mathcal{A}_{\text {out }}$. Denote by $\Pi_{\Phi}$ the permutation of tensor factors induced by $\Phi: E \rightarrow S$,

$$
\begin{equation*}
\Pi_{\Phi}: \mathcal{A}_{E} \rightarrow \mathcal{A}_{S} \tag{3.34}
\end{equation*}
$$

Using this, we define

$$
\begin{align*}
\mathcal{K} & :=\left[\mathcal{A}_{E} \xrightarrow{\mathcal{Z}} \mathcal{A}_{E} \xrightarrow{\Pi_{\Phi}} \mathcal{A}_{S} \xrightarrow{\mathcal{F}} \mathbb{I}\right],  \tag{3.35}\\
\mathcal{L} & :=\left[\mathcal{A}_{\text {in }} \xrightarrow{\mathcal{C}} \mathcal{A}_{E} \otimes \mathcal{A}_{\text {out }} \xrightarrow{\mathcal{K} \otimes \mathrm{id}_{\mathcal{A}_{\text {out }}}} \mathcal{A}_{\text {out }}\right] . \tag{3.36}
\end{align*}
$$

6. Let $\Pi_{\text {in }}$ and $\Pi_{\text {out }}$ denote the permutation of tensor factors induced by the maps $\beta_{\text {in }}$ and $\beta_{\text {out }}$ respectively:

$$
\begin{gather*}
\Pi_{i n}: Z_{A}(\rho)=\bigotimes_{x \in X} Z_{\rho_{x}} \rightarrow \bigotimes_{b \in B_{i n}} Z_{\lambda_{b}}  \tag{3.37}\\
\Pi_{\text {out }}: \bigotimes_{c \in B_{\text {out }}} Z_{\mu_{c}} \rightarrow \bigotimes_{y \in Y} Z_{\sigma_{y}}=Z_{A}(\sigma) \tag{3.38}
\end{gather*}
$$

Using these permutations and the embedding and projection maps $\iota_{\lambda}, \pi_{\lambda}$ from (3.9) we construct the morphisms linking the action of $Z_{A}$ on objects to the tensor products $\mathcal{A}_{\text {in/out }}$ :

$$
\begin{align*}
\mathcal{E}_{\text {in }} & :=\left[Z_{A}(\rho) \xrightarrow{\Pi_{i n}} \bigotimes_{b \in B_{\text {in }}} Z_{\lambda_{b}} \xrightarrow{\bigotimes_{b \in B_{\text {in }}} \lambda_{b}} \mathcal{A}_{\text {in }}\right]  \tag{3.39}\\
\mathcal{E}_{\text {out }} & :=\left[\mathcal{A}_{\text {out }} \xrightarrow{\otimes_{c \in B_{\text {out }}} \pi_{\mu_{c}}} \bigotimes_{c \in B_{\text {out }}} Z_{\mu_{c}} \xrightarrow{\Pi_{\text {out }}} Z_{A}(\sigma)\right] \tag{3.40}
\end{align*}
$$

We have now gathered all ingredients to define the action of $Z_{A}$ on morphisms:

$$
\begin{equation*}
Z_{A}(\Sigma):=\left[Z_{A}(\rho) \xrightarrow{\mathcal{E}_{\text {in }}} \mathcal{A}_{\text {in }} \xrightarrow{\mathcal{L}} \mathcal{A}_{\text {out }} \xrightarrow{\mathcal{E}_{\text {out }}} Z_{A}(\sigma)\right] . \tag{3.41}
\end{equation*}
$$

Theorem 3.8. Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element $\tau$ and with $N^{r}=\mathrm{id}_{A}$.

1. The morphism defined in (3.41) is independent of the choice of the marked PLCW decomposition and the assignment $V$.
2. The state-sum construction yields a symmetric monoidal functor $Z_{A}: \mathcal{B o r d}_{2}^{r} \rightarrow \mathcal{S}$ whose action on objects and morphisms is given by (3.23) and (3.41), respectively.

The proof of this theorem works by reducing to the corresponding statement for triangulations and is given in Appendix A.5.

Remark 3.9. The above construction yields a TFT on the category of closed $r$-spin bordisms, where the complete boundary of the $r$-spin bordisms is parametrised, so the parametrised boundary is a closed manifold. One can define a different $r$-spin bordism category, called the open-closed $r$-spin bordism category, where only a one dimensional submanifold of the boundary of $r$-spin surfaces is parametrised. The subcategory of the latter generated by the open cup, the open pair of pants and their duals is called the open $r$-spin bordism category. In [Ster] a TFT on open $r$-spin bordisms was constructed using $\Lambda_{r}$-Frobenius algebras [DK, Prop. I.41] which are Frobenius algebras whose Nakayama automorphism $N$ satisfies $N^{r}=\mathrm{id}$.

### 3.4 Evaluation of state-sum TFTs on generating $r$-spin bordisms

In this section we apply the state-sum construction from Theorem 3.8 to pairs of pants and discs with $r$-spin structure. On the one hand, these bordisms generate $\mathcal{B o r d} d_{2}^{r}$, and on the other hand, we will recover the algebra structure of the $\mathbb{Z}_{r}$-graded center $Z^{r}(A)$ of $A$ in this way. Finally, we evaluate $Z_{A}$ on a connected bordism of genus $g$ with only ingoing boundary components.

## Pair of pants as multiplication

Consider the $r$-spin 3-holed sphere parametrised as in Section 2.5 with 2 ingoing boundary components $B_{\text {in }}=\left\{u_{1}, u_{2}\right\}$ and 1 outgoing boundary component $B_{\text {out }}=\left\{u_{3}\right\}$ between $r$-spin objects $\rho:\left\{x_{1}, x_{2}\right\} \rightarrow \mathbb{Z}_{r}$ and $\sigma:\{y\} \rightarrow \mathbb{Z}_{r}$ with $\beta_{\text {in }}\left(x_{i}\right)=u_{i}(i=1,2)$ and $\beta_{\text {out }}(y)=u_{3}$. Let $\lambda_{1}:=\lambda_{u_{1}}, \lambda_{2}:=\lambda_{u_{2}}$ and $\lambda_{3}:=\mu_{u_{3}}$. Then by (2.19) $R_{u_{1}}=\lambda_{1}-1$, $R_{u_{2}}=\lambda_{2}-1$ and $R_{u_{3}}=1-\lambda_{3}$. Substituting these and $\chi\left(\Sigma_{0,3}\right)=2-0-3=-1$ in (2.23) gives

$$
\begin{equation*}
\lambda_{1}+\lambda_{2} \equiv \lambda_{3} \quad(\bmod r) \tag{3.42}
\end{equation*}
$$

Denote this $r$-spin bordism by

$$
S_{1,2}\left(u_{1}, u_{2}, u_{3}, \lambda_{1}, \lambda_{2}\right):=\Sigma_{0,3}\left(u_{1}, u_{2}, u_{3}, \lambda_{1}-1, \lambda_{2}-1,1-\lambda_{3}\right): \rho \rightarrow \sigma,
$$

(cf. (2.26)). The sets $S, E$ are (see Figure 7)

$$
\begin{align*}
S & =\{(f, k) \mid k=1, \ldots, 9\} \simeq\{1, \ldots, 9\}  \tag{3.43}\\
E & =\left\{\left(u_{1}, r\right),\left(u_{2}, r\right),\left(r_{1}, l\right),\left(r_{1}, r\right),\left(r_{2}, l\right),\left(r_{2}, r\right),\left(r_{3}, l\right),\left(r_{3}, r\right),\left(u_{3}, r\right)\right\} \\
& \simeq\{1, \ldots, 9\} \tag{3.44}
\end{align*}
$$

where in (3.44) the isomorphism is given by the order of elements of $E$ as listed. We have one inner vertex $v_{0}$ and 3 boundary vertices $v_{1}, v_{2}$ and $v_{3}$, with $v_{3}$ placed on the outgoing boundary component. We set

$$
\begin{equation*}
V\left(v_{0}\right):=\left(r_{2}, r\right), \quad V\left(v_{1}\right):=\left(u_{1}, r\right), \quad V\left(v_{2}\right):=\left(u_{2}, r\right) . \tag{3.45}
\end{equation*}
$$

Following the steps of the state-sum construction we get:
2. For the various edge indices

$$
\mathcal{C}=N^{-u_{1}-1} \otimes N^{-u_{2}-1} \otimes g_{r_{1}} \otimes g_{r_{2}} \otimes g_{r_{3}} \otimes g_{u_{3}}
$$

from (3.29). Recall from Notation 2.17 that the same symbols denote edges and the assigned edge indices.
3. For the inner vertex and the ingoing vertices we set

$$
z=\left(\tau^{-1}\right)^{\otimes 2} \otimes \eta^{\otimes 3} \otimes\left(\tau^{-1}\right) \otimes \eta^{\otimes 3}
$$

from (3.31) according to the map $V$ in (3.45).
4. For the single 9 -gon $\mathcal{F}=\varepsilon \circ \mu^{(9)}$ from (3.33).
5. The permutation is $\Pi_{\Phi}=(12543)(89)$ from (3.34) where we use the cycle notation for the permutation of tensor factors. After a calculation using associativity of the product and the last equation of (3.3), the morphism $\mathcal{L}$ in (3.36) is

$$
\begin{equation*}
\left[A \otimes A \xrightarrow{P_{\lambda_{1}} \circ N^{-u_{1}-1} \otimes P_{\lambda_{2}} \circ N^{-u_{2}-1}} A \otimes A \xrightarrow{\mu} A \xrightarrow{P_{\lambda_{1}+\lambda_{2}} \circ N^{u_{3}+1}} A\right] \tag{3.46}
\end{equation*}
$$

6. For the in- and outgoing boundary components we get $\mathcal{E}_{\text {in }}=\iota_{\lambda_{1}} \otimes \iota_{\lambda_{2}}$ and $\mathcal{E}_{\text {out }}=\pi_{\lambda_{1}+\lambda_{2}}$ from (3.39) and (3.40), since the permutations induced by $\beta_{\text {in }}$ and $\beta_{\text {out }}$ from (3.37) and (3.38) are identities. Also note that $\rho_{x_{1}}=\lambda_{1}$, etc. Finally by composing $\mathcal{L}$ with $\mathcal{E}_{\text {in }}$ and $\mathcal{E}_{\text {out }}$ as in (3.41) we obtain

$$
\begin{align*}
& Z_{A}\left(S_{1,2}\left(u_{1}, u_{2}, u_{3}, \lambda_{1}, \lambda_{2}\right)\right) \\
= & {\left[Z_{\lambda_{1}} \otimes Z_{\lambda_{2}} \xrightarrow{N_{\lambda_{1}}^{-u_{1}} \otimes N_{\lambda_{2}}^{-u_{2}}} Z_{\lambda_{1}} \otimes Z_{\lambda_{2}} \xrightarrow{\mu_{\lambda_{1}, \lambda_{2}}} Z_{\lambda_{1}+\lambda_{2}} \xrightarrow{N_{\lambda_{3}}^{u_{3}}} Z_{\lambda_{1}+\lambda_{2}}\right] . } \tag{3.47}
\end{align*}
$$

Observe that $Z_{A}\left(S_{1,2}\left(0,0,0, \lambda_{1}, \lambda_{2}\right)\right)=\mu_{\lambda_{1}, \lambda_{2}}$ from (3.13).

## Cup as unit

Consider a disk with outgoing boundary. By Corollary 2.20, we get a unique $r$-spin structure for boundary parametrisation, namely $\mu=0$. Note that the map $\beta_{\text {out }}$ is unique. Using the notation in (2.26) we write $S_{1,0}:=\Sigma_{0,1}(u, 0): \emptyset \rightarrow \rho$. with $\rho:\{*\} \rightarrow \mathbb{Z}_{r} \rho_{*}=0$. However, since the $r$-spin structure is actually independent of $u$ we may as well set $u=0$. We have

$$
\begin{align*}
& S=\{(f, k) \mid k=1,2,3\} \simeq\{1,2,3\},  \tag{3.48}\\
& E=\left\{\left(r_{1}, l\right),\left(r_{1}, r\right),\left(u_{1}, r\right)\right\} \simeq\{1,2,3\} . \tag{3.49}
\end{align*}
$$

There is an inner vertex $v_{0}$ and an outgoing boundary vertex $v_{1}$, and we set

$$
\begin{equation*}
V\left(v_{0}\right):=\left(u_{1}, r\right) . \tag{3.50}
\end{equation*}
$$

By the state-sum construction one has
2. For the 2 edges $\mathcal{C}=g_{r_{1}} \otimes g_{u_{1}}$ from (3.29).
3. For the inner vertex $z=\eta^{\otimes 2} \otimes \tau^{-1}$ from (3.31).
4. For the single 3-gon $\mathcal{F}=\varepsilon \circ \mu^{(3)}$ from (3.33).
5. The permutation is $\Pi_{\Phi}=$ (23) from (3.34). Putting the above together according to (3.36) we get

$$
\begin{equation*}
\mathcal{L}=P_{0} \circ \eta . \tag{3.51}
\end{equation*}
$$

6. For the (empty) in- and outgoing boundary components we get $\mathcal{E}_{\text {in }}=\mathrm{id}_{\mathbb{I}}$ and $\mathcal{E}_{\text {out }}=\pi_{0}$ from (3.39) and (3.40). From (3.41) we finally get

$$
\begin{equation*}
Z_{A}\left(S_{1,0}\right)=\left[\mathbb{I} \xrightarrow{\eta} A \xrightarrow{\pi_{0}} Z_{0}\right] . \tag{3.52}
\end{equation*}
$$

Observe that $\left[\mathbb{I} \xrightarrow{Z_{A}\left(S_{1,0}\right)} Z_{0} \xrightarrow{e_{0}} \oplus_{\lambda \in \mathbb{Z}_{r}} Z_{\lambda}\right]=\bar{\eta}$ from (3.15).

## Pair of pants as comultiplication

Consider a 3 -holed sphere with the parametrisation as above, just with in- and outgoing boundary components exchanged, i.e. $\lambda_{1}, \lambda_{2}$ stand for outgoing boundary components, $\lambda_{3}$ for the ingoing etc. Then from (2.23) one has:

$$
\begin{equation*}
\lambda_{1}+\lambda_{2}-2 \equiv \lambda_{3} \quad(\bmod r) \tag{3.53}
\end{equation*}
$$

Denote this $r$-spin surface with parametrised boundary by

$$
S_{2,1}\left(u_{1}, u_{2}, u_{3}, \lambda_{1}, \lambda_{2}\right):=\Sigma_{0,3}\left(u_{1}, u_{2}, u_{3}, 1-\lambda_{1}, 1-\lambda_{2}, \lambda_{3}-1\right): \sigma \rightarrow \rho,
$$

(cf. (2.26)). The morphism $\mathcal{L}$ in (3.36) assigned to it by the state-sum construction is

$$
\begin{equation*}
\left[A \xrightarrow{P_{\lambda_{1}+\lambda_{2}-2} \circ N^{-u_{3}-1}} A \xrightarrow{\Delta \circ(\tau \cdot(-))} A \otimes A \xrightarrow{P_{\lambda_{1}} \circ N^{u_{1}+1} \otimes P_{\lambda_{2}} \circ N^{u_{2}+1}} A \otimes A\right] \tag{3.54}
\end{equation*}
$$

and from (3.41) one obtains

$$
\begin{align*}
& Z_{A}\left(S_{2,1}\left(u_{1}, u_{2}, u_{3}, \lambda_{1}, \lambda_{2}\right)\right)= \\
& {\left[Z_{\lambda_{1}+\lambda_{2}-2} \xrightarrow{N_{\lambda_{1}+\lambda_{2}-2}^{-u_{3}}} Z_{\lambda_{1}+\lambda_{2}-2} \xrightarrow{\Delta_{\lambda_{1}, \lambda_{2}}} Z_{\lambda_{1}} \otimes Z_{\lambda_{2}} \xrightarrow{N_{\lambda_{1}}^{u_{1}} \otimes N_{\lambda_{2}}^{u_{2}}} Z_{\lambda_{1}} \otimes Z_{\lambda_{2}}\right] .} \tag{3.55}
\end{align*}
$$

Observe that $Z_{A}\left(S_{2,1}\left(0,0,0, \lambda_{1}, \lambda_{2}\right)\right)=\Delta_{\lambda_{1}, \lambda_{2}}$ from (3.21). While the above morphism is defined also for $r=0$, as was remarked in Section 3.2 one can sum these morphisms only in the case when $r \neq 0$, in which case one obtains (3.20).

## Cap as counit

Consider an $r$-spin disk with ingoing boundary. By Corollary 2.20, the boundary parametrisation has $\lambda=2$ and the $r$-spin structure is independent of the edge indices. Denote this $r$-spin surface with parametrised boundary with $S_{0,1}:=\Sigma_{0,1}(0,2): \sigma \rightarrow \emptyset$, (cf. (2.26)), with $\sigma:\{*\} \rightarrow \mathbb{Z}_{r} \sigma_{*}=2$. By the state-sum construction one has

$$
\begin{equation*}
Z_{A}\left(S_{0,1}\right)=\left[Z_{2} \xrightarrow{\iota_{2}} A \xrightarrow{\left(\tau^{-1} \cdot(-)\right)} A \xrightarrow{\varepsilon} \mathbb{I}\right] \tag{3.56}
\end{equation*}
$$

Observe that $\left[\oplus_{\lambda \in \mathbb{Z}_{r}} Z_{\lambda} \xrightarrow{p_{2}} Z_{2} \xrightarrow{Z_{A}\left(S_{1,0}\right)} \mathbb{I}\right]=\bar{\varepsilon}$ from (3.22).
We collect the above computations for $Z_{A}$ evaluated on generators in the following proposition:
Proposition 3.10. Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element $\tau$ and with $N^{r}=\mathrm{id}_{A}$, and let $Z_{A}$ be the $r$-spin $T F T Z_{A}$ defined in Theorem 3.8. The the $\mathbb{Z}_{r}$-graded center $Z^{r}(A)$ is equal to $\bigoplus_{\lambda \in \mathbb{Z}_{r}} Z_{A}(\lambda)$ with product and unit (restricted to the corresponding graded components) given by $Z_{A}\left(S_{1,2}\left(0,0,0, \lambda_{1}, \lambda_{2}\right)\right)$ and $Z_{A}\left(S_{1,0}\right)$, respectively. For $r>0$, we obtain an equality of Frobenius algebras.

For $r=2$, the above relation between state spaces and the $\mathbb{Z}_{r}$-graded center was already observed in [MS].

## Connected $r$-spin bordisms

Finally, let us evaluate $Z_{A}$ on a general connected $r$-spin bordism with only ingoing boundary components, that is, on $\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)$ in the notation of (2.26). Write


Using the decomposition of $\Sigma_{g, b}$ from Figure $7 a$ ), a straightforward computation along the same lines as above gives the following proposition.
Proposition 3.11. Let $\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)$ denote the $r$-spin surface of Definition 2.10 with only ingoing boundary components. Then

$$
\begin{equation*}
Z_{A}\left(\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right)=\varepsilon \circ\left(\tau^{-1} \cdot(-)\right) \circ \prod_{i=1}^{g} \varphi\left(s_{i}, t_{i}\right) \circ \mu^{(b)} \circ \bigotimes_{j=1}^{b}\left(N^{-u_{j}-1} \circ \iota_{\lambda_{j}}\right) . \tag{3.58}
\end{equation*}
$$

## 4 Action of the mapping class group

Since the TFT is defined on diffeomorphism classes of $r$-spin surfaces with parametrised boundary, it is of natural interest to calculate these classes. Bundles related by homotopic underlying surface diffeomorphisms are isomorphic, therefore studying diffeomorphism classes of $r$-spin surfaces is the same as finding the orbits of the mapping class


Figure 12: Dehn twists along the following loops provide a choice of generators of the MCG.
a) Loops in $\Sigma_{0, b}:\left\{\partial_{i}, h_{i j} \mid i, j=1, \ldots, b, i \neq j\right\}$. Here, $\partial_{j}$ denotes the boundary component $j$ and, for $b \geq 2, h_{i j}$ denotes the connected sum of $\partial_{i}$ and $\partial_{j}$. The connected sum is taken with respect to a choice of points on each loop and a path between these points, so that the result is as shown in the figure. Note that $h_{i j}=h_{j i}$.
b) Loops in $\Sigma_{g, b}$ for $g \geq 1$ : $\left\{\partial_{i}, h_{i j}\right\}$ as before, and $\left\{f_{i}, a_{l}, b_{l}, d_{x} \mid i=1, \ldots, b, l=1, \ldots, g, x=\right.$ $1, \ldots, g-1\}$. Here, $f_{i}$ denotes the connected sum of $\partial_{i}$ and $b_{1} ; d_{x}$ denotes the connected sum of $b_{x}$ and $-b_{x+1}$ and occurs only if $g \geq 2$.
group acting on the set of isomorphism classes of $r$-spin structures on fixed surfaces. Here by mapping class group (MCG) we mean diffeomorphisms of the surface which restrict to the identity on the boundary, up to smooth homotopy which fix the boundary, see [FM, Sec. 2.1].

We consider $\Sigma_{g, b}$, a connected surface of genus $g$ with $b$ boundary components. Generators of the MCG of $\Sigma_{g, b}$ are given by Dehn twists along loops in $\Sigma_{g, b}$ as shown in Figure 12. For $g=0$ this can be shown combining [FM, Thm.4.9, Prop. 3.19 and Sect.9.3]; for $g \geq 1$ this is shown in [FM, Sect. 4.4.4].

We will compute the action of these generators on the set $\mathcal{R}^{r}\left(\Sigma_{g, b}\right)_{\lambda, \mu}$ of isomorphism classes of $r$-spin structures with parametrised boundary for given maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}$ and $\mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$ in terms of the parametrisation given in Section 2.5. We will then use these results in Section 6 in order to prove our main result, Theorem 1.2.
Lemma 4.1. Consider a surface with a PLCW decomposition which has a cylinder inside decomposed into a square with identified opposite edges as in Figure 13. A Dehn twist around the edge labeled by $t$ sends $t \mapsto t+s$ and does not change the other edge labels.

Proof. First refine the decomposition of Figure $13 a$ ) as in Figure $14 a$ ). This gives an isomorphic $r$-spin structure to the original one by Proposition 2.16. Pulling back the $r$ spin structure along the induced action of the Dehn twists yields the $r$-spin surface shown


Figure 13: Action of a Dehn twist along the edge labeled by $t$. The two vertical edges labelled by $s$ are identified.


Figure 14: Pulling back the $r$-spin structure along an (inverse) Dehn twist along the edge labeled by $t: a$ ) insert the diagonal edge labelled $0 ; b$ ) carry out a Dehn-twist along the upper horizontal edge labelled $t ; c$ ) apply a deck transformation to the top right triangle to change the label of the diagonal edge to 0 .
in Figure 14 b). Now apply Part 3 of Lemma 2.11 on the upper triangle to obtain Figure $14 c$ ). Remove the middle edge by Proposition 2.16 to get the $r$-spin surface described by Figure $13 b$ ).

Lemma 4.2. Recall the parametrisation of $r$-spin structures on $\Sigma_{g, b}$ from (2.26) and Figure 7. Let $l$ denote a loop in $\Sigma_{g, b}$ and let $D_{l}$ denote the isomorphism of $r$-spin surfaces induced by a Dehn twist around $l$. We write

$$
\begin{equation*}
D_{l}\left(\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, R_{j}\right)\right)=\Sigma_{g, b}\left(s_{i}^{\prime}, t_{i}^{\prime}, u_{j}^{\prime}, R_{j}\right) \tag{4.1}
\end{equation*}
$$

Then the action of Dehn twists along the loops shown in Figure 12 is as listed in the following table (only the parameters that change are listed):

| loop | effect on parameters |  |  |
| :---: | :--- | :--- | :--- |
| $\partial_{j}$ | $u_{j}^{\prime}=u_{j}-R_{j}$ |  |  |
| $h_{i j}$ | $u_{i}^{\prime}=u_{i}+R_{i}+R_{j}+1 \quad$ and $\quad u_{j}^{\prime}=u_{j}+R_{i}+R_{j}+1$ |  |  |
| $a_{i}$ | $s_{i}^{\prime}=s_{i}-t_{i}$ |  |  |
| $b_{i}$ | $t_{i}^{\prime}=t_{i}-s_{i}$ |  |  |
| $f_{j}$ | $u_{j}^{\prime}=u_{j}+s_{1}+1+R_{j}$ | and | $t_{1}^{\prime}=t_{1}-s_{1}-1-R_{j}$ |
| $d_{i}$ | $t_{i}^{\prime}=t_{i}+s_{i+1}-s_{i}+1$ | and | $t_{i+1}^{\prime}=t_{i+1}-s_{i+1}+s_{i}-1$ |



Figure 15: The loop $f_{2}$ (above, in 2 segments, between edges $r_{2}$ and $s_{1}$ ) and the loop $d_{g-1}$ (below, in 4 segments, between edges $t_{g-1}, s_{g-1}, t_{g}$ and $s_{g}$ ) on the PLCW decomposition.


Figure 16: Calculation of the Dehn twist along the loop $f_{2} . a$ ) The cylinder along the loop $f_{2}$. The empty dot denotes the boundary vertex, the full dot the inner vertex. The vertical edges labeled by $p$ are identified. b) Move the marking to the $s$ edge (Lemma 2.11 (2)), shift the labels on the left square by $s+1$ (Lemma 2.11 (3)), and remove the middle edge (Proposition 2.16). $c$ ) Add an edge between two opposite corners with edge index 0. d) Move the markings and flip the middle edge orientation. e) Apply a Dehn twist along the top horizontal edges (marked $s$ and 0 ). f) Apply a deck transformation to the top left triangle and move right marking. $g$ ) Remove the diagonal edge, insert new vertical edge. $h$ ) Shift edge indices on left square, move left marking.


Figure 17: Calculation of a Dehn twist along the loop $d_{g-1}$. The left-most and right-most vertical edges are identified in all figures. a) The cylinder along the loop $d_{g-1}$. b) Move the markings, flip the " $p$ " edge orientation and shift the edge indices on the 3 rectangles on the left. c) Remove the 3 inner edges and add a new edge. d) Do a Dehn twist along the $d_{g-1}$ loop. e) Shift the edge indices on the upper triangle by $s-p-1$; flip the orientation of the middle edge and then remove the middle edge; put back 3 edges. f) Move the markings, flip the first and fourth edge orientation. $g$ ) Shift the edge indices on the 3 rectangles on the left.


Figure 18: Part of the PLCW decomposition after a Dehn twist along the loop $d_{g-1}$. We shift the edge indices by $Q:=s-p-1$ on the following polygons: on the 2 triangles marked by dots; on the rectangle with edge labels $Q-1, p,-1$ and $t$; on the triangle below with edge labels -1 , $q$ and 0 ; on the rectangle below with edge labels $0, Q,-1$ and $t$; on the triangle below with edge labels $Q, p$ and $t$.


Figure 19: The loop $h_{12}$ (in two segments between the edges $r_{1}$ and $r_{2}$ ) in the PLCW decomposition of $\Sigma_{g, b}$ and a cylinder around it.


Figure 20: Dehn twist along the loop $\left.h_{12} . a\right)$ Take the cylinder from Figure 19. b) After changing the marking one obtains a similar cylinder as in Figure $16 a)$. c) Do the same steps as in Figure 16 to apply the Dehn twist. d) Change back the marking.

Proof. - $a_{i}, b_{i}$ : For the loops $a_{i}$ and $b_{i}$ the statement is a direct consequence of Lemma 4.1. For example for $a_{1}$ split the edge $t_{1}$ in two by inserting a vertex, then insert an edge parallel to $s_{1}$. Then apply the lemma and remove the previously added edge and vertex.

- $\partial_{j}$ : For the loop $\partial_{j}$ the statement follows along the same lines, together with (2.24).
- $f_{j}$ : We prove the statement for $f_{j}$ in the example $j=2$. Let $s:=s_{1}$ and $p:=r_{2}$. First find the curve $f_{2}$ on the polygon decomposition, insert the dotted edges parallel to the loop $f_{2}$ using Proposition 2.16 and change orientations using Lemma 2.11 (1) as in Figure 15. We need to consider the part of the decomposition which is a cylinder glued together from two rectangles as shown in Figure $16 a)$. Then proceed with the sequence of steps shown in Figure 16. Finally apply a deck transformation by $-p-s-1$ on the rectangle bounded by the edges with edge index $t_{1}, p+s+1, u_{2}$ and $p+s$. The result is a decomposition as on Figure 15 with $t_{1}$ replaced by $t_{1}-p-s-1$ and $u_{2}$ replaced by $u_{2}+p+s+1$. Now remove the newly added edges via Proposition 2.16 (flipping the edges labelled -1 ) to arrive to the statement.
- $d_{i}$ : We treat the case $i=g-1$ as an example by applying a similar argument as before. Let $s:=s_{g-1}, p:=s_{g}, t:=t_{g-1}$ and $q:=t_{g}$. First add vertices and dotted edges parallel to the loop $d_{g-1}$ as shown in Figure 15. We will concentrate on the cylinder cut out by these edges, as shown in Figure 17 a). Proceed along the steps shown in Figure 17, after which one is left with the marked PLCW decomposition shown in Figure 18. Let $Q:=s-p-1$ and shift edge indices by $Q$ according to the steps in Figure 18. This amounts to

$$
t \mapsto t-Q \quad \text { and } \quad q \mapsto q+Q
$$

after removing the newly added edges and vertices.

- $h_{i j}$ : We show the computation for $i=1, j=2$ as an example, for other values of $i$ and $j$ the argument is the same. Add vertices and dotted edges parallel to the loop $h_{12}$ as shown in Figure 19. We then follow the steps in Figure 20. As the last step, one shifts the edge indices by $r_{1}+r_{2}+1=R_{1}+R_{2}+1$ by a deck transformation on the square which has edges $u_{1}$ and $u_{2}$ on opposite sides.


## $5 r$-spin TFT computing the Arf-invariant

In this section we give an example for the state-sum construction of $r$-spin TFTs, namely for the two-dimensional Clifford algebra in super vector spaces, and we compute its value on connected $r$-spin bordisms (Section 5.1). We then recall the definition of the Arf invariant for $r$-spin surfaces and observe that the TFT obtained from the Clifford algebra computes this invariant (Section 5.2).

## $5.1 \quad r$-spin TFT from a Clifford algebra

Let $r \in \mathbb{Z}_{\geq 0}$ be even and let $k$ be a field not of characteristic 2 . Let $C \ell \in \mathcal{S V}$ ect be the Clifford algebra with one odd generator $\theta$, i.e. $C \ell=k \oplus k \theta$ with $\theta^{2}=1$. We turn $C \ell$ into
a Frobenius algebra via

$$
\begin{equation*}
\varepsilon(1)=2, \quad \varepsilon(\theta)=0, \quad \Delta(1)=\frac{1}{2}(1 \otimes 1+\theta \otimes \theta), \quad \Delta(\theta)=\frac{1}{2}(\theta \otimes 1+1 \otimes \theta) . \tag{5.1}
\end{equation*}
$$

Lemma 5.1. For the Frobenius algebra $C \ell$ the following hold.

1. $\tau=\mu \circ \Delta \circ \eta=\eta$, hence $C \ell$ has invertible window element.
2. The Nakayama automorphism is given by $N\left(\theta^{m}\right)=(-1)^{m} \theta^{m}$.
3. For $\lambda \in \mathbb{Z}_{r}, P_{\lambda}\left(\theta^{m}\right)=\frac{1}{2}\left[1+(-1)^{\lambda-m}\right] \theta^{m}$, hence $Z_{\lambda}=k \theta^{\lambda}$.
4. The morphism $\varphi_{s, t}$ from (3.57) is given by $\varphi_{s, t}=\frac{1}{2}(-1)^{(s+1)(t+1)} \mathrm{id}_{C \ell}$.

Proof. 1. $\tau(1)=\mu \circ \Delta \circ \eta(1)=\mu\left(\frac{1}{2}\left(\theta^{n} \otimes \theta^{-n}\right)\right)=1=\eta(1)$. Its inverse is $\eta$.
2. $N(1)=1$ in any Frobenius algebra. We calculate $N(\theta)$ in steps:

$$
\theta \mapsto \theta \otimes(1 \otimes 1+\theta \otimes \theta) / 2 \mapsto(1 \otimes \theta \otimes 1-\theta \otimes \theta \otimes \theta) / 2 \mapsto-\theta
$$

3. We calculate $P_{\lambda}\left(\theta^{m}\right)$ in steps according to (3.4):

$$
\begin{aligned}
\theta^{m} & \mapsto \frac{1}{2}\left(\theta^{m} \otimes 1+\theta^{m-1} \otimes \theta\right) \mapsto \frac{1}{2}\left(\theta^{m} \otimes 1+(-1)^{1-\lambda} \theta^{m-1} \otimes \theta\right) \\
& \mapsto \frac{1}{2}\left(1 \otimes \theta^{m}+(-1)^{m-\lambda} \theta \otimes \theta^{m-1}\right) \mapsto \frac{1}{2} \theta^{m}\left(1+(-1)^{m-\lambda}\right)
\end{aligned}
$$

We see that if $\lambda$ and $m$ have the same parity this is the identity, otherwise this is zero, i.e. $P_{\lambda}$ is a projection onto $k . \theta^{\lambda}$.
4. We calculate $\varphi_{s, t}\left(\theta^{m}\right)$ in steps according to (3.57):

$$
\begin{aligned}
\theta^{m} & \mapsto \frac{1}{2} \sum_{n=0}^{1} \theta^{m-n} \otimes \theta^{n} \mapsto \frac{1}{4} \sum_{n, p=0}^{1} \theta^{m-n} \otimes \theta^{n-p} \otimes \theta^{p} \\
& \mapsto \frac{1}{4} \sum_{n, p=0}^{1}(-1)^{(s+1)(n-p)+(t+1) p} \theta^{m-n} \otimes \theta^{n-p} \otimes \theta^{p} \\
& \mapsto \frac{1}{4} \sum_{n, p=0}^{1}(-1)^{(s+1)(n-p)+(t+1) p+(n-p) p} \theta^{m-n} \otimes \theta^{p} \otimes \theta^{n-p} \\
& \mapsto \frac{1}{4} \theta^{m} \sum_{n, p=0}^{1}(-1)^{(s+1)(n-p)+(t+1) p+(n-p) p} \\
& =\frac{1}{4} \theta^{m} \sum_{n, p=0}^{1}(-1)^{(s+1+p)(t+1+n-p)-(s+1)(t+1)}=\frac{1}{2} \theta^{m}(-1)^{(s+1)(t+1)},
\end{aligned}
$$

where at the last step we execute first the summation over $n$ for a fixed $p$ and notice that we either get 0 or 2 .

Let $Z_{C \ell}$ denote the TFT from Theorem 3.8 given by the Frobenius algebra $C \ell$ and recall from Section 2.5 the $r$-spin structure with parametrised boundary $\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)$ with only ingoing boundary components and where $g+b \geq 1$. By calculating (3.58) in Proposition 3.11 and using (2.23) we get the following proposition.
Proposition 5.2. The value of the TFT $Z_{C l}$ is

$$
\begin{equation*}
Z_{C \ell}\left(\sum_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right)\left(\theta^{\lambda_{1}} \otimes \cdots \otimes \theta^{\lambda_{b}}\right)=2^{1-g}(-1)^{\sum_{n=1}^{g}\left(s_{n}+1\right)\left(t_{n}+1\right)+\sum_{j=1}^{b-1}\left(u_{j}-u_{b}\right) \lambda_{j}} \tag{5.2}
\end{equation*}
$$

The following corollary will be used to distinguish 2 MCG orbits on $\mathcal{R}^{r}\left(\Sigma_{g, b}\right)_{\lambda, \mu}$ for $g \geq 2$ and $r$ even.
Corollary 5.3. Assume that $g \geq 1$ or that $b \geq 1$ and at least one of the $\lambda_{j}$ 's is odd (by (2.23) in this case $b \geq 2$ and at least two $\lambda_{j}$ 's are odd). Then the following map is surjective:

$$
\begin{align*}
\mathcal{R}^{r}\left(\Sigma_{g, b}\right)_{\lambda} & \rightarrow\{+1,-1\} \\
{\left[\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right] } & \mapsto 2^{g-1} \cdot Z_{C \ell}\left(\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right)\left(\theta^{\lambda_{1}} \otimes \cdots \otimes \theta^{\lambda_{b}}\right) \tag{5.3}
\end{align*}
$$

Remark 5.4. 1. One can show, using a similar argument as in [Nov, Sec. 6.5], that for any choice of Frobenius algebra $A \in \mathcal{V}$ ect with invertible window element and with $N^{r}=\operatorname{id}_{A}$ the TFT $Z_{A}$ of Section 3.3 is independent of the $r$-spin structure. The idea is that if there exists a symmetric Frobenius algebra structure on an algebra $A$, then $Z_{A}$ is independent of the $r$-spin structure for every other Frobenius algebra structure on $A$ as well.
2. Let $r$ be a positive integer and let us consider the category of $\mathbb{Z}_{r}$-graded $k$-vector spaces $\mathcal{V e c t}_{\mathbb{Z}_{r}}$. By using the correspondence between braided monoidal structures on $\mathcal{V} e c t_{\mathbb{Z}_{r}}$ and quadratic forms on $\mathbb{Z}_{r}[J S]$ (see [FRS, App. A] for a review) one can check that for odd $r$ there is only one symmetric monoidal structure on $\mathcal{V}$ ect $\mathbb{Z}_{r}$. For even $r$ there are two: the trivial one inherited from $\mathcal{V}$ ect and the non-trivial one given by the super grading.
3. One may wonder whether taking $\mathcal{V e c t}_{\mathbb{Z}_{r}}$ with some choice of symmetric monoidal structure would yield more examples of $r$-spin TFTs than what one can find with target Vect or $\mathcal{S}$ Vect. Part 2 shows that this is not so: All symmetric monoidal structures on $\mathcal{V}$ ect $\mathbb{Z}_{r}$ are inherited from $\mathcal{V}$ ect or $\mathcal{S}$ Vect (and only from the former for $r$ odd). Thus all algebras $A \in \mathcal{V}$ ect $_{\mathbb{Z}_{r}}$ as in Theorem 3.8 are also algebras in $\mathcal{V}$ ect, respectively $\mathcal{S V}$ ect, with the same properties, and produce the same results in the state-sum construction.

### 5.2 The $r$-spin Arf-invariant

Let $\Sigma$ be a compact $r$-spin surface with parametrised boundary with maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}$ and $\mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$. By a curve in $\Sigma$ we mean a smooth immersion $\gamma:[0,1] \rightarrow \Sigma$ (i.e. $\gamma$ has nowhere vanishing derivative), and which is either closed, or which starts and ends on


Figure 21: Two arcs $p, q \in A(\gamma)$ of a curve $\gamma$ on a face $f$. Here $f_{p}=f_{q}=f, \hat{s}_{e_{p}}=-s_{e_{p}}-1$, $\hat{s}_{e_{q}}=s_{e_{q}}, \hat{\delta}_{f_{p}}^{p}=+1$ and $\hat{\delta}_{f_{q}}^{q}=0$.
the boundary of $\Sigma$. In the former case we require in addition that the tangent vectors at the start and end point agree: $\frac{d}{d t} \gamma(0)=\frac{d}{d t} \gamma(1)$. In the latter case we require that the start and end points are the images of $1 \in \mathbb{S}_{1} \subset \mathbb{C}$ under the boundary parametrisation maps and that the tangent vector of the curve is the same as the tangent vector of the boundary curve. Two curves $\gamma_{0}$ and $\gamma_{1}$ with $\gamma_{0}(0)=\gamma_{1}(0)$ and $\gamma_{0}(1)=\gamma_{1}(1)$ are homotopic if there is a homotopy $(s, t) \mapsto \gamma_{s}(t)$ between them, such that for each $s, \gamma_{s}$ is a curve in the above sense. In particular, since $\frac{d}{d t} \gamma$ must remain nonzero everywhere along the homotopy, one cannot "pull straight" a loop in the curve.

Pick a lift $\gamma_{F}:[0,1] \rightarrow F_{G L} \Sigma$ of $\gamma$ to the oriented frame bundle by taking the tangent vector of $\gamma$ (which is non-zero since $\gamma$ is an immersion) and adding another non-zero and non-parallel vector such that the orientation induced by them agrees with the orientation of the surface. Such a lift of a curve in $\Sigma$ to $F_{G L} \Sigma$ is unique up to homotopy, see e.g. [Nov, p.26]. Also, if two curves in $\Sigma$ are homotopic, then their lifts to $F_{G L} \Sigma$ are homotopic as well.

Consider a disc $D$ around 1 in $\mathbb{C}^{\times}$with $r$-spin structure $D^{\kappa}$ given by the restriction of $\mathbb{C}^{\kappa}$ for $\kappa \in \mathbb{Z}_{r}$ as in Example 2.3. As on a contractible surface, all $r$-spin structures are isomorphic (see e.g. [Nov, Lem.3.10]), there is an isomorphism of $r$-spin structures $D^{0} \rightarrow D^{\kappa}$. In fact, there are exactly $r$ such isomorphisms, and we pick the one which acts as the identity on the fibre over 1 (by Example 2.3, the fibre and projection over $1 \in \mathbb{C}^{\times}$ agree for all $\mathbb{C}^{\kappa}$ ). This construction will be needed to assign a holonomy to curves between different boundary components.

Recall that $P_{\widetilde{G L}} \Sigma$ is a principal $\mathbb{Z}_{r}$ bundle over $F_{G L} \Sigma$. Pick a lift $\tilde{\gamma}:[0,1] \rightarrow P_{\widetilde{G L}} \Sigma$ of $\gamma_{F}$ to the $r$-spin bundle. Since the fibers of $P_{\widetilde{G L}} \Sigma \xrightarrow{p} F_{G L} \Sigma$ are discrete, this lift is unique after fixing it at one point and homotopic curves in $F_{G L} \Sigma$ lift to homotopic curves in $P_{\widetilde{G L}} \Sigma$. If $\gamma$ is a closed curve let $\zeta(\gamma) \in \mathbb{Z}_{r}$ denote the holonomy of $\tilde{\gamma}$ at $\gamma(0)=\gamma(1)$. If $\gamma$ is not closed, use the isomorphism $D^{0} \rightarrow D^{\kappa}$ from above to identify the fibers $\mathbb{Z}_{r}$ over the startand end-point of $\gamma_{F}$, and let again $\zeta(\gamma) \in \mathbb{Z}_{r}$ denote the resulting holonomy of $\tilde{\gamma}$.

We now explain how to compute these holonomies in terms of the combinatorial description of $r$-spin structures. Take a decorated PLCW decomposition of $\Sigma$ with edge index assignment $s$ and consider the $r$-spin structure $\Sigma(s, \lambda, \mu)$ given by Definition 2.10. We may
assume the PLCW decomposition to be fine enough so that its edges split $\gamma$ into a set of $\operatorname{arcs} A(\gamma)$ as in Figure 21. Then for every $a \in A(\gamma)$ there is a face $f_{a} \in \Sigma_{2}$ containing $a$ and an edge $e_{a}$ in the boundary of $f_{a}$ where the arc $a$ leaves the face $f_{a}$ (see Figure 21). Let us assume that $e_{a}$ is not a boundary edge. For $s_{e_{a}}$ the edge index of the edge $e_{a}$ let $\hat{s}_{e_{a}}^{a}=s_{e_{a}}$ if the edge $e_{a}$ and $a$ cross positively, and $\hat{s}_{e_{a}}^{a}=-s_{e_{a}}-1$ otherwise (see again Figure 21 for conventions). Let $\hat{\delta}_{f_{a}}^{a}=+1$ if the clockwise vertex of the marked edge of the face $f_{a}$ is on the right side of $a$ (before glueing the edges) and $\hat{\delta}_{f_{a}}^{a}=0$ otherwise. If $\gamma$ is not a closed curve, let $e_{\text {start }}$ (resp. $e_{\text {end }}$ ) denote the boundary edge where $\gamma$ starts (resp. ends), and let $s_{e_{\text {start }}}$ (resp. $s_{e_{\text {end }}}$ ) be its edge index. Recall that at the starting (ending) point of $\gamma$ the tangent vector of $\gamma$ is parallel to the boundary edge. Set $\hat{s}_{e_{\text {start }}}:=-s_{e_{\text {start }}}-1$ if the edge $e_{\text {start }}$ and the tangent vector point in the same direction and $\hat{s}_{e_{\text {start }}}:=s_{e_{\text {start }}}$ otherwise. Set $\hat{s}_{e_{\text {end }}}:=s_{e_{\text {end }}}$ if the edge $e_{\text {end }}$ and the tangent vector point in the same direction and $\hat{s}_{e_{\text {end }}}:=-s_{e_{\text {end }}}-1$ otherwise.

The proof of the following lemma relies on the relation to triangulations introduced in Appendix A and is given in Appendix A.6.
Lemma 5.5. Let $\gamma$ be a curve in $\Sigma$. Then:

1. If $\gamma$ bounds a disc $\mathbb{D}$ embedded in $\Sigma, \zeta(\gamma)=1$ if $\gamma$ is oriented counter-clockwise around the boundary of $\mathbb{D}$ and $\zeta(\gamma)=-1$ otherwise.
2. If $\gamma^{\prime}$ is a curve homotopic to $\gamma$ then $\zeta\left(\gamma^{\prime}\right)=\zeta(\gamma)$;
3. We have

$$
\zeta(\gamma)=\sum_{a \in A(\gamma)}\left(\hat{s}_{e_{a}}^{a}+\hat{\delta}_{f_{a}}^{a}\right)+ \begin{cases}0 & ; \gamma \text { is closed }  \tag{5.4}\\ \hat{S}_{e_{\text {start }}}+1 & ; \gamma \text { is not closed } .\end{cases}
$$

Note that in Part 3, in case the curve goes from boundary to boundary, the edge index of the boundary edge where the endpoint of the curve lies is included in the sum over $A(\gamma)$.

Let $g+b \geq 1$ and consider a compact connected surface $\Sigma_{g, b}$ of genus $g$ with $b$ boundary components with parametrised ingoing boundary and fix a set of curves in the surface $\Sigma_{g, b}$ as shown in Figure $22 a$ ). Let us consider a marked PLCW decomposition of $\Sigma_{g, b}$ as in Section 2.5 and recall the corresponding $r$-spin structure $\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)$ from (2.26).
Corollary 5.6. The holonomies of the curves in Figure 22 are

$$
\begin{equation*}
\zeta\left(a_{i}\right)=s_{i}, \quad \zeta\left(b_{i}\right)=t_{i}, \quad \zeta\left(c_{j}\right)=u_{j}-u_{b}+1, \quad \zeta\left(\partial_{j}\right)=1-\lambda_{j} \tag{5.5}
\end{equation*}
$$

for $i=1, \ldots, g$ and $j=1, \ldots, b-1$.
Proof. We only show the calculation of the latter two holonomies. There is only one face, let us denote it by $f$. The tangent vectors of the edge $u_{b}$ and the loop $c_{j}$ point in the same direction and the loop starts at this edge $\left(u_{b}=e_{\text {start }}\right)$, therefore $\hat{s}_{u_{b}}^{f}=\hat{s}_{e_{\text {start }}}=-u_{b}-1$; the tangent vectors of the edge $u_{j}$ and the loop $c_{j}$ point in the same direction and the loop
a)

b)


Figure 22: a) A set of curves in $\Sigma_{g, b}:\left\{a_{i}, b_{i}, c_{j} \mid i=1, \ldots, g ; j=1, \ldots, b-1\right\}$.b) These curves in the PLCW decomposition of $\Sigma_{g, b}$ (cf. Figure 7). The bigger arrows on the edges show the marked edge: $r_{1}$ for $b>0$ and $s_{1}$ for $b=0$.
ends at this edge $\left(u_{j}=e_{\text {end }}\right)$, therefore $\hat{s}_{u_{j}}^{f}=\hat{s}_{e_{\text {end }}}=u_{j}$; the clockwise vertex determined by the marked edge of the face $f$ is on the right side of the curve $c_{j}$ so $\hat{\delta}_{f}^{c_{j}}=1$. Taking the sum of all these we get $\zeta\left(c_{j}\right)=u_{j}+1-u_{b}-1+1=u_{j}-u_{b}+1$. The edge $r_{j}$ and the loop $\partial_{j}$ cross negatively and the clockwise vertex is on the right side of the loop, so we get $\zeta\left(\partial_{j}\right)=-r_{j}-1+1=1-\lambda_{j}$.

Definition 5.7 ([Ran, Sec. 2.4] and [GG, Sec.5]). Let $r \geq 0$ be even. The $r$-spin Arfinvariant of the $r$-spin surface $\Sigma_{g, b}$ is

$$
\begin{equation*}
\operatorname{Arf}\left(\Sigma_{g, b}\right)=\sum_{i=1}^{g}\left(\zeta\left(a_{i}\right)+1\right) \cdot\left(\zeta\left(b_{i}\right)+1\right)+\sum_{j=1}^{b-1}\left(\zeta\left(c_{j}\right)+1\right) \cdot\left(\zeta\left(\partial_{j}\right)+1\right) \quad(\bmod 2) \tag{5.6}
\end{equation*}
$$

Notice that for $r$ even, $r$-spin structures naturally factorise through 2-spin structures. Therefore it makes sense to talk about the Arf-invariant of them, which was introduced for 2 -spin structures [Joh]. $\operatorname{Arf}\left(\Sigma_{g, b}\right)$ is invariant under the action of the mapping class group of $\Sigma_{g, b}$, which has been proven in [Ran, Prop. 2.8] and [GG, Lem. 7]. We provide a different proof of this result in the corollary to the following theorem.
Theorem 5.8. The TFT $Z_{C l}$ computes the $r$-spin Arf-invariant:

$$
\begin{equation*}
Z_{C \ell}\left(\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right)\left(\theta^{\lambda_{1}} \otimes \cdots \otimes \theta^{\lambda_{b}}\right)=2^{1-g} \cdot(-1)^{\operatorname{Arf}\left(\Sigma_{g, b}\left(s_{i}, t_{i}, u_{j}, \lambda_{j}-1\right)\right)} \tag{5.7}
\end{equation*}
$$

Proof. This is immediate from Proposition 5.2, Corollary 5.6 and Definition 5.7.

Since the morphisms in $\mathcal{B}$ ord $d_{2}^{r}$ are diffeomorphism classes of $r$-spin bordisms (rel boundary), we get (cf. Remark 1.3 (2)):
Corollary 5.9. The $r$-spin Arf invariant is constant on mapping class group orbits.

## 6 Counting mapping class group orbits

In this section we present the proof of Part 3 of Theorem 1.2. As we advertised it in Part 3 of Remark 1.3, we give an explicit expression for the number of mapping class group orbits of $r$-spin structures on $\Sigma_{0, b}$, i.e. $\left|O_{0}(r)\right|$, depending on the value of $r$ and the $R_{i}$ 's. Our proof follows the same ideas used in [Ran, GG] to count orbits. Specifically, in [Ran, Thm. 2.9] the number of orbits is given for $r=2$ in case $g=1, b>0$ and for general $r>0$ in case $g \geq 2, b>0$ and in [GG, Prop. 5] it is given for general $r>0$, $g \geq 0$ and $b=0$. Indeed, the authors of [Ran, GG] also calculate how the lifts of Dehn twists act on the isomorphism classes of $r$-spin structures in terms of a parametrisation and use these operations to reduce the parametrisation to a simpler form. Our proof uses the combinatorial model we introduced in Section 2.3, which is is different from the parametrisation used in [Ran, GG], and we add the missing cases $g=0, b>0$ and $g=1$, $b>0$ for arbitrary $r \geq 0$.

Recall that a non-negative integer $d \in \mathbb{Z}_{\geq 0}$ is a divisor of $r$ if there exists an integer $n$ such that $d \cdot n=r$. In particular, with this definition every non-negative integer is a divisor of 0 . As before we denote by $\operatorname{gcd}(a, b) \in \mathbb{Z}_{\geq 0}$ the non-negative integer that generates the ideal generated by $a$ and $b$ in $\mathbb{Z}$.

The $g=0$ case
For $g=0$ the MCG is generated by Dehn twists along loops $\partial_{j}$ and $h_{i j}$ shown in Figure 12. The cases $b=0$ and $b=1$ have been treated in Lemma 2.18 and in Corollary 2.20, so let us assume $b \geq 2$.

- Recall from Proposition 2.19 that the set of isomorphism classes of $r$-spin structures is given by $\prod_{i=1}^{b} \mathbb{Z}_{r} /\langle G\rangle$, where $G=(1,1, \ldots, 1)$.
- By applying Lemma 4.2 for the loop $\partial_{j}, \Sigma_{0, b}\left(u_{j}, R_{j}\right)$ and $\Sigma_{0, b}\left(u_{j}^{\prime}, R_{j}\right)$ are in the same orbit if

$$
\begin{equation*}
u_{j} \equiv u_{j}^{\prime}+R_{j} \quad(\bmod r) \tag{6.1}
\end{equation*}
$$

- By applying Lemma 4.2 for the loop $h_{i j}, \Sigma_{0, b}\left(u_{j}, R_{j}\right)$ and $\Sigma_{0, b}\left(u_{j}^{\prime}, R_{j}\right)$ are in the same orbit if

$$
\begin{align*}
u_{j} & \equiv u_{j}^{\prime}+R_{i}+R_{j}+1 \\
u_{i} & \equiv u_{i}^{\prime}+R_{i}+R_{j}+1 \tag{6.2}
\end{align*} \quad(\bmod r)
$$

Let $n \in \mathbb{Z}_{\geq 0}$ and let $\hat{R}_{i}, \hat{H}_{i j} \in \prod_{i=1}^{b} \mathbb{Z}_{n}$ for $i, j=1, \ldots, b, i \neq j$ have components $\left(\hat{R}_{i}\right)_{k}=\delta_{i, k} R_{i},\left(\hat{H}_{i j}\right)_{i}=\left(\hat{H}_{i j}\right)_{j}=R_{i}+R_{j}+1$ and $\left(\hat{H}_{i j}\right)_{k}=0$ for $k \neq i, j$. Let us define the quotient group

$$
\begin{equation*}
O_{0}(n):=\left(\mathbb{Z}_{n}\right)^{b} /\left\langle\hat{R}_{i}, \hat{H}_{i j}, G\right\rangle \tag{6.3}
\end{equation*}
$$

By construction, the set $O_{0}(r)$ is in bijection with orbits under the action of the MCG on isomorphism classes of $r$-spin structures. This proves the $g=0$ case of Part 3 of Theorem 1.2. Notice that in case $b=2, O_{0}(r)$ can be computed explicitly by hand:

$$
\begin{equation*}
O_{0}(r)=\mathbb{Z}_{\operatorname{gcd}\left(R_{1}, r\right)} \times \mathbb{Z}_{\operatorname{gcd}\left(R_{2}, r\right)} /\langle G\rangle \simeq \mathbb{Z}_{\operatorname{gcd}\left(R_{1}, r\right)} \tag{6.4}
\end{equation*}
$$

since by $(2.23) R_{1}+R_{2} \equiv 0(\bmod r)$. We continue with computing the order of the set $O_{0}(r)$.

## Explicit count of MCG orbits in the $g=0$ case

Proposition 6.1. The number of orbits $\left|O_{0}(r)\right|$ of the mapping class group on the set of isomorphism classes of $r$-spin structures on $\Sigma_{0, b}$ with $b \geq 2$ and with boundary parameters $R_{j}, j=1, \ldots, b$, is:

- $r=0$ :
* $\left|O_{0}(0)\right|=\infty$ if $b=2$ and $R_{1}=R_{2}=0$,
* $\left|O_{0}(0)\right|=\operatorname{gcd}\left(2\left(R_{j}+1\right)\left(R_{k}+1\right), R_{k}\left(R_{k}+1\right) \mid j, k \neq i, j \neq k\right)$ if $b \geq 3$ and $R_{i}=0$ for an $i \in\{1, \ldots, b\}$,
* $\left|O_{0}(0)\right|=\left|O_{0}\left(R_{1} \cdot R_{2} \cdots R_{b}\right)\right|$ else.
- $r>0:$ Let $r=p_{1}^{\alpha_{1}} \ldots p_{L}^{\alpha_{L}}, \alpha_{i}>0$, be the prime decomposition of $r$. Then $\left|O_{0}(r)\right|=$ $\prod_{i=1}^{L}\left|O_{0}\left(p_{i}^{\alpha_{i}}\right)\right|$, with $\left|O_{0}\left(p_{i}^{\alpha_{i}}\right)\right|$ as computed in Lemma 6.3.

In the following we give the proof of this proposition. ${ }^{1}$ Let us first suppose that $r>0$ and let $r=p_{1}^{\alpha_{1}} \ldots p_{L}^{\alpha_{L}}$ be the prime factorisation of $r$. The following lemma, whose proof is elementary, allows one to consider each $p_{i}^{\alpha_{i}}$ separately.
Lemma 6.2. Let $\varphi:\left(\mathbb{Z}_{n}\right)^{b} \rightarrow\left(\mathbb{Z}_{p_{1}^{\alpha_{1}}}\right)^{b} \times \cdots \times\left(\mathbb{Z}_{p_{L}^{\alpha_{L}}}\right)^{b}$ be the isomorphism of abelian groups provided by the Chinese Remainder Theorem. Let $U=\left\langle u_{1}, \ldots, u_{N}\right\rangle$ be the subgroup of $\left(\mathbb{Z}_{n}\right)^{b}$ generated by $N$ elements $u_{m} \in\left(\mathbb{Z}_{n}\right)^{b}$, and let $V=\varphi(U)$ be its image. Then $V$ is generated by the $L N$ elements $u_{m}^{(l)}, l=1, \ldots, L, m=1, \ldots, N$, whose components in $\left(\mathbb{Z}_{p_{i}^{\alpha_{i}}}\right)^{b}$ are

$$
\left(u_{m}^{(l)}\right)_{i}= \begin{cases}u_{m} & \bmod p_{i}^{\alpha_{i}}  \tag{6.5}\\ 0 & i=l \\ 0 & i \neq l\end{cases}
$$

[^1]This lemma allows us to write $O_{0}(r)=\prod_{i=1}^{L} O_{0}\left(p_{i}^{\alpha_{i}}\right)$. The key observation is the next lemma.
Lemma 6.3. Let $\alpha \in \mathbb{Z}_{>0}$ and $p$ a prime number. Then the order of $O_{0}\left(p^{\alpha}\right)$ is given as follows:

| $p$ | $\#\left\{R_{k}\right.$ divisible by $\left.p\right\}$ | $\left\|O_{0}\left(p^{\alpha}\right)\right\|$ |
| :---: | :---: | :--- |
| 2 | 0 or odd | 1 |
|  | 2 | $\operatorname{gcd}\left(2^{\alpha},\left\{R_{k}\right.\right.$ s.t. $\left.2 \mid R_{k}\right\},\left\{R_{k}+1\right.$ s.t. $\left.\left.2 \nmid R_{k}\right\}\right)$ |
| $>2$ | $>2$ and even | 2 |
|  | 2 | $\operatorname{gcd}\left(p^{\alpha},\left\{R_{k}\right.\right.$ s.t. $\left.p \mid R_{k}\right\},\left\{R_{k}+1\right.$ s.t. $\left.\left.p \nmid R_{k}\right\}\right)$ |
|  | $\neq 2$ | 1 |

Proof. We start by rewriting

$$
\begin{equation*}
O_{0}\left(p^{\alpha}\right)=\left(\prod_{i=1}^{b} \mathbb{Z}_{\operatorname{gcd}\left(R_{i}, p^{\alpha}\right)}\right) /\left\langle H_{i j}, G\right\rangle=\left(\prod_{i=1}^{b} \mathbb{Z}_{p^{\beta_{i}}}\right) /\left\langle H_{i j}, G\right\rangle \tag{6.6}
\end{equation*}
$$

where we defined $\beta_{i} \in\{0,1, \ldots, \alpha\}$ via $\operatorname{gcd}\left(R_{i}, p^{\alpha}\right)=p^{\beta_{i}}$. Let $H_{i j} \in \prod_{i=1}^{b} \mathbb{Z}_{p^{\beta_{i}}}$ for $i, j=$ $1, \ldots, b, i \neq j$ have components $\left(H_{i j}\right)_{i}=R_{j}+1,\left(H_{i j}\right)_{j}=R_{i}+1$ and $\left(H_{i j}\right)_{k}=0$ for $k \neq i, j$. That is, $\hat{H}_{i j}=H_{i j}$ in $\prod_{i=1}^{b} \mathbb{Z}_{p^{\beta_{i}}}$.

Note that $R_{i}=S_{i} p^{\beta_{i}}$ for some integer $S_{i}$ (which may still be divisible by $p$ ). Now if some $R_{i}$ is not divisible by $p$, then $\beta_{i}=0$ and so this factor can be omitted from the above product. Let

$$
\begin{equation*}
I \subset\{1,2, \ldots, b\} \tag{6.7}
\end{equation*}
$$

consist of elements $i$ for which $\beta_{i}>0$. The generators $H_{i j}$ now split into two sets, namely $H_{i j}$ with $i, j \in I, i \neq j$, and $H_{i}^{(k)}$, with $i \in I, k \notin I$, whose only non-zero component is the $i$ 'th one, which is equal to $R_{k}+1$. We arrive at

$$
\begin{equation*}
O_{0}\left(p^{\alpha}\right)=\left(\prod_{i \in I} \mathbb{Z}_{p^{\beta_{i}}}\right) /\left\langle H_{i j}, H_{i}^{(k)}, G\right\rangle \tag{6.8}
\end{equation*}
$$

Pick a pair $i, j \in I$ with $i \neq j$ and such that $\beta_{i} \leq \beta_{j}$. Then in $A:=\mathbb{Z}_{p^{\beta_{i}}} \times \mathbb{Z}_{p^{\beta_{j}}}$ we have $H_{i j}=\left(1, R_{i}+1\right)$. Since $R_{i}$ is divisible by $p, R_{i}+1$ is not, and hence is invertible modulo $p^{\beta_{j}}$. Let $q \in \mathbb{Z}$ be such that $q\left(R_{i}+1\right) \equiv 1 \bmod p^{\beta_{j}}$. Since $\beta_{i} \leq \beta_{j}$ this implies also that $q \equiv 1 \bmod p^{\beta_{i}}\left(\right.$ as $\left.R_{i} \equiv 0 \bmod p^{\beta_{i}}\right)$. Altogether, in $A$ we have $q\left(1, R_{i}+1\right)=(1,1)$. Conversely, $\left(R_{i}+1\right)(1,1)=\left(1, R_{i}+1\right)$ in $A$, and so we can replace the generator $H_{i j}$ by $\bar{H}_{i j} \in \prod_{k \in I} \mathbb{Z}_{p^{\beta_{k}}}$ which has entries 0 everywhere except for in positions $i, j \in I$, where it has entry 1. The group $O_{0}\left(p^{\alpha}\right)$ can thus be written as

$$
\begin{equation*}
O_{0}\left(p^{\alpha}\right)=\left(\prod_{i \in I} \mathbb{Z}_{p^{\beta_{i}}}\right) /\left\langle\bar{H}_{i j}, H_{i}^{(k)}, G\right\rangle=\left(\prod_{i \in I} \mathbb{Z}_{p^{\beta_{i}^{\prime}}}\right) /\left\langle\bar{H}_{i j}, G\right\rangle \tag{6.9}
\end{equation*}
$$

where now, for $i \in I$,

$$
\begin{equation*}
p^{\beta_{i}^{\prime}}=\operatorname{gcd}\left(p^{\alpha}, R_{i},\left\{R_{k}+1\right\}_{k \notin I}\right) \tag{6.10}
\end{equation*}
$$

At this point we distinguish cases by the number of elements in $I$ :

- $|I|=0,1$ : In this case $G$ already generates the group and so $O_{0}\left(p^{a}\right)=\{0\}$.
- $|I|=2$ : Then $I=\{i, j\}$ for some $i \neq j$ and $\bar{H}_{i j}=G$. Thus $O_{0}\left(p^{a}\right) \cong \mathbb{Z}_{\gamma}$ with $\gamma=\operatorname{gcd}\left(p^{\alpha},\left\{R_{k}\right\}_{k \in I},\left\{R_{k}+1\right\}_{k \notin I}\right)$.
- $|I| \geq 3$ : Then $\bar{H}_{12}+\bar{H}_{13}-\bar{H}_{23}$ has 2 at the first component and 0 everywhere else. This means that one can take the first component $(\bmod 2)$, and by a similar argument also for all the other components, that is

$$
\begin{equation*}
O_{0}\left(p^{\alpha}\right)=\{0\} \quad \text { if } p \neq 2 . \tag{6.11}
\end{equation*}
$$

If $p=2$, first note that if $|I|$ is odd, then using $G$ one can generate a 1 in any one component, with zeros in all other components, so furthermore

$$
\begin{equation*}
O_{0}\left(2^{\alpha}\right)=\{0\} \quad \text { if }|I| \text { odd } \tag{6.12}
\end{equation*}
$$

If $|I|$ is even, by the above argument we can take every entry $(\bmod 2)$, and it is then easy to see that there are exactly two orbits:

$$
\begin{equation*}
O_{0}\left(2^{\alpha}\right) \cong \mathbb{Z}_{2} \quad \text { if }|I| \text { even } \tag{6.13}
\end{equation*}
$$

Next we turn to the case $r=0$.
Lemma 6.4. If $R_{i} \neq 0$ for every $i=1, \ldots, b$ then $O_{0}(0)=O_{0}\left(R_{1} \cdot R_{2} \cdots R_{b}\right)$. If $R_{i}=0$ for some $i$ and $b=2$ then the order of $O_{0}(0)$ is infinite. If $R_{i}=0$ for some $i$ and $b \geq 3$ then the order of $O_{0}(0)$ is $\operatorname{gcd}\left(2\left(R_{j}+1\right)\left(R_{k}+1\right), R_{k}\left(R_{k}+1\right) \mid j, k \neq i, j \neq k\right)$.

Proof. Let us assume that $R_{i} \neq 0$ for every $i=1, \ldots, b$. Then observe that

$$
\begin{equation*}
O_{0}(0)=\prod_{i=1}^{b} \mathbb{Z}_{R_{i}} /\left\langle H_{i j}, G\right\rangle=O_{0}\left(R_{1} \cdot R_{2} \cdots R_{b}\right) \tag{6.14}
\end{equation*}
$$

by $\operatorname{gcd}\left(R_{1} \cdot R_{2} \cdots R_{b}, R_{i}\right)=R_{i}$ as in (6.6).
Assume that there is an $i_{0}$ such that $R_{i_{0}}=0$. If $b=2$ then using (2.23) we see that $R_{1}=R_{2}=0$. Hence $\hat{R}_{1}=\hat{R}_{2}=0, \hat{H}_{12}=G$ and so (6.3) reduces to $O_{0}(0)=\mathbb{Z}^{2} /\langle G\rangle \cong \mathbb{Z}$.

Suppose now that $b \geq 3$. For simplicity we take $i_{0}=1$. Note that the element $\sum_{j>1} H_{1 j}-G$ has $-1+\sum_{j>1}\left(R_{j}+1\right)=-2+\sum_{j=1}^{b}\left(R_{j}+1\right)$ as the first component and 0 everywhere else. But by (2.23) we have that $\sum_{i=1}^{b}\left(R_{i}+1\right)=2$, so that we get $G=\sum_{j>1} H_{1 j}$, i.e. the generator $G$ is redundant. Furthermore, the following elements are in the subgroup $\left\langle H_{i j}\right\rangle$ of $\mathbb{Z} \times \prod_{j=2}^{b} \mathbb{Z}_{R_{j}}$, for $i, j>1, i \neq j$ :

$$
\begin{align*}
R_{j} H_{1 j} & =\left(R_{j}\left(R_{j}+1\right), 0, \ldots, 0\right) \\
\left(R_{i}+1\right) H_{1 j}+\left(R_{j}+1\right) H_{1 i}-H_{i j} & =\left(2\left(R_{i}+1\right)\left(R_{j}+1\right), 0, \ldots, 0\right) \tag{6.15}
\end{align*}
$$

Write $g=\operatorname{gcd}\left(2\left(R_{i}+1\right)\left(R_{j}+1\right), R_{j}\left(R_{j}+1\right) \mid i, j>1, i \neq j\right)$ and consider the map

$$
\begin{equation*}
\phi: O_{0}(0)=\prod_{i=1}^{b} \mathbb{Z}_{R_{i}} /\left\langle H_{i j}\right\rangle \longrightarrow \mathbb{Z}_{g} \quad, \quad\left(a_{1}, \ldots, a_{b}\right) \mapsto a_{1}-\sum_{j=2}^{b}\left(R_{j}+1\right) a_{j} \tag{6.16}
\end{equation*}
$$

Note that this map is indeed well-defined on the quotient and is a surjection. The map $\psi: \mathbb{Z}_{g} \rightarrow \prod_{i=1}^{b} \mathbb{Z}_{R_{i}} /\left\langle H_{i j}\right\rangle, m \mapsto(m, 0, \ldots, 0)$ is equally well defined thanks to the elements in the subgroup listed in (6.15). By construction, $\phi \circ \psi=\mathrm{id}$. We now show that $\psi \circ \phi=\mathrm{id}$. The composition maps

$$
\begin{equation*}
\left(a_{1}, \ldots, a_{b}\right) \mapsto\left(\left[a_{1}-\sum_{j=2}^{b}\left(R_{j}+1\right) a_{j}\right], 0, \ldots, 0\right) \tag{6.17}
\end{equation*}
$$

By adding $\sum_{j=2}^{b} a_{j} H_{1, j}$ we get back $\left(a_{1}, \ldots, a_{b}\right)$. Thus $\left|O_{0}(0)\right|=g$.
This completes the proof of Proposition 6.1, i.e. the explicit count of MCG orbits in the $g=0$ case mentioned in Part 3 of Remark 1.3.

## The $g=1$ case

By Lemma 4.2, the set of MCG orbits is in bijection with

$$
\begin{equation*}
O_{1}(r):=\left(\mathbb{Z}_{r}^{2} \times \prod_{i=1}^{b} \mathbb{Z}_{\operatorname{gcd}\left(R_{i}, r\right)}\right) / T \tag{6.18}
\end{equation*}
$$

the set of orbits under the action a group $T$ generated by the following affine-linear transformations. Write an element of the above product as

$$
\begin{equation*}
\vec{x}=\left(s, t ; u_{1}, \ldots, u_{b}\right) \tag{6.19}
\end{equation*}
$$

Then $T$ is generated by the transformations (recall that $d_{i}$ in Lemma 4.2 only appears for $g>1$ )

$$
\begin{array}{rlrl}
T_{G}(\vec{x}) & =\left(s, t ; u_{1}+1, \ldots, u_{n}+1\right) \\
T_{a}(\vec{x}) & =\left(s-t, t ; u_{1}, \ldots, u_{b}\right) \\
T_{b}(\vec{x}) & =\left(s, t-s ; u_{1}, \ldots, u_{b}\right) \\
T_{f_{j}}(\vec{x}) & =\left(s, t-s-1-R_{j} ; u_{1}, \ldots, u_{j}+s+1, \ldots, u_{b}\right) & & \\
T_{h_{i j}}(\vec{x}) & =\left(s, t ; u_{1}, \ldots, u_{i}+R_{j}+1, \ldots, u_{j}+R_{i}+1, \ldots, u_{b}\right) & ; 1 \leq i \leq b,  \tag{6.20}\\
\end{array}
$$

It will be convenient to replace $T_{f_{j}}$ by $T_{j}:=T_{b}^{-1} T_{f_{j}}$ which acts as

$$
\begin{equation*}
T_{j}(\vec{x})=\left(s, t-\left(R_{j}+1\right) ; u_{1}, \ldots, u_{j}+s+1, \ldots, u_{b}\right) . \tag{6.21}
\end{equation*}
$$

Another convenient combination of generators is

$$
\begin{equation*}
T_{S}:=T_{a} T_{b}^{-1} T_{a}(\vec{x})=\left(-t, s ; u_{1}, \ldots, u_{b}\right) \tag{6.22}
\end{equation*}
$$

Note that $T_{a}$ and $T_{b}$ give an action of $S L(2, \mathbb{Z})$ on $\mathbb{Z}_{r}^{2}$. The orbits of this action are parametrised by divisors of $r$ :

Lemma 6.5. Let $D_{r}$ denote the set of divisors of $r$. The map $D_{r} \rightarrow \mathbb{Z}_{r}^{2} / S L(2, \mathbb{Z}), d \mapsto$ $[(0, d)]$, is a bijection.

Proof. Surjectivity: Let $(s, t) \in \mathbb{Z}^{2}$ be arbitrary and let $g:=\operatorname{gcd}(s, t)$ and $d:=\operatorname{gcd}(r, g)$, in particular $d \in D_{r}$. We can find $u, v \in \mathbb{Z}$ such that $u s+v t=g$ and $x, y \in \mathbb{Z}$ such that $x r+y g=d$. Consider the elements

$$
A=\left(\begin{array}{cc}
t / g & -s / g  \tag{6.23}\\
u & v
\end{array}\right) \quad \text { and } \quad B=\left(\begin{array}{cc}
g / d & -r / d \\
x & y
\end{array}\right)
$$

in $S L(2, \mathbb{Z})$. They satisfy $A[(s, t)]=[(0, g)]=[(r, g)]$ and $B[(r, g)]=[(0, d)]$ in $\mathbb{Z}_{r}^{2}$. So we have that $B A[(s, t)]=[(0, d)]$.
Injectivity: Let $d, d^{\prime} \in D_{r}$ and assume that $(0, d)$ and $\left(0, d^{\prime}\right)$ lie on the same $S L(2, \mathbb{Z})$-orbit. That is, there is an $A \in S L(2, \mathbb{Z})$ such that $A(0, d)=\left(0, d^{\prime}\right)$ holds in $\mathbb{Z}_{r}^{2}$. It follows that there is an integer $a \in \mathbb{Z}$ such that $a d \equiv d^{\prime}(\bmod r)$. Conversely, there is an integer $a^{\prime}$ such that $a^{\prime} d^{\prime} \equiv d(\bmod r)$. These relations, together with the fact that $d$ and $d^{\prime}$ are divisors of $r$, show that the ideal $\left\langle r, d, d^{\prime}\right\rangle$ in $\mathbb{Z}$ generated by $r, d, d^{\prime}$ is equal to $\langle d\rangle$ and equal to $\left\langle d^{\prime}\right\rangle$. But then $d= \pm d^{\prime}$, and since both are non-negative, we have $d=d^{\prime}$.

To analyse the set of orbits $O_{1}(r)$ we distinguish three cases by the number of boundary components.

- $b=0$ : In this case $T$ is generated by $T_{a}$ and $T_{b}$ only and we can directly use Lemma 6.5 to conclude that $\left|O_{1}(r)\right|=\left|D_{r}\right|$.
- $b=1$ : In this case, $T_{G}\left(s, t ; u_{1}\right)=\left(s, t ; u_{1}+1\right)$, which removes the factor $\mathbb{Z}_{\operatorname{gcd}\left(R_{1}, r\right)}$ in (6.18). The remaining non-trivial generators acting now on $\mathbb{Z}_{r}^{2}$ are $T_{a}, T_{b}$ and $T_{1}(s, t)=$ ( $s, t-\left(R_{1}+1\right)$ ). But by (2.23) we have $R_{1}+1=0$, and so $T_{1}$ also acts trivially. This reduces us to the $b=0$ case and we again have $\left|O_{1}(r)\right|=\left|D_{r}\right|$.
- $b \geq 2$ : The generators $T_{G}$ and $T_{h_{i j}}$ commute with all generators. Let $U \subset T$ be the subgroup generated by $T_{G}$ and $T_{h_{i j}}$ and write

$$
\begin{equation*}
A:=\left(\mathbb{Z}_{r}^{2} \times \prod_{i=1}^{b} \mathbb{Z}_{\operatorname{gcd}\left(R_{i}, r\right)}\right) / U \tag{6.24}
\end{equation*}
$$

Note that the quotient by $U$ amounts to dividing out a subgroup, and so $A$ is still an abelian group. In the following, we will consider the action of $T$ on $A$. By construction, we have $A / T=O_{1}(r)$.

For all $i \neq j$ we have

$$
\begin{equation*}
X_{i j}(\vec{x}):=T_{j}^{R_{i}+1} T_{i}^{R_{j}+1}(\vec{x})=\left(s, t-2\left(R_{i}+1\right)\left(R_{j}+1\right) ; u_{1}^{\prime}, \ldots, u_{b}^{\prime}\right), \tag{6.25}
\end{equation*}
$$

where $u_{i}^{\prime}=u_{i}+(s+1)\left(R_{j}+1\right), u_{j}^{\prime}=u_{j}+(s+1)\left(R_{i}+1\right)$ and $u_{k}^{\prime}=u_{k}$ for $k \neq i, j$. We can set $u_{i}^{\prime}$ and $u_{j}^{\prime}$ back to $u_{i}$ and to $u_{j}$ respectively by acting with $T_{h_{i j}}^{-s-1}$, and so in $A$ we just have that

$$
\begin{equation*}
X_{i j}(\vec{x})=\left(s, t-2\left(R_{i}+1\right)\left(R_{j}+1\right) ; u_{1}, \ldots, u_{b}\right) . \tag{6.26}
\end{equation*}
$$

Condition (2.23) now reads $\sum_{i=1}^{b}\left(R_{i}+1\right)=0$. Using this, we compute the iterated composition

$$
\begin{equation*}
\prod_{i=1, i \neq j}^{b} X_{i j}(\vec{x})=\left(s, t+2\left(R_{j}+1\right)\left(R_{j}+1\right) ; u_{1}, \ldots, u_{b}\right) . \tag{6.27}
\end{equation*}
$$

The $R_{j}$ 's power of $T_{j}$ acts as $T_{j}^{R_{j}}(\vec{x})=\left(s, t-R_{j}\left(R_{j}+1\right) ; u_{1}, \ldots, u_{b}\right)$, so that altogether we find elements $Y_{j} \in T$ which act on $A$ as

$$
\begin{equation*}
Y_{j}(\vec{x})=\left(s, t-2\left(R_{j}+1\right) ; u_{1}, \ldots, u_{b}\right) . \tag{6.28}
\end{equation*}
$$

The cases $R_{j}$ even and $R_{j}$ odd behave differently:

- $R_{j}$ even: the action of $T_{j}^{R_{j}}$ can be obtained as a power of $Y_{j}$,
- $R_{j}$ odd: an appropriate combination of $T_{j}^{R_{j}}$ and $Y_{j}$ maps $\vec{x}$ to $\left(s, t+R_{j}+1 ; u_{1}, \ldots, u_{b}\right)$.

We define

$$
P_{i}:= \begin{cases}2\left(R_{i}+1\right) & ; R_{i} \text { even }  \tag{6.29}\\ R_{i}+1 & ; R_{i} \text { odd }\end{cases}
$$

and $g:=\operatorname{gcd}\left(r, P_{1}, \ldots, P_{b}\right)$. With this notation, $T$ contains an element that maps $(s, t ; \vec{u})$ to $\left(s, t+P_{j} ; \vec{u}\right), j=1, \ldots, b$. By conjugating with $T_{S}$ from (6.22) one furthermore obtains a group element that maps $(s, t ; \vec{u})$ to $\left(s+P_{j}, t ; \vec{u}\right)$. We are therefore reduced to considering the $T$-orbits in

$$
\begin{equation*}
A^{\prime}:=\left(\mathbb{Z}_{g}^{2} \times \prod_{i=1}^{b} \mathbb{Z}_{\operatorname{gcd}\left(R_{i}, r\right)}\right) / U \tag{6.30}
\end{equation*}
$$

As before, $A^{\prime}$ is an abelian group, and by construction we have $A^{\prime} / T=O_{1}(r)$.
The above expression for $g$ can be simplified. Indeed, the number of times a prime $p \geq 3$ divides $P_{i}$ is equal to the number of times it divides $R_{i}+1$, as the presence of a factor of 2 makes no difference. For the prime $p=2$ note that 2 divides $P_{i}$ exactly once if $R_{i}$ is even, and at least once if $R_{i}$ is odd. One easily checks that with

$$
\begin{equation*}
g^{\prime}=\operatorname{gcd}\left(r, R_{1}+1, \ldots, R_{b}+1\right) \tag{6.31}
\end{equation*}
$$

we have

$$
g= \begin{cases}2 g^{\prime} & ; r \text { even and at least one } R_{i} \text { even },  \tag{6.32}\\ g^{\prime} & ; \text { else }\end{cases}
$$

At this point it is easy to give a lower bound on the number of orbits: Consider the projection $A^{\prime} \rightarrow \mathbb{Z}_{g^{\prime}}^{2},(s, t, \vec{u}) \mapsto(s, t)$, with $g^{\prime}$ as in (6.31). On $\mathbb{Z}_{g^{\prime}}^{2}$ the generators $T_{G}, T_{j}$, $T_{h_{i j}}$ all act trivially, so that we obtain a surjection

$$
\begin{equation*}
A^{\prime} / T \longrightarrow \mathbb{Z}_{g^{\prime}}^{2} / S L(2, \mathbb{Z}) \tag{6.33}
\end{equation*}
$$

By Lemma 6.5, the right hand side consists of $D_{g^{\prime}}$ many orbits. Altogether,

$$
\begin{equation*}
\left|O_{1}(r)\right| \geq\left|D_{g^{\prime}}\right| \tag{6.34}
\end{equation*}
$$

We now give an upper bound for the number of orbits. From Lemma 6.5 we know that each orbit contains a representative

$$
\begin{equation*}
\vec{y}:=\left(0, d ; u_{1}, \ldots, u_{b}\right) \tag{6.35}
\end{equation*}
$$

where $d$ now is a divisor of $g$. Since irrespective of the parity of $R_{j}$, adding $2\left(R_{j}+1\right)$ to the second entry of $A^{\prime}$ acts trivially, we have $T_{j}^{2}(\vec{y})=\left(0, d ; u_{1}, \ldots, u_{j}+2, \ldots, u_{b}\right)$.

We now go through various cases depending on the parity of $r$ and the $R_{j}$ :

- $r$ odd: In this case $\operatorname{gcd}\left(r, R_{j}\right)$ is odd for all $j$, and the above shift by 2 can be replaced by a shift by 1 , so that each orbit in $A^{\prime} / T$ contains an element $(0, d ; 0, \ldots, 0)$. Furthermore, by (6.32) we have $g=g^{\prime}$ and so the lower bound (6.34) is strict.
- $r$ even: Define $J \subset\{1,2, \ldots, b\}$ as

$$
\begin{equation*}
J=\left\{j \mid R_{j} \text { even }\right\} \tag{6.36}
\end{equation*}
$$

Suppose that $j \notin J$, i.e. that $R_{j}$ is odd. Then adding $R_{j}+1$ acts trivially on the second component in $A^{\prime}$, and so $T_{j}(\vec{y})=\left(0, d ; u_{1}, \ldots, u_{j}+1, \ldots, u_{b}\right)$. In this way one can set all entries $u_{j}$ of $\vec{u}$ to zero for which $j \neq J$. Depending on the number of even $R_{j}$ 's, we see different behaviour:
$\triangleright|J|=0$ : All $R_{j}$ are odd and hence $g=g^{\prime}$ and each orbit contains a representative $(0, d ; 0, \ldots, 0)$. Thus the lower bound (6.34) is strict.
$\triangleright|J|$ odd: This case cannot occur as $\sum_{j=1}^{b}\left(R_{j}+1\right)=0$ by (2.23). Indeed, taking this mod 2 and using that $R_{j}+1$ is even for $j \notin J$ shows that $0 \equiv \sum_{j \in J}\left(R_{j}+1\right) \equiv$ $|J|(\bmod 2)$.
$\triangleright|J| \geq 2$ even: We already know that every entry of $\vec{u}$ can be reduced mod 2 . Let $i, j \in J$ with $i \neq j$. Applying the generator $T_{h_{i j}}$ and reducing mod 2 shows that we can find an element of $U$ that maps $\vec{y}$ to $\left(0, d ; u_{1}, \ldots, u_{i}+1, \ldots, u_{j}+1, \ldots, u_{d}\right)$. Without loss of generality let us assume that $1 \in J$. Using the above shifts, and the mod 2 reduction we have anyway, we can transform $(0, d ; \vec{u})$ to one of

$$
\begin{equation*}
(0, d ; 0,0, \ldots, 0) \quad \text { or } \quad(0, d ; 1,0, \ldots, 0) \tag{6.37}
\end{equation*}
$$

Furthermore, acting with $T_{1}$ shows that, for $\varepsilon \in\{0,1\}$,

$$
\begin{equation*}
(0, d ; \varepsilon, 0, \ldots, 0) \quad \text { and } \quad\left(0, d+\left(R_{1}+1\right) ; \varepsilon+1(\bmod 2), 0, \ldots, 0\right) \tag{6.38}
\end{equation*}
$$

lie on the same $T$-orbit.
By definition, $d$ is a divisor of $g=2 g^{\prime}$. But using (6.38) on a given orbit we can always find a representative of the form (6.37) where $d$ is actually a divisor of $g^{\prime}$. Indeed, in the present case $g^{\prime}$ is odd, and so if $d$ divides $g$ but not $g^{\prime}$, it must be even, and $d+R_{1}+1$ is odd. As in the proof of Lemma 6.5 we can use
the $S L(2, \mathbb{Z})$ action to replace $d+R_{1}+1$ by $\operatorname{gcd}\left(0, d+R_{1}+1,2 g^{\prime}\right)$ which is odd and hence a divisor of $g^{\prime}$.
From the surjection (6.33) we know that different divisors $d$ of $g^{\prime}$ lie on different orbits of $T$.
It remains to show that for each $d \in D_{g^{\prime}}$, the two elements in (6.37) lie on distinct orbits. We can assume without loss of generality that all boundary components are ingoing by changing the corresponding $R_{j}$ to $-R_{j}$, since by this operation we do not change the parity. With this assumption we use Proposition 5.2 for $\Sigma_{1, b}\left(0, d, \varepsilon, 0, \ldots, 0, R_{1}, \ldots, R_{b}\right)$. One computes the RHS of (5.2) to be

$$
\begin{equation*}
(-1)^{d+1+\varepsilon \cdot\left(R_{1}+1\right)} \tag{6.39}
\end{equation*}
$$

Since $R_{1}+1$ is odd, different values of $\varepsilon$ produce different signs.
Altogether, we have shown that in the present case, the number of orbits is

$$
\begin{equation*}
\left|O_{1}(r)\right|=2\left|D_{g^{\prime}}\right| \tag{6.40}
\end{equation*}
$$

This proves the $g=1$ case of Part 3 of Theorem 1.2.

The $g \geq 2$ case
For $g \geq 2$ one can set $s_{i}=0$ for every $i=1, \ldots, g$ as before and by using Lemma 4.2 for the loops $\partial_{j}$ and $f_{j}$ one can set $u_{j}=0$ for every $j=1, \ldots, b$. Then using the lemma for the loops $d_{i}$ one can set $t_{i}=0$ for $i=2, \ldots, g$. Let us focus on $\left(s_{1}, t_{1}, s_{2}, t_{2}\right)$ and apply the lemma for the following loops:

$$
(0, t, 0,0) \xrightarrow{\text { loop } d_{1}}(0, t-1,0,+1) \xrightarrow{\text { loops } a_{2}, b_{2}}(0, t-1,0,-1) \xrightarrow{\text { loop } d_{1}}(0, t-2,0,0)
$$

This shows that there are at most 2 orbits for $r$ even and 1 orbit for $r$ odd. Again, as in the $g=1$ case we assume that all boundary components are ingoing. By Corollary 5.3 there are at least 2 orbits for $r$ even.

This completes the proof of the $g \geq 2$ case of Part 3 of Theorem 1.2 and thereby the proof of the entire theorem.

## A From triangulations to PLCW decompositions

By a triangulation of a surface we mean a smooth simplicial complex for the surface such that each boundary component consists of 3 edges and 3 vertices. In [Nov] a combinatorial description of $r$-spin surfaces was given using triangulations. The purpose of this appendix is to show how to obtain the combinatorial description of $r$-spin surfaces using PLCW decomposition of Section 2 from triangulations.


Figure 23: The 3 edges and 3 vertices of a boundary component together with the additional marking of one edge. The curly arrow shows the orientation of the boundary component and the empty vertex shows the ending vertex of the additionally marked edge.

## A. $1 \quad r$-spin surfaces with triangulations

Let us summarise the results of [Nov]. More precisely let us look at the differences between that formalism and the formalism developed in Section 2.

Let $\Sigma$ be a marked triangulation of a surface with parametrised boundary, i.e. every edge has an orientation and an edge index and every face has a marked edge. Let us assume that all boundary components are ingoing and recall the notions of Section 2.3. Put an additional marking on one of the edges of each boundary component $b$. The induced orientation of the boundary component gives a starting and and ending vertex of this additionally marked edge, see Figure 23. For a boundary vertex $u$ let $\alpha_{u}:=+1$ if it is an ending vertex for the additionally marked edge and $\alpha_{u}:=0$ otherwise. We furthermore assume that the orientation of boundary edges agrees with the induced orientation of the boundary components. The marking is called admissible for a given map $\tilde{\lambda}: \pi_{0}(\partial \Sigma) \rightarrow \mathbb{Z}_{r}$ $b \mapsto \tilde{\lambda}_{b}$, if the following hold for every inner vertex $v$ and every boundary vertex $u$ on a boundary component $b$.

$$
\begin{array}{ll}
\sum_{e \in \partial^{-1}(v)} \hat{s}_{e} \equiv D_{v}-N_{v}+1 & (\bmod r) \\
\sum_{e \in \partial^{-1}(u)} \hat{s}_{e} \equiv D_{u}-N_{u}+1+\alpha_{u} \cdot\left(1-\tilde{\lambda}_{b}\right) & (\bmod r) \tag{A.2}
\end{array}
$$

Here, $D_{v / u}, N_{v / u}$ and $\hat{s}_{e}$ are defined as in Section 2.3. According to the construction in [Nov, Sec.4.8] we proceed as follows:

- Define an $r$-spin structure on $\Sigma$ minus edges and vertices by giving the interior of the faces the $r$-spin structure $\mathbb{C}^{0}$.
- Define transition functions for every pair of faces fixed by the edge indices to extend the above to $\Sigma$ minus vertices.
- There is a unique $r$-spin structure $\Sigma(\underline{s})$ extending to the vertices if and only if the edge index assignment is admissible. Extend the $r$-spin structure to the vertices.
- The $r$-spin boundary parametrisation map is the inclusion of the $r$-spin collars according to the map $\tilde{\lambda}$. The inclusions map $1 \in \mathbb{C}^{\tilde{\lambda}}$ to the boundary vertex determined by the extra marking of the given boundary component.


## A. 2 Distinguishing in- and outgoing boundary components

The glueing of $r$-spin surfaces with parametrised boundary is defined as follows. First for every $\kappa \in \mathbb{Z}_{r}$ we specify an $r$-spin lift $I_{\varepsilon}^{\kappa}: \mathbb{C}^{\kappa} \rightarrow \mathbb{C}^{2-\kappa}\left(\tilde{s}^{\varepsilon}\right.$ in [Nov, Eqn. (3.35)]) of the map $z \mapsto z^{-1}$ given by an element $\varepsilon \in \mathbb{Z}_{r}$. Take two boundary components with $r$-spin structure on a neighbourhood of these components $\mathbb{C}^{\kappa}$ and $\mathbb{C}^{2-\kappa}$. We can glue these boundary components along their $r$-spin boundary parametrisation composed with $I_{\varepsilon}^{\kappa}$.

To define outgoing boundary components we precompose the above boundary parametrisations with $I_{\varepsilon}^{\kappa}$ for outgoing boundary components. (For convenience we will choose $\varepsilon=0$, as different choices of $\varepsilon$ can be seen as composition with different $r$-spin cylinders.) Then one can glue $r$-spin boundary components along in- and outgoing boundary parametrisations as described in Section 2.1. We now give more details on the construction.

Let $\Sigma$ be an $r$-spin surface with ingoing $r$-spin boundary parametrisation

$$
\tilde{\varphi}: \bigsqcup_{b \in \pi_{0}(\partial \Sigma)} U_{b}^{\tilde{\lambda}_{b}} \rightarrow \Sigma
$$

for a map $\tilde{\lambda}: \pi_{0}(\partial \Sigma) \rightarrow \mathbb{Z}_{r}$ which maps $b \mapsto \tilde{\lambda}_{b}$. In order to distinguish in- and outgoing boundary components we first fix two sets $B_{\text {in }}, B_{\text {out }} \subset \pi_{0}(\partial \Sigma)$ as in Section 2.1. Let $\hat{I}: \mathbb{Z}_{r} \rightarrow \mathbb{Z}_{r}$ be the map $x \mapsto 2-x$. We define maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}$ and $\mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$ by $\lambda:=\left.\tilde{\lambda}\right|_{B_{\text {in }}}$ and $\mu:=\left.\hat{I} \circ \tilde{\lambda}\right|_{B_{\text {out }}}$. For the in- and outgoing $r$-spin boundary parametrisations we set

$$
\varphi_{\text {in }}:=\left.\tilde{\varphi}\right|_{\bigsqcup_{b \in B_{\text {in }}} U_{b}^{\lambda_{b}}} \quad \text { and } \quad \varphi_{\text {out }}:=\left.\tilde{\varphi}\right|_{\sqcup_{c \in B_{\text {out }}} U_{c}^{\lambda_{c}}} \circ\left(\bigsqcup_{c \in B_{\text {out }}} I_{0}^{\lambda_{c}}\right)
$$

respectively. The admissibility condition (A.2) needs to be changed since we are parametrising outgoing boundary components $c \in B_{\text {out }}$ with $\mathbb{C}^{2-\tilde{\lambda}_{c}}$ instead of with $\mathbb{C}^{\tilde{\lambda}_{c}}$. This means that for a vertex $u$ on an outgoing boundary component $c$ the factor $\alpha_{u}$ needs to be -1 instead of +1 , since $1-\left(2-\tilde{\lambda}_{c}\right)=-\left(1-\tilde{\lambda}_{c}\right)$.

## A. 3 Refining PLCW decompositions of $r$-spin surfaces

By a series of radial subdivisions we mean radially subdividing the 1-cells and then the 2-cells, see Figure 24. This means splitting each edge in two by adding a vertex, adding a vertex to the interior of each face and adding edges between this new vertex and all other vertices of this face. The following lemmas follow from straightforward calculations.
Lemma A.1. Let $L$ be a PLCW decomposition obtained by a series of radial subdivisions from a PLCW decomposition $K$ with admissible marking. Assign to new edges the markings, orientations and edge labels as shown in Figure 24. The vertex conditions (A.1) and (A.2) are satisfied at the old and new vertices.


Figure 24: New edge indices after a series of radial subdivision. The new edge connecting the new vertex in the middle with the vertex which was in the clockwise direction of the marked side of the face (cf. Figure 3) has edge index -2 , all other new edges inside the face have edge index -1 . The admissibility conditions (A.1) and (A.2) at the vertices remain unchanged at the old vertices and they are satisfied at the new vertices.


Figure 25: Refinement at a boundary component $b$. Edges without labels have edge index -1 , the edges between $v_{0}$ and $v_{1}$ with edge label $r_{b}$ are identified.

Since we assumed that every boundary component consists of a single vertex and a single edge, applying two series of radial subdivisions gives four vertices and four edges on each boundary component. In order to get a triangulation we will modify this refinement as follows.
Lemma A.2. Let $L$ be a marked PLCW decomposition obtained by applying the steps in Lemma A. 1 twice on another PLCW decomposition $K$ with admissible marking. Add 7 triangles at each boundary component and assign the marking to the new edges as shown in Figure 25 and put the extra markings on edges on boundary components so that the ending vertex is $v_{0}$ in Figure 25. Then the conditions (A.1) and (A.2) hold at old and new vertices.

Sketch of proof. Let us assume that at each boundary component there are only two edges
connecting to the single vertex: the boundary edge and another one coming from the interior of the surface. In such a situation the refinement is shown in Figure 25. The conditions (A.1) and (A.2) can be checked by hand at every vertex.

If there are boundary components where more edges connect to the boundary vertex from the interior in the original PLCW decomposition, checking the conditions (A.1) and (A.2) is similar, but we omit the figure here.

We now have all the ingredients needed to define an $r$-spin structure with $r$-spin boundary parametrisation using the tools developed by [Nov]. We proceed as follows.

- Take a surface with parametrised boundary and a marked PLCW decomposition with some edge indices $s$ and maps $\lambda: B_{\text {in }} \rightarrow \mathbb{Z}_{r}$ and $\mu: B_{\text {out }} \rightarrow \mathbb{Z}_{r}$.
- Refine this marked PLCW decomposition as described in Lemma A.2. This is a triangulation by [Kir, Thm. 6.3].

The new marking obtained this way is admissible in the sense of [Nov] (i.e. (A.1) and (A.2) hold) if and only if the marking of the original PLCW decomposition is admissible in the sense of Section 2.3 (i.e. (2.20) and (2.21) hold).

Definition A.3. Let $\Sigma(s, \lambda, \mu)$ denote the $r$-spin structure on $\Sigma$ obtained by the above steps.

## A. 4 Proofs for Section 2

Proof of Lemma 2.11. Operation 1 follows directly from part 2 of [Nov, Lem.4.11].
For Operation 3 do a deck transformation [Nov, Part 1 of Lem. 4.11] on all triangles inside the polygon.

For Operation 2 first notice that moving the marking of a polygon to the next clockwise edge amounts to changing the edge indices as in Figure 26. This is done by a deck transformation on all filled triangles.

It is a straightforward calculation to show that these operations commute with each other.

Proof of Theorem 2.13. Let $\Sigma$ be a surface with PLCW decomposition. Let $\Sigma^{\prime}$ the same surface, but now with a triangulation as obtained by a two-fold series of radial subdivisions as in Section A.3. For clarity, in this proof we will write $\underline{\Sigma}$ for the surface without decomposition underlying both $\Sigma$ and $\Sigma^{\prime}$.

In [Nov, Sec. 4.8] $\mathcal{M}\left(\Sigma^{\prime}\right)_{\tilde{\lambda}}^{\text {triang }}$ the set of admissible markings for a fixed triangulation of $\underline{\Sigma}$ with only ingoing boundary components and fixed map $\tilde{\lambda}$ has been defined along with a similar equivalence relation as $\sim_{f i x}$, which we denote by $\sim_{\text {fix }}^{\text {triang }}$. [Nov, Thm. 4.18] gives the isomorphism from the quotient of this set by $\sim_{\text {fix }}^{\text {triang }}$ to $\mathcal{R}^{r}(\underline{\Sigma})_{\tilde{\lambda}}$ the isomorphism classes of $r$-spin structures. By a simple reparametrisation as in Section A. 2 one obtains from this the set of admissible markings for in- and outgoing boundary components $\mathcal{M}\left(\Sigma^{\prime}\right)_{\lambda, \mu}^{\text {triang }}$


Figure 26: Shifting the marking on a face of a PLCW decomposition clockwise. All unlabeled edges have index -1. a) Part of a face of a marked PLCW decomposition showing the marked edge. b) The corresponding triangulation after two series of radial subdivisions. c) Execute a deck transformation on the 12 filled triangles. d) The PLCW decomposition with shifted marked edge which produces the triangulation shown in $c$ ).
and the set of isomorphism classes of $r$-spin structures with in- and outgoing boundary components $\mathcal{R}^{r}(\underline{\Sigma})_{\lambda, \mu}$. Thus we get a bijection

$$
\begin{equation*}
\mathcal{M}\left(\Sigma^{\prime}\right)_{\lambda, \mu}^{\text {triang }} / \sim_{\text {fix }}^{\text {triang }} \xrightarrow{f} \mathcal{R}^{r}(\underline{\Sigma})_{\lambda, \mu} \tag{A.3}
\end{equation*}
$$

Let us denote by $\alpha: \mathcal{M}(\Sigma)_{\lambda, \mu}^{\mathrm{PLCW}} \rightarrow \mathcal{M}\left(\Sigma^{\prime}\right)_{\lambda, \mu}^{\text {triang }}$ the map that sends a marked PLCW decomposition to its refinement according to Section A.3. Since the generators of the equivalence relation $\sim_{\text {fix }}$ are built up from generators of the equivalence relation $\sim_{\text {fix }}^{\text {triang }}$ (see the proof of Lemma 2.11 above), we get a well defined map

$$
\begin{equation*}
\mathcal{M}(\Sigma)_{\lambda, \mu}^{\mathrm{PLCW}} / \sim_{\text {fix }} \xrightarrow{\bar{\alpha}} \mathcal{M}\left(\Sigma^{\prime}\right)_{\lambda, \mu}^{\text {triang }} / \sim_{\text {fix }}^{\text {triang }} \tag{A.4}
\end{equation*}
$$

By construction the composition of the maps (A.3) and (A.4) is the map (2.22) in the statement of the theorem. It therefore remains to show that $\bar{\alpha}$ is a bijection.
$\bar{\alpha}$ is surjective: Let $\left(m^{\prime}, o^{\prime}, s^{\prime}\right)$ be an admissible marking of $\Sigma^{\prime}$. As a first step, use the relation $\sim_{\text {fix }}^{\text {triang }}$ to change the edge markings $m^{\prime}$ and orientations $o^{\prime}$ to the form prescribed in Section A.3, resulting in a marking $\left(m^{\prime \prime}, o^{\prime \prime}, s^{\prime \prime}\right)$. Next follow the algorithm described in Figure 27 to bring all edge indices of $\Sigma^{\prime}$ in the interior of faces of $\Sigma$ to the form shown in Figure 25. Denote the resulting marking by $\left(m^{\prime \prime}, o^{\prime \prime}, \tilde{s}\right)$. Let $e$ be an interior edge of $\Sigma$


Figure 27: To convert the edge indices of all edges of the triangulation in the interior of some face of the PLCW decomposition to the form shown in Figure 25 apply the following algorithm to all faces: I) Pick a triangle in area I; proceeding clockwise around the vertex, use deck transformations on each triangle to bring the edge index of each edge radiating from the central vertex to the prescribed value ( -1 or -2 ); note that the final edge in this procedure automatically has the correct index due to the admissibility condition around the central vertex. II) Pick a triangle $t$ in region II which shares an edge with region I but whose neighbour $t^{\prime}$ in anti-clockwise direction of region II does not. Use a deck transformation on $t$ to set the edge index of the edge on the boundary of region I to the value in Figure 25; proceed clockwise around region II setting the edge index between two triangles of region II to the correct value; the edge indices between II and I are determined by the admissibility condition (and so automatically as stated in Figure 25); finally, the edge between $t^{\prime}$ and $t$ has the correct value by the admissibility condition around the vertex between region I and II shared by $t$ and $t^{\prime}$. III) If the face in question has a boundary component, then in region III one proceeds in the same way as in region II.
and let $e_{1}, \ldots, e_{4}$ be the edges of $\Sigma^{\prime}$ which cover $e$, and $v_{12}, v_{23}, v_{34}$ the three additional vertices on $e$. The admissibility condition around $v_{12}, v_{23}, v_{34}$ implies that the edge indices on $e_{1}, \ldots, e_{4}$ must all be equal. The same argument shows that edge indices on boundary components are all equal. This shows that ( $m^{\prime \prime}, o^{\prime \prime}, \tilde{s}$ ) lies in the image of $\alpha$.
$\bar{\alpha}$ is injective: Let $(m, o, s),\left(m^{\prime}, o^{\prime}, s^{\prime}\right) \in \mathcal{M}(\Sigma)_{\lambda, \mu}^{\mathrm{PLCW}}$ such that $\bar{\alpha}[(m, o, s)]=\bar{\alpha}\left[\left(m^{\prime}, o^{\prime}, s^{\prime}\right)\right]$, i.e. $\alpha(m, o, s) \sim_{\text {fix }}^{\text {triang }} \alpha\left(m^{\prime}, o^{\prime}, s^{\prime}\right)$. Notice that Lemma 2.12 and Remark 2.14 apply to marked triangulations as well. This means that we can assume that the marked edges and the edge orientations agree ( $m=m^{\prime}$ and $o=o^{\prime}$ ) for the PLCW decomposition and the triangulation as well. Furthermore, $\alpha(m, o, s)$ and $\alpha\left(m, o, s^{\prime}\right)$ are related by a series $D$ of deck transformation on the triangulation: $D(\alpha(m, o, s))=\alpha\left(m, o, s^{\prime}\right)$.

Write $\delta_{\Delta}(k)$ for a deck transformation by $k$ units on the triangle $\Delta$ of the triangulation of $\Sigma^{\prime}$. Deck transformations on different triangles commute, so we can write the sequence of deck transformations as $D=\prod_{\Delta} \delta_{\Delta}\left(k_{\Delta}\right)$. It is not hard to see that the identity $D(\alpha(m, o, s))=\alpha\left(m, o, s^{\prime}\right)$ requires the $k_{\Delta}$ for all $\Delta$ belonging to a given face of the PLCW-decomposition of $\Sigma$ to be equal. But this precisely means that $D$ can be written as a product of deck transformations on the PLCW-decomposition of $\Sigma$, i.e.


Figure 28: Pachner 3-1 move


Figure 29: Pachner 2-2 move
$(m, o, s) \sim_{\text {fix }}\left(m^{\prime}, o^{\prime}, s^{\prime}\right)$.
In the following we are going to give some tools that relate different marked triangulations and marked PLCW decompositions which parametrise isomorphic $r$-spin structures. First we recall [Nov, Prop. 4.19 and 4.20].
Lemma A.4. Let $\Sigma$ and $\Sigma^{\prime}$ be two $r$-spin surfaces with triangulation and with the same underlying surface related by a Pachner 3-1 or 2-2 move as in Figure 28 and 29. Then these two $r$-spin structures are isomorphic.

We define the $T_{n}$-moves for $n \geq 2$ as in Figure 30, which takes a $2 n$-gon glued together from $2 n$ triangles to a $2 n$-gon glued together from $2(n-1)$ triangles.
Lemma A.5. The $T_{n}$ move induces an isomorphism of $r$-spin structures.
Proof. First we show that one can obtain the $T_{n}$ move on a triangulation without any


Figure 30: $T_{n}$ move for $n \geq 2$. We remove or add the filled triangles.


Figure 31: $T_{2}$ move without marking


Figure 32: Induction step for the $T_{n+1}$ move
marking by a series of Pachner moves by induction on $n$. For $n=2$ do a Pachner 2-2 move and then a Pachner 3-1 move as in Figure 31. Now assume that the statement holds for $n$ and show for $n+1$. First we do two Pachner 2-2 moves and then apply a $T_{n}$ move as in Figure 32 to get exactly the $T_{n+1}$ move.

Since the Pachner moves in Figures 28 and 29 only change the marking locally, it is enough to check how the marking can possibly change near the vertices that are touched by these moves. If one calculates (2.20) for these vertices before and after a $T_{n}$ move one sees that the marking can only change according to Figure 30.

Lemma A.6. Removing a univalent vertex (whose edge was not marked) induces an isomorphism of $r$-spin structures.

Proof. When we remove an edge from a PLCW decomposition we need to compare the associated triangulation with marking from Definition A. 3 and then use the above defined moves to go from one to the other. The part of the triangulations that need to be transformed into one another together with the transformation steps are shown in Figure 33.

Proof of Proposition 2.16.
Move $b$ ) in Figure 5 for $v \neq v^{\prime}$ : As in the proof of Lemma A. 6 we need to compare the marked triangulations associated to the marked PLCW decompositions. The part of the triangulations that need to be transformed into one another together with the transformation steps are shown in Figure 34.

Since we did local moves which induce isomorphisms of $r$-spin structures, it is enough to check how the edge indices will change at those vertices which have been touched by the above moves. These vertices are marked with a circle. Observe that at the vertices $v_{l}$,


Figure 33: The part of the triangulations that need to be transformed into one another in case of removing or adding an univalent vertex $v$ with its edge. The dotted edges have edge index -2 , all other unlabeled edges have edge index -1 . The orientation of the edges is left implicit, cf. Definition A.3. We need to remove the 24 numbered triangles from the middle, we proceed by removing them in pairs. We use the $T_{n}$ moves consecutively: first remove the two triangles marked by 1 , then the two triangles marked by 2 , etc until finally removing the two triangles marked by 12 .
$v_{m}$ and $v_{r}$ one does not get any condition on $s$. The vertices $v_{l}^{\text {up }}, v_{l}^{\text {down }}, v_{r}^{\text {up }}$ and $v_{r}^{\text {down }}$ get identified with others.

Assume that the vertices $v$ and $v^{\prime}$ are distinct and that $s_{i}^{\prime}=s_{i}(i=1, \ldots, 4)$. At these two vertices one obtains $s \equiv 0(\bmod r)$.
Move a) in Figure 5: When removing a bivalent vertex as in Figure 5 a), a similar argument applies.
Move b) in Figure 5 for $v=v^{\prime}$ : Indeed, look at the original PLCW decomposition and assume that the vertices $v$ and $v^{\prime}$ are the same. Insert a bivalent vertex on the edge, remove one of the two new edges by the above and then the univalent vertex with its edge using Lemma A.6. Again, one obtains $s \equiv 0(\bmod r)$.

This completes the proof of the proposition.


Figure 34: The part of the triangulations that need to be transformed into one another in case of removing or adding an edge between the vertices $v$ and $v^{\prime}$ (cf. Figure $5 b$ )). The edges between $v, v_{l}^{\text {up }}$ and $v_{m}^{\text {up }}$ have edge index -2 , all other edges without edge index have edge index -1 . We need to remove the 24 triangles from the middle, of which 12 has been numbered in pairs. We use the $T_{n}$ moves consecutively: first remove the two triangles marked by 1 , then the two triangles marked by 2 , etc until finally removing the two triangles marked by 6 . Then do the same thing again for the mirror pairs.

## A. 5 Proof of Theorem 3.8

For Part 1 a direct computation shows that the morphism assigned to a PLCW decomposition and the morphism assigned to the triangulation obtained by the refinement of the PLCW decomposition are the same. One needs to use that multiplication with the $\tau^{-1}$ 's in the state-sum construction amount to canceling the "bubbles" $\mu \circ \Delta$. Independence of the choice of the function $V$ follows from the fact that $\tau$ is a central element.

Next we check independence from the triangulation and from the choice of marking (for a given $r$-spin structure). Let us assume that $\Sigma$ has $b$ ingoing and no outgoing boundary components. Let $T_{A}(\Sigma)$ denote the morphism in $\mathcal{S}$ assigned to $\Sigma$ using a triangulation by the state sum construction of [Nov]. Note that we get three tensor factors of $A$ for each boundary component, since each boundary component consists of three edges. Now we explain how to reduce $A^{\otimes 3}$ to $A$ for each boundary component. Recall that we used the notation (13) for the cyclic permutation of the first and third tensor factors. Composing $T_{A}(\Sigma)$ with $\bigotimes_{i=1}^{b}(13) \circ\left(\Delta \otimes \operatorname{id}_{A}\right) \circ \Delta \circ\left(\tau^{-2} \cdot(-)\right) \circ \iota_{\lambda_{i}}$, we obtain the morphism in (3.41). To show this we use that the factors of $\tau^{-1}$ remove the "bubbles" $\mu \circ \tau$. If $\Sigma$ has outgoing boundary components, it is easy to see that composing with appropriate factors of $\Gamma_{i, j, \varepsilon}$ maps of [Nov, Sec. 5.4] and $\pi_{\lambda_{i}} \circ \mu^{(3)} \circ(13)$ again yields the morphism in (3.41). Independence of the details of the triangulation is shown in [Nov, Thm. 5.10]. This latter theorem also states that $T_{A}(\Sigma)=T_{A}\left(\Sigma^{\prime}\right)$ for isomorphic $r$-spin surfaces $\Sigma$ and $\Sigma^{\prime}$, so that the assignment $Z_{A}: \mathcal{B o r d}_{2}^{r} \rightarrow \mathcal{S}$ is well defined on morphisms.


Figure 35: Detail of a face with interior edges of a refined PLCW decomposition with the segment $p \in A(\gamma)$ of the curve $\gamma$ crossing it. Using Part 2, we can assume that the segment of the curve crosses as shown in the figure. All edge indices without edge labels are -1 . Notice that when crossing the dotted area, the lift of the curve does not pick up any of the $\omega_{e}$ contributions.


Figure 36: The different values of $\kappa_{e}$ for different positions of the crossing curve segment. The edge $e$ is where the line segment leaves the triangle.

For Part 2 functoriality can now be seen easily from the above discussion and by using [Nov, Prop. 5.11], since the embeddings and projectors $\iota_{\lambda_{i}}$ and $\pi_{\lambda_{i}}$ compose to $P_{\lambda_{i}}$, which can be omitted due to [Nov, Prop. 5.13]. Monoidality and symmetry follow directly from the construction. This completes the proof of Theorem 3.8.

## A. 6 Proof of Lemma 5.5

Part 1 does not involve the marked PLCW decomposition and is shown in [Nov, Lem. 3.12].
Part 2 follows directly from the discussion in the main text: homotopic curves in $\Sigma$ (in the sense described in the beginning of Section 5.2) have homotopic lifts in $F_{G L} \Sigma$ and homotopic curves in $F_{G L} \Sigma$ have the same lifts in $P_{\widetilde{G L}} \Sigma$ after fixing them at the same starting point.

For Part 3, we are going to calculate the holonomy by summing up the contributions for all $\operatorname{arcs} A(\gamma)$ as in (5.4).

The contribution for $p \in A(\gamma)$ can be computed as follows. Take the face $f_{p}$ which $p$
a)

b)


Figure 37: Detail of two (not necessarily different) faces with two boundary edges of a refined PLCW decomposition where the curve $\gamma$ starts (b) and ends (a), i.e. at the image of $1 \in \mathbb{C}^{\times}$ under the boundary parametrisation. All edge indices are -1 unless otherwise noted.
crosses and take its refinement to a triangulation as in Section A.3. Let us first assume that this face has only inner edges, as in Figure 35. Let $e_{f_{p}}$ be the edge where $p$ leaves the face $f_{p}$. The contribution of $p$ can now be calculated by summing up for each triangle the " $\omega_{e}$ " contributions of [Nov, Section 4.7]. For a given triangle $t$ and edge $e$, where the curve leaves $t$, the contribution is $\omega_{e}=\hat{s}_{e}+\kappa_{e}$ by [Nov, (4.33)], where $\kappa_{e}$ is given in Figure 36.

First the curve crosses 3 triangles, which give a contribution of

$$
(0+0)+(0+0)+(0+0)=0 .
$$

Notice that when afterwards crossing the dotted area, the lift of the curve does not pick up any of contributions: for every group of 4 triangles the contribution is

$$
(-2+1)+(0+1)+(-2+1)+(0+1)=0 .
$$

If the marked edge of the face $f_{p}$ is on the right side of $p$ with respect to the orientation of $f_{p}$ then the curve has crossed the corresponding edge with edge label -2 and the contribution is

$$
\hat{\delta}_{f_{p}}^{p}=1
$$

Finally the curve crosses 6 triangles, which give a contribution of

$$
(-2+1)+(0+1)+(-2+1)+(-1+1)+(-1+1)+\left(\hat{s}_{e_{p}}+1\right)=\hat{s}_{e_{p}} .
$$

This proves the formula (5.4) if $\gamma$ is a closed curve.
If the curve $\gamma$ starts and ends on the boundary of the surface then we take it into account as follows. The parts of the triangulation where $\gamma$ starts and ends is shown in Figure 37. As described in the main text we have $r$-spin isomorphisms $D^{\kappa} \rightarrow D^{0}$ of some neighbourhoods of the starting and ending point of $\gamma$, both sending these two points to $1 \in D^{0} \subset \mathbb{C}^{0}$. Under these isomorphisms the neighbourhood of 1 , together with a part of $\gamma$ and the boundary edges is shown in Figure 38. This way we can handle $\gamma$ as a closed curve and by Part 3 we can modify the curve by a homotopy as in Figure 38, so that it crosses the edges $e_{\text {start }}$ and $e_{\text {end }}$.


Figure 38: Detail of $D^{0}$ with the image of the identification of the neighbourhoods of the starting and ending point of $\gamma$. The circle denotes the two boundary components mapped onto each other. We obtain a closed curve which, by using Part 2, we are allowed to change by a homotopy to the dotted curve. This allows us to compute the holonomy in terms of the " $\omega_{e}$ " contributions as before.

We can now calculate the contribution of these crossed triangles as before. The curve first crosses the boundary triangle in Figure 38 picking up the contribution

$$
\hat{s}_{e_{\text {start }}}+1
$$

Then it crosses the two triangles in Figure 37 b) picking up the contribution

$$
(0+0)+(-1+0)
$$

After crossing inner edges finally it crosses the two triangles in Figure 37 a), using Figure 38, picking up the contribution

$$
(0+1)+\left(0+\hat{s}_{e_{\text {end }}}\right) .
$$

Summing up the above contributions, we get formula (5.4).
This completes the proof of Lemma 5.5.

## References

[BT] J.W. Barrett and S.O.G. Tavares, Two-dimensional state sum models and spin structures. Comm. Math. Phys. 336 (2015) 63-100 [1312.7561 [math. QA]].
[Dav] A. Davydov, Centre of an algebra. Adv. Math. 225 (2010) 319-348 [0908. 1250 [math.CT]].
[DK] T. Dyckerhoff and M. Kapranov. Crossed simplicial groups and structured surfaces. In T. Pantev, C. Simpson, B. Toën, M. Vaquié, and G. Vezzosi, editors, Stacks and Categories in Geometry, Topology, and Algebra, volume 643, AMS (2015) [1403.5799 [math.AT]].
[FM] B. Farb and D. Margalit. A Primer on Mapping Class Groups. Princeton Mathematical Series. Princeton University Press (2012).
[FRS] J. Fuchs, I. Runkel, and C. Schweigert, TFT construction of RCFT correlators: III: simple currents. Nucl. Phys. B 694 (2004) 277-353 [hep-th/0403157].
[FS] J. Fuchs and C. Stigner, On Frobenius algebras in rigid monoidal categories. Arab. J. Sci. Eng. 33-2C (2008) 175-191 [0901. 4886 [math.CT]].
[GG] H. Geiges and J. Gonzalo, Generalised spin structures on 2-dimensional orbifolds. Osaka J. Math. 49 (2012) 449-470 [1004. 1979 [math.GT]].
[GK] D. Gaiotto and A. Kapustin, Spin TQFTs and fermionic phases of matter. Int. J. Mod. Phys. A 31 (2016) 1645044 [1505. 05856 [cond-mat.str-el]].
[Gun] S. Gunningham, Spin Hurwitz numbers and topological quantum field theory. Geom. Topol. 20 (2016) 1859-1907 [1201.1273 [math.QA]].
[Hus] D. Husemoller. Fibre Bundles. Graduate Texts in Mathematics. Springer, 3rd edition (1994).
[Joh] D. Johnson, Spin structures and quadratic forms on surfaces. J. London Math. Soc. 2 (1980) 365-373.
[JS] A. Joyal and R. Street, Braided tensor categories. Adv. Math. 102 (1993) 20-78.
[Kir] A. Kirillov, Jr., On piecewise linear cell decompositions. Algebr. Geom. Topol. 12 (2012) 95-108 [1009. 4227 [math. GT]].
[Koc] J. Kock. Frobenius Algebras and 2D Topological Quantum Field Theories. Cambridge University Press (2004).
[LP] A.D. Lauda and H. Pfeiffer, State sum construction of two-dimensional open-closed topological quantum field theories. J. Knot Theor. Ramif. 16 (2007) 1121-1163 [math/0602047 [math.QA]].
[Lut] F.H. Lutz, Triangulated Manifolds with Few Vertices: Combinatorial Manifolds. math/0506372 [math.CO].
[MS] G.W. Moore and G. Segal, D-branes and K-theory in 2D topological field theory. hep-th/0609042.
[Mun] J.R. Munkres. Elementary Differential Topology, volume 54 of Annals of Mathematics Studies. Princeton University Press (1966).
[Nov] S. Novak. Lattice topological field theories in two dimensions. PhD thesis, Universität Hamburg, 2015, http://ediss.sub.uni-hamburg.de/volltexte/2015/ 7527.
[NR] S. Novak and I. Runkel, State sum construction of two-dimensional topological quantum field theories on spin surfaces. J. Knot Theor. Ramif. 24 (2015) 1550028 [1402.2839 [math.QA]].
[Ran] O. Randal-Williams, Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces. J. Topol. 7 (2014) 155-186 [1001.5366 [math.GT]].
[Stee] N.E. Steenrod. The Topology of Fibre Bundles, volume 14 of Princeton Mathematical Series. Princeton University Press (1951).
[Ster] W.H. Stern, Structured Topological Field Theories via Crossed Simplicial Groups. 1603.02614 [math.CT].
[Wit] E. Witten. Algebraic Geometry Associated with Matrix Models of Two Dimensional Gravity. In L.R. Goldberg and A.V. Phillips, editors, Topological Methods in Modern Mathematics, pages 235-269. Publish or Perish, Inc. (1993).


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