Involutive categories, colored *-operads and quantum field theory

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Abstract

Involutive category theory provides a flexible framework to describe involutive structures on algebraic objects, such as anti-linear involutions on complex vector spaces. Motivated by the prominent role of involutions in quantum (field) theory, we develop the involutive analogs of colored operads and their algebras, named colored *-operads and *-algebras. Central to the definition of colored *-operads is the involutive monoidal category of symmetric sequences, which we obtain from a general product-exponential 2-adjunction whose right adjoint forms involutive functor categories. Using a novel criterion for trivializability of involutive structures, we show that the involutive monoidal category of symmetric sequences admits a very simple description. For *-algebras over *-operads we obtain involutive analogs of the usual change of color and operad adjunctions. As an application, we turn the colored operads for algebraic quantum field theory into colored *-operads. The simplest instance is the associative *-operad, whose *-algebras are unital and associative *-algebras.

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1 Introduction and summary

In ordinary category theory, an involution on an object $c \in \mathbf{C}$ of a category \mathbf{C} is an endomorphism $i: c \to c$ that squares to the identity, i.e. $i^2 = \mathrm{id}_c$. Unfortunately, this concept is too rigid to describe many examples of interest. For instance, given an associative and unital *-algebra A over \mathbb{C} , e.g. the algebra of observables of a quantum system, the involution $*: A \to A$ on its underlying vector space is *not* an endomorphism in the category of complex vector spaces, but rather a complex *anti-linear* map.

Involutive categories [BM09, Egg11, Jac12] were developed in order to introduce the flexibility required to resolve this insufficiency. Their definition is a particular instance of the "microcosm principle" of Baez and Dolan [BD98], which states that certain algebraic structures can be defined in any category equipped with a categorified version of the same structure. Hence, an involutive category is a category C equipped with an endofunctor $J: \mathbf{C} \to \mathbf{C}$ that squares to the identity endofunctor id_C, up to a given natural isomorphism $j : id_{C} \to J^{2}$ which has to satisfy certain coherence conditions (cf. Definition 2.1). In an involutive category (\mathbf{C}, J, j) , one can introduce a more flexible concept of involution on an object $c \in \mathbf{C}$, which is given by a **C**-morphism $*: c \to Jc$ satisfying $(J^*) * = j_c$ as morphisms from c to J^2c (cf. Definition 2.14). Such objects (homotopy fixed points, as a matter of fact) are called self-conjugates in [Jac12], involutive objects in [Egg11] and *-objects in [BM09]. We shall follow the latter terminology because it seems the most natural one to us. If a category is equipped with its trivial involutive structure $J = id_{\mathbf{C}}$ and $j = id_{idc}$ (cf. Example 2.2), then *-objects are just endomorphisms squaring to the identity, i.e. the ordinary involutions mentioned above. This framework, however, becomes much richer and flexible by allowing for non-trivial involutive structures: For example, endowing the category of complex vector spaces $\operatorname{Vec}_{\mathbb{C}}$ with the involutive structure given by the endofunctor that assigns to a complex vector space V its complex conjugate vector space \overline{V} , the complex anti-linear map underlying a *-algebra may be regarded as a *-object * : $A \to \overline{A}$ in this involutive category (cf. Examples 2.3 and 2.17).

The observables of a quantum system form a unital and associative *-algebra over \mathbb{C} . This shows the relevance of involutive categories for general quantum theory, quantum field theory and also noncommutative geometry. Our main motivation for this paper stems precisely from these areas and more specifically from our recent operadic approach to algebraic quantum field theory [BSW17]. There the axioms of algebraic quantum field theory [HK64, BFV03] are encoded in a colored operad and generalized to richer target categories, such as chain complexes and other symmetric monoidal categories, which are central in modern approaches to quantum gauge theories [CG17, BSS15, BS17, BSW17, Yau18]. For their physical interpretation, however, it is essential that quantum systems such as quantum field theories come equipped with involutions. These enable us to perform the GNS construction and recover the usual probabilistic interpretation of quantum theory. We refer to [Jac12] for a generalization of the GNS construction to involutive symmetric monoidal categories.

The purpose of this paper is to combine the theory of colored operads and that of involutive

categories, resulting in what we shall call colored *-operads. Despite of our quite concrete motivation, we believe that working out the theory of colored *-operads in full generality provides an interesting and valuable addition to the largely unexplored field of involutive category theory. On the one hand, our constructions naturally lead to interesting new structures such as involutive functor categories, which have not been discussed in the literature. On the other hand, our study of involutive structures on the category of symmetric sequences, which is a monoidal category that does not admit a braiding, provides an interesting example of an involutive monoidal category in the sense of [Jac12], but not in the sense of [BM09, Egg11], see Remark 4.8 for details. This shows that Jacobs' definition of involutive monoidal categories is the one suitable to develop the theory of colored *-operads, consequently we shall use this one in our paper.

The outline of the paper is as follows: Sections 2 and 3 contain a brief review of involutive categories and involutive (symmetric) monoidal categories following mostly [Jac12]. We shall in particular emphasize and further develop the 2-categorical aspects of this theory, including the 2-functorial behavior of the assignments of the categories of *-objects and *-monoids. For the sake of concreteness, we also describe the most relevant constructions and definitions arising this way in fully explicit terms. Theorems 2.23 and 3.17 establish simple criteria that are useful to detect whether an involutive ((symmetric) monoidal) category is isomorphic to one with a trivial involutive structure. In Section 4 we show that the category of colored symmetric sequences, which underlies colored operad theory, carries a canonical involutive monoidal structure in the sense of [Jac12], but not in the sense of [BM09, Egg11]. The relevant involutive structure is obtained by employing a general construction, namely exponentiation of involutive categories, which results in involutive structures on functor categories. Colored *-operads with values in any bicomplete involutive closed symmetric monoidal category (\mathbf{M}, J, j) are defined in Section 5 as *-monoids in our involutive monoidal category of colored symmetric sequences. In Proposition 5.4 we shall prove that the resulting category is isomorphic to the category of ordinary colored operads with values in the category of *-objects in (\mathbf{M}, J, j) . In Section 6 we introduce and study the category of *-algebras over colored *-operads. In particular, we prove that a change of color and operad induces an adjunction between the associated categories of *-algebras, which generalizes the corresponding crucial and widely used result from ordinary to involutive category theory. Finally, in Section 7 we endow the algebraic quantum field theory operads constructed in [BSW17] with a canonical order-reversing structure of colored *-operads and provide a characterization of the corresponding categories of *-algebras. As a simple example, we obtain a *-operad structure on the associative operad and show that its *-algebras behave like *-algebras over $\mathbb C$ in the sense that the involution reverses the order of multiplication $(a b)^* = b^* a^*$. It is essential to emphasize that this order-reversal is encoded in our *-operad structure. This is radically different from the approach of [BM09, Egg11], whose definition of an involutive monoidal category prescribes that the endofunctor J reverses the monoidal structure up to natural isomorphism, thus recovering unital and associative *-algebras over \mathbb{C} directly as *-monoids in Vec_{\mathbb{C}}.

Notations: We denote categories by boldface letters like \mathbf{C} , \mathbf{D} and \mathbf{E} . Objects in categories are indicated by $c \in \mathbf{C}$ and we write $\mathbf{C}(c, c')$ for the set of morphisms from c to c' in \mathbf{C} . Functors are denoted by symbols like $F : \mathbf{C} \to \mathbf{C}'$ or $X : \mathbf{D} \to \mathbf{C}$. Natural transformations are denoted by symbols like $\zeta : F \to G$ or $\alpha : X \to Y$. Given functors $K : \mathbf{D}' \to \mathbf{D}, X : \mathbf{D} \to \mathbf{C}$ and $J : \mathbf{C} \to \mathbf{C}'$, we denote their composition simply by juxtaposition $JXK : \mathbf{D}' \to \mathbf{C}'$. Given also a natural transformation $\alpha : X \to Y$ of functors $X, Y : \mathbf{D} \to \mathbf{C}$, we denote by

$$J\alpha K : JXK \longrightarrow JYK$$
 (1.1a)

the whiskering of J, α and K. Explicitly, $J\alpha K$ is the natural transformation with components

$$(J\alpha K)_{d'} = J\alpha_{Kd'} : JXKd' \longrightarrow JYKd'$$
, (1.1b)

for all $d' \in \mathbf{D}'$. For $\beta: Y \to Z$ another natural transformation, one easily confirms that

$$(J\beta K) (J\alpha K) = J(\beta \alpha) K : JXK \longrightarrow JZK \quad , \tag{1.2}$$

where (vertical) composition of natural transformations is also denoted by juxtaposition. We shall need some basic elements of (strict) 2-category theory, for which we refer to [KS74].

2 Involutive categories

This section contains a brief review of involutive categories. We shall mostly follow the definitions and conventions of Jacobs [Jac12] and refer to this paper for more details and some of the proofs. We strongly emphasize and also develop further the 2-categorical aspects of involutive category theory established in [Jac12], which will be relevant for the development of our present paper. When it comes to notations and terminology, we sometimes prefer the work of Beggs and Majid [BM09] and the one of Egger [Egg11].

2.1 Basic definitions and properties

Definition 2.1. An *involutive category* is a triple (\mathbf{C}, J, j) consisting of a category \mathbf{C} , an endofunctor $J : \mathbf{C} \to \mathbf{C}$ and a natural isomorphism $j : \mathrm{id}_{\mathbf{C}} \to J^2$ satisfying

$$jJ = Jj : J \longrightarrow J^3 \quad . \tag{2.1}$$

Example 2.2. For any category \mathbf{C} , the triple $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$ defines an involutive category. We call this the *trivial involutive category* over \mathbf{C} .

Example 2.3. Let $\operatorname{Vec}_{\mathbb{C}}$ be the category of complex vector spaces. Consider the endofunctor $\overline{(-)}: \operatorname{Vec}_{\mathbb{C}} \to \operatorname{Vec}_{\mathbb{C}}$ that assigns to any $V \in \operatorname{Vec}_{\mathbb{C}}$ its complex conjugate vector space $\overline{V} \in \operatorname{Vec}_{\mathbb{C}}$ and to any \mathbb{C} -linear map $f: V \to W$ the canonically induced \mathbb{C} -linear map $\overline{f}: \overline{V} \to \overline{W}$. Notice that $\overline{(-)} = \operatorname{id}_{\operatorname{Vec}_{\mathbb{C}}}$, hence the triple $(\operatorname{Vec}_{\mathbb{C}}, \overline{(-)}, \operatorname{id}_{\operatorname{id}_{\operatorname{Vec}_{\mathbb{C}}}})$ is an involutive category. ∇

Example 2.4. Let \mathfrak{C} be any non-empty set and $\Sigma_{\mathfrak{C}}$ the associated groupoid of \mathfrak{C} -profiles. The objects of $\Sigma_{\mathfrak{C}}$ are finite sequences $\underline{c} = (c_1, \ldots, c_n)$ of elements in \mathfrak{C} , including also the empty sequence $\emptyset \in \Sigma_{\mathfrak{C}}$. We denote by $|\underline{c}| = n$ the length of the sequence. The morphisms of $\Sigma_{\mathfrak{C}}$ are right permutations $\sigma : \underline{c} \to \underline{c}\sigma := (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$, with $\sigma \in \Sigma_{|\underline{c}|}$ in the symmetric group on $|\underline{c}|$ letters. We define an endofunctor Rev : $\Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$ as follows: To an object $\underline{c} = (c_1, \ldots, c_n) \in \Sigma_{\mathfrak{C}}$ it assigns the reversed sequence

$$\operatorname{Rev}(\underline{c}) := \underline{c} \rho_{|c|} := (c_n, \dots, c_1) \quad , \tag{2.2a}$$

where $\rho_{\underline{|c|}} \in \Sigma_{\underline{|c|}}$ denotes the order-reversal permutation. To a $\Sigma_{\mathfrak{C}}$ -morphism $\sigma : \underline{c} \to \underline{c}\sigma$ it assigns the right permutation

$$\operatorname{Rev}(\sigma) := \rho_{|\underline{c}|} \sigma \rho_{|\underline{c}|} : \operatorname{Rev}(\underline{c}) \longrightarrow \operatorname{Rev}(\underline{c}\sigma) \quad , \tag{2.2b}$$

where we also used that $|\underline{c}\sigma| = |\underline{c}|$. Notice that $\operatorname{Rev}^2 = \operatorname{id}_{\Sigma_{\mathfrak{C}}}$, hence the triple $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ is an involutive category. ∇

The following very useful result appears in [Jac12, Lemma 1].

Lemma 2.5. For every involutive category (\mathbf{C}, J, j) , the endofunctor $J : \mathbf{C} \to \mathbf{C}$ is self-adjoint, *i.e.* $J \dashv J$. As a consequence, J preserves all limits and colimits that exist in \mathbf{C} .

Definition 2.6. An *involutive functor* $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ consists of a functor $F : \mathbf{C} \to \mathbf{C}'$ and a natural transformation $\nu : FJ \to J'F$ satisfying

$$F = F$$

$$F_{j} \downarrow \qquad \qquad \downarrow j'F$$

$$FJ^{2} \xrightarrow{\nu J} J'FJ \xrightarrow{J'\nu} J'^{2}F$$

$$(2.3)$$

An involutive natural transformation $\zeta : (F, \nu) \to (G, \chi)$ between involutive functors $(F, \nu), (G, \chi) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is a natural transformation $\zeta : F \to G$ satisfying

$$FJ \xrightarrow{\zeta J} GJ$$

$$\nu \downarrow \qquad \qquad \downarrow \chi$$

$$J'F \xrightarrow{J'\zeta} J'G$$

$$(2.4)$$

Proposition 2.7. Involutive categories, involutive functors and involutive natural transformations form a 2-category **ICat**.

Remark 2.8. Let us describe the 2-category structure on ICat explicitly.

- (i) For any involutive category (\mathbf{C}, J, j) , the identity involutive functor is given by $(\mathrm{id}_{\mathbf{C}}, \mathrm{id}_J) : (\mathbf{C}, J, j) \to (\mathbf{C}, J, j)$.
- (ii) Given two involutive functors $(F,\nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ and $(F', \nu') : (\mathbf{C}', J', j') \to (\mathbf{C}'', J'', j'')$, their composition is given by

$$(F',\nu')(F,\nu) := (F'F,(\nu'F)(F'\nu)) : (\mathbf{C},J,j) \longrightarrow (\mathbf{C}'',J'',j'') \quad .$$
(2.5)

(iii) Vertical/horizontal composition of involutive natural transformations is given by vertical/horizontal composition of their underlying natural transformations. (It is easy to verify that the latter compositions define involutive natural transformations.) \triangle

The following technical lemma is proven in [Jac12, Lemma 2].

Lemma 2.9. For every involutive functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$, the natural transformation $\nu : FJ \to J'F$ is a natural isomorphism.

As in any 2-category, there exists the concept of adjunctions in the 2-category ICat.

Definition 2.10. An involutive adjunction

$$(L,\lambda) : (\mathbf{C},J,j) \longleftrightarrow (\mathbf{D},K,k) : (R,\rho)$$
 (2.6)

consists of two involutive functors $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ and $(R, \rho) : (\mathbf{D}, K, k) \to (\mathbf{C}, J, j)$ together with two involutive natural transformations $\eta : (\mathrm{id}_{\mathbf{C}}, \mathrm{id}_J) \to (R, \rho) (L, \lambda)$ (called *unit*) and $\epsilon : (L, \lambda) (R, \rho) \to (\mathrm{id}_{\mathbf{D}}, \mathrm{id}_K)$ (called *counit*) that satisfy the triangle identities

$$(R,\rho) \xrightarrow{\eta(R,\rho)} (R,\rho) (L,\lambda) (R,\rho) \qquad (L,\lambda) \xrightarrow{(L,\lambda)\eta} (L,\lambda) (R,\rho) (L,\lambda) \qquad (2.7)$$

$$\downarrow^{id_{(R,\rho)}} (R,\rho) \epsilon \qquad \downarrow^{id_{(L,\lambda)}} (L,\lambda) \xrightarrow{(L,\lambda)\eta} (L,\lambda) (R,\rho) (L,\lambda) \qquad (2.7)$$

We also denote involutive adjunctions simply by $(L, \lambda) \dashv (R, \rho)$.

Remark 2.11. Applying the forgetful 2-functor **ICat** \rightarrow **Cat**, every involutive adjunction $(L, \lambda) \dashv (R, \rho)$ defines an ordinary adjunction $L \dashv R$ in the 2-category of categories **Cat**. Notice that an involutive adjunction is the same thing as an ordinary adjunction $L \dashv R$ (between categories equipped with an involutive structure) whose functors L and R are equipped with involutive structures that are compatible with the unit and counit in the sense that the latter become of involutive natural transformations. This alternative point of view will be useful in Corollary 4.9 and Theorem 6.6 below, where we make use of the construction in the following proposition.

Proposition 2.12. Let $(R, \rho) : (\mathbf{D}, K, k) \to (\mathbf{C}, J, j)$ be an involutive functor and suppose that $L : \mathbf{C} \to \mathbf{D}$ is a left adjoint to the functor $R : \mathbf{D} \to \mathbf{C}$. Define a natural transformation λ by

$$\begin{array}{cccc}
LJ & & & \lambda \\
LJ\eta & & & & & \\
LJRL & & & & & \\
& & & & & \\
LJRL & & & & & \\
\end{array} (2.8)$$

where $\eta : id_{\mathbf{C}} \to RL$ and $\epsilon : LR \to id_{\mathbf{D}}$ are the unit and counit of the adjunction $L \dashv R$. Then $(L, \lambda) \dashv (R, \rho)$ is an involutive adjunction.

Proof. The above diagram defines a natural transformation λ because ρ is a natural isomorphism, cf. Lemma 2.9. A slightly lengthy diagram chase shows that $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ is an involutive functor. Furthermore, by the definition of λ , the natural transformations η and ϵ are involutive natural transformations.

Remark 2.13. Even though we will not need it in the following, let us briefly mention that the dual of Proposition 2.12 also holds true: Let $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ be an involutive functor and suppose that $R : \mathbf{D} \to \mathbf{C}$ is a right adjoint to the functor $L : \mathbf{C} \to \mathbf{D}$. Then $(L, \lambda) \dashv (R, \rho)$ is an involutive adjunction for ρ defined by

where $\eta : \mathrm{id}_{\mathbf{C}} \to RL$ and $\epsilon : LR \to \mathrm{id}_{\mathbf{D}}$ are the unit and counit of the adjunction $L \dashv R$.

2.2 *-objects

Definition 2.14. A *-object in an involutive category (\mathbf{C}, J, j) is a **C**-morphism * : $c \to Jc$ satisfying

$$c \xrightarrow{*} Jc \qquad (2.10)$$

$$j_c \xrightarrow{} J^2c$$

A *-morphism $f: (*: c \to Jc) \to (*': c' \to Jc')$ is a C-morphism $f: c \to c'$ satisfying

 $\begin{array}{c}c & \xrightarrow{f} & c' \\ * \downarrow & \downarrow & \downarrow *' \\ Jc & \xrightarrow{Jf} & Jc' \end{array}$ (2.11)

We denote the category of *-objects in (\mathbf{C}, J, j) by *-**Obj** (\mathbf{C}, J, j) .

Remark 2.15. For any *-object $(*: c \to Jc) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$, the **C**-morphism $*: c \to Jc$ is an isomorphism with inverse given by $j_c^{-1} J * : Jc \to c$.

Example 2.16. Consider the trivial involutive category $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$ from Example 2.2. A *-object consists of an object $c \in \mathbf{C}$ equipped with a **C**-endomorphism $*: c \to c$ satisfying $*^2 = \mathrm{id}_c$, i.e. an object equipped with an involution. ∇

Example 2.17. Consider the involutive category $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{id}_{\mathbf{Vec}_{\mathbb{C}}}})$ from Example 2.3. A *object consists of a complex vector space V equipped with a complex *anti-linear* map $*: V \to V$ satisfying $*^2 = \mathrm{id}_V$. ∇

Example 2.18. Consider the involutive category $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ from Example 2.4. A *-object consists of a \mathfrak{C} -profile $\underline{c} = (c_1, \ldots, c_n)$ equipped with a right permutation $*: \underline{c} \to \operatorname{Rev}(\underline{c}) = \underline{c} \rho_{|\underline{c}|}$ satisfying $*\rho_{|\underline{c}|} * \rho_{|\underline{c}|} = e \in \Sigma_{|\underline{c}|}$, where e denotes the identity permutation. In particular, any object $\underline{c} \in \Sigma_{\mathfrak{C}}$ carries a canonical *-object structure given by $\rho_{|\underline{c}|} : \underline{c} \to \underline{c} \rho_{|\underline{c}|}$. The assignment $\underline{c} \mapsto (\rho_{|\underline{c}|} : \underline{c} \to \underline{c} \rho_{|\underline{c}|})$ defines a functor $\rho : \Sigma_{\mathfrak{C}} \to *-\operatorname{Obj}(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ that is a section of the forgetful functor $U : *-\operatorname{Obj}(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}}) \to \Sigma_{\mathfrak{C}}$.

For any involutive category (\mathbf{C}, J, j) , there exists a forgetful functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ specified by $(*: c \to Jc) \mapsto c$. If the category \mathbf{C} has coproducts, we can define for any object $c \in \mathbf{C}$ a morphism

$$F(c) := \left(c \sqcup Jc \cong Jc \sqcup c \xrightarrow{\operatorname{id} \sqcup j_c} Jc \sqcup J^2 c \cong J(c \sqcup Jc) \right)$$
(2.12)

in **C**, where in the last step we used that J preserves coproducts because of Lemma 2.5. One can easily check that (2.12) defines a *-object in (**C**, J, j), i.e. $F(c) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$. Another direct computation shows

Proposition 2.19. Let (\mathbf{C}, J, j) be an involutive category that admits coproducts. The assignment $c \mapsto F(c)$ given by (2.12) naturally extends to a functor $F : \mathbf{C} \to *-\mathbf{Obj}(\mathbf{C}, J, j)$, which is a left adjoint of the forgetful functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$.

Remark 2.20. [Jac12, Lemma 5] shows that $*-\mathbf{Obj}(\mathbf{C}, J, j)$ inherits all limits and colimits that exist in **C**. These are preserved by the forgetful functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$.

As noted in [Jac12, Lemma 6], the assignment of the categories of *-objects extends to a 2-functor

*-Obj : ICat
$$\longrightarrow$$
 Cat . (2.13)

Concretely, this 2-functor is given by the following assignment:

- an involutive category (\mathbf{C}, J, j) is mapped to its category of *-objects *-**Obj** (\mathbf{C}, J, j) ;
- an involutive functor (F,ν) : $(\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped to the functor *-**Obj** (F,ν) : *-**Obj** $(\mathbf{C}, J, j) \to$ *-**Obj** (\mathbf{C}', J', j') that acts on objects as

$$*-\mathbf{Obj}(F,\nu)\big(*:c\to Jc\big) := \left(Fc\xrightarrow{F*}FJc\xrightarrow{\nu_c}J'Fc\right)$$
(2.14)

and on morphisms as F;

• an involutive natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the natural transformation $*-\mathbf{Obj}(\zeta) : *-\mathbf{Obj}(F, \nu) \to *-\mathbf{Obj}(G, \chi)$ with components $*-\mathbf{Obj}(\zeta)_{(*:c\to Jc)} := \zeta_c$, for all $(*: c \to Jc) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$.

Recalling the trivial involutive categories from Example 2.2, we obtain another 2-functor

$$triv : Cat \longrightarrow ICat \quad . \tag{2.15}$$

Concretely, this 2-functor assigns to a category \mathbf{C} the trivial involutive category $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$, to a functor $F : \mathbf{C} \to \mathbf{C}'$ the involutive functor $(F, \mathrm{id}_F) : (\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}) \to (\mathbf{C}', \mathrm{id}_{\mathrm{C}'}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}'}})$, and to a natural transformation $\zeta : F \to G$ the involutive natural transformation $\zeta : (F, \mathrm{id}_F) \to (G, \mathrm{id}_G)$.

Theorem 2.21. The 2-functors (2.13) and (2.15) form a 2-adjunction

triv : Cat
$$\longrightarrow$$
 ICat : *-Obj . (2.16)

The unit $\eta : id_{Cat} \to *-Obj$ triv and counit $\epsilon : triv *-Obj \to id_{ICat}$ 2-natural transformations are stated explicitly in the proof below.

Proof. The component at $\mathbf{C} \in \mathbf{Cat}$ of the 2-natural transformation η is the functor

$$\eta_{\mathbf{C}} : \mathbf{C} \longrightarrow *-\mathbf{Obj}(\operatorname{triv}(\mathbf{C}))$$
(2.17)

that equips objects with their identity involution (cf. Example 2.16), i.e. $c \mapsto (\mathrm{id}_c : c \to c)$. The component at $(\mathbf{C}, J, j) \in \mathbf{ICat}$ of the 2-natural transformation ϵ is the involutive functor

$$\epsilon_{(\mathbf{C},J,j)} = (U,\nu) : \operatorname{triv}\left(*\operatorname{-\mathbf{Obj}}(\mathbf{C},J,j)\right) \longrightarrow (\mathbf{C},J,j) \quad , \tag{2.18}$$

where $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ is the forgetful functor $(*: c \to Jc) \mapsto c$ and its involutive structure $\nu : U \to JU$ is the natural transformation defined by the components $\nu_{(*:c\to Jc)} = *: c \to Jc$, for all $(*: c \to Jc) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$. An elementary check shows that η and ϵ are indeed 2-natural transformations that satisfy the triangle identities, hence (2.16) is a 2-adjunction with unit η and counit ϵ .

Remark 2.22. Notice that both **Cat** and **ICat** carry a Cartesian monoidal structure, which is concretely given by the product categories $\mathbf{C} \times \mathbf{D}$ in **Cat** and the product involutive categories $(\mathbf{C}, J, j) \times (\mathbf{D}, K, k) = (\mathbf{C} \times \mathbf{D}, J \times K, j \times k)$ in **ICat**. Because *-**Obj** is a right adjoint functor, it follows that there are canonical isomorphisms

$$*-\mathbf{Obj}((\mathbf{C}, J, j) \times (\mathbf{D}, K, k)) \cong *-\mathbf{Obj}(\mathbf{C}, J, j) \times *-\mathbf{Obj}(\mathbf{D}, K, k) \quad ,$$
(2.19)

for all involutive categories (\mathbf{C}, J, j) and (\mathbf{D}, K, k) .

We conclude this section with a useful result that allows us to detect involutive categories carrying a trivial involutive structure.

Theorem 2.23. Let (\mathbf{C}, J, j) be an involutive category. Any section $* : \mathbf{C} \to *-\mathbf{Obj}(\mathbf{C}, J, j)$ of the forgetful functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ canonically determines an **ICat**-isomorphism between (\mathbf{C}, J, j) and the trivial involutive category $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$. In particular, if a section of U exists, then the involutive categories (\mathbf{C}, J, j) and $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$ are isomorphic.

Proof. A section $*: \mathbb{C} \to *-\mathbf{Obj}(\mathbb{C}, J, j)$ of U assigns to each $c \in \mathbb{C}$ a $*-\mathbf{object} *_c : c \to Jc$ and to each \mathbb{C} -morphism $f: c \to c'$ a $*-\mathbf{morphism}$



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Notice that this diagram implies that $*_c$ are the components of a natural transformation *: $\mathrm{id}_{\mathbf{C}} \to J$. It is straightforward to check that $(\mathrm{id}_{\mathbf{C}}, *) : (\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}) \to (\mathbf{C}, J, j)$ is an involutive functor, which is invertible via the involutive functor $(\mathrm{id}_{\mathbf{C}}, *^{-1}) : (\mathbf{C}, J, j) \to (\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$. \Box

Corollary 2.24. The involutive category $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ of \mathfrak{C} -profiles equipped with reversal as involutive structure (cf. Examples 2.4 and 2.18) is isomorphic to the trivial involutive category $(\Sigma_{\mathfrak{C}}, \operatorname{id}_{\Sigma_{\mathfrak{C}}}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$.

3 Involutive structures on monoidal categories

In this section we introduce involutive (symmetric) monoidal categories and *-monoids therein. We again shall follow mostly the definitions and conventions of Jacobs [Jac12]. Our main goal is to clarify and work out the 2-functorial behavior of the assignment of the categories of *-objects and monoids to involutive (symmetric) monoidal categories. To fix our notations, we start with a brief review of some basic aspects of (symmetric) monoidal categories and monoids therein.

3.1 (Symmetric) monoidal categories and monoids

Recall that a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category \mathbf{C} , a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object $I \in \mathbf{C}$ and three natural isomorphisms

$$\alpha : \otimes (\otimes \times \mathrm{id}_{\mathbf{C}}) \longrightarrow \otimes (\mathrm{id}_{\mathbf{C}} \times \otimes) \quad , \tag{3.1a}$$

$$\lambda : I \otimes (-) \longrightarrow \mathrm{id}_{\mathbf{C}} \quad , \tag{3.1b}$$

$$\rho: (-) \otimes I \longrightarrow \mathrm{id}_{\mathbf{C}} \quad , \tag{3.1c}$$

which satisfy the pentagon and triangle identities. We follow the usual abuse of notation and often denote a monoidal category by its underlying category **C**. The associator α and the unitors λ and ρ will always be suppressed. Given two monoidal categories **C** and **C'**, a *(lax) monoidal functor* from **C** to **C'** is a triple (F, F_2, F_0) consisting of a functor $F : \mathbf{C} \to \mathbf{C'}$, a natural transformation

$$F_2 : \otimes' (F \times F) \longrightarrow F \otimes \quad , \tag{3.2a}$$

and a $\mathbf{C}'\text{-morphism}$

$$F_0: I' \longrightarrow FI$$
 , (3.2b)

which are required to satisfy the usual coherence conditions involving the associators and unitors. We often denote a monoidal functor by its underlying functor $F : \mathbf{C} \to \mathbf{C}'$. A monoidal natural transformation $\zeta : F \to G$ between monoidal functors $F = (F, F_2, F_0)$ and $G = (G, G_2, G_0)$ is a natural transformation $\zeta : F \to G$ satisfying



Proposition 3.1. Monoidal categories, (lax) monoidal functors and monoidal natural transformations form a 2-category MCat.

A symmetric monoidal category is a monoidal category \mathbf{C} together with a natural isomorphism called *braiding*

$$\tau : \otimes \longrightarrow \otimes^{\mathrm{op}} := \otimes \sigma \tag{3.4}$$

from the tensor product to the opposite tensor product, where $\sigma : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is the flip functor $(c_1, c_2) \mapsto (c_2, c_1)$, which satisfies the hexagon identities and the symmetry constraint



We often denote a symmetric monoidal category by its underling category **C**. A symmetric monoidal functor is a monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ that preserves the braidings, i.e.

$$\otimes' (F \times F) \xrightarrow{\tau' (F \times F)} \otimes' \sigma(F \times F) = \otimes' (F \times F) \sigma$$

$$F_2 \downarrow \qquad \qquad \downarrow F_2 \sigma$$

$$F \otimes \xrightarrow{F \tau} F \otimes \sigma$$

$$(3.6)$$

commutes. A *symmetric monoidal natural transformation* is just a monoidal natural transformation between symmetric monoidal functors.

Proposition 3.2. Symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal natural transformations form a 2-category **SMCat**.

Definition 3.3. A monoid in a (symmetric) monoidal category **C** is a triple (M, μ, η) consisting of an object $M \in \mathbf{C}$ and two **C**-morphisms $\mu : M \otimes M \to M$ (called *multiplication*) and $\eta :$ $I \to M$ (called *unit*) satisfying the associativity and unitality axioms. A monoid morphism $f : (M, \mu, \eta) \to (M', \mu', \eta')$ is a **C**-morphism $f : M \to M'$ preserving multiplications and units. We denote the category of monoids in **C** by **Mon**(**C**).

The assignment of the categories of monoids extends to a 2-functor

$$Mon : (S)MCat \longrightarrow Cat$$
 . (3.7)

Concretely, this 2-functor is given by the following assignment:

- a (symmetric) monoidal category **C** is mapped to its category of monoids **Mon**(**C**);
- a (symmetric) monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ is mapped to the functor $\mathbf{Mon}(F) : \mathbf{Mon}(\mathbf{C}) \to \mathbf{Mon}(\mathbf{C}')$ that acts on objects as

$$\mathbf{Mon}(F)(M,\mu,\eta) := \left(FM, FM \otimes' FM \xrightarrow{F_{2M,M}} F(M \otimes M) \xrightarrow{F\mu} FM, I' \xrightarrow{F_0} FI \xrightarrow{F\eta} FM\right)$$
(3.8)

and on morphisms as F;

• a (symmetric) monoidal natural transformation $\zeta : F \to G$ is mapped to the natural transformation $\mathbf{Mon}(\zeta) : \mathbf{Mon}(F) \to \mathbf{Mon}(G)$ with components $\mathbf{Mon}(\zeta)_{(M,\mu,\eta)} := \zeta_M$, for all $(M, \mu, \eta) \in \mathbf{Mon}(\mathbf{C})$.

3.2 Involutive (symmetric) monoidal categories

The following definition of an involutive (symmetric) monoidal category is due to [Jac12]. We prefer this definition over the one in [Egg11, BM09] as it has the advantage that the category of *-objects inherits a monoidal structure (cf. [Jac12, Proposition 1] and Proposition 3.15 in the present paper). This has interesting consequences for the theory of involutive monads in [Jac12] and the developments in our present paper.

Definition 3.4. An *involutive (symmetric) monoidal category* is a triple (\mathbf{C}, J, j) consisting of a (symmetric) monoidal category \mathbf{C} , a (symmetric) monoidal endofunctor $J = (J, J_2, J_0) : \mathbf{C} \to \mathbf{C}$ and a (symmetric) monoidal natural isomorphism $j : \mathrm{id}_{\mathbf{C}} \to J^2$ satisfying

$$jJ = Jj : J \longrightarrow J^3 \quad . \tag{3.9}$$

The following statement is proven in [Jac12, Lemma 7].

Lemma 3.5. For any involutive (symmetric) monoidal category, the (symmetric) monoidal endofunctor $J = (J, J_2, J_0) : \mathbb{C} \to \mathbb{C}$ is strong, i.e. $J_2 : \otimes (J \times J) \to J \otimes$ and $J_0 : I \to JI$ are isomorphisms.

Remark 3.6. Let us emphasize again and more clearly that our Definition 3.4 of involutive (symmetric) monoidal categories agrees with the one of Jacobs [Jac12]. The definitions in [BM09] and [Egg11] are different because their analog of J_2 is order-reversing, i.e. a natural isomorphism $\otimes^{\text{op}} (J \times J) \rightarrow J \otimes$. The reason why we consider order-preserving J_2 as in [Jac12] is that this is better suited for our development of involutive operad theory, cf. Remark 4.8 below. \triangle

Remark 3.7. The condition for $j : id_{\mathbb{C}} \to J^2$ to be a (symmetric) monoidal natural transformation explicitly means that the diagrams



commute. One may reinterpret these diagrams as follows: The left diagram states that (\otimes, J_2) : $(\mathbf{C}, J, j) \times (\mathbf{C}, J, j) \rightarrow (\mathbf{C}, J, j)$ is an involutive functor on the product involutive category $(\mathbf{C}, J, j) \times (\mathbf{C}, J, j) = (\mathbf{C} \times \mathbf{C}, J \times J, j \times j)$, see also Remark 2.22. The right diagram states that $(J_0 : I \rightarrow JI) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$ is a *-object in (\mathbf{C}, J, j) . These two structures allow us to endow the functor $I \otimes (-) : \mathbf{C} \rightarrow \mathbf{C}$ with an involutive structure $I \otimes J(-) \rightarrow J(I \otimes (-))$ defined by the components

$$I \otimes Jc \xrightarrow{J_0 \otimes \mathrm{id}} JI \otimes Jc \xrightarrow{J_{2I,c}} J(I \otimes c) \quad , \tag{3.11}$$

for all $c \in \mathbf{C}$. An analogous statement holds true for the functor $(-) \otimes I : \mathbf{C} \to \mathbf{C}$. The axioms for the (symmetric) monoidal structure on J can then be reinterpreted as the equivalent property that the associator and unitors (as well as the braiding in the symmetric case) are involutive natural transformations.

Summing up, we obtain an equivalent description of an involutive (symmetric) monoidal category in terms of the following data: An involutive category (\mathbf{C}, J, j) , an involutive functor $(\otimes, J_2) : (\mathbf{C}, J, j) \times (\mathbf{C}, J, j) \to (\mathbf{C}, J, j)$, a *-object $(J_0 : I \to JI) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$ and involutive natural transformations for the associator and unitors (as well as the braiding in the symmetric case), which satisfy analogous axioms as those for (symmetric) monoidal categories. This alternative point of view is useful for (3.16) and (3.17) below.

Example 3.8. For any (symmetric) monoidal category \mathbf{C} , the triple (\mathbf{C} , id_{\mathbf{C}}, id_{id_{$\mathbf{C}}$}), with id_{\mathbf{C}} the identity (symmetric) monoidal functor and id_{id_{$\mathbf{C}}</sub> the identity (symmetric) monoidal natural transformation, defines an involutive (symmetric) monoidal category. We call this the$ *trivial involutive (symmetric) monoidal category* $over <math>\mathbf{C}$.</sub></sub></sub>

Example 3.9. Let us equip the category of complex vector spaces $\mathbf{Vec}_{\mathbb{C}}$ with its standard symmetric monoidal structure where \otimes is the usual tensor product, $I = \mathbb{C}$ is the ground field and τ is given by the flip maps $\tau_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$. The endofunctor $\overline{(-)} : \mathbf{Vec}_{\mathbb{C}} \to \mathbf{Vec}_{\mathbb{C}}$ from Example 2.3 can be promoted to a symmetric monoidal functor by using the canonical maps $\overline{(-)}_{2V,W} : \overline{V} \otimes \overline{W} \to \overline{V \otimes W}$ and complex conjugation $\overline{(-)}_0 : \mathbb{C} \to \overline{\mathbb{C}}$. The resulting triple $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{id}_{\mathbf{Vec}_{\mathbb{C}}}})$ is an involutive symmetric monoidal category. ∇

Example 3.10. Recall the groupoid of \mathfrak{C} -profiles $\Sigma_{\mathfrak{C}}$ from Example 2.4. The category $\Sigma_{\mathfrak{C}}$ may be equipped with the symmetric monoidal structure given by concatenation of \mathfrak{C} -profiles, i.e. $\underline{c} \otimes \underline{d} = (c_1, \ldots, c_n, d_1, \ldots, d_m), I = \emptyset$ is the empty \mathfrak{C} -profile and $\tau_{\underline{c},\underline{d}} := \tau \langle |\underline{c}|, |\underline{d}| \rangle : \underline{c} \otimes \underline{d} \to \underline{d} \otimes \underline{c}$ is the block transposition. The reversal endofunctor Rev : $\Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$ can be promoted to a symmetric monoidal functor by using

$$\operatorname{Rev}_{2c,d} := \tau \langle |\underline{c}|, |\underline{d}| \rangle : \operatorname{Rev}(\underline{c}) \otimes \operatorname{Rev}(\underline{d}) \longrightarrow \operatorname{Rev}(\underline{c} \otimes \underline{d})$$
(3.12)

and $\operatorname{Rev}_0 := \operatorname{id}_{\emptyset} : \emptyset \to \operatorname{Rev}(\emptyset) = \emptyset$. The resulting triple $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ is an involutive symmetric monoidal category. ∇

Definition 3.11. An involutive (symmetric) monoidal functor (F, ν) : $(\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ consists of a (symmetric) monoidal functor $F = (F, F_2, F_0) : \mathbf{C} \to \mathbf{C}'$ and a (symmetric) monoidal natural transformation $\nu : FJ \to J'F$ satisfying the analog of diagram (2.3) in Definition 2.6.

An involutive (symmetric) monoidal natural transformation $\zeta : (F, \nu) \to (G, \chi)$ between involutive (symmetric) monoidal functors $(F, \nu), (G, \chi) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is a natural transformation $\zeta : F \to G$ that is both involutive and (symmetric) monoidal.

Proposition 3.12. Involutive (symmetric) monoidal categories, involutive (symmetric) monoidal functors and involutive (symmetric) monoidal natural transformations form a 2-category I(S)MCat.

Remark 3.13. The condition for the natural transformation $\nu : FJ \to J'F$ to be monoidal explicitly means that the diagrams



commute. From the perspective established in Remark 3.7, one may reinterpret these diagrams as follows: The left diagram states that F_2 is an involutive natural transformation

$$F_2 : (\otimes', J_2') ((F, \nu) \times (F, \nu)) \longrightarrow (F, \nu) (\otimes, J_2)$$

$$(3.14)$$

of involutive functors from $(\mathbf{C}, J, j) \times (\mathbf{C}, J, j)$ to (\mathbf{C}', J', j') . The right diagram states that F_0 defines a morphism

$$F_0 : (J'_0 : I' \to J'I') \longrightarrow *-\mathbf{Obj}(F,\nu)(J_0 : I \to JI)$$
(3.15)

in the category $*-\mathbf{Obj}(\mathbf{C}', J', j')$ of $*-\mathbf{objects}$ in (\mathbf{C}', J', j') .

Summing up, we obtain an equivalent description of an involutive (symmetric) monoidal functor in terms of the following data: An involutive functor (F,ν) : $(\mathbf{C}, J, j) \rightarrow (\mathbf{C}', J', j)$, an involutive natural transformation F_2 as in (3.14) and a *-morphism F_0 as in (3.15), which satisfy axioms analogous to those for a (symmetric) monoidal functor. This alternative point of view is useful for (3.20) below.

Remark 3.14. Let us summarize Remarks 3.7 and 3.13 by one slogan: Involutive (symmetric) monoidal categories are the same things as (symmetric) monoidal involutive categories. \triangle

Let (\mathbf{C}, J, j) be an involutive (symmetric) monoidal category and consider its category of *-objects *-**Obj** (\mathbf{C}, J, j) . Making use of the 2-functor *-**Obj** : **ICat** \rightarrow **Cat** given in (2.13), we

may equip the category $*-\mathbf{Obj}(\mathbf{C}, J, j)$ with a (symmetric) monoidal structure. Concretely, the tensor product functor is given by

where the vertical isomorphism was explained in Remark 2.22 and the involutive functor (\otimes, J_2) in Remark 3.7. The unit object

$$(J_0: I \to JI) \in *-\mathbf{Obj}(\mathbf{C}, J, j)$$

$$(3.17)$$

is the *-object constructed in Remark 3.7. The associator and unitors (as well as the braiding in the symmetric case) are obtained by applying the 2-functor *-**Obj** to the associator and unitors (as well as the braiding in the symmetric case) of (\mathbf{C}, J, j) , which makes sense because Remark 3.7 shows that these are involutive natural transformations. Let us also mention that the tensor product of two *-objects (* : $c \rightarrow Jc$), (*' : $c' \rightarrow Jc'$) \in *-**Obj**(\mathbf{C}, J, j) explicitly reads as

$$(*: c \to Jc) \otimes (*': c' \to Jc') = \left(c \otimes c' \xrightarrow{* \otimes *'} Jc \otimes Jc' \xrightarrow{J_{2c,c'}} J(c \otimes c') \right) \quad . \tag{3.18}$$

Summing up, we have proven

Proposition 3.15. Let (\mathbf{C}, J, j) be an involutive (symmetric) monoidal category. Then the category of *-objects *-**Obj**(\mathbf{C}, J, j) is a (symmetric) monoidal category with tensor product (3.16) and unit object (3.17). Moreover, if (\mathbf{C}, J, j) is also closed, i.e. it has internal homs, then *-**Obj**(\mathbf{C}, J, j) is closed too (cf. [Jac12, Proposition 1]).

The assignment of the (symmetric) monoidal categories of *-objects extends to a 2-functor

*-Obj :
$$I(S)MCat \rightarrow (S)MCat$$
 , (3.19)

which we shall denote with an abuse of notation by the same symbol as the 2-functor in (2.13). Concretely, this 2-functor is given by the following assignment:

- an involutive (symmetric) monoidal category (C, J, j) is mapped to the (symmetric) monoidal category *-Obj(C, J, j) given in Proposition 3.15;
- an involutive (symmetric) monoidal functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped to the (symmetric) monoidal functor

$$*-\mathbf{Obj}(F,\nu) : *-\mathbf{Obj}(\mathbf{C},J,j) \longrightarrow *-\mathbf{Obj}(\mathbf{C}',J',j')$$
(3.20a)

with underlying functor as in (2.13) and (symmetric) monoidal structure given by

*-**Obj**
$$(F)_2 := *-Obj(F_2)$$
, *-**Obj** $(F)_0 := F_0$, (3.20b)

where F_2 and F_0 should be interpreted according to Remark 3.13;

• an involutive (symmetric) monoidal natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the (symmetric) monoidal natural transformation determined by (2.13).

Remark 3.16. Notice that the 2-functor $*-\text{Obj} : I(S)MCat \rightarrow (S)MCat$ given in (3.19) is a lift of the 2-functor $*-\text{Obj} : ICat \rightarrow Cat$ given in (2.13) along the forgetful 2-functors forget_ \otimes : $I(S)MCat \rightarrow ICat$ and forget_ \otimes : $(S)MCat \rightarrow Cat$ that forget the (symmetric) monoidal

structures. More precisely, using the explicit descriptions of our 2-functors, one easily confirms that the diagram



of 2-categories and 2-functors commutes (on the nose).

We conclude this section with a useful result that generalizes Theorem 2.23 to the (symmetric) monoidal setting. Let us first notice that the forgetful functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ satisfies $\otimes (U \times U) = U \otimes$ and $U(J_0 : I \to JI) = I$, hence it can be promoted to a (symmetric) monoidal functor via the trivial (symmetric) monoidal structure $U_2 = \mathrm{id}_{U\otimes}$ and $U_0 = \mathrm{id}_I$.

Theorem 3.17. Let (\mathbf{C}, J, j) be an involutive (symmetric) monoidal category. Any (symmetric) monoidal section $* : \mathbf{C} \to *-\mathbf{Obj}(\mathbf{C}, J, j)$ of the forgetful (symmetric) monoidal functor $U : *-\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ canonically determines an $\mathbf{I}(\mathbf{S})\mathbf{MCat}$ -isomorphism between (\mathbf{C}, J, j) and the trivial involutive (symmetric) monoidal category $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$. In particular, if such a section of U exists, then the involutive (symmetric) monoidal categories (\mathbf{C}, J, j) and $(\mathbf{C}, \mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{id}_{\mathbf{C}}})$ are isomorphic.

Proof. Using that the (symmetric) monoidal structure on U is trivial, i.e. $U_2 = id_{U\otimes}$ and $U_0 = id_I$, and also that U is a faithful functor, one observes that the (symmetric) monoidal structure on the (symmetric) monoidal section $*: \mathbb{C} \to *-\mathbf{Obj}(\mathbb{C}, J, j)$ is necessarily trivial. The proof then proceeds analogously to the one of Theorem 2.23.

Corollary 3.18. The involutive symmetric monoidal category $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ of \mathfrak{C} -profiles equipped with reversal as involutive structure (cf. Example 3.10) is isomorphic to the trivial involutive symmetric monoidal category $(\Sigma_{\mathfrak{C}}, \operatorname{id}_{\Sigma_{\mathfrak{C}}}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$.

Proof. By Theorem 3.17, it is sufficient to construct a symmetric monoidal section $\rho = (\rho, \rho_2, \rho_0)$: $\Sigma_{\mathfrak{C}} \to *-\mathbf{Obj}(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ of the forgetful symmetric monoidal functor U. Taking the underlying functor as in Example 2.18, i.e. $\rho : \underline{c} \mapsto (\rho_{|\underline{c}|} : \underline{c} \to \underline{c}\rho_{|\underline{c}|})$ with the order-reversal permutations $\rho_{|\underline{c}|} \in \Sigma_{|\underline{c}|}$, one easily checks that $\otimes(\rho \times \rho) = \rho \otimes$ and $\rho(\emptyset) = (\operatorname{id}_{\emptyset} : \emptyset \to \emptyset) = (\operatorname{Rev}_0 : \emptyset \to \operatorname{Rev}(\emptyset))$. We choose the trivial symmetric monoidal structure $\rho_2 = \operatorname{id}_{\rho \otimes}$ and $\rho_0 = \operatorname{id}_{\emptyset}$.

3.3 *-monoids

Let us recall the 2-functors $Mon : (S)MCat \rightarrow Cat$ given in (3.7), *-Obj : $ICat \rightarrow Cat$ given in (2.13) and its lift *-Obj : $I(S)MCat \rightarrow S(M)Cat$ given in (3.19). The aim of this subsection is to describe a 2-functor $Mon : I(S)MCat \rightarrow ICat$ that lifts $Mon : (S)MCat \rightarrow Cat$ to the involutive setting, such that the diagram



of 2-categories and 2-functors commutes (on the nose). We then define *-monoids in terms of the diagonal 2-functor *-Mon : $I(S)MCat \rightarrow Cat$ in this square.

Let us start with describing the 2-functor

$$Mon: I(S)MCat \longrightarrow ICat$$
(3.23)

that lifts (3.7) to the involutive setting in some detail:

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• an involutive (symmetric) monoidal category (\mathbf{C}, J, j) is mapped to the involutive category

$$\mathbf{Mon}(\mathbf{C}, J, j) := (\mathbf{Mon}(\mathbf{C}), \mathbf{Mon}(J), \mathbf{Mon}(j)) \in \mathbf{ICat}$$
(3.24)

given by evaluating the 2-functor (3.7) on the (symmetric) monoidal category \mathbf{C} , on the (symmetric) monoidal endofunctor $J : \mathbf{C} \to \mathbf{C}$ and on the (symmetric) monoidal natural isomorphism $j : \mathrm{id}_{\mathbf{C}} \to J^2$;

• an involutive (symmetric) monoidal functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped to the involutive functor

$$\mathbf{Mon}(F,\nu) := (\mathbf{Mon}(F),\mathbf{Mon}(\nu)) : \mathbf{Mon}(\mathbf{C},J,j) \longrightarrow \mathbf{Mon}(\mathbf{C}',J',j')$$
(3.25)

given by evaluating the 2-functor (3.7) on the (symmetric) monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ and on the (symmetric) monoidal natural transformation $\nu : FJ \to J'F$;

• an involutive (symmetric) monoidal natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the involutive natural transformation

$$\mathbf{Mon}(\zeta) : \mathbf{Mon}(F,\nu) \longrightarrow \mathbf{Mon}(G,\chi)$$
 (3.26)

given by evaluating the 2-functor (3.7) on ζ .

Lemma 3.19. The diagram (3.22) of 2-categories and 2-functors commutes (on the nose).

Proof. This is an elementary check using the explicit definitions of the 2-functors given in (3.7), (2.13), (3.19) and (3.23).

Definition 3.20. The 2-functor *-Mon : $I(S)MCat \rightarrow Cat$ is defined as the diagonal 2-functor in the commutative square (3.22), i.e.



For an involutive (symmetric) monoidal category (\mathbf{C}, J, j) , we call *-**Mon** (\mathbf{C}, J, j) the category of *-monoids in (\mathbf{C}, J, j) .

Remark 3.21. Let (\mathbf{C}, J, j) be an involutive (symmetric) monoidal category. We provide an explicit description of the objects and morphisms in the associated category of *-monoids *-**Mon** (\mathbf{C}, J, j) , which we shall call *-monoids and *-monoid morphisms. Unpacking Definition 3.20, one obtains that a *-monoid is a quadruple $(M, \mu, \eta, *) \in *-\mathbf{Mon}(\mathbf{C}, J, j)$ consisting of an object $M \in \mathbf{C}$ and three \mathbf{C} -morphisms $\mu : M \otimes M \to M, \eta : I \to M$ and $* : M \to JM$, which satisfy the following conditions:

- (1) (M, μ, η) is a monoid in the (symmetric) monoidal category C;
- (2) $*: M \to JM$ is a *-object in the involutive category (\mathbf{C}, J, j) ;
- (3) these two structures are compatible in the sense that the diagrams



in \mathbf{C} commute.

As a consequence of Lemma 3.19, these conditions have two equivalent interpretations which correspond to the counterclockwise and clockwise paths in the commutative diagram (3.27): The first option is to regard $*: (M, \mu, \eta) \to \mathbf{Mon}(J)(M, \mu, \eta)$ as a *-object in the involutive category $\mathbf{Mon}(\mathbf{C}, J, j) \in \mathbf{ICat}$. The second option is to regard $\eta: (J_0: I \to JI) \to (*: M \to JM)$ and $\mu: (*: M \to JM) \otimes (*: M \to JM) \to (*: M \to JM)$ as the structure maps of a monoid in the (symmetric) monoidal category *-**Obj**(\mathbf{C}, J, j) \in (**S**)**MCat**.

A *-monoid morphism $f : (M, \mu, \eta, *) \to (M', \mu', \eta', *')$ is a C-morphism $f : M \to M'$ that preserves both the monoid structures and *-involutions.

Example 3.22. Let us consider a *-monoid $(A, \mu, \eta, *)$ in the involutive symmetric monoidal category ($\operatorname{Vec}_{\mathbb{C}}, \overline{(-)}, \operatorname{id}_{\operatorname{id}_{\operatorname{Vec}_{\mathbb{C}}}}$) from Example 3.9. In particular, the triple (A, μ, η) is an associative and unital algebra over \mathbb{C} with multiplication $a b = \mu(a \otimes b)$ and unit $\mathbb{1} = \eta(1)$. By Example 2.17, * is a complex anti-linear automorphism of A that squares to the identity, i.e. $a^{**} = a$. The compatibility conditions in (3.28) state that $\mathbb{1}^* = \mathbb{1}$ and $(a b)^* = a^* b^*$. We would like to emphasize that the latter condition *is not* the usual axiom for associative and unital *-algebras over \mathbb{C} , which is given by order-reversal $(a b)^* = b^* a^*$. As a consequence, our concept of *-monoids given in Definition 3.20 *does not* include the usual associative and unital *-algebras over \mathbb{C} are recovered as *-algebras over a suitable *-operad, which provides a sufficiently flexible framework to implement order-reversal $(a b)^* = b^* a^*$.

4 Involutive structures on colored symmetric sequences

Colored operads can be defined as monoids in the monoidal category of colored symmetric sequences, see e.g. [Yau16, WY18, BSW17] and below for a brief review. Let $\mathfrak{C} \in \mathbf{Set}$ be any non-empty set and \mathbf{M} any bicomplete closed symmetric monoidal category. (We denote the monoidal structure on \mathbf{M} by \otimes and I, and the internal hom by $[-, -] : \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$.) The category of \mathfrak{C} -colored symmetric sequences with values in \mathbf{M} is defined as the functor category

$$\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) := \mathbf{M}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}} \quad , \tag{4.1}$$

where $\Sigma_{\mathfrak{C}}$ is the groupoid of \mathfrak{C} -profiles defined in Example 2.4 and the set \mathfrak{C} is regarded as a discrete category. Given $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, we write

$$X(^t_c) \in \mathbf{M} \tag{4.2a}$$

for the evaluation of this functor on objects $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ and

$$X(\sigma) : X(\underline{c}) \longrightarrow X(\underline{c}) \longrightarrow (4.2b)$$

for its evaluation on morphisms $\sigma : (\underline{c}, t) \to (\underline{c}\sigma, t)$ in $\Sigma_{\mathfrak{C}} \times \mathfrak{C}$.

The category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ can be equipped with the following monoidal structure: The tensor product is given by the *circle product* \circ : $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \times \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \rightarrow \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. Concretely, the circle product of $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is defined by the coend

$$(X \circ Y) \begin{pmatrix} t \\ \underline{c} \end{pmatrix} := \int^{\underline{a}} \int^{(\underline{b}_1, \dots, \underline{b}_m)} \Sigma_{\mathfrak{C}} (\underline{b}_1 \otimes \dots \otimes \underline{b}_m, \underline{c}) \otimes X \begin{pmatrix} t \\ \underline{a} \end{pmatrix} \otimes Y \begin{pmatrix} a_1 \\ \underline{b}_1 \end{pmatrix} \otimes \dots \otimes Y \begin{pmatrix} a_m \\ \underline{b}_m \end{pmatrix} \quad , \qquad (4.3)$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. Two remarks are in order: (1) This expression makes use of the symmetric monoidal structure on $\Sigma_{\mathfrak{C}}$ that we described in Example 3.10. (2) The tensor product between the Hom-set $\Sigma_{\mathfrak{C}}(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}) \in \mathbf{Set}$ and the object $X(\frac{t}{a}) \in \mathbf{M}$ is given by the canonical **Set**-tensoring of \mathbf{M} , i.e. $S \otimes m := \coprod_{s \in S} m$ for any $S \in \mathbf{Set}$ and $m \in \mathbf{M}$. The *circle unit* is the object $I_{\circ} \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ defined by

$$I_{\circ}\left(\frac{t}{\underline{c}}\right) := \Sigma_{\mathfrak{C}}(t,\underline{c}) \otimes I \quad , \tag{4.4}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. The following result is well known, see e.g. [BSW17, Section 3.1.1].

Proposition 4.1. (SymSeq_{\mathfrak{C}}(M), \circ , I_{\circ}) is a right closed monoidal category.

The aim of this section is to transfer these structures and results to the setting of involutive categories.

4.1 Product-exponential 2-adjunction

Because the category of symmetric sequences (4.1) is defined as a functor category, we shall start with developing a notion of functor categories in the involutive setting. For this we will first recall the relevant structures for ordinary category theory from a perspective that easily generalizes to involutive category theory.

Let us denote by $\operatorname{Cat} \times \operatorname{Cat}$ the 2-category with objects given by pairs (\mathbf{C}, \mathbf{D}) of categories, morphisms given by pairs (F, G) of functors and 2-morphisms given by pairs (ζ, ξ) of natural transformations, and all compositions given component-wise. (We use the symbol \times to denote the above product 2-category because we reserve the symbol \times for the 2-functors defined below.) Notice that taking products of categories, functors and natural transformations defines a 2-functor

$$\times : \operatorname{Cat} \widetilde{\times} \operatorname{Cat} \longrightarrow \operatorname{Cat} \quad . \tag{4.5}$$

Let us denote by $\mathbf{Cat}^{\mathrm{op}}$ the opposite 2-category, i.e. morphisms $\mathbf{C} \to \mathbf{D}$ are functors $F : \mathbf{D} \to \mathbf{C}$ going in the opposite direction and 2-morphisms are not reversed. We define the exponential 2-functor

$$(-)^{(-)} : \operatorname{Cat}^{\operatorname{op}} \widetilde{\times} \operatorname{Cat} \longrightarrow \operatorname{Cat}$$

$$(4.6)$$

as follows:

- a pair (**D**, **C**) of categories is mapped to the functor category **C**^{**D**};
- a pair $(G : \mathbf{D}' \to \mathbf{D}, F : \mathbf{C} \to \mathbf{C}')$ of functors is mapped to the functor $F^G : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}'^{\mathbf{D}'}$ that acts on objects and morphisms as

$$F^{G}(X: \mathbf{D} \to \mathbf{C}) := (FXG: \mathbf{D}' \to \mathbf{C}') , \qquad (4.7a)$$

$$F^G(\alpha: X \to Y) := (F\alpha G: FXG \to FYG) ;$$
 (4.7b)

• a pair $(\xi : G \to G', \zeta : F \to F')$ of natural transformations is mapped to the natural transformation $\zeta^{\xi} : F^G \to F'^{G'}$ with components given by any of the two compositions in the commutative square

$$\begin{array}{c} FXG \xrightarrow{\zeta XG} F'XG \\ FX\xi & \searrow \\ FX\xi & \searrow \\ FXG' \xrightarrow{\zeta XG'} F'XG' \end{array}$$
(4.8)

for all $X \in \mathbf{C}^{\mathbf{D}}$.

The two 2-functors \times and $(-)^{(-)}$ are related by a family of 2-adjunctions.

Proposition 4.2. For every $D \in Cat$, there is a 2-adjunction

$$(-) \times \mathbf{D} : \mathbf{Cat} \rightleftharpoons \mathbf{Cat} : (-)^{\mathbf{D}}$$
 . (4.9)

Proof. The component at $\mathbf{C} \in \mathbf{Cat}$ of the unit 2-natural transformation $\eta : \mathrm{id}_{\mathbf{Cat}} \to ((-) \times \mathbf{D})^{\mathbf{D}}$ is given by the functor

$$\eta_{\mathbf{C}} : \mathbf{C} \longrightarrow (\mathbf{C} \times \mathbf{D})^{\mathbf{D}}$$

$$(4.10)$$

that assigns to $c \in \mathbf{C}$ the inclusion functor $\eta_{\mathbf{C}}(c) : \mathbf{D} \to \mathbf{C} \times \mathbf{D}$ specified by $d \mapsto (c, d)$. The component at $\mathbf{C} \in \mathbf{Cat}$ of the counit 2-natural transformations $\epsilon : (-)^{\mathbf{D}} \times \mathbf{D} \to \mathrm{id}_{\mathbf{Cat}}$ is given by the evaluation functor

$$\epsilon_{\mathbf{C}} : \mathbf{C}^{\mathbf{D}} \times \mathbf{D} \longrightarrow \mathbf{C} \quad , \tag{4.11}$$

that assigns to $(X, d) \in \mathbf{C}^{\mathbf{D}} \times \mathbf{D}$ the object $Xd \in \mathbf{C}$. The triangle identities are a straightforward check.

Because of their 2-functoriality, our constructions above can be immediately extended to involutive category theory. Concretely, using the 2-functor (4.5), we define the product 2-functor

$$\times : \mathbf{ICat} \times \mathbf{ICat} \longrightarrow \mathbf{ICat}$$
(4.12)

in the involutive setting as follows:

• a pair of involutive categories is mapped to the involutive category

$$(\mathbf{C}, J, j) \times (\mathbf{D}, K, k) := (\mathbf{C} \times \mathbf{D}, J \times K, j \times k) \quad ; \tag{4.13}$$

• a pair of involutive functors is mapped to the involutive functor

$$(F,\nu) \times (G,\chi) := (F \times G, \nu \times \chi) \quad ; \tag{4.14}$$

• a pair of involutive natural transformations is mapped to the involutive natural transformation $\zeta \times \xi$.

Similarly, using the 2-functor (4.6), we define the exponential 2-functor

$$(-)^{(-)} : \mathbf{ICat}^{\mathrm{op}} \widetilde{\times} \mathbf{ICat} \longrightarrow \mathbf{ICat}$$

$$(4.15)$$

in the involutive setting as follows:

• a pair of involutive categories is mapped to the involutive category

$$(\mathbf{C}, J, j)^{(\mathbf{D}, K, k)} := \left(\mathbf{C}^{\mathbf{D}}, J^{K}, j^{k}\right) \quad ; \tag{4.16}$$

• a pair of involutive functors is mapped to the involutive functor

$$(F,\nu)^{(G,\chi)} := (F^G,\nu^{\chi^{-1}}) ; \qquad (4.17)$$

• a pair of involutive natural transformations is mapped to the involutive natural transformation ζ^{ξ} .

Analogously to Proposition 4.2, one can prove

Proposition 4.3. For every $(\mathbf{D}, K, k) \in \mathbf{ICat}$, there is a 2-adjunction

$$(-) \times (\mathbf{D}, K, k) : \mathbf{ICat} \rightleftharpoons \mathbf{ICat} : (-)^{(\mathbf{D}, K, k)}$$
 (4.18)

4.2 Involutive colored symmetric sequences

Let (\mathbf{M}, J, j) be an involutive closed symmetric monoidal category, which we assume to be bicomplete, and $\mathfrak{C} \in \mathbf{Set}$ a non-empty set of colors. Consider the groupoid of \mathfrak{C} -profiles $\Sigma_{\mathfrak{C}}$ from Examples 2.4, 2.18 and 3.10, where it was also shown that it may be equipped with the structure of an involutive symmetric monoidal category $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$. Regarding the set \mathfrak{C} as a discrete category, we may endow it with the trivial involutive structure from Example 2.2, i.e. $\operatorname{triv}(\mathfrak{C}) = (\mathfrak{C}, \operatorname{id}_{\mathfrak{C}}, \operatorname{id}_{\operatorname{id}_{\mathfrak{C}}})$. Using the product 2-functor (4.12), we may form the involutive category $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}}) \times \operatorname{triv}(\mathfrak{C}) \in \mathbf{ICat}$.

Definition 4.4. Let $\mathfrak{C} \in \mathbf{Set}$ be a non-empty set. The *involutive category of* \mathfrak{C} -colored symmetric sequences with values in a bicomplete involutive closed symmetric monoidal category (\mathbf{M}, J, j) is defined via the exponential 2-functor (4.15) by

$$\left(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J^{\operatorname{Rev}\times\operatorname{id}_{\mathfrak{C}}}, j^{\operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}\times\mathfrak{C}}}\right) := \left(\mathbf{M}, J, j\right)^{\left(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}}\right) \times \operatorname{triv}(\mathfrak{C})} \in \mathbf{ICat} \quad , \qquad (4.19)$$

where the underlying category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ was introduced in (4.1).

The involutive category of \mathfrak{C} -colored symmetric sequences in Definition 4.4 turns out to be isomorphic to a more convenient and simpler involutive category: Recalling Corollary 3.18, there exists an **ISMCat**-isomorphism between $(\Sigma_{\mathfrak{C}}, \operatorname{Rev}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$ and the trivial involutive symmetric monoidal category triv $(\Sigma_{\mathfrak{C}}) = (\Sigma_{\mathfrak{C}}, \operatorname{id}_{\Sigma_{\mathfrak{C}}}, \operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}})$. Using also 2-functoriality of the product and exponential, we obtain

Corollary 4.5. There exists an ICat-isomorphism

$$\left(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J^{\operatorname{Rev}\times\operatorname{id}_{\mathfrak{C}}}, j^{\operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}\times\mathfrak{C}}}\right) \cong \left(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J^{\operatorname{id}_{\Sigma_{\mathfrak{C}}\times\mathfrak{C}}}, j^{\operatorname{id}_{\operatorname{id}_{\Sigma_{\mathfrak{C}}}\times\mathfrak{C}}}\right) \quad .$$
(4.20)

Remark 4.6. In the remaining part of this paper, we shall always work with the simpler but isomorphic version of the involutive category of \mathfrak{C} -colored symmetric sequences given in Corollary 4.5. To simplify notations, we shall write

$$(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*) := (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J^{\mathrm{id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}}, j^{\mathrm{id}_{\mathrm{id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}}})$$
 (4.21)

Concretely, the endofunctor $J_* := J^{\mathrm{id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}} : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is post-composing with $J : \mathbf{M} \to \mathbf{M}$, i.e. $X \mapsto JX$, and the natural isomorphism $j_* := j^{\mathrm{id}_{\mathrm{id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}} : \mathrm{id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \to J_*^2$ has components $j_{*X} := jX$ given by whiskering the natural isomorphism $j : \mathrm{id}_{\mathbf{M}} \to J^2$ and the functor $X : \Sigma_{\mathfrak{C}} \times \mathfrak{C} \to \mathbf{M}$, for all $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$.

We now show that the involutive category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ given in (4.21) may be promoted to an involutive monoidal category, extending the monoidal structure of Proposition 4.1 to the involutive setting. Recalling Definition 3.4, this amounts to endowing the endofunctor $J_* : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ with the structure of a monoidal functor such that $j_* : \mathrm{id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \to J^2_*$ becomes a monoidal natural isomorphism. We first define the natural transformation $J_{*2} : \circ (J_* \times J_*) \to J_* \circ$ in terms of the components

for all $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. For the horizontal arrows we used the definition of the circle product (4.3) and the fact that $J : \mathbf{M} \to \mathbf{M}$ is self-adjoint (cf. Lemma

2.5), hence it preserves coends and the **Set**-tensoring. In the right vertical arrow we denoted by J_2^m the *m*-fold iteration of the natural transformation $J_2 : \otimes (J \times J) \to J \otimes$ corresponding to the involutive symmetric monoidal category (\mathbf{M}, J, j) . We next define the **SymSeq**_{\mathfrak{C}}(**M**)-morphism $J_{*0} : I_{\circ} \to J_*I_{\circ}$ for the circle unit (4.4) by

$$I_{\circ}\begin{pmatrix} t \\ c \end{pmatrix} \xrightarrow{J_{*0}} (J_*I_{\circ})\begin{pmatrix} t \\ c \end{pmatrix}$$

$$\downarrow \cong$$

$$\Sigma_{\mathfrak{C}}(t,\underline{c}) \otimes I \xrightarrow{\mathrm{id} \otimes J_0} \Sigma_{\mathfrak{C}}(t,\underline{c}) \otimes JI$$

$$(4.23)$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. For the right vertical arrow we used again that $J : \mathbf{M} \to \mathbf{M}$ is self-adjoint and hence it preserves the **Set**-tensoring. In the bottom horizontal arrow $J_0 : I \to JI$ denotes the morphism corresponding to the involutive symmetric monoidal category (\mathbf{M}, J, j) .

Theorem 4.7. The involutive category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ of (4.21) becomes an involutive right closed monoidal category when the underlying category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is equipped with the circle monoidal structure of Proposition 4.1 and the underlying endofunctor J_* is equipped with the monoidal functor structure (J_{*2}, J_{*0}) of (4.22) and (4.23).

Proof. It is straightforward to confirm that (J_*, J_{*2}, J_{*0}) : $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, as defined in (4.21), (4.22) and (4.23), is a monoidal endofunctor with respect to the circle monoidal structure and that the natural isomorphism $j_* : \mathrm{id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \to J_*^2$ is monoidal. \Box

Remark 4.8. Because $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is in general a *non-symmetric* monoidal category, the *nonreversing* notion of involutive structure due to [Jac12] (see also Definition 3.4) and the *reversing* one considered in [Egg11, BM09] are a priori inequivalent. This is indeed the case: While Theorem 4.7 equips the monoidal category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ with a *non-reversing* involutive structure, one cannot obtain a *reversing* one as this requires to specify isomorphisms $J_*X \circ J_*Y \cong J_*(Y \circ X)$, which in general do not exist by the following argument: Assume that $I \cong \emptyset$ in \mathbf{M} (e.g. $\mathbf{M} =$ $\mathbf{Vec}_{\mathbb{C}}$) and that the set \mathfrak{C} has cardinality ≥ 2 . Define $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ by setting

$$X(\underline{t}) = \left(\Sigma_{\mathfrak{C}}(t, t_0) \times \Sigma_{\mathfrak{C}}(t_0, \underline{c})\right) \otimes I \quad , \qquad Y(\underline{t}) = \Sigma_{\mathfrak{C}}(\emptyset, \underline{c}) \otimes I \quad , \tag{4.24}$$

for some fixed $t_0 \in \mathfrak{C}$. Recalling (4.3) we obtain

$$(X \circ Y) \begin{pmatrix} t \\ \underline{c} \end{pmatrix} \cong \Sigma_{\mathfrak{C}}(t, t_0) \otimes Y \begin{pmatrix} t \\ \underline{c} \end{pmatrix} , \qquad (Y \circ X) \begin{pmatrix} t \\ \underline{c} \end{pmatrix} \cong Y \begin{pmatrix} t \\ \underline{c} \end{pmatrix} .$$
(4.25)

Since $J_*X \cong X$ and $J_*Y \cong Y$, we find for $t \neq t_0$ that $(J_*X \circ J_*Y) \begin{pmatrix} t \\ \emptyset \end{pmatrix} \cong \emptyset \not\cong I \cong J_*(Y \circ X) \begin{pmatrix} t \\ \emptyset \end{pmatrix}$. This counterexample explains why the *non-reversing* involutive structures defined by [Jac12] are better suited for developing the theory of colored *-operads than the *reversing* ones of [Egg11, BM09]. \triangle

Many interesting constructions in colored operad theory arise from changing the underlying set of colors, see e.g. [BSW17] for examples inspired by quantum field theory. We shall now generalize the relevant constructions to the setting of involutive category theory.

Any map $f : \mathfrak{C} \to \mathfrak{D}$ of non-empty sets induces a functor $f : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{D}}$ between the associated groupoids of profiles. Concretely, we have that $\underline{c} = (c_1, \ldots, c_n) \mapsto f(\underline{c}) = (f(c_1), \ldots, f(c_n))$. This functor may be equipped with the obvious involutive symmetric monoidal structure such that it defines an involutive symmetric monoidal functor

$$(f, \mathrm{id}_f) : (\Sigma_{\mathfrak{C}}, \mathrm{id}_{\Sigma_{\mathfrak{C}}}, \mathrm{id}_{\mathrm{id}_{\Sigma_{\mathfrak{C}}}}) \longrightarrow (\Sigma_{\mathfrak{D}}, \mathrm{id}_{\Sigma_{\mathfrak{D}}}, \mathrm{id}_{\mathrm{id}_{\Sigma_{\mathfrak{D}}}})$$
 (4.26)

Moreover, regarding \mathfrak{C} and \mathfrak{D} as discrete categories, we obtain an involutive functor (denoted by the same symbol)

$$(f, \mathrm{id}_f) : (\mathfrak{C}, \mathrm{id}_{\mathfrak{C}}, \mathrm{id}_{\mathrm{id}_{\mathfrak{C}}}) \longrightarrow (\mathfrak{D}, \mathrm{id}_{\mathfrak{D}}, \mathrm{id}_{\mathrm{id}_{\mathfrak{D}}})$$

$$(4.27)$$

between the associated trivial involutive categories. Using the product and exponential 2-functors (cf. (4.12) and (4.15)), we may exponentiate the identity $id_{(\mathbf{M},J,j)} = (id_{\mathbf{M}}, id_J)$ involutive functor by the product involutive functor $(f, id_f) \times (f, id_f)$ to obtain an involutive functor

$$(f^*, \mathrm{id}_{f^*J_*}) : (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) \longrightarrow (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$$

$$(4.28)$$

describing the pullback along f of \mathfrak{D} -colored symmetric sequences to \mathfrak{C} -colored symmetric sequences. (Notice that $f^*J_* = J_*f^*$ as functors from $\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$ to $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ because J_* is a pullback.)

Corollary 4.9. For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, there exists an involutive adjunction (cf. Definition 2.10)

$$(f_!, \lambda_f) : (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*) \iff (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) : (f^*, \mathrm{id}_{f^*J_*}) \quad .$$
(4.29)

Proof. By left Kan extension, the functor f^* has a left adjoint $f_!$. The involutive structure λ_f on $f_!$ is the one described in Proposition 2.12, which implies that we have an involutive adjunction.

Our pullback functor $f^* : \operatorname{SymSeq}_{\mathfrak{D}}(\mathbf{M}) \to \operatorname{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ may be equipped with the following canonical monoidal structure, see also [BSW17, Section 3.1.3] for more details. The components of the natural transformation $f_2^* : \circ^{\mathfrak{C}}(f^* \times f^*) \to f^* \circ^{\mathfrak{D}}$ are specified by

for all $X, Y \in \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$ and all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. The horizontal arrows are the canonical inclusions into the coend and the left vertical arrow denotes the action of the functor $f : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{D}}$ on Hom-sets. The $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ -morphism $f_0^* : I_{\circ}^{\mathfrak{C}} \to f^*I_{\ast}^{\mathfrak{D}}$ is defined similarly by

$$f_0^* : I_\circ^{\mathfrak{C}} \binom{t}{\underline{c}} = \Sigma_{\mathfrak{C}}(t, \underline{c}) \otimes I \xrightarrow{f \otimes \mathrm{id}} \Sigma_{\mathfrak{D}} (f(t), f(\underline{c})) \otimes I = f^* (I_\circ^{\mathfrak{D}}) \binom{t}{\underline{c}} \quad , \tag{4.31}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$.

Theorem 4.10. For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, the involutive functor $(f^*, \mathrm{id}_{f^*J_*}) : (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) \to (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ of (4.28) becomes an involutive monoidal functor when equipped with the monoidal structure (f_2^*, f_0^*) of (4.30) and (4.31).

Proof. By Definition 3.11, it remains to prove that $id_{J_*f^*} : J_*f^* \to f^*J_* = J_*f^*$ is a monoidal natural transformation, which is clearly the case.

4.3 *-objects

We conclude this section by describing rather explicitly the monoidal category

*-Obj
$$($$
SymSeq $_{\mathfrak{C}}(\mathbf{M}), J_*, j_*) \in \mathbf{MCat}$ (4.32)

of *-objects in the involutive monoidal category of symmetric sequences. Given any *-object $(*: X \to J_*X) \in *-\mathbf{Obj}(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$, we consider its components at $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ and observe that this is precisely the same data as a symmetric sequence with values in $*-\mathbf{Obj}(\mathbf{M}, J, j)$, which is a bicomplete closed symmetric monoidal category, cf. Proposition 3.15 and Remark 2.20. Similarly, one observes that a morphism in (4.32) is the same data as a morphism in $\mathbf{SymSeq}_{\mathfrak{C}}(*-\mathbf{Obj}(\mathbf{M}, J, j))$, which means that these two categories are canonically isomorphic. We now show that this isomorphism is compatible with the monoidal structures.

Proposition 4.11. The canonical identification above defines an isomorphism

$$*-\mathbf{Obj}(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*) \cong \mathbf{SymSeq}_{\mathfrak{C}}(*-\mathbf{Obj}(\mathbf{M}, J, j))$$
(4.33)

of monoidal categories.

Proof. It remains to prove that our canonical isomorphism of categories is monoidal, i.e. that tensor products and units are preserved up to coherent isomorphisms. Given two objects $* : X \to J_*X$ and $*' : Y \to J_*Y$ in *-Obj(SymSeq_c(M), J_*, j_*), their tensor product reads as

$$(*: X \to J_*X) \circ (*': Y \to J_*Y) = \left(X \circ Y \xrightarrow{*\circ*'} J_*X \circ J_*Y \xrightarrow{(J_{*2})_{X,Y}} J_*(X \circ Y)\right) \quad . \quad (4.34)$$

By a brief calculation one shows that the composed morphism on the right-hand side of this equation is induced by functoriality of coends and **Set**-tensoring via the family of maps

$$X^{\binom{t}{\underline{a}}} \otimes \bigotimes_{i=1}^{m} Y^{\binom{a_i}{\underline{b}_i}} \xrightarrow{* \otimes \bigotimes_{i=1}^{m} *'} JX^{\binom{t}{\underline{a}}} \otimes \bigotimes_{i=1}^{m} JY^{\binom{a_i}{\underline{b}_i}} \xrightarrow{J_2^m} J(X^{\binom{t}{\underline{a}}} \otimes \bigotimes_{i=1}^{m} Y^{\binom{a_i}{\underline{b}_i}}) \quad .$$
(4.35)

Notice that (4.35) is the tensor product $(*: X(\frac{t}{a}) \to JX(\frac{t}{a})) \otimes \bigotimes_{i=1}^{m} (*': Y(\frac{b_i}{b_i}) \to JY(\frac{b_i}{b_i}))$ in *-**Obj**(**M**, *J*, *j*). Because *J* preserves coends and the **Set**-tensoring, we obtain the natural isomorphism relating the tensor products on both sides of (4.33). A similar construction provides the isomorphism relating the units.

5 Colored *-operads

In the following let $\mathfrak{C} \in \mathbf{Set}$ be any non-empty set. Let us briefly recall the notion of \mathfrak{C} -colored operads, see e.g. [BSW17, Definition 3.2.1].

Definition 5.1. The category of \mathfrak{C} -colored operads with values in a bicomplete closed symmetric monoidal category \mathbf{M} is the category of monoids (cf. Definition 3.3) in the monoidal category **SymSeq** $_{\mathfrak{C}}(\mathbf{M})$ (cf. (4.1), (4.3) and (4.4)), i.e.

$$\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) := \mathbf{Mon} (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}))$$
 . (5.1)

Using the concepts and techniques that we have developed so far in this paper, the above definition admits the following natural generalization to involutive category theory.

Definition 5.2. The category of \mathfrak{C} -colored *-operads with values in a bicomplete involutive closed symmetric monoidal category (\mathbf{M}, J, j) is the category of *-monoids (cf. Definition 3.20) in the involutive monoidal category (**SymSeq** $\mathfrak{c}(\mathbf{M}), J_*, j_*$) (cf. Theorem 4.7), i.e.

$$*-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) := *-\mathbf{Mon}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big) \quad .$$
(5.2)

Remark 5.3. It is worth to specialize Remark 3.21 to the present case. We observe that a \mathfrak{C} -colored *-operad is a quadruple $(\mathcal{O}, \gamma, \mathbb{1}, *)$ consisting of a \mathfrak{C} -colored symmetric sequence $\mathcal{O} \in$ **SymSeq**_{\mathfrak{C}}(**M**) and three **SymSeq**_{\mathfrak{C}}(**M**)-morphisms $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ (called *operadic composition*), $\mathbb{1} : I_{\circ} \to \mathcal{O}$ (called *operadic unit*) and $* : \mathcal{O} \to J_*\mathcal{O}$ (called *-involution), which satisfy the following conditions:

(1) $(\mathcal{O}, \gamma, \mathbb{1})$ is a monoid in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_{\circ})$, i.e. the diagrams



in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commute;

(2) $*: \mathcal{O} \to J_*\mathcal{O}$ is a *-object in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$, i.e. the diagram

$$\mathcal{O} \xrightarrow{*} J_* \mathcal{O} \tag{5.4}$$
$$\downarrow_{J_*^2 \mathcal{O}}$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commutes;

(3) these two structures are compatible, i.e. the diagrams

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commute.

In particular, there exist two equivalent interpretations of a colored *-operad: The first option is to regard $(\mathcal{O}, \gamma, \mathbb{1})$ as an ordinary \mathfrak{C} -colored operad valued in (\mathbf{M}, J, j) , equipped with an operad morphism $* : (\mathcal{O}, \gamma, \mathbb{1}) \to \mathbf{Mon}(J_*)(\mathcal{O}, \gamma, \mathbb{1})$. The second option is to regard $* : \mathcal{O} \to J_*\mathcal{O}$ as a *-object in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$, equipped with the structure of a monoid consisting of the *-morphisms $\gamma : (* : \mathcal{O} \to J_*\mathcal{O}) \circ (* : \mathcal{O} \to J_*\mathcal{O}) \to (* : \mathcal{O} \to J_*\mathcal{O})$ and $\mathbb{1} : (J_{*0} : I_\circ \to J_*I_\circ) \to$ $(* : \mathcal{O} \to J_*\mathcal{O}).$

Proposition 5.4. The category of \mathfrak{C} -colored *-operads with values in a bicomplete involutive closed symmetric monoidal category (\mathbf{M}, J, j) is isomorphic to the category of \mathfrak{C} -colored operads with values in the bicomplete closed symmetric monoidal category *- $\mathbf{Obj}(\mathbf{M}, J, j)$, i.e. there exists an isomorphism

$$* - \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \cong \mathbf{Op}_{\mathfrak{C}}(* - \mathbf{Obj}(\mathbf{M}, J, j))$$

$$(5.6)$$

of categories.

Proof. This is proven by the following chain of **Cat**-isomorphisms

$$*-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) = \mathbf{Mon} \left(*-\mathbf{Obj}(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)\right)$$

$$\cong \mathbf{Mon} \left(\mathbf{SymSeq}_{\mathfrak{C}}(*-\mathbf{Obj}(\mathbf{M}, J, j))\right) = \mathbf{Op}_{\mathfrak{C}}(*-\mathbf{Obj}(\mathbf{M}, J, j)) \quad , \quad (5.7)$$

where in the first step we used Definitions 5.2 and 3.20, in the second step Proposition 4.11 and in the last step Definition 5.1. $\hfill \Box$

Remark 5.5. Proposition 5.4 may be summarized by the following slogan: Colored *-operads are the same things as colored operads in *-objects. \triangle

We shall now study the behavior of colored *-operads under changing the underlying set of colors. Let $f : \mathfrak{C} \to \mathfrak{D}$ be a map between non-empty sets. By Theorem 4.10, we obtain an involutive monoidal functor $(f^*, \mathrm{id}_{f^*J_*}) : (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) \to (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$. As a consequence of 2-functoriality of *-**Mon** : **IMCat** \to **Cat** (cf. Definition 3.20) and the definition of colored *-operads (cf. Definition 5.2), we obtain

Proposition 5.6. For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, there exists a functor

$$f^* := *-\mathbf{Mon}(f^*, \mathrm{id}_{f^*J_*}) : *-\mathbf{Op}_{\mathfrak{D}}(\mathbf{M}, J, j) \longrightarrow *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \quad , \tag{5.8}$$

which we call the pullback functor.

Using the pullback functor, we may define the category of *-operads with varying colors.

Definition 5.7. We denote by *-**Op**(**M**, *J*, *j*) the category of *colored* *-*operads* with values in (**M**, *J*, *j*). The objects are pairs ($\mathfrak{C}, \mathcal{O}$) consisting of a non-empty set $\mathfrak{C} \in \mathbf{Set}$ and a \mathfrak{C} -colored *-operad $\mathcal{O} \in *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$. The morphisms are pairs $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ consisting of a map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets and a *-**Op**_{\mathfrak{C}}(\mathbf{M}, J, j)-morphism $\phi : \mathcal{O} \to f^*\mathcal{P}$.

Remark 5.8. There exists a projection functor π : *-**Op**(**M**, J, j) \rightarrow **Set**, given explicitly by $(\mathfrak{C}, \mathcal{O}) \mapsto \mathfrak{C}$, whose fiber $\pi^{-1}(\mathfrak{C})$ over $\emptyset \neq \mathfrak{C} \in$ **Set** is isomorphic to the category *-**Op**_{\mathfrak{C}}(**M**, J, j) of \mathfrak{C} -colored *-operads.

6 *-algebras over colored *-operads

A convenient description of algebras over colored operads is in terms of algebras over their associated monads, see e.g. [WY18, BSW17]. Let us briefly review the relevant constructions before generalizing them to the setting of involutive categories.

Let $\mathfrak{C} \in \mathbf{Set}$ be a non-empty set of colors. Recall that the category of \mathfrak{C} -colored objects with values in \mathbf{M} is the functor category $\mathbf{M}^{\mathfrak{C}}$. We may equivalently regard $\mathbf{M}^{\mathfrak{C}}$ as the full subcategory of $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ consisting of all functors $X : \Sigma_{\mathfrak{C}} \times \mathfrak{C} \to \mathbf{M}$ such that $X \begin{pmatrix} t \\ c \end{pmatrix} = \emptyset$, for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ with length $|\underline{c}| \geq 1$. We introduce the notation $X_t := X \begin{pmatrix} t \\ \emptyset \end{pmatrix}$, for all $t \in \mathfrak{C}$.

Given any \mathfrak{C} -colored operad $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$, the endofunctor $\mathcal{O} \circ (-) : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ restricts to an endofunctor

$$\mathcal{O} \circ (-) : \mathbf{M}^{\mathfrak{C}} \longrightarrow \mathbf{M}^{\mathfrak{C}}$$

$$(6.1)$$

on the category of colored objects. Because \mathcal{O} is by definition a monoid in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, with multiplication γ and unit 1, it follows that (6.1) canonically carries the structure of a monad in the category $\mathbf{M}^{\mathfrak{C}}$. We refer to [MacL98, Chapter VI] for details on monad theory, see also [BSW17, Section 2.4] for a very brief review. Concretely, the structure natural transformations $\gamma : \mathcal{O} \circ (\mathcal{O} \circ (-)) \rightarrow \mathcal{O} \circ (-)$ and $1 : \mathrm{id}_{\mathbf{M}^{\mathfrak{C}}} \rightarrow \mathcal{O} \circ (-)$, which we denote with abuse of notation by the same symbols as the operadic composition and unit, are given by the components

$$\begin{array}{cccc} \mathcal{O} \circ (\mathcal{O} \circ X) & \xrightarrow{\gamma_X} \mathcal{O} \circ X & X & \xrightarrow{\mathbb{1}_X} \mathcal{O} \circ X \\ \cong \uparrow & & \cong \uparrow & & & \\ (\mathcal{O} \circ \mathcal{O}) \circ X & & & I_\circ \circ X \end{array}$$
(6.2)

for all $X \in \mathbf{M}^{\mathfrak{C}}$.

Definition 6.1. The category $\operatorname{Alg}(\mathcal{O})$ of algebras over a \mathfrak{C} -colored operad $\mathcal{O} \in \operatorname{Op}_{\mathfrak{C}}(\mathbf{M})$ is the category of algebras over the monad $\mathcal{O} \circ (-) : \mathbf{M}^{\mathfrak{C}} \to \mathbf{M}^{\mathfrak{C}}$. Concretely, an object of $\operatorname{Alg}(\mathcal{O})$ is a pair (A, α) consisting of an object $A \in \mathbf{M}^{\mathfrak{C}}$ and an $\mathbf{M}^{\mathfrak{C}}$ -morphism $\alpha : \mathcal{O} \circ A \to A$ such that $\alpha (\mathcal{O} \circ \alpha) = \alpha \gamma_A$ and $\alpha \mathbb{1}_A = \operatorname{id}_A$. An $\operatorname{Alg}(\mathcal{O})$ -morphism $\varphi : (A, \alpha) \to (B, \beta)$ is an $\mathbf{M}^{\mathfrak{C}}$ -morphism $\varphi : A \to B$ that preserves the structure maps, i.e. $\beta (\mathcal{O} \circ \varphi) = \varphi \alpha$.

The assignment of the categories of algebras to colored operads is functorial

$$Alg : Op(M)^{op} \longrightarrow Cat$$
 (6.3)

with respect to the category $\mathbf{Op}(\mathbf{M})$ of colored operads with varying colors. Concretely, given any $\mathbf{Op}(\mathbf{M})$ -morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, i.e. a map of non-empty sets $f : \mathfrak{C} \to \mathfrak{D}$ together with an $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ -morphism $\phi : \mathcal{O} \to f^*\mathcal{P}$, we define a functor

$$(f,\phi)^* := \mathbf{Alg}(f,\phi) : \mathbf{Alg}(\mathcal{P}) \longrightarrow \mathbf{Alg}(\mathcal{O})$$
 (6.4a)

by setting

$$(f,\phi)^*(A,\alpha) := \left(f^*A, \ \mathcal{O} \circ f^*A \xrightarrow{\phi \circ \mathrm{id}} f^*\mathcal{P} \circ f^*A \xrightarrow{(f_2^*)_{\mathcal{P},A}} f^*(\mathcal{P} \circ A) \xrightarrow{f^*\alpha} f^*A\right) \quad , \quad (6.4\mathrm{b})$$

for all \mathcal{P} -algebras $(A, \alpha : \mathcal{P} \circ A \to A) \in \mathbf{Alg}(\mathcal{P})$. (The natural transformation f_2^* was defined in (4.30).) Furthermore, as a consequence of the adjoint lifting theorem [Bor94, Chapter 4.5], it follows that the functor $(f, \phi)^*$ admits a left adjoint, i.e. we obtain an adjunction

$$(f,\phi)_!$$
: $\mathbf{Alg}(\mathcal{O}) \longleftrightarrow \mathbf{Alg}(\mathcal{P}) : (f,\phi)^*$, (6.5)

for every $\mathbf{Op}(\mathbf{M})$ -morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$. See for example [BM07, BSW17] for further details and also [BSW17] for applications of these adjunctions to quantum field theory.

We develop now a generalization of these definitions and constructions to the setting of involutive categories. Let (\mathbf{M}, J, j) be a bicomplete involutive closed symmetric monoidal category. The involutive analog of the category of \mathfrak{C} -colored objects is obtained by using the exponential 2-functor (4.15) to form $(\mathbf{M}, J, j)^{\operatorname{triv}(\mathfrak{C})} \in \mathbf{ICat}$. Notice that the full subcategory embedding $\mathbf{M}^{\mathfrak{C}} \hookrightarrow \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ can be equipped with an obvious involutive structure, thus providing an \mathbf{ICat} -isomorphism between $(\mathbf{M}, J, j)^{\operatorname{triv}(\mathfrak{C})}$ and the involutive category obtained by restricting the involutive structure on $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ to the full subcategory $\mathbf{M}^{\mathfrak{C}} \subseteq \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. In the following we shall always suppress this isomorphism and identify the involutive categories

$$(\mathbf{M}^{\mathfrak{C}}, J_*, j_*) \cong (\mathbf{M}, J, j)^{\operatorname{triv}(\mathfrak{C})} \quad .$$
(6.6)

Given a \mathfrak{C} -colored *-operad $\mathcal{O} \in *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ in the sense of Definition 5.2 (see also Remark 5.3 for a more explicit description), we obtain an involutive endofunctor

$$\left(\mathcal{O}\circ(-),\nu\right):\left(\mathbf{M}^{\mathfrak{C}},J_{*},j_{*}\right)\longrightarrow\left(\mathbf{M}^{\mathfrak{C}},J_{*},j_{*}\right)\tag{6.7a}$$

with the natural transformation $\nu : \mathcal{O} \circ J_*(-) \to J_*(\mathcal{O} \circ (-))$ defined by the components

$$\mathcal{O} \circ J_* X \xrightarrow{\nu_X} J_* (\mathcal{O} \circ X)$$

$$\xrightarrow{\text{*oid}} J_* \mathcal{O} \circ J_* X$$

$$(6.7b)$$

for all $X \in \mathbf{M}^{\mathfrak{C}}$, where $*: \mathcal{O} \to J_*\mathcal{O}$ denotes the *-involution on \mathcal{O} .

Proposition 6.2. Given any \mathfrak{C} -colored *-operad $(\mathcal{O}, \gamma, \mathbb{1}, *) \in *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$, the components given in (6.2) define involutive natural transformations $\gamma : (\mathcal{O} \circ (-), \nu) (\mathcal{O} \circ (-), \nu) \to (\mathcal{O} \circ (-), \nu)$ and $\mathbb{1} : (\mathrm{id}_{\mathbf{M}^{\mathfrak{C}}}, \mathrm{id}_{J_*}) \to (\mathcal{O} \circ (-), \nu)$ for the involutive endofunctor (6.7). In the terminology of [Jac12, Definition 7], the triple $((\mathcal{O} \circ (-), \nu), \gamma, \mathbb{1})$ is an involutive monad in $(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$.

Proof. This statement is analogous [Jac12, Example 3 (i)] and may be proven by a slightly lengthy diagram chase argument. \Box

The category of algebras $\operatorname{Alg}(\mathcal{O})$ (cf. Definition 6.1) over (the underlying colored operad of) a \mathfrak{C} -colored *-operad $\mathcal{O} \in *-\operatorname{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ can be equipped with a canonical involutive structure

$$(\operatorname{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}) \in \operatorname{ICat}$$
, (6.8)

see also [Jac12, Proposition 3] for a similar construction. Concretely, the endofunctor $J_{\mathcal{O}}$: $\mathbf{Alg}(\mathcal{O}) \to \mathbf{Alg}(\mathcal{O})$ acts on objects $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$ as

$$J_{\mathcal{O}}(A,\alpha) := \left(J_*A, \ \mathcal{O} \circ J_*A \xrightarrow{* \circ \mathrm{id}} J_*\mathcal{O} \circ J_*A \xrightarrow{(J_{*2})_{\mathcal{O},A}} J_*(\mathcal{O} \circ A) \xrightarrow{J_*\alpha} J_*A\right)$$
(6.9)

and on morphisms as J_* . The natural transformation $j_{\mathcal{O}} : \mathrm{id}_{\mathbf{Alg}(\mathcal{O})} \to J_{\mathcal{O}}^2$ is defined by the components $j_{\mathcal{O}(A,\alpha)} := j_{*A}$, for all $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$. This allows us to introduce the concept of *-algebras over colored *-operads.

Definition 6.3. The category of *-algebras over a \mathfrak{C} -colored *-operad $\mathcal{O} \in *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ is defined by evaluating the 2-functor *-**Obj** : **ICat** \rightarrow **Cat** (cf. (2.13)) on the involutive category of \mathcal{O} -algebras (6.8), i.e.

$$*-\mathbf{Alg}(\mathcal{O}) := *-\mathbf{Obj}(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}) \quad .$$
(6.10)

Remark 6.4. Unpacking this definition, we obtain that a *-algebra over $\mathcal{O} \in *-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ is a triple $(A, \alpha, *_A) \in *-\mathbf{Alg}(\mathcal{O})$ consisting of a \mathfrak{C} -colored object $A \in \mathbf{M}^{\mathfrak{C}}$ and two $\mathbf{M}^{\mathfrak{C}}$ -morphisms $\alpha : \mathcal{O} \circ A \to A$ and $*_A : A \to J_*A$, which satisfy the following conditions:

- (1) $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$ is an algebra over the \mathfrak{C} -colored operad \mathcal{O} ;
- (2) $(*_A : A \to J_*A) \in *-\mathbf{Obj}(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$ is a *-object in the involutive category $(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$;
- (3) these two structures are compatible, i.e. the diagram

$$\begin{array}{cccc} \mathcal{O} \circ A & & \alpha & & & & \\ \mathcal{O} \circ *_{A} & & & & \downarrow \\ \mathcal{O} \circ J_{*}A & \xrightarrow{\ast \text{oid}} & J_{*}\mathcal{O} \circ J_{*}A & & & \downarrow \\ & & & \downarrow \\ \end{array} \xrightarrow{\alpha} & & & \downarrow \\ & & \downarrow \\ & & \downarrow \\ & & \downarrow \\ \mathcal{O} \circ J_{*}A & \xrightarrow{\ast \text{oid}} & J_{*}\mathcal{O} \circ J_{*}A & & \downarrow \\ & & & \downarrow \\$$

in $\mathbf{M}^{\mathfrak{C}}$ commutes.

A *-algebra morphism $\varphi : (A, \alpha, *_A) \to (B, \beta, *_B)$ is an $\mathbf{M}^{\mathfrak{C}}$ -morphism $\varphi : A \to B$ preserving the structure maps and *-involutions, i.e. $\beta (\mathcal{O} \circ \varphi) = \varphi \alpha$ and $*_B \varphi = (J_* \varphi) *_A$.

Similarly to (6.3), we observe that the assignment of the involutive categories of algebras to colored *-operads is functorial

$$Alg : *-Op(M, J, j)^{op} \longrightarrow ICat$$
 (6.12)

with respect to the category *- $\mathbf{Op}(\mathbf{M}, J, j)$ of colored *-operads with varying colors (cf. Definition 5.7). Concretely, this functor assigns to a *- $\mathbf{Op}(\mathbf{M}, J, j)$ -morphism $(f, \phi) : (\mathbf{C}, \mathcal{O}) \to (\mathbf{D}, \mathcal{P})$ the involutive functor

$$\left((f,\phi)^*, \mathrm{id}_{(f,\phi)^* J_{\mathcal{P}}}\right) : \left(\mathbf{Alg}(\mathcal{P}), J_{\mathcal{P}}, j_{\mathcal{P}}\right) \longrightarrow \left(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\right) \quad , \tag{6.13}$$

which is given by equipping the pullback functor (6.4) with the trivial involutive structure $\operatorname{id}_{(f,\phi)^* J_{\mathcal{P}}} : (f,\phi)^* J_{\mathcal{P}} \to J_{\mathcal{O}}(f,\phi)^* = (f,\phi)^* J_{\mathcal{P}}$. (Showing that $J_{\mathcal{O}}(f,\phi)^* = (f,\phi)^* J_{\mathcal{P}}$ requires a brief check.) As a consequence of (6.12) and (2-)functoriality of *-**Obj** : **ICat** \to **Cat** (cf. (2.13)), we obtain that also the assignment of the categories of *-algebras (cf. Definition 6.3) to colored *-operad is functorial

*-Alg : *-Op(M,
$$J, j)^{\text{op}} \longrightarrow \text{Cat}$$
 . (6.14)

Given any *-**Op**(**M**, J, j)-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, we denote the corresponding functor simply by

$$(f,\phi)^* := *-\mathbf{Alg}(f,\phi) : *-\mathbf{Alg}(\mathcal{P}) \longrightarrow *-\mathbf{Alg}(\mathcal{O}) \quad . \tag{6.15}$$

Concretely, it is given by evaluating the 2-functor $*-Obj : ICat \rightarrow Cat$ given in (2.13) on the involutive functor (6.13).

Remark 6.5. Recalling Proposition 5.4, there exists an isomorphism

>

$$*-\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \cong \mathbf{Op}_{\mathfrak{C}}(*-\mathbf{Obj}(\mathbf{M}, J, j))$$

$$(6.16)$$

between the category of colored *-operads with values in (\mathbf{M}, J, j) and the category of ordinary colored operads with values in *-**Obj** (\mathbf{M}, J, j) . This isomorphism clearly extends to the categories of colored (*-)operads with varying colors. As a consequence, there exists a second option for assigning categories of *-algebras to colored *-operads, which is given by the lower path in the diagram



where Alg denotes the functor given in (6.3). Similarly to [Jac12, Proposition 3], one can prove that the diagram (6.17) commutes up to a natural isomorphism, hence the second option for assigning the categories of *-algebras is equivalent to our original definition in (6.14).

We would like to emphasize that the main reason why the diagram in (6.17) commutes is that the conditions (1-3) in Remark 6.4 admit two equivalent interpretations: The first option is to regard $(A, \alpha) \in \operatorname{Alg}(\mathcal{O})$ as an algebra over the \mathfrak{C} -colored operad \mathcal{O} and $*_A : (A, \alpha) \to J_{\mathcal{O}}(A, \alpha)$ as an $\operatorname{Alg}(\mathcal{O})$ -morphism. One observes that $(*_A : (A, \alpha) \to J_{\mathcal{O}}(A, \alpha)) \in *-\operatorname{Obj}(\operatorname{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}})$ is a *-object in the involutive category $(\operatorname{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}})$, which recovers our original Definition 6.3 and hence the upper path in the diagram (6.17). The second option is to regard $(*_A : A \to J_*A) \in *-\operatorname{Obj}(\mathbf{M}, J, j)^{\mathfrak{C}}$ as a \mathfrak{C} -colored object in $*-\operatorname{Obj}(\mathbf{M}, J, j)$ and $\alpha : (*: \mathcal{O} \to J_*\mathcal{O}) \circ (*_A : A \to J_*A) \to (*_A : A \to J_*A)$ as a $*-\operatorname{Obj}(\mathbf{M}, J, j)^{\mathfrak{C}}$ -morphism. One observes that this defines an algebra over \mathcal{O} , regarded as an object in $\operatorname{Op}_{\mathfrak{C}}(*-\operatorname{Obj}(\mathbf{M}, J, j))$, which recovers the lower path in the diagram (6.17).

We conclude this section by noticing that (6.13) equips the right adjoint functor $(f, \phi)^*$: $\mathbf{Alg}(\mathcal{P}) \to \mathbf{Alg}(\mathcal{O})$ of the adjunction (6.5) with an involutive structure. Hence, applying Proposition 2.12, we obtain a canonical involutive structure $\lambda_{(f,\phi)} : (f,\phi)_! J_{\mathcal{O}} \to J_{\mathcal{P}}(f,\phi)_!$ on the left adjoint functor $(f,\phi)_! : \mathbf{Alg}(\mathcal{O}) \to \mathbf{Alg}(\mathcal{P})$ together with an involutive adjunction

$$\left((f,\phi)_{!},\lambda_{(f,\phi)}\right):\left(\mathbf{Alg}(\mathcal{O}),J_{\mathcal{O}},j_{\mathcal{O}}\right) \longleftrightarrow \left(\mathbf{Alg}(\mathcal{P}),J_{\mathcal{P}},j_{\mathcal{P}}\right):\left((f,\phi)^{*},\mathrm{id}_{(f,\phi)^{*}J_{\mathcal{P}}}\right) \quad . \tag{6.18}$$

Because 2-functors preserve adjunctions, we may apply the 2-functor $*-Obj : ICat \rightarrow Cat$ to the involutive adjunction (6.18) in order to obtain an adjunction

$$(f,\phi)_!$$
 : *-Alg(\mathcal{O}) \rightleftharpoons *-Alg(\mathcal{P}) : $(f,\phi)^*$ (6.19)

between the categories of *-algebras. Summing up, we have proven

Theorem 6.6. Associated to every *-**Op**(**M**, J, j)-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, there is an involutive adjunction (6.18) between the involutive categories of algebras and an adjunction (6.19) between the categories of *-algebras.

7 Algebraic quantum field theory *-operads

As an application of the concepts and techniques developed in this paper, we study the family of colored operads arising in algebraic quantum field theory [BSW17] within the setting of involutive category theory. The main motivation for promoting these colored operads to colored *-operads is due to quantum physics: A quantum mechanical system is described not only by an associative and unital algebra over \mathbb{C} , but rather by an associative and unital *-algebra A over \mathbb{C} . Here the relevant type of *-algebras is the reversing one, i.e. $(a b)^* = b^* a^*$. The additional structure given by the complex anti-linear *-involution is essential for quantum physics: It enters the GNS

construction that is crucial to recover the usual probabilistic interpretation of quantum theory in terms of Hilbert spaces.

Throughout this section we let (\mathbf{M}, J, j) be any bicomplete involutive closed symmetric monoidal category. In traditional quantum field theory, one would choose the example given by complex vector spaces $(\mathbf{Vec}_{\mathbb{C}}, (-), \mathrm{id}_{\mathrm{id}_{\mathbf{Vec}_{\mathbb{C}}}})$, see Examples 2.3, 2.17 and 3.9 for details. More modern approaches to quantum gauge theories, however, have lead to the concept of homotopical quantum field theory and crucially rely on using different and richer target categories, such as chain complexes and other monoidal model categories, see e.g. [BSS15, BS17, BSW17, Yau18] for algebraic quantum field theory and also [CG17] for similar developments in factorization algebras. Hence, it is justified to present our constructions with this high level of generality.

Let us provide a very brief review of the algebraic quantum field theory operads constructed in [BSW17]. We refer to this paper for more details and the physical motivations.

Definition 7.1. An orthogonality relation on a small category \mathbf{C} is a subset $\perp \subseteq \text{Mor} \mathbf{C}_t \times_t$ Mor \mathbf{C} of the set of pairs of \mathbf{C} -morphisms with coinciding target that is symmetric, i.e. $(f_1, f_2) \in \perp$ implies $(f_2, f_1) \in \perp$, and stable under post- and pre-composition, i.e. $(f_1, f_2) \in \perp$ implies $(gf_1, gf_2) \in \perp$ and $(f_1h_1, f_2h_2) \in \perp$ for all composable \mathbf{C} -morphisms g, h_1 and h_2 . We call elements $(f_1, f_2) \in \perp$ orthogonal pairs and also write $f_1 \perp f_2$. A pair (\mathbf{C}, \perp) consisting of a small category \mathbf{C} and an orthogonality relation \perp on \mathbf{C} is called an orthogonal category.

Remark 7.2. Examples of orthogonal categories that are relevant to quantum field theory can be found in [BSW17, Section 4.6]. Informally, one can think of **C** as a suitable category of spacetimes and spacetime embeddings, equipped with the orthogonality relation \perp encoding the causal relations between regions of a spacetime.

Let (\mathbf{C}, \perp) be an orthogonal category and denote by \mathbf{C}_0 the set of objects of \mathbf{C} . To define the algebraic quantum field theory operad associated to (\mathbf{C}, \perp) it is convenient to introduce the following notations: Given $\underline{c} = (c_1, \ldots, c_n) \in \Sigma_{\mathbf{C}_0}$ and $t \in \mathbf{C}$, we denote by $\mathbf{C}(\underline{c}, t) := \prod_{i=1}^n \mathbf{C}(c_i, t)$ the product of Hom-sets. Its elements will be denoted by symbols like $f = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{c}, t)$.

Definition 7.3. Let (\mathbf{C}, \perp) be an orthogonal category. The algebraic quantum field theory operad of type (\mathbf{C}, \perp) with values in \mathbf{M} is the \mathbf{C}_0 -colored operad $\mathcal{O}_{(\mathbf{C}, \perp)} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ defined as follows:

(a) For any $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$, we set

$$\mathcal{O}_{(\mathbf{C},\perp)}\left(\frac{t}{\underline{c}}\right) := \left(\Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c},t)\right) / \sim_{\perp} \otimes I \in \mathbf{M} \quad , \tag{7.1}$$

where the equivalence relation is as follows: $(\sigma, \underline{f}) \sim_{\perp} (\sigma', \underline{f}')$ if and only if (1) $\underline{f} = \underline{f}'$ and (2) the right permutation $\sigma \sigma'^{-1} : \underline{f} \sigma^{-1} \to \underline{f} {\sigma'}^{-1}$ is generated by transpositions of adjacent orthogonal pairs (cf. [BSW17, Definition 4.9]).

(b) For any $\Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$ -morphism $\sigma' : (\underline{c}, t) \to (\underline{c}\sigma', t)$, we set

$$\mathcal{O}_{(\mathbf{C},\perp)}(\sigma') : \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} t \\ \underline{c} \end{pmatrix} \longrightarrow \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} t \\ \underline{c}\sigma' \end{pmatrix}$$
(7.2)

to be the **M**-morphism induced by the map of sets $[\sigma, \underline{f}] \mapsto [\sigma\sigma', \underline{f}\sigma']$ via functoriality of the **Set**-tensoring.

(c) The operadic composition is determined by the M-morphisms

$$\gamma : \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} t \\ \underline{a} \end{pmatrix} \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} a_{i} \\ \underline{b}_{i} \end{pmatrix} \longrightarrow \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} t \\ \underline{b}_{1} \otimes \cdots \otimes \underline{b}_{m} \end{pmatrix}$$
(7.3a)

induced by the maps of sets

$$[\sigma, \underline{f}] \otimes \bigotimes_{i=1}^{m} [\sigma_i, \underline{g}_i] \longmapsto [\sigma(\sigma_1, \dots, \sigma_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)]$$
(7.3b)

via functoriality of the **Set**-tensoring. Here $\sigma(\sigma_1, \ldots, \sigma_m) = \sigma \langle |\underline{b}_{\sigma^{-1}(1)}|, \ldots, |\underline{b}_{\sigma^{-1}(m)}| \rangle$ ($\sigma_1 \oplus \cdots \oplus \sigma_m$) denotes the group multiplication in $\Sigma_{|\underline{b}_1|+\cdots+|\underline{b}_m|}$ of the corresponding block permutation and block sum permutation, and $\underline{f}(\underline{g}_1, \ldots, \underline{g}_m) = (f_1 g_{11}, \ldots, f_m g_{m|\underline{b}_m|})$ is given by composition of **C**-morphisms.

(d) The operadic unit is determined by the M-morphisms

$$1: I \longrightarrow \mathcal{O}_{(\mathbf{C},\perp)} \begin{pmatrix} t \\ t \end{pmatrix}$$
(7.4)

induced by the maps of sets $* \mapsto (e, id_t)$, where $e \in \Sigma_1$ is the group unit, via functoriality of the **Set**-tensoring.

The following results are proven in [BSW17, Proposition 4.11 and Theorem 4.27].

Theorem 7.4. For any orthogonal category (\mathbf{C}, \perp) , Definition 7.3 defines a \mathbf{C}_0 -colored operad $\mathcal{O}_{(\mathbf{C},\perp)} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$. Furthermore, there exists an isomorphism

$$\operatorname{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \cong \operatorname{Mon}(\mathbf{M})^{(\mathbf{C},\perp)}$$
(7.5)

between the category of $\mathcal{O}_{(\mathbf{C},\perp)}$ -algebras and the category of \perp -commutative functors from \mathbf{C} to \mathbf{M} . Concretely, the latter is the full subcategory of the functor category $\mathbf{Mon}(\mathbf{M})^{\mathbf{C}}$ consisting of all functors $\mathfrak{A}: \mathbf{C} \to \mathbf{Mon}(\mathbf{M})$ for which the diagrams

in **M** commute, for all orthogonal pairs $f_1 \perp f_2$. Here μ_c (respectively μ_c^{op}) denotes the (opposite) multiplication on the monoid $\mathfrak{A}(c)$.

Example 7.5. For physically relevant examples inspired by quantum field theory we refer to [BSW17, Section 4.6]. For the purpose of our present paper, let us just mention that our construction includes two very simple special cases:

- 1. Consider the terminal category $\mathbf{C} = \{*\}$ equipped with the empty orthogonality relation $\perp = \emptyset$. Then $\mathcal{O}_{\{*\},\emptyset\}}$ is the associative operad and hence its algebras are monoids in \mathbf{M} , i.e. $\mathbf{Alg}(\mathcal{O}_{\{\{*\},\emptyset\}}) \cong \mathbf{Mon}(\mathbf{M})$.
- 2. Consider the terminal category $\mathbf{C} = \{*\}$ equipped with its (unique) non-trivial orthogonality relation $\perp = \{(\mathrm{id}_*, \mathrm{id}_*)\}$. Then $\mathcal{O}_{(\{*\}, \{(\mathrm{id}_*, \mathrm{id}_*\})}$ is the commutative operad and hence its algebras are commutative monoids in \mathbf{M} , i.e. $\mathbf{Alg}(\mathcal{O}_{(\{*\}, \{(\mathrm{id}_*, \mathrm{id}_*\})}) \cong \mathbf{CMon}(\mathbf{M})$. ∇

We will now endow $\mathcal{O}_{(\mathbf{C},\perp)} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ with the structure of a colored *-operad. According to Remark 5.3, this amounts to equipping the symmetric sequence underlying $\mathcal{O}_{(\mathbf{C},\perp)}$ with the structure of a *-object in the involutive monoidal category $(\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M}), J_*, j_*)$ that is compatible with the operadic compositions and units. Let us define a $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M})$ -morphism $*: \mathcal{O}_{(\mathbf{C},\perp)} \to J_*\mathcal{O}_{(\mathbf{C},\perp)}$ by setting, for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$,

$$\begin{array}{ccc}
\mathcal{O}_{(\mathbf{C},\perp)}\left(\underline{c}\right) & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ \left(\Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c},t)\right)/\sim_{\perp} \otimes I & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\$$

to be the **M**-morphism induced by the map of sets $\rho_{|\underline{c}|} : [\sigma, \underline{f}] \mapsto [\rho_{|\underline{c}|}\sigma, \underline{f}]$, where $\rho_{|\underline{c}|} \in \Sigma_{|\underline{c}|}$ is the order-reversal permutation from Example 2.4, and the **M**-morphism $J_0 : I \to JI$. (For the right vertical arrow recall that J is self-adjoint, hence it preserves the **Set**-tensoring.) Evidently, (7.7) is equivariant with respect to the action of permutations given in Definition 7.3 (b), hence it defines a **SymSeq**_{C₀}(**M**)-morphism. It is, moreover, straightforward to verify that $(*: \mathcal{O}_{(\mathbf{C}, \perp)}) \to J_*\mathcal{O}_{(\mathbf{C}, \perp)}) \in *-\mathbf{Obj}(\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M}), J_*, j_*)$ is a *-object by using that $\rho_{|\underline{c}|}^2 = e$ is the identity permutation and that $j : \mathrm{id}_{\mathbf{M}} \to J^2$ is by hypothesis a monoidal natural transformation.

Proposition 7.6. Endowing the colored operad $(\mathcal{O}_{(\mathbf{C},\perp)}, \gamma, \mathbb{1}) \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ from Definition 7.3 with the *-involution *: $\mathcal{O}_{(\mathbf{C},\perp)} \to J_*\mathcal{O}_{(\mathbf{C},\perp)}$ defined in (7.7) yields a colored *-operad

$$\left(\mathcal{O}_{(\mathbf{C},\perp)},\gamma,\mathbb{1},*\right) \in * \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M},J,j) \quad .$$
(7.8)

Proof. It remains to check the compatibility conditions in Remark 5.3 (3). This is a straightforward calculation using standard permutation identities, given e.g. in [BSW17, Lemma 4.6]. \Box

Let us now study the *-algebras over the colored *-operad $\mathcal{O}_{(\mathbf{C},\perp)} \in *-\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}, J, j)$ defined in Proposition 7.6. Using the explicit description explained in Remark 6.4, these are triples $(A, \alpha, *_A)$ consisting of an algebra (A, α) over $\mathcal{O}_{(\mathbf{C},\perp)}$ together with a compatible *-involution $*_A : A \to J_*A$. Using Theorem 7.4 to identity (A, α) with a \perp -commutative functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}(\mathbf{M})$, the *involution $*_A : A \to J_*A$ is identified with a family of **M**-morphism

$$*_c : \mathfrak{A}(c) \longrightarrow J\mathfrak{A}(c) , \qquad (7.9)$$

for all $c \in \mathbf{C}$, which as a consequence of Remark 6.4 (3) has to satisfy the following basic conditions:

(1) Compatibility with monoid structure: For all $c \in \mathbf{C}$,

where μ_c (respectively μ_c^{op}) is the (opposite) multiplication on $\mathfrak{A}(c) \in \mathbf{Mon}(\mathbf{M})$, and

where η_c is the unit on $\mathfrak{A}(c) \in \mathbf{Mon}(\mathbf{M})$.

(2) Compatibility with functor structure: For all C-morphisms $f: c \to c'$,

$$\begin{aligned} \mathfrak{A}(c) & \xrightarrow{*_{c}} J\mathfrak{A}(c) & (7.11) \\ \mathfrak{A}(f) & \downarrow J\mathfrak{A}(f) \\ \mathfrak{A}(c') & \xrightarrow{*_{c'}} J\mathfrak{A}(c') \end{aligned}$$

Example 7.7. Consider the simplest example given by $(\mathbf{C}, \perp) = (\{*\}, \emptyset)$, cf. Example 7.5. Then $\mathcal{O}_{(\{*\}, \emptyset)}$ is the associative operad and Proposition 7.6 defines a *-operad structure on it. For later convenience, let us denote the corresponding category of *-algebras by

*-
$$Mon_{rev}(M, J, j) := *-Alg(\mathcal{O}_{(\{*\}, \emptyset)})$$
 . (7.12)

Using our concrete description from above, an object in this category is a quadruple $(A, \mu, \eta, *)$ consisting of a monoid $(A, \mu, \eta) \in \mathbf{Mon}(\mathbf{M})$ together with a *-involution $* : A \to JA$ satisfying the compatibility conditions in (7.10). (The conditions in (7.11) are vacuous because we consider the discrete category $\{*\}$ in this example.) Comparing these structures to *-monoids, cf. Remark 3.21, we observe that they are very similar, up to the appearance of the opposite multiplication in (7.10). This order-reversal of the multiplication under *-involution, which results from our *-operad structure (7.7), motivates our notation *- $\mathbf{Mon}_{rev}(\mathbf{M}, J, j)$.

As a very concrete example, and referring back to Example 3.22, let us consider the involutive symmetric monoidal category ($\operatorname{Vec}_{\mathbb{C}}, \overline{(-)}, \operatorname{id}_{\operatorname{id}_{\operatorname{Vec}_{\mathbb{C}}}}$) from Example 3.9. In this case (7.12) describes the category of order-reversing associative and unital *-algebras over \mathbb{C} , i.e. $(a \ b)^* = b^* a^*$, which is of major relevance for (traditional) quantum physics. ∇

In general, we have the following explicit characterization of *-algebras over the colored *operad $\mathcal{O}_{(\mathbf{C},\perp)} \in *-\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}, J, j)$ defined in Proposition 7.6.

Proposition 7.8. For any orthogonal category, there exists an isomorphism

$$*-\operatorname{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \cong *-\operatorname{Mon}_{\operatorname{rev}}(\mathbf{M},J,j)^{(\mathbf{C},\perp)}$$
(7.13)

between the category of *-algebras over $\mathcal{O}_{(\mathbf{C},\perp)}$ and the category of \perp -commutative functors from \mathbf{C} to the category of order-reversing *-monoids in (\mathbf{M}, J, j) , cf. Example 7.7.

Proof. This is an immediate consequence of Theorem 7.4 together with (7.10) and (7.11).

The assignment $(\mathbf{C}, \perp) \mapsto \mathcal{O}_{(\mathbf{C}, \perp)}$ of our colored *-operads is functorial

$$\mathcal{O}: \mathbf{OrthCat} \longrightarrow *-\mathbf{Op}(\mathbf{M}, J, j)$$
(7.14)

on the category of orthogonal categories, where a morphism $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ is a functor preserving the orthogonality relations in the sense of $F(\perp) \subseteq \perp'$. Together with Theorem 6.6, this implies

Corollary 7.9. Associated to every **OrthCat**-morphism $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ there is an adjunction

$$\mathcal{O}_{F!} : *-\mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \iff *-\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}',\perp')}) : \mathcal{O}_{F}^{*} \quad .$$
(7.15)

Remark 7.10. Such adjunctions have plenty of quantum field theoretic applications, see e.g. [BSW17, Section 5] and also [BDS17]. The results of this section show that these adjunctions are also available in the involutive setting, which is crucial to describe the order-reversing associative and unital *-algebras appearing in quantum field theory. \triangle

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