DECOMPOSING EDGE-COLOURED COMPLETE SYMMETRIC DIGRAPHS INTO MONOCHROMATIC PATHS

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Abstract. Confirming and extending a conjecture by Guggiari, we show that every countable $(r+1)$-edge-coloured complete symmetric digraph containing no directed paths of edge-length ℓ_i for any colour $i \leq r$ can be covered by $\prod_{i \leq r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.

1. Introduction

For $n \in \mathbb{N}$ we write $[n] = \{1, 2, \ldots, n\}$. Let K_n be the complete undirected graph with vertex set $[n]$, and $\vec{K}_n$ be the complete symmetric digraph on $[n]$, i.e. the directed graph where every edge of K_n appears in both its orientations. A *tournament* of order n is a complete antisymmetric digraph on $[n]$, i.e. a directed graph in which every edge of K_n appears in precisely one of its possible orientations. Similarly, let $K_{\mathbb{N}}$ and $\vec{K}_{\mathbb{N}}$ be the complete graph and the complete symmetric digraph on the positive integers respectively.

Let G be a digraph. A sequence $v_1, v_2, \ldots, v_n$ of vertices such that there is an oriented edge $\vec{e}_i = (v_i, v_{i+1}) \in E(G)$ from v_i to v_{i+1} for all $i \in [n-1]$ is a *directed* path in G of *length* $n-1$ (in this paper, the length of a path always refers to its edge-length). A sequence $v_1, v_2, v_3, \ldots$ satisfying the above conditions for all $i \in \mathbb{N}$ is a *one-way infinite directed path*. The term *directed path* may refer to both finite and one-way infinite directed paths.

The *upper density* of a set $A \subseteq \mathbb{N}$ is defined as

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [n]|}{n}.$$

DeBiasio and McKenney [1] have recently shown that for every $\varepsilon > 0$, there exists a 2-edge-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic directed path has upper density less than ε. Answering a question of the above-named authors [1, Problem 8.3], Guggiari [3] constructed a 2-edge-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic path has upper density 0, but observed that if one restricts the maximal length of directed paths in the first colour, then there must be monochromatic paths in the second colour with non-vanishing upper density. More generally:

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Theorem 1 (Guggiari)**.** *For any edge colouring $c\colon E(\vec{K}_{\mathbb{N}}) \to [r+1]$ for which there are no directed paths of length ℓ_i in colour i for any $i \in [r]$, there is a monochromatic directed path in colour $r+1$ with upper density at least $\prod_{i\leq r} \ell_i^{-1}$.*

After establishing the upper-density result, Guggiari concludes her paper with the following conjecture:

Conjecture 1 (Guggiari)**.** *Take any 2-edge-colouring of $\vec{K}_{\mathbb{N}}$ that does not contain a red directed path of length ℓ. Then the vertices of $\vec{K}_{\mathbb{N}}$ can be covered by at most ℓ vertex-disjoint blue directed paths.*

Clearly, such a partition result would imply the corresponding upper density result, for if one has a partition into at most ℓ vertex-disjoint directed paths, then by the pigeon hole principle, one of the paths must have upper density at least $1/\ell$.

The purpose of this note is to confirm Guggiari's conjecture, and furthermore to extend it to all finite edge-colourings of $\vec{K}_{\mathbb{N}}$, with possibly more than two colours.

Theorem 2. *Let G be a complete symmetric digraph, either finite or countably infinite. Then for every $(r+1)$-edge-colouring for which there is no monochromatic directed path of edge-length ℓ_i in colour i for any $i \in [r]$, the vertex set of G can be covered by $\prod_{i\leq r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.*

Our proof contains three new ideas. First, following a proof strategy of Loh [5], we show in Section 2 that edge-coloured tournaments without long monochromatic paths are somewhat small. Second, in Section 3, we show that for a finitely edge-coloured $\vec{K}_n$, the number k of monochromatic paths in the last colour needed to cover the vertex set gives rise to a subtournament of $\vec{K}_n$ of order k where the last colour does not occur.

These two observations will then be used as follows: In Section 4, following the proof strategy by Guggiari [3], we construct a well-behaved finite partition $\mathcal{U}$ of $\mathbb{N}$ such that each partition classes can be covered by a monochromatic path in the last colour. The third and final new piece is in Section 5 to construct a subtournament of $\vec{K}_{\mathbb{N}}$ not using the last colour which (almost) spans $\mathcal{U}$ — allowing us to apply observations one and two to bound the number of partition classes in $\mathcal{U}$, giving rise to a decomposition into few monochromatic paths in the last colour.

We remark that for the infinite case $\vec{K}_{\mathbb{N}}$, the upper bound of $\prod_{i\leq r} \ell_i$ in Theorem 2 is best possible by the example in [3, Figure 5]. It would be interesting to know whether in the finite case the bound can be improved.

2. Monochromatic paths in finite edge-coloured tournaments

Before investigating complete symmetric digraphs, we begin our discussion with a result on longest monochromatic paths in finitely edge-coloured tournaments.

One of the main results in a recent paper by Loh [5] is that every r-edge coloured tournament of order n contains a monochromatic directed path of length at least

$\sqrt[r]{n} - 1$.[1] The same proof method as in [5, Theorem 1.1], with only minor modifications, also implies the following imbalanced version of this result:

Theorem 3. *Let T be a finite tournament and $c\colon E(T) \to [r]$ an edge-colouring. Moreover, assume that there is no directed path of length ℓ_i in colour i for $i \in [r]$. Then T has order at most $\prod_{i \le r} \ell_i$.*

The proof relies on the following theorem, which, as noted in [5], has been independently discovered in [2, 4, 6, 7].

Theorem 4 (Gallai-Hasse-Roy-Vitaver)**.** *Let G be an undirected graph with chromatic number k. No matter how its edges are oriented, the resulting directed graph contains a directed path of length at least $k - 1$.* □

Proof of Theorem 3. Let T be a tournament of order n, and consider an arbitrary edge-colouring $c\colon E(T) \to [r]$ of T such that there is no directed path of length ℓ_i in colour i for any $i \in [r]$.

This colouring induces an edge-coloring of the underlying undirected complete graph K_n, which partitions K_n into r edge-disjoint subgraphs $G_1, G_2, \ldots, G_r$, each corresponding to a color class in the tournament. If $\chi(G)$ denotes the chromatic number of a graph G, then Theorem 4 and our assumptions on the various monochromatic path lengths imply that $\chi(G_i) \le \ell_i$ for all i. By considering the product colouring on K_n, it follows that

$$n = \chi(K_n) \le \prod_{i \le r} \chi(G_i) \le \prod_{i \le r} \ell_i. \qquad \square$$

3. Decomposing finite symmetric complete digraphs

In this section, we use Theorem 3 to prove our main decomposition theorem in the finite case. As a corollary to its proof, we make the useful structural observation that the number k of monochromatic paths in the last colour needed to cover the vertex set gives rise to a subtournament of $\vec{K}_n$ of order k where the last colour does not occur—an observation which will be crucial further below in the proof our main result for the infinite case.

Theorem 5. *For any edge-colouring $c\colon E(\vec{K}_n) \to [r+1]$ such that there is no directed path of edge-length ℓ_i in colour i for $i \in [r]$, the vertex set of $\vec{K}_n$ can be covered by $\prod_{i \le r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.*

Proof. Let k be the smallest integer such that the vertex set of $\vec{K}_n$ can be covered by monochromatic pairwise disjoint directed paths in colour $(r+1)$. Choose a sequence $\mathcal{P} := (P_1, \ldots, P_k)$ of such monochromatic paths satisfying that $(|P_1|, \ldots, |P_k|)$ is minimal with respect to the lexicographical order of $\mathbb{N}^k$. Let us write v_i for the first vertex on P_i for $i \le k$. Then (v_i, v_j) has some colour $\le r$, whenever $i < j$. Otherwise, the path system

$$(P_1, \ldots, \mathring{v}_i P_i, \ldots, v_i P_j, \ldots, P_k)$$

[1] Loh defines the length of a path to be its *vertex-length*, and so we have to add an additional '-1' in our statements, translating between edge- and vertex-length.

would contradict the minimality of $\mathcal{P}$. Hence, the set $\{(v_i, v_j) \colon i < j\}$ defines a tournament T on $\{v_1, \dots, v_k\}$ with edge colours in $[r]$. By Theorem 3, we have $k \leq \prod_{i \leq r} \ell_i$, completing the proof. $\square$

As a consequence to the proof of Theorem 5 we obtain the following corollary:

Corollary 6. *Let $c \colon E(\vec{K}_n) \to [r+1]$ be an edge-colouring of $\vec{K}_n$. If k is the smallest number of pairwise disjoint monochromatic directed paths in colour $r+1$ needed to cover the vertex set of $\vec{K}_n$, then $\vec{K}_n$ contains a tournament of order k with edge-colours in $[r]$ as a subgraph.* $\square$

4. A partition result

A non-trivial digraph G is said to be *strongly $<\aleph_0$-connected* if for every two vertices v and w of G, there are infinitely many independent directed $v-w$ paths, and also infinitely many independent directed $w-v$ paths (equivalently: after deleting any finite number of vertices from $G - \{v, w\}$, the vertices v and w lie in the same strongly connected component).

In this section we gather two technical results. The first can be extracted from the proof of [3, Theorem 1.3].

Lemma 7. *Every countable strongly $<\aleph_0$-connected digraph can be covered by a single one-way infinite directed path.* $\square$

The second of our technical results is inspired by the techniques used in the proofs of [3, Theorem 1.3 & 1.5], but requires some modifications, and therefore we give the complete proof.

Lemma 8. *For every edge-colouring $c \colon E(\vec{K}_{\mathbb{N}}) \to [r+1]$ of $\vec{K}_{\mathbb{N}}$ such that there is no directed path of edge-length ℓ_i in colour i for $i \in [r]$, there is a finite partition $\mathcal{U}$ of $\mathbb{N}$ such that the edges coloured with colour $r+1$ induce a strongly $<\aleph_0$-connected subgraph on every non-singleton partition class of $\mathcal{U}$.*

Proof. For $k = 1, 2, \dots, r+1$ we will recursively define finite partitions $\mathcal{U}_k$ of $\mathbb{N}$ such that

(1) $\mathcal{U}_k$ refines $\mathcal{U}_{k-1}$,
(2) for every $U \in \mathcal{U}_k$ and every vertex $u \in U$, the set $\{u' \in U \colon c(u', u) < k\}$ has cardinality at most $\sum_{j<k} \ell_j$,
(3) for every non-singleton partition class $U \in \mathcal{U}_k$, all vertices $u \in U$ satisfy that $\{u' \in U \colon c(u, u') \geq k\}$ is infinite.

Before we describe the recursive construction, let us see that $\mathcal{U} := \mathcal{U}_{r+1}$ is as desired. Let $U \in \mathcal{U}$ be a non-singleton partition class, and let $v \neq w \in U$. To see that U is $<\aleph_0$-connected, let us write $N^+_{r+1}(v)$ and $N^-_{r+1}(v)$ for the out- and in-neighbourhood of v in colour $r+1$ respectively. By property (3), we have $N^+_{r+1}(v) \cap U$ is infinite, and by property (2), $N^-_{r+1}(w) \cap U$ is cofinite in U. Therefore,

$$N^+_{r+1}(v) \cap N^-_{r+1}(w) \cap U$$

is infinite, and so there are infinitely many independent monochromatic directed $v-w$-paths in U (each of length 2) in colour $r+1$.

We now proceed with the recursive construction of the $\mathcal{U}_k$. For $k = 1$, properties $(1)-(3)$ are trivially satisfied for $\mathcal{U}_1 := \{\mathbb{N}\}$. Now, assume that $\mathcal{U}_k$ has already been defined. For $0 \leq i < \ell_k$ let A_i consist of those vertices $a \in \mathbb{N}$ such that the longest k-coloured directed path in $\vec{K}_{\mathbb{N}}$ with first vertex a has length i. Then define $\mathcal{A} := \{A_i \colon 0 \leq i < \ell_k\} \setminus \{\emptyset\}$.

Claim 1. $\mathcal{A}$ is a finite partition of $\mathbb{N}$ and for every partition class $A \in \mathcal{A}$ and every vertex $a \in A$ we have $\{a' \in A \colon c(a', a) = k\}$ has cardinality at most ℓ_k.

Proof of Claim 1. Cf. [3, Theorem 1.3, Claim 1]. Since $\vec{K}_{\mathbb{N}}$ contains no directed monochromatic path in colour k of length ℓ_k or bigger, it is clear that $\mathcal{A}$ is indeed a finite partition. Suppose for some $a \in A \in \mathcal{A}$ we have $|N^-_k(a) \cap A| > \ell_k$. Consider a longest monochromatic path P in $\vec{K}_{\mathbb{N}}$ in colour k with first vertex a. Then $(N^-_k(a) \cap A) \setminus P \neq \emptyset$, and so for any $a' \in (N^-_k(a) \cap A) \setminus P$, the path $P' = a'P$ is a strictly longer monochromatic path in colour k starting at an element in the same partition class $A \in \mathcal{A}$, contradicting the definition of $\mathcal{A}$. $\Diamond$

Now consider the smallest common refinement $\mathcal{V} := \{U \cap A \colon U \in \mathcal{U}_k, A \in \mathcal{A}\}$.

Claim 2. For every $V \in \mathcal{V}$ and every vertex $v \in V$, the set $\{v' \in V \colon c(v', v) < k+1\}$ has cardinality at most $\sum_{j<k+1} \ell_j$.

Proof of Claim 2. Fix $v \in V \in \mathcal{V}$. Then $V = U \cap A$ for some $U \in \mathcal{U}_k$ and $A \in \mathcal{A}$. By property (2), we have $\{u' \in U \colon c(u', v) < k\}$ has cardinality at most $\sum_{j<k} \ell_j$, and by Claim 1 we have $\{a' \in A \colon c(a', v) = k\}$ has cardinality at most ℓ_k. Hence, $\{v' \in U \cap A \colon c(v', v) < k + 1\}$ has cardinality at most $\sum_{j<k+1} \ell_j$. $\Diamond$

For $V \in \mathcal{V}$, let us define $X(V) = \{v \in V \colon |\{v' \in V \colon c(v, v') \geq k + 1\}| \text{ is finite}\}$.

Claim 3. For every infinite partition class of $V \in \mathcal{V}$ the set $X(V)$ has cardinality at most $\sum_{j<k+1} \ell_j$.

Proof of Claim 3. Cf. [3, Theorem 1.3, Claim 2]. Indeed, consider some infinite $V \in \mathcal{V}$ and suppose for a contradiction that there is a finite subset $X \subseteq X(V)$ with $|X| > \sum_{j<k+1} \ell_j$. Because V is infinite, there is a vertex:

$$w \in V \setminus \bigcup_{x \in X} \{v' \in V \colon c(x, v') \geq k + 1\}.$$

Then $c(x, w) < k + 1$ for all $x \in X$, and so it follows from Claim 2 applied to the vertex $w \in V$ that

$$\sum_{j<k+1} \ell_j < |X| \leq |\{v' \in V \colon c(v', w) < k + 1\}| \leq \sum_{j<k+1} \ell_j,$$

a contradiction. $\Diamond$

Finally, let $\mathcal{S}$ be the collection of singletons of the form $\{v\}$ for which v is either part of a finite partition class of $\mathcal{V}$, or is contained in a set $X(V)$ for some $V \in \mathcal{V}$. By induction assumption and Claim 3, we know that $\mathcal{S}$ is finite.

Claim 4. $\mathcal{U}_{k+1} := \mathcal{S} \cup \{V \setminus X(V) \colon V \in \mathcal{V} \text{ is infinite}\}$ satisfies properties $(1)-(3)$.

Proof of Claim 4. By construction, $\mathcal{U}_{k+1}$ is a finite partition of $\mathbb{N}$ such that every non-singleton partition class is infinite. Property (1) for $\mathcal{U}_{k+1}$ is obvious. Property (2) follows from Claim 2, as $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{V}$. Finally, for property (3) consider some infinite $U \in \mathcal{U}_{k+1}$. Then $U = V \setminus X(V)$ for some infinite $V \in \mathcal{V}$. By definition of $X(V)$, it follows that for all $u \in U$ the

$$\{v \in V : c(u,v) \geq k+1\}$$

is infinite, and therefore, as $X(V)$ is finite by Claim 3, we also have that

$$\{v \in V \setminus X(V) : c(u,v) \geq k+1\}$$

is infinite, which verifies property (3) for the partition class $U \in \mathcal{U}_{k+1}$. $\Diamond$

Thus, we see that $\mathcal{U}_{k+1}$ is as required, completing the recursive construction. $\square$

5. Decomposing countably infinite symmetric complete digraphs

Theorem 9. *For every edge-colouring $c\colon E(\vec{K}_{\mathbb{N}}) \to [r+1]$ of $\vec{K}_{\mathbb{N}}$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$, the vertex set $\mathbb{N}$ can be covered by $\prod_{i \leq r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.*

Proof. By Lemma 8 there exists a finite partition $\mathcal{U}$ of the vertex set $\mathbb{N}$ such that every partition class is either a singleton or strongly $<\aleph_0$-connected in colour $r+1$. Choose such a $\mathcal{U}$ with minimal cardinality.

Claim 1. Let $U, U' \in \mathcal{U}$ be infinite partition classes. Moreover, suppose that M is a maximal directed $U - U'$ matching in colour $r+1$ and M' is a maximal directed $U' - U$ matching in colour $r+1$. Then M or M' is finite.

Proof of Claim 1. Suppose for a contradiction that $U, U' \in \mathcal{U}$ are both infinite partition classes such that there exist infinite directed matchings M and M' as in Claim 1. We aim to show that $U \cup U'$ is strongly $<\aleph_0$-connected in colour $r+1$, contradicting the minimal choice of $\mathcal{U}$.

Towards this, fix vertices $v \in U$ and $w \in U'$. Since U is $<\aleph_0$-connected in colour $r+1$ we find a directed $v - A$ fan in colour $r+1$ for some infinite $A \subseteq \{u \in U : (u,u') \in M\}$. Let $B = \{u' \in U' : u \in A, (u,u') \in M\}$. Similarly, since U' is $<\aleph_0$-connected in colour $r+1$, there also exists a directed $B' - w$ fan for some infinite $B' \subset B$. Combining these two fans with suitable edges from M shows that there are infinitely many independent monochromatic directed $v - w$-paths in colour $r+1$ in $U \cup U'$.

Applying the same argument to the matching M', one also finds infinitely many independent monochromatic directed $w - v$-paths in colour $r+1$ in $U \cup U'$. $\Diamond$

Recall that we write $N^+_{r+1}(v)$ and $N^-_{r+1}(v)$ for the out- and in-neighbourhood respectively of a vertex $v \in \mathbb{N}$ in colour $r+1$.

Claim 2. Let $U, U' \in \mathcal{U}$ be partition classes such that $U = \{u\}$ is a singleton and U' is infinite. Then $N^+_{r+1}(u) \cap U'$ or $N^-_{r+1}(u) \cap U'$ is finite.

Proof of Claim 2. Suppose for a contradiction that there are partition classes $U, U' \in \mathcal{U}$ with $U = \{u\}$ and U' infinite such that there are infinite sets $\vec{E}$ and $\vec{E}'$ of directed $u - U'$ and $U' - u$ edges respectively in colour $r+1$. Again, we claim that $U \cup U'$ is strongly $<\aleph_0$-connected in colour $r+1$, contradicting the minimal choice of $\mathcal{U}$.

Towards this, fix a vertex $w \in U'$. Let $A = \{u' \in U' \colon (u, u') \in \vec{E}\}$. Since U' is strongly $<\aleph_0$-connected in colour $r+1$, there is a directed $A' - w$ fan in colour $r+1$ for some infinite $A' \subseteq A$. Combining the fan with suitable edges from $\vec{E}$ shows that there are infinitely many independent monochromatic directed $u - w$-paths in colour $r+1$ in $U \cup U'$.

Applying the same argument to set $\vec{E}'$ of edges, one also finds infinitely many independent monochromatic directed $w - u$-paths in colour $r+1$ in $U \cup U'$. $\Diamond$

Let $\mathcal{S}$ be the set of singletons in $\mathcal{U}$ and $S := \bigcup \mathcal{S}$. Furthermore, let k be the smallest number of pairwise disjoint monochromatic directed paths in colour $r+1$ needed, to cover S in $\vec{K}_{\mathbb{N}}[S]$ with regard to the colouring induced by c. By Corollary 6, there exists a sub-tournament T' of order k in $\vec{K}_{\mathbb{N}}[S]$ with edge-colours in $[r]$. Let $\mathcal{S}' := \{\{v\} \colon v \in V(T')\}$.

We now define a digraph H with vertex set $\mathcal{S}' \cup (\mathcal{U} \backslash \mathcal{S})$, where we insert a direct edge $\vec{e} = (U, U') \in E(H)$ and define a finite set $W_{\vec{e}} \subset \mathbb{N}$ for every such edge, if

- (†1) both U and U' are infinite and there exists a maximal finite directed $U - U'$ matching $M_{\vec{e}}$ in colour $r+1$. In this case, define $W_{\vec{e}} := V[M_{\vec{e}}]$ to be the set of vertices covered by $M_{\vec{e}}$.
- (†2) If $U = \{u\}$ is a singleton, U' is infinite and $W_{\vec{e}} := N^+_{r+1}(u) \cap U'$ is finite.
- (†3) If U is infinite, $U' = \{u'\}$ is a singleton and $W_{\vec{e}} := N^-_{r+1}(u') \cap U$ is finite.
- (†4) Or if $U = \{u\}$ and $U' = \{u'\}$ are both singletons and $(u, u') \in E(T')$. In this case, we put $W_{\vec{e}} := \emptyset$.

Then we define

$$W = \bigcup_{\vec{e} \in E(H)} W_{\vec{e}}$$

which by construction is a finite subset of the vertex set $\mathbb{N}$ of $\vec{K}_{\mathbb{N}}$.

Choose a vertex $x_U \in U \setminus W$ for every partition class $U \in \mathcal{S}' \cup (\mathcal{U} \backslash \mathcal{S})$ and write $X = \{x_U \colon U \in \mathcal{S}' \cup (\mathcal{U} \backslash \mathcal{S})\}$. Since W is finite, and by construction has non-trivial intersection only with the infinite partition classes in $\mathcal{U} \backslash \mathcal{S}$, this is always possible.

Claim 3. The digraph $\vec{K}_{\mathbb{N}}[X]$ induced by the vertices of X contains a spanning sub-tournament T with edge colours in $[r]$.

Proof of Claim 3. We show that for every pair of vertices $x, x' \in X$, one of the directed edges (x, x') or (x', x) has a colour in $[r]$. Suppose $x \in U \in \mathcal{U}$ and $x' \in U' \in \mathcal{U}$. Since T' is a tournament with edge colours in $[r]$, we may assume that one of U and U' infinite, and so it follows from Claim 1 and 2 that (U, U') or (U', U) is a directed edge of H. By symmetry, let us assume that $\vec{e} := (U, U') \in E(H)$.

- If U and U' are both infinite, then (x, x') has a colour in $[r]$ because otherwise, as $x, x' \notin W$, the directed matching $M_{\vec{e}} \cup \{(x, x')\}$ contradicts the maximality of $M_{\vec{e}}$ in (†1).

- If $U = \{u\}$ is a singleton and U' is infinite, then (x, x') has a colour in $[r]$ because otherwise, $x' \in N^+_{r+1}(u) \cap U' \subseteq W$ by (†2).
- If U is infinite and $U' = \{u'\}$ is a singleton, then (x, x') has a colour in $[r]$ because otherwise, $x \in N^-_{r+1}(u') \cap U \subseteq W$ by (†3). $\diamondsuit$

Let T be a tournament as in Claim 3. Since c induces an r-edge colouring of T such that there are no directed paths of length ℓ_i for any $i \in [r]$, it follows from Theorem 3 that $|\mathcal{S}'| + |\mathcal{U}\backslash\mathcal{S}| = |T| \leq \prod_{i\leq r} \ell_i$. But now we have $k = |\mathcal{S}'|$ pairwise disjoint monochromatic directed paths in colour $r+1$ in $\vec{K}_{\mathbb{N}}[S]$ covering S and—by Lemma 7—we can cover each infinite partition class $U \in \mathcal{U}\backslash\mathcal{S}$ with a single one-way infinite monochromatic directed paths in colour $r+1$. Thus, we have covered the vertex set of $\vec{K}_{\mathbb{N}}$ by $k + |\mathcal{U} \setminus \mathcal{S}| \leq \prod_{i\leq r} \ell_i$ many monochromatic directed paths in colour $r+1$, completing the proof of the theorem. $\square$

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