A STRICTLY COMMUTATIVE MODEL FOR THE COCHAIN ALGEBRA OF A SPACE

BIRGIT RICHTER AND STEFFEN SAGAVE

ABSTRACT. Using a Day convolution product on diagrams of chain complexes indexed by the category of finite sets and injections $\mathcal{I}$ allows one to model E_∞ differential graded algebras by strictly commutative objects, called commutative $\mathcal{I}$-dgas. In this note we introduce a functor $A^{\mathcal{I}}$ from simplicial sets to commutative $\mathcal{I}$-dgas that is a commutative lift of the usual cochain algebra functor. In particular, $A^{\mathcal{I}}$ gives rise a new construction of the E_∞ dga of cochains. Our approach is motivated by the functor A_{PL} of polynomial forms on simplicial sets used in rational homotopy theory. The functor $A^{\mathcal{I}}$ shares many properties of A_{PL}, and can be viewed as a generalization of A_{PL} that works over arbitrary commutative ground rings.

1. Introduction

The cochains $C(X;k)$ on a space X with values in a commutative ring k form a differential graded algebra whose cohomology is the singular cohomology $H^*(X;k)$ of X. The multiplication of $C(X;k)$ induces the cup product on $H^*(X;k)$. Over the rationals, $C(X;\mathbb{Q})$ is quasi-isomorphic to the commutative differential graded algebra $A_{\mathrm{PL}}(X)$ of polynomial forms on X, which is a very powerful tool in rational homotopy theory [Sul77, BG76]. For general k, there is no cdga which is quasi-isomorphic to $C(X;k)$, for example because the Steenrod operations witness the non-commutativity of $C(X;\mathbb{F}_p)$. However, $C(X;k)$ is always commutative up to coherent homotopy. This can be encoded using the language of operads [May72]: the multiplication of $C(X;k)$ extends to the action of an E_∞ operad in chain complexes turning $C(X;k)$ into an E_∞ dga. This additional structure is important because Mandell showed that the cochain functor $C(-;\mathbb{Z})$ to E_∞ dgas classifies simply connected spaces of finite type up to weak equivalence [Man06].

One can describe homotopy coherent commutative multiplications on chain complexes using diagram categories instead of operads. Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{m} = \{1\ldots,m\}, m \geq 0$, with the convention that $\mathbf{0}$ is the empty set. Morphisms in $\mathcal{I}$ are the injections. Concatenation in $\mathcal{I}$ and the tensor product of chain complexes of k-modules give rise to a symmetric monoidal product $\boxtimes$ on the category $\mathrm{Ch}_k^{\mathcal{I}}$ of $\mathcal{I}$-diagrams in Ch_k. A *commutative $\mathcal{I}$-dga* is a commutative monoid in $(\mathrm{Ch}_k^{\mathcal{I}},\boxtimes)$ or, equivalently, a lax symmetric monoidal functor $\mathcal{I} \to \mathrm{Ch}_k$. Equipped with suitable model structures, the category of commutative $\mathcal{I}$-dgas $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is Quillen equivalent to the category of E_∞ dgas [RS17, §9]. This is analogous to the situation in spaces, where commutative monoids in $\mathcal{I}$-diagrams of spaces are equivalent to E_∞ spaces [SS12, §3].

Chasing the E_∞ dga of cochains $C(X;k)$ on a space X through the chain of Quillen equivalences relating E_∞ dgas and commutative $\mathcal{I}$-dgas shows that $C(X;k)$ can be represented by a commutative $\mathcal{I}$-dga. The purpose of this paper is to construct a direct point set level model $A^{\mathcal{I}}(X)$ for the quasi-isomorphism type of commutative $\mathcal{I}$-dgas determined by $C(X;k)$.

Date: January 4, 2018.
2010 *Mathematics Subject Classification.* Primary 55P48; Secondary 16E45, 18D50, 55P62.

If E is a commutative $\mathcal{I}$-dga, then its Bousfield–Kan homotopy colimit $E_{h\mathcal{I}}$ has a canonical action of the Barratt–Eccles operad, which is an E_∞ operad built from the symmetric groups. The commutative $\mathcal{I}$-dga $A^{\mathcal{I}}(X)$ thus gives rise to an E_∞ dga $A^{\mathcal{I}}(X)_{h\mathcal{I}}$ which can be compared to the usual cochains without referring to model structures.

Theorem 1.1. *The contravariant functors $X \mapsto A^{\mathcal{I}}(X)_{h\mathcal{I}}$ and $X \mapsto C(X;k)$ from simplicial sets to E_∞ dgas are naturally quasi-isomorphic.*

We prove the theorem using Mandell's uniqueness result for cochain theories [Man02, Main Theorem]. Since the definition of $A^{\mathcal{I}}$ does not rely on the existing constructions of E_∞ structures on cochains, the theorem implies that our approach provides an alternative construction of the E_∞ dga $C(X;k)$. If k is a field of characteristic 0, then there is a natural quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \to A_{\mathrm{PL}}(X)$ relating our approach to the classical polynomial forms (see Theorem 5.9).

The passage through commutative $\mathcal{I}$-dgas has the advantage that we do not need to lift the action of the acyclic Eilenberg–Zilber operad to the action of an actual E_∞ operad as done by Mandell [Man02, §5] based on work of Hinich–Schechtman [HS87], and it also avoids the elaborate combinatorial arguments used by Berger–Fresse [BF04]. Another approach to capture the commutativity of $C(X;k)$ has been pursued by Karoubi [Kar09] who introduces a notion of *quasi-commutative* dgas that is based on a certain reduced tensor product, constructs a quasi-commutative model for the cochains, and uses Mandell's results to relate it to ordinary cochains.

Since it is often easier to work with strictly commutative objects rather than E_∞ objects, we also expect that the commutative $\mathcal{I}$-dga $A^{\mathcal{I}}(X)$ will be a useful replacement of the E_∞ dga $C(X;k)$ in applications. For instance, iterated bar constructions for E_∞ algebras as developed in [Fre11] are rather involved whereas iterated bar construction for commutative monoids are straightforward. Commutative $\mathcal{I}$-dgas are tensored over simplicial sets whereas enrichments for E_∞ monoids are more complicated because the coproduct is not just the underlying monoidal product. This allows for constructions such as higher order Hochschild homology [Pir00] for commutative $\mathcal{I}$-dgas.

1.2. **Outline of the construction.** Our chain complexes are homologically graded so that cochains are concentrated in non-positive degrees. We model spaces by simplicial sets and consider the singular complex of a topological space if necessary.

The functor $A_{\mathrm{PL}}\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}}$ of polynomial forms used in rational homotopy theory (see e.g. [BG76, §1]) motivates our definition of $A^{\mathcal{I}}$. We recall that A_{PL} arises by Kan extending the functor $A_{\mathrm{PL},\bullet}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ sending $[p]$ in Δ to the algebra of polynomial differential forms

$$A_{\mathrm{PL},p} = \Lambda(t_0,\dots,t_p;dt_0,\dots,dt_p)/(t_0+\dots+t_1 = 1, dt_0+\dots+dt_p = 0)\ .$$

Here Λ is the free graded commutative algebra over $\mathbb{Q}$, the generators t_i have degree 0, and the dt_i have degree -1 (in our homological grading). Setting $d(t_i) = dt_i$ extends to a differential that turns $A_{\mathrm{PL},q}$ into a commutative dga, and addition of the t_i and insertion of 0 define the simplicial structure of $A_{\mathrm{PL},\bullet}$.

Let $\mathbb{C}D^0$ be the free commutative $\mathbb{Q}$-dga on the chain complex D^0 with $(D^0)_i = 0$ if $i \neq 0,-1$ and $d_0\colon (D^0)_0 \to (D^0)_{-1}$ being $\mathrm{id}_{\mathbb{Q}}$. Moreover, let S^0 in $\mathrm{Ch}_{\mathbb{Q}}$ be the monoidal unit, *i.e.*, the chain complex with a copy of $\mathbb{Q}$ concentrated in degree 0. Sending $1 \in (\mathbb{C}D^0)_0$ to either 1 or 0 in $\mathbb{Q}$ defines two commutative $\mathbb{C}D^0$-algebra structures on S^0 that we denote by S^0_0 and S^0_1. One can now check that the simplicial $\mathbb{Q}$-cdga $A_{\mathrm{PL},\bullet}$ is isomorphic to the two sided bar construction

$$B_\bullet(S^0_0, \mathbb{C}D^0, S^0_1) = \big([p] \mapsto S^0_0 \otimes (\mathbb{C}D^0)^{\otimes p} \otimes S^0_1\big)$$

whose face maps are provided by the algebra structures on S^0_1 and S^0_0 and the multiplication of $\mathbb{C}D^0$, and whose degeneracy maps are induced by unit of $\mathbb{C}D^0$.

While polynomial differential forms appear to have no obvious counterpart in commutative $\mathcal{I}$-dgas, their description in terms of a two sided bar construction easily generalizes to commutative $\mathcal{I}$-dgas over an arbitrary commutative ground ring k. For this we consider the left adjoint

$$\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}\colon \mathrm{Ch}_k \to \mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}], \qquad A \mapsto \left(\mathbf{m} \mapsto \bigoplus_{s\geq 0}\left(\left(\bigoplus_{\mathcal{I}(\mathbf{1}^{\sqcup s},\mathbf{m})} A^{\otimes s}\right)/\,\Sigma_s\right)\right)$$

to the evaluation of a commutative $\mathcal{I}$-dga at the object $\mathbf{1}$ in $\mathcal{I}$ and recall that the unit $U^{\mathcal{I}}$ in $\mathrm{Ch}^{\mathcal{I}}_k$ is the constant $\mathcal{I}$-diagram on the unit S^0 in Ch_k. As above, we form $\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}D^0$, observe that $U^{\mathcal{I}}$ gives rise to two commutative $\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}D^0$ algebras $U^{\mathcal{I}}_0$ and $U^{\mathcal{I}}_1$, and define $A^{\mathcal{I}}_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}]$ to be the two sided bar construction

$$B_\bullet(U^{\mathcal{I}}_0, \mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}(D^0), U^{\mathcal{I}}_1) = \left([p] \mapsto U^{\mathcal{I}}_0 \boxtimes (\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}(D^0))^{\boxtimes p} \boxtimes U^{\mathcal{I}}_1\right)\ .$$

At this point it is central to work with strictly commutative objects since the multiplication map of an E_∞ object is typically not an E_∞ map. It is also important to use $\mathbf{1}$ rather than $\mathbf{0}$ in the above left adjoint since this ensures that $A^{\mathcal{I}}_p(\mathbf{m})$ is contractible. This is related to J. Smith's insight that one has to use a *positive* model structures for commutative symmetric ring spectra.

Via Kan extension and restriction along the canonical functor $\Delta^{\mathrm{op}} \to \mathrm{sSet}^{\mathrm{op}}$, this $A^{\mathcal{I}}_\bullet$ gives rise to functors $A^{\mathcal{I}}\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}]$ and $K^{\mathcal{I}}\colon \mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}]^{\mathrm{op}} \to \mathrm{sSet}$. More explicitly, the evaluation of $A^{\mathcal{I}}(X)$ at $\mathcal{I}$-degree $\mathbf{m}$ and chain complex level q is the k-module of simplicial set morphisms $\mathrm{sSet}(X, A^{\mathcal{I}}_\bullet(\mathbf{m})_q)$. The functors $A^{\mathcal{I}}$ and $K^{\mathcal{I}}$ are contravariant right adjoint in the sense that there are natural isomorphisms $\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}](E, A^{\mathcal{I}}(X)) \cong \mathrm{sSet}(X, K^{\mathcal{I}}(E))$.

1.3. **Homotopical analysis of $A^{\mathcal{I}}$.** We equip simplicial sets with the standard model structure and the category of commutative $\mathcal{I}$-dgas $\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}]$ with the positive $\mathcal{I}$-model structure making it Quillen equivalent to E_∞ dgas.

Theorem 1.4. *Both $A^{\mathcal{I}}$ and $K^{\mathcal{I}}$ send cofibrations to fibrations and acyclic cofibrations to acyclic fibrations. They induce functors $K^{\mathcal{I}}_{\mathbb{R}}\colon \mathrm{Ho}(\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}])^{\mathrm{op}} \to \mathrm{Ho}(\mathrm{sSet})$ and $A^{\mathcal{I}}_{\mathbb{R}}\colon \mathrm{Ho}(\mathrm{sSet})^{\mathrm{op}} \to \mathrm{Ho}(\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}])$ that are related by a natural isomorphism*

$$\mathrm{Ho}(\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}])(E, A^{\mathcal{I}}_{\mathbb{R}}(X)) \cong \mathrm{Ho}(\mathrm{sSet})(X, K^{\mathcal{I}}_{\mathbb{R}}(E))\ .$$

A similar result for $A_{\mathrm{PL}}\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}}$ has been established by Bousfield–Gugenheim [BG76, §8]. Mandell [Man02, §4] constructed an analogous adjunction between simplicial sets and E_∞ dgas using the E_∞ structure on cochains as input.

Since all simplicial sets are cofibrant, the statement of Theorem 1.4 implies that each $A^{\mathcal{I}}(X)$ is positive fibrant. Writing $\mathcal{I}_+$ for the full subcategory of $\mathcal{I}$ on objects $\mathbf{m}$ with $|\mathbf{m}| \geq 1$, this means that each morphism $\mathbf{m} \to \mathbf{n}$ in $\mathcal{I}_+$ induces a quasi-isomorphisms $A^{\mathcal{I}}(X)(\mathbf{m}) \to A^{\mathcal{I}}(X)(\mathbf{n})$. Hence each chain complex $A^{\mathcal{I}}(X)(\mathbf{m})$ with $\mathbf{m}$ in $\mathcal{I}_+$ captures the quasi-isomorphism type of the cochains $C(X;k)$. Since the positive $\mathcal{I}$-model structure is the left Bousfield localization of a positive level model structure, it also follows that weak homotopy equivalences $X \to Y$ induce quasi-isomorphisms $A^{\mathcal{I}}(Y)(\mathbf{m}) \to A^{\mathcal{I}}(X)(\mathbf{m})$ for $\mathbf{m}$ in $\mathcal{I}_+$.

Analogous to the corresponding statement about A_{PL}, the proof of the theorem is based on the observation that the the simplicial sets $A^{\mathcal{I}}_\bullet(\mathbf{m})_q$ obtained by fixing an $\mathbf{m}$ in $\mathcal{I}_+$ and a chain level q are contractible.

1.5. **Notations and conventions.** Throughout the paper, k denotes a commutative ring with unit, and Ch_k denotes the category of unbounded homologically graded chain complexes of k-modules.

1.6. **Organization.** In Section 2 we study homotopy colimits of commutative $\mathcal{I}$-dgas. Section 3 provides the construction of the functor $A^{\mathcal{I}}$. We review model structures on $\mathcal{I}$-chain complexes and commutative $\mathcal{I}$-dgas in Section 4. In Section 5 we establish the homotopical properties of $A^{\mathcal{I}}$, prove a comparison to the usual cochains disregarding multiplicative structures, and prove Theorem 1.4. In the final Section 6, we prove the E_∞ comparison from Theorem 1.1 as Theorem 6.2.

2. Homotopy colimits of $\mathcal{I}$-chain complexes

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{m} = \{1, \ldots, m\}$ for $m \geq 0$ and with morphisms the injective maps. In this section we study multiplicative properties of the homotopy colimit functor for $\mathcal{I}$-diagrams of chain complexes.

Definition 2.1. An *$\mathcal{I}$-chain complex* is a functor $\mathcal{I} \to \mathrm{Ch}_k$, and $\mathrm{Ch}_k^{\mathcal{I}}$ denotes the resulting functor category.

For each $\mathbf{m}$ in $\mathcal{I}$ there is an adjunction $F^{\mathcal{I}}_{\mathbf{m}} \colon \mathrm{Ch}_k \rightleftarrows \mathrm{Ch}_k^{\mathcal{I}} \colon \mathrm{Ev}_{\mathbf{m}}$ with right adjoint the evaluation functor $\mathrm{Ev}_{\mathbf{m}}(P) = P(\mathbf{m})$ and left adjoint

$$F^{\mathcal{I}}_{\mathbf{m}} \colon \mathrm{Ch}_k \to \mathrm{Ch}_k^{\mathcal{I}}, \quad A \mapsto \left(\mathbf{n} \mapsto \bigoplus\nolimits_{\mathcal{I}(\mathbf{m},\mathbf{n})} A\right) \ . \tag{2.1}$$

The functor $F^{\mathcal{I}}_{\mathbf{0}}$ is isomorphic to the constant functor since $\mathbf{0}$ is initial in $\mathcal{I}$.

2.2. **Homotopy colimits.** Our next aim is to define Bousfield–Kan style homotopy colimits for $\mathcal{I}$-diagrams of chain complexes. For the subsequent multiplicative analysis, we fix notation and conventions about bicomplexes.

Definition 2.3. Let $\mathrm{Ch}_k(\mathrm{Ch}_k)$ be the category of chain complexes in Ch_k. Its objects are $\mathbb{Z}\times\mathbb{Z}$-graded k-modules $(Y_{p,q})_{p,q\in\mathbb{Z}}$ with k-linear *horizontal differentials*, $d_h \colon Y_{p,q} \to Y_{p-1,q}$, and k-linear *vertical differentials*, $d_v \colon Y_{p,q} \to Y_{p,q-1}$, such that

$$d_h \circ d_h = 0 = d_v \circ d_v \text{ and } d_v \circ d_h = d_h \circ d_v.$$

A morphism $g \colon Y \to Z$ in $\mathrm{Ch}_k(\mathrm{Ch}_k)$ is a family $(g_{p,q} \colon Y_{p,q} \to Z_{p,q})_{p,q\in\mathbb{Z}\times\mathbb{Z}}$ of k-linear maps that commute with the horizontal and vertical differentials, *i.e.*,

$$d_h \circ g_{p,q} = g_{p-1,q} \circ d_h \text{ and } d_v \circ g_{p,q} = g_{p,q-1} \circ d_v$$

for all $p, q \in \mathbb{Z}$.

Since we require horizontal and vertical differentials to commute, an additional sign is needed to form the total complex:

Definition 2.4. Let Y be an object in $\mathrm{Ch}_k(\mathrm{Ch}_k)$. Its *associated total complex* $\mathrm{Tot}(Y)$ is the chain complex with $\mathrm{Tot}(Y)_n = \bigoplus_{p+q=n} Y_{p,q}$ in chain degree $n \in \mathbb{Z}$ and with differential $d_{\mathrm{Tot}}(y) = d_h(y) + (-1)^p d_v(y)$ for every homogeneous $y \in E_{p,q}$.

Let sCh_k be the category of simplicial objects in Ch_k.

Definition 2.5. For $A \in \mathrm{sCh}_k$ we denote by $C_*(A)$ the chain complex in chain complexes with $(C_*(A))_{p,q} = A_{p,q}$. We define the horizontal differential on $C_*(A)$, $d_h \colon A_{p,q} \to A_{p-1,q}$, as

$$d_h = \sum_{i=0}^{p} (-1)^i d_i$$

where the d_i are the simplicial face maps of A. The vertical differential on $C_*(A)$ is given by the differential d^A on A.

As the d_i's commute with d^A, this gives indeed a chain complex in chain complexes whose horizontal part is concentrated in non-negative degrees.

Construction 2.6. Let $P\colon \mathcal{I} \to \mathrm{Ch}_k$ be an $\mathcal{I}$-chain complex. The *simplicial replacement* of P is the simplicial chain complex $\mathrm{srep}(P)\colon \Delta^{\mathrm{op}} \to \mathrm{Ch_k}$ given in simplicial degree $[p]$ by

$$\mathrm{srep}(P)[p] = \bigoplus_{(\mathbf{n_0}\xleftarrow{\alpha_1}\dots\xleftarrow{\alpha_p}\mathbf{n_p})\in N(\mathcal{I})_p} P(\mathbf{n_p}) \ .$$

The last face map maps the copy of $P(\mathbf{n_p})$ indexed by $(\alpha_1,\dots,\alpha_p)$ via $P(\alpha_p)$ to the copy of $P(\mathbf{n_{p-1}})$ indexed by $(\alpha_1,\dots,\alpha_{p-1})$. The other face and degeneracy maps are induced by the identity on $P(\mathbf{n_p})$ and corresponding simplicial structure maps of the nerve $\mathcal{N}(\mathcal{I})$ of $\mathcal{I}$.

The homotopy colimit functor $(-)_{h\mathcal{I}}\colon \mathrm{Ch}_k^{\mathcal{I}} \to \mathrm{Ch}_k$ is defined by

$$P_{h\mathcal{I}} = \mathrm{Tot}\ C_*(\mathrm{srep}(P)) \ .$$

A bicomplex spectral sequence argument shows that $P_{h\mathcal{I}} \to Q_{h\mathcal{I}}$ is a quasi-isomorphism if each $P(\mathbf{m}) \to Q(\mathbf{m})$ is a quasi-isomorphism. There is a canonical map $P_{h\mathcal{I}} \to \mathrm{colim}_{\mathcal{I}} P$, and one can show by cell induction that it is a quasi-isomorphism if P is cofibrant in the projective level model structure on $\mathrm{Ch}_k^{\mathcal{I}}$. Together this shows that $P_{h\mathcal{I}}$ is a model for the homotopy colimit of P. A more elaborate argument that shows that $P_{h\mathcal{I}}$ is a corrected homotopy colimit can be found in [RG14].

2.7. **Commutative $\mathcal{I}$-dgas.** The ordered concatenation of ordered sets $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$ equips $\mathcal{I}$ with a symmetric strict monoidal structure that has $\mathbf{0}$ as a strict unit and the block permutations as symmetry isomorphisms. If $P, Q\colon \mathcal{I} \to \mathrm{Ch}_k$ are $\mathcal{I}$-chain complexes, then the left Kan extension of

$$\mathcal{I} \times \mathcal{I} \xrightarrow{P\times Q} \mathrm{Ch}_k \times \mathrm{Ch}_k \xrightarrow{\otimes} \mathrm{Ch}_k$$

along $\sqcup\colon \mathcal{I}\times\mathcal{I} \to \mathcal{I}$ provides an $\mathcal{I}$-chain complex $P \boxtimes Q$. This defines a symmetric monoidal product $\boxtimes$ on $\mathrm{Ch}_k^{\mathcal{I}}$ with unit the constant $\mathcal{I}$-space $U^{\mathcal{I}} = F_0^{\mathcal{I}}(S^0)$.

Definition 2.8. A *commutative $\mathcal{I}$-dga* is a commutative monoid in $(\mathrm{Ch}_k^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}})$, *i.e.*, a lax symmetric monoidal functor $(\mathrm{Ch}_k^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}}) \to (\mathrm{Ch}_k, \otimes, S^0)$. The resulting category of commutative $\mathcal{I}$-dgas is denoted by $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$.

We write $\mathbb{C}\colon \mathrm{Ch}_k^{\mathcal{I}} \rightleftarrows \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]\colon U$ for the adjunction with right adjoint the forgetful functor and left adjoint the free functor $\mathbb{C}$ given by

$$\mathbb{C}(P) = \textstyle\bigoplus_{s\geq 0} P^{\boxtimes s}/\Sigma_s. \qquad (2.2)$$

The definition of $\boxtimes$ as a left Kan extension implies the existence of a natural isomorphism $F_{\mathbf{n_1}}^{\mathcal{I}}(A^1) \boxtimes F_{\mathbf{n_2}}^{\mathcal{I}}(A^2) \cong F_{\mathbf{n_1}\sqcup\mathbf{n_2}}^{\mathcal{I}}(A^1 \otimes A^2)$. This shows that in the case $P = F_{\mathbf{1}}^{\mathcal{I}}(A)$, we have an isomorphism $F_{\mathbf{1}}^{\mathcal{I}}(A)^{\boxtimes s} \cong F_{\mathbf{1}^{\sqcup s}}^{\mathcal{I}}(A^{\otimes s})$ of Σ_s-equivariant objects where Σ_s acts on the target by permuting both the $\otimes$-powers of A and the index set of the sum. The commutative $\mathcal{I}$-dga $\mathbb{C}(F_{\mathbf{1}}^{\mathcal{I}}(A))$ will be of particular importance for us, and we note that the above implies

$$\mathbb{C}(F_{\mathbf{1}}^{\mathcal{I}}(A))(\mathbf{m}) \cong \textstyle\bigoplus_{s\geq 0}\left(\left(\bigoplus_{\mathcal{I}(\mathbf{1}^{\sqcup s},\mathbf{m})} A^{\otimes s}\right) / \Sigma_s\right). \qquad (2.3)$$

2.9. **Homotopy colimits of commutative $\mathcal{I}$-dgas.** We will now construct an operad action on the homotopy colimit of a commutative $\mathcal{I}$-dga. Our construction involves a symmetric monoidal structure on simplicial chain complexes:

Definition 2.10. Let A and B be two simplicial chain complexes. Their tensor product $A\hat{\otimes}B$ is the simplicial chain complex with

$$\bigoplus_{\ell+m=n} A_{p,\ell} \otimes B_{p,m}$$

in simplicial degree p and chain degree n. The simplicial structure maps act coordinatewise and the differential $d^{\hat{\otimes}}$ is

$$d^{\hat{\otimes}}(a \otimes b) = d(a) \otimes b + (-1)^{\ell} a \otimes d(b)$$

for $a \otimes b \in A_{p,\ell} \otimes B_{p,m}$. The symmetry isomorphism $c\colon A\hat{\otimes}B \to B\hat{\otimes}A$ sends a homogeneous element $a \otimes b$ as above to $(-1)^{\ell\cdot m} b \otimes a$.

We denote by $\widetilde{\Sigma}_s$ the translation category of the symmetric group Σ_s. Its objects are elements $\sigma \in \Sigma_s$ and $\tau \in \Sigma_s$ is the unique morphism from σ to $\tau \circ \sigma$ in $\widetilde{\Sigma}_s$. Since there is exactly one morphism between each pair of objects, we get a functor

$$\widetilde{\Sigma}_s \times \widetilde{\Sigma}_{j_1} \times \cdots \times \widetilde{\Sigma}_{j_s} \to \widetilde{\Sigma}_{j_1+\cdots+j_s} \tag{2.4}$$

by specifying that $(\sigma; \tau_1, \ldots, \tau_s)$ is sent to $\tau_{\sigma^{-1}(1)} \sqcup \ldots \sqcup \tau_{\sigma^{-1}(s)}$. The action (2.4) is associative, unital, and symmetric. It turns the collection of categories $(\widetilde{\Sigma}_n)_{n\geq 0}$ into an operad $\widetilde{\Sigma}$ in the category cat of small categories. For the next definition, we use that the nerve functor $\mathcal{N}\colon \mathrm{cat} \to \mathrm{sSet}$ and the k-linearization $k\{-\}\colon \mathrm{sSet} \to \mathrm{sMod}_k$ are strong symmetric monoidal and that the associated chain complex functor $C_*\colon \mathrm{sMod}_k \to \mathrm{Ch}_k$ is lax symmetric monoidal (compare Proposition 2.16 below).

Definition 2.11. The Barratt–Eccles operad is the E_∞ operad $\mathcal{E}$ in Ch_k with $\mathcal{E}_n = C_*(k\{\mathcal{N}(\widetilde{\Sigma}_n)\})$ and operad structure induced by the functor (2.4).

The commutativity operad $\mathcal{C}$ in Ch_k is the operad with $\mathcal{C}_n = k$ concentrated in chain complex level 0. The operad $\mathcal{E}$ admits a canonical operad map $\mathcal{E} \to \mathcal{C}$ which is a quasi-isomorphism in each level. Moreover, $\mathcal{E}_n$ is a free $k[\Sigma_n]$-module for each n. Thus $\mathcal{E}$ is an E_∞ operad in Ch_k in the terminology of [Man02, Definition 4.1].

Only applying the nerve to $\widetilde{\Sigma}$ defines an operad in sSet that is more commonly referred to as the Barratt–Eccles operad. It is well known that the latter operad acts on the nerve of a permutative category [May74, Theorem 4.9]. The next lemma recalls the underlying action of $\widetilde{\Sigma}$ for the permutative category $\mathcal{I}$.

Lemma 2.12. *The operad $\widetilde{\Sigma}$ in* cat *acts on $\mathcal{I}$. On objects σ in $\widetilde{\Sigma}_n$ and $\mathbf{m_i}$ in $\mathcal{I}$, the action is given by $(\sigma; \mathbf{m_1}, \ldots, \mathbf{m_n}) \mapsto \mathbf{m}_{\sigma^{-1}(\mathbf{1})} \sqcup \ldots \sqcup \mathbf{m}_{\sigma^{-1}(\mathbf{n})}$.*

Proof. This is a special case of [May74, Lemmas 4.3 and 4.4]. Functoriality in morphisms of $\widetilde{\Sigma}_n$ uses the symmetry isomorphism of $\mathcal{I}$ while the functoriality in $\mathcal{I}$ is the evident one. □

The next result is our main motivation for considering the Barratt–Eccles operad. It is analogous to the result about $\mathcal{I}$-diagrams in spaces established in [Sch09, Proposition 6.5].

Theorem 2.13. *For every commutative $\mathcal{I}$-dga E, the chain complex $E_{h\mathcal{I}}$ has a natural action of the Barratt–Eccles operad $\mathcal{E}$.*

Proof. We can view the simplicial k-module $k\{\mathcal{N}(\widetilde{\Sigma}_n)\}$ as a simplicial chain complex concentrated in degree 0. The operad structure of $\widetilde{\Sigma}$ turns these simplicial k-modules into an operad in sMod_k and in $s\mathrm{Ch}_k$. We construct an action

$$k\{\mathcal{N}(\widetilde{\Sigma}_s)\}\hat{\otimes}\,\mathrm{srep}(E)^{\hat{\otimes} s} \to \mathrm{srep}(E).$$

It is enough to specify the action of a q-simplex $(\sigma_1, \ldots, \sigma_q)$ in $\mathcal{N}(\widetilde{\Sigma}_s)$ on a collection of elements $(\alpha^i_1, \ldots, \alpha^i_q; x^i)$ in $\mathrm{srep}(E)[q]_{p_i}$ where $\alpha^i_j\colon \mathbf{n^i_j} \to \mathbf{n^i_{j-1}}$ is a map in $\mathcal{I}$ and x^i is an element in $E(\mathbf{n^i_q})_{p_i}$. On the indices $(\alpha^i_1, \ldots, \alpha^i_q)$ for the sums in the simplicial replacement, we use the action of $(\sigma_1, \ldots, \sigma_q)$ provided by the previous lemma. As element in $E(\mathbf{n_q}^{\sigma_\mathbf{q}^{-1}(\mathbf{1})} \sqcup \ldots \sqcup \mathbf{n_q}^{\sigma_\mathbf{q}^{-s}(\mathbf{1})})_{p_1+\cdots+p_s}$ we take the product $x^{\sigma_q^{-1}(1)} \cdots x^{\sigma_q^{-1}(s)}$. Since E is commutative, this does indeed define an operad action in sCh_k. By

Propositions 2.16 and 2.17 below, the composite $\mathrm{Tot}\, C_*$ is lax symmetric monoidal. Hence it follows that $\mathcal{E}$ acts on $E_{h\mathcal{I}}$. □

2.14. **Monoidality of C_* and** Tot**.** It remains to verify the monoidal properties of C_* and Tot that were used in the proof of Theorem 2.13.

Definition 2.15. Let Y and Z be two objects in $\mathrm{Ch}_k(\mathrm{Ch}_k)$. Their tensor product is $Y \otimes Z$ is the object in $\mathrm{Ch}_k(\mathrm{Ch}_k)$ with

$$(Y \otimes Z)_{p,q} = \bigoplus_{a_1+a_2=p} \bigoplus_{b_1+b_2=q} Y_{a_1,b_1} \otimes Z_{a_2,b_2}$$

and differentials $d_h^{\otimes}(y \otimes z) = d_h(y) \otimes z + (-1)^{a_1} y \otimes d_h(z)$ and $d_v^{\otimes}(y \otimes z) = d_v(y) \otimes z + (-1)^{b_1} y \otimes d_v(z)$. The symmetry isomorphism $\tau \colon Y \otimes Z \to Z \otimes Y$ sends a homogeneous element $y \otimes z \in Y_{a_1,b_1} \otimes Z_{a_2,b_2}$ to $(-1)^{a_1a_2+b_1b_2} z \otimes y$.

Proposition 2.16. *The functor* $C_* \colon \mathrm{sCh}_k \to \mathrm{Ch}_k(\mathrm{Ch}_k)$ *is lax symmetric monoidal.*

Proof. As in [ML63, Theorem VIII.8.8] we denote (p, q)-shuffles as two disjoint subsets $\mu_1 < \ldots < \mu_p$ and $\nu_1 < \ldots < \nu_q$ of $\{0, \ldots, p + q - 1\}$. For simplicial chain complexes A and B we define maps

$$\mathrm{sh}_{A,B} \colon C_*(A) \otimes C_*(B) \to C_*(A \hat{\otimes} B)$$

that turn C_* into a lax symmetric monoidal functor: If $a \otimes b$ is a homogeneous element in $A_{r_1,r_2} \otimes B_{s_1,s_2}$ we set

$$\mathrm{sh}_{A,B}(a \otimes b) = \sum_{(\mu,\nu)} \mathrm{sgn}(\mu, \nu) s_{\nu_{s_1}} \circ \ldots \circ s_{\nu_1}(a) \otimes s_{\mu_{r_1}} \circ \ldots \circ s_{\mu_1}(b).$$

Here, the sum runs over all (r_1, s_1)-shuffles (μ, ν) and $\mathrm{sgn}(\mu, \nu)$ denotes the signum of the associated permutation.

As the simplicial structure maps of A and B commute with d^A and d^B, it follows that sh commutes with the vertical differential. The proof that the horizontal differential is compatible with sh is the same as for sh in the context of simplicial modules.

It remains to show that sh turns C_* into a lax symmetric monoidal functor, *i.e.*, we have to show that

$$C_*(c) \circ \mathrm{sh}(a \otimes b) = \mathrm{sh} \circ \tau(a \otimes b) \tag{2.5}$$

for any homogeneous element $a \otimes b \in A_{r_1,r_2} \otimes B_{s_1,s_2}$. As $\tau(a \otimes b) = (-1)^{r_1s_1+r_2s_2} b \otimes a$, the right-hand side of equation (2.5) is

$$\sum_{(\xi,\zeta)} (-1)^{r_1s_1+r_2s_2} \mathrm{sgn}(\xi, \zeta) s_{\zeta_{s_1}} \circ \ldots \circ s_{\zeta_1}(b) \otimes s_{\xi_{r_1}} \circ \ldots \circ s_{\xi_1}(a)$$

with (ξ, ζ) being (s_1, r_1)-shuffles, whereas the left-hand side of the equation gives

$$(-1)^{r_2s_2} \sum_{(\mu,\nu)} \mathrm{sgn}(\mu, \nu) s_{\mu_{r_1}} \circ \ldots \circ s_{\mu_1}(b) \otimes s_{\nu_{s_1}} \circ \ldots \circ s_{\nu_1}(a)$$

because τ introduces the sign $(-1)^{r_2s_2}$. Precomposing with the permutation that exchanges the blocks $0 < \ldots < r_1 - 1$ and $r_1 < \ldots < r_1 + s_1 - 1$ gives a bijection between the summation indices and introduces the sign $(-1)^{r_1s_1}$. Hence the two sides agree. □

Proposition 2.17. *The functor* Tot *is strong symmetric monoidal.*

Proof. Spelling out what $\mathrm{Tot}(Y)\otimes\mathrm{Tot}(Z)$ is in degree n we obtain

$$(\mathrm{Tot}(Y)\otimes\mathrm{Tot}(Z))_n \cong \bigoplus_{r_1+r_2+s_1+s_2=n} Y_{r_1,r_2}\otimes Z_{s_1,s_2}\ ,$$

and we send a homogeneous element $y\otimes z\in Y_{r_1,r_2}\otimes Z_{s_1,s_2}$ to the element

$$(-1)^{r_2s_1} y\otimes z \in \mathrm{Tot}(Y\otimes Z)_n \cong \bigoplus_{r_1+s_1+r_2+s_2} Y_{r_1,r_2}\otimes Z_{s_1,s_2}.$$

This gives isomorphisms

$$\varphi_{Y,Z}\colon\ \mathrm{Tot}(Y)\otimes\mathrm{Tot}(Z)\to\mathrm{Tot}(Y\otimes Z)$$

that are associative. It is clear that Tot respects the unit up to isomorphism.

The maps $\varphi_{Y,Z}$ are compatible with the differential: Let $y\otimes z$ be a homogeneous element in $Y_{r_1,r_2}\otimes Z_{s_1,s_2}$. The composition $d_{\mathrm{Tot}}\circ\varphi$ applied to $y\otimes z$ gives

$$\begin{aligned} d_{\mathrm{Tot}}\circ\varphi(y\otimes z) &= (-1)^{r_2s_1} d_h^{\otimes}(y\otimes z) + (-1)^{r_2s_1}(-1)^{r_1+s_1} d_v^{\otimes}(y\otimes z)\\ &= (-1)^{r_2s_1} d_h(y)\otimes z + (-1)^{r_2s_1+r_1} y\otimes d_h(z)\\ &\quad + (-1)^{r_2s_1+r_1+s_1} d_v(y)\otimes z + (-1)^{r_2s_1+r_1+s_1+r_2} y\otimes d_v(z). \end{aligned}$$

First applying the differential to $y\otimes z$ and then φ yields

$$\begin{aligned} &\varphi(d_{\mathrm{Tot}}(y)\otimes z + (-1)^{r_1+r_2} y\otimes d_{\mathrm{Tot}}(z))\\ =&\varphi(d_h(y)\otimes z + (-1)^{r_1} d_v(y)\otimes z + (-1)^{r_1+r_2} y\otimes d_h(z) + (-1)^{r_1+r_2+s_1} y\otimes d_v(z))\\ =&(-1)^{r_2s_1} d_h(y)\otimes z + (-1)^{r_1+(r_2-1)s_1} d_v(y)\otimes z + (-1)^{r_1+r_2+r_2(s_1-1)} y\otimes d_h(z)\\ &+ (-1)^{r_1+r_2+s_1+r_2s_1} y\otimes d_v(z) \end{aligned}$$

thus both terms agree.

We denote the symmetry isomorphism in the category of chain complexes by χ. Then

$$\varphi\circ\chi(e\otimes f) = \varphi((-1)^{(r_1+r_2)(s_1+s_2)} f\otimes e) = (-1)^{r_1s_1+r_2s_2+s_1r_2+2s_2r_1} f\otimes e$$

and this is equal to

$$\mathrm{Tot}(\tau)\circ\varphi(e\otimes f) = \mathrm{Tot}(\tau)((-1)^{r_2s_1} e\otimes f) = (-1)^{r_2s_1+r_1s_1+r_2s_2} f\otimes e. \qquad \square$$

Remark 2.18. One can also consider a symmetric monoidal structure on $\mathrm{Ch}_k(\mathrm{Ch}_k)$ with the same underlying tensor product but with symmetry isomorphism

$$y\otimes z\mapsto (-1)^{(r_1+r_2)(s_1+s_2)} z\otimes y$$

for homogeneous elements $y\otimes z\in Y_{r_1,r_2}\otimes Z_{s_1,s_2}$. Then one can take φ in Proposition 2.17 to be the identity. However, this symmetry isomorphism is *not* compatible with the shuffle transformation from the proof of Proposition 2.16.

Remark 2.19. For a simplicial chain complex A one can also consider a normalized object $N(A)\in\mathrm{Ch}_k(\mathrm{Ch}_k)$ where one divides out by the subobject generated by degenerate elements. As the simplicial structure maps commute with the differential of A, this is well-defined, and the proof of Proposition 2.16 can be adapted as in [ML63, Corollary VIII.8.9] to show that the functor $N\colon \mathrm{sCh}_k\to\mathrm{Ch}_k(\mathrm{Ch}_k)$ is also lax symmetric monoidal. Consequently, one can also use N instead of C_* in the definition of the Barratt–Eccles operad $\mathcal{E}$ and the homotopy colimit $P_{h\mathcal{I}}$ so that Theorem 2.13 remains valid.

3. Cochain functors with values in $\mathcal{I}$-chain complexes

In this section we construct the functor $A^{\mathcal{I}}$ discussed in the introduction and a version of the ordinary cochains with values in $\mathcal{I}$-chain complexes.

3.1. **Adjunctions induced by simplicial objects.** We briefly recall an ubiquitous construction principle for adjunctions that we will later apply to simplicial objects in the categories of commutative $\mathcal{I}$-dgas and $\mathcal{I}$-chain complexes.

Construction 3.2. Let $D_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{D}$ be a simplicial object in a complete category $\mathcal{D}$. Passing to opposite categories, $D_\bullet$ gives rise to a functor $\widetilde{D}^\bullet \colon \Delta \to \mathcal{D}^{\mathrm{op}}$. Since $\mathcal{D}$ is complete, $\mathcal{D}^{\mathrm{op}}$ is cocomplete. Hence restriction and left Kan extension along $\Delta \to \mathrm{sSet}$, $[p] \mapsto \Delta^p$ define an adjunction

$$\widetilde{D} \colon \mathrm{sSet} \rightleftarrows \mathcal{D}^{\mathrm{op}} \colon K_D \ .$$

Writing $D \colon \mathrm{sSet}^{\mathrm{op}} \to \mathcal{D}$ for the opposite of $\widetilde{D}$, this implies that for a simplicial set X and an object E of $\mathcal{D}$, we have a natural isomorphism

$$\mathcal{D}(E, D(X)) = \mathcal{D}^{\mathrm{op}}(\widetilde{D}(X), E) \cong \mathrm{sSet}(X, K_D(E)) \tag{3.1}$$

exhibiting D and K_D as *contravariant right adjoint* functors. Unraveling definitions, the contravariant functors K_D and D are given by $K_D(E)_\bullet = \mathcal{D}(E, D_\bullet)$ and $D(X) = \lim_{\Delta^p \to X} D_p$ where the limit is taken over the category of elements of X. In the special case $\mathcal{D} = \mathrm{Set}$, writing X as a colimit of representable functors indexed over its category of elements provides a natural bijection $D(X) \cong \mathrm{sSet}(X, D)$.

The functor D extends the original functor $D_\bullet$ in that there is a natural isomorphism $D_\bullet \cong D(\Delta^\bullet)$. The construction is also functorial in $D_\bullet$, *i.e.*, a natural transformation $D_\bullet \to D'_\bullet$ of functors $\Delta^{\mathrm{op}} \to \mathcal{D}$ induces a natural transformation $D \to D'$ of functors $(\mathrm{sSet})^{\mathrm{op}} \to \mathcal{D}$.

We note an immediate consequence of having the adjunction $(\widetilde{D}, K_D)$.

Lemma 3.3. *The functor D takes colimits in* sSet *to limits in $\mathcal{D}$, and K_D takes colimits in $\mathcal{D}$ to limits in* sSet. □

When $D_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is a simplicial object in commutative $\mathcal{I}$-dgas, we may apply Construction 3.2 both to $D_\bullet$ and to its composite $D'_\bullet = UD'_\bullet$ with the forgetful functor $U \colon \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}] \to \mathrm{Ch}_k^{\mathcal{I}}$. Since the extensions of $D_\bullet$ and $D'_\bullet$ to functors on sSet are defined by limit constructions and U commutes with limits, we have a natural isomorphism $U(D(X)) \cong D'(X)$ for a simplicial set X. The adjoints K_D and $K_{D'}$ are related by a natural isomorphism $K_{D'} \cong K_D \circ \mathbb{C} \colon (\mathrm{Ch}_k^{\mathcal{I}})^{\mathrm{op}} \to \mathrm{sSet}$. An analogous remark applies to simplicial objects of algebras in $\mathrm{Ch}_k^{\mathcal{I}}$ over a more general operad than the commutativity operad.

For $D_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, the fact that $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}] \to \mathrm{Set}$, $E \mapsto E(\mathbf{m})_q$ commutes with limits implies that the underlying set of $D(X)(\mathbf{m})(q)$ is $\mathrm{sSet}(X, D_\bullet(\mathbf{m})_q)$. The pointwise k-module structure, differentials and multiplications on these sets give rise to the commutative $\mathcal{I}$-dga structure on $D(X)$.

3.4. **The commutative $\mathcal{I}$-dga version of polynomial forms.** Composing the left adjoints in the adjunctions $(F_{\mathbf{1}}^{\mathcal{I}}, \mathrm{Ev}_{\mathbf{1}})$ and $(\mathbb{C}, U)$ introduced in (2.1) and (2.2) provides a left adjoint $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}} \colon \mathrm{Ch}_k \to \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ made explicit in (2.3). We are particularly interested in the commutative $\mathcal{I}$-dga $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0)$. For an element $i \in k$, the k-module map $k \to k = \mathrm{Ev}_1(U^{\mathcal{I}})_0$ determined by $1 \mapsto i$ gives rise to a map $\varepsilon_i \colon \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0) \to U^{\mathcal{I}}$. We write $U_0^{\mathcal{I}}$ and $U_1^{\mathcal{I}}$ for the two commutative $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0)$-algebras resulting from the elements $0, 1 \in k$.

Definition 3.5. We let $A_\bullet^{\mathcal{I}} \colon \Delta^{\mathrm{op}} \to \mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ be the simplicial commutative $\mathcal{I}$-dga given by the two-sided bar construction

$$[p] \mapsto A_p^{\mathcal{I}} = B_p(U_0^{\mathcal{I}}, \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0), U_1^{\mathcal{I}}) = U_0^{\mathcal{I}} \boxtimes \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0)^{\boxtimes p} \boxtimes U_1^{\mathcal{I}} \ . \tag{3.2}$$

As with the space level version (see e.g. [May72]), the outer face maps are provided by the algebra structures of $U_0^{\mathcal{I}}$ and $U_1^{\mathcal{I}}$, the inner face come from the multiplication of $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0)$, and the degeneracy maps are induced by its unit.

To make this simplicial object more explicit, we write D_r^0 for the chain complex with copies of k on generators r in degree 0 and on dr in degree -1 and 0 elsewhere. Its non-zero differential is $d(a{\cdot}r) = a{\cdot}dr$. Since $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}$ is left adjoint and $U^{\mathcal{I}}$ is the unit for $\boxtimes$, commuting $\mathbb{C}F_{1}^{\mathcal{I}}$ with coproducts provides an isomorphism of commutative $\mathcal{I}$-dgas

$$A_p^{\mathcal{I}} \cong \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D_{r_1(p)} \oplus \cdots \oplus D_{r_p(p)})$$

where the generators $r_1(p), \dots, r_p(p)$ correspond to the p copies of $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0)$. By adjunction, maps $f\colon \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D_{r_1(p)} \oplus \cdots \oplus D_{r_p(p)}) \to E$ in $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ correspond to families of elements $f(r_1(p)), \dots, f(r_p(p)) \in E(\mathbf{1})_0$.

We now set $r_0(p) = 0$ and define $r_{p+1}(p)$ to be the image of 1 under the map

$$k = U^{\mathcal{I}}(\mathbf{1}) \to \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D_{r_1(p)} \oplus \cdots \oplus D_{r_p(p)})(\mathbf{1})$$

induced by the unit. With this notation, the simplicial structure maps of the two sided bar construction (3.2) are determined by requiring

$$d_i(r_j(p)) = \begin{cases} r_j(p-1) & \text{if } j \le i \\ r_{j-1}(p-1) & \text{if } j > i, \end{cases} \qquad s_i(r_j(p)) = \begin{cases} r_j(p+1) & \text{if } j \le i \\ r_{j+1}(p+1) & \text{if } j > i\ . \end{cases}$$

For a simplicial k-module $Z\colon \Delta^{\mathrm{op}} \to \mathrm{Mod}_k$, *extra degeneracies* are a family of k-linear maps $s_{p+1}\colon Z_p \to Z_{p+1}$ satisfying $d_{p+1}s_{p+1} = \mathrm{id}_{Z_p}$ if $p \ge 0$, $d_i s_{p+1} = s_p d_i\colon Z_p \to Z_p$ if $p \ge 1$ and $0 \le i \le p$, and $0 = d_0 s_1\colon Z_0 \to Z_0$. The presence of extra degeneracies implies that Z is contractible to 0 (in the sense that $Z \to 0$ is a weak equivalence in sMod_k) since the maps $(-1)^{p+1}s_{p+1}$ define a contracting homotopy for the chain complex $C_*(Z)$.

The following lemma is the technical backbone for our homotopical analysis of the prolongation $A^{\mathcal{I}}$ of $A_\bullet^{\mathcal{I}}$ in Section 5. It is analogous to [BG76, Proposition 1.1].

Lemma 3.6. *For all $q \in \mathbb{Z}$ and all positive objects $\mathbf{m}$ in $\mathcal{I}$, the simplicial k-module $A_{\bullet,q}^{\mathcal{I}}(\mathbf{m})$ is contractible to* 0.

Remark 3.7. The statement of the lemma does not hold for $\mathbf{m} = \mathbf{0}$ since $A_{\bullet,0}(\mathbf{m})$ is the constant simplicial object on k, which is not contractible.

Proof of Lemma 3.6. We decompose the sum over s in (2.2) as $\mathbb{C}(P) = U^{\mathcal{I}} \oplus \mathbb{N}(P)$ where $\mathbb{N}(P) = \bigoplus_{s\ge1} P^{\boxtimes s}/\Sigma_s$ is the free non-unital $\mathcal{I}$-dga on the $\mathcal{I}$-chain complex P. For $P = F_{\mathbf{1}}^{\mathcal{I}}(D_{r_1(p)} \oplus \cdots \oplus D_{r_p(p)})$, this gives a decomposition of $A_p^{\mathcal{I}}$ that we use to define the extra degeneracy s_{p+1}. Restricted to the summand $\mathbb{N}(P)$, the maps s_{p+1} for varying $\mathbf{m}$ will form a map of non-unital $\mathcal{I}$-dgas. By the universal property of the free non-unital $\mathcal{I}$-dgas, this map is determined by setting $s_{p+1}(r_j(p)) = r_j(p{+}1)$ for all $1 \le j \le p$. On the summand $U^{\mathcal{I}}$ in the decomposition of $\mathbb{C}(P)$, the map s_{p+1} will neither be a chain map nor be functorial in $\mathcal{I}$. To define s_{p+1} on the copy of k in $\mathcal{I}$-degree $\mathbf{m}$, we let $r_{p+1}(p)$ be its generator, choose a map $\iota\colon \mathbf{1} \to \mathbf{m}$, and define $s_{p+1}(r_{p+1}(p)) = r_{p+1}^{\iota}(p{+}1)$ where the index of the latter generator indicates that it lives in the summand with indices $s = 1$ and $\iota \in \mathcal{I}(\mathbf{1}^{\sqcup 1}, \mathbf{m})$ of

$$\mathbb{N}(P) = \bigoplus\nolimits_{s\ge1} \left(\left(\bigoplus\nolimits_{\mathcal{I}(\mathbf{1}^{\sqcup s},\mathbf{m})} (D_{r_1(p)} \oplus \cdots \oplus D_{r_p(p)})^{\otimes s} \right) / \, \Sigma_s \right)\ .$$

It remains to check that the s_{p+1} provide extra degeneracies. In simplicial degree 0, the relation $d_0 s_1 = 0$ holds because d_0 sends the generator $r_1^{\iota}(1)$ to 0. Now assume $p \ge 0$. We again use the sum decomposition $\mathbb{C}(P) = U^{\mathcal{I}} \oplus \mathbb{N}(P)$ and the universal property of the free non-unital $\mathcal{I}$-dga to see that it is enough to check the compatibilites on the generators. The relation $d_{p+1}s_{p+1} = \mathrm{id}$ holds on the generators $r_1(p), \ldots, r_p(p)$ since

$$d_{p+1}s_{p+1}r_j(p) = d_{p+1}r_j(p+1) = r_j(p).$$

For the generator $r_{p+1}(p)$, we have

$$d_{p+1}s_p(r_{p+1}(p)) = d_{p+1}(r^{\iota}_{p+1}(p+1)) = r_{p+1}(p).$$

Here we use that the restriction of d_{p+1} to the $s = 1$ summand is the sum over all $\alpha\colon \mathbf{1} \to \mathbf{m}$ of the maps that send the generator $r^{\alpha}_{p+1}(p)$ of the respective copy of k to $1 = r_{p+1}(p)$. Now let $p \geq 1$ and $0 \leq i \leq p$. For $r_1(p), \ldots, r_p(p)$, the relations $d_i s_{p+1} = s_p d_i$ hold since s_{p+1} only raises the index p in generators by 1 while d_i does not depend on p. For $r_{p+1}(p)$, we have

$$d_i s_{p+1}(r_{p+1}(p)) = d_i(r^{\iota}_{p+1}(p+1)) = r^{\iota}_p(p) = s_p(r_p(p-1)) = s_p d_i(r_{p+1}(p)). \quad \square$$

3.8. **Ordinary cochains.** Let $C(X;k)$ be the cochains with values in k on the simplicial set X, viewed as a homologically graded chain complex concentrated in non-positive degrees. (At this point, we disregard its cup product structure.) So for $q \geq 0$, we have $C_{-q}(X;k) = \mathrm{Set}(X_q, k)$ with the pointwise k-module structure and differential induced by the face maps of X. The cochains on the standard n-simplices assemble to a functor $C_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Ch}_k$, $[p] \mapsto C(\Delta^p;k)$. The following lemma is well known (see e.g. [FHT01, Lemma 10.11 and Lemma 10.12(ii)]).

Lemma 3.9. *(i) The extension of $C_\bullet$ to a functor* sSet $\to \mathrm{Ch}_k$ *resulting from Construction 3.2 is naturally isomorphic to $C(-;k)$.*
(ii) For all $q \in \mathbb{Z}$, the simplicial k-module $C_{\bullet,q} = C(\Delta^\bullet;k)_q$ is contractible to 0.

Proof. For (i), we note that the description of the extension as $\lim_{\Delta^p \to X} C(\Delta^p;k)$ implies that there is a natural map from $C(X;k)$. Writing X as a colimit of representable functors over its category of elements, the evaluation of this map at q is a bijection since taking maps into k turns colimits into limits.

For (ii), we only need to consider the case $q \leq 0$, set $n = -q$ and define

$$s_{p+1}\colon C_q(\Delta^p;k) \to C_q(\Delta^{p+1};k)$$

on $f\colon (\Delta^p)_n \to k$ as follows: We set $s_{p+1}(f)\colon (\Delta^{p+1})_n \to k$ to be 0 on all n-simplices not in the image of $d^{p+1}\colon \Delta^p \to \Delta^{p+1}$ and require that $s_{p+1}(f)$ restricts to f on the last face. Identifying Δ^{p+1}_n with $\Delta([n],[p+1])$, this means that $s_{p+1}(f)(d^{p+1}\alpha') = f(\alpha')$ and $s_{p+1}(f)(\alpha) = 0$ if $p+1 \in \alpha([n])$. Then for $\beta\colon [n] \to [p]$, the equation $d_{p+1}(s_{p+1}(f))(\beta) = \beta$ holds by definition, and $d_0 s_1 = 0$ in simplicial degree 0 is also immediate. Now assume $p \geq 1$. If β has p in its image, then $d_i s_{p+1}(f)(\beta) = 0 = s_p d_i(f)(\beta)$. Otherwise, we must have $\beta = d^p\beta'$ and thus

$$\begin{aligned} d_i s_{p+1}(f)(d^p\beta') &= s_{p+1}(f)(d^i d^p \beta') = s_{p+1}(f)(d^{p+1}d^i\beta') \\ &= f(d^i\beta') = (d_i f)(\beta') = s_p d_i(f)(d^p\beta'). \end{aligned} \quad \square$$

For later use, we lift $C_\bullet$ to $\mathcal{I}$-chain complexes by defining

$$C^{\mathcal{I}}_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Ch}^{\mathcal{I}}_k, \quad [p] \mapsto F^{\mathcal{I}}_{\mathbf{0}}(C(\Delta^p;k)) \ .$$

Corollary 3.10. *(i) The extension $C^{\mathcal{I}}$ of $C^{\mathcal{I}}_\bullet$ to a functor* sSet $\to \mathrm{Ch}^{\mathcal{I}}_k$ *resulting from Construction 3.2 is naturally isomorphic to $X \mapsto F^{\mathcal{I}}_{\mathbf{0}}C(X;k)$.*
(ii) For all $q \in \mathbb{Z}$ and $\mathbf{m}$ in $\mathcal{I}$, the simplicial k-module $C^{\mathcal{I}}_{\bullet,q}(\mathbf{m}) = F^{\mathcal{I}}_{\mathbf{0}}(C(\Delta^\bullet;k)_q)$ is contractible to 0. $\square$

4. Homotopy theory of $\mathcal{I}$-chain complexes and commutative $\mathcal{I}$-dgas

In this section we review basic results about model category structures on $\mathcal{I}$-chain complexes and commutative $\mathcal{I}$-dgas. Much of this is motivated by (and analogous to) the corresponding results for space valued functors developed in [SS12, §3].

We continue to consider the category of unbounded chain complexes Ch_k. For $q \in \mathbb{Z}$, we as usual write S^q for the chain complex with k concentrated in degree q, and D^q for the chain complex with $(D^q)_i = k$ if $i \in \{q, q-1\}$, with $(D^q)_i = 0$ for all other i, and with $d_q = \mathrm{id}_k$. We equip Ch_k with the projective model structure whose weak equivalences are the quasi-isomorphisms and whose fibrations are the level-wise surjections [Hov99, Theorem 2.3.11]. It has the inclusions $S^{q-1} \hookrightarrow D^q$ as generating cofibrations and the maps $0 \to D^q$ as generating acyclic cofibrations.

4.1. **Level model structures.** We call an object $\mathbf{m}$ of $\mathcal{I}$ *positive* if $|\mathbf{m}| \geq 1$ and write $\mathcal{I}_+$ for the full subcategory of positive objects in $\mathcal{I}$.

A map $f\colon P \to Q$ in $\mathrm{Ch}_k^{\mathcal{I}}$ is an *absolute* (resp. *positive*) *level equivalence* if $f(\mathbf{m})$ is a quasi-isomorphism for all $\mathbf{m}$ in $\mathcal{I}$ (resp. all $\mathbf{m}$ in $\mathcal{I}_+$). It is an *absolute* (resp. *positive*) *level fibration* if $f(\mathbf{m})$ is a fibration for all $\mathbf{m}$ in $\mathcal{I}$ (resp. all $\mathbf{m}$ in $\mathcal{I}_+$). An *absolute* (resp. *positive*) *level cofibration* is a map with the left lifting property with respect to all maps which are both absolute (resp. positive) level fibrations and equivalences.

Proposition 4.2. *These maps define an* absolute *and a* positive level model structures *on* $\mathrm{Ch}_k^{\mathcal{I}}$. *Both model structures are proper and combinatorial.*

Proof. This follows from standard model category arguments, compare [SS12, Proposition 6.7]. Alternatively, one may invoke [PS, Theorem 3.2.5]. □

The cofibrations in these level model structures are the retracts of relative cell complexes of the form $F^{\mathcal{I}}_{\mathbf{m}}(S^{q-1} \hookrightarrow D^q)$ with $\mathbf{m}$ in $\mathcal{I}$ (resp. $\mathcal{I}_+$) and $q \in \mathbb{Z}$. Here $F^{\mathcal{I}}_{\mathbf{m}}$ is the free functor defined in (2.1).

4.3. **$\mathcal{I}$-model structures.** We now again use the homotopy colimit $P_{h\mathcal{I}}$ from Construction 2.6. A map $P \to Q$ in $\mathrm{Ch}_k^{\mathcal{I}}$ is an *$\mathcal{I}$-equivalence* if it induces a quasi-isomorphism $P_{h\mathcal{I}} \to Q_{h\mathcal{I}}$. Moreover, an $\mathcal{I}$-chain complex P is *absolute* (resp. *positive*) $\mathcal{I}$-fibrant if $\alpha_*\colon P(\mathbf{m}) \to P(\mathbf{n})$ is a quasi-isomorphism for all $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ (resp. $\mathcal{I}_+$).

Proposition 4.4. *The absolute (resp. positive) level model structure on* $\mathrm{Ch}_k^{\mathcal{I}}$ *admits a left Bousfield localization with fibrant objects the absolute (resp. positive) $\mathcal{I}$-fibrant objects. The weak equivalences in these two model structures coincide, and they are given by the $\mathcal{I}$-equivalences.*

Proof. Under the identification of $\mathcal{I}$-diagrams with generalized symmetric spectra (see [RS17, Proposition 9.1] or [PS, Proposition 3.3.9]), the existence of the model structures and the fact that they have the same weak equivalences follows from [PS, Theorem 3.3.4]. An alternative construction of the absolute $\mathcal{I}$-model structure results from [Dug01, Theorem 5.2]. Since the weak equivalences in the latter case are the maps that induce weak equivalences on the corrected homotopy colimits, the claim about $\mathcal{I}$-equivalences follows. □

We call these two model structures the *absolute* and *positive $\mathcal{I}$-model structures* on $\mathrm{Ch}_k^{\mathcal{I}}$ and their fibrations *absolute* and *positive $\mathcal{I}$-fibrations.*

Remark 4.5. Analogous to [SS12, Proposition 6.16], one can also give a direct construction of the $\mathcal{I}$-model structures without relying on an abstract existence theorem for left Bousfield localizations. This has been done by Joachimi [Joa11], and has the advantage of providing an explicit characterization of the $\mathcal{I}$-fibrations by a homotopy pullback condition like in [SS12, Proposition 3.2].

Corollary 4.6. *Let $f\colon P \to Q$ be a map between positive $\mathcal{I}$-fibrant objects.*

(i) If f is an $\mathcal{I}$-equivalence, then it is also a positive level equivalence.
(ii) If f is a positive level fibration, then it is also a positive $\mathcal{I}$-fibration.

Proof. This follows from [Hir03, Theorem 3.2.13 and Proposition 3.3.16]. □

We also note that the quasi-isomorphism type of $P_{h\mathcal{I}}$ can easily be read off for positive $\mathcal{I}$-fibrant P:

Lemma 4.7. *If P is positive $\mathcal{I}$-fibrant in $\mathrm{Ch}_k^{\mathcal{I}}$ and $\mathbf{m}$ is positive, then the inclusion of $\mathbf{m}$ in $\mathcal{I}$ induces a natural quasi-isomorphism $P(\mathbf{m}) \to P_{h\mathcal{I}}$.*

Proof. This follows for example from [Dug01, Proposition 5.4] since the inclusion $\mathcal{I}_+ \to \mathcal{I}$ is homotopy cofinal and $\mathcal{I}_+$ has contractible classifying space. □

As another consequence of [Dug01, Theorem 5.2], we note that the adjunction $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Ch}_k^{\mathcal{I}} \rightleftarrows \mathrm{Ch}_k\colon \mathrm{const}_{\mathcal{I}}$ is a Quillen equivalence when $\mathrm{Ch}_k^{\mathcal{I}}$ is equipped with the absolute or positive $\mathcal{I}$-model structure. In particular, the composite of

$$(\mathrm{const}_{\mathcal{I}} A)_{h\mathcal{I}} \to \mathrm{colim}_{\mathcal{I}}\, \mathrm{const}_{\mathcal{I}} A \to A \tag{4.1}$$

is always a quasi-isomorphism, and each P in $\mathrm{Ch}_k^{\mathcal{I}}$ is related by a zig-zag of $\mathcal{I}$-equivalences

$$\mathrm{const}_{\mathcal{I}}\, \mathrm{colim}_{\mathcal{I}}(P^{\mathrm{cof}}) \leftarrow P^{\mathrm{cof}} \to P \tag{4.2}$$

to a constant $\mathcal{I}$-diagram $\mathrm{colim}_{\mathcal{I}}(P)^{\mathrm{cof}}$ where $P^{\mathrm{cof}} \to P$ is a cofibrant replacement.

We record the following lemma for later use.

Lemma 4.8. *If $(P_j)_{j\in J}$ is a family of $\mathcal{I}$-chain complexes, then the canonical map*

$$\Big(\prod\nolimits_{j\in J} P_j\Big)_{h\mathcal{I}} \to \prod\nolimits_{j\in J} (P_j)_{h\mathcal{I}}$$

is a quasi-isomorphism provided that all P_j are positive $\mathcal{I}$-fibrant.

Proof. Arbitrary products of weak equivalences between fibrant objects in a model category are weak equivalences. Therefore, using that (4.2) is a zig-zag of $\mathcal{I}$-equivalences between positive $\mathcal{I}$-fibrant objects under our assumptions allows us to assume that each P_j is of the form $\mathrm{const}_{\mathcal{I}} A_j$. Forming the adjoint of the isomorphism $\prod_{j\in J} \mathrm{const}_{\mathcal{I}} A_j \xrightarrow{\cong} \mathrm{const}_{\mathcal{I}}\Big(\prod_{j\in J} A_j\Big)$ under the Quillen equivalence $(\mathrm{colim}_{\mathcal{I}}, \mathrm{const}_{\mathcal{I}})$ shows that the composite in

$$\Big(\prod\nolimits_{j\in J} \mathrm{const}_{\mathcal{I}} A_j\Big)_{h\mathcal{I}} \to \prod\nolimits_{j\in J} (\mathrm{const}_{\mathcal{I}} A_j)_{h\mathcal{I}} \xrightarrow{\sim} \prod\nolimits_{j\in J} A_j$$

is a quasi-isomorphism. Since the second map is a product of quasi-isomorphisms, the claim follows by two-out-of-three. □

4.9. **Commutative $\mathcal{I}$-dgas.** Although essentially only our formulation of Theorem 1.4 depends on the existence of a lifted model structure on $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, the following result is the main motivation for working with commutative $\mathcal{I}$-dgas.

Theorem 4.10. *The category $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ admits a* positive $\mathcal{I}$-model structure *where a map is an weak equivalence (or fibration) if the underlying map in the positive $\mathcal{I}$-model structure on $\mathrm{Ch}_k^{\mathcal{I}}$ is. With this model structure, $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is Quillen equivalent to the the category of E_∞ dgas and to the category of commutative Hk-algebra spectra.*

Proof. The existence of the model structure follows from [PS, Theorem 3.4.1], and the relation to commutative Hk-algebra spectra is the content of [RS17, Theorem 9.5]. □

For later use we note that the commutative $\mathcal{I}$-dga $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(A)$ from (2.3) has the following homotopical feature:

Lemma 4.11. *Let A be a cofibrant acyclic chain complex. Then each $(\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(A))(\mathbf{m})$ is cofibrant in Ch_k, and the unit $U^{\mathcal{I}} \to \mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(A)$ is an absolute level equivalence.*

Proof. We show that the summands in (2.3) indexed by $s \geq 1$ are acyclic and cofibrant. When $|\mathbf{m}| < s$, the indexing set $\mathcal{I}(\mathbf{1}^{\sqcup s}, \mathbf{m})$ is empty. If $|\mathbf{m}| \geq s$, the Σ_s-action on the set of injections $\mathcal{I}(\mathbf{1}^{\sqcup s}, \mathbf{m})$ is free. If R is a set of representatives of the orbits, we get an isomorphism $\bigoplus_R A^{\otimes s} \cong (\bigoplus_{\mathcal{I}(\mathbf{1}^{\sqcup s}, \mathbf{m})} A^{\otimes s})/\Sigma_s$, and the claim follows because each $A^{\otimes s}$ is cofibrant and acyclic. □

5. Comparison of cochain functors

We now define a simplicial $\mathcal{I}$-chain complex $B_\bullet^{\mathcal{I}}$ by setting $B_p^{\mathcal{I}} = A_p^{\mathcal{I}} \boxtimes C_p^{\mathcal{I}}$ in simplicial level p and using the $\boxtimes$-products of the simplicial structure maps of $A^{\mathcal{I}}$ and $C^{\mathcal{I}}$ as simplicial structure maps for $B_\bullet^{\mathcal{I}}$. There is a natural isomorphism

$$B_p^{\mathcal{I}}(\mathbf{m}) = (A_p^{\mathcal{I}} \boxtimes F_{\mathbf{0}}^{\mathcal{I}}(C_p))(\mathbf{m}) \cong A_p^{\mathcal{I}}(\mathbf{m}) \otimes C_p \tag{5.1}$$

that results from the definition of $\boxtimes$ as a left Kan extension.

The unit maps $U^{\mathcal{I}} \to C^{\mathcal{I}}$ and $U^{\mathcal{I}} \to A^{\mathcal{I}}$ induce a chain

$$A_\bullet^{\mathcal{I}} \to B_\bullet^{\mathcal{I}} \leftarrow C_\bullet^{\mathcal{I}} \tag{5.2}$$

of maps of simplicial objects in $\mathrm{Ch}_k^{\mathcal{I}}$. By Construction 3.2, this chain gives rise to a chain of natural transformations $A^{\mathcal{I}} \to B^{\mathcal{I}} \leftarrow C^{\mathcal{I}}$ of functors $(\mathrm{sSet})^{\mathrm{op}} \to \mathrm{Ch}_k^{\mathcal{I}}$.

Theorem 5.1. *For every simplicial set X, the maps $A^{\mathcal{I}}(X) \to B^{\mathcal{I}}(X) \leftarrow C^{\mathcal{I}}(X)$ are positive level equivalences between positive $\mathcal{I}$-fibrant objects.*

We prove the theorem at the end of the section.

Corollary 5.2. *If $X \to Y$ is a weak homotopy equivalence of simplicial sets, then $A^{\mathcal{I}}(Y) \to A^{\mathcal{I}}(X)$ is a positive level equivalence between positive $\mathcal{I}$-fibrant objects.*

Proof. The map $C^{\mathcal{I}}(Y) \to C^{\mathcal{I}}(X)$ is an $\mathcal{I}$-equivalence since $C(Y) \to C(X)$ is a quasi-isomorphism of chain complexes by the homotopy invariance of singular homology. By the theorem, the claim about $A^{\mathcal{I}}(Y) \to A^{\mathcal{I}}(X)$ follows. □

Lemma 5.3. *The maps in* (5.2) *are absolute level equivalences between absolute $\mathcal{I}$-fibrant objects when evaluated in simplicial degree p.*

Proof. Let $\mathbf{m}$ be an object in $\mathcal{I}$. By Lemma 4.11 the map $S^0 = U^{\mathcal{I}}(\mathbf{m}) \to A_p^{\mathcal{I}}(\mathbf{m})$ is a quasi-isomorphism between cofibrant and fibrant objects and thus even a chain homotopy equivalence. The map $S^0 \to C(\Delta^p) = C_p$ is a quasi-isomorphism by the known computation of $H^*(\Delta^p; k)$. Applying $F_{\mathbf{0}}^{\mathcal{I}}$, it provides an absolute level equivalence $U^{\mathcal{I}} \to C_p^{\mathcal{I}}$. By (5.1), we can decompose $U^{\mathcal{I}}(\mathbf{m}) \to B_p^{\mathcal{I}}(\mathbf{m})$ as

$$S^0 \to C_p \xrightarrow{\cong} S^0 \otimes C_p \to A_p^{\mathcal{I}} \otimes C_p \ .$$

We already showed that the first map is is a quasi-isomorphism. The last one is a quasi-isomorphism since $- \otimes C_p$ preserves chain homotopy equivalences. The $\mathcal{I}$-chain complexes $A_p^{\mathcal{I}}$, $B_p^{\mathcal{I}}$, and $C_p^{\mathcal{I}}$ are absolute $\mathcal{I}$-fibrant for each $p \geq 0$ since they absolute level equivalent to $U^{\mathcal{I}}$ and $U^{\mathcal{I}} = \mathrm{const}_{\mathcal{I}} S^0$ is absolute $\mathcal{I}$-fibrant. □

Lemma 5.4. *For all $q \in \mathbb{Z}$ and all positive objects $\mathbf{m}$ in $\mathcal{I}$, the simplicial k-module $(B_{\bullet,q}^{\mathcal{I}})(\mathbf{m})$ is contractible to* 0.

Proof. From (5.1) we get an isomorphism $B^{\mathcal{I}}_{\bullet,q} \cong \bigoplus_{r+s=q} A^{\mathcal{I}}_{\bullet,r} \otimes C_{\bullet,s}$ of simplicial k-modules. The sum over the tensor products of the extra degeneracies for $A^{\mathcal{I}}_{\bullet,r}(\mathbf{m})$ and $C_{\bullet,s}(\mathbf{m})$ from Lemma 3.6 and Lemma 3.9(ii) provide extra degeneracies for $(B^{\mathcal{I}}_{\bullet,q})(\mathbf{m})$. □

Lemma 5.5. *Let $D_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Ch}^{\mathcal{I}}_k$ be a simplicial object in $\mathcal{I}$-chain complexes such that for all $q \in \mathbb{Z}$ and all positive objects $\mathbf{m}$ in $\mathcal{I}$, the simplicial k-module $(D_{\bullet,q})(\mathbf{m})$ is contractible to 0. Then for all $p \geq 0$, the boundary inclusion $\partial\Delta^p \to \Delta^p$ induces a positive level fibration $D(\Delta^p) \to D(\partial\Delta^p)$.*

Proof. A map in $\mathrm{Ch}^{\mathcal{I}}_k$ is a positive level fibration if and only if it has the right lifting property against the maps $(U \to V) = F^{\mathcal{I}}_{\mathbf{m}}(0 \to D^q)$ with $\mathbf{m}$ positive and $q \in \mathbb{Z}$. By the adjunction (3.1), the lifting property for $U \to V$ and $D(\Delta^p) \to D(\partial\Delta^p)$ is equivalent to the lifting property for $\partial\Delta^p \to \Delta^p$ and $K_D(V) \to K_D(U)$. Inspecting the definition of K_D, it follows that asking the latter lifting property for all $p \geq 0$ is equivalent to asking the map of simplicial sets $\mathrm{Ch}^{\mathcal{I}}_k(V, D_\bullet) \to \mathrm{Ch}^{\mathcal{I}}_k(U, D_\bullet)$ to be an acyclic Kan fibration. Since $(F^{\mathcal{I}}_{\mathbf{m}}, \mathrm{Ev}_{\mathbf{m}})$ is an adjunction and since morphisms in Ch_k out of D^q correspond to level q elements, the assumption that $U \to V$ is $F^{\mathcal{I}}_{\mathbf{m}}(0 \to D^q)$ implies that $\mathrm{Ch}^{\mathcal{I}}_k(V, D_\bullet) \to \mathrm{Ch}^{\mathcal{I}}_k(U, D_\bullet)$ is isomorphic to $(D_{\bullet,q})(\mathbf{m}) \to 0$. The source of this map is contractible by assumption and a Kan complex because it is the underlying simplicial set of a simplicial k-module. Hence $(D_{\bullet,q})(\mathbf{m}) \to 0$ is an acyclic Kan fibration. □

Remark 5.6. When $(D_{\bullet,0})(\mathbf{0})$ is not contractible and $U \to V$ is $F^{\mathcal{I}}_{\mathbf{0}}(0 \to D^0)$, the map $\mathrm{Ch}^{\mathcal{I}}_k(V, D_\bullet) \to \mathrm{Ch}^{\mathcal{I}}_k(U, D_\bullet)$ considered in the previous proof is not an acyclic Kan fibration. Thus $D(\Delta^p) \to D(\partial\Delta^p)$ is not an absolute level fibration. In view of Remark 3.7, this shows that $A^{\mathcal{I}}(\Delta^p) \to A^{\mathcal{I}}(\partial\Delta^p)$ is not an absolute level fibration.

Proposition 5.7. *Let $D_\bullet \to D'_\bullet$ be a natural transformation of functors $\Delta^{\mathrm{op}} \to \mathrm{Ch}^{\mathcal{I}}_k$. Suppose that for all $p \geq 0$, the map $D_p \to D'_p$ is a positive level equivalence between positive $\mathcal{I}$-fibrant objects and that for all $q \in \mathbb{Z}$ and all positive objects $\mathbf{m}$ in $\mathcal{I}$, the simplicial k-modules $(D_{\bullet,q})(\mathbf{m})$ and $(D'_{\bullet,q})(\mathbf{m})$ are contractible. Then for every simplicial set X, the map $D(X) \to D'(X)$ is a positive level equivalence between positive $\mathcal{I}$-fibrant objects.*

Proof. As usual, this is proved by cell induction. Let us first assume that for all $p \geq 0$, the map $D(\partial\Delta^p) \to D'(\partial\Delta^p)$ is a positive level equivalence between positive $\mathcal{I}$-fibrant objects. Any simplicial set X can be written as a cell complex $X = \mathrm{colim}_{\lambda<\kappa} X_\lambda$ built from attaching cells of the form $\partial\Delta^p \to \Delta^p$. The functor D takes the inclusion $\partial\Delta^p \to \Delta^p$ to a positive level fibration by Lemma 5.5. Since we assume that $D(\partial\Delta^p)$ and $D_p \cong D(\Delta^p)$ are positive $\mathcal{I}$-fibrant, it follows from Corollary 4.6 that $D(\Delta^p) \to D(\partial\Delta^p)$ is a positive $\mathcal{I}$-fibration. The same holds for D'. Since both D and D' take colimits to limits by Lemma 3.3, the coglueing lemma in the positive level model structure and the fact that base change preserves $\mathcal{I}$-fibrations shows that $D(X) \to D'(X)$ arises as a limit of pointwise positive level equivalences between inverse systems of positive $\mathcal{I}$-fibrations. Hence it is a positive level equivalence between positive $\mathcal{I}$-fibrant objects.

Since $\partial\Delta^p$ only has non-degenerate simplices in dimensions strictly less than p, an analogous induction over the dimension of $\partial\Delta^p$ shows the remaining claim about $D(\partial\Delta^p) \to D'(\partial\Delta^p)$. □

Proof of Theorem 5.1. Combining Lemma 3.6, Corollary 3.10(ii), Lemma 5.4, and Lemma 5.3, the two maps $A^{\mathcal{I}} \to B^{\mathcal{I}}$ and $C^{\mathcal{I}} \to B^{\mathcal{I}}$ satisfy the hypotheses of Proposition 5.7. □

We can now also prove Theorem 1.4 from the introduction:

Proof of Theorem 1.4. The adjunction $(A^{\mathcal{I}}, K^{\mathcal{I}})$ arises from $A^{\mathcal{I}}_{\bullet}$ by applying Construction 3.2. Lemma 3.6 and Lemma 5.5 show that $A^{\mathcal{I}}$ sends cofibrations to positive level fibrations and thus to positive $\mathcal{I}$-fibrations. Corollary 5.2 implies that $A^{\mathcal{I}}$ sends weak homotopy equivalences to positive level equivalences and thus to $\mathcal{I}$-equivalences. The rest is an immediate consequence of the self-duality of model structures with respect to the passage to opposite categories and the adjunction isomorphisms (3.1). □

5.8. **The relation to polynomial forms.** By adjunction, the canonical map $D^0 \to (\mathrm{const}_{\mathcal{I}}\, \mathbb{C}D^0)(\mathbf{1})$ induces a map $\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}(D^0) \to \mathrm{const}_{\mathcal{I}}\, \mathbb{C}D^0$ of commutative $\mathcal{I}$-dgas. Using the description of $A_{\mathrm{PL},\bullet}$ as a two sided bar construction outlined in the introduction, it induces a map $A^{\mathcal{I}}_{\bullet} \to \mathrm{const}_{\mathcal{I}}\, A_{\mathrm{PL},\bullet}$ in $\mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}]$ and thus a natural map $A^{\mathcal{I}}(X) \to \mathrm{const}_{\mathcal{I}}\, A_{\mathrm{PL}}(X)$ on the extensions to simplicial sets.

Theorem 5.9. *Let k be a field of characteristic 0. Then $A^{\mathcal{I}}(X) \to \mathrm{const}_{\mathcal{I}}\, A_{\mathrm{PL}}(X)$ is a positive level equivalence. It induces a quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \to A_{\mathrm{PL}}(X)$ that is an $\mathcal{E}$-algebra map if we view the cdga $A_{\mathrm{PL}}(X)$ as an $\mathcal{E}$-algebra by restricting along the canonical operad map from $\mathcal{E}$ to the commutativity operad.*

Proof. In characteristic zero the homology groups of $(D^0)^{\otimes n}/\Sigma_n$ are isomorphic to the coinvariants $H_*(D^0)^{\otimes n}/\Sigma_n$ and the latter term is trivial for $n \geq 1$ because D^0 is acyclic. Therefore $\mathbb{C}F^{\mathcal{I}}_{\mathbf{1}}(D^0) \to \mathrm{const}_{\mathcal{I}}\, \mathbb{C}D^0$ is a positive level equivalence. The claim about general X follows from Proposition 5.7 and the contractibility property of $A_{\mathrm{PL},\bullet}$ established in [BG76, Proposition 1.1]. Applying $(-)_{h\mathcal{I}}$ to this positive level equivalence and composing with the natural quasi-isomorphism (4.1) gives the quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \to A_{\mathrm{PL}}(X)$. To see that it is an $\mathcal{E}$-algebra map, we note that it follows from the definitions that (4.1) is an $\mathcal{E}$-algebra map when evaluated on a cdga. □

6. Comparison of E_∞ structures

Let $\mathcal{E}$ be the Barratt–Eccles operad introduced in Definition 2.11. We now define $A\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}_k[\mathcal{E}]$ to be the composite $A = (A^{\mathcal{I}})_{h\mathcal{I}}$ of the functor $A^{\mathcal{I}}$ from the previous section and the functor $(-)_{h\mathcal{I}}\colon \mathrm{Ch}^{\mathcal{I}}_k[\mathcal{C}] \to \mathrm{Ch}_k[\mathcal{E}]$ resulting from Theorem 2.13. The following proposition shows that A is a *cochain theory* in the sense of [Man02].

Proposition 6.1. *The functor $A\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}_k[\mathcal{E}]$ has the following properties.*

(i) It sends weak equivalence of simplicial sets to quasi-isomorphisms.
(ii) For a sub-simplicial set $Y \subseteq X$, the induced map from $\mathrm{hofib}(A(X/Y) \to A(*))$ to $\mathrm{hofib}(A(X) \to A(Y))$ is a quasi-isomorphism.
(iii) For a family $(X_j)_{j\in J}$ of simplicial sets indexed by a set J, the canonical map $A(\coprod_{j\in J} X_j) \to \prod_{j\in J} A(X_j)$ is a quasi-isomorphism.
(iv) It satisfies $H_0(A(*)) \cong k$ and $H_n(A(*)) \cong 0$ if $n \neq 0$.

Proof. Part (i) follows from Corollary 5.2, part (iv) is an immediate consequence of Theorem 5.1, and part (iii) follows from Lemma 4.8 because $A^{\mathcal{I}}$ takes coproducts in sSet to products of fibrant objects in $\mathrm{Ch}^{\mathcal{I}}_k$.

For (ii), we view X/Y as the pushout of $* \leftarrow Y \to X$. The functor $A^{\mathcal{I}}$ sends this pushout to a pullback diagram

$$\begin{array}{ccc} A^{\mathcal{I}}(X/Y) & \longrightarrow & A^{\mathcal{I}}(X) \\ \downarrow & & \downarrow \\ A^{\mathcal{I}}(*) & \longrightarrow & A^{\mathcal{I}}(Y) \end{array} \tag{6.1}$$

We need to show that we get a homotopy cartesian square after applying $(-)_{h\mathcal{I}}$ to (6.1). Since all objects in (6.1) are positive fibrant, it follows from Lemma 4.7 that the resulting square of homotopy colimits over $\mathcal{I}$ is quasi-isomorphic to the square obtained from (6.1) by evaluating at **1**. The latter square is homotopy cartesian since it is a pullback in which the vertical maps are fibrations. □

Let $\mathcal{E}^{\mathrm{cof}}$ be a cofibrant E_∞ operad in the sense of [Man02, Definition 4.2]. Then there exists an operad map $\mathcal{E}^{\mathrm{cof}} \to \mathcal{E}$ to the Barratt–Eccles operad [Man02, Lemma 4.5], and by restricting along $\mathcal{E}^{\mathrm{cof}} \to \mathcal{E}$ we may view A as a functor to $\mathcal{E}^{\mathrm{cof}}$-algebras. On the other hand, the cosimplicial normalization functor for the category $\mathrm{Ch}_k[\mathcal{E}^{\mathrm{cof}}]$ provided by [Man02, Theorem 5.8] allows one to lift the ordinary cochain functor $C\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}_k$ to a functor with values in $\mathrm{Ch}_k[\mathcal{E}^{\mathrm{cof}}]$ (compare [Man02, §1]). We are now in a situation where [Man02, Main Theorem] applies:

Theorem 6.2. *The functor $A\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}_k[\mathcal{E}^{\mathrm{cof}}]$ is naturally quasi-isomorphic to the singular cochain functor $C\colon \mathrm{sSet}^{\mathrm{op}} \to \mathrm{Ch}_k[\mathcal{E}^{\mathrm{cof}}]$.* □

Remark 6.3. It is well-known how to express the cup-i products on the singular cohomology of spaces using the Barratt-Eccles operad, see for instance [BF04, Theorem 2.1.1]. This way the $\mathcal{E}$-algebra structure on $A(X) = A^{\mathcal{I}}(X)_{h\mathcal{I}}$ gives rise to cup-i products, and the previous theorem shows that they are equivalent to the usual cup-i products on the cochain algebra.

References

[BF04] C. Berger and B. Fresse, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 135–174.

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94.

[Dug01] D. Dugger, *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc. **353** (2001), no. 12, 5003–5027 (electronic).

[FHT01] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.

[Fre11] B. Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, Algebr. Geom. Topol. **11** (2011), no. 2, 747–838.

[Hir03] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[HS87] V. A. Hinich and V. V. Schechtman, *On homotopy limit of homotopy algebras*, K-theory, arithmetic and geometry (Moscow, 1984–1986), 1987, pp. 240–264.

[Joa11] R. Joachimi, *About a Quillen equivalence between chain complexes and diagrams of chain complexes*, 2011. Diplomarbeit, Universität Bonn.

[Kar09] M. Karoubi, *Cochaînes quasi-commutatives en topologie algébrique*, Pure and Applied Mathematics Quaterly **5** (2009), no. 1, 1 –68.

[Man02] M. A. Mandell, *Cochain multiplications*, Amer. J. Math. **124** (2002), no. 3, 547–566.

[Man06] ———, *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 213–246.

[ML63] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[May72] J. P. May, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics Vol. 271, Springer-Verlag, Berlin-New York, 1972.

[May74] ———, *E_∞ spaces, group completions, and permutative categories*, New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), 1974, pp. 61 –93.

[PS] D. Pavlov and J. Scholbach, *Symmetric operads in abstract symmetric spectra.* to appear in the Journal of the Institute of Mathematics of Jussieu, `arXiv:1410.5699`.

[Pir00] T. Pirashvili, *Hodge decomposition for higher order Hochschild homology*, Ann. Sci. École Norm. Sup. **33** (2000), no. 4, 151 –179.

[RS17] B. Richter and B. Shipley, *An algebraic model for commutative $H\mathbb{Z}$-algebras*, Algebr. Geom. Topol. **17** (2017), no. 4, 2013–2038.

[RG14] B. Rodríguez González, *Realizable homotopy colimits*, Theory Appl. Categ. **29** (2014), no. 22, 609 –634.

[Sch09] C. Schlichtkrull, *Thom spectra that are symmetric spectra*, Doc. Math. **14** (2009), 699–748.

[SS12] S. Sagave and C. Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. **231** (2012), no. 3-4, 2116–2193.

[Sul77] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269 –331.

Universität Hamburg, Germany
E-mail address: birgit.richter@uni-hamburg.de

Radboud Universiteit Nijmegen, The Netherlands
E-mail address: s.sagave@math.ru.nl