MODIFIED TRACE IS A SYMMETRISED INTEGRAL

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Abstract. A modified trace for a finite $\Bbbk$-linear pivotal category is a family of linear forms on endomorphism spaces of projective objects which has cyclicity and so-called partial trace properties. We show that a non-degenerate modified trace defines a compatible with duality Calabi-Yau structure on the subcategory of projective objects. The modified trace provides a meaningful generalisation of the categorical trace to non-semisimple categories and allows to construct interesting topological invariants. We prove, that for any finite-dimensional unimodular pivotal Hopf algebra over a field $\Bbbk$, a modified trace is determined by a symmetric linear form on the Hopf algebra constructed from an integral. More precisely, we prove that shifting with the pivotal element defines an isomorphism between the space of right integrals, which is known to be 1-dimensional, and the space of modified traces. This result allows us to compute modified traces for all simply laced restricted quantum groups at roots of unity.

1. Introduction

This paper establishes a one-to-one correspondence between two *a priori* very different notions in the theory of finite-dimensional pivotal Hopf algebras. One of them is the well-known linear form on the Hopf algebra H, called *integral*, and the other is a certain trace function on the category of projective H-modules, called *modified trace*. Let us introduce both of them.

Integral. The integral or dually cointegral can be thought as analogs of the Haar measure on a compact group and the invariant $\sum_{g\in G} g$ in the group algebra of a finite group, respectively. If non-zero, they generate one-dimensional ideals in the algebra and its dual. The integral has important topological applications. It plays the role of a Kirby color in the Hennings construction [He] of 3-manifold invariants generalizing those of Reshetikhin-Turaev.

Let $H = (H, m, \mathbf{1}, \Delta, \epsilon, S)$ be a Hopf algebra over a field $\Bbbk$. A *right integral* on H is a linear form $\boldsymbol{\mu}\colon H \to \Bbbk$ satisfying

$$(\boldsymbol{\mu} \otimes \mathrm{id})\Delta(x) = \boldsymbol{\mu}(x)\mathbf{1} \quad \text{for any} \quad x \in H. \tag{1.1}$$

Analogously, a *left integral* $\boldsymbol{\mu}^l \in H^*$ satisfies

$$(\mathrm{id} \otimes \boldsymbol{\mu}^l)\Delta(x) = \boldsymbol{\mu}^l(x)\mathbf{1} \quad \text{for any} \quad x \in H. \tag{1.2}$$

If H is finite-dimensional, the space of solutions of these equations is known to be 1-dimensional. A *pivotal* Hopf algebra is a pair $(H, \boldsymbol{g})$, where the pivot $\boldsymbol{g} \in H$ is a group-like element implementing S^2, i.e. $S^2(x) = \boldsymbol{g}x\boldsymbol{g}^{-1}$ for any $x \in H$.

A *symmetrised right* integral $\boldsymbol{\mu}_{\boldsymbol{g}}$ on $(H, \boldsymbol{g})$ is defined by

$$\boldsymbol{\mu}_{\boldsymbol{g}}(x) := \boldsymbol{\mu}(\boldsymbol{g}x) \quad \text{for any} \quad x \in H \ . \tag{1.3}$$

Analogously, a *symmetrised left* integral is

$$\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(x) := \boldsymbol{\mu}^l(\boldsymbol{g}^{-1}x) \quad \text{for any} \quad x \in H \ . \tag{1.4}$$

We call a pivotal Hopf algebra $(H, \boldsymbol{g})$ *unibalanced* if its symmetrised right integral is also left.

Dually, a left (resp. right) *cointegral* in H is an element $\boldsymbol{c} \in H$ such that $x\boldsymbol{c} = \epsilon(x)\boldsymbol{c}$ (resp. $\boldsymbol{c}x = \epsilon(x)\boldsymbol{c}$) for all $x \in H$. Non-trivial right and left cointegrals are unique up to scalar [LS]. We call a Hopf algebra *unimodular* if its right cointegral is also left.

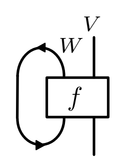
In the unimodular case, the symmetrised integrals define *symmetric* linear forms on H, i.e.

$$\boldsymbol{\mu}_{\boldsymbol{g}}(xy) = \boldsymbol{\mu}_{\boldsymbol{g}}(yx) \quad \text{and} \quad \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(xy) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(yx), \tag{1.5}$$

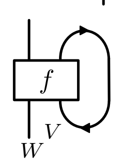
which are also non-degenerate (compare with Proposition 4.4 below).

Modified trace. Our second main player is the *modified trace* introduced in [GPV, GKP]. Unlike the integral, it is defined on the category of modules and motivated by topology. For braided pivotal categories, the modified trace allows a non-zero evaluation of the Reshetikhin-Turaev type invariants on links colored with projective objects, even if the category is not semisimple. We will work with pivotal categories without braiding assumptions and refer to Section 3 for detailed definitions and graphical conventions.

Let $\mathcal{C}$ be a $\Bbbk$-linear pivotal category. Given $V, W \in \mathcal{C}$ and $f \in \mathrm{End}_{\mathcal{C}}(W \otimes V)$, let $\mathrm{tr}^l_W(f)$ and $\mathrm{tr}^r_V(f)$ be the left and right *partial* traces defined as follows

(1.6) $\mathrm{tr}^l_W(f) = (\mathrm{ev}_W \otimes \mathrm{id}_V) \circ (\mathrm{id}_{W^*} \otimes f) \circ (\widetilde{\mathrm{coev}}_W \otimes \mathrm{id}_V) \ =$ 

$\in \ \mathrm{End}_{\mathcal{C}}(V)$,

(1.7) $\mathrm{tr}^r_V(f) = (\mathrm{id}_W \otimes \widetilde{\mathrm{ev}}_V) \circ (f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_W \otimes \mathrm{coev}_V) \ =$ 

$\in \ \mathrm{End}_{\mathcal{C}}(W)$.

The main example of a pivotal category used in this paper is the category H-mod of finite-dimensional left modules over a pivotal Hopf algebra $(H, \boldsymbol{g})$. In H-mod the left (co)evaluation morphisms are those for vector spaces while the right ones are defined using the pivot.

Setting $W = \mathbf{1}$ in (1.7) and assuming $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \Bbbk$, we get the definition of the (right) categorical trace

$$\mathrm{tr}^{\mathcal{C}}_V(f) := \widetilde{\mathrm{ev}}_V \circ (f \otimes \mathrm{id}) \circ \mathrm{coev}_V \ \in \ \Bbbk. \tag{1.8}$$

Analogously, assuming $V = \mathbf{1}$ in (1.6), we get its left version ${}^{\mathcal{C}}\mathrm{tr}_V(f)$.

We assume now that tensor product in $\mathcal{C}$ is exact and let $\mathsf{Proj}(\mathcal{C})$ be the tensor ideal of projective objects in $\mathcal{C}$. A *right (left) modified trace* on $\mathsf{Proj}(\mathcal{C})$ is a family of linear functions

$$\{\mathsf{t}_P\colon \operatorname{End}_{\mathcal{C}}(P) \to \Bbbk\}_{P\in\mathsf{Proj}(\mathcal{C})} \tag{1.9}$$

satisfying cyclicity and right (left) partial trace properties formulated below.

Cyclicity: If $P, P' \in \mathsf{Proj}(\mathcal{C})$ then for any morphisms $f\colon P \to P'$ and $g\colon P' \to P$

$$\mathsf{t}_P(g\circ f) = \mathsf{t}_{P'}(f\circ g)\ . \tag{1.10}$$

Right partial trace property: If $P \in \mathsf{Proj}(\mathcal{C})$ and $V \in \mathcal{C}$ then

$$\mathsf{t}_{P\otimes V}(f) = \mathsf{t}_P\big(\mathrm{tr}^r_V(f)\big) \tag{1.11}$$

for any $f \in \operatorname{End}_{\mathcal{C}}(P\otimes V)$.

Left partial trace property: If $P \in \mathsf{Proj}(\mathcal{C})$ and $V \in \mathcal{C}$ then

$$\mathsf{t}_{V\otimes P}(f) = \mathsf{t}_P\big(\mathrm{tr}^l_V(f)\big) \tag{1.12}$$

for any $f \in \operatorname{End}_{\mathcal{C}}(V\otimes P)$.

A left and right modified trace will be called *modified trace*.

It is then clear from the definition that the right categorical trace is also a right modified trace, and analogously for the left. The trace $\mathrm{tr}^{\mathcal{C}}$ is non-zero on $\mathsf{Proj}(\mathcal{C})$ if and only if $\mathcal{C}$ is semisimple. However, there are many examples of non-semisimple categories where a non-zero modified trace exists, and even non-degenerate, which we discuss below.

We call a right (left) modified trace t *non-degenerate* if the pairings

$$\operatorname{Hom}_{\mathcal{C}}(M,P)\times\operatorname{Hom}_{\mathcal{C}}(P,M)\to\Bbbk \quad , \quad (f,g)\mapsto \mathsf{t}_P(f\circ g)\ , \tag{1.13}$$

are non-degenerate for all $P \in \mathsf{Proj}(\mathcal{C})$ and $M \in \mathcal{C}$.[1]

For our main example $\mathcal{C} = H$-mod, $\mathsf{Proj}(\mathcal{C}) = H$-pmod is the full subcategory of projective H-modules.

Let us motivate the definition of the modified trace from a different perspective.

Modified trace and Calabi-Yau structure. Let $\mathcal{D}$ be a $\Bbbk$-linear category equipped with a family of *trace maps*, i.e. $\Bbbk$-linear maps

$$\{t_V\colon \operatorname{End}_{\mathcal{D}}(V)\to\Bbbk\}_{V\in\mathcal{D}} \tag{1.14}$$

satisfying the trace relation (or cyclicity)

$$t_V(g\circ f) = t_W(f\circ g)$$

for any $f\colon V\to W$ and $g\colon W\to V$ in $\mathcal{D}$. We say that $\mathcal{D}$ is *Calabi-Yau* if the following pairings

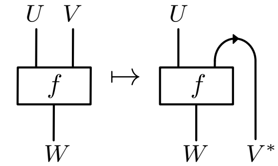
$$\operatorname{Hom}_{\mathcal{D}}(V,W)\times\operatorname{Hom}_{\mathcal{D}}(W,V)\to\Bbbk \quad , \quad (f,g)\mapsto t_W(f\circ g) \tag{1.15}$$

[1] We note that M is not necessarily projective and so the cyclicity property does not generally applies here.

are non-degenerate for all $V, W \in \mathcal{D}$.

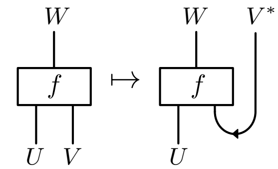
In any $\Bbbk$-linear pivotal category $\mathcal{D}$ we have the following duality isomorphisms:

$$d^{\cap}\colon \quad \begin{array}{c} \mathrm{Hom}_{\mathcal{D}}(W, U\otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(W\otimes V^*, U) \\ f \mapsto (\mathrm{id}_U \otimes \widetilde{\mathrm{ev}}_V)\circ(f\otimes \mathrm{id}_{V^*}) \end{array} ,$$



(1.16)

$$d_{\cup}\colon \quad \begin{array}{c} \mathrm{Hom}_{\mathcal{D}}(U\otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(U, W\otimes V^*) \\ f \mapsto (f\otimes \mathrm{id}_{V^*})\circ(\mathrm{id}_U \otimes \mathrm{coev}_V) \end{array} ,$$



Let $\mathcal{D}$ be a $\Bbbk$-linear pivotal category. We call a Calabi-Yau stucture on $\mathcal{D}$ *compatible with duality on the right* if the following diagram commutes, for all $U, V, W \in \mathcal{D}$,

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(U\otimes V, W)\times \mathrm{Hom}_{\mathcal{D}}(W, U\otimes V) \arrow[r, "\circ"] \arrow[d, "d_{\cup}\ \ d^{\cap}"] & \mathrm{End}_{\mathcal{D}}(U\otimes V) \arrow[d, "t_{U\otimes V}"'] \\ & \Bbbk \\ \mathrm{Hom}_{\mathcal{D}}(U, W\otimes V^*)\times \mathrm{Hom}_{\mathcal{D}}(W\otimes V^*, U) \arrow[r, "\circ"] & \mathrm{End}_{\mathcal{D}}(U) \arrow[u, "t_U"] \end{tikzcd} \tag{1.17}$$

We analogously define Calabi-Yau stucture on $\mathcal{D}$ compatible with duality on the *left*, see more details in Section 3. It is now easy to check that the right partial trace condition (1.11) formulated for the family (1.14) with $\mathcal{D} = \mathsf{Proj}(\mathcal{C})$ implies commutativity of (1.17), and similarly for the left property. We give a proof that the inverse is also true, in Theorem 3.3.

Main results. The previous discussion together with Theorem 3.3 imply that a non-degenerate modified trace on $\mathsf{Proj}(\mathcal{C})$ is nothing else but a Calabi-Yau structure on $\mathsf{Proj}(\mathcal{C})$ compatible with duality. For a finite-dimensional pivotal Hopf algebra H, such Calabi-Yau structure on H-pmod is uniquely determined by the non-degenerate symmetric linear form $\mathsf{t}_H\colon \mathrm{End}_H(H)\to\Bbbk$ associated with the left regular representation. This is proven in Proposition 2.4 and Theorem 2.6 in a more general setting.

We are now ready to formulate our main result.

Theorem 1. *Let $(H, \boldsymbol{g})$ be a finite-dimentional unimodular pivotal Hopf algebra over a field $\Bbbk$. Then the space of right (left) modified traces on* H-pmod *is equal to the space of symmetrised right (left) integrals, and hence is 1-dimensional. Moreover, the right modified trace on* H-pmod *is non-degenerate and determined by*

$$\mathsf{t}_H(f) = \boldsymbol{\mu}_{\boldsymbol{g}}\big(f(\mathbf{1})\big) \quad \textit{for any} \quad f\in \mathrm{End}_H(H)\ . \tag{1.18}$$

Analogously, the left modified trace is non-degenerate and determined by

$$\mathsf{t}_H(f) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}\big(f(\mathbf{1})\big) \quad \textit{for any} \quad f \in \mathrm{End}_H(H) \ . \tag{1.19}$$

In particular, H is unibalanced if and only if the right modified trace is also left.

In the language of Calabi-Yau categories, Theorem 1 can be reformulated as follows.

Corollary 1.1. *If H is a finite-dimensional unimodular pivotal Hopf algebra, then the space of Calabi-Yau structures on H*-pmod *compatible with duality on the right (left) is one dimensional.*

To the best of our knowledge, Theorem 1 is the first result relating modified traces with general concepts in the theory of Hopf algebras. The power of this theorem is in the generality of its assumptions. So far the existence and uniqueness of the modified trace was proven in [GR] for finite pivotal and braided categories with a non-degenerate monodromy (called factorisable), see also [GKP, Cor. 3.2.1] for a more technical statement. The equality of the right and left modified traces was known in the ribbon case only. However, Theorem 1 does not require braiding and allows to compute the modified trace in all cases where the integral and pivot are known. We give few infinite families of unimodular Hopf algebras with explicit formulas for the integral and pivots.

To prove Theorem 1 we first show that the right partial trace property for the regular representation implies the general property in (1.11), and similarly for the left property. This is the context of the so-called Reduction Lemma that is proven in Section 3 in the general context of finite pivotal $\Bbbk$-linear categories.

Then we study the centralizer algebras $\mathrm{End}_H(H^{\otimes k})$ for $k \geq 1$. In Section 5 for any n-dimensional Hopf algebra H, we construct an explicit algebra isomorphism between $\mathrm{End}_H(H\otimes H)$ and $\mathrm{Mat}_{n,n}(H^{\mathrm{op}})$, which allows us to reduce the right partial trace property to the defining relation for the symmetrised right integral. The proof uses graphical calculus.

It is worth to mention the following consequence of Theorem 1.

Proposition 1.2. *Let H be a finite-dimensional unimodular pivotal Hopf algebra over a field $\Bbbk$. The right categorical trace $\mathrm{tr}^{\mathcal{C}}_H$ and its left version ${}^{\mathcal{C}}\mathrm{tr}_H$ are non zero if and only if H*-mod *is semisimple and in this case coincide up to a scalar with the trace maps*

$$f \mapsto \boldsymbol{\mu}_{\boldsymbol{g}}\big(f(\mathbf{1})\big) \qquad \textit{and} \qquad f \mapsto \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}\big(f(\mathbf{1})\big) \ , \tag{1.20}$$

respectively, where $f \in \mathrm{End}_H(H)$.

In Section 4 we give a Hopf-theoretic proof of Proposition 1.2 without using Theorem 1.

Proposition 1.2 shows that the symmetrised integral $\boldsymbol{\mu}_{\boldsymbol{g}}$ provides a non-trivial generalisation of the categorical trace for non-semisimple categories H-pmod. In this case, the categorical trace is identically zero, however the symmetrised integral (or rather the corresponding modified trace) is not. In particular, for a finite group G and its group algebra over $\Bbbk = \mathbb{F}_p$, the symmetrised integral, which is in this case just the integral, defines a non-degenerate trace compatible with duality on the category of projective $\mathbb{F}_p[G]$-modules, even in the case when the characteristic p divides the order of the group. This is a surprising application of our theorem to the classical modular representation theory, that will be discussed in more details in Section 4.

In Section 7, we consider finite-dimensional Lusztig quantum groups at roots of unity in the simply laced cases and give explicit formulas for their integral, cointegral, symmetrised integral, and hence an explicit expression for the modified trace t_H. We expect similar formulas to hold in general type.

In type A_1, using Theorem 1 together with formulas for minimal idempotents given in [GT], we obtain an alternative derivation of [BBG] formulas for the modified trace for all endomorphisms of indecomposable projectives. This illustrates how the modified trace can be explicitly computed from the symmetrised integral.

In [BBG], combining the modified trace on the finite-dimensional restricted quantum $\mathfrak{sl}(2)$ at a root of unity with the Hennings construction, a *logarithmic Hennings* invariant was defined for any 3-manifold with a colored link inside. An interesting feature of this construction is that it works for a not necessarily quasi-triangular Hopf algebra. The results of this paper suggest that the invariants of [BBG] can be extended to finite-dimensional Lusztig quantum groups at a root of unity which might not allow braiding.

The paper is organised as follows. In Section 2, we collect results on traces in finite abelian categories. In Section 3, we study a relationhsip between the modified trace and Calabi-Yau structures in finite pivotal categories and prove Reduction Lemma. In Section 4, after recalling standard facts from the theory of Hopf algebras, we study properties of symmetrised integrals, in particular we show that they provide a non-degenerate symmetric pairing between the center $Z(H)$ and $\mathrm{HH}_0(H)$, and then prove Proposition 1.2. Section 5 contains a detailed analysis of the centralizer algebras on tensor powers of the regular representation. Section 6 contains our proof of Theorem 1. Section 7 provides an application of our main theorem to restricted quantum groups of types ADE: we compute the modified trace via a calculation of $\boldsymbol{\mu_g}$. Then in Section 8 we provide more detailed analysis for $\mathfrak{sl}_2$ case. Finally, Appendices contain proofs of several lemmas.

Acknowledgements. The authors are grateful to NCCR SwissMap for generous support and to Nathan Geer, Bertrand Patureau, Marco de Renzi and Ingo Runkel for helpful discussions. The authors are also thankful to the organizers of conference “Invariants in low-dimensional geometry & topology” in Toulouse in May, 2017, where a substantial part of this work was done. CB and AMG also thank Institute of Mathematics in Zurich University for kind hospitality during 2017. AMG is supported by CNRS and also thanks the Humboldt Foundation for a partial financial support.

2. Traces on finite categories

Throughout this section A is a finite-dimensional $\Bbbk$-algebra. Our aim is to show that any symmetric linear form t on A determines a family of trace functions on A-pmod

$$\{t_P\colon \operatorname{End}_A(P) \to \Bbbk\}_{P\in A\text{-pmod}}\ , \tag{2.1}$$

i.e. linear maps satisfying cyclicity (1.10). We will also show that if $t \in A^*$ is non-degenerate, then the traces (2.1) are non-degenerate in the sense of (1.13).

General Setting. We assume that $\Bbbk$ is a field and $\mathcal{C}$ is an additive category. We call $\mathcal{C}$ $\Bbbk$-linear if $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is a vector space over $\Bbbk$ for all $X, Y \in \mathcal{C}$ and the composition of morphisms is $\Bbbk$-bilinear. All categories used in this paper are assumed to be $\Bbbk$-linear.

An abelian category $\mathcal{C}$ is called finite if it is equivalent to the category A-mod of finite-dimensional left A-modules for some finite-dimensional $\Bbbk$-algebra A. In other words, $\mathcal{C}$ is abelian and has finitely many isomorphism classes of simples, length of any object is finite, it has enough projectives and Hom spaces are finite-dimensional. An algebra A can be constructed as $\operatorname{End}_{\mathcal{C}}(G)$ for a projective generator $G \in \mathcal{C}$, see e.g. [DK]. Then, the equivalence functor $\operatorname{Hom}_{\mathcal{C}}(-,G)\colon \mathcal{C} \to A$-mod sends G to the regular representation A. Therefore, without loss of generality in this section we will assume that $\mathcal{C} = A$-mod. We will also use the notation A-pmod for the full subcategory of projective A-modules.

We first show that a family of traces (2.1) on A-pmod defines a symmetric linear form on A. Let us denote by A^{op} the algebra with the opposite multiplication.

Lemma 2.1. *We have the isomorphism of algebras*

$$r\colon A^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_A(A) \tag{2.2}$$

given by

$$r(x) = r_x, \qquad r^{-1}(f) = f(\mathbf{1}) \tag{2.3}$$

where by r_x we denote the right multiplication with x.

Proof. It is straightforward to check that the maps r and r^{-1} defined in (2.2) are inverse to each other. Moreover, for any $x, y \in A$ we have $r(xy) = r_y r_x$ and for any $f, g \in \operatorname{End}_A(A)$, $r^{-1}(gf) = (gf)(\mathbf{1}) = f(\mathbf{1})g(\mathbf{1})$, where in the last equality we used the intertwining property of g. □

Suppose we are given trace functions (2.1). Then, in particular, for the regular A-module A, we have the trace function $t_A\colon \operatorname{End}_A(A) \to \Bbbk$. Lemma 2.1 shows that t_A defines a symmetric linear form t on A^{op}. Since the flip of multiplication is irrelevant in the argument of a symmetric form, we have $t \in A^*$.

To argue that the converse is also true: given a symmetric form $t \in A^*$ we can extend it uniquely to a family of traces on A-pmod, we will need a categorical notion of the 0^{th}-Hochschild homology.

Traces of categories. The 0^{th}-Hochschild homology or trace of a $\Bbbk$-linear category $\mathcal{C}$ is defined by

$$\mathrm{HH}_0(\mathcal{C}) := \frac{\bigoplus_{X\in\mathcal{C}} \mathrm{End}_{\mathcal{C}}(X)}{[\mathcal{C},\mathcal{C}]} \tag{2.4}$$

where

$$[\mathcal{C},\mathcal{C}] := \mathrm{Span}\{f\circ g - g\circ f \mid f\in \mathrm{Hom}_{\mathcal{C}}(X,Y),\ g\in\mathrm{Hom}_{\mathcal{C}}(Y,X),\ X,Y\in\mathcal{C}\}\ .$$

The image of $f\in\mathrm{End}_{\mathcal{C}}(X)$ in $\mathrm{HH}_0(\mathcal{C})$ will be called its trace class and denoted by $[X,f]$ or simply by $[f]$.

In particular, 0^{th}-Hochschild homology of an algebra A (viewed as a category with one object) is

$$\mathrm{HH}_0(A) := \frac{A}{[A,A]} \quad \text{with} \quad [A,A] = \mathrm{Span}\{xy-yx \mid x,y\in A\}. \tag{2.5}$$

Again the image of $x\in A$ in $\mathrm{HH}_0(A)$ will be called its trace class and denoted by $[x]$.

Actually, $\mathrm{HH}_0(A)$ and $\mathrm{HH}_0(A\text{-pmod})$ are isomorphic. To show this we will need some preparation.

Lemma 2.2. *For any projective A-module P there exists a decomposition of the identity:*

$$\mathrm{id}_P = \sum_{i\in I} a_i\circ \mathrm{id}_A\circ b_i \tag{2.6}$$

for some finite set I and morphisms $a_i\colon A\to P$ and $b_i\colon P\to A$.

Proof. Recall that any finitely generated projective A-module P splits as a direct sum of indecomposables ones:

$$P\simeq \bigoplus_{i\in I} P_i\ , \tag{2.7}$$

for a finite indexing set I. Here several direct summands can be isomorphic. We further observe that each indecomposable P_i can be realised as a direct summand in A, since the regular module A is a projective generator of A-pmod. We have therefore an injective map $x_i\colon P_i\hookrightarrow A$ and a surjective map $y_i\colon A\twoheadrightarrow P_i$. Fixing these maps such that $y_i\circ x_i = \mathrm{id}_{P_i}$ we can define $a_i\colon A\to P$ and $b_i\colon P\to A$ as the compositions

$$a_i\colon A\xrightarrow{y_i} P_i\hookrightarrow P \quad \text{and} \quad b_i\colon P\twoheadrightarrow P_i\xrightarrow{x_i} A\ . \tag{2.8}$$

They clearly satisfy (2.6). □

We note that the decomposition of id_P in (2.6) is not unique, and we provide several examples.

Example. For $P = A$, we can use the trivial decomposition $a_1 = b_1 = \mathrm{id}_A$. However we can make another choice, the one corresponding to a_i and b_i in the proof of Lemma 2.2: $a_i = b_i \colon x \mapsto x\pi_i$, for $x \in A$ and here π_i is the primitive idempotent corresponding to the direct summand P_i in the decomposition $A = \oplus_{i=1}^l P_i$. The identity (2.6) is clearly satisfied because $\sum_{i=1}^l \pi_i = \mathbf{1}$.
In a more general case of $P = A^{\oplus m}$, we also have two natural decompositions. For the first one, we set $a_j \colon A \hookrightarrow A^{\oplus m}$ and $b_j \colon A^{\oplus m} \twoheadrightarrow A$ such that $b_i \circ a_j = \delta_{i,j}\,\mathrm{id}_A$, then (2.6) holds. For the other choice, let V be the m-dimensional multiplicity space with a basis e_j, $1 \le j \le m$, and we can then define for each pair $i = (k,j)$ the maps a_i, b_i as

$$\begin{aligned} a_{(k,j)} \colon A &\to A^{\oplus m} , & b_{(k,j)} \colon A^{\oplus m} &\to A , \\ x &\mapsto x\pi_k \otimes e_j , & x \otimes e_n &\mapsto \delta_{n,j}\, x\pi_k , \end{aligned} \tag{2.9}$$

for $x \in A$, and $1 \le k \le l$ and $1 \le n, j \le m$. It is then straightforward to check the identity (2.6) on $x \otimes e_n$ for any $x \in A$ and $1 \le n \le m$.

Proposition 2.3. *For a finite-dimensional algebra A, there is an isomorphism*

$$\begin{aligned} \Phi : \mathrm{HH}_0(A) &\xrightarrow{\sim} \mathrm{HH}_0(A\text{-pmod}) , \\ [x] &\mapsto [r_x] \end{aligned} \tag{2.10}$$

with the inverse map

$$\begin{aligned} \Psi : \mathrm{HH}_0(A\text{-pmod}) &\xrightarrow{\sim} \mathrm{HH}_0(A) , \\ [P,f] &\mapsto \sum_{i \in I} [(b_i \circ f \circ a_i)(\mathbf{1})] , \end{aligned} \tag{2.11}$$

for any sets $\{a_i \colon A \to P\}_{i\in I}$ and $\{b_i \colon P \to A\}_{i \in I}$ satisfying (2.6)*.*

We provide the proof in Appendix A for completeness.

Proposition 2.4. *A symmetric linear form t on a finite-dimensional algebra A extends uniquely to a family of trace maps $\{t_P \colon \mathrm{End}_A(P) \to \Bbbk\}_{P \in A\text{-pmod}}$ where*

$$t_P(f) = \sum_{i=1}^k t\big((b_i \circ f \circ a_i)(\mathbf{1})\big) , \qquad f \in \mathrm{End}_A(P) , \tag{2.12}$$

for a given decomposition of id_P *as in* (2.6)*. In particular, we have*

$$t_A(r_x) = t(x) , \qquad x \in A . \tag{2.13}$$

Proof. We first note that there is a bijection between linear forms on $\mathrm{HH}_0(A\text{-pmod})$ and families of trace maps $\{t_P \colon \mathrm{End}_A(P) \to \Bbbk\}_{P \in A\text{-pmod}}$ such that $t_P(f) = l([P,f])$ for a linear form l. A symmetric linear form $t\colon A \to \Bbbk$ provides a linear form on $\mathrm{HH}_0(A)$ which we also denote by t. By Proposition 2.3, this defines a linear form on $\mathrm{HH}_0(A\text{-pmod})$ by the formula

$$t_P(f) = t \circ \Psi\big([f]\big) \tag{2.14}$$

for any $f \in \mathrm{End}_A(P)$ and Ψ given in (2.11). Since Ψ is an isomorphism and it does not depend on the choice of the decomposition of id_P, we have the existence and uniqueness of the extension. Finally, the equality (2.13) is straightforward after using (2.10). □

We remark that a result similar to Proposition 2.4 was also proven in [GR, proof of Prop. 5.8 (1)] (however in the case of non-degenerate traces).

Example. We assume here that $P = A^{\oplus m}$ and demonstrate the use of the formula (2.12). The algebra of A-invariant endomorphisms of $A^{\oplus m}$ can be rewritten as a matrix algebra:

$$\mathrm{End}_A(A^{\oplus m}) \cong \mathrm{Mat}_{m,m}(A^{\mathrm{op}}) \tag{2.15}$$

where $\mathrm{Mat}_{m,m}$ is the $m \times m$ matrix algebra and we used Lemma 2.1. With notation as in (2.9), the isomorphism (2.15) sends a matrix (h_{ij}) to the endomorphism $x \otimes e_j \mapsto \sum_{r=1}^m x h_{rj} \otimes e_r$. Let us choose a_i and b_i as in (2.9). From (2.12), we then obtain the unique extension $t_{A^{\oplus m}}$ of the symmetric form t

$$t^{\oplus m}(h) := t_{A^{\oplus m}}(h) = \sum_{i=1}^m t(h_{ii}) \ , \qquad h \in \mathrm{End}_A(A^{\oplus m}) \ , \tag{2.16}$$

where we used cyclicity of t and on RHS we identified h with the corresponding element in $\mathrm{Mat}_{m,m}(A^{\mathrm{op}})$ under the isomorphism in (2.15).

Remark 2.5. For an indecomposable projective A-module P, we can reformulate Proposition 2.4 in the following way. Let us fix an injection $j\colon P \hookrightarrow A$ and projection $p\colon A \twoheadrightarrow P$ such that $p \circ j = \mathrm{id}_P$ – this identity provides a decomposition as in (2.6). Then $j \circ p \in \mathrm{End}_A(A)$ is right multiplication by a primitive idempotent π, and so $t_P(\mathrm{id}_P) = t(\pi)$.
If $P \in A$-pmod is not necessarily indecomposable, then it can be realised as a direct summand of $A^{\oplus m}$ for some finite $m \in \mathbb{Z}_{>0}$, i.e. we have injective and surjective maps:

$$j_P\colon\ P \hookrightarrow A^{\oplus m} \ , \qquad p_P\colon\ A^{\oplus m} \twoheadrightarrow P \tag{2.17}$$

such that the composition $p_P \circ j_P$ is identity on P and $j_P \circ p_P$ is an idempotent in $\mathrm{End}_A(A^{\oplus m})$. We then get a decomposition of the form (2.6) with

$$\tilde{a}_{(k,j)}\colon A \xrightarrow{a_{(k,j)}} A^{\oplus m} \xrightarrow{p_P} P \quad \text{and} \quad \tilde{b}_{(k,j)}\colon P \xrightarrow{j_P} A^{\oplus m} \xrightarrow{b_{(k,j)}} A \ , \tag{2.18}$$

while $a_{(k,j)}$ and $b_{(k,j)}$ are defined as in (2.9). Then (2.12) for the choice (2.18) gives the following expression for t_P:

$$t_P\colon\ f \mapsto t^{\oplus m}\big(j_P \circ f \circ p_P\big) \ , \tag{2.19}$$

with $t^{\oplus m}$ defined in (2.16). For certain proofs below it will be more convenient to use the decomposition $\mathrm{id}_P = p_P \circ j_P$ instead of (2.6) and this expression of t_P. It is a consequence of Proposition 2.4 that the map (2.19) does not depend on the choices we made in the construction.

Non-degeneracy. Let us prove the equivalence of the different notions of non-degeneracy.

For a finite-dimensional algebra A over a field $\Bbbk$, we call a linear form $t \in A^*$ *non-degenerate* if the associated bilinear pairing $(x, y) \mapsto t(xy)$ is non-degenarate, i.e. $t(xy) = 0$ for all $x \in A$ implies $y = 0$.

Theorem 2.6. *For a finite-dimensional algebra A with a symmetric linear form $t \in A^*$ the following three statements are equivalent:*

(1) *t is non-degenerate.*
(2) A-pmod *is Calabi-Yau with t_P defined by* (2.19).
(3) *The pairings* (1.13)

$$\mathrm{Hom}_A(M, P) \times \mathrm{Hom}_A(P, M) \to \Bbbk \quad , \quad (f, g) \mapsto t_P(f \circ g)$$

*are non-degenerate for all $P \in A$-*pmod *and $M \in A$-*mod.

Proof. The equivalence of the first two statements was proven in [GR, Prop. 5.8]. Since the third statement is the strongest, it is enough to show that it follows from the first one. For that we need to show that for any $f : M \to P$ there exists a non-zero map $g : P \to M$ such that $t_P(f \circ g) \neq 0$. The idea is to use non-degeneracy of the linear form $t^{\oplus m}$. Let us fix a projective cover P_M of M with the canonical surjective map $\pi_M : P_M \twoheadrightarrow M$. Since any projective module is a direct summand of a projective generator, say $A^{\oplus m}$ for some m, we have surjective and injective maps:

$$p_M : A^{\oplus m} \twoheadrightarrow P_M \quad \text{and} \quad j_M : P_M \hookrightarrow A^{\oplus m}.$$

Let us consider the surjective map $\tilde{p}_M = \pi_M \circ p_M : A^{\oplus m} \twoheadrightarrow M$. By assumption f is non-zero and therefore the composition $j_P \circ f \circ \tilde{p}_M \in \mathrm{End}_A(A^{\oplus m})$ is non-zero too, because $\tilde{p}_M$ is surjective and j_P is injective. Since $t^{\oplus m}$ is non-degenerate, there should be non-zero $\tilde{g} \in \mathrm{End}_A(A^{\oplus m})$ such that

$$t^{\oplus m}\big((j_P \circ f \circ \tilde{p}_M) \circ \tilde{g}\big) \neq 0 \, . \tag{2.20}$$

We set $g = \tilde{p}_M \circ \tilde{g} \circ j_P : P \to M$ and check using (2.19) the non-degeneracy of t_P:

$$\begin{aligned} t_P(f \circ g) &= t^{\oplus m}\big(j_P \circ f \circ (\tilde{p}_M \circ \tilde{g} \circ j_P) \circ p_P\big) \\ &= t^{\oplus m}\big(j_P \circ p_P \circ j_P \circ f \circ \tilde{p}_M \circ \tilde{g}\big) = t^{\oplus m}\big(j_P \circ f \circ \tilde{p}_M \circ \tilde{g}\big) \neq 0 \end{aligned} \tag{2.21}$$

where in the second equality we used cyclicity of $t^{\oplus m}$ and in the third the identity $p_P \circ j_P = \mathrm{id}_P$, and finally we used (2.20). This also shows that the map g is non-zero. This calculation finishes the proof of non-degeneracy of the family t_P. □

3. Modified trace and Calabi-Yau structure

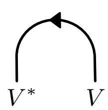
In this section for a finite pivotal category $\mathcal{C}$ we prove Reduction Lemma and show that a Calabi-Yau structure on $\mathsf{Proj}(\mathcal{C})$ provides a non-degenerate modified trace if and only if a compatibility between the Calabi-Yau structure and duality holds. Recall that $\mathsf{Proj}(\mathcal{C})$ denotes the tensor ideal of projective modules in $\mathcal{C}$.

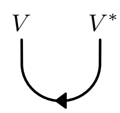
Pivotal structure. A category $\mathcal{C}$ is pivotal if $\mathcal{C}$ is a monoidal category with left duality equipped with a monoidal natural isomorphism $\delta\colon \mathrm{id}_{\mathcal{C}} \to (-)^{**}$ between the identity functor and the double duality functor and the corresponding isomorphisms satisfy $\delta_{V^*} = (\delta_V^*)^{-1}$ for $V \in \mathcal{C}$.

The pivotal structure allows to define right duality. Right dual objects are identified with the left ones, and the right (co)evaluation maps are defined as

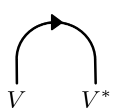
(3.1)
$$\begin{aligned} \widetilde{\mathrm{ev}}_V &:= \mathrm{ev}_{V^*} \circ (\delta_V \otimes \mathrm{id}_{V^*})\colon \qquad V \otimes V^* \to \mathbf{1}\ , \\ \widetilde{\mathrm{coev}}_V &:= (\mathrm{id}_{V^*} \otimes \delta_V^{-1}) \circ \mathrm{coev}_{V^*}\colon \quad \mathbf{1} \to V^* \otimes V\ . \end{aligned}$$

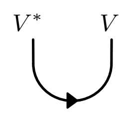
For the left and right (co)evaluation maps we will use the following diagrammatical notations:

(3.2) $\mathrm{ev}_V =$ 

, $\mathrm{coev}_V =$ 

,

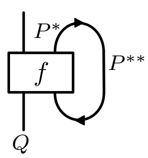
$\widetilde{\mathrm{ev}}_V =$ 

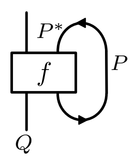
, $\widetilde{\mathrm{coev}}_V =$ 

.

We recall the definition of the right and left partial traces in (1.7). They have the following property.

Lemma 3.1. *Let $\mathcal{C}$ be a pivotal category and $Q, P \in \mathcal{C}$, we have then the equality*

(3.3) 

$=$ 

for any $f \in \mathrm{End}_{\mathcal{C}}(Q \otimes P^*)$, and similarly for the left partial trace of f.

Proof. We factorise $\mathrm{id}_{P^{**}} = \delta_P \circ \delta_P^{-1}$ using pivotal isomorphisms and use (3.1) to reverse arrows. $\square$

We call an abelian category $\mathcal{C}$ *finite pivotal* if $\mathcal{C}$ is a finite tensor category in the sense of [EGNO], i.e. (1) if $\mathcal{C}$ is finite as an abelian category, (2) if it is a rigid monoidal category with $\Bbbk$-bilinear and bi-exact tensor product functor, and (3) if its tensor unit is simple; and if $\mathcal{C}$ has a pivotal structure.

Reduction Lemma. Let us prove Reduction Lemma mentioned in Introduction, which says that to verify the right or left partial trace property, it is enough to check it on a projective generator. Below is the exact statement, recall also Proposition 2.4.

Lemma 3.2. *Given a finite pivotal category $\mathcal{C}$ and a projective generator $G \in \mathcal{C}$, a symmetric linear form $t \in A^*$, where $A := \mathrm{End}_{\mathcal{C}}(G)$, extends to a right modified trace on $\mathsf{Proj}(\mathcal{C})$ if and only if*

$$t_{G\otimes G}(f) = t_G\big(\mathrm{tr}^r_G(f)\big), \qquad \text{for all} \quad f \in \mathrm{End}_{\mathcal{C}}(G\otimes G). \tag{3.4}$$

Analogously, t extends to a left modified trace on $\mathsf{Proj}(\mathcal{C})$ if and only if

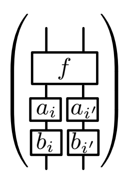
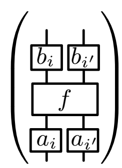
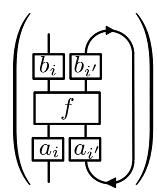
$$t_{G\otimes G}(f) = t_G\big(\mathrm{tr}^l_G(f)\big), \qquad \text{for all} \quad f \in \mathrm{End}_{\mathcal{C}}(G\otimes G). \tag{3.5}$$

Proof. Only one direction is not obvious. By Proposition 2.4, the symmetric form $t \in A^*$ extends uniquely to a family of linear maps $t_P\colon \mathrm{End}_{\mathcal{C}}(P) \to \Bbbk$, for $P \in \mathsf{Proj}(\mathcal{C})$, which satisfies the cyclicity property. We need to check the right partial trace property.

We first prove (1.11) for a pair of projective objects. Assume $P, P' \in \mathsf{Proj}(\mathcal{C})$ and $f \in \mathrm{End}_{\mathcal{C}}(P\otimes P')$. We have finite sum decompositions of the identities as in (2.6):[2]

$$\mathrm{id}_P = \sum_{i\in I} a_i \circ \mathrm{id}_G \circ b_i\ , \qquad \mathrm{id}_{P'} = \sum_{i'\in I'} a_{i'} \circ \mathrm{id}_G \circ b_{i'}\ . \tag{3.6}$$

We can now calculate $t_{P\otimes P'}(f)$ in terms of $t_{G\otimes G}$ by inserting these identities and using the cyclicity. Indeed,

(3.7) $t_{PP'}(f) \ = \ t_{PP'}$  $\overset{\text{cycl.}}{=}\ t_{GG}$  $\overset{(3.4)}{=}\ t_G$  $\overset{*}{=}\ t_P\big(\mathrm{tr}^r_{P'}(f)\big)$

where we omit the tensor product symbol in the index of t for brevity, and the summation is assumed over the repeated indices, i.e. over $i \in I$ and $i' \in I'$. In the step $(*)$ we used first the standard manipulations with dual maps to move $b_{i'}$ around the loop and then applied (3.6), and finally applied the cyclicity property of t_G using again (3.6).

We have thus established the right partial trace property of t in the case where both objects are projective. Now assume $P \in \mathsf{Proj}(\mathcal{C})$ and $V \in \mathcal{C}$. Then we set $\hat{P} := P\otimes V$ which is in $\mathsf{Proj}(\mathcal{C})$ due to exactness of the tensor product. For $f \in \mathrm{End}_{\mathcal{C}}(P\otimes V)$, let $\mathsf{A} \in \mathrm{Hom}_{\mathcal{C}}(P\otimes P^*, \hat{P}\otimes\hat{P}^*)$ and $\mathsf{B} \in \mathrm{Hom}_{\mathcal{C}}(\hat{P}\otimes\hat{P}^*, P\otimes P^*)$ be defined as in Figure 1. Using the right partial trace property for projective objects established in (3.7), we get

$$\begin{aligned} t_{P\otimes P^*}(\mathsf{B}\circ\mathsf{A}) &= t_P\big(\mathrm{tr}^r_{P^*}(\mathsf{B}\circ\mathsf{A})\big) \overset{*}{=} t_P\big(\mathrm{tr}^r_V(f)\big)\ , \\ t_{\hat{P}\otimes\hat{P}^*}(\mathsf{A}\circ\mathsf{B}) &= t_{\hat{P}}\big(\mathrm{tr}^r_{\hat{P}^*}(\mathsf{A}\circ\mathsf{B})\big) \overset{*}{=} t_{P\otimes V}(f)\ , \end{aligned} \tag{3.8}$$

[2] Here, we use the projective generator G instead of the regular module A as we work in $\mathcal{C}$, recall that the equivalence functor $\mathrm{Hom}_{\mathcal{C}}(-, G)$ between $\mathcal{C}$ and A-mod sends G to A.

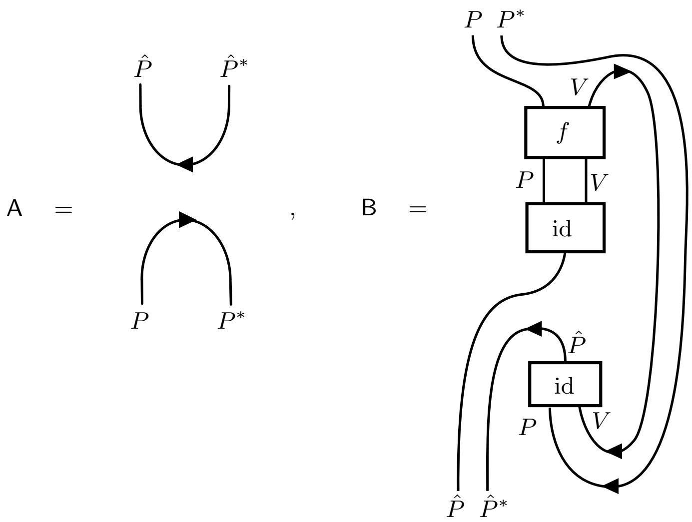
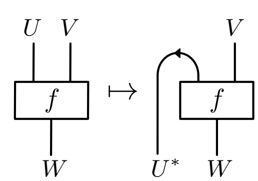


FIGURE 1. Morphisms A and B.

where in steps $(*)$ we used first Lemma 3.1 and then simple manipulations with the diagrams, like the zig-zag indentity for the left duality. Using the cyclicity equation $t_{P\otimes P^*}(\mathsf{B}\circ\mathsf{A}) = t_{\hat{P}\otimes\hat{P}^*}(\mathsf{A}\circ\mathsf{B})$ and comparing both the lines in (3.8) we finally get the equality $t_{P\otimes V}(f) = t_P(\mathrm{tr}^r_V(f))$. The proof for the left modified trace goes along similar lines after reflecting all diagrams on a vertical line. $\square$

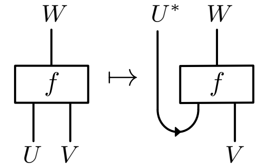
Duality and Calabi-Yau structure. We now recall that in any pivotal category $\mathcal{D}$ we have the isomorphisms, for $U, V, W \in \mathcal{D}$,

$$
{}^{\cap}d\colon \quad \begin{array}{c} \mathrm{Hom}_{\mathcal{D}}(W, U\otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(U^*\otimes W, V) \\ f \mapsto (\mathrm{ev}_U \otimes \mathrm{id}_V)\circ(\mathrm{id}_{U^*}\otimes f) \end{array} \quad ,
$$



(3.9)

$$
{}_{\cup}d\colon \quad \begin{array}{c} \mathrm{Hom}_{\mathcal{D}}(U\otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(V, U^*\otimes W) \\ f \mapsto (\mathrm{id}_{U^*}\otimes f)\circ(\widetilde{\mathrm{coev}}_U \otimes \mathrm{id}_V) \end{array} \quad ,
$$



that are defined analogously to (1.16), with the duality maps on the left side.

Calabi-Yau (CY) structure on $\mathcal{D}$ compatible with duality on the right was introduced before diagram (1.17). Similarly, we say that a CY structure on $\mathcal{D}$ is *compatible with duality on the* *left* if the following diagram commutes for all $U, V, W \in \mathcal{D}$:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(U \otimes V, W) \times \mathrm{Hom}_{\mathcal{D}}(W, U \otimes V) \arrow[r, "\circ"] \arrow[d, "\cup d"] \arrow[d, "\cap d"] & \mathrm{End}_{\mathcal{D}}(U \otimes V) \arrow[d, "t_{U\otimes V}"'] \\
 & \Bbbk \\
\mathrm{Hom}_{\mathcal{D}}(V, U^* \otimes W) \times \mathrm{Hom}_{\mathcal{D}}(U^* \otimes W, V) \arrow[r, "\circ"] & \mathrm{End}_{\mathcal{D}}(V) \arrow[u, "t_V"]
\end{tikzcd}
\tag{3.10}
$$

Theorem 3.3. *Let $\mathcal{C}$ be a $\Bbbk$-linear finite pivotal category. A Calabi-Yau structure on $\mathsf{Proj}(\mathcal{C})$ is compatible with duality on the right (left) if and only if the corresponding trace maps are non-degenerate and have the right (left) partial trace property.*

Proof. We prove the right case only, the left one is similar. The one direction is an easy check. Indeed, assume t is a non-degenerate right modified trace on $\mathsf{Proj}(\mathcal{C})$, and $a \in \mathrm{Hom}_{\mathcal{C}}(U \otimes V, W)$ and $b \in \mathrm{Hom}_{\mathcal{C}}(W, U \otimes V)$, for $U, V, W \in \mathsf{Proj}(\mathcal{C})$, then the top-right side of diagram (1.17) gives $\mathsf{t}_{U\otimes V}(b \circ a)$ while the left-bottom part gives $\mathsf{t}_U\big(\mathrm{tr}^r_V(b \circ a)\big)$. Then using (1.11) we conclude that diagram (1.17) commutes for $\mathcal{D} = \mathsf{Proj}(\mathcal{C})$.

It remains to show the necessary condition. Let $\{t_P \,|\, P \in \mathsf{Proj}(\mathcal{C})\}$ be CY structure on $\mathsf{Proj}(\mathcal{C})$ compatible with duality on the right. We need to establish the right partial trace property (1.11). By Reduction Lemma 3.2, it is enough to consider the case where $U = V = G$ for G a projective generator. Let us also fix $W = G \otimes G$ and choose $b = \mathrm{id}_{G\otimes G}$ and any $a \in \mathrm{End}_{\mathcal{C}}(G \otimes G)$. Then by the assumption and using the previous calculation, commutativity of the diagram (1.17) gives the equality $t_{G\otimes G}(a) = t_G\big(\mathrm{tr}^r_G(a)\big)$ which by Reduction Lemma 3.2 implies that t is a right modified trace. □

4. Pivotal Hopf algebras

In this section, we first recall standard facts from theory of finite-dimensional Hopf algebras which will be needed later and then prove Proposition 1.2. The main reference is the book [Ra]. In what follows, H will be a finite-dimensional Hopf algebra over a field $\Bbbk$ with the unit $\mathbf{1}$, multiplication μ, counit ϵ, coproduct Δ, and antipode S. In this case, the antipode is invertible [IR]. In addition, we show that if H is a unimodular pivotal Hopf algebra, then H-pmod admits a non-degenerate and unique up-to-scalar right modified trace, or equivalently a Calabi-Yau structure compatible with duality on the right, and a similar statement for the left property.

Pivot. We will say that an element $g \in H$ is *group-like* if $\Delta(g) = g \otimes g$. It follows [Ka, Prop. III.3.7] that g is invertible, $S(g) = g^{-1}$ and $\epsilon(g) = 1$.

Definition 4.1. A group-like element $\boldsymbol{g} \in H$ is called a *pivot* if

$$S^2(x) = \boldsymbol{g} x \boldsymbol{g}^{-1}, \qquad \text{for all} \quad x \in H. \tag{4.1}$$

The pair $(H, \boldsymbol{g})$ of a Hopf algebra H and a pivot $\boldsymbol{g}$ is called a *pivotal* Hopf algebra.

A pivot $\boldsymbol{g}$ in a Hopf algebra, if it exists, is not necessarily unique. For a group-like element z in the center of H, the product $z\boldsymbol{g}$ is also a pivot. We will therefore indicate the choice of a pivot explicitly by the notation $(H, \boldsymbol{g})$.

Examples. Let G be a finite group. Then its group algebra $\Bbbk[G]$ is a finite-dimensional pivotal Hopf algebra with $\boldsymbol{g} = \mathbf{1}$.
Ribbon Hopf algebras defined e.g. in [Tu] are pivotal Hopf algebras. The canonical choice of a pivot is given by $\boldsymbol{g} = \boldsymbol{u}\boldsymbol{v}^{-1}$, where $\boldsymbol{u} = \mu \circ (S \otimes \mathrm{id})(R_{21})$ is the canonical Drinfeld element, and $\boldsymbol{v}$ is the ribbon element.
Many more examples can be constructed as follows. Any Hopf algebra H can be extended to a pivotal Hopf algebra as follows [AAGTV, Sec. 2.1]. Recall that S is invertible and order of S^2 is finite. Let G be the cyclic group generated by S^2 and set $\boldsymbol{g} = S^2$. We can then consider the smash product of H with $\Bbbk G$. The result is a pivotal Hopf algebra with the pivot $\boldsymbol{g}$.

Symmetrised left and right integrals. For any pivotal Hopf algebra $(H, \boldsymbol{g})$ with the right integral $\boldsymbol{\mu}$, the *symmetrised* right integral $\boldsymbol{\mu}_{\boldsymbol{g}}$ is defined by $\boldsymbol{\mu}_{\boldsymbol{g}}(x) := \boldsymbol{\mu}(\boldsymbol{g}x)$, for $x \in H$. Applying (1.1) for $\boldsymbol{g}x$ we get the relation for $\boldsymbol{\mu}_{\boldsymbol{g}}$:

$$(\boldsymbol{\mu}_{\boldsymbol{g}} \otimes \boldsymbol{g})\Delta(x) = \boldsymbol{\mu}_{\boldsymbol{g}}(x)\mathbf{1}\ . \tag{4.2}$$

We note that relation (4.2) defines $\boldsymbol{\mu}_{\boldsymbol{g}}$ uniquely (up to a scalar) because of up-to-scalar uniqueness of $\boldsymbol{\mu}$ and invertibility of the pivot $\boldsymbol{g}$.

Analogously, the *symmetrised left integral* is defined by $\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(x) := \boldsymbol{\mu}^l(\boldsymbol{g}^{-1}x)$ for any $x \in H$. Applying (1.2) for $\boldsymbol{g}^{-1}x$ we get the defining relation for the symmetrised left integral:

$$(\boldsymbol{g}^{-1} \otimes \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}})\Delta(x) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(x)\mathbf{1} \quad \text{for any} \quad x \in H\ . \tag{4.3}$$

We note that the spaces of left and right integrals are not necessarily equal. We have a simple lemma.

Lemma 4.2. *The left integral can be chosen as $\boldsymbol{\mu}^l(x) = \boldsymbol{\mu}(S(x))$.*

Proof. From (1.1) we have $(\boldsymbol{\mu} \otimes \mathrm{id})\Delta(S(x)) = \boldsymbol{\mu}(S(x))\mathbf{1}$ for any $x \in H$. Using the identity $(S \otimes S)\Delta^{\mathrm{op}}(x) = \Delta(S(x))$ we get

$$(\boldsymbol{\mu} \circ S \otimes S)\Delta^{\mathrm{op}}(x) = (S \otimes \boldsymbol{\mu} \circ S)\Delta(x) = \boldsymbol{\mu}(S(x))\mathbf{1}$$

Applying S^{-1} to both sides of the last equality and using $S^{-1}(\mathbf{1}) = \mathbf{1}$, we obtain that $\boldsymbol{\mu} \circ S$ satisfies the defining equation for a left integral. $\square$

Example. If H is semisimple with $S^2 = \mathrm{id}$, then $\boldsymbol{\mu} = \boldsymbol{\mu}_{\boldsymbol{g}} = \boldsymbol{\mu}^l = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$ is the character of the regular representation [Ra, Prop. 10.7.4].

Proposition 4.3 ([Ra1])**.** *Let H be a finite-dimensional Hopf algebra. Then right and left integrals are non-degenerate linear forms.*

Proof. Let us first prove the non-degeneracy of $\boldsymbol{\mu}$. For any $h \in H$ we set $\boldsymbol{\mu}_h(-) := \boldsymbol{\mu}(h \cdot -)$. By [Ra, Theorem 10.2.2(e)], H^* is a free H-module with basis $\{\boldsymbol{\mu}\}$, where the action by $a \in H$ sends $\boldsymbol{\mu}$ to $\boldsymbol{\mu}_{S(a)}$. This means that for any non-zero $b \in H$, there exist b' such that $\boldsymbol{\mu}(bb') \neq 0$, since S is bijective. This proves that left kernel of $\boldsymbol{\mu}$ is trivial. Since H is finite-dimensional, $\boldsymbol{\mu}$ is non-degenerate. Non-degeneracy of $\boldsymbol{\mu}^l$ follows from Lemma 4.2 and non-degeneracy of $\boldsymbol{\mu}$. □

Unimodular Hopf algebras. A *right cointegral* in H is an element $\boldsymbol{c} \in H$ such that

$$x\boldsymbol{c} = \epsilon(x)\boldsymbol{c} \ , \qquad \text{for all } \ x \in H \ . \tag{4.4}$$

Similarly, a *left cointegral* is defined by the equation $\boldsymbol{c}x = \epsilon(x)\boldsymbol{c}$. Non-zero right and left cointegrals exist in any finite-dimensional Hopf algebra and are unique up to scalar multiple [LS]. A Hopf algebra is called *unimodular* if its right cointegral is also left. In this case, we call the cointegral *two-sided*.

It is shown in [Hu, Theorem 2] that existence of a non-degenerate symmetric linear form on H implies unimodularity. The argument is as follows. Let $\boldsymbol{c}$ and $\boldsymbol{c}'$ be respectively right and left cointegrals. With respect to a non-degenerate symmetric linear form, both $\boldsymbol{c}$ and $\boldsymbol{c}'$ belong to the orthogonal complement of $\mathrm{Ker}(\epsilon \colon H \to \Bbbk)$, which is 1-dimensional. Let us show the converse.

Proposition 4.4. *For a unimodular pivotal Hopf algebra $(H, \boldsymbol{g})$, the symmetrised right and left integrals define non-degenerate symmetric linear forms on H.*

Proof. By Proposition 4.3, the forms $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^l$ are non-degenerate. The shift of the left or right integral by an invertible element preserves this property. Hence, $\boldsymbol{\mu}_{\boldsymbol{g}}$ and $\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$ are also non-degenerate. By [Ra, Thm. 10.5.4 (e)] we have

$$\boldsymbol{\mu}(xy) = \boldsymbol{\mu}\big(S^2(y)x\big) \tag{4.5}$$

since in the unimodular case the distinguished group-like element of H^* is the counit ϵ. Similarly, we have

$$\boldsymbol{\mu}^l\left(S^{-2}(y)x\right) = \boldsymbol{\mu}\left(S\left(S^{-2}(y)x\right)\right) = \boldsymbol{\mu}\left(S(x)S^{-1}(y)\right) = \boldsymbol{\mu}\left(S(y)S(x)\right) = \boldsymbol{\mu}^l(xy) \tag{4.6}$$

where we applied Lemma 4.2 for the first and last, and (4.5) for the third equalities.

By an easy computation, we check that $\boldsymbol{\mu}_{\boldsymbol{g}}$ is symmetric:

$$\boldsymbol{\mu}_{\boldsymbol{g}}(xy) = \boldsymbol{\mu}(\boldsymbol{g}xy) = \boldsymbol{\mu}(S^2(y)\boldsymbol{g}x) = \boldsymbol{\mu}(\boldsymbol{g}yx) = \boldsymbol{\mu}_{\boldsymbol{g}}(yx)$$

where we used (4.5) and $S^2(y) = \boldsymbol{g}y\boldsymbol{g}^{-1}$. Similarly, using (4.6) we get

$$\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(xy) = \boldsymbol{\mu}^l(\boldsymbol{g}^{-1}xy) = \boldsymbol{\mu}^l(S^{-2}(y)\boldsymbol{g}^{-1}x) = \boldsymbol{\mu}^l(\boldsymbol{g}^{-1}yx) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(yx) \ .$$

□

By the previous proposition 4.4 we thus have two non-degenerate symmetric forms on a unimodular pivotal H, given by the symmetrised left and right integrals. By Proposition 2.4 and Theorem 2.6 they define two Calabi-Yau structures on H-pmod. In other words we have

Corollary 4.5. *The symmetric forms $\boldsymbol{\mu}_{\boldsymbol{g}}$ and $\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$ make a unimodular pivotal Hopf algebra $(H, \boldsymbol{g})$ a symmetric Frobenius algebra.*

We recall now definition (2.5) of 0^{th}-Hochschild homology $\mathrm{HH}_0(H)$ of an algebra H.

Proposition 4.6. *A right symmetrised integral on a unimodular pivotal Hopf algebra H gives a non-degenerate symmetric pairing between the center $Z(H)$ and $\mathrm{HH}_0(H)$:*

$$(z, h) \mapsto \boldsymbol{\mu}_{\boldsymbol{g}}(zh) \ , \qquad z \in Z(H) \ , \ h \in \mathrm{HH}_0(H) \ . \tag{4.7}$$

Similarly, a left symmetrised integral gives a non-degenerate symmetric pairing.

Proof. We first recall that a linear form f on $\mathrm{HH}_0(H)$ satisfies $f(ab - ba) = 0$, for $a, b \in H$, or defines a symmetric linear form on H. For a given non-degenerate symmetric form t, we have an isomorphism between the center and the space $\mathrm{Ch}(H)$ of symmetric forms on H, see e.g. [Br, Lem. 2.5]:

$$Z(H) \xrightarrow{\sim} \mathrm{Ch}(H) \ , \qquad z \mapsto t(z-) \ . \tag{4.8}$$

By Proposition 4.4, we can choose $t = \boldsymbol{\mu}_{\boldsymbol{g}}$, and therefore any linear form f on $\mathrm{HH}_0(H)$ can be written as $\boldsymbol{\mu}_{\boldsymbol{g}}(z-)$ for an appropriate $z \in Z(H)$. This is equivalent to non-degeneracy of the pairing (4.7). The proof for a left symmetrised integral is similar. □

Unibalanced Hopf algebras. We first recall that a right integral generates a one-dimensional right ideal of H^*, which is also a left ideal on $(H^*)^{\mathrm{op}}$, by the argument in [Ra, p. 306] we have

$$(\mathrm{id} \otimes \boldsymbol{\mu})\Delta(x) = \boldsymbol{\mu}(x)\boldsymbol{a}, \tag{4.9}$$

for a certain $\boldsymbol{a} \in H$ called *comodulus* which is group-like. Multiplying (4.9) with $\boldsymbol{a}^{-1}$ and evaluating at $\boldsymbol{a}x$, we see that the left and right integrals are related by the comodulus:

$$\boldsymbol{\mu}^l(x) = \boldsymbol{\mu}(\boldsymbol{a}x). \tag{4.10}$$

Recall that in Lemma 4.2 we had another choice for $\boldsymbol{\mu}^l(x)$ using the antipode. Let us show that these two choices agree.

Proposition 4.7. *We have the equality $\boldsymbol{\mu}(S(x)) = \boldsymbol{\mu}(\boldsymbol{a}x)$.*

Proof. By Lemma 4.2 and (4.10), both $\boldsymbol{\mu}(S(x))$ and $\boldsymbol{\mu}(\boldsymbol{a}x)$ are left integrals. Then we clearly have $\boldsymbol{\mu}(S(x)) = \lambda\boldsymbol{\mu}(\boldsymbol{a}x)$, for some $\lambda \in \Bbbk^\times$, because the left integral is unique up to a scalar. To compute the proportionality coefficient it is enough to evaluate both forms $\boldsymbol{\mu}(S(-))$ and $\boldsymbol{\mu}(\boldsymbol{a}-)$ on one element, we choose it to be the left cointegral $\boldsymbol{c}$. Without loss of generality, we will assume $\boldsymbol{\mu}(\boldsymbol{c}) = 1$, see [Ra, Thm. 10.2.2 (b)]. Then by [Ra, Eq. (10.4)] we also have $\boldsymbol{\mu}(S(\boldsymbol{c})) = 1$. Therefore,

$$1 = \boldsymbol{\mu}(S(\boldsymbol{c})) = \lambda\boldsymbol{\mu}(\boldsymbol{a}\boldsymbol{c}) = \lambda\epsilon(\boldsymbol{a})\boldsymbol{\mu}(\boldsymbol{c}) = \lambda\epsilon(\boldsymbol{a}) \ . \tag{4.11}$$

Recall that $\boldsymbol{a}$ is group-like and so $\epsilon(\boldsymbol{a}) = 1$, and therefore $\lambda = 1$ from the above equality. □

A pivotal Hopf algebra $(H, \boldsymbol{g})$ is called *unibalanced* if its right symmetrised integral is also left. For a given right integral, let us choose the left integral as $\boldsymbol{\mu}^l = \boldsymbol{\mu} \circ S$ (compare in Lemma 4.2). Then in the unibalanced case we have the equality

$$\boldsymbol{\mu}_{\boldsymbol{g}} = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}} \ . \tag{4.12}$$

Indeed, we have $\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}} = \lambda\boldsymbol{\mu}_{\boldsymbol{g}}$ for some $\lambda \in \mathbb{C}^\times$ and to compute λ we evaluate the symmetrised integrals on $\boldsymbol{c}$. We note that by [Ra, Eq. (10.4)] $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^l$ take same non zero value on $\boldsymbol{c}$, say $\boldsymbol{\mu}(\boldsymbol{c}) = \boldsymbol{\mu}^l(\boldsymbol{c}) = a \in \mathbb{C}^\times$. Then, we have $a = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(\boldsymbol{c}) = \lambda\boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{c}) = a\lambda$, and so $\lambda = 1$.

We have the following characterisation of the unibalanced case in terms of the comodulus $\boldsymbol{a}$.

Lemma 4.8. *A pivotal Hopf algebra $(H, \boldsymbol{g})$ is unibalanced if and only if $\boldsymbol{a} = \boldsymbol{g}^2$.*

Proof. Assume first that $\boldsymbol{a} = \boldsymbol{g}^2$. Then evaluating (4.9) on $\boldsymbol{g}x$ we get

$$(\boldsymbol{g}^{-1} \otimes \boldsymbol{\mu}_{\boldsymbol{g}})\Delta(x) = \boldsymbol{\mu}_{\boldsymbol{g}}(x)\mathbf{1} \ . \tag{4.13}$$

which is the defining relation for the symmetrised left integral, and therefore $\boldsymbol{\mu}_{\boldsymbol{g}} = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$.

For the other direction, assume now $(H, \boldsymbol{g})$ is unibalanced, then applying (4.10) to $\boldsymbol{g}^{-1}x$ and using (4.12) we get the equality

$$\boldsymbol{\mu}\big((\boldsymbol{a}\boldsymbol{g}^{-1} - \boldsymbol{g})x\big) = 0 \ , \qquad \text{for any} \ \ x \in H \ . \tag{4.14}$$

By Proposition 4.3, $\boldsymbol{\mu}$ is non-degenerate. Therefore, the equality (4.14) holds if and only if $\boldsymbol{a}\boldsymbol{g}^{-1} = \boldsymbol{g}$. □

Quantum groups at roots of unity provide many examples of unimodular and unibalanced pivotal Hopf algebras, see details in Section 7.

Pivotal structure on H-mod. For a pivotal Hopf algebra $(H, \boldsymbol{g})$, each object V in H-mod has a left dual $V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ with the H action defined by $(hf)(x) = f(S(h)x)$, $f \in V^*$, $h, x \in H$, while the action by $\boldsymbol{g}$ corresponds to the natural isomorphism δ between the identity functor on H-mod and the double duality functor $(-)^{**}$. More precisely, we have the family of isomorphisms

$$\delta_V \colon V \to V^{**} \ , \qquad \delta_V = \boldsymbol{g} \circ \delta^{\mathbf{vect}}_V, \quad V \in H\text{-mod} \ , \tag{4.15}$$

where $\delta^{\mathbf{vect}}$ is the standard pivotal structure in the category $\mathbf{vect}_{\Bbbk}$: $\delta^{\mathbf{vect}}_V(v) = \langle -, v\rangle$, for the underlying vector space V, $v \in V$ and $\langle -,-\rangle$ is the pairing between V^* and V. The isomorphisms (4.15) are obviously natural and monoidal, and satisfy $\delta_{V^*} = (\delta^*_V)^{-1}$. We have therefore H-mod is pivotal.

In H-mod, we have the standard left duality morphisms. Assume $\{v_j \,|\, j \in J\}$ is a basis of V and $\{v^*_j \,|\, j \in J\}$ is the dual basis of V^*, then

$$\begin{aligned} &\mathrm{ev}_V : \; V^* \otimes V \to \Bbbk, && \text{given by} \quad f \otimes v \mapsto f(v), \\ &\mathrm{coev}_V : \; \Bbbk \to V \otimes V^*, && \text{given by} \quad 1 \mapsto \sum_{j\in J} v_j \otimes v^*_j. \end{aligned} \tag{4.16}$$

The pivot $\boldsymbol{g}$ allows to define the right duality morphisms as follows

$$\begin{aligned} &\widetilde{\mathrm{ev}}_V : \; V \otimes V^* \to \Bbbk, && \text{given by} \quad v \otimes f \mapsto f(\boldsymbol{g}v) \\ &\widetilde{\mathrm{coev}}_V : \; \Bbbk \to V^* \otimes V, && \text{given by} \quad 1 \mapsto \sum_i v^*_i \otimes \boldsymbol{g}^{-1} v_i \ , \end{aligned} \tag{4.17}$$

where we used the combination of (3.1) and (4.15).

We recall the (right) categorical trace (1.8) which is in our case

$$\mathrm{tr}^{H\text{-mod}}_V(f) := \widetilde{\mathrm{ev}}_V \circ (f \otimes \mathrm{id}) \circ \mathrm{coev}_V(1) \ , \tag{4.18}$$

for any $V \in H$-mod and $f \in \mathrm{End}_H(V)$. With the definitions above we have

$$\mathrm{tr}^{H\text{-mod}}_V(f) = \mathrm{tr}_V(l_{\boldsymbol{g}} \circ f) \tag{4.19}$$

where $\mathrm{tr}_V(f)$ is the usual trace of the endomorphism f of V. The trace (4.19) is often called *quantum trace*. Analogously, we can define the left categorical trace

$$^{H\text{-mod}}\mathrm{tr}_V(f) := \mathrm{ev}_V \circ (\mathrm{id} \otimes f) \circ \widetilde{\mathrm{coev}}_V(1)$$

for any $V \in H$-mod and $f \in \mathrm{End}_H(V)$. Then we compute

$$^{H\text{-mod}}\mathrm{tr}_V(f) = \sum_i v^*_i \big(f(\boldsymbol{g}^{-1} v_i)\big) = \mathrm{tr}_V(f \circ l_{\boldsymbol{g}^{-1}}). \tag{4.20}$$

We note that the left and right traces are related. Indeed, using Lemma 3.1 for $Q = \mathbf{1}$, $P = V$, we have the relation

$$^{H\text{-mod}}\mathrm{tr}_V(f) = \mathrm{tr}^{H\text{-mod}}_{V^*}(f^*) \ . \tag{4.21}$$

We are now ready to prove Proposition 1.2.

Proof of Proposition 1.2. We will assume that the right integral $\boldsymbol{\mu}$ and the cointegral $\boldsymbol{c}$ satisfy $\boldsymbol{\mu}(\boldsymbol{c}) = 1$. From [Ra, Thm. 10.4.1], for any $f \in \mathrm{End}_H(H)$, we then have

$$\mathrm{tr}_H(f) = \boldsymbol{\mu}\big(S(\boldsymbol{c}'')f(\boldsymbol{c}')\big) \tag{4.22}$$

and

$$\mathrm{tr}_H(f) = \boldsymbol{\mu}\big(S(f(\boldsymbol{c}''))\boldsymbol{c}'\big) \ . \tag{4.23}$$

We use here Sweedler's notation with implicit sum: $\Delta(\boldsymbol{c}) = \boldsymbol{c}' \otimes \boldsymbol{c}''$. From Lemma 2.2, any $f \in \mathrm{End}_H(H)$ is right multiplication by $x = f(1)$, i.e. $f = r_x$. The right categorical trace for $f = r_x$ is obtained from (4.22) as follows:

$$\begin{aligned}\mathrm{tr}_H(l_{\boldsymbol{g}} \circ f) &= \boldsymbol{\mu}\big(S(\boldsymbol{c}'')\boldsymbol{g}\boldsymbol{c}'x\big) = \boldsymbol{\mu}\big(S(\boldsymbol{c}'')S^2(\boldsymbol{c}')\boldsymbol{g}x\big)\\ &= \boldsymbol{\mu}\big(S(S(\boldsymbol{c}')\boldsymbol{c}'')\boldsymbol{g}x\big) = \epsilon(\boldsymbol{c})\boldsymbol{\mu}_{\boldsymbol{g}}(x)\ .\end{aligned}$$

We similarly get the left categorical trace using (4.23)

$$\begin{aligned}\mathrm{tr}_H(l_{\boldsymbol{g}^{-1}} \circ f) &= \boldsymbol{\mu}\big(S(\boldsymbol{g}^{-1}\boldsymbol{c}''x)\boldsymbol{c}'\big) = \boldsymbol{\mu}\big(S(x)S(\boldsymbol{c}'')\boldsymbol{g}\boldsymbol{c}'\big)\\ &= \boldsymbol{\mu}\big(S(x)\boldsymbol{g}S^{-1}(\boldsymbol{c}'')\boldsymbol{c}'\big) = \boldsymbol{\mu}\big(S(x)\boldsymbol{g}S^{-1}(S(\boldsymbol{c}')\boldsymbol{c}'')\big)\\ &= \epsilon(\boldsymbol{c})\boldsymbol{\mu}\big(S(\boldsymbol{g}^{-1}x)\big) = \epsilon(\boldsymbol{c})\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(x)\ ,\end{aligned}$$

where the last equality comes from the formula for a left integral in Lemma 4.2. By [Ra, Cor. 10.3.3] $\epsilon(\boldsymbol{c})$ is non-zero if and only if the algebra H is semisimple. This shows that the categorical traces agree with (1.20) up to a non-zero scalar if and only if H-mod is semisimple. □

From Proposition 1.2, we conclude that in the non-semisimple case $\mathrm{tr}_H(l_{\boldsymbol{g}}r_x)$ is zero for all $x \in H$, while $\boldsymbol{\mu}_{\boldsymbol{g}}(x)$ is not. This naturally suggests that $\boldsymbol{\mu}_{\boldsymbol{g}}$ provides a non-trivial generalisation of the categorical trace for the tensor ideal of projective H-modules, recall Lemma 3.2 for the case $G = H$. Such a generalisation indeed exists and is given by the (right) modified trace – this is the content of our Theorem 1. The proof is rather long and requires more preparation, we delegate it to Section 6.

Remark. Proposition 1.2 can also be deduced directly from Theorem 1. Indeed, the right symmetrised integral $\boldsymbol{\mu}_{\boldsymbol{g}}$ gives a non-zero right modified trace on H, which is unique up to a scalar. As we mentioned in Introduction, the right categorical trace is also a right modified trace. However, the right categorical trace is non-zero on $H \in H$-pmod if and only if H-pmod is semisimple, see e.g. [GR, Rem. 4.6], or equivalently if and only if H-mod is semisimple. Therefore, the two traces agree if and only if H is semisimple as an algebra. Similar argument applies for the left categorical trace.

It is interesting to note an application of Theorem 1 in the classical context – to the modular representation theory of finite groups. Let G be a finite group and consider its group algebra $\mathbb{F}_p[G]$ over the field $\Bbbk = \mathbb{F}_p$ when the characteristic p divides the order of the group. It is a unimodular pivotal Hopf algebra with $\boldsymbol{g} = \mathbf{1}$ and the two sided cointegral is $\boldsymbol{c} = \sum_{g\in G} g$. So, the symmetrised integral in this case is just the integral and it provides a non-degenerate modified trace on the subcategory of projective $\mathbb{F}_p[G]$-modules. To our knowledge, such modified traces were not observed in this generality. However we should also mention that existence and non-degeneracy of the modified trace in the finite characteristic case was proven in [GR] in the case of Drinfeld doubles of $\mathbb{F}_p[G]$ and under an extra technical assumption, which did not work e.g. in the case of abelian p-groups.

As another corollary of Theorem 1 and Theorem 3.3 we conclude this section with the following (c.f. Corollary 1.1).

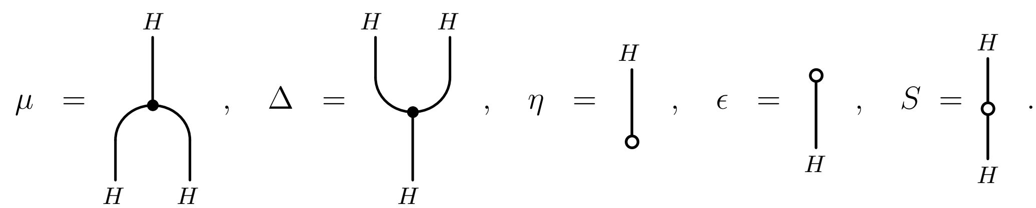
Corollary 4.9. *Let* $(H, \boldsymbol{g})$ *be a unimodular pivotal Hopf algebra. Then* H-pmod *admits a unique up-to-scalar CY structure compatible with duality on the right, and a possibly different CY structure compatible with duality on the left. The CY structure on* H-pmod *is compatible with duality on the right and the left if and only if* H *is unibalanced.*

5. Decomposition of tensor powers of the regular representation

In this section for any finite-dimensional Hopf algebra H, we decompose tensor powers of the regular representation and describe the centralizer algebras $\mathrm{End}_H(H^{\otimes k})$ explicitly. Then we generalise these results to $\mathrm{End}_{H\otimes W}$ for any $W \in H$-mod. We will need these endomorphism algebras to prove our main theorem in next Section 6.

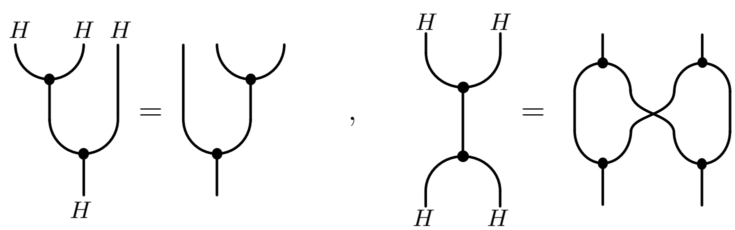
Diagrammatics for Hopf algebras. We will use the following diagrams for the structural maps corresponding to the Hopf algebra data:

(5.1)



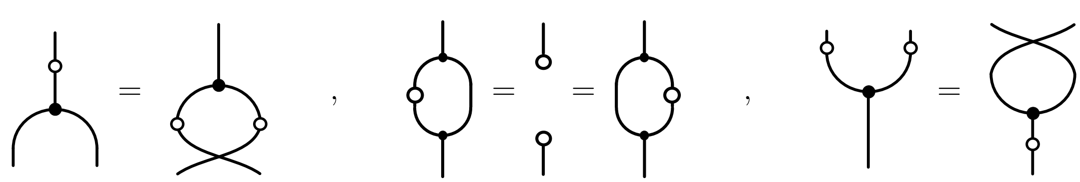
We note that these are maps in the category $\mathbf{vect}_{\Bbbk}$ of finite-dimensional vector spaces over $\Bbbk$. Here is a list of graphical identities corresponding to the Hopf algebra axioms we use extensively below:

(5.2)



where the first is for coassociativity, the second says that Δ is an algebra map, and the antipode axioms (here, we skip labels H for brevity)

(5.3)



where the first and third say that S is an anti-algebra and anti-coalgebra map, respectively. The axioms involving unit and counit are rather clear and we omit them.

The case of $H^{\otimes 2}$. Let us denote by ${}_\epsilon H$ the vector space underlying H equipped with the trivial action of H, i.e. for $m \in {}_\epsilon H$ and $h \in H$ we have $hm = \epsilon(h)m$. As a H-module, ${}_\epsilon H$ is isomorphic to $\dim H$ copies of the trivial representation. We use Sweedler's notation with implicit sum: $\Delta(h) = h' \otimes h''$.

Theorem 5.1. *We have for all $h \in H$ and $m \in {}_\epsilon H$*

(*a*) *the map*

$$\begin{array}{rrcl} \phi: & H \otimes {}_\epsilon H & \to & H \otimes H \\ & h \otimes m & \mapsto & h' \otimes h'' m \end{array} \tag{5.4}$$

is an isomorphism of H-modules whose inverse is

$$\begin{array}{rrcl} \psi: & H \otimes H & \to & H \otimes {}_\epsilon H \\ & x \otimes y & \mapsto & x' \otimes S(x'')y \ ; \end{array} \tag{5.5}$$

(*b*) *the map*

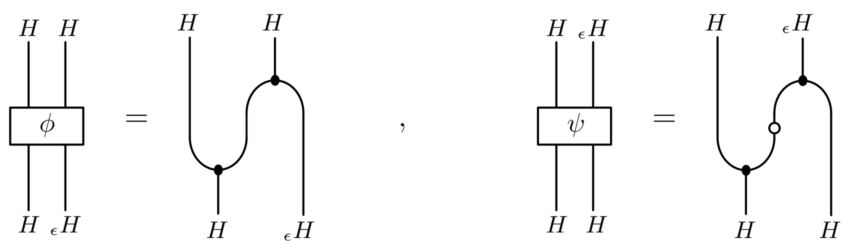
$$\begin{array}{rrcl} \phi^l: & {}_\epsilon H \otimes H & \to & H \otimes H \\ & m \otimes h & \mapsto & h'm \otimes h'' \end{array} \tag{5.6}$$

is an isomorphism of H-modules whose inverse is

$$\begin{array}{rrcl} \psi^l: & H \otimes H & \to & {}_\epsilon H \otimes H \\ & x \otimes y & \mapsto & S^{-1}(y')x \otimes y'' \ . \end{array} \tag{5.7}$$

In what follows we will use graphical calculation. Recall our conventions for Hopf algebras in Section 5. Then, for the maps ϕ and ψ we have the expressions

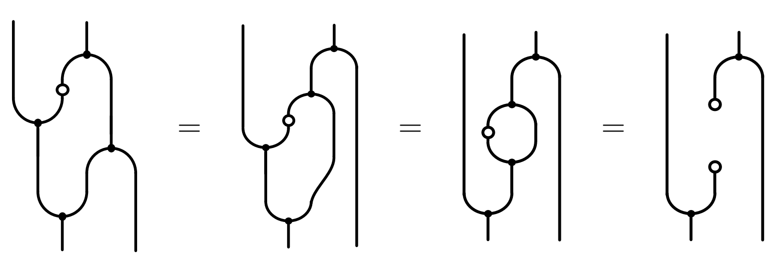
(5.8)



and similarly for ϕ^l and ψ^l.

Proof. We begin with the part (a) and first check that ψ is left inverse to ϕ, we thus compute the composition

(5.9) $\psi \circ \phi$ =

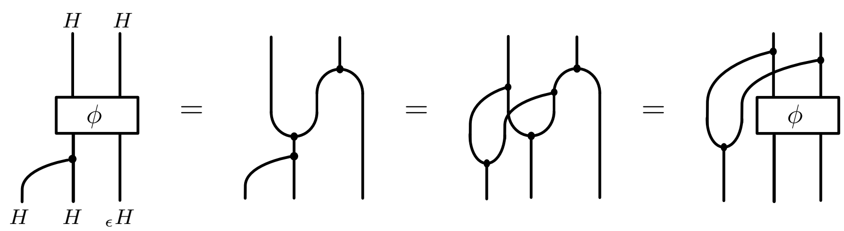


= $\mathrm{id}_{H \otimes {}_\epsilon H}$

where we used coassociativity of the coproduct in the third equality, and then the antipode axiom. Since the left and right inverses of a linear endomorphism of a finite-dimensional space are always equal, we also have $\phi \circ \psi = \mathrm{id}_{H\otimes H}$.

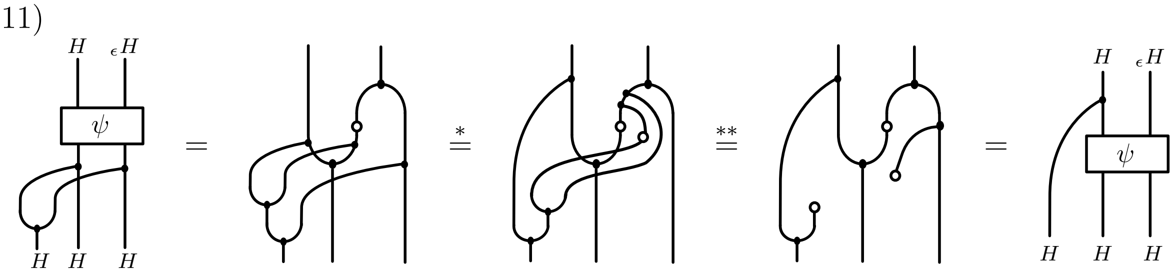
Then we check that ϕ intertwines the corresponding H actions:

(5.10)



where we used the property of coproduct being an algebra map and associativity of multiplication. We also show explicitly the source and target labels, H in this case, only on LHS for brevity. Clearly, the inverse map of an intertwiner is automatically an intertwiner. Therefore, it proves that ψ is an intertwiner as well. However, we also provide a direct argument (as it illustrates better the graphical manipulations we use often below):

(5.11)



where in the step (*) we used coassociativity of the coproduct and that the antipode is an algebra anti-homomorphism. In step (**) we used the associativity of multiplication and the antipode axiom from (5.3), then in the last step the unit and counit properties.

The part b) is proven in an analogous way. □

From Theorem 5.1 we obtain two corollaries: the first is about an explicit decomposition of $H \otimes H$ while the second contains a description of the centraliser algebra of the H-action on $H \otimes H$. First, we need a little preparation. Let us fix a basis B of H, it is a finite set. We introduce then two families of intertwining maps:

(5.12)
$$\begin{aligned} g_y\colon H \to H \otimes {}_\epsilon H \ , &\qquad h \mapsto h \otimes y \ , &\qquad y \in B \ , \\ f_y\colon H \otimes {}_\epsilon H \to H \ , &\qquad h \otimes u \mapsto \delta_{u,y} h \ , &\qquad u, y \in B \ , \end{aligned}$$

where δ is the Kronecker symbol, and the last map we extend linearly to the whole space $H \otimes {}_\epsilon H$. It is clear that $f_{y'} \circ g_y = \delta_{y',y}\mathrm{id}_H$ and $g_y \circ f_y$ is an idempotent for each $y \in B$. The intertwining property of g_y and f_y is very straightforward to see. From this and from the isomorphisms established in Theorem 5.1 we have the following corollary.

Corollary 5.2. *Let H be the regular module of a Hopf algebra H and B be a basis of H. We have then the decomposition*

$$H \otimes H \;\cong\; \bigoplus_{y \in B} H_y \tag{5.13}$$

where each direct summand H_y is the regular H-module and the corresponding idempotent e_y is given by the composition

$$e_y = \iota_y \circ \pi_y \tag{5.14}$$

with the monomorphisms

$$\iota_y \colon H \to H \otimes H \ , \qquad h \mapsto \phi \circ g_y(h) = h' \otimes h'' y \ , \quad y \in B \tag{5.15}$$

and the epimorphisms

$$\pi_y \colon H \otimes H \to H \ , \qquad h \otimes u \mapsto f_y \circ \psi(h \otimes u) \ , \quad y \in B \ , \ u \in H \ . \tag{5.16}$$

In other words, the image of ι_y is H_y in (5.13) and π_y is identity on H_y.

Proof. The direct sum decomposition (5.13) clearly follows from Theorem 5.1 where the corresponding isomorphisms ϕ and $\psi = \phi^{-1}$ are given. That ι_y is an intertwiner is clear from the definition $\iota_y := \phi \circ g_y$ as the composition of two intertwining maps. And the same applies to π_y. The idempotent property of $e_y = \phi \circ g_y \circ f_y \circ \phi^{-1}$ follows from that of $g_y \circ f_y$. The image of e_y is $H_y \subset H \otimes H$ and e_y is identity on H_x if and only if $x = y$ for $x, y \in B$. This finishes the proof. $\square$

From (5.14), we also note the equalities

$$e_y e_x = \delta_{y,x} e_y \ , \quad x, y \in B \tag{5.17}$$

and

$$\sum_{y \in B} e_y = \mathrm{id}_{H \otimes H} \ . \tag{5.18}$$

Before formulating the second corollary of Theorem 5.1, let us recall that for any n-dimensional $\Bbbk$-algebra A there is a natural isomorphism $\mathrm{Mat}_{n,n}(A) \cong A \otimes \mathrm{Mat}_{n,n}(\Bbbk)$, where $\mathrm{Mat}_{n,n}$ is the $n \times n$ matrix algebra.

Corollary 5.3. *For any n-dimensional Hopf algebra H, there is an algebra isomorphism*

$$\mathrm{End}_H(H \otimes H) \cong \mathrm{Mat}_{n,n}(H^{\mathrm{op}}) \ , \tag{5.19}$$

Hence, any element $\boldsymbol{f} \in \mathrm{End}_H(H \otimes H)$ is parametrised by the triple (h, v, γ), for $h, v \in H$ and $\gamma \in H^*$, where

$$\boldsymbol{f}(h, v, \gamma) := \phi \circ f(h, v, \gamma) \circ \psi : \quad H \otimes H \to H \otimes H \tag{5.20}$$

with

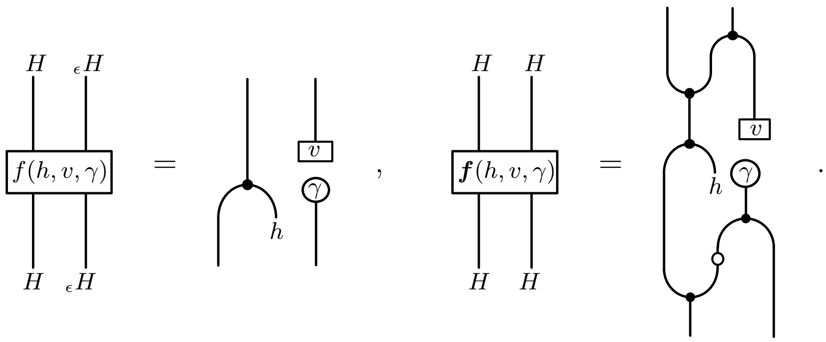
$$f(h, v, \gamma) \colon \ x \otimes y \mapsto \gamma(y) \cdot (xh) \otimes v \ , \qquad x \in H, \ y \in {}_{\epsilon}H \ . \tag{5.21}$$

Their product is the composition with

$$f(h_1, v_1, \gamma_1) \circ f(h_2, v_2, \gamma_2) = \gamma_1(v_2) f(h_2 h_1, v_1, \gamma_2)\ . \tag{5.22}$$

Here is the graphical presentation of the maps $f(h, v, \gamma)$ and $\boldsymbol{f}(h, v, \gamma)$:

(5.23)



Proof. We first recall the decomposition (5.13) where the multiplicity space is the vector space underlying H. We will denote it $M := H$ in order to distinguish from the regular module H. We have then isomorphisms[3]

$$\mathrm{End}_H(H \otimes H) \cong \mathrm{End}_H(H \otimes_{\Bbbk} M) \cong \mathrm{Hom}_H(H \otimes_{\Bbbk} M \otimes_{\Bbbk} M^*, H)\ , \tag{5.24}$$

where in the last isomorphism we used the duality maps ev_M and coev_M. We note that RHS of (5.24) is obviously isomorphic to $\mathrm{End}_H(H) \otimes_{\Bbbk} \mathrm{Mat}_{n,n}(\Bbbk)$ with $n = \dim H$. Then by Lemma 2.1 we get an isomorphism of vector spaces in (5.19). Let us describe this isomorphism explicitly. First, we construct the isomorphism

$$\Phi\colon\ H^{\mathrm{op}} \otimes \big(M \otimes_{\Bbbk} M^*\big) \xrightarrow{\sim} \mathrm{End}_H(H \otimes_{\Bbbk} M)\ , \tag{5.25}$$

$$h \otimes v \otimes \gamma \mapsto f(h, v, \gamma) \tag{5.26}$$

with $f(h, v, \gamma)$ from (5.21). It is straightforward to check that $f(h, v, \gamma)$ is an intertwiner. The inverse to the map Φ is defined as follows. Elements in $\mathrm{End}_H(H \otimes_{\Bbbk} M)$ are of the form

$$g = r_h \otimes s\colon\ x \otimes y \mapsto xh \otimes s(y)\ , \tag{5.27}$$

where $s \in \mathrm{End}_{\Bbbk}(M)$ and we used that g has to intertwine the regular H-action and that by Lemma 2.1 such intertwiner is given by right multiplication r_h with an element $h \in H$. Recall the isomorphism $M \otimes_{\Bbbk} M^* \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(M)$ that sends $v \otimes \gamma$ to the operator $\gamma(-)v$. Then it is straightforward to check that $\Phi^{-1} : g \mapsto h \otimes \sum_{v,u \in B} s_{vu} v \otimes u^*$, where $(s_{vu})_{v,u \in B}$ is the matrix of the linear map s. Finally, conjugating the image of Φ by ϕ, i.e. sending $h \otimes v \otimes \gamma$ to $\boldsymbol{f} := \phi \circ f(h, v, \gamma) \circ \phi^{-1}$, gives explicitly the isomorphism (5.19).

We show next that the map Φ is also an algebra map. The multiplication on $M \otimes_{\Bbbk} M^*$ is

$$\big(M \otimes_{\Bbbk} M^*\big) \otimes \big(M \otimes_{\Bbbk} M^*\big) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_M \otimes \mathrm{id}} M \otimes_{\Bbbk} M^* \tag{5.28}$$

[3]Using $\otimes_{\Bbbk}$ we distinguish the tensor product of vector spaces from the one for H-modules.

or explicitly (which is the standard matrix multiplicaition in $\mathrm{Mat}_{n,n}(\Bbbk)$)

$$(v_1 \otimes \gamma_1) \cdot (v_2 \otimes \gamma_2) = \gamma_1(v_2) v_1 \otimes \gamma_2 \ . \tag{5.29}$$

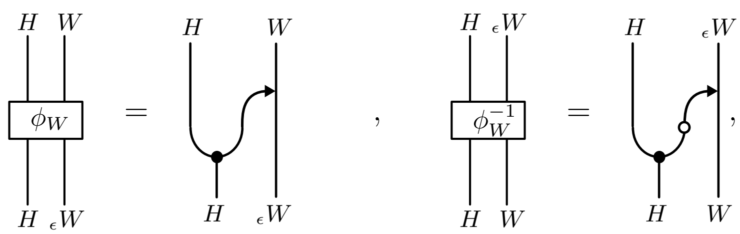
The source of Φ is then the product of two algebras H^{op} and $M \otimes_\Bbbk M^*$. In the image space of Φ, the multiplication is given by the composition (5.22), as follows from the definition of $f(h, v, \gamma)$. Then using (5.29) it is easy to see that multiplication in $H^{\mathrm{op}} \otimes (M \otimes_\Bbbk M^*)$ agrees with the one in $\mathrm{End}_H(H \otimes_\Bbbk M)$. By conjugating with ϕ, the latter algebra is isomorphic to $\mathrm{End}_H(H \otimes H)$. This finishes our proof. □

The general case of $H \otimes W$. We study here the more general case of the product $H \otimes W$ for any $W \in H$-mod. The generalisation of Theorem 5.1 is straightforward. Let us denote by ${}_\epsilon W$ the vector space underlying W equipped with the trivial action of H, i.e. for $w \in {}_\epsilon W$ and $h \in H$ we have the action $h.w = \epsilon(h) w$ (the initial action on W is $w \mapsto hw$, without the dot).

Theorem 5.4. *Let H be a finite-dimensional Hopf algebra H and $W \in H$-mod. We then have the isomorphisms of H-modules*

$$\phi_W \colon H \otimes {}_\epsilon W \to H \otimes W \ , \qquad \phi_W^{-1} \colon H \otimes W \to H \otimes {}_\epsilon W \tag{5.30}$$

which are given graphically as

(5.31) 

where the arrow denotes the H-action on W. In particular, we have an algebra isomorphism

$$H^{\mathrm{op}} \otimes \mathrm{Mat}_{m,m}(\Bbbk) \xrightarrow{\sim} \mathrm{End}_H(H \otimes W) \ , \qquad m = \dim(W) \ , \tag{5.32}$$

which sends $h \otimes A$ to the intertwining map

$$x \otimes w \mapsto (x'h)' \otimes (x'h)'' m_A\big(S(x'')w\big) \ , \qquad x \in H,\ w \in W \ , \tag{5.33}$$

and m_A here is the operator, $m_A \in \mathrm{End}_\Bbbk({}_\epsilon W)$, corresponding to the matrix A.

Proof. The proof of (5.30) and (5.31) literally repeats the one for part (a) of Theorem 5.1, where ${}_\epsilon H$ is replaced by ${}_\epsilon W$ and the multiplication on the right factor is replaced by the action of H on W. The proof of the isomorphism (5.32) similarly repeats the one for Corollary 5.3, and the explicit map (5.33) follows from (5.20) where the second tensor factor is replaced by W while ϕ and ψ are replaced by ϕ_W and ϕ_W^{-1}, respectively. □

Similarly to part (b) of Theorem 5.1, we have the isomorphisms $\phi_W^l \colon {}_\epsilon W \otimes H \to W \otimes H$ and its inverse as in (5.6) and (5.7), repsectively, where again the multiplication should be replaced by the H-action on W.

Replacing W in the previous theorem with a tensor power of the regular representation we get the following result.

Corollary 5.5. *Let $k \geq 2$.*

a) The map

$$\begin{array}{rrcl} \phi_k : & H \otimes {}_\epsilon H^{\otimes k-1} & \to & H \otimes H^{\otimes k-1} = H^{\otimes k} \\ & h \otimes x & \mapsto & h' \otimes h'' x \ , \end{array} \tag{5.34}$$

where the action of h'' on $x \in {}_\epsilon H^{\otimes k-1}$ is via repeated coproduct, is an isomorphism of H-modules whose inverse is

$$\begin{array}{rrcl} \psi_k : & H \otimes H^{\otimes k-1} & \to & H \otimes {}_\epsilon H^{\otimes k-1} \\ & h \otimes x & \mapsto & h' \otimes S(h'') x \ ; \end{array} \tag{5.35}$$

b) We have an isomorphism of algebras

$$H^{\mathrm{op}} \otimes \mathrm{Mat}_{n^{k-1}, n^{k-1}}(\Bbbk) \cong \mathrm{End}_H(H^{\otimes k}) \ , \tag{5.36}$$

which associates to $h \otimes A$ the intertwinner

$$x \otimes y \mapsto (x'h)' \otimes (x'h)'' m_A(S(x'')y) \ ,$$

where m_A is the operator corresponding to the matrix A.

6. Proof of Theorem 1

We have now all the necessary ingredients to prove our main theorem. We start with a reformulation of Reduction Lemma 3.2 adapted to our current setting.

Corollary 6.1. *Given a unimodular pivotal Hopf algebra $(H, \boldsymbol{g})$, a symmetric linear function $\mathsf{t} \in H^*$ extends to a right modified trace on H-*pmod *if and only if for all $f \in \mathrm{End}_H(H \otimes H)$*

$$\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H\big(\mathrm{tr}^r_H(f)\big) \ . \tag{6.1}$$

*Analogously, t extends to a left modified trace on H-*pmod *if and only if*

$$\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H\big(\mathrm{tr}^l_H(f)\big) \ , \qquad \textit{for all} \quad f \in \mathrm{End}_H(H \otimes H) \ . \tag{6.2}$$

Corollary 6.1 allows us to restrict the analysis to the regular module and its tensor powers, and therefore we can use the results of the previous section.

The proof of Theorem 1 is divided into three steps.

Step 1: $\boldsymbol{\mu_g}$ provides right modified trace. We first show that the symmetrised right integral $\boldsymbol{\mu_g}$ provides the right modified trace. By Proposition 4.4 and by the assumption that H is unimodular, $\boldsymbol{\mu_g}$ is a symmetric form on H.

By Corollary 6.1 it is enough to check $\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H(\mathrm{tr}^r_H(f))$ with $\mathsf{t}_H(f) = \boldsymbol{\mu_g}(f(\mathbf{1}))$ for any $f \in \mathrm{End}_H(H)$. Let us rewrite LHS of the last equation as

$$\mathsf{t}_{H\otimes H}(f) = \sum_{y\in B} \mathsf{t}_{H\otimes H}(f \circ e_y) = \sum_{y\in B} \mathsf{t}_H(\pi_y \circ f \circ \iota_y) \ , \tag{6.3}$$

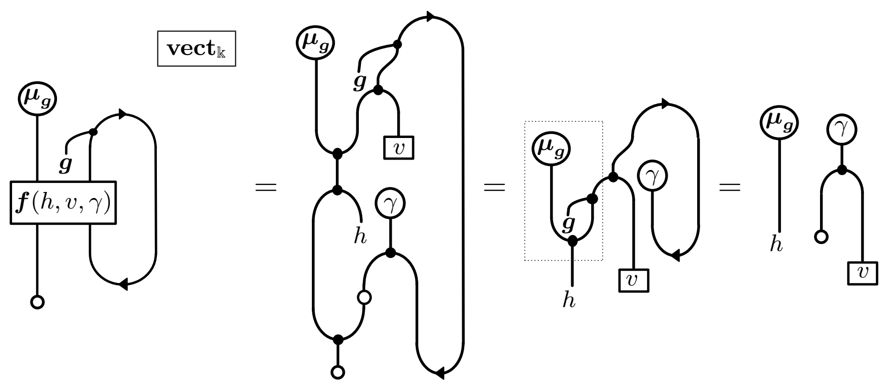
where B is a basis in H. Here, we first inserted the identity (5.18), then used Corollary 5.2 and cyclicity of t_H. Therefore the equation we have to check is

$$\sum_{y\in B} \boldsymbol{\mu_g}\big(\pi_y \circ \boldsymbol{f}(h,v,\gamma) \circ \iota_y(\mathbf{1})\big) = \boldsymbol{\mu_g}\Big(\mathrm{tr}^r_H\big(\boldsymbol{f}(h,v,\gamma)\big)(\mathbf{1})\Big) \ , \quad h \in H,\ v \in B,\ \gamma \in H^*. \tag{6.4}$$

Recall that by Corollary 5.3 any element $f \in \mathrm{End}_H(H \otimes H)$ is of the form $\boldsymbol{f}(h,v,\gamma)$ defined in (5.20). From Corollary 5.2, we have that $\iota_y = \phi \circ g_y(h)$, $\pi_y = f_y \circ \psi$, $\psi = \phi^{-1}$ and

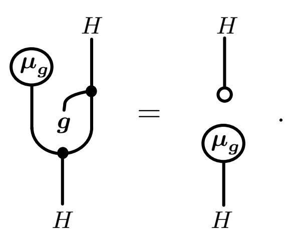
$$\text{LHS of (6.4)} = \sum_{y\in B} \boldsymbol{\mu_g}\big(f_y \circ f(h,v,\gamma) \circ g_y(\mathbf{1})\big) = \sum_{y\in B} \gamma(y)\boldsymbol{\mu_g}\big(f_y(h\otimes v)\big) = \gamma(v)\boldsymbol{\mu_g}(h) \ ,$$

where we also used (5.12). It remains to compute the RHS of (6.4). Using the graphical expression for $\boldsymbol{f}(h,v,\gamma)$ in (5.23), we get[4]

(6.5) RHS of (6.4) = 

,

where for the first equality we use the definition of the partial trace in (1.7) and formulas (4.16)-(4.17) for the left coevaluation coev_H and the right evaluation $\tilde{\mathrm{ev}}_H$ maps; in the second equality we substitute the explicit expression (5.23) for $\boldsymbol{f}(h,v,\gamma)$; the third equality is obvious; then in the fourth equality we replace the part of the diagram inside the dashed rectangle by the (defining) relation (4.2) for the symmetrised integral $\boldsymbol{\mu_g}$ which is diagrammatically written as

(6.6) 

.

[4]We emphasize here by $\mathbf{vect}_\Bbbk$ in the box that the diagrams, as maps from $\Bbbk$ to $\Bbbk$, are morphisms in $\mathbf{vect}_\Bbbk$, so in particular evaluation and coevaluation maps are those from $\mathbf{vect}_\Bbbk$ (the evaluation map in $\mathbf{Rep}\,H$ was already resolved by using the pivotal element $\boldsymbol{g}$).

We finally see that RHS of (6.4) also equals $\gamma(v)\boldsymbol{\mu_g}(h)$, as we got for LHS of (6.4). Therefore the equality (6.4) is true indeed for all $h \in H$, $v \in B$, and $\gamma \in H^*$ and thus for all endomorphisms of $H \otimes H$. This proves that the symmetric form $\boldsymbol{\mu_g}$ satisfies the right partial trace condition, and thus provides a right modified trace for the ideal of projective H-modules.

Step 2: Right modified trace is symmetrised integral. We now turn to the proof for the opposite direction. Assume we have a right modified trace, and hence the symmetric form t_P on $\mathrm{End}_H P$ for any projective P, in particular the symmetric forms on $\mathrm{End}_H H$ and $\mathrm{End}_H(H \otimes H)$. They satisfy $\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H(\mathrm{tr}^r_H(f))$, or equivalently

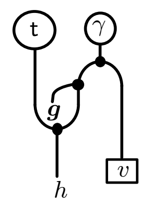
$$\sum_{y\in B} \mathsf{t}_H\big(\pi_y \circ \boldsymbol{f}(h,v,\gamma) \circ \iota_y\big) = \mathsf{t}_H\Big(\mathrm{tr}^r_H\big(\boldsymbol{f}(h,v,\gamma)\big)\Big) , \tag{6.7}$$

for all $h \in H$, $v \in B$, $\gamma \in H^*$. By the same arguments as in Step 1, we get $\gamma(v)\mathsf{t}_H(r_h)$ for LHS of (6.7), where r_h is the right multiplication with h, which we can rewrite

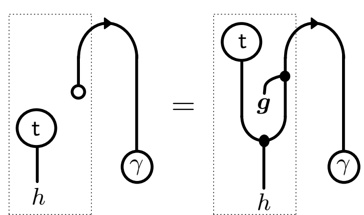
$$\text{LHS of (6.7)} \;=\; \gamma(v)\mathsf{t}(h) \qquad \text{where} \qquad \mathsf{t}(h) := \mathsf{t}_H(r_h) \tag{6.8}$$

is the image of t_H under the isomorphism in Lemma 2.1, i.e. t is a symmetric form on H. We will further work with t only.

Repeating now calculation in (6.5) for the symmetric form t, RHS of (6.7) takes the form:

(6.9) RHS of (6.7) $=$  ,

where we used the relation $\mathsf{t}_H(f) = \mathsf{t}\big(f(\mathbf{1})\big)$. Combining results (6.8) and (6.9) for the both sides and setting $v = \mathbf{1}$, we get for any $\gamma \in H^*$ and $h \in H$ the equality

(6.10)  .

As it is true for all $\gamma \in H^*$ we get the corresponding equality for the arguments of γ – the part of the diagram inside the dashed rectangles – and this agrees with (6.6). In other words, t satisfies the defining relation for the symmetrised right integral, i.e.

$$(\mathsf{t} \otimes \boldsymbol{g})\Delta(h) = \mathsf{t}(h)\mathbf{1} , \qquad h \in H . \tag{6.11}$$

We thus conclude that t, or equivalently the right modified trace t_H, is a symmetrised right integral. As the latter is non-zero and unique up to a scalar, and the right modified trace on H-mod is determined by its value on H by Corollary 6.1, we conclude that a non-zero right modified trace on H-pmod exists (under the assumptions of Theorem 1) and is *unique* up to scalar.

Step 3: Non-degeneracy, left and balanced cases. By Proposition 4.4 and Theorem 2.6 the right modified trace defined by $\boldsymbol{\mu}_{\boldsymbol{g}}$ is non-degenerate. This finishes the proof of Theorem 1 in the right case.

The proof for the left modified trace is completely analogous to the previous one. For example, to show that the left symmetrised integral provides the left modified trace, it is enough to check the left partial trace property $\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H\left(\mathrm{tr}^l_H(f)\right)$ for $\boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$ which is

$$\sum_{y\in B} \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}\big(\pi_y \circ \boldsymbol{f}(h,v,\gamma) \circ \iota_y(\mathbf{1})\big) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}\Big(\mathrm{tr}^l_H\big(\boldsymbol{f}(h,v,\gamma)\big)(\mathbf{1})\Big) \ , \tag{6.12}$$

for all $h \in H$, $v \in B$ and $\gamma \in H^*$. Computations similar to those in (6.5) reduce this equality to (4.13), i.e.

$$(\boldsymbol{g}^{-1} \otimes \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}})\Delta(x) = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}(x)\mathbf{1} \ ,$$

which is the defining relation for the symmetrised left integral.

Clearly, whenever H is unibalanced, left and right symmetrised integrals can be properly normalised such that they agree, e.g. by choosing $\boldsymbol{\mu}^l = \boldsymbol{\mu} \circ S$. Therefore, the corresponding left and right modified traces agree too.

This finishes the proof of Theorem 1.

7. Quantum Groups of types ADE

In this section we study finite-dimensional quantum groups at roots of unity as defined in [L1, Sec. 5][5] in the simply laced case. We compute their right and left integrals and cointegrals, check that they are unibalanced and give a formula for the modified trace on the regular representation. Here, the quantum parameter $q \in \Bbbk$ is a root of 1, whose square has order $p \geq 2$.

Definition. For $n \geq 1$, let $A = (a_{ij})$ be an indecomposable positive definite symmetric Cartan matrix of type A_n, D_n or E_n, and $\mathfrak{g}$ denote the corresponding Lie algebra, with associated pairing denoted by $(\cdot|\cdot)$. In particular $a_{ii} = 2$ for $1 \leq i \leq n$, and $a_{ij} = a_{ji} \in \{0, -1\}$ for $1 \leq i < j \leq n$. The $\Bbbk$-algebra $\overline{U}_q\,\mathfrak{g}$ is generated by $K_i^{\pm 1}$, E_i and F_i, $1 \leq i \leq n$, with relations, for all i, j:

$$\begin{aligned} K_iK_i^{-1} &= K_i^{-1}K_i = \mathbf{1}, & K_iK_j &= K_jK_i, \\ K_iE_jK_i^{-1} &= q^{a_{ij}}E_j, & K_iF_jK_i^{-1} &= q^{-a_{ij}}F_j, \\ [E_i,F_j] &= \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}}, & & \\ E_iE_j = E_jE_i,\ \ F_iF_j &= F_jF_i, & &\text{if } a_{ij} = 0, \end{aligned} \tag{7.1}$$

[5]We use the opposite coproduct compared to the one in [L1].

$$
\begin{aligned}
E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2&=0, && \text{if } a_{ij}=-1,\\
F_i^2F_j-(q+q^{-1})F_iF_jF_i+F_jF_i^2&=0, && \text{if } a_{ij}=-1,\\
E_i^p=F_i^p&=0, && K_i^{2p}=\mathbf{1}\ .
\end{aligned}
$$

The algebra $\overline{U}_q\,\mathfrak{g}$ is a Hopf algebra where the coproduct, counit and antipode are defined as

$$
\begin{aligned}
\Delta(E_i)&=1\otimes E_i+E_i\otimes K_i, && \varepsilon(E_i)=0, && S(E_i)=-E_iK_i^{-1},\\
\Delta(F_i)&=K_i^{-1}\otimes F_i+F_i\otimes 1, && \varepsilon(F_i)=0, && S(F_i)=-K_iF_i,\\
\Delta(K_i)&=K_i\otimes K_i, && \varepsilon(K_i)=1, && S(K_i)=K_i^{-1}.
\end{aligned}
\tag{7.2}
$$

Let L be the root lattice, with $\mathbb{Z}$-basis denoted by α_i, $1\le i\le n$. We denote by Δ_+ the set of positive roots, by $N=|\Delta_+|$ its cardinality, and by ρ half the sum of the positive roots. The formulas for N and the sum of positive roots 2ρ in different types are given below (compare with [B, Ch. VI]):

	N	2ρ
$A_n, n\ge 1$	$\frac{n(n+1)}{2}$	$\sum_{i=1}^n i(n-i+1)\alpha_i$
$D_n, n\ge 4$	$n(n-1)$	$\sum_{i=1}^n (2in-i(i+1))\alpha_i$
E_6	36	see [B, Plate V]
E_7	63	see [B, Plate VI]
E_8	120	see [B, Plate VII]

PBW basis. Let $\mathcal{W}$ be the Weyl group generated by the simple reflexions s_i, $1\le i\le n$. It is a finite Coxeter group. Its basic structural properties we use here can be found in [B]. For $w\in\mathcal{W}$ we denote by $l(w)$ the length of a reduced expression in the generators s_i. Let us choose a reduced expression of the longest element of $\mathcal{W}$,

$$w_0=s_{i_1}s_{i_2}\dots s_{i_N}\ , \tag{7.3}$$

in the simple refexions s_i, $1\le i\le n$. To get an ordered list of positive roots [B, Sec. VI.1.6, Cor. 2] we set

$$\beta_1=\alpha_{i_1}\ ,\quad \beta_2=s_{i_1}(\alpha_{i_2})\ ,\quad \beta_3=s_{i_1}s_{i_2}(\alpha_{i_3})\ ,\quad \dots\ ,\quad \beta_N=s_{i_1}\dots s_{i_{N-1}}(\alpha_{i_N})\ . \tag{7.4}$$

For $1\le i\le n$, let T_i be an algebra automorphism of $\overline{U}_q\,\mathfrak{g}$ which acts on generators K_j, E_j, and F_j by

$$
\begin{aligned}
T_i(K_j)&=K_i^{-a_{ij}}K_j\ , && T_i(E_i)=-F_iK_i\ , && T_i(F_i)=-K_i^{-1}E_i\ ,\\
T_i(E_j)&=E_j\ , && T_i(F_j)=F_j\ , && \text{if } a_{ij}=0\ ,\\
T_i(E_j)&=-E_iE_j+q^{-1}E_jE_i\ , && T_i(F_j)=qF_iF_j-F_jF_i\ , && \text{if } a_{ij}=-1\ .
\end{aligned}
\tag{7.5}
$$

The root vectors are then defined by, see [L1] or [Ja, Ch. 8],

(7.6)

$$E_{\beta_1}=E_{i_1}\ ,\quad E_{\beta_2}=T_{i_1}(E_{i_2})\ ,\quad E_{\beta_3}=T_{i_1}T_{i_2}(E_{i_3})\ ,\quad \dots\ ,\quad E_{\beta_N}=T_{i_1}\dots T_{i_{N-1}}(E_{i_N})\ \ ,$$

$$F_{\beta_1} = F_{i_1}\ , \quad F_{\beta_2} = T_{i_1}(F_{i_2})\ , \quad F_{\beta_3} = T_{i_1}T_{i_2}(F_{i_3})\ , \quad \dots\ , \quad F_{\beta_N} = T_{i_1}\dots T_{i_{N-1}}(F_{i_N})\ .$$

Example. For $A_2 = \mathfrak{sl}(3,\mathbb{C})$ there are two reduced decompositions of the longest element $w_0 = s_1s_2s_1$ and $w_0 = s_2s_1s_2$. The corresponding sequences of positive root vectors are

$$E_1\ , \quad T_1(E_2) = -E_1E_2 + q^{-1}E_2E_1\ , \quad T_1T_2(E_1) = E_2$$

and

$$E_2\ , \quad T_2(E_1) = -E_2E_1 + q^{-1}E_1E_2\ , \quad T_2T_1(E_2) = E_1\ .$$

The algebra automorphisms T_i satisfy the braid relations

$$\begin{aligned} T_i \circ T_j &= T_j \circ T_i && \text{if } a_{ij} = 0\ , \\ T_i \circ T_j \circ T_i &= T_j \circ T_i \circ T_j && \text{if } a_{ij} = -1\ . \end{aligned} \tag{7.7}$$

For a given $w \in \mathcal{W}$ and a reduced decomposition $w = s_{j_1}\dots s_{j_m}$ there is an algebra automorphism $T_w = T_{j_1} \circ \dots \circ T_{j_m}$. The relations (7.7) assert that T_w depends only on the element w and not on its decomposition.

The algebra $\overline{U}_q\,\mathfrak{g}$ has L-grading denoted by wt and defined on generators by $\mathsf{wt}(E_i) = \alpha_i$, $\mathsf{wt}(F_i) = -\alpha_i$ and $\mathsf{wt}(K_i) = 0$. We also define $\mathsf{wt}(E_iE_j) = \alpha_i + \alpha_j$, etc. This makes the algebra graded, because relations are homogeneous. We will use the following lemma.

Lemma 7.1. *For any root β, the root vectors E_β and F_β have L-grading $\mathsf{wt}(E_\beta) = \beta$ and $\mathsf{wt}(F_\beta) = -\beta$, respectively.*

When q is not a root of unity this known lemma can be established using the adjoint action of the Cartan elements [KS, Ch. 6, Prop. 23]. For completeness we give in Appendix B a proof of the stronger statement in the next lemma for all non-zero values of q.

Lemma 7.2. *Assume that for a pair (w,i), with $w \in \mathcal{W}$ and $1 \le i \le n$, we have $l(ws_i) = l(w) + 1$. Then $w(\alpha_i) \in \Delta_+$ and $T_w(E_i)$ has L-grading $\mathsf{wt}\big(T_w(E_i)\big) = w(\alpha_i)$. And similarly, $\mathsf{wt}\big(T_w(F_i)\big) = -w(\alpha_i)$.*

Recall that the root vectors are obtained from a reduced decomposition of the longest word (7.3). Lemma 7.1 is obtained by applying Lemma 7.2 to $(s_{i_1}\dots s_{i_{k-1}}, i_k)$, $1 \le k \le N$, and using (7.4).

Introducing $I = \{0,1,\dots,2p-1\}$, $J = \{0,1,\dots,p-1\}$, we can now construct a PBW basis of $\overline{U}_q\,\mathfrak{g}$ [L1, Section 5.8]

$$B_{m^-,m,m^+} = \prod_{\beta\in\Delta_+} F_\beta^{m_\beta^-} \prod_{i=1}^{n} K_i^{m_i} \prod_{\beta\in\Delta_+} E_\beta^{m_\beta^+} \tag{7.8}$$

indexed by $m \in I^n$ and $m^\pm \in J^{\Delta_+}$, or in other words $m^\pm = (m^\pm_\beta)$ is a map from Δ_+ to J. We will use the notation $m^\pm_k$ for $m^\pm_{\beta_k}$ where β_k is the k-th root defined in (7.4). We denote by $B^*_{m^-,m,m^+}$ the dual basis in $\left(\overline{U}_q\,\mathfrak{g}\right)^*$ defined by

$$\langle B^*_{m^-,m,m^+}, B_{\tilde{m}^-,\tilde{m},\tilde{m}^+}\rangle = \delta_{m^-,\tilde{m}^-}\delta_{m,\tilde{m}}\delta_{m^+,\tilde{m}^+} \ .$$

7.1. **Main result.** We are now in position to present the main result of this section.

Theorem 7.3. *a) The Hopf algebra $\overline{U}_q\,\mathfrak{g}$ is unimodular with the cointegral*

$$\boldsymbol{c} = \prod_{i=1}^{n}\left(\sum_{m=1}^{2p} K_i^m\right)\prod_{\beta\in\Delta_+} F_\beta^{p-1}\prod_{\beta\in\Delta_+} E_\beta^{p-1} \ . \tag{7.9}$$

b) The Hopf algebra $\overline{U}_q\,\mathfrak{g}$ is pivotal with pivots

$$\boldsymbol{g}_\varepsilon = K_{2\rho}\prod_{i=1}^{n} K_i^{p\varepsilon_i} \ , \qquad \varepsilon\in\{0,1\}^{\times n} \tag{7.10}$$

and it is unibalanced for any choice of ε, with the corresponding symmetrised integral

$$\boldsymbol{\mu}_{\boldsymbol{g}} = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}} = B^*_{(p-1)^{\Delta_+},p\varepsilon,(p-1)^{\Delta_+}} \ . \tag{7.11}$$

Here $(p-1)^{\Delta_+}$ is the constant map on Δ_+ with value $p-1$.

Before giving a proof, we first note that as a consequence of Theorem 1 the formula in (7.11) computes the modified trace t for endomorphisms of the regular representation. We also note that for type A_n and with slightly different version of the quantum group, a cointegral and an integral were computed in [GW]. Our proof for the cointegral goes along the lines in [GW, Thm. 2.1.5], however in our case it requires the following lemma on commutation relations whose proof is in Appendix C.

Lemma 7.4. *For $1\le j<k\le N$, we have in $\overline{U}_q\,\mathfrak{g}$ the commutation relation for the root vectors, with β_j defined in* (7.4)*,*

$$E^{p-1}_{\beta_{j+1}}E^{p-1}_{\beta_{j+2}}\ldots E^{p-1}_{\beta_k}E_{\beta_j} = q^{(p-1)(\beta_j|\beta_{j+1}+\cdots+\beta_k)}E_{\beta_j}E^{p-1}_{\beta_{j+1}}E^{p-1}_{\beta_{j+2}}\ldots E^{p-1}_{\beta_k} \ . \tag{7.12}$$

Proof of Thm. 7.3. We first prove the part a). We begin with computing cointegrals for the Borel subalgebras. For brevity, we will use the notation $\overline{U}_q := \overline{U}_q\,\mathfrak{g}$

Let $\overline{U}_q^-$ be the negative Borel subalgebra with the basis $B_{m^-,m,0}$, $m^-\in J^{\Delta_+}$ and $m\in I^n$, it is also a Hopf subalgebra. And similarly for the positive $\overline{U}_q^+$ with the basis B_{0,m,m^+}, $m\in I^n$ and $m^+\in J^{\Delta_+}$.

We claim that

$$\boldsymbol{c}^- = \prod_{i=1}^{n}\left(\sum_{m=1}^{2p} K_i^m\right)\prod_{\beta\in\Delta_+} F_\beta^{p-1} \tag{7.13}$$

is a left cointegral for $\overline{U}_q^-$. Indeed,

$$K_i\boldsymbol{c}^- = \boldsymbol{c}^- = \epsilon(K_i)\boldsymbol{c}^- \quad \text{ for } 1\le i\le n \ . \tag{7.14}$$

From Lemma 7.1 we see that $\prod_{\beta\in\Delta_+} F_\beta^{p-1}$ has the minimal possible L-degree $-(p-1)2\rho$. Therefore we have

$$F_i \cdot \prod_{\beta\in\Delta_+} F_\beta^{p-1} = 0 \ . \tag{7.15}$$

We can then check

$$F_i \boldsymbol{c}^- = 0 = \epsilon(F_i)\boldsymbol{c}^- \ , \quad \text{for } 1 \le i \le n \ , \tag{7.16}$$

because moving F_j through the Cartan part of $\boldsymbol{c}^-$ just replaces K_i by $q^{a_{ij}}K_i$ and the most non-trivial part is the equality (7.15). Hence for all $x \in \overline{U}_q^-$, we have

$$x\boldsymbol{c}^- = \epsilon(x)\boldsymbol{c}^-, \tag{7.17}$$

and so $\boldsymbol{c}^-$ is indeed a left cointegral in $\overline{U}_q^-$. We similarly get that

$$\boldsymbol{c}^+ = \prod_{\beta\in\Delta_+} E_\beta^{p-1} \prod_{i=1}^{n} \left(\sum_{m=1}^{2p} K_i^m \right)$$

is a right cointegral in $\overline{U}_q^+$.

We know that $\overline{U}_q$ has a non-zero left cointegral $\boldsymbol{c}$, unique up to normalisation. Moreover there exists a group-like element $\alpha \in \overline{U}_q^*$, called the modulus, such that

$$\boldsymbol{c}x = \alpha(x)\boldsymbol{c} \quad \text{for all } x \in \overline{U}_q \ , \tag{7.18}$$

see [Ra, Eq. (10.8)]. Using the basis (7.8) in $\overline{U}_q$, we see that $\overline{U}_q$ is a free left module over $\overline{U}_q^-$ with basis $B_{0,0,m^+}$ with $m^+ \in J^{\Delta_+}$. Let us write $\boldsymbol{c}$ in this basis

$$\boldsymbol{c} = \sum_{m^+} \boldsymbol{c}_{m^+} B_{0,0,m^+} \quad \text{with} \quad \boldsymbol{c}_{m^+} \in \overline{U}_q^- . \tag{7.19}$$

Using (7.18) we get

$$\boldsymbol{c}E_i = \alpha(E_i)\boldsymbol{c} = 0 \quad \text{for } 1 \le i \le n \ . \tag{7.20}$$

Here, the vanishing is because the modulus α is group-like and hence $\alpha(E_i^p) = \alpha(E_i)^p$, but $E_i^p = 0$ and so $\alpha(E_i) = 0$. We therefore have that for all root vectors E_{β_j}

$$\sum_{m^+} \boldsymbol{c}_{m^+} B_{0,0,m^+} E_{\beta_j} = 0 \ . \tag{7.21}$$

We show by induction on $\nu = N - j$ that here $\boldsymbol{c}_{m^+} = 0$ if $m_l^+ < p-1$ for some $l \ge j$.

Let us denote by $\tau_j(m^+)$ the result of increasing the j-th component of m^+ by 1. We have that $B_{0,0,\tau_j(m^+)}$ is zero if $m_j^+ = p-1$ and is a PBW basis element otherwise.

We begin with $\nu = 0$, the equation (7.21) for $j = N$ then gives

$$\sum_{m^+} \boldsymbol{c}_{m^+} B_{0,0,\tau_N(m^+)} = 0 \ , \tag{7.22}$$

where only terms with $m_N^+ < p-1$ contribute. As the corresponding elements $B_{0,0,\tau_N(m^+)}$ are linearly independent over $\overline{U}_q^-$, we have $\boldsymbol{c}_{m^+} = 0$ if $m_N^+ < p-1$. This is the first step of induction.

By the induction hypothesis at $\nu = N-j$ we assume $\boldsymbol{c}_{m^+} = 0$ in (7.21) if $m_l^+ < p-1$ for some $l \geq j$. Then, equation (7.21) for $\nu = N-j+1$ gives

$$\sum_{\substack{m^+ \\ m_j^+ = \cdots = m_N^+ = p-1}} \boldsymbol{c}_{m^+} B_{0,0,m^+} E_{\beta_{j-1}} = 0 \ .$$

Using the commutation relation (7.12), we obtain

$$\sum_{\substack{m^+ \\ m_j^+ = \cdots = m_N^+ = p-1}} \boldsymbol{c}_{m^+} B_{0,0,\tau_{j-1}(m^+)} = 0 \ .$$

We deduce as before $\boldsymbol{c}_{m^+} = 0$ if $m_{j-1}^+ < p-1$ and this finishes the proof by induction.

As the equality (7.21) is true for all root vectors, we have thus obtained that only the term with $m^+ = (p-1)^{\Delta_+}$ contributes to (7.19). We obtain that the left cointegral has the form

$$\boldsymbol{c} = \boldsymbol{c}_{(p-1)^{\Delta_+}} B_{0,0,(p-1)^{\Delta_+}} \ , \qquad \text{with} \qquad \boldsymbol{c}_{(p-1)^{\Delta_+}} \in \overline{U}_q^- \ . \tag{7.23}$$

Recall that $\boldsymbol{c}$ is a left cointegral by assumption, therefore we have the equality

$$x\boldsymbol{c} = \epsilon(x)\boldsymbol{c} \qquad \text{for all } x \in \overline{U}_q^- \ . \tag{7.24}$$

Using that $\overline{U}_q$ is a free module over $\overline{U}_q^-$, we get

$$x\boldsymbol{c}_{(p-1)^{\Delta_+}} = \epsilon(x)\boldsymbol{c}_{(p-1)^{\Delta_+}} \qquad \text{for all } x \in \overline{U}_q^- \ . \tag{7.25}$$

We have that $\boldsymbol{c}_{(p-1)^{\Delta_+}}$ is a left cointegral in $\overline{U}_q^-$, i.e. it is proportional to $\boldsymbol{c}^-$ from (7.13). This shows that $\boldsymbol{c}$ is proportional to $\boldsymbol{c}^- B_{0,0,(p-1)^{\Delta_+}}$ which is the formula in (7.9).

We now show that $\boldsymbol{c}$ is two-sided. Indeed, for the right multiplication on $\boldsymbol{c}$ we have

$$\boldsymbol{c}K_i = \boldsymbol{c} \ , \qquad \boldsymbol{c}E_i = \alpha(E_i)\boldsymbol{c} = 0 \ , \qquad \boldsymbol{c}F_i = \alpha(F_i)\boldsymbol{c} = 0 \ , \quad \text{for } 1 \leq i \leq n \ ,$$

where the first equality is due to the relation (7.1) and we used explicit expression (7.9), for the second and third equalities we first used (7.18) and then the fact that the modulus α vanishes on E_i and F_i because α is group-like and $E_i^p = F_i^p = 0$. We have thus shown that $\boldsymbol{c}$ is a two-sided cointegral which implies unimodularity of $\overline{U}_q$.

Now we prove part b). To verify the defining relation for the right integral $\boldsymbol{\mu}$ we will need a formula for coproduct of PBW basis elements. Let

$$K_\beta = \prod_{i=1}^n K_i^{n_i} \qquad \text{for} \quad \beta = \sum n_i \alpha_i \ .$$

For the root vectors E_β, for $\beta \in \Delta_+$, the coproduct can be written as follows [Ja, Sec. 4.12]

$$\Delta(E_\beta) = E_\beta \otimes K_\beta + 1 \otimes E_\beta + \sum_\nu x_\nu \otimes y_\nu \tag{7.26}$$

where x_ν and y_ν are PBW elements $B_{0,m,m^+} \in \overline{U}_q^+$ with non-zero m^+ and such that $\mathsf{wt}(x_\nu) + \mathsf{wt}(y_\nu) = \beta$. We similarly have

$$\Delta(F_\beta) = F_\beta \otimes 1 + K_\beta^{-1} \otimes F_\beta + \sum_\nu x_\nu \otimes y_\nu \tag{7.27}$$

where x_ν and y_ν are now PBW elements $B_{m^-,m,0} \in \overline{U}_q^-$ with non-zero m^- and such that $\mathsf{wt}(x_\nu) + \mathsf{wt}(y_\nu) = -\beta$. More generally, for the coproduct of a PBW basis element (7.8), we have

$$\begin{aligned} \Delta(B_{m^-,m,m^+}) =& B_{m^-,m,m^+} \otimes K_{\mathsf{wt}(B_{0,0,m^+})} \prod_{i=1}^n K_i^{m_i} \\ & + K_{\mathsf{wt}(B_{m^-,0,0})} \prod_{i=1}^n K_i^{m_i} \otimes B_{m^-,m,m^+} + \sum_\nu x_\nu \otimes y_\nu \end{aligned} \tag{7.28}$$

where x_ν and y_ν are in the span of PBW elements $B_{\tilde{m}^-,\tilde{m},\tilde{m}^+}$ where all components of $\tilde{m}^-$ (resp. $\tilde{m}^+$) are lower or equal to those of m^- (resp. m^+), and at least one of them is strictly lower.

Let $M := \big(M_i\big)_{1 \le i \le n}$ be the coordinates of the sum of positive roots in basis of simple roots:

$$2\rho = \sum_{\beta \in \Delta_+} \beta = \sum_{i=1}^n M_i \alpha_i \ .$$

The corresponding Cartan element is $K_{2\rho} = \prod_{i=1}^n K_i^{M_i}$.

Let us now verify that

$$\boldsymbol{\mu} = B^*_{(p-1)^{\Delta_+},(p+1)M,(p-1)^{\Delta_+}} \tag{7.29}$$

satisfies the defining relation for the right integral

$$(\boldsymbol{\mu} \otimes \mathrm{id})\Delta(x) = \boldsymbol{\mu}(x)\mathbf{1} \ . \tag{7.30}$$

For PBW elements B_{m^-,m,m^+} where at least one $m^\pm_\beta$ is lower than $p-1$, using (7.28) we see that both sides of this equation give 0. For $B_{(p-1)^{\Delta_+},m,(p-1)^{\Delta_+}}$, we get

$$\Delta\big(B_{(p-1)^{\Delta_+},m,(p-1)^{\Delta_+}}\big) = B_{(p-1)^{\Delta_+},m,(p-1)^{\Delta_+}} \otimes K_{2\rho}^{p-1} \prod_{i=1}^n K_i^{m_i} + \text{ other terms.} \tag{7.31}$$

Here, $\boldsymbol{\mu} \otimes \mathrm{id}$ vanishes on the "other terms". If $m \neq (p+1)M$ we again get 0 on both sides of (7.30). In the remaining case with $m = (p+1)M$, we have $K_{2\rho}^{p-1} \prod_{i=1}^n K_i^{(p+1)M_i} = \mathbf{1}$ which shows that the equality (7.30) holds indeed.

We now compute the comodulus $\boldsymbol{a}$ using the defining equation (4.9). Using

$$\boldsymbol{\mu}\big(B_{(p-1)^{\Delta_+},(p+1)M,(p-1)^{\Delta_+}}\big) = 1 \ ,$$

we obtain the formula

$$\boldsymbol{a} = (\mathrm{id}\otimes\boldsymbol{\mu})\Delta\big(B_{(p-1)^{\Delta_+},(p+1)M,(p-1)^{\Delta_+}}\big) \ . \tag{7.32}$$

Taking now into account the second term on RHS of (7.28), we have

$$\Delta\big(B_{(p-1)^{\Delta_+},(p+1)M,(p-1)^{\Delta_+}}\big) = K_{2\rho}^{1-p}\prod_{i=1}^{n} K_i^{(p+1)M_i}\otimes B_{(p-1)^{\Delta_+},(p+1)M,(1-p)^{\Delta_+}} + \ \text{other terms} \ .$$

From this, we deduce the value of the comodulus

$$\boldsymbol{a} = K_{2\rho}^{1-p}\prod_{i=1}^{n} K_i^{(p+1)M_i} = K_{2\rho}^{2} \ . \tag{7.33}$$

We study next group-like square roots of $\boldsymbol{a}$, these are $\boldsymbol{g}_\varepsilon = K_{2\rho}\prod_{i=1}^n K_i^{p\varepsilon_i}$, with $\varepsilon\in\{0,1\}^{\times n}$. We check on generators that each $\boldsymbol{g}_\varepsilon$ implements S^2, and so a pivot. Indeed, for $1\le i\le n$,

$$\begin{aligned}
\boldsymbol{g}_\varepsilon K_i\,\boldsymbol{g}_\varepsilon^{-1} &= K_i = S^2(K_i) \ ,\\
\boldsymbol{g}_\varepsilon E_i\,\boldsymbol{g}_\varepsilon^{-1} &= K_iE_iK_i^{-1} = S^2(E_i) \ ,\\
\boldsymbol{g}_\varepsilon F_i\,\boldsymbol{g}_\varepsilon^{-1} &= K_iF_iK^{-1} = S^2(F_i) \ .
\end{aligned}$$

Therefore, the Hopf algebra $\overline{U}_q$ is pivotal with a pivot $\boldsymbol{g}_\varepsilon = K_{2\rho}\prod_{i=1}^n K_i^{p\varepsilon_i}$ for any $\varepsilon\in\{0,1\}^{\times n}$. We then get formula (7.11) for the right symmetrised integral. By Lemma 4.8, $(\overline{U}_q,\boldsymbol{g}_\varepsilon)$ is unibalanced for any choice of ε because $\boldsymbol{a}=\boldsymbol{g}_\varepsilon^2$, or the right symmetrised integral is also left. Moreover, we have $\boldsymbol{\mu}_{\boldsymbol{g}} = \boldsymbol{\mu}^l_{\boldsymbol{g}^{-1}}$ and so (7.11) holds for the left symmetrised integral too. $\square$

8. Modified trace for the restricted quantum $\mathfrak{sl}_2$

Here, we apply results of the previous section to type A_1 and demonstrate how the modified trace for indecomposable projectives can be explicitly computed from the symmetrised integral. For this we will use an explicit basis of Hom-spaces between indecomposable projectives constructed in [FGST]. The quantum group in type A_1, for the choice $q=e^{i\pi/p}$ and $p\ge 2$, is known as *restricted* quantum $\mathfrak{sl}_2$, and will be denoted by $\overline{U}_q\,\mathfrak{sl}_2$. In [BBG] the modified trace on all endomorphisms of indecomposable projectives in $\overline{U}_q\,\mathfrak{sl}_2$-pmod was computed and then extended to the regular representation $\overline{U}_q\,\mathfrak{sl}_2$. Here we do the converse: we reprove [BBG] formulas starting with the symmetrised integral. In this section, we set $[k]=\dfrac{q^k-q^{-k}}{q-q^{-1}}$ and $[m]! = \prod_{k=1}^m [k]$, and $\left[{m \atop k}\right] = \frac{[m]!}{[k]![m-k]!}$, for k and m positive integers.

Symmetrised integral. We will work with the choice of pivot $\boldsymbol{g} := \boldsymbol{g}_{\varepsilon=1} = K^{p+1}$, recall (7.10). In the PBW basis of $\overline{U}_q\,\mathfrak{sl}_2$, the right integral is given by

$$\boldsymbol{\mu}(F^iE^mK^n) = \eta\,\delta_{i,p-1}\delta_{m,p-1}\delta_{n,p+1}$$

where η is a non-zero normalising coefficient. Then our (right) symmetrised integral is

$$\boldsymbol{\mu}_{\boldsymbol{g}}(F^iE^mK^n) = \eta\,\delta_{i,p-1}\delta_{m,p-1}\delta_{n,0}\ . \tag{8.1}$$

Basis for the center $Z(\overline{U}_q\,\mathfrak{sl}_2)$. Recall that the center of $\overline{U}_q\,\mathfrak{sl}_2$ is $3p-1$ dimensional. The basis of $Z(\overline{U}_q\,\mathfrak{sl}_2)$ consists of the central idempotents $\boldsymbol{e}_s$ and nilpotent elements $\boldsymbol{w}^{\pm}_s$. The formulas for these elements in the PBW basis were given in [GT]: [6]

(8.2)

$$\begin{aligned}
\boldsymbol{w}^+_s &= \zeta_s\sum_{n=0}^{s-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1}([i]!)^2q^{j(s-1-2n)}\begin{bmatrix}s-n+i-1\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}F^{p-1-i}E^{p-1-i}K^j,\\
\boldsymbol{w}^-_s &= \zeta_s\sum_{n=0}^{p-s-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1}(-1)^{i+j}([i]!)^2q^{j(p-s-1-2n)}\begin{bmatrix}p-s-n+i-1\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}F^{p-1-i}E^{p-1-i}K^j,\\
\boldsymbol{e}_0 &= \zeta_0\sum_{n=0}^{p-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1}(-1)^{i+j}([i]!)^2q^{j(p-1-2n)}\begin{bmatrix}p-n+i-1\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}F^{p-1-i}E^{p-1-i}K^j,\\
\boldsymbol{e}_p &= \zeta_p\sum_{n=0}^{p-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1}([i]!)^2q^{j(p-1-2n)}\begin{bmatrix}p-n+i-1\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}F^{p-1-i}E^{p-1-i}K^j,\\
\boldsymbol{e}_s &= \frac{q^s+q^{-s}}{[s]^2}(\boldsymbol{w}^+_s+\boldsymbol{w}^-_s)\\
&\quad+\zeta_s\sum_{m=0}^{p-2}\sum_{j=0}^{2p-1}\Big(\sum_{n=0}^{s-1}q^{j(s-1-2n)}\mathsf{B}^+_{n,p-1-m}(s)+\sum_{k=0}^{p-s-1}q^{j(-s-1-2k)}\mathsf{B}^-_{k,p-1-m}(p-s)\Big)F^mE^mK^j,
\end{aligned}$$

where $\mathsf{B}^{\pm}_{n,m}$ are non-zero numbers and we set

$$\begin{aligned}
\zeta_s &= \frac{(-1)^{p-s-1}}{2p}\frac{[s]^2}{([p-1]!)^2}\ , \qquad 1\le s\le p-1\ ,\\
\zeta_0 &= \frac{(-1)^{p-1}}{2p}\frac{1}{([p-1]!)^2}\ , \qquad \zeta_p = \frac{1}{2p}\frac{1}{([p-1]!)^2}\ .
\end{aligned}\tag{8.3}$$

The symmetrised integral from (8.1) has the following values on the central basis elements (8.2):

$$\begin{aligned}
&\boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{w}^+_s) = s\eta\zeta_s\ , \qquad \boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{w}^-_s) = (p-s)\eta\zeta_s\ ,\\
&\boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{e}_s) = (-1)^sp\eta(q^s+q^{-s})\zeta_0\ ,\\
&\boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{e}_p) = p\eta\zeta_p\ , \qquad \boldsymbol{\mu}_{\boldsymbol{g}}(\boldsymbol{e}_0) = p\eta\zeta_0\ .
\end{aligned}\tag{8.4}$$

[6]We used here a relation with Radford basis in the center: the formulas are extracted from Section 3.2.7, Propositions C.4 and C.5.1 in [GT].

Extension of $\boldsymbol{\mu_g}$ to $\overline{U}_q\,\mathfrak{sl}_2$-pmod. Here, we compute the modified trace[7] on endomorphisms of indecomposable projective $\overline{U}_q\,\mathfrak{sl}_2$-modules. We recall now our result in Theorem 1 on the modified trace t, and also note that for evaluating t_P on endomorphisms f of P it is enough to consider only corresponding trace classes $[f]$. For this, we will also recall a basis in

$$\mathrm{HH}_0 := \mathrm{HH}_0\big(\overline{U}_q\,\mathfrak{sl}_2\text{-pmod}\big)\ .$$

Indecomposable projective $\overline{U}_q\,\mathfrak{sl}_2$-modules are classified up to isomorphism in [FGST]: they are precisely the projective covers $\mathcal{P}^{\pm}_s$ of the simple modules where $1 \le s \le p$. In particular, $\mathcal{P}^{\pm}_p$ is a simple module with highest weight $\pm q^{p-1}$. The module $\mathcal{P}^{+}_1$ is the projective cover of the trivial one. The non-trivial morphisms between indecomposable projective modules are listed below:

- the endomorphism ring $\mathrm{End}_{\overline{U}_q\,\mathfrak{sl}_2}(\mathcal{P}^{\pm}_s)$ is one dimensional for $s = p$ and two dimensional with basis $\{\mathrm{id}_{\mathcal{P}^{\pm}_s}, x^{\pm}_s\}$, for $1 \le s \le p-1$,
- the Hom-spaces $\mathrm{Hom}_{\overline{U}_q\,\mathfrak{sl}_2}(\mathcal{P}^{+}_s, \mathcal{P}^{-}_{p-s})$ and $\mathrm{Hom}_{\overline{U}_q\,\mathfrak{sl}_2}(\mathcal{P}^{-}_s, \mathcal{P}^{+}_{p-s})$ are two dimensional with respective bases $\{a^{+}_s, b^{+}_s\}$ and $\{a^{-}_s, b^{-}_s\}$, for $1 \le s \le p-1$.

It is proven in [BBG], that the images of $x^{\epsilon}_s = b^{-\epsilon}_{p-s}a^{\epsilon}_s$ and $x^{-\epsilon}_{p-s} = a^{\epsilon}_s b^{-\epsilon}_{p-s}$ in HH_0 coincide, i.e. $[x^{\epsilon}_s] = [x^{-\epsilon}_{p-s}]$ for any $1 \le s \le p-1$. A basis of HH_0 consists of trace classes of identities of indecomposable projectives $[\mathrm{id}_{\mathcal{P}^{\pm}_s}]$, $1 \le s \le p$, and trace classes of nilpotent elements $[x^{+}_s]$, $1 \le s \le p-1$.

In order to compute the modified trace t on the above basis in HH_0, we need primitive idempotents. Let us first define the projectors onto q^n-eigenspace of K:

$$\pi_n = \frac{1}{2p}\sum_{j=0}^{2p-1} q^{-nj}K^j. \tag{8.5}$$

The primitive (non-central) idempotents are then

$$I_{n,s} = \pi_n \boldsymbol{e}_s, \qquad 1 \le n \le 2p, \quad 1 \le s \le p-1, \quad n - s = 1 \bmod 2. \tag{8.6}$$

Finally, $x^{\pm}_s$ is equal to the action of the central element $\boldsymbol{w}^{+}_s$ on $\mathcal{P}^{\pm}_s$, so that we have

$$\mathsf{t}_{\mathcal{P}^{\pm}_s}(x^{\pm}_s) = \boldsymbol{\mu_g}(I_{\pm s-1,s}\,\boldsymbol{w}^{\pm}_s)\ . \tag{8.7}$$

Recall Remark 2.5 explaining how to express a modified trace on an indecomposable projective via the modified trace on the regular representation given by the symmetric form $\boldsymbol{\mu_g}$. Inserting the primitive idempotents $I_{s-1,s}$ into the arguments of $\boldsymbol{\mu_g}$ in (8.4), we get

$$\begin{aligned} &\boldsymbol{\mu_g}(I_{s-1}\boldsymbol{w}^{+}_s) = \eta\zeta_s\ , \qquad \boldsymbol{\mu_g}(I_{p-s-1}\boldsymbol{w}^{-}_s) = \eta\zeta_s\ , \\ &\boldsymbol{\mu_g}(I_{s-1}\boldsymbol{e}_s) = \eta(-1)^s(q^s+q^{-s})\zeta_0\ , \\ &\boldsymbol{\mu_g}(I_{p-1}\boldsymbol{e}_p) = \eta\zeta_p\ , \qquad \boldsymbol{\mu_g}(I_{2p-1}\boldsymbol{e}_0) = \eta\zeta_0\ . \end{aligned} \tag{8.8}$$

[7] We recall that by Theorem 7.3 it is both right and left.

This gives the following values for modified trace on our basis in HH_0:

	$[\mathrm{id}_{\mathcal{P}_p^+}]$	$[\mathrm{id}_{\mathcal{P}_p^-}]$	$[x_s^+] = [x_{p-s}^-]$	$[\mathrm{id}_{\mathcal{P}_s^+}]$	$[\mathrm{id}_{\mathcal{P}_{p-s}^-}]$
t	$\eta\zeta_p$	$\eta\zeta_0$	$\eta\zeta_s$	$\eta(-1)^s(q^s+q^{-s})\zeta_0$	$\eta(-1)^s(q^s+q^{-s})\zeta_0$
t for $\eta=\zeta_0^{-1}$	$(-1)^{p-1}$	1	$(-1)^s[s]^2$	$(-1)^s(q^s+q^{-s})$	$(-1)^s(q^s+q^{-s})$

where the second row is normalisation free, while the third row recovers the results of [BBG] with the normalisation choice $\eta = \zeta_0^{-1} = (-1)^{p-1}2p([p-1]!)^2$.

Appendix A. Proof of Proposition 2.3

From the definitions of $\mathrm{HH}_0(A)$ and $\mathrm{HH}_0(A\text{-pmod})$ the map $x \mapsto r_x$ induces a linear map $\Phi\colon \mathrm{HH}_0(A) \to \mathrm{HH}_0(A\text{-pmod})$ on the corresponding classes. We need to construct its inverse. By Lemma 2.2, for $P \in A$-pmod we have a decomposition:

$$\mathrm{id}_P = \sum_{i=1}^{k} a_i \circ \mathrm{id}_A \circ b_i \ , \quad \text{with} \quad b_i\colon P \to A,\ a_i\colon A \to P \ . \tag{A.1}$$

Let us define a map $\psi_P\colon \mathrm{End}_A(P) \to \mathrm{HH}_0(A)$ by

$$\psi_P(f) := \sum_i \left[(b_i \circ f \circ a_i)(\mathbf{1})\right] \ . \tag{A.2}$$

We will check that the map

$$\begin{aligned} \Psi\colon\ \mathrm{HH}_0(A\text{-pmod}) &\xrightarrow{\sim} \mathrm{HH}_0(A) \\ [P,f] &\mapsto \psi_P(f) \end{aligned} \tag{A.3}$$

is well-defined, i.e. it does not depend on the choice of the decomposition (A.1) and descends on the class of f in $\mathrm{HH}_0(A\text{-pmod})$.

Assume we have another decomposition $\mathrm{id}_P = \sum_{i'} a'_{i'} \circ \mathrm{id}_A \circ b'_{i'}$, with the associated map

$$\psi'_P(f) = \sum_{i'} [(b'_{i'} \circ f \circ a'_{i'})(\mathbf{1})] \ .$$

Inserting the identity (A.1), we have

$$\begin{aligned} \psi'_P(f) &= \sum_{i,i'} \left[(b'_{i'} \circ f \circ a_i \circ b_i \circ a'_{i'})(\mathbf{1})\right] \\ &= \sum_{i,i'} \left[(b_i \circ a'_{i'})(\mathbf{1})\, (b'_{i'} \circ f \circ a_i)(\mathbf{1})\right] \end{aligned} \tag{A.4}$$

where we applied the algebra isomorphism from Lemma 2.1 to the composition of A-endomorphisms $(b'_{i'} \circ f \circ a_i)$ and $(b_i \circ a'_{i'})$. Similarly,

$$\begin{aligned} \psi_P(f) &= \sum_{i,i'} \left[(b_i \circ a'_{i'} \circ b'_{i'} \circ f \circ a_i)(\mathbf{1})\right] \\ &= \sum_{i,i'} \left[(b'_{i'} \circ f \circ a_i)(\mathbf{1})\, (b_i \circ a'_{i'})(\mathbf{1})\right] \end{aligned} \tag{A.5}$$

which is equal to the second line in (A.4) because the summands are classes in $\mathrm{HH}_0(A)$. We thus get the equality $\psi'_P(f) = \psi_P(f) \in \mathrm{HH}_0(A)$.

Let us now show that the family

$$\{\psi_P\colon \mathrm{End}_A(P) \to \mathrm{HH}_0(A) \,|\, P \in A\text{-pmod}\}$$

has cyclicity property. Let $f\colon P \to P'$ and $g\colon P' \to P$, and id_P as in (A.1) and let $\mathrm{id}_{P'} = \sum_{i'} a'_{i'} \circ \mathrm{id}_A \circ b'_{i'}$. We then have

$$\begin{aligned}
\psi_{P'}(f \circ g) &= \sum_{i,i'} [(b'_{i'} \circ f \circ a_i \circ \mathrm{id}_A \circ b_i \circ g \circ a'_{i'})(\mathbf{1})] \\
&= \sum_{i,i'} [(b_i \circ g \circ a'_{i'})(\mathbf{1})\,(b'_{i'} \circ f \circ a_i)(\mathbf{1})] \\
&= \sum_{i,i'} [(b'_{i'} \circ f \circ a_i)(\mathbf{1})\,(b_i \circ g \circ a'_{i'})(\mathbf{1})] \\
&= \sum_{i,i'} [(b_i \circ g \circ a'_{i'} \circ b'_{i'} \circ f \circ a_i)(\mathbf{1})] \\
&= \psi_P(g \circ f)
\end{aligned} \tag{A.6}$$

where we again used the algebra isomorphism in Lemma 2.1. From this cyclicity property, we see that the map ψ_P does not depend on representatives f in the class $[f] \in \mathrm{HH}_0(A\text{-pmod})$, for $f \in \mathrm{End}_A(P)$. Therefore, the map Ψ in (A.3) is well-defined.

To see that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{HH}_0(A)}$ we have to check that the composition $[x] \mapsto [r_x] \mapsto \psi_A(r_x)$ is identity. Note that here we use only $P = A$ component in the quotient (2.4). Using the trivial decomposition of id_A from (A.1), we indeed get the expected identity, and so Ψ is a left inverse of Φ.

To show that Ψ is also a right inverse of Φ, assume $P \in A$-pmod and $f \in \mathrm{End}_A(P)$. Then Ψ maps $[P, f]$ to the class of $x = \sum_i (b_i \circ f \circ a_i)(\mathbf{1}) \in A$. We note that the corresponding endomorphism of A by right multiplication with x is $r_x = \sum_i (b_i \circ f \circ a_i)$. And by cyclicity we have $[r_x] = [f] \in \mathrm{HH}_0(A\text{-pmod})$. We thus get $\Phi \circ \Psi = \mathrm{id}_{\mathrm{HH}_0(A\text{-pmod})}$, which completes the proof of the proposition.

Appendix B. Proof of Lemma 7.2

The fact that $w(\alpha_i)$ is a positive root if $l(ws_i) = l(w) + 1$ follows from [B, VI.1.6, Cor. 2]. We will prove the formula for $\mathsf{wt}(T_w(E_i))$ by induction on the length $l(w) = \nu \geq 0$. A proof for $\mathsf{wt}(T_w(F_i))$ works similarly. For $\nu = 0$, w is the unit element and the statement holds by definition of the L-grading. We suppose that the statement holds for $\nu \geq 0$, i.e. that $T_w(E_i)$ has L-grading $\mathsf{wt}\big(T_w(E_i)\big) = w(\alpha_i)$ if $l(w) \leq \nu$ and $l(ws_i) = l(w) + 1$.

Let $w \in \mathcal{W}$ be an element with length $l(w) = \nu + 1$ and i be such that $l(ws_i) = \nu + 2$. Recall that $w(\alpha_i) \in \Delta_+$. We claim that there exists $j \neq i$ such that $w(\alpha_j)$ is a negative root.

This follows from [B, Sec. V.4.4, Thm 1], indeed if w permutes the positive roots, then w fixes the positive chamber $C = \{x \in L \mid (\alpha_i|x) > 0, 1 \leq i \leq n\}$ and hence is identity. Let us choose such j. Recall that $l(ws_j) = l(w)+1$ would imply that $w(\alpha_j)$ is a positive root, hence we have that $l(ws_j) < \nu + 2$. From the defining relations, multiplication with s_j changes the length by ± 1, we then clearly have $l(ws_j) \neq l(w)$, therefore $l(ws_j) = \nu$. Denote by $\langle s_i, s_j \rangle \subset \mathcal{W}$ the subgroup generated by s_i and s_j. The idea is to use elements from the orbit $w\langle s_i, s_j\rangle$ to construct an appropriate pair (w', k) to which the induction hypothesis applies. For a given choice of j above, we have 3 cases: $a_{ij} = 0$ or if $a_{ij} = -1$ then ws_js_i might have length $\nu \pm 1$. We analyse all of these cases:

Case 1: $a_{ij} = 0$. We can choose $(w', k) = (ws_j, i)$. Indeed, $l(w') = \nu$ and since $l(ws_i) = \nu + 2$ then $w's_i = ws_is_j$ has length $\nu + 1$, and so we can apply the induction hypothesis. We then get $T_w(E_i) = (T_{w'} \circ T_j)(E_i) = T_{w'}(E_i)$ because $T_j(E_i) = E_i$, see (7.5). Using that $s_j(\alpha_i) = \alpha_i$ we get $\mathsf{wt}\big(T_w(E_i)\big) = w'(\alpha_i) = w(\alpha_i)$.

Case 2a: $a_{ij} = -1$ and $l(ws_js_i) = \nu + 1$. We choose $w' = ws_j$ and to both (w', i), (w', j) the induction hypothesis applies. We have $T_j(E_i) = -E_iE_j + q^{-1}E_jE_i$, $s_j(\alpha_i) = \alpha_i + \alpha_j$, hence

$$\begin{aligned}\mathsf{wt}(T_w(E_i)) &= \mathsf{wt}(T_{w'} \circ T_j(E_i)) = \mathsf{wt}(T_{w'}(E_i)) + \mathsf{wt}(T_{w'}(E_j)) \\ &= w'(\alpha_i) + w'(\alpha_j) = (w' \circ s_j)(\alpha_i) = w(\alpha_i) \ ,\end{aligned} \tag{B.1}$$

where we used that $T_{w'}$ is an automorphism of the algebra and that wt makes the algebra graded.

Case 2b: $a_{ij} = -1$ and $l(ws_js_i) = \nu - 1$. We choose $w' = ws_js_i$ and check that $l(w's_j) = l(ws_is_js_i) = \nu$ because on one side it is at most ν and on the other side it is at least $l(ws_i) - l(s_js_i) = \nu$. Therefore, we can apply the induction hypothesis to (w', j). We have $(T_i \circ T_j)(E_i) = E_j$ and $(s_js_i)(\alpha_j) = \alpha_i$, hence

$$\mathsf{wt}(T_w(E_i)) = \mathsf{wt}(T_{w'}(E_j)) = w'(\alpha_j) = w(\alpha_i) \ .$$

This finishes the proof.

Appendix C. Proof of Lemma 7.4

We will use the following result [Xi, Thm. 2.3][8] stated for $1 \leq j \leq k$, and $1 \leq a, b \leq p-1$:

$$E^a_{\beta_k} E^b_{\beta_j} = q^{ab(\beta_j|\beta_k)} E^b_{\beta_j} E^a_{\beta_k} + \sum_{\substack{0 \leq a_j, a_{j+1}, \dots, a_k \leq p-1 \\ a_j < b \ , \ a_k < a}} \rho(a_j, \dots, a_k) E^{a_j}_{\beta_j} E^{a_{j+1}}_{\beta_{j+1}} \dots E^{a_k}_{\beta_k} \tag{C.1}$$

[8] We note that in [Xi, Thm. 2.3] a commutation formula is given for divided powers, and we just rewrite it for our choice of powers of E_β.

where the coefficients $\rho(a_j, \ldots, a_k) \in \Bbbk$ vanish if the corresponding monomials do not have the expected L-grading:

$$\rho(a_j, \ldots, a_k) = 0 \quad \text{if} \quad a_j\beta_j + a_{j+1}\beta_{j+1} + \cdots + a_k\beta_k \neq b\beta_j + a\beta_k \ . \tag{C.2}$$

We prove the lemma by induction on $\nu = k - j$.

Let us consider the case $\nu = 1$. The formula (C.1) gives

$$E_{\beta_{j+1}}^{p-1} E_{\beta_j} = q^{(p-1)(\beta_j|\beta_{j+1})} E_{\beta_j} E_{\beta_{j+1}}^{p-1} \ , \tag{C.3}$$

where we used that the second term in (C.1) vanishes because of the condition (C.2), which is in our case

$$a_{j+1}\beta_{j+1} \neq \beta_j + (p-1)\beta_{j+1} \ , \tag{C.4}$$

holds for all $a_{j+1} < p - 1$. Equality (C.3) shows that (7.12) is true for $k - j = 1$.

Assume the induction hypothesis that for $1 \leq \nu < N$ the formula (7.12) is true if $k-j \leq \nu$. We consider the case where $k - j = \nu + 1$. From (C.1), we get

$$E_{\beta_k}^{p-1} E_{\beta_j} = q^{(p-1)(\beta_j|\beta_k)} E_{\beta_j} E_{\beta_k}^{p-1} + \sum_{\substack{0 \leq a_{j+1}, \ldots, a_k \leq p-1 \\ a_k < p-1}} \rho(0, a_{j+1}, \ldots, a_k) E_{\beta_{j+1}}^{a_{j+1}} \ldots E_{\beta_k}^{a_k} \ . \tag{C.5}$$

We then use the condition (C.2) on vanishing coefficients $\rho(0, a_{j+1}, \ldots, a_k)$, which is in our case

$$a_{j+1}\beta_{j+1} + \cdots + a_k\beta_k \neq \beta_j + (p-1)\beta_k \ .$$

We see that it certainly holds if all the integers $a_{j+1}, \ldots, a_{k-1}$ are zero – in this case we get the inequality $a_k\beta_k \neq \beta_j + (p-1)\beta_k$, similar to (C.4). Therefore, for *non*-vanishing coefficients ρ in the sum (C.5) we have to necessarily assume that at least one of the integers $a_{j+1}, \ldots, a_{k-1}$ is non zero. Let l be the smallest index for which a_l is non zero. We have $j+1 \leq l < k$ hence $|k - l| < \nu$. The induction hypothesis gives us commutation relation for the root vector E_{β_l}, and we get

$$E_{\beta_l}^{p-1} E_{\beta_{l+1}}^{p-1} \ldots E_{\beta_{k-1}}^{p-1} E_{\beta_l} = q^{(p-1)(\beta_l|\beta_{l+1}+\cdots+\beta_{k-1})} E_{\beta_l}^{p-1} E_{\beta_l} E_{\beta_{l+1}}^{p-1} \ldots E_{\beta_{k-1}}^{p-1} = 0 \ . \tag{C.6}$$

This gives the following vanishing result for terms in the sum (C.5) corresponding to non-zero coefficients $\rho(0, a_{j+1}, \ldots, a_k)$:

$$E_{\beta_{j+1}}^{p-1} E_{\beta_{j+2}}^{p-1} \ldots E_{\beta_{k-1}}^{p-1} E_{\beta_l} = 0 \ ,$$

and therefore these terms do not contribute while moving E_{β_j} to the left in LHS of (7.12). We have thus obtained

$$E_{\beta_{j+1}}^{p-1} E_{\beta_{j+2}}^{p-1} \ldots E_{\beta_k}^{p-1} E_{\beta_j} = q^{(p-1)(\beta_j|\beta_k)} E_{\beta_{j+1}}^{p-1} E_{\beta_{j+2}}^{p-1} \ldots E_{\beta_{k-1}}^{p-1} E_{\beta_j} E_{\beta_k}^{p-1} \ .$$

Using again the induction hypothesis, we move E_{β_j} to the left using (C.1) and get the expected formula (7.12), which completes the proof.

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