Profinite separation systems

Reinhard Diestel Jakob Kneip

December 22, 2017

Abstract

Separation systems are posets with additional structure that form an abstract setting in which tangle-like clusters in graphs, matroids and other combinatorial structures can be expressed and studied.

This paper offers some basic theory about infinite separation systems and how they relate to the finite separation systems they induce. They can be used to prove tangle-type duality theorems for infinite graphs and matroids, which will be done in future work that will build on this paper.

1 Introduction

This paper is a sequel to, and assumes familiarity with, an earlier paper [2] in which finite abstract separation systems were introduced. The profinite separation systems introduced here, and the results proved, will form the basis for our proof of the tangle duality theorem for infinite graphs [1], as well as for a more comprehensive study of profinite *tree sets* [8].

Abstract separation systems were introduced in [2] to lay the foundations for a comprehensive study of how tangles, originally introduced by Robertson and Seymour [10] in the course of their graph minors project, can be generalized to capture, and relate, various other types of highly cohesive regions in graphs, matroids and other combinatorial structures. The basic idea behind tangles is that they describe such a region indirectly: not by specifying which objects, such as vertices or edges, belong to it, but by setting up a system of pointers on the entire structure that point towards this region. The advantage of this indirect approach is that such pointers can locate such a highly cohesive region even when it is a little fuzzy – e.g., when for every low-order separation of a graph or matroid 'most' of the region will lie on one side or the other, so that this separation can be oriented towards it and become a pointer, but each individual vertex or edge (say) can lie on the 'wrong' side of *some* such separation. (The standard example is that of a large grid in a graph: for every low-oder separation of the graph, most of the grid will lie on one side, but each of its vertices lies on the 'other' side of the small separator that consists of just its four neighbours.)

This indirect approach can be applied also to capture fuzzy clusters in settings very different from graphs; see [4, 5, 6] for more discussion.

Robertson and Seymour [10] proved two major theorems about tangles: the *tangle-tree theorem*, which shows how the tangles in a graph can be pairwise separated by a small set of nested separations (which organize the graph into a tree-like shape), and the *tangle duality* theorem which tells us that if the graph has no tangle of a given 'order' then the entire graph has corresponding tree-like structure. Both these theorems can be proved in the general settings of abstract separation systems [5, 4], and then be applied in the above scenarios.

Unlike the tangle-tree theorem, the tangle duality theorem can be extended without unforeseen obstacles to infinite graphs and matroids. This too is most conveniently done in the setting of abstract separation systems. The infinite separation systems needed for this are *profinite* – which means, roughly, that they are completely determined by their finite subsystems. This helps greatly with the proof: it allows us to use the abstract tangle duality theorems from [5] on the finite subsystems, and then extend this by compactness [1].

Although compactness lies at the heart of these extensions, they are not entirely straightforward: one needs to know how exactly a profinite separation system is related to its finite subsystems – not just that these, in principle, determine it. This paper examines and solves these questions. It does so in the general setting of arbitrary profinite separation systems, of which the separations of infinite graphs and matroids (no matter how large) are examples.[1]

Our paper is organized as follows. The basic definitions and facts about abstract separation systems are not repeated here; instead, the reader is referred to [2]. In Section 2 we cite just a couple of lemmas that will be used throughout, and define homomorphisms between separation systems; these will be needed to set up the inverse systems on which our profinite separation systems will be based. A brief reminder of inverse limits of finite sets is offered in Section 3. Profinite separation systems, including a natural topology on them, are then introduced rigorously in Section 4. Section 5 contains most of our work: it gives a detailed analysis of how, in 'nested' profinite separation systems, the properties of the separation system itself are related to the properties of the finite separation systems of which it is an inverse limit. In Section 6, finally, we compare the combinatorial properties of nested separation systems with their topological properties; this has implications on the tools, topological or combinatorial, that can be used to handle profinite nested separation systems when these are applied.

2 Separation Systems

For definitions and basic properties of abstract separation systems we refer to [2] and [3]. In addition, we call a separation *co-small* if its inverse is small.

We shall be using the following two lemmas from [2]:

Lemma 2.1 (Extension Lemma). [2, Lemma 4.1] *Let S be a set of unoriented separations, and let P be a consistent partial orientation of S.*

(i) *P extends to a consistent orientation O of S if and only if no element of P is co-trivial in S.*

(ii) *If $\vec{p}$ is maximal in P, then O in* (i) *can be chosen with $\vec{p}$ maximal in O if and only if $\vec{p}$ is nontrivial in $\vec{S}$.*

(iii) *If S is nested, then the orientation O in* (ii) *is unique.*

Nested separation systems without degenerate or trivial elements are known as *tree sets*. A subset $\sigma \subseteq \vec{S}$ *splits* a nested separation system $\vec{S}$ if $\vec{S}$ has a

[1]Note that whether or not two vertex sets A, B covering a graph G form a separation is determined by its subgraphs of order 2: it is if and only if G has no edge from $A \setminus B$ to $B \setminus A$.

consistent orientation O such that $O \subseteq \lceil \sigma \rceil$ and σ is precisely the set of maximal elements of O. Conversely, a consistent orientation O of $\vec{S}$ *splits* (at σ) if it is contained in the down-closure of the set σ of its maximal elements.

The consistent orientations of a finite nested separation system can be recovered from its splitting subsets by taking their down-closures, but infinite separation systems can have consistent orientations without any maximal elements.

Lemma 2.2. [2, Lemmas 4.4, 4.5] *The splitting subsets of a nested separation system $\vec{S}$ without degenerate elements are proper stars. Their elements are neither trivial nor co-trivial in $\vec{S}$. If S has a degenerate element s, then $\{\vec{s}\}$ is the unique splitting subset of $\vec{S}$.*

The splitting stars of the edge tree set $\vec{E}(T)$ of a tree T, for example, are the sets $\vec{F_t}$ of edges at a node t all oriented towards t. By this correspondence, the nodes of T can be recovered from its edge tree set; if T is finite, its nodes correspond bijectivly to its consistent orientations.

Given two separation systems R, S, a map $f\colon \vec{R} \to \vec{S}$ is a *homomorphism* of separation systems if it commutes with their involutions and respects the ordering on $\vec{R}$. Formally, we say that f *commutes with the involutions* of $\vec{R}$ and $\vec{S}$ if $\big(f(\vec{r}\,)\big)^* = f(\overleftarrow{r})$ for all $\vec{r} \in \vec{R}$. It *respects the ordering* on $\vec{R}$ if $f(\vec{r_1}) \le f(\vec{r_2})$ whenever $\vec{r_1} \le \vec{r_2}$. Note that the condition for f to be order-respecting is not 'if and only if': we allow that $f(\vec{r_1}) \le f(\vec{r_2})$ also for incomparable $\vec{r_1}, \vec{r_2} \in R$. Furthermore f need not be injective. It can therefore happen that $\vec{r_1} < \vec{r_2}$ but $f(\vec{r_1}) = f(\vec{r_2})$, so f need not preserve strict inequality. A bijective homomorphism of separation systems whose inverse is also a homomorphism is an *isomorphism*.

3 Inverse limits of sets

Let $(P, \le)$ be a *directed* poset, one such that for all $p, q \in P$ there is an $r \in P$ such that $p, q \le r$. A family $\mathcal{X} = (\, X_p \mid p \in P \,)$ of sets X_p is an *inverse system* if it comes with maps $f_{qp}\colon X_q \to X_p$, defined for all $q > p$, which are compatible in that $f_{rp} = f_{qp} \circ f_{rq}$ whenever $p < q < r$.[2] If all the f_{qp} are surjective, we call $\mathcal{X}$ a *surjective inverse system*. Any inverse system $(\, Y_p \mid p \in P \,)$ with $Y_p \subseteq X_p$ and maps $f_{qp} \restriction Y_q$ is a *restriction* of $\mathcal{X}$.

A family $(\, x_p \mid p \in P \,)$ with $x_p \in X_p$ such that $f_{qp}(x_q) = x_p$ for all $p < q$ is a *limit* of $\mathcal{X}$. The set $\varprojlim \mathcal{X}$ of all limits of $\mathcal{X}$ is the *inverse limit* of $\mathcal{X}$. If each of the X_p carries a topology, then $\varprojlim \mathcal{X}$ is a subspace of the product space $\prod_{p \in P} X_p$, and we give $\mathcal{X}$ the subspace topology of the product topology on this space. In this paper, all the sets X_p will be finite and carry the discrete topology. This makes the maps f_{qp} continuous and $\prod_{p \in P} X_p$ compact, so $\mathcal{X}$ is compact too since it is closed in $\prod_{p \in P} X_p$.

The following folklore 'compactness theorem' ensures that limits of such inverse systems exist; see, e.g., [9] for an introduction to inverse limits including its standard short proof.

Lemma 3.1. *The inverse limit of any inverse system of non-empty discrete finite sets is non-empty, Hausdorff and compact.*

[2] If desired, this can be extended to all $p \le q \le r$ in P by letting $f_{pp} = \mathrm{id}_{X_p}$ for all $p \in P$.

Thus, in any inverse system of finite sets we can pick one element from each of them so that these choices commute with the maps f_{qp}. When P is the set of natural numbers, this becomes König's familiar infinity lemma.

The following lemma, whose proof is straightforward from the definitions, translates an inverse system of sets to one of their power sets. It also finds a restriction of the original system whose maps are onto:

Lemma 3.2. *Let $\mathcal{X} = (\, X_p \mid p \in P\,)$ be an inverse system with maps f_{qp}.*

(i) *The family $2^{\mathcal{X}} := (\, 2^{X_p} \mid p \in P\,)$ is an inverse system with respect to $f_{qp}\colon 2^{X_q} \to 2^{X_p}$, where f_{qp} maps $Z \subseteq X_q$ to $\{\, f_{qp}(z) \mid z \in Z\,\} \subseteq X_p$.*

(ii) *Every limit $(\, Z_p \mid p \in P\,) \in \varprojlim 2^{\mathcal{X}}$ is itself an inverse system with maps $f_{qp} \restriction Z_q$: a surjective restriction of $\mathcal{X}$.* □

The topology on $\varprojlim 2^{\mathcal{X}}$ can be used to topologize the power set of $X := \varprojlim \mathcal{X}$, as follows. Every set $Y \subseteq X$ gives rise to the element $(\, Y_p \mid p \in P\,)$ of $\varprojlim 2^{\mathcal{X}}$, with $Y_p \subseteq X_p$ defined as $Y_p := \{\, y_p \in X_p \mid \exists\, (\, x_p \mid p \in P\,) \in Y\colon x_p = y_p\}$ for every $p \in P$. We endow the power set 2^X of X with the topology that makes this map $Y \mapsto (\, Y_p \mid p \in P\,)$ from 2^X to $\varprojlim 2^{\mathcal{X}}$ continuous.

4 Profinite separation systems

For this entire section, let P be a directed poset and $\mathcal{S} = (\, \vec{S_p} \mid p \in P\,)$ a family of finite separation systems $(\vec{S_p}, \le, {}^*)$ with homomorphisms $f_{qp}\colon \vec{S_p} \to \vec{S_q}$. Then $\vec{S} := \varprojlim \mathcal{S}$ becomes a separation system $(\vec{S}, \le, {}^*)$ if we define

- $(\, \vec{r_p} \mid p \in P\,) \le (\, \vec{s_p} \mid p \in P\,) \;:\Leftrightarrow\; \forall p \in P\colon \vec{r_p} \le \vec{s_p}$ and
- $(\, \vec{s_p} \mid p \in P\,)^* := (\, \overleftarrow{s_p} \mid p \in P\,)$.

We call $(\vec{S}, \le, {}^*)$ the *inverse limit* of the separation systems $(\vec{S_p}, \le, {}^*)$. Separation systems that are isomorphic to the inverse limit of some finite separation systems are *profinite.*

For example, let V be any set, $(\vec{S}_V, \le, {}^*)$ the system of set separations of V, and suppose that P is the set of finite subsets of V ordered by inclusion. For each $p \in P$ assume that $(\vec{S_p}, \le, {}^*)$ is the system of all the set separations of p, and that the f_{qp} are restrictions: that for $q > p$ and $\vec{s_q} = (A, B) \in \vec{S_q}$ we have

$$f_{qp}(\vec{s_q}) = \vec{s_q} \restriction p := (A \cap p, B \cap p).$$

Then every limit $\vec{s} = (\, \vec{s_p} \mid p \in P\,) \in \vec{S} = \varprojlim \mathcal{S}$ defines an oriented separation (A, B) of V by setting $A := \bigcup_{p \in P} A_p$ and $B := \bigcup_{p \in P} B_p$ for $\vec{s_p} =: (A_p, B_p)$. This (A, B) is the unique separation of V such that $(A \cap p, B \cap p) = \vec{s_p}$ for all p, and $\vec{s} \mapsto (A, B)$ is easily seen to be an isomorphism between the separation systems $\vec{S}$ and $\vec{S}_V$.

Although we shall mostly work with abstract separation systems and their inverse limits in this paper, it will be useful to keep this example in mind. To help support this intuition, we shall usually write the maps f_{qp} as restrictions, i.e., write

$$\vec{s_q} \restriction p := f_{qp}(\vec{s_q})$$

whenever $\vec{s_q} \in \vec{S_q}$ and $q > p$, as well as

$$\vec{s} \restriction p := \vec{s_p}$$

whenever $\vec{s} = (\vec{s_p} \mid p \in P) \in \vec{S}$, even when the separations considered are not separations of sets. Similarly, given $s \in S$, or $s \in S_q$ with $q \geq p$, we write $s \restriction p := \{\vec{s_p}, \overleftarrow{s_p}\}$ for the separation in S_p with orientations $\vec{s_p} := \vec{s} \restriction p$ and $\overleftarrow{s_p} := \overleftarrow{s} \restriction p$.

Given a subset $O \subseteq \vec{S}$ or a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of such subsets, we write

$$O \restriction p := \{\, \vec{s} \restriction p \mid \vec{s} \in O \,\} \quad \text{and} \quad \mathcal{F} \restriction p := \{\, O \restriction p \mid O \in \mathcal{F} \,\}.$$

We shall refer to these as the *projections* of $\vec{s_q}$, $\vec{s}$, O and $\mathcal{F}$ to p. Note that $\vec{S} \restriction p = \vec{S_q} \restriction p \subseteq \vec{S_p}$ for all p and $q \geq p$, with equality if our inverse system $\mathcal{S}$ is surjective.

Even if our inverse system $(\,\vec{S_p} \mid p \in P\,)$ is not surjective, it induces the surjective inverse system $(\,\vec{S} \restriction p \mid p \in P\,)$, whose inverse limit is again $\vec{S}$ with the same topology. When we consider a given profinite separation system in this paper and need to work with a concrete inverse system of which it is the inverse limit, we can therefore always choose this inverse system to be surjective.

By Lemma 3.1, our separation system $\vec{S} = \varprojlim \mathcal{S}$ is a compact topological space. Every subset O of $\vec{S}$ induces a restriction of $\mathcal{S}$, the inverse system $(\,O \restriction p \mid p \in P\,)$. While clearly $O \subseteq \varprojlim\, (\,O \restriction p \mid p \in P\,)$, this inclusion can be strict. In fact, $\varprojlim\, (\,O \restriction p \mid p \in P\,)$ is precisely the topological closure $\overline{O}$ of O in $\vec{S}$:

Lemma 4.1. (i) *The topological closure in $\vec{S}$ of a set $O \subseteq \vec{S}$ is the set of all limits $\vec{s} = (\,\vec{s_p} \mid p \in P\,)$ with $\vec{s_p} \in O \restriction p$ for all p.*

(ii) *A set $O \subseteq \vec{S}$ is closed in $\vec{S}$ if and only if there are sets $O_p \subseteq \vec{S_p}$, with $f_{qp} \restriction O_q \subseteq O_p$ whenever $p < q \in P$, such that $O = \varprojlim\, (\,O_p \mid p \in P\,)$.*

Proof. Assertion (i) is straightforward from the definition of the product topology on $\vec{S}$. The forward implication of (ii) follows from (i) with $O_p = O \restriction p$. For the other direction, note that $\varprojlim\, (\,O_p \mid p \in P\,)$ is a compact subspace of the Hausdorff space $\vec{S}$ (Lemma 3.1). $\square$

An immediate consequence of Lemma 4.1 is that every separation system $\tau \subseteq \vec{S}$ that is closed in $\vec{S}$ is itself profinite and can therefore be studied independently of its ambient system $\vec{S}$. We shall later have to deal in particular with nested subsystems of given separation systems. We shall therefore study profinite nested separation systems and tree sets first (Section 5).

Lemma 4.1 will be our main tool for checking whether orientations of separation systems are closed. Let us illustrate how this is typically done.

Example 4.2. Consider a ray $G = v_1 e_1 v_2 e_2 v_3 \ldots$, with end ω say, and let $\vec{R}$ be the separation system consisting of all separations of order $< k$ of the graph G for some fixed integer $k \geq 2$. Let P be the set of all finite subsets of $V = V(G)$, and for $p \in P$ let $\vec{S_p}$ be the set of separations of order $< k$ of the subgraph $G[p]$. With restriction as bonding maps, this makes $\mathcal{S} = (\vec{S_p} \mid p \in P)$ into an inverse system, with $\vec{R} \simeq \vec{S} = \varprojlim \mathcal{S}$ via $(A, B) \mapsto (\,(A \cap p\,,\, B \cap p) \mid p \in P\,)$ for all $(A, B) \in \vec{R}$, as earlier.

Now let O be the orientation of R 'towards ω', the set of all separations $(A, B) \in \vec{R}$ such that $G[B]$ contains a tail of G. Note that since k is finite, O is well defined and consistent. Clearly, $(V, \emptyset) \notin O$. It is also straightforward to check from first principles that $(V, \emptyset)$ lies in the closure of O.

Instead, let us use Lemma 4.1 to prove this. We just have to find separations $\vec{s_p} \in O \restriction p$ such that $\vec{s} = (\,\vec{s_p} \mid p \in P\,)$ corresponds to $(V, \emptyset)$ under our isomorphism between $\vec{R}$ and $\vec{S}$. Let $\vec{s_p} := (p, \emptyset)$ for each $p \in P$. These $\vec{s_p}$ lie in $O \upharpoonright p$, because $\vec{s_p} = \vec{r} \upharpoonright p$ for $\vec{r} = (p, V \smallsetminus p) \in O$. But we also have $\vec{s_p} = (V, \emptyset) \upharpoonright p$ for each p, so $(V, \emptyset)$ corresponds to $\vec{s} = (\, \vec{s_p} \mid p \in P\,)$ as desired. $\square$

In Example 4.2 we expressed $(V, \emptyset)$, via the isomorphism $\vec{R} \to \vec{S}$, as the limit $\vec{s} = (\, \vec{s_p} \mid p \in P\,)$ of its projections $\vec{s_p} = (p, \emptyset)$. It can also be expressed as a limit in $\vec{R}$ itself in various ways. For example, $(V, \emptyset)$ is the supremum in $\vec{R}$ of the chain $C = \{\, (A_n, B_n) \mid n \in \mathbb{N}\,\} \subseteq O$ where $A_n := \{\, v_i \mid i < n\,\}$ and $B_n := \{\, v_i \mid i \geq n\,\}$. Topologically, $(V, \emptyset)$ lies in the closure in $\vec{S}$ not only of O but also of C.

Our next lemma generalizes these observations to our abstract $\vec{S}$: chains in $\vec{S}$ always have a supremum, which lies in their topological closure.

Lemma 4.3. *If $C \subseteq \vec{S}$ is a non-empty chain, then C has a supremum and an infimum in $\vec{S}$. Both these lie in the closure of C in $\vec{S}$.*

Proof. We prove the assertion about suprema; the proof for infima is analogous.

For each $p \in P$, let $\vec{s_p}$ be the maximum of the finite chain $C \upharpoonright p$ in $\vec{S_p}$. If these $\vec{s_p}$ are compatible, in the sense that $\vec{s} := (\, \vec{s_p} \mid p \in P\,) \in \vec{S}$, then $\vec{s}$ is clearly the supremum of C, and it will lie in its closure by Lemma 4.1.

To show that the $\vec{s_p}$ are compatible, suppose that $\vec{s_q} \upharpoonright p \neq \vec{s_p}$ for some $p < q$. Then $\vec{s_q} \upharpoonright p < \vec{s_p}$, since $\vec{s_q} \upharpoonright p$ lies in $(C \upharpoonright q) \upharpoonright p = C \upharpoonright p$, whose maximum is $\vec{s_p}$. Pick any $\vec{r} \in C$ with $\vec{r} \upharpoonright p = \vec{s_p}$. Then $\vec{r} \upharpoonright q \neq \vec{s_q}$ and hence $\vec{r} \upharpoonright q < \vec{s_q}$, while $f_{qp}(\vec{r} \upharpoonright q) = \vec{r} \upharpoonright p = \vec{s_p} > \vec{s_q} \upharpoonright p = f_{qp}(\vec{s_q})$. Hence f_{qp} does not respect the ordering on $\vec{S_q}$, which contradicts our assumption that it is a homomorphism of separation systems. $\square$

A direct consequence of Lemma 4.3 is that, in closed subsets of $\vec{S}$, every separation lies below some maximal element and above some minimal element:

Lemma 4.4. *If $O \subseteq \vec{S}$ is closed in $\vec{S}$, then for every $\vec{s} \in O$ there exist in O some $\vec{r} \leq \vec{s}$ that is minimal in O and some $\vec{t} \geq \vec{s}$ that is maximal in O.*

Proof. Let C be a maximal chain in O containing $\vec{s}$. By Lemma 4.3, C has an infimum $\vec{r}$ and a supremum $\vec{t}$ in $\vec{S}$. By the lemma these lie in $\overline{C}$, and hence in O since O is closed. Then $\vec{r}$ is minimal and $\vec{t}$ is maximal in O, by the maximality of C, and $\vec{r} \leq \vec{s} \leq \vec{t}$ as desired. $\square$

Recall that we shall be interested in tree sets $\tau \subseteq \vec{S}$, and in particular in their consistent orientations. As we noted earlier, the consistent orientations O of τ can be retrieved as $O = \lceil \sigma \rceil$ from just their sets σ of maximal elements, the splitting stars of τ—but only if every $\vec{r} \in O$ lies below some maximal element of O. Lemma 4.4 says that this is the case when O is closed in $\vec{S}$:

Corollary 4.5. *A consistent orientation of a nested separation system $\tau \subseteq \vec{S}$ splits if it is closed in $\vec{S}$.* $\square$

In Section 6 we shall see that Lemma 4.5 does not have a direct converse: profinite tree sets can have consistent orientations that split but are not closed in these tree sets.

5 Profinite nested separation systems

Let us again consider a profinite separation system $\vec{S} = \varprojlim (\vec{S_p} \mid p \in P) \neq \emptyset$ with bonding maps f_{qp}, fixed throughout this section. Our aim in this section is to study the relationship between $\vec{S}$ and the projections $\vec{S_p}$. This will enable us later to lift theorems about finite separation systems to profinite ones.

We are particularly interested in the cases where the $\vec{S_p}$ (and hence $\vec{S}$, as we shall see) are nested; we shall then study the interplay between the splitting stars of $\vec{S}$ and those of the $\vec{S_p}$. Understanding this interplay will be crucial to lifting tangle duality or tangle-tree theorems about finite separation systems to profinite ones, as both these types of theorem are about tree sets and their splitting stars.

We begin our study with some basic observations. Since the $\vec{S_p}$ form an inverse system of separations, and $\vec{S}$ is its inverse limit, we have

$$\vec{r} \leq \vec{s} \;\Rightarrow\; \vec{r} \restriction p \leq \vec{s} \restriction p \tag{1}$$

for all $\vec{r}, \vec{s} \in \vec{S_q}$ with $q \geq p$ or $\vec{r}, \vec{s} \in \vec{S}$. In particular, projections of nested sets of separations are nested, and projections of stars are stars. Thus if our inverse system $(\vec{S_p} \mid p \in P)$ is surjective and $\vec{S}$ is nested, then so are all the $\vec{S_p}$.

Projections also commute with inversion: $\overleftarrow{s} \restriction p = (\vec{s} \restriction p)^*$ for all $\vec{s}$ as above. Hence projections of small separations are small, and projections of trivial separations are trivial if p is large enough to distinguish them from their witness:

Lemma 5.1. *Let $\vec{r} = (\vec{r_p} \mid p \in P) \in \vec{S}$ and $\vec{s} = (\vec{s_p} \mid p \in P) \in \vec{S}$ be such that $\vec{r}$ is trivial in $\vec{S}$ and s witnesses this. Then there exists $p \in P$ such that, for all $q \geq p$ in P, the separation $\vec{r_q}$ is trivial in $\vec{S_q}$ and s_q witnesses this.*

Proof. Since s witnesses the triviality of $\vec{r}$, we have $\vec{r} \notin \{ \vec{s}, \overleftarrow{s} \}$. Hence there exists $p \in P$ such that $\vec{r_q} \notin \{ \vec{s_q}, \overleftarrow{s_q} \}$ for all $q \geq p$. For all such q we have $\vec{r_q} < \vec{s_q}$ as well as $\vec{r_q} < \overleftarrow{s} \restriction q = \overleftarrow{s_q}$ by (1), so s_q witnesses the triviality of $\vec{r_q}$ in $\vec{S_q}$. ☐

Let us see to what extent these observations, starting with (1), have a converse. Suppose we know that the projections in $\vec{S_p}$ of some elements of $\vec{S}$ or of $\vec{S_q}$ with $p < q$ are small, trivial, nested or splitting stars: can we infer that these elements themselves are small, trivial, nested or splitting stars in $\vec{S}$ or $\vec{S_q}$?

The direct converse of (1) will usually fail: two separations of a set, for example, can happily cross even if their restrictions to some small subset are nested. If $\vec{S}$ is nested, however, we have a kind of converse of (1):

Lemma 5.2. *Let $\vec{r}, \vec{s}$ be elements of a nested set $\tau \subseteq \vec{S}$, and let $p \in P$. Assume that $r \restriction p \neq s \restriction p$, that neither $\vec{r} \restriction p$ nor $\overleftarrow{s} \restriction p$ is trivial in $\tau \restriction p$, and that $\vec{r} \restriction p \leq \vec{s} \restriction p$. Then $r \neq s$ and $\vec{r} \leq \vec{s}$.*

Proof. Our assumption of $r \restriction p \neq s \restriction p$ implies $r = \{ \vec{r}, \overleftarrow{r} \} \neq \{ \vec{s}, \overleftarrow{s} \} = s$ by (1). If orientations of r and s were related in any way other than as $\vec{r} \leq \vec{s}$ then, by (1), their projections to p would be related correspondingly. This would contradict our assumption of $\vec{r} \restriction p \leq \vec{s} \restriction p$ as either $\vec{r} \restriction p$ or $\overleftarrow{s} \restriction p$ would then be trivial in $\tau \restriction p$, or $\vec{r} \restriction p = \vec{s} \restriction p$. Since τ is nested, however, r and s must have comparable orientations. The only possibility left, then, is that $\vec{r} \leq \vec{s}$ as desired. ☐

Lemma 5.2 says that any ordering between two fixed projections of limits $\vec r, \vec s \in \vec S$ lifts to $\vec r$ and $\vec s$ themselves if we assume that these are nested. Our next lemma says that this will be the case as soon as the $\vec S_p$ are nested:

Lemma 5.3. *If every $\vec S_p$ is nested, then so is $\vec S$.*

Proof. If $r, s \in S$ are not nested, then $\vec r \not\le \vec s$ for all four choices of orientations $\vec r$ of r and $\vec s$ of s. Each of these four statement is witnessed by some $p \in P$, in that $\vec r \restriction p \not\le \vec s \restriction p$. Since P is a directed poset, there exists some $q \in P$ that is larger than these four p. By (1), then, $\vec r \restriction q \not\le \vec s \restriction q$ for all choices of orientations $\vec r$ of r and $\vec s$ of s. But these $\vec r \restriction q$ and $\vec s \restriction q$ are the four orientations of $r \restriction q$ and $s \restriction q$, since the involution in $\vec S$ commutes with that in $\vec S_q$ via the projection $\vec S \to \vec S_q$. Hence $\vec S_q$ is not nested, contrary to assumption. $\square$

The fact that projections of small separations are small also has a converse at the limit:

Lemma 5.4. *If $\vec r = (\vec r_p \mid p \in P) \in \vec S$ is such that each $\vec r_p$ is small, then $\vec r$ is small.*

Proof. As projections commute with the involutions *, we have $\overleftarrow r = (\overleftarrow{r}_p \mid p \in P)$. Hence if $\vec r_p \le \overleftarrow r_p$ for all $p \in P$, then $\vec r \le \overleftarrow r$ by definition of $\le$ on $\vec S$. $\square$

Lemma 5.4 says that being small is a property that cannot disappear suddenly at the limit. But it is not clear that regular profinite separation systems, those without small elements, are inverse limits of finite regular separation systems. Such finite systems can, however, be found:

Proposition 5.5.

(i) *Every inverse limit of finite regular separation systems is regular.*

(ii) *Every profinite regular separation system is an inverse limit of finite regular separation systems.*

Proof. Assertion (i) follows immediately from (1).

For (ii) assume that $\vec S = \varprojlim (\vec S_p \mid p \in P)$ is regular. If some $p_0 \in P$ exists for which $\vec S_{p_0}$ is regular then, by (1), every $\vec S_p$ with $p \ge p_0$ is regular, and $\vec S = \varprojlim (\vec S_p \mid p \ge p_0)$ as desired.

Suppose now that no such p_0 exists, i.e., that no $\vec S_p$ is regular. For each $p \in P$ let $X_p \ne \emptyset$ be the set of small separations in $\vec S_p$. Then $(X_p \mid p \in P)$ is a restriction of $(\vec S_p \mid p \in P)$, since $f_{qp}(X_q) \subseteq X_p$ for all $p < q$ by (1). By Lemma 3.1 it has a non-empty inverse limit X. Pick $\vec x \in X$. Then also $\vec x \in \vec S$, and by construction every projection of $\vec x$ is small. But then Lemma 5.4 implies that $\vec x$ is small, contrary to our assumption that $\vec S$ is regular. $\square$

Interestingly, we cannot replace the word 'small' in Lemma 5.4 with 'trivial'. It is true that if all projections of $\vec r \in \vec S$ are trivial then there exists $s \in S$ such that both $\vec r \in \vec s$ and $\vec r \in \overleftarrow s$. But this s cannot, in general, be chosen distinct from r: it may happen that $\vec s = \vec r$ is the only choice. In that case, we have said no more than that $\vec r$ is small—which we know already from Lemma 5.4.

Let us call $\vec r = (\vec r_p \mid p \in P) \in \vec S$ *finitely trivial* in $\vec S$ if for every $p \in P$ there exists $q \ge p$ in P such that $\vec r_q$ is trivial in $\vec S \restriction q$. By Lemma 5.4, any such $\vec r$ will be small, since if $\vec r_q$ is trivial and hence small, then $\vec r_p$ too will be small, by (1), if $p < q$. However, even if $\vec{S}$ is a tree set and $\vec{r}$ is maximal in a closed consistent orientation of $\vec{S}$ it can happen that $\vec{r} \in \vec{S}$ is finitely trivial (but not trivial), even with $\vec{r_p}$ trivial in $\vec{S} \upharpoonright p$ for all $p \in P$. Here is an example:

Example 5.6. Let $P = \{1, 2, \dots\}$. For each $p \in P$ let σ_p be a proper star with p elements, and $\vec{S_p}$ the separation system consisting of σ_p and the respective inverses as well as a separation $s_p = \{\vec{s_p}, \overleftarrow{s_p}\}$, where $\vec{s_p}$ is trivial in $\vec{S_p}$ with exactly one element $\vec{t_p}$ of σ_p witnessing this. Let $f_{p+1,p} \colon \vec{S}_{p+1} \to \vec{S_p}$ be the homomorphism which maps $\vec{s}_{p+1}$ and $\vec{t}_{p+1}$ to $\vec{s_p}$, maps $\sigma_{p+1} \setminus \{\vec{t}_{p+1}\}$ bijectively to σ_p, and is defined on the inverses of these separations so as to commute with the inversion. These maps induce bonding maps $f_{qp} \colon \vec{S_q} \to \vec{S_p}$ by concatenation.

Let $\vec{S} := \varprojlim\, (\vec{S_p} \mid p \in P)$. Then $\vec{s} := (\,\vec{s_p} \mid p \in P\,) \in \vec{S}$ is the only separation in $\vec{S}$ whose projections to p do not meet σ_p, for any $p \in P$. It is easy to see that every $r \in S \setminus \{s\}$ has an orientation $\vec{r}$ such that $\vec{r} \upharpoonright p = \vec{t_p}$ for some p. Then $\vec{r} \upharpoonright p' = \vec{s_{p'}}$ for any $p' < p$, and $\vec{r} \upharpoonright q \in \sigma_q \setminus \{\vec{t_q}\}$ for all $q > p$. So this p is in fact unique given r, and conversely $\vec{r}$ is the only element of $\vec{S}$ such that $\vec{r} \upharpoonright p = \vec{t_p}$; let us denote this $\vec{r}$ by $\vec{r_p}$.

We have shown that $\vec{S}$ consists of the star $\sigma := \{\,\vec{s}\,\} \cup \{\,\vec{r_p} \mid p \in P\,\}$ plus inverses. By construction $\vec{s}$ is finitely trivial; in fact, all of its projections are trivial. However $\vec{s}$ is not trivial in $\vec{S}$, because no r_p could witness this: as $\vec{r_p} \upharpoonright q \in \sigma_q \setminus \{\vec{t_q}\}$ for $q > p$, and $\vec{t_q}$ is the only element of σ_q witnessing the triviality of $\vec{s_q}$, the fact that $\vec{s_q} \le \overleftarrow{t_q} \le \overleftarrow{r_p} \upharpoonright q$ (since σ_q is a star containing both $\vec{t_q}$ and $\vec{r_p} \upharpoonright q$) implies that $\vec{s_q} \not\le \vec{r_p} \upharpoonright q$ and hence $\vec{s} \not\le \vec{r_p}$ by (1).

Thus, $\vec{s}$ is a maximal element of the star σ, which is a closed consistent orientation of $\vec{S}$. $\square$

Note how the separation $\vec{s}$ in Example 5.6 escapes being trivial even though all its projections are trivial, because the witnesses $\vec{t_p}$ of the triviality of those projections do not lift to a common element of $\vec{S}$, which would then witness the triviality of $\vec{s}$. Rather, the $\vec{t_p}$ are projections of pairwise different separations $\vec{r_p} \in \vec{S}$, none of which can alone witness the triviality of $\vec{s}$.

The fact that σ can be a splitting star containing a finitely trivial separation, even though splitting stars cannot contain trivial separations (Lemma 2.2), thus seems to rest on the fact that σ is infinite. This is indeed the case:

Lemma 5.7. *Assume that $\vec{S}$ is nested, and let $\sigma = \{\,\vec{r}, \vec{s_1}, \dots, \vec{s_n}\}$ be any finite splitting star of $\vec{S}$ (where $n \ge 0$).*

(i) *No element of σ is finitely trivial in $\vec{S}$.*

(ii) *If $\sigma = \{\,\vec{r}\,\}$, then $\vec{r} \upharpoonright p$ is nontrivial in $\vec{S} \upharpoonright p$ for every $p \in P$.*

Proof. Let O be a consistent orientation of $\vec{S}$ that splits at σ.

For (i) suppose without loss of generality that $\vec{r}$ is finitely trivial in $\vec{S}$. Let $P' \subseteq P$ be the set of all $p \in P$ for which $\vec{r} \upharpoonright p$ is trivial in $\vec{S} \upharpoonright p$, and for each $p \in P'$ pick some $s(p) \in S$ such that $s(p) \upharpoonright p$ witnesses the triviality of $\vec{r} \upharpoonright p$ in $\vec{S} \upharpoonright p$. Let $\vec{s}\,(p)$ be the orientation of $s(p)$ that lies in O.

As $\vec{r} \upharpoonright p < \vec{s}\,(p) \upharpoonright p$ by the choice of $s(p)$ we cannot have $\vec{s}\,(p) \le \overleftarrow{r}$, by (1). Hence $\vec{s}\,(p) \le \overleftarrow{s_i}$ for some i. As σ is finite, there is some $i \in \{1, \dots, n\}$ for which the set P'' of all $p \in P'$ with $\vec{s}\,(p) \le \overleftarrow{s_i}$ is cofinal in P' (and hence in P). But now we have $\vec{r} \upharpoonright p < \vec{s}\,(p) \upharpoonright p \le \overleftarrow{s_i} \upharpoonright p$ for every $p \in P''$, and hence $\vec{r} \upharpoonright p \le \overleftarrow{s_i} \upharpoonright p$ for all $p \in P$. But this means that $\vec{r} \le \overleftarrow{s_i}$, which contradicts the fact that σ is a proper star (Lemma 2.2).

For (ii) let $p \in P$ be given and suppose that $\vec{r} \restriction p$ is trivial in $\vec{S} \restriction p$. Let $\vec{s} \in O$ be such that $s \restriction p$ witnesses the triviality of $\vec{r} \restriction p$ in $\vec{S} \restriction p$. Then $\vec{r} \restriction p < \vec{s} \restriction p$. But as $\{\vec{r}\}$ splits $\vec{S}$, witnessed by $O \ni \vec{s}$, we also have $\vec{s} \le \vec{r}$ and hence $\vec{s} \restriction p \le \vec{r} \restriction p$, by (1). This is a contradiction. $\square$

Our situation when lifting theorems about finite separation systems to profinite ones will be that we know the $\vec{S_p}$ and wish to study $\vec{S}$. In particular, we shall be interested in its closed consistent orientations, given those of the $\vec{S_p}$.

Let us begin by addressing a local question in this context. Suppose an orientation O of $\vec{S}$ that we would like to be consistent is in fact inconsistent, and that this is witnessed by an inconsistent pair $\{\overleftarrow{s}, \vec{s'}\} \subseteq \vec{S}$. Will this show in the projections of $\overleftarrow{s_p}$ and $\vec{s_p'}$? More specifically, if we start with a consistent orientation O_p of S_p (without degenerate elements) and take as O the unique orientation of S that projects to a subset of O_p, will O be consistent?

Given an inconsistent pair $\{\overleftarrow{s}, \vec{s'}\} \subseteq \vec{S}$, consider $\overleftarrow{r_p} := \overleftarrow{s} \restriction p$ and $\vec{r_p'} := \vec{s'} \restriction p$ for some $p \in P$. If $r_p \neq r_p'$ then $\{\overleftarrow{r_p}, \vec{r_p'}\}$ is also inconsistent, by (1). But it can also happen that $r_p = r_p'$. In that case we would expect that $\overleftarrow{r_p}$ and $\vec{r_p'}$ are inverse to each other, i.e. that $\vec{r_p} = \vec{r_p'}$. This is indeed what usually happens, and it will not cause us any problems. (In our example, only one of $\overleftarrow{r_p}$ and $\vec{r_p'}$ lies in O_p, so only one of $\overleftarrow{s}$ and $\vec{s'}$ will be adopted for O.) However it can also happen that $\overleftarrow{s}$ and $\vec{s'}$, despite being inconsistent, induce the same orientation $\overleftarrow{r_p} = \vec{r_p'}$ in $\vec{S_p}$:

Example 5.8. Let V be a disjoint union of four sets A, B, C, X. Let $\vec{S}$ be the system of set separations of V, and consider its projections to $U := C \cup X$. Let $\vec{s} = (X \cup A, C \cup X \cup B)$ and $\vec{s'} = (C \cup X \cup A, X \cup B)$. Then $\vec{s} < \vec{s'}$, so $\{\overleftarrow{s}, \vec{s'}\}$ is inconsistent, but their projections $\overleftarrow{r_p} = \vec{r_p'} = (C \cup X, X)$ to U conincide.

In the above example, notice that $\vec{r_p}$ is small. This will, in fact, always be the case in this situation, since $\vec{s} \le \vec{s'}$ because $\overleftarrow{s}, \vec{s'}$ are inconsistent, and hence $\vec{r_p} \le \vec{r_p'} = \overleftarrow{r_p}$ by (1). If these $\vec{r_p} \in \vec{S_p}$ are the projections of some $\vec{r} = (\vec{r_p} \mid p \in P)$, then $\vec{r}$ too will be small, by Lemma 5.4.

Our next lemma relates the inverse $\overleftarrow{r}$ of such an $\vec{r} \in \vec{S}$ to those inconsistent pairs $\overleftarrow{s}, \vec{s'} \in \vec{S}$ that also project to its projections $\overleftarrow{r_p}$. Recall from the end of Section 3 the definition of the topology on $2^{\mathcal{X}}$, which we need below for $\mathcal{X} = \vec{S}$.

Lemma 5.9. *Let $\vec{r} = (\vec{r_p} \mid p \in P) \in \vec{S}$ be such that for every $p \in P$ there exists an inconsistent set $\{\overleftarrow{s_p}, \vec{s_p'}\} \subseteq \vec{S}$ such that $\overleftarrow{s_p} \restriction p = \overleftarrow{r_p} = \vec{s_p'} \restriction p$. Then $\vec{r}$ is small, and $\{\overleftarrow{r}\}$ is a limit point in $2^{\vec{S}}$ of the set $\{\{\overleftarrow{s_p}, \vec{s_p'}\} \mid p \in P\}$.*

Proof. For each $p \in P$ the set $\{\overleftarrow{s_p}, \vec{s_p'}\} \subseteq \vec{S}$ inconsistent, so $\vec{s_p} \restriction p \le \vec{s_p'} \restriction p$ by (1) and hence $\vec{r_p} = \vec{s_p} \restriction p \le \vec{s_p'} \restriction p = \overleftarrow{r_p}$. By Lemma 5.4 this implies that $\vec{r}$ is small.

To show that $\{\overleftarrow{r}\}$ is a limit point of the set $\{\{\overleftarrow{s_p}, \vec{s_p'}\} \mid p \in P\}$, consider any fixed $p \in P$. Then the point $\sigma_p = \{\overleftarrow{s_p}, \vec{s_p'}\}$ of $2^{\vec{S}}$ lies in the basic neighbourhood of the point $\{\overleftarrow{r}\}$ in $2^{\vec{S}}$ given by p, the set of all $\sigma \subseteq \vec{S}$ such that $\sigma \restriction p = \{\overleftarrow{r}\} \restriction p$, since $\overleftarrow{s_p} \restriction p = \overleftarrow{r_p} = \vec{s_p'} \restriction p$. $\square$

Let us return to the question of what we can deduce about $\vec{S}$ from information we have about the $\vec{S_p}$. We saw in Lemma 5.3 that if all the $\vec{S_p}$ are nested then so is $\vec{S}$. Then the closed consistent orientations of $\vec{S}$, which we shall be interested in most, are essentially determined by its splitting stars (see

Section 6). Ideally, then, the stars σ splitting $\vec{S}$ should be the limits (as sets) of the stars splitting the $\vec{S_p}$.

Before we prove two lemmas saying essentially this, let us see why it cannot quite be true as stated. For a start, σ might consist of two separations $\vec{s_1}, \vec{s_2} \in \vec{S}$ whose projections $\vec{s_1} \upharpoonright p$ and $\vec{s_2} \upharpoonright p$ to some $p \in P$ are inverse to each other (i.e., are the two orientations of the same unoriented separation in S_p), and hence cannot lie in a common consistent orientation of S_p. More fundamentally recall that, by Lemma 2.2, splitting stars contain no trivial separations. But separations $\vec{s} \in \vec{S}$ that are not trivial in $\vec{S}$ may well have projections $\vec{s} \upharpoonright p$ that are trivial in $\vec{S_p}$, as we saw in Example 5.6. Hence we cannot expect $\sigma \upharpoonright p$ to split $\vec{S_p}$ just because σ splits $\vec{S}$.

Our next lemma shows that these two examples are essentially the only ones. The first can be overcome by choosing p large enough, while in the second we only have to delete the trivial separations in $\sigma \upharpoonright p$ to obtain a splitting star of $\vec{S_p}$.

Recall that $\vec{S}^\circ$ denotes the *essential core* of $\vec{S}$, the separation system obtained from $\vec{S}$ by deleting its trivial and co-trivial elements. Given a set σ of oriented separations, let us write $\overleftarrow{\sigma} := \{ \overleftarrow{s} \mid \vec{s} \in \sigma \}$.

Lemma 5.10. *Assume that* $(\vec{S_p} \mid p \in P)$ *is surjective, and that the* $\vec{S_p}$ *and hence* $\vec{S}$ *are nested. Assume further that* $\sigma \subseteq \vec{S}$ *splits* $\vec{S}$*, witnessed by the consistent orientation* O *of* $\vec{S}$*. Then either* $\sigma = \{\vec{r}\}$ *with* $\overleftarrow{r}$ *finitely trivial in* $\vec{S}$*, or there exists* $p_0 \in P$ *such that, for all* $p \geq p_0$*, the set* $\sigma_p := (\sigma \upharpoonright p) \cap \vec{S}_p^\circ$ *splits* $\vec{S_p}$*, witnessed by the consistent orientation* $O_p := (O \upharpoonright p) \setminus \overleftarrow{\sigma}_p$ *of* $\vec{S_p}$*.*

Proof. Consider first the case that $|\sigma| = 1$, i.e., that σ has the form $\{\vec{r}\}$ for some $\vec{r} = (\vec{r_p} \mid p \in P) \in \vec{S}$. By Lemma 5.7, $\vec{r}$ cannot be finitely trivial, and if $\overleftarrow{r}$ is finitely trivial we are done by assumption. Thus we may assume that neither $\vec{r}$ nor $\overleftarrow{r}$ is finitely trivial in $\vec{S}$. Then there exists $p_0 \in P$ such that $\vec{r_p} \in \vec{S}_p^\circ$ for all $p \geq p_0$. We claim that this p_0 is as desired.

To see this, observe that $\vec{r}$ is the greatest element of O, which by (1) implies that $\vec{r_p}$ is the greatest element of O_p. As all orientations with a greatest element are consistent, it thus suffices to show that O_p is an orientation of $\vec{S_p}$. So suppose there is a nondegenerate $s \in \vec{S_p}$ for which $\vec{s}, \overleftarrow{s} \in O_p$. Then $\vec{s} < \vec{r_p}$ and $\overleftarrow{s} < \vec{r_p}$, so $\vec{r_p}$ is co-trivial in $\vec{S_p}$, contrary to the choice of p_0.

Let us now consider the case that $|\sigma| \geq 2$. We shall first show that there exists some $p_0 \in P$ such that all $O \upharpoonright p$ with $p \geq p_0$ are consistent orientations of $\vec{S_p}$. We then show that this p_0 is as desired.

By the surjectivity of the bonding maps we have $\vec{S_p} = \vec{S} \upharpoonright p$ for each $p \in P$, so $O \upharpoonright p$ contains at least one of $\vec{s_p}$ and $\overleftarrow{s_p}$ for each $s_p \in \vec{S_p}$. It thus suffices to find p_0 so that $O \upharpoonright p$ is antisymmetric and consistent for all $p \geq p_0$.

For every $p \in P$ let

$$I_p := \{ (\vec{r_p}, \vec{s_p}) \mid \vec{r_p}, \vec{s_p} \in O \upharpoonright p \text{ and } \overleftarrow{r_p} \leq \vec{s_p} \}.$$

Note that $O \upharpoonright p$ is a consistent orientation of $\vec{S_p}$ if (and only if) I_p is empty: any pair $(\vec{r_p}, \vec{s_p})$ of separations in $O \upharpoonright p$ that witnesses that $O \upharpoonright p$ is inconsistent lies in I_p, and so does any pair witnessing that $O \upharpoonright p$ is not antisymmetric. Furthermore, for any $q > p$, if $(\vec{r_q}, \vec{s_q})$ lies in I_q then $(\vec{r_q} \upharpoonright p, \vec{s_q} \upharpoonright p)$ lies in I_p. Thus if I_{p_0} is empty for some $p_0 \in P$ then I_p is empty for all $p \geq p_0$.

Suppose I_p is non-empty for all $p \in P$. The family $(I_p \mid p \in P)$ forms an inverse system with bonding maps borrowed from $(\vec{S_p} \mid p \in P)$, which by

Lemma 3.1 has a non-empty inverse limit. Let $(\vec r, \vec s) = (\,(\vec{r_p}, \vec{s_p}) \mid p \in P\,)$ be an element of this inverse limit. Then $\vec r = (\,\vec{r_p} \mid p \in P\,)$ and $\vec s = (\,\vec{s_p} \mid p \in P\,)$ are elements of $\vec S$ satisfying $\overleftarrow{r} \le \vec s$. By Lemma 6.3,[3] O is closed in $\vec S$, giving $\vec r, \vec s \in O$ by Lemma 4.1(i) since $\vec{r_p}, \vec{s_p} \in O \upharpoonright p$ for all p.

But this contradicts the fact that O is a consistent orientation of $\vec S$. Indeed, if $r \neq s$ then $\vec r$ and $\vec s$ witness its inconsistency. If $r = s$ with $\vec r = \vec s$, then $\overleftarrow{r} \le \vec s = \vec r$, so $\vec r$ is co-small. Choose $\vec{s'} \in \sigma$ so that $\vec s \le \vec{s'}$. Then every $\vec t \in \sigma \setminus \{\vec{s'}\}$ (which exists, since $|\sigma| \ge 2$) is trivial, as $\vec t < \overleftarrow{s'} \le \overleftarrow{s} \le \vec s$, which contradicts Lemma 2.2. And finally, if $r = s$ with $\vec r = \overleftarrow{s}$, then r must be degenerate as O is antisymmetric, but in that case $\sigma = \{\vec r\}$ by Lemma 2.2, contradicting $|\sigma| \ge 2$. This completes our proof that there exists $p_0 \in P$ such that $O \upharpoonright p$ is a consistent orientation of $\vec{S_p}$ for all $p \ge p_0$.

Let us now show that this p_0 is as claimed. For this we only need to check, for all $p \ge p_0$, that $\sigma_p := (\sigma \upharpoonright p) \cap \vec{S_p}^{\circ}$ is the set of maximal elements of $O \upharpoonright p$: since $O \upharpoonright p$ is a consistent orientation of $\vec{S_p}$, by the choice of p_0, and finite, it clearly splits at the set τ_p of its maximal elements.

To see that τ_p is contained in σ_p let $\vec{r_p} \in \tau_p$ be given and pick some $\vec r \in O$ with $\vec r \upharpoonright p = \vec{r_p}$. As O splits at σ, there is some $\vec s \in \sigma$ such that $\vec r \le \vec s$. But then $\vec{r_p} \le \vec s \upharpoonright p$ by (1), and hence $\vec{r_p} = \vec s \upharpoonright p$ by the maximality of $\vec{r_p}$ in $O \upharpoonright p$. Thus, $\vec{r_p} \in \sigma \upharpoonright p$. But this implies $\vec{r_p} \in \sigma_p$, since $\tau_p \subseteq \vec{S_p}^{\circ}$ by Lemma 2.2.

For the converse inclusion let $\vec{s_p} \in \sigma_p$ be given and pick $\vec s \in \sigma$ with $\vec s \upharpoonright p = \vec{s_p}$. If $\vec{s_p}$ is maximal in $O \upharpoonright p$ then $\vec{s_p} \in \tau_p$ and there is nothing to show. Otherwise there exists $\vec{t_p} \in \tau_p$ with $t_p \neq s_p$ and $\vec{s_p} \le \vec{t_p}$. As seen above we can find $\vec t \in \sigma$ with $\vec t \upharpoonright p = \vec{t_p}$. Then $\vec s \neq \vec t$ and hence $\vec s \le \overleftarrow{t}$ by the star property, which implies $\vec{s_p} \le \overleftarrow{t_p}$ by (1). But then t_p witnesses that $\vec{s_p}$ is trivial in $\vec{S_p}$ and hence does not lie in σ_p, contrary to our assumption. $\square$

Next, let us prove a local converse of Lemma 5.10: that every star splitting an element $\vec{S_p}$ of an inverse system $(\,\vec{S_p} \mid p \in P\,)$ of nested separation systems is induced, modulo the deletion of trivial and co-trivial separations, by some splitting star in every $\vec{S_q}$ with $q > p$.

Lemma 5.11. *Assume that $(\,\vec{S_p} \mid p \in P\,)$ is surjective, that each of the $\vec{S_p}$ is nested and contains no degenerate separations, and let $p, q \in P$ with $p \le q$ be given. If $\sigma_p \subseteq \vec{S_p}$ splits $\vec{S_p}$, then $\vec{S_q}$ is split by a set σ such that $(\sigma \upharpoonright p) \cap \vec{S_p}^{\circ} = \sigma_p$.*

Proof. Let O_p be a consistent orientation of $\vec{S_p}$ of which σ_p is the set of maximal elements. By Lemma 2.2, σ_p is a star in $\vec{S_p}^{\circ}$ (which it clearly also splits). Let

$$O_q := \{\, \vec s \in \vec{S_q} \mid \vec s \upharpoonright p \in O_p \,\}\,.$$

As $\vec{S_p}$ contains no degenerate element, O_q is antisymmetric: an orientation of $\vec{S_q}$.

Here our proof splits into two cases: that σ_p has a co-small element or not. We first assume it does not.

Let us show that O_q is consistent. Suppose it is not. Then there are $\overleftarrow{r}, \vec{r'} \in O_q$ such that $\vec r < \vec{r'}$ (and $r \neq r'$). By (1) we have $\vec r \upharpoonright p \le \vec{r'} \upharpoonright p$. If $r \upharpoonright p \neq r' \upharpoonright p$ then this violates the consistency of O_p, which contains $\overleftarrow{r} \upharpoonright p$ and $\vec{r'} \upharpoonright p$ because $\overleftarrow{r}, \vec{r'} \in O_q$. Hence $r \upharpoonright p = r' \upharpoonright p$, with orientations $\overleftarrow{r} \upharpoonright p = \vec{r'} \upharpoonright p$ since O_p contains both but is antisymmetric. Thus, $\vec r \upharpoonright p = \overleftarrow{r'} \upharpoonright p$ is small, while $\vec{r'} \upharpoonright p \in O_p$ lies below ($\le$) some $\vec s \in \sigma_p$. But then we must have $\vec{r'} \upharpoonright p = \vec s$ as

[3]. . . in whose proof we shall not use Lemma 5.10

otherwise $\overleftarrow{s} < \overleftarrow{r'} \restriction p \le \vec{r'} \restriction p$ is trivial, which contradicts our assumption that $\vec{s} \in \sigma_p \subseteq \vec{S}^{\circ}_p$. Hence $\vec{r'} \restriction p = \vec{s} \in \sigma_p$ is co-small, contradicting our assumption.

We have shown that O_q is a consistent orientation of $\vec{S}_q$. Let σ be the set of its maximal elements, a splitting star of $\vec{S}_q$. Let us show that $(\sigma \restriction p) \cap \vec{S}^{\circ}_p = \sigma_p$. For a proof of $(\sigma \restriction p) \cap \vec{S}^{\circ}_p \subseteq \sigma_p$ consider any $\vec{s} \in \sigma$ for which $\vec{s} \restriction p \in \vec{S}^{\circ}_p$. As $\vec{s} \in O_q$ we have $\vec{s} \restriction p \in O_p$. Hence if $\vec{s} \restriction p \notin \sigma_p$ then $\vec{s} \restriction p < \vec{s'} \restriction p \in \sigma_p$ for some $\vec{s'} \in \vec{S}_q$. (Here we use that f_{qp} is surjective.) Note that $\vec{s'} \in O_q$, by definition of O_q. As $\vec{s} \restriction p$ and $\vec{s'} \restriction p$ both lie in O_p, they cannot be inverse to each other, so $s \restriction p \ne s' \restriction p$. Recall that $\vec{s} \restriction p$ is nontrivial by assumption. As O_p is consistent, $\vec{s'} \restriction p \in O_p$ cannot be co-trivial in $\vec{S}_p$. (Apply Lemma 2.1(i) with $P = \{\vec{s'} \restriction p\}$.) Hence $\vec{s} < \vec{s'}$ by Lemma 5.2, which contradicts the maximality of $\vec{s}$ in O_q as an element of σ.

To show the converse inclusion $\sigma \restriction p \supseteq \sigma_p$ (note that $\sigma_p \subseteq \vec{S}^{\circ}_p$ by Lemma 2.2), consider any $\vec{r} \in \sigma_p$ and choose $\vec{s} \in O_q$ maximal with $\vec{s} \restriction p = \vec{r}$. We need to show that $\vec{s} \in \sigma$, i.e., that $\vec{s}$ is maximal in O_q. By (1), any $\vec{s'} > \vec{s}$ in O_q satisfies $\vec{s'} \restriction p \ge \vec{s} \restriction p$. As $\vec{s'} \restriction p \in O_p$ by definition of O_q, and $\vec{s} \restriction p = \vec{r}$ is maximal in O_p as an element of σ_p, we must have equality: $\vec{s'} \restriction p = \vec{s} \restriction p$. But this contradicts the choice of $\vec{s}$ in O_q. Hence such an $\vec{s'}$ does not exist, so $\vec{s} \in \sigma$ as desired. This completes our proof that $(\sigma \restriction p) \cap \vec{S}^{\circ}_p = \sigma_p$, and hence of the lemma, in the case that σ_p contains no co-small separation.

Let us now assume that σ_p does contain a co-small separation, $\vec{s_p}$ say. Then the star σ_p cannot contain any other separations, as any such separation $\vec{r} < \overleftarrow{s_p} \le \vec{s_p}$ would be trivial and thus contradict Lemma 2.2.

Let $M \subseteq O_q$ be the set of all $\vec{s} \in \vec{S}_q$ with $\vec{s} \restriction p = \vec{s_p}$. As before, for any inconsistent pair $\overleftarrow{r}, \vec{r'} \in O_q$ we have $\overleftarrow{r} \restriction p = \vec{r'} \restriction p \in \sigma_p = \{\vec{s_p}\}$ and this separation is co-small, giving $\overleftarrow{r}, \vec{r'} \in M$.

Let us show that no minimal element $\vec{r'}$ of M can be co-trivial in $\vec{S}_q$. Any witness r for this would have orientations $\overleftarrow{r}, \vec{r} < \vec{r'}$. One of them, $\overleftarrow{r}$ say, would be in O_q. Then $\overleftarrow{r}, \vec{r'}$ are an inconsistent pair in O_q, giving $\overleftarrow{r}, \vec{r'} \in M$ as above. But now $\overleftarrow{r} < \vec{r'}$ contradicts the minimality of $\vec{r'}$ in M.

Similarly, no $\vec{r} \in M$ can be trivial in $\vec{S}_q$. Indeed, suppose $s \in S_q$ witnesses the triviality of $\vec{r}$. Then $\vec{r} < \vec{s}$ as well as $\vec{r} < \overleftarrow{s}$, which implies $\vec{r} \restriction p \le \vec{s} \restriction p$ as well as $\vec{r} \restriction p \le \overleftarrow{s} \restriction p$ by (1). One of these, $\vec{s} \restriction p$ say, lies in O_p. Then $\vec{s_p} = \vec{r} \restriction p = \vec{s} \restriction p$ by the maximality of $\vec{s_p}$ in O_p. But now $\vec{s_p} = \vec{s} \restriction p = \vec{r} \restriction p \le \overleftarrow{s} \restriction p$, so $\vec{s_p}$ is small as well as, by assumption, co-small, and hence degenerate. This contradicts our assumptions about $\vec{S}_p$.

As M has a minimal element, we thus have $M' := M \cap \vec{S}^{\circ}_q \ne \emptyset$; let $\vec{s_1} \in M'$ be maximal. Then $\vec{s_1}$ is also maximal in $O'_q := O_q \setminus (M \setminus \{\vec{s_1}\}) \subseteq O_q$: for any $\vec{s} \in O_q$ with $\vec{s_1} \le \vec{s}$ we have $\vec{s_p} = \vec{s_1} \restriction p \le \vec{s} \restriction p \in O_p$, with equality by the maximality of $\vec{s_p}$ in O_p, and hence $\vec{s} \in M$. Now O'_q is a partial orientation of S_q, and it is consistent, since it has only one element in M. By Lemma 2.1, O'_q therefore extends to a (unique) consistent orientation O of $\vec{S}_q$ in which $\vec{s_1}$ is maximal. Let $\sigma \ni \vec{s_1}$ be the set of all the maximal elements of O. Then σ is a splitting star of $\vec{S}_q$.

To complete our proof we have to show that $(\sigma \restriction p) \cap \vec{S}^{\circ}_p = \sigma_p$. We have $\sigma_p = \{\vec{s_p}\} \subseteq \sigma \restriction p \cap \vec{S}^{\circ}_p$, since $\vec{s_p} = \vec{s_1} \restriction p$ and $\vec{s_1} \in \sigma$. For the converse inclusion let $\vec{s} \in \sigma$ be arbitrary; we show that if $\vec{s} \restriction p \ne \vec{s_p}$ then $\vec{s} \restriction p$ is trivial in $\vec{S}_p$. By (1) and the star property we have $\vec{s} \restriction p \le \overleftarrow{s_1} \restriction p = \overleftarrow{s_p} \le \vec{s_p}$, so $\vec{s} \restriction p$ is trivial unless either $\vec{s} \restriction p = \vec{s_p}$ or $\vec{s} \restriction p = \overleftarrow{s_p}$. In the first case there is nothing to show. In the second case we have $\overleftarrow{s} \in M$ with $\vec{s_1} \le \overleftarrow{s}$. The inequality is strict, since

$\vec{s_1}$ and $\vec{s}$ are distinct elements of the asymmetric set O. Hence $\overleftarrow{s} \notin M'$ by the choice of $\vec{s_1}$, and thus $\overleftarrow{s} \in M \setminus M'$. Now $\overleftarrow{s}$ cannot be trivial in $\vec{S_q}$, since this would make $\vec{s_1} < \overleftarrow{s}$ trivial too. Hence $\overleftarrow{s}$ is co-trivial in $\vec{S_q}$. But then $\vec{s}$ is trivial and thus cannot be maximal in O (Lemma 2.1(ii)), a contradiction. □

Recall that our aim was, broadly, to show that the splitting stars σ of $\vec{S}$ are precisely the limits of the splitting stars of the projections $\vec{S_p}$. In Lemma 5.10 we showed that, except for the pathological case that σ is a 'finitely co-trivial' singleton, this is indeed true for all large enough p. In Lemma 5.11 we proved only a local converse of this: we did not show that the splitting stars of $\vec{S_p}$ are induced by splitting stars of $\vec{S}$, but by splitting stars of $\vec{S_q}$ for all $q > p$. This is what we shall need in [1].

Here, then, is a more direct converse of Lemma 5.10:

Proposition 5.12. *Assume that $(\vec{S_p} \mid p \in P)$ is surjective, and that each of the $\vec{S_p}$ is nested and contains no degenerate separations. Let $p \in P$ be given. If $\sigma_p \subseteq \vec{S_p}$ splits $\vec{S_p}$ and contains no co-small separation, then $\vec{S}$ is split by a set σ such that $(\sigma \restriction p) \cap \vec{S_p}^{\circ} = \sigma_p$.*

Proof. Let O_p be the consistent orientation of $\vec{S_p}$ witnessing that σ_p splits $\vec{S_p}$, and let $O := \{ \vec{s} \in \vec{S} \mid \vec{s} \restriction p \in O_p \}$. As in the proof of Lemma 5.11, first case, O is a consistent orientation of $\vec{S}$. Also as before, its set σ of maximal elements satisfies $(\sigma \restriction p) \cap \vec{S_p}^{\circ} = \sigma_p$. But in order to show that σ splits $\vec{S}$ we must prove that every element of O lies below some element of σ. This follows from Lemma 4.4 if O is closed in $\vec{S}$. But by Lemma 4.1 and the definition of O, the topological closure of O in $\vec{S}$ is O itself. So O is indeed closed in $\vec{S}$. □

What happens in Proposition 5.12 if σ_p does contain a co-small separation, $\vec{s_p}$ say? Then $\sigma_p = \{\vec{s_p}\}$ as before. Suppose $\vec{S}$ has a splitting set σ as desired, one such that $(\sigma \restriction p) \cap \vec{S_p}^{\circ} = \sigma_p$, witnessed by the consistent orientation O of $\vec{S}$, say. Then σ has an element $\vec{s}$ in $M := \{ \vec{s} \in \vec{S} \mid \vec{s} \restriction p = \vec{s_p} \}$. Note that $\vec{s} \in \sigma$ is neither trivial nor co-trivial, by Lemma 2.2. Let us show that $\vec{s}$ is maximal in $M \cap \vec{S}^{\circ}$, indeed in the set M' of elements of M that are not co-trivial in $\vec{S}$.

Suppose M' has an element $\vec{s'} > \vec{s}$. As $\vec{s'} \notin O$ by the maximality of $\vec{s}$ in O, we have $\overleftarrow{s'} \in O$. Then σ has an element $\overleftarrow{t}$ such that $\overleftarrow{s'} \le \overleftarrow{t}$. If $\overleftarrow{t} = \vec{s}$ then $\overleftarrow{s'} \le \vec{s} < \vec{s'}$. Since $\overleftarrow{s'}$ is not trivial, this can happen only with equality $\overleftarrow{s'} = \vec{s}$ [2, Lemma 2.4]. But then $\vec{s}$ is small, and hence so is $\vec{s} \restriction p = \vec{s_p}$, by (1). Since $\vec{s_p}$ is also co-small it is degenerate, a contradiction. Hence $\overleftarrow{t} \neq \vec{s}$. As $\overleftarrow{t}$ and $\vec{s}$ lie in the star σ, we have $\vec{s} \le \vec{t}$, as well as $\vec{t} \le \vec{s'}$ by choice of $\overleftarrow{t}$. But then $\vec{s} \le \vec{t} \le \vec{s'}$ and thus, by (1), $\vec{s_p} \le \vec{t} \restriction p \le \vec{s_p}$ with equality. In particular, $\overleftarrow{t} \restriction p \in \vec{S_p}^{\circ}$, and hence $\overleftarrow{t} \restriction p \in (\sigma \restriction p) \cap \vec{S_p}^{\circ} = \sigma_p = \{\vec{s_p}\}$, since $\overleftarrow{t} \in \sigma$. But now $\vec{t} \restriction p = \vec{s_p} = \overleftarrow{t} \restriction p$, contradicting our assumption that $\vec{S_p}$ has no degenerate elements. This completes the proof that $\vec{s'}$ does not exist, and hence that our arbitrary $\vec{s} \in \sigma \cap M$ is maximal in $M' \supseteq M \cap \vec{S}^{\circ}$.

The upshot of all this is that if σ_p contains a co-small separation, we can only hope to find a splitting star σ of $\vec{S}$ inducing σ_p, in the sense that $(\sigma \restriction p) \cap \vec{S_p}^{\circ} = \sigma_p$, if the set $M \cap \vec{S}^{\circ}$ has a maximal element. However, while M is closed in $\vec{S}$ and thus has maximal elements above all its elements (Lemma 4.4), this need not be the case for $M \cap \vec{S}^{\circ}$. Indeed, it is even possible that every $\vec{S_p}$ contains such a splitting singleton $\{\vec{s_p}\}$ for which the conclusion of Proposition 5.12 fails; so we cannot even get this conclusion for sufficiently large $p \in P$.

6 Orienting profinite nested separation systems

In the last section we examined how the splitting stars of a nested profinite separation system $\vec{S} = \varprojlim (\vec{S_p} \mid p \in P)$ relate to those of its projections. Let us now look at the consistent orientations which these splitting stars induce.

If $\vec{S}$ is finite, all its consistent orientations O are induced by a splitting star σ, in the sense that $O = \lceil \sigma \rceil \setminus \overleftarrow{\sigma}$ where σ is the set of maximal elements of O.

This is not the case for profinite separation systems in general. For example, the orientations of the edge tree set $\vec{E}(T)$ of an infinite tree T towards some fixed end of T are consistent but do not split. Note that these orientations also fail to be closed in the topology of the inverse limit: by Corollary 4.5, every closed consistent orientation of $\vec{S}$ splits. This raises the question of whether the closed consistent orientations of a nested profinite separation system $\vec{S}$ are precisely those that split. Such a purely combinatorial description of the closed consistent orientations of $\vec{S}$, and in addition one that makes no reference to the inverse system $(\vec{S_p} \mid p \in P)$, would certainly be useful.

Unfortunately, though, even tree sets can have consistent orientations that split but are not closed. Here is an example that is not a tree set, but typical:

Example 6.1. Let P be the set of all integers $p \geq 3$. For each $p \in P$ let $\vec{S_p}$ be the separation system consisting of a star σ_p with p elements, plus their inverses, where σ_p contains two separations $\vec{s_p}$ and $\vec{r_p}$ such that $\sigma_p \setminus \{\vec{s_p}, \vec{r_p}\}$ is a proper star, $\vec{s_p} \leq \overleftarrow{s_p}$, and $\vec{r_p} < \vec{t}$ (as well as $\vec{r_p} < \overleftarrow{t}$) for all $\vec{t} \in \sigma_p \setminus \{\vec{r_p}\}$. For each p pick an element $\vec{t_p}$ of $\sigma_p \setminus \{\vec{s_p}, \vec{r_p}\}$.

Let $f_{p+1,p} \colon \vec{S}_{p+1} \to \vec{S_p}$ be the homomorphism which maps $\vec{s}_{p+1}$ and $\vec{t}_{p+1}$ to $\vec{s_p}$, maps $\vec{r}_{p+1}$ to $\vec{r_p}$, maps $\sigma_{p+1} \setminus \{\vec{s}_{p+1}, \vec{r}_{p+1}, \vec{t}_{p+1}\}$ bijectively to $\sigma_p \setminus \{\vec{s_p}, \vec{r_p}\}$, and is defined on the inverses of these separations so as to commute with the inversion. These maps induce bonding maps $f_{qp} \colon \vec{S_q} \to \vec{S_p}$ by concatenation.

Let $\vec{S} := \varprojlim (\vec{S_p} \mid p \in P)$. Then $\vec{S}$ consists of an infinite star σ as well as the respective inverses, and we have $\sigma \restriction p = \sigma_p$ for all $p \in P$. The separation $\vec{s} = (\vec{s_p} \mid p \in P)$ lies in σ and is small, as $\vec{s_p} \leq \overleftarrow{s_p}$ for all $p \in P$ by construction. However $\vec{s}$ is not trivial in $\vec{S}$: we have $\vec{s} \not< \vec{r}$ for $\vec{r} := (\vec{r_p} \mid p \in P) \in \sigma$, since $\vec{r_p} < \vec{s_p}$ for all p; and for every other $\vec{t} \in \sigma$ with $\vec{t} \neq \vec{s}$ there is a unique $p \in P$ such that $\vec{t} \restriction p = \vec{t_p}$, and for this p we have $\vec{s_p} \not< \vec{t_p}$. Hence $\vec{s} \not< \vec{t}$ in $\vec{S}$.

For future use, note that $\vec{r}$ is trivial in $\vec{S}$, witnessed by s.

Consider the orientation $O := \{ \overleftarrow{s} \} \cup (\sigma \setminus \{ \vec{s} \})$ of $\vec{S}$. By the star property of σ we have $\vec{t} \leq \overleftarrow{s}$ for all $\vec{t} \in \sigma$ with $\vec{t} \neq \vec{s}$, so $\overleftarrow{s}$ is the greatest element of O. As $\overleftarrow{s}$ is not co-trivial in $\vec{S}$, as seen above, this means that O is consistent and splits at $\{ \overleftarrow{s} \}$.

Let us now show that O is not closed in $\vec{S}$ by proving that $\vec{s}$ lies in the closure of O in $\vec{S}$. By Lemma 4.1 we need to show that $\vec{s_p} \in O \restriction p$ for each $p \in P$. So let $p \in P$ be given and consider the (unique) element $\vec{t} \in \sigma \setminus \{ \vec{s} \}$ for which $\vec{t} \restriction (p+1) = \vec{t}_{p+1}$. Then $\vec{t} \restriction p = \vec{s_p}$ by the definition of $f_{p+1,p}$ and hence $\vec{s_p} \in O \restriction p$.

Therefore the orientation O of $\vec{S}$ splits but is not closed in $\vec{S}$. $\square$

The next two lemmas show that every orientation O of $\vec{S}$ that splits but fails to be closed must more or less look like Example 6.1. First we show that if O has a greatest element then this must be co-small:

Lemma 6.2. *Let O be a consistent orientation of $\vec{S}$ with a greatest element $\vec{m}$. Then either O is closed in $\vec{S}$, or $\vec{m}$ is co-small but not co-trivial and $\overleftarrow{m}$ lies in the closure of O.*

Proof. Note first that $\vec{m}$ cannot be co-trivial, because consistent orientations of separation systems cannot have co-trivial elements (Lemma 2.1 (i)). Suppose that O fails to be closed, that is, there is some nondegenerate $\vec{s} \in O$ for which $\overleftarrow{s}$ lies in the closure of O in $\vec{S}$. By Lemma 4.1 there exists for every $p \in P$ some $\vec{r} = \vec{r}(p) \in O$ such that $\overleftarrow{s} \upharpoonright p = \vec{r} \upharpoonright p$. As $\vec{m}$ is the greatest element of O we have $\vec{r} \leq \vec{m}$, and hence $\overleftarrow{s} \upharpoonright p = \vec{r} \upharpoonright p \leq \vec{m} \upharpoonright p$ for every p, giving $\overleftarrow{s} \leq \vec{m}$ in $\vec{S}$. If $s \neq m$ this contradicts the consistency of O. So $s = m$, and hence $\vec{s} = \vec{m}$ since these lie in O. Thus, $\vec{s} = \vec{m} \geq \overleftarrow{s}$ is co-small. □

If a consistent orientation of $\vec{S}$ has two or more maximal elements, then these cannot be co-small. Indeed, since $\vec{S}$ is nested, the only way such elements could be incomparable would be that they are inconsistent. Thus if such an orientation splits, we would expect it to be closed in $\vec{S}$. This is indeed the case:

Lemma 6.3. *Any consistent orientation O of $\vec{S}$ that splits at a star of order at least 2 is closed in $\vec{S}$.*

Proof. We verify the premise of Lemma 4.1 (ii), with $O_p := O \upharpoonright p$ for all p.

Suppose that $O \neq O' := \varprojlim (O_p \mid p \in P)$. Then O' contains some $\overleftarrow{s} \notin O$, so $\vec{s} \in O$. This $\vec{s}$ lies below some maximal element $\vec{m}$ of O. Let $\vec{n}$ be another maximal element of O. As $\vec{m}$ and $\vec{n}$ are consistent and incomparable, but have comparable orientations, they point towards each other. In particular, $\vec{n} \leq \overleftarrow{m}$. As $\vec{m}$ and $\vec{n}$ both lie in O they cannot be inverse to each other, so $\vec{n} < \overleftarrow{m}$. Hence there exists $q \in P$ such that $\vec{n} \upharpoonright p < \overleftarrow{m} \upharpoonright p$ for all $p \geq q$.

Consider any $p \geq q$. As $\overleftarrow{s} \in O'$ there exists $\vec{r} = \vec{r}(p) \in O$ with $\vec{r} \upharpoonright p = \overleftarrow{s} \upharpoonright p$. This $\vec{r}$ lies below a maximal element of O. But we cannot have $\vec{r} \leq \vec{n}$, since in that case $\vec{n} \upharpoonright p \geq \vec{r} \upharpoonright p = \overleftarrow{s} \upharpoonright p \geq \overleftarrow{m} \upharpoonright p$, contradicting the fact that $p \geq q$.

Therefore $\vec{r}$ lies below some other maximal element of O and hence, like that element, points towards $\vec{n}$: we have $\vec{r} \leq \overleftarrow{n}$. Thus, $\overleftarrow{n} \upharpoonright p \geq \vec{r} \upharpoonright p = \overleftarrow{s} \upharpoonright p \geq \overleftarrow{m} \upharpoonright p$. As this inequality holds for each $p \geq q$ we have $\vec{n} \leq \vec{m}$ in $\vec{S}$, contrary to the maximality of $\vec{n} \neq \vec{m}$ in O. □

Lemmas 6.2 and 6.3 together show that a consistent orientation of $\vec{S}$ that splits but is not closed in $\vec{S}$ has a very particular form: it must have a co-small greatest element that is not co-trivial. Rather than describing such orientations directly, let us see if we can characterize those $\vec{S}$ that admit such an orientation.

Let us call a nested separation system $\vec{S}$ *normal* if the consistent orientations of $\vec{S}$ that split are precisely those that are closed in $\vec{S}$.

Theorem 6.4. *If all small separations in $\vec{S}$ are trivial, then $\vec{S}$ is normal. In particular, regular profinite tree sets are normal.*

Proof. Let us assume that all small separations in $\vec{S}$ are trivial and show that $\vec{S}$ is normal. By Corollary 4.5, all closed consistent orientations of $\vec{S}$ split. Conversely, let O be a consistent orientation of $\vec{S}$ that splits, say at σ. If σ is empty then so is $\vec{S}$, so O is closed in $\vec{S}$. If $|\sigma| \geq 2$ then O is closed by Lemma 6.3. And if $|\sigma| = 1$, then O is closed by Lemma 6.2. □

Theorem 6.4 leaves us with the problem to characterize the normal nested separation systems among those that do contain nontrivial small separations. Interestingly, there can be no such characterization, at least not in terms of separation systems alone: normality is not an invariant of separation systems!

Indeed, our next example shows that we can have isomorphic profinite nested separations systems of which one is normal and the other is not:

Example 6.5. Let $G = (V, E)$ be a countably infinite star with centre z and $x \neq z$ another vertex of G. Let $\vec{S}_G$ be the separation system consisting of $(\{z\}, V)$ and all separations of the form $(\{y, z\}, V - y)$ for $y \neq z$, plus inverses. Let $(A, B) \leq (C, D)$ if either $A \subseteq C$ and $B \supseteq D$, or $(A, B) = (\{x, z\}, V - x)$ and $(C, D) = (V - x, \{z, x\})$.

Let Q be the set of all finite subsets of V that contain x and z. For all $q \in Q$ let $\vec{S}_q$ be the set of all $(A, B) \restriction q := (A \cap q\,,\, B \cap q)$ with $(A, B) \in \vec{S}_G$. Given $(A_q, B_q), (C_q, D_q) \in \vec{S}_q$ let $(A_q, B_q) \leq (C_q, D_q)$ if and only if there are $(A, B) \leq (C, D)$ in $\vec{S}_G$ such that $(A, B) \restriction q = (A_q, B_q)$ and $(C, D) \restriction q = (C_q, D_q)$.

Then $\vec{S}_G = \varprojlim\, (\, \vec{S}_q \mid q \in Q\,)$. Note that $\vec{S}_G$ contains only two co-small separations: $(V - x, \{z, x\})$ and $(V, \{z\})$. The first of these is a splitting singleton star, since its down-closure (minus its inverse) orients $\vec{S}_G$ consistently: the orientation consists of all (A, B) with $x \in B$. This orientation of $\vec{S}_G$ is clearly closed in $\vec{S}_G$. In particular, the inverse $(\{x, z\}, V - x)$ of its greatest element does not lie in its closure.

The other co-small separation, $(\{z\}, V)$, is co-trivial. By Lemmas 6.2 and 6.3, therefore, all splitting consistent orientations of $\vec{S}_G$ are closed, so $\vec{S}_G$ is normal.

However, $\vec{S}_G$ is isomorphic to the separation system $\vec{S}$ of Example 6.1, which has a consistent orientation that splits but is not closed, and hence is not normal. Indeed, both separation systems consist of a countably infinite star (plus inverses) that contains one small but nontrivial separation, and one trivial separation with all the other separations as witnesses, and has no further relations. □

Example 6.5 may serve as a reminder that the topology on a profinite separation system $\vec{S}$ depends, by definition, on the inverse system of which $\vec{S}$ is the inverse limit. It shows that this dependence is not just formal:

Corollary 6.6. *The topologies of isomorphic profinite nested separation systems can differ.* □

In our particular case of Examples 6.1 and 6.5, the difference in the topologies hinged on the question of whether the small but nontrivial element of an infinite star lies in the closure of the rest of that star. In Theorem 6.4 we noted that if $\vec{S}$ contains no nontrivial small separations it is normal.

In the rest of this section we shall prove a converse of this: if $\vec{S}$ contains no trivial (small) separation, i.e., is a tree set, it fails to be normal as soon as it contains an infinite star. The reason, interestingly, is that *every* maximal infinite star contains a small separation that lies in the closure of its other separations [7]. The only way in which this is compatible with normality is that that separation is not only small but in fact trivial.

Lemma 6.7. *If $\vec{S}$ is a tree set containing an infinite star, then $\vec{S}$ is not normal.*

Proof. Let $\sigma \subseteq \vec{S}$ be an infinite star. Like all stars, σ is consistent [2]. It is also antisymmetric, since otherwise all but two of its elements would be trivial. The inverses of the elements of σ, therefore, are pairwise inconsistent.

Our plan is to find a small separation $\vec{s_0} \in \vec{S}$ that lies in the closure of $\sigma' := \sigma \setminus \{\vec{s_0}\}$. By Lemma 2.1 we can extend $\{\overleftarrow{s_0}\}$ to a consistent orientation O of $\vec{S}$. Since the inverses of the separations in σ' are pairwise inconsistent, O contains all but at most one element of σ'.[4] Like σ', therefore, O will have $\vec{s_0}$ in its closure, and hence not be closed. As $\overleftarrow{s_0}$ is co-small it will be maximal in $\vec{S}$, since any larger separation would be co-trivial in $\vec{S}$. Since every two separations in O have comparable orientations, the maximality of $\overleftarrow{s_0}$ in $\vec{S}$ implies that O lies in its down-closure and hence splits at $\{\overleftarrow{s_0}\}$. Thus, O will be a consistent orientation of $\vec{S}$ that splits but is not closed in $\vec{S}$, completing our proof that $\vec{S}$ is not normal.

Let us now show that such an $\vec{s_0}$ exists. For every $p \in P$ let $\vec{S}'_p \subseteq \vec{S}_p$ be the set of all $\vec{s} \in \vec{S}_p$ for which there are infinitely many $\vec{r} \in \sigma$ with $\vec{r} \restriction p = \vec{s}$. As σ is infinite but $\vec{S}_p$ is finite, $\vec{S}'_p$ is non-empty. Note that $(\, \vec{S}'_p \mid p \in P\,)$ is a restriction[5] of $(\, \vec{S}_p \mid p \in P\,)$. Pick $\vec{s_0} \in \varprojlim\, (\, \vec{S}'_p \mid p \in P\,)$. Then for each $p \in P$ there are distinct $\vec{r}, \vec{r}\,' \in \sigma \setminus \{\vec{s_0}\}$ such that $\vec{r} \restriction p = \vec{s_0} \restriction p = \vec{r}\,' \restriction p$, so $\vec{s_0}$ lies in the closure of $\sigma \setminus \{\vec{s_0}\}$. Furthermore by the star property we have $\vec{r} \le \overleftarrow{r}\,'$ and hence $\vec{s_0} \restriction p \le \overleftarrow{s_0} \restriction p$ by (1), so $\vec{s_0}$ is small by Lemma 5.4. □

The converse of Lemma 6.7 holds too:

Lemma 6.8. *If $\vec{S}$ is a tree set but not normal, it contains an infinite star.*

Proof. Assume that $\vec{S}$ is not normal, and let O be a consistent orientation of $\vec{S}$ that witnesses this: one that splits but is not closed in $\vec{S}$. By Lemma 6.2 and 6.3, O has a greatest element $\vec{m}$ with $\vec{m} \ge \overleftarrow{m}$ and $\overleftarrow{m}$ in the closure of O. Suppose that $\vec{S}$ contains no infinite star. We will find some $p \in P$ for which $(\overleftarrow{m} \restriction p) \notin O \restriction p$, contradicting the fact that $\overleftarrow{m}$ lies in the closure of O.

Let M be the set of minimal elements of $\vec{S}$. Then M is a star in $\vec{S}$, and therefore finite by assumption. Note that $\overleftarrow{m} \in M \setminus O$, since any $\overleftarrow{s} < \overleftarrow{m}$ would have $\vec{s} > \vec{m}$ contradict the definition of $\vec{m}$, but $M \setminus \{\overleftarrow{m}\} \subseteq \lceil \vec{m} \rceil \setminus \{\overleftarrow{m}\} = O$. By Lemma 4.4 every separation in $\vec{S}$ lies above an element of M.

No $\vec{s} \in M$ is trivial in $\vec{S}$ with witness m. Hence for every $\vec{s} \in M$ with $s \neq m$ there is a $p(\vec{s}) \in P$ such that $s \restriction p \neq m \restriction p$ and $\vec{s} \restriction p$ is not trivial with witness $m \restriction p$ for all $p \ge p(\vec{s})$. By (1) we have $\vec{s} \restriction p < \vec{m} \restriction p$ for all $p \ge p(\vec{s})$. Pick $p \in P$ large enough that $\overleftarrow{m} \restriction p \neq \vec{m} \restriction p$ and $p \ge p(\vec{s})$ for all $\vec{s} \in M$ with $s \neq m$. We shall show that $(\overleftarrow{m} \restriction p) \notin O \restriction p$.

Suppose to the contrary that there exists $\vec{t} \in O$ such that $\vec{t} \restriction p = \overleftarrow{m} \restriction p$. Pick $\vec{s} \in M$ with $\vec{s} \le \vec{t}$. Then $\vec{s} \le \vec{t} \le \vec{m}$, where the last inequality is strict since $\vec{t} \restriction p = \overleftarrow{m} \restriction p \neq \vec{m} \restriction p$ by the choice of p. In particular, $\vec{s} \neq \vec{m}$. But we also have $\vec{s} \neq \overleftarrow{m}$: otherwise $\vec{s} = \overleftarrow{m} < \vec{t} < \vec{m}$, where the first inequality is strict since $\vec{t} \in O$ while $\overleftarrow{m} \notin O$, which makes $\overleftarrow{m}$ trivial with witness t, a contradiction. Thus, $\vec{s} \notin \{\vec{m}, \overleftarrow{m}\}$ and hence $s \neq m$.

By the choice of p, this implies that even $s \restriction p \neq m \restriction p$, yet that $\vec{s} \restriction p$ is not trivial with witness $m \restriction p$. But it is, since $\vec{s} \restriction p \le \vec{m} \restriction p$ (by $\vec{s} \le \vec{t} \le \vec{m}$) as well as $\vec{s} \restriction p \le \vec{t} \restriction p = \overleftarrow{m} \restriction p$ (by the choice of $\vec{t}$), a contradiction. □

[4] In fact, all do – but this is harder to show [7].

[5] As an inverse system of sets, not of separation systems: the $\vec{S}'_p$ need not be closed under taking inverses.

Lemmas 6.7 and 6.8 together amount to a characterization of the profinite tree sets that are normal. It turns out that, for tree sets, normality is more restrictive than it might have appeared at first glance:

Theorem 6.9. *A profinite tree set is normal if and only if it contains no infinite star.* □

References

[1] N. Bowler, R. Diestel, and F. Mazoit. Tangle-tree duality in infinite graphs. In preparation.

[2] R. Diestel. Abstract separation systems. To appear in *Order* (2017), DOI 10.1007/s11083-017-9424-5, arXiv:1406.3797v5.

[3] R. Diestel. Tree sets. To appear in *Order* (2017), DOI 10.1007/s11083-017-9425-4, arXiv:1512.03781.

[4] R. Diestel, F. Hundertmark, and S. Lemanczyk. Profiles of separations: in graphs, matroids, and beyond. arXiv:1110.6207, to appear in *Combinatorica.*

[5] R. Diestel and S. Oum. Tangle-tree duality in abstract separation systems. arXiv:1701.02509, 2017.

[6] R. Diestel and G. Whittle. Tangles and the Mona Lisa. arXiv:1603.06652.

[7] Jakob Kneip. Tree sets and tree-like spaces. Masterarbeit Universität Hamburg, 2016.

[8] Jakob Kneip. Profinite tree sets. In preparation, 2017.

[9] L. Ribes and P. Zalesskii. *Profinite Groups.* Springer, 2010.

[10] N. Robertson and P.D. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory (Series B)*, 52:153–190, 1991.