DECOMPOSITION OF DEGENERATE GROMOV-WITTEN INVARIANTS

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ABSTRACT. We prove a decomposition formula of logarithmic Gromov-Witten invariants in a degeneration setting. A one-parameter log smooth family $X \to B$ with singular fibre over $b_0 \in B$ yields a family $\mathscr{M}(X/B, \beta) \to B$ of moduli stacks of stable logarithmic maps. We give a virtual decomposition of the fibre of this family over b_0 in terms of rigid tropical curves. This generalizes one aspect of known results in the case that the fibre X_{b_0} is a normal crossings union of two divisors. We exhibit our formulas in explicit examples.

CONTENTS

1. INTRODUCTION

1.1. **Statement of results.** One of the main goals of logarithmic Gromov–Witten theory is to develop new formulas relating the Gromov–Witten invariants of a smooth variety X_η to invariants of a degenerate variety X_0.

Consider a logarithmically smooth and projective morphism $X \to B$, where B is a logarithmically smooth curve having a single point $b_0 \in B$ where the logarithmic structure is nontrivial. In the language of [KKMSD73, AK00], this is the same as saying that the underlying schemes $\underline{X}$ and $\underline{B}$ are provided with a toroidal structure such that $\underline{X} \to \underline{B}$ is a toroidal morphism, and $\underline{\{b_0\}} \subset \underline{B}$ is the toroidal divisor. One defines as in [GS13], see also [Che14, AC14], an algebraic

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stack $\mathscr{M}(X/B,\beta)$ parametrizing *stable logarithmic maps* $f: C \to X$ *with discrete data* $\beta = (g, A, u_{p_1}, \ldots, u_{p_k})$ from logarithmically smooth curves to X. Here

- g is the genus of C,
- A is the homology class $\underline{f}_*[\underline{C}]$, which we assume is supported on fibres of $\underline{X} \to \underline{B}$ and
- $u_{p_1}, \ldots, u_{p_k}$ are the *logarithmic types* or *contact orders* of the marked points with the logarithmic strata of X.

Writing $\underline{\beta} = (g, k, A)$ for the non-logarithmic discrete data, there is a natural morphism $\mathscr{M}(X/B,\beta) \to \mathscr{M}(\underline{X}/\underline{B},\underline{\beta})$ "forgetting the logarithmic structures", which is proper and representable [ACMW17, Theorem 1.1.1]. The map $\mathscr{M}(X/B,\beta) \to \mathscr{M}(\underline{X}/\underline{B},\underline{\beta})$ is in fact finite, see [Wis16a, Corollary 1.2]. There is also a natural morphism $\mathscr{M}(X/B,\beta) \to B$, and we denote its fibre over $b \in B$ by $\mathscr{M}_\beta(X_b/b)$.

Since $X \to B$ is logarithmically smooth there is a perfect relative logarithmic obstruction theory $\mathbf{E}^\bullet \to \mathbf{L}_{\mathscr{M}(X/B,\beta)\,/\,\mathrm{Log}_B}$ giving rise to a virtual fundamental class $[\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$ and to Gromov–Witten invariants.

An immediate consequence of the formalism is the following (this is indicated after [GS13, Theorem 0.3]):

Theorem 1.1.1 (Logarithmic deformation invariance). *For any point* $\{b\} \overset{j_b}{\hookrightarrow} B$ *one has*
$$j_b^![\mathscr{M}(X/B,\beta)]^{\mathrm{virt}} = [\mathscr{M}(X_b/b,\beta)]^{\mathrm{virt}}.$$

This implies, in particular, that Gromov–Witten invariants of X_b agree with those of $X_0 = X_{b_0}$, and it is important to describe invariants of X_0 in simpler terms.

The main result here is the following:

Theorem 1.1.2 (The logarithmic decomposition formula). *Suppose the morphism* $X_0 \to b_0$ *is logarithmically smooth and* X_0 *is simple. Then*

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{\tau\,\in\,\Omega} m_\tau \cdot \sum_{\mathbf{A}\vdash A} (i_{\tau,\mathbf{A}})_*[\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)]^{\mathrm{virt}} \tag{1.1.1}$$

See Definition 2.1.5 for the notion of simple logarithmic structures. The notation $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$, m_τ and $i_{\tau,\mathbf{A}}$ is briefly explained as follows.

Given discrete data β we define below a finite set $\Omega = \{\tau\}$, which we describe in at least two equivalent ways. First, Ω is the set of isomorphism classes τ of *rigid tropical curves* of type β in the polyhedral complex $\Delta(X)$. A second description comes in the proof: an element $\tau \in \Omega$ is a ray in the tropical moduli space $M^{\mathrm{trop}}(\Sigma(X)/\Sigma(B))$ surjecting to $\Sigma(B) = \mathbb{R}_{\geq 0}$.

Each element $\tau \in \Omega$ comes with a multiplicity $m_\tau \in \mathbb{Z}$; in terms of the latter description as a ray, m_τ is the index of the image of the lattice of τ inside the lattice $\mathbb{N}$ of $\Sigma(B)$.

The notation $\mathbf{A} \vdash A$ stands for a partition of the curve class A into classes $\mathbf{A}(v), v \in V(G)$, where G is the graph underlying τ.

The moduli stack $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ is constructed as a stack quotient

$$\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta) = [\mathscr{M}_{\tilde{\tau},\mathbf{A}}(X_0/b_0,\beta)/\operatorname{Aut}(\tilde{\tau},\mathbf{A})]\,,$$

where $\mathscr{M}_{\tilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ parametrizes logarithmic maps marked by a rigidified version $\tilde{\tau}$ of τ, where the dual graph of each curve is enriched by a contraction to a fixed graph G_τ. Finally, there is a canonical map $i_{\tau,\mathbf{A}} : \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta) \to \mathscr{M}(X_0/b_0,\beta)$ forgetting the marking by τ.

The following is thus an equivalent formulation of the formula (1.1.1) which is useful for studying the splitting formula and applications:

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{(\tilde{\tau},\mathbf{A})\,:\,\mathbf{A}\vdash A} \frac{m_{\tilde{\tau}}}{|\operatorname{Aut}(\tilde{\tau},\mathbf{A})|}(i_{\tilde{\tau},\mathbf{A}})_*\,[\mathscr{M}_{\tilde{\tau},A}(X_0/b_0,\beta)]^{\mathrm{virt}}$$

(see Theorem 4.8.1).

Remark 1.1.3. In general, the sum over $(\tilde{\tau},\mathbf{A})$ will be infinite, but because the moduli space is of finite type, all but a finite number of the moduli spaces $\mathscr{M}_{\tilde{\tau},A}(X_0/b_0,\beta)$ will be empty. In practice one uses the balancing condition [GS13], Proposition 1.15 to control how curves can break up into strata of X_0. This is carried out in some of the examples in §6.

These theorems form the first two steps towards a general logarithmic degeneration formula. In many cases this is sufficient for meaningful computations, as we show in Section 6. These results have precise analogies with results in [Li02], as explained in §6.1. Theorem 1.1.1 is a generalization of [Li02], Lemma 3.10, while Theorem 1.1.2 is a generalization of part of [Li02], Corollary 3.13. There, what is written as $\mathfrak{M}(\mathfrak{Y}_1^{rel} \cup \mathfrak{Y}_2^{rel},\eta)$ plays the role of what is written here as $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$.

What is missing in our more general situation is any implication that the moduli stack $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ can be described in terms of relative invariants of individual irreducible components of X_0. Indeed, this will not be the case, and we give an example in §6.2 in which X_0 has three components meeting normally, with one triple point. We give a log curve contributing to the Gromov-Witten invariant which has a component contracting to the triple point, and this curve cannot be broken up into relative curves on the three irreducible components of X_0.

In fact, a new theory is needed to give a more detailed description of the moduli spaces $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ in terms of pieces of simpler curves. In future work, we will define *punctured curves* which replace the relative curves of Jun Li's gluing formula. Crucially, we will explain how punctured curves can be glued together to describe the moduli spaces $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$.

The results described here are also analogous to results of Brett Parker proved in his category of exploded manifolds. Theorem 1.1.1 is analogous to [Par11], Theorem 3.7, while Theorem 1.1.2 is analogous to part of [Par11], Theorem 4.6.

The aim in proving a full gluing formula is a full logarithmic analogue of that theorem.

The structure of the paper is as follows. In §2, we review various aspects of logarithmic Gromov-Witten theory, with a special emphasis on the relationship with tropical geometry. While this point of view was present in [GS13], we make it more explicit here, and in particular discuss tropicalization in a sufficient degree of generality as needed here.

§3 proves a first version of the degeneration formula. A toric morphism $X_\Sigma \to \mathbb{A}^1$ from a toric variety X_Σ has fibre over 0 easily described in terms of the corresponding map of fans $\Sigma \to \mathbb{R}_{\geq 0}$. There is a one-to-one correspondence between irreducible components of the fibre and rays of Σ mapping surjectively to $\mathbb{R}_{\geq 0}$, and their multiplicity is determined by the integral structure of this map of rays, see Proposition 3.1.1. The main point is that the equality of Weil divisors (3.1.1) then generalizes to "virtually log smooth" morphisms, in this case the morphism $\mathscr{M}(X/B,\beta) \to B$. Thus one obtains in Proposition 3.4.2 a first version of a decomposition formula, decomposing $\mathscr{M}(X_0/B_0,\beta)$ into "virtual components" $\mathscr{M}_m(X_0/B_0,\beta)$, which can be thought of as reduced unions of those components of $\mathscr{M}(X_0/B_0,\beta)$ appearing with multiplicity m.

§4 then refines this decomposition. In analogy with the purely toric case, we obtain a further decomposition in terms of rays (representing "virtual divisors" of $\mathscr{M}(X/B,\beta)$) in the tropical version of the moduli space of curves, $M^{\mathrm{trop}}(\Sigma(X)/\Sigma(B))$. These rays are interpreted as parameterizing rigid tropical curves, leading to the proof of Theorem 1.1.2.

The second part of the paper turns to some simple applications of the theory. While there are quite a few theoretical papers on log Gromov-Witten invariants, there is still a gap in the literature as far as explicit calculations are concerned. In §5 we explain some simple methods for constructing examples of stable logarithmic maps, building on work of Nishinou and Siebert in [NS06]. This allows us to give some explicit examples of the decomposition formula in the final section of the paper.

1.2. **Acknowledgements.** That there are analogies with Parker's work is not an accident: we received a great deal of inspiration from his work and had many fruitful discussions with Brett Parker. We also benefited from discussions with Steffen Marcus, Dhruv Ranganathan, Ilya Tyomkin, Martin Ulirsch and Jonathan Wise.

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1.3. **Convention.** All logarithmic schemes and stacks we consider here are fine and saturated and defined over an algebraically closed field $\Bbbk$ of characteristic 0. We will usually only consider toric monoids, i.e., monoids of the form $P = P_{\mathbb{R}} \cap M$ for $M \cong \mathbb{Z}^n$, $P_{\mathbb{R}} \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ a rational polyhedral cone. For P a toric monoid, we write

$$P^\vee = \mathrm{Hom}(P, \mathbb{N}), \quad P^* = \mathrm{Hom}(P, \mathbb{Z}).$$

1.4. **Notation.**

$\Bbbk$	the base field, usually algebraically closed of characteristic zero
$\mathcal{M}$	a fine and saturated logarithmic structure
$\overline{\mathcal{M}}$	the characteristic (or ghost) monoid
$X \to B$	logarithmically smooth family
Log_B	stack of fine and saturated logarithmic structures over B
$\mathcal{D}_m \subset \mathrm{Log}_B$	divisor corresponding to multipliciation by m (Definition 3.3.1)
$b_0 \in B$	special point, with induced structure of logarithmic point
$X_0 \subset X$	fibre $X_0 = X \times_B \{b_0\}$
$H_2(X)$	group of degree data, e.g. $H_2(X, \mathbb{Z})$ for $\Bbbk = \mathbb{C}$
A	a curve class $A \in H_2(X)$
g	genus of a curve
k	number of marked points
$f : C \to X$	a stable logarithmic map
$\mathbf{p}$	marked points on C, $\mathbf{p} = (p_1, \ldots, p_k)$
u_{p_i}	type or contact order at marked point p_i (§2.3.6(ii))
q	a node $q \in C$
$\underline{\beta}$	discrete data for a usual stable map: $\underline{\beta} = (g, k, A)$
β	discrete data for a stable logarithmic map: $\beta = (\underline{\beta}, u_{p_1}, \ldots, u_{p_k})$
β'	discrete data for a logarithmic map: $\beta' = (g, u_{p_1}, \ldots, u_{p_k})$
$\mathscr{M}(X/B, \beta)$	moduli stack of stable logarithmic maps of X relative to B
$\mathscr{M}(\underline{X}/\underline{B}, \underline{\beta})$	moduli stack of stable maps of the underlying $\underline{X}$ relative to $\underline{B}$
$\mathscr{M}(X_0/b_0, \beta)$	moduli stack of stable logarithmic maps of the fibre X_0
$\mathcal{X}$	the 1-dimensional Artin stack $\mathcal{A}_X \times_{\mathcal{A}} B$ (§2.2)
$\mathcal{X}_0$	the fibre $\mathcal{X} \times_B b_0$ of $\mathcal{X}$
$\mathfrak{M}_B, \mathfrak{M}_{b_0}$	moduli of logarithmic curves over B, respectively b_0
$\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$	moduli of logarithmic maps
$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$	moduli of logarithmic maps with multiplicity m
$\Sigma(X)$	the cone complex of X (§2.1.2)
$\Delta(X)$	the polyhedral complex of X (§2.1.3)

G	a graph
$E(G)$	set of edges of G
$V(G)$	set of vertices of G
$L(G)$	set of legs of G
ℓ	a length function $\ell : E(G) \to \mathbb{R}_{>0}$
Γ	a tropical curve $\Gamma = (G, \mathbf{g}, \ell)$ (Definition 2.5.2)
$\tilde{\tau}$	the combinatorial type of a decorated tropical curve in $\Sigma(X)$ (Definition 2.5.3)
τ	isomorphism class of $\tilde{\tau}$
$\mathbf{A} \vdash A$	a partition of the curve class A
$\mathscr{M}_{\tilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$	moduli of stable logarithmic maps marked by $\tilde{\tau}, \mathbf{A}$
$\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta)$	moduli of stable logarithmic maps marked by the isomorphism class $\tau, \mathbf{A}$
$\mathscr{M}(X/B, \beta, \sigma)$	moduli of stable logarithmic maps with point constraints
$O^\dagger$	the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$
$Y^\dagger$	$:= Y \times O^\dagger$
$\mathscr{M}(Y^\dagger/b_0, \beta, \sigma)$	moduli of stable logarithmic maps with point constraints
$\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$	moduli of marked stable logarithmic maps with point constraints

Part 1. Theory

2. Preliminaries

2.1. Cone complexes associated to logarithmic stacks.

2.1.1. *The category of cones.* We consider the category of rational polyhedral cones, wich we denote by **Cones**. The objects of **Cones** are pairs $\sigma = (\sigma_{\mathbb{R}}, N)$ where $N \cong \mathbb{Z}^n$ is a lattice and $\sigma_{\mathbb{R}} \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a top-dimensional strictly convex rational polyhedral cone. A morphism of cones $\varphi : \sigma_1 \to \sigma_2$ is a homomorphism $\varphi : N_1 \to N_2$ which takes $\sigma_{1\mathbb{R}}$ into $\sigma_{2\mathbb{R}}$. Such a morphism is a *face morphism* if it identifies $\sigma_{1\mathbb{R}}$ with a face of $\sigma_{2\mathbb{R}}$ and N_1 with a saturated sublattice of N_2. If we need to specify that N is associated to σ we write N_σ instead.

2.1.2. *Generalized cone complexes.* Recall from [ACP15] that a *generalized cone complex* is a topological space with a presentation as the colimit of an arbitrary finite diagram in the category **Cones** with all morphisms being face morphisms. If Σ denotes a generalized cone complex, we write $\sigma \in \Sigma$ if σ is a cone in the diagram yielding Σ, and write $|\Sigma|$ for the underlying topological space. A morphism of generalized cone complexes $f : \Sigma \to \Sigma'$ is a continuous map $f : |\Sigma| \to |\Sigma'|$ such that for each $\sigma_{\mathbb{R}} \in \Sigma$, the induced map $\sigma \to |\Sigma'|$ factors through a morphism $\sigma \to \sigma' \in \Sigma'$.

Note that two generalized cone complexes can be isomorphic yet not have the same presentation. In particular, [ACP15, Proposition 2.6.2] gives a good choice of presentation, called a *reduced* presentation. This presentation has the property that every face of a cone in the diagram is in the diagram, and every isomorphism in the diagram is a self-map.

2.1.3. *Generalized polyhedral complexes.* We can similarly define a *generalized polyhedral complex*, where in the above set of definitions pairs $(\sigma_{\mathbb{R}}, N)$ live in the category **Poly** of rationally defined polyhedra. This is more general than cones, as any cone σ is in particular a polyhedron (usually unbounded). For example, an affine slice of a fan is a polyhedral complex.

2.1.4. *The tropicalization of a logarithmic scheme.* Now let X be a Zariski fs log scheme of finite type. For the generic point η of a stratum of X, its characteristic monoid $\overline{\mathcal{M}}_{X,\eta}$ defines a dual monoid $(\overline{\mathcal{M}}_{X,\eta})^\vee := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{N})$ lying in a group $(\overline{\mathcal{M}}_{X,\eta})^* := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{Z})$, see Section 1.3, hence a dual cone

$$\sigma_\eta := \big((\overline{\mathcal{M}}_{X,\eta})^\vee_{\mathbb{R}}, (\overline{\mathcal{M}}_{X,\eta})^*\big).$$

If η is a specialization of η', then there is a well-defined generization map $\overline{\mathcal{M}}_{X,\eta} \to \overline{\mathcal{M}}_{X,\eta'}$ since we assumed X is a Zariski logarithmic scheme. Dualizing, we obtain a face morphism $\sigma_{\eta'} \to \sigma_\eta$. This gives a diagram of cones indexed by strata of X with face morphisms, and hence gives a generalized cone complex $\Sigma(X)$. We call this the *tropicalization* of X, following [GS13], Appendix B. [1]

This construction is functorial: given a morphism of log schemes $f : X \to Y$, one obtains from the map $f^\flat : f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ an induced map of generalized cone complexes $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$.

Definition 2.1.5. We say X is *monodromy free* if X is a Zariski log scheme and for every $\sigma \in \Sigma(X)$, the natural map $\sigma \to |\Sigma(X)|$ is injective on the interior of any face of σ, see [GS13, Definition B.2]. We say X is *simple* if the map is injective on every σ.

As remarked in [GS13], we can then define the generalized cone complex associated with a finite type logarithmic stack X, in particular allowing for logarithmic schemes X in the étale topology. One can always find a cover $X' \to X$ in the smooth topology with X' a union of simple log schemes, and with $X'' = X' \times_X X'$, we define $\Sigma(X)$ to be the colimit of $\Sigma(X'') \rightrightarrows \Sigma(X')$. The resulting generalized cone complex is independent of the choice of cover. This process is explicitly carried out in [ACP15] and [Uli15].

Examples 2.1.6. (1) If X is a toric variety with the canonical toric logarithmic structure, then $\Sigma(X)$ is abstractly the fan defining X. It is missing the embedding of $|\Sigma(X)|$ as a fan in a vector space $N_{\mathbb{R}}$, and should be viewed as a piecewise linear object.

[1] This terminology differs slightly from that of [Uli17], where the tropicalization is a canonically defined map to the compactified cone complex. Hopefully this will not cause confusion.

(2) Let $\Bbbk$ be a field and $X = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$ the standard log point with $\mathcal{M}_X = \Bbbk^\times \times \mathbb{N}$. Then $\Sigma(X)$ consists of the ray $\mathbb{R}_{\geq 0}$.

(3) Let C be a curve with an étale logarithmic structure with the property that $\overline{\mathcal{M}}_C$ has stalk $\mathbb{N}^2$ at any geometric point, but has monodromy of the form $(a,b) \mapsto (b,a)$, so that the pull-back of $\overline{\mathcal{M}}_C$ to a double cover $C' \to C$ is constant but $\overline{\mathcal{M}}_C$ is only locally constant. Then $\Sigma(C)$ can be described as the quotient of $\mathbb{R}^2_{\geq 0}$ by the automorphism $(a,b) \mapsto (b,a)$. If we use the reduced presentation, $\Sigma(C)$ has one cone each of dimensions 0, 1 and 2.

See [GS13, Example B.1] for a further example which is Zariski but not monodromy-free.

2.2. **Artin fans.** Let W be a fine and saturated algebraic log stack. We are quite permissive with algebraic stacks, as delineated in [Ols03, (1.2.4)–(1.2.5)], since we need to work with stacks with non-separated diagonal. An Artin stack logarithmically étale over $\operatorname{Spec} \Bbbk$ is called an *Artin fan*.

The logarithmic structure of W is encoded by a morphism $W \to \mathrm{Log}$, see [Ols03]. For many purposes this needs to be refined, since different strata of W may map to the same point of Log, and we wish to distinguish strata. Following preliminary notes written by two of us (Chen and Gross), the paper [AW13] introduces a canonical Artin fan $\mathcal{A}_W$ associated to a logarithmically smooth fs log scheme W. This was generalized in [ACMW17, Prop. 3.1.1]:

Theorem 2.2.1. *Let X be a logarithmic algebraic stack which is locally connected in the smooth topology. Then there is an initial factorization of the map $X \to \mathrm{Log}$ through a strict étale morphism $\mathcal{A}_X \to \mathrm{Log}$ which is representable by algebraic spaces.*

There is a more explicit description of $\mathcal{A}_X$ in terms of the cone complex $\Sigma(X)$. For any cone $\sigma \subseteq N_{\mathbb{R}}$, let $P = \sigma^\vee \cap M$ be the corresponding monoid. We write

$$\mathcal{A}_\sigma = \mathcal{A}_P := [\operatorname{Spec} \Bbbk[P] / \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}]]. \tag{2.2.1}$$

This stack carries the standard toric logarithmic structure coming from the global chart $P \to \Bbbk[P]$. We then have:

Proposition 2.2.2. *Let X be a logarithmic Deligne-Mumford stack, with cone complex $\Sigma(X)$ a colimit of a diagram of cones $s : I \to$ **Cones**. Then $\mathcal{A}_X$ is the colimit as sheaves over* Log *of the corresponding diagram of sheaves given by $I \ni i \mapsto \mathcal{A}_{s(i)}$.*

Proof. This follows from the construction of $\mathcal{A}_X$ in [ACMW17], Proposition 3.1.1. ♠

Remark 2.2.3. Unlike $\Sigma(X)$, the formation of $\mathcal{A}_X$ is not functorial for all logarithmic morphisms $Y \to X$. This is a result of the fact that the morphism $Y \to \mathrm{Log}$ is not the composition $Y \to X \to \mathrm{Log}$, unless $Y \to X$ is strict. Note also that not all Artin fans $\mathcal{Y}$ are of the form $\mathcal{A}_X$, since $\mathcal{Y} \to \mathrm{Log}$ may fail to be representable.

Lemma 2.2.4. *Suppose X is a log smooth scheme with Zariski log structure. Then $\mathcal{A}_X$ admits a Zariski open covering $\{\mathcal{A}_\sigma \subset \mathcal{A}_X\}$.*

Proof. Since X has Zariski log structure, we may select a covering $\{U \to X\}$ of Zariski open sets such that $U \to \mathcal{A}_{\sigma_U}$ is the Artin fan. On the other hand, denote by $\tilde{U}$ the image of the composition $U \to X \to \mathcal{A}_X$. By the log smoothness of X, the morphism $X \to \mathcal{A}_X$ is smooth, hence $\tilde{U} \subset \mathcal{A}_X$ is an open substack.

It remains to show that $\tilde{U}$ is the Artin fan of U. By [AW13, Section 2.3 and Definition 2.3.2(2)], this amounts to show that $\tilde{U}$ parameterizes the connected components of the fibres of $U \to \mathrm{Log}$. Since both $X \to \mathrm{Log}$ and $U \to \mathrm{Log}$ are smooth morphisms between reduced stacks, it suffices to consider each geometric point $T \to \mathrm{Log}$. Since $U \subset X$ is Zariski open, $U_T = T \times_{\mathrm{Log}} U \subset X_T = T \times_{\mathrm{Log}} X$ is also Zariski open. Thus, for each connected component $V \subset U_T$, there is a unique connected component $V' \subset X_T$ containing V as a Zariski open dense set. As the set of connected components of X_T is parameterized by $T \times_{\mathrm{Log}} \mathcal{A}_X$, we observe that the set of connected components of U_T is parameterized by the subscheme $T \times_{\mathrm{Log}} \tilde{U} \subset T \times_{\mathrm{Log}} \mathcal{A}_X$. ♠

Lemma 2.2.5. *Suppose X is a log smooth scheme with Zariski log structure. Then any morphism $X \to \mathcal{A}_\tau$ has a canonical factorization through $\mathcal{A}_X \to \mathcal{A}_\tau$.*

Proof. By Lemma 2.2.4, we may select a Zariski covering $\mathcal{C} := \{\mathcal{A}_\sigma \subset \mathcal{A}_X\}$, hence a Zariski covering $\{U_\sigma := \mathcal{A}_\sigma \times_{\mathcal{A}_X} X \subset X\}$. We may assume that if $\sigma' \subset \sigma$ is a face, then $\mathcal{A}_{\sigma'} \subset \mathcal{A}_\sigma \subset \mathcal{A}_X$ is also in $\mathcal{C}$.

Locally, the morphism $U_\sigma \to \mathcal{A}_\tau$ induces a morphism $\tau^\vee \to \Gamma(U_\sigma, \overline{\mathcal{M}}_{U_\sigma}) = \sigma^\vee$, hence a canonical $\phi_\sigma : \mathcal{A}_\sigma \to \mathcal{A}_\tau$ through which $U_\sigma \to \mathcal{A}_\tau$ factors.

To see the local construction glues, observe that the intersection $\mathcal{A}_{\sigma_1} \cap \mathcal{A}_{\sigma_2}$ of two Zariski charts in $\mathcal{C}$ is again covered by elements in $\mathcal{C}$. It suffices to verify that $\phi_{\sigma_1}, \phi_{\sigma_2}$ agree on $\mathcal{A}_{\sigma'} \in \mathcal{C}$ if $\mathcal{A}_{\sigma'} \subset \mathcal{A}_{\sigma_1} \cap \mathcal{A}_{\sigma_2}$. Taking global sections, we observe that the composition $\tau^\vee \to \Gamma(U_{\sigma_i}, \overline{\mathcal{M}}_{U_{\sigma_i}}) \to \Gamma(U_{\sigma'}, \overline{\mathcal{M}}_{U_{\sigma'}}) = (\sigma')^\vee$ is independent of $i = 1, 2$ as they are determined by the restriction of $U_{\sigma_i} \to \mathcal{A}_\tau$ to the common Zariski open $U_{\sigma'}$. Hence $\phi_{\sigma_1}|_{U_{\sigma'}} = \phi_{\sigma_2}|_{U_{\sigma'}}$. ♠

Proposition 2.2.6. *Let $X \to Y$ be a morphism of log schemes. Suppose X is log smooth with Zariski log structure. Then there is a canonical morphism $\mathcal{A}_X \to \mathcal{A}_Y$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{A}_X & \longrightarrow & \mathcal{A}_Y \end{array}$$

Proof. Since the to be constructed arrow $\mathcal{A}_X \to \mathcal{A}_Y$ is canonical, the statement can be checked locally on $\mathcal{A}_Y$ by étale descent. Passing to an étale chart, we may assume that $\mathcal{A}_Y = \mathcal{A}_\tau$, and apply Lemma 2.2.5. ♠

The following proposition, while strictly speaking not needed for this paper, is very useful for developing an intuition of what it means to give a log morphism to an Artin fan.

Proposition 2.2.7. *Let X be a Zariski fs log scheme log smooth over* $\operatorname{Spec} \Bbbk$. *Then $\mathcal{A}_X$ represents the functor on fs log schemes*

$$T \mapsto \operatorname{Hom}(\Sigma(T), \Sigma(X)),$$

the latter being the set of morphisms of cone complexes.

Proof. Step I. Description of $\mathcal{A}_X$. By Lemma 2.2.4, we may select a Zariski covering $\mathcal{C} := \{\mathcal{A}_\sigma \subset \mathcal{A}_X\}$, hence a Zariski covering $\{U_\sigma := \mathcal{A}_\sigma \times_{\mathcal{A}_X} X \subset X\}$. We may assume that if $\sigma' \subset \sigma$ is a face, then $\mathcal{A}_{\sigma'} \subset \mathcal{A}_\sigma \subset \mathcal{A}_X$ is also in $\mathcal{C}$.

Thus $\Sigma(X)$ can be presented by the collection of cones $\{\sigma\}$ glued along face maps $\sigma' \to \sigma$.

In particular, this shows that $\Sigma(X) = \Sigma(\mathcal{A}_X)$. Since Σ is functorial, there is then a map $\operatorname{Hom}(T, \mathcal{A}_X) \to \operatorname{Hom}(\Sigma(T), \Sigma(X))$. We need to construct the inverse.

Step II. T is atomic. Suppose T has unique closed stratum T_0 and a global chart $P \to \mathcal{M}_T$ inducing an isomorphism $P \cong \overline{\mathcal{M}}_{T,\bar{t}}$ at some point $\bar{t} \in T_0$ - in the language of [AW13, Defnition 2.2.4] the logarithmic scheme T is *atomic*. Then with $\tau := \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$, $\Sigma(T) = \tau$.

Using the presentation of $\Sigma(X)$ described in Step I, a map $\alpha : \Sigma(T) \to \Sigma(X)$ has image $\alpha(\tau) \subseteq \sigma_i \in \Sigma(X)$ for some i. Observe that $\operatorname{Hom}(T, \mathcal{A}_{\sigma_i}) = \operatorname{Hom}(Q_i, \Gamma(T, \overline{\mathcal{M}}_T))$ by [Ols03], Prop. 5.17. Now $\Gamma(T, \overline{\mathcal{M}}_T) = P$, and giving a homomorphism $Q_i \to P$ is equivalent to giving a morphism of cones $\tau \to \sigma_i$. Thus $\operatorname{Hom}(T, \mathcal{A}_{\sigma_i}) = \operatorname{Hom}(\tau, \sigma_i)$. In particular, α induces a composed map $T \to \mathcal{A}_{\sigma_i} \subseteq \mathcal{A}_X$, yielding the desired inverse map $\operatorname{Hom}(\Sigma(T), \Sigma(X)) \to \operatorname{Hom}(T, \mathcal{A}_X)$.

Step III. T general. In general T has an étale cover $\{T_i\}$ by atomic logarithmic schemes, and each $T_{ij} := T_i \times_T T_j$ also has such a covering $\{T_{ij}^k\}$ by atomic logarithmic schemes. This gives a presentation $\coprod \Sigma(T_{ij}^k) \rightrightarrows \coprod \Sigma(T_i)$ of $\Sigma(T)$. In particular, a morphism of cone complexes $\Sigma(T) \to \Sigma(X)$ induces morphisms $\Sigma(T_i) \to \Sigma(X)$ compatible with the maps $\Sigma(T_{ij}^k) \to \Sigma(T_i), \Sigma(T_j)$. Thus we obtain unique morphisms $T_i \to \mathcal{A}_X$ compatible with the morphisms $T_{ij}^k \to T_i, T_j$, inducing a morphism $T \to \mathcal{A}_X$. ♠

2.3. Stable logarithmic maps and their moduli.

2.3.1. *Definition.* We fix a log morphism $X \to B$ with the logarithmic structure on X being defined in the Zariski topology. Recall from [GS13], [Che14] and [AC14]:

Definition 2.3.2. A *stable logarithmic map* $(C/W, \mathbf{p}, f)$ is a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ W & \longrightarrow & B \end{array} \tag{2.3.1}$$

where

(1) $\pi : C \to W$ is a proper logarithmically smooth and integral morphism of log schemes together with a tuple of sections $\mathbf{p} = (p_1, \ldots, p_k)$ of $\underline{\pi}$ such that every geometric fibre of π is a reduced and connected curve, and if $U \subset \underline{C}$ is the non-critical locus of $\underline{\pi}$ then $\overline{\mathcal{M}}_C|_U \cong \underline{\pi}^*\overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^k p_{i*}\mathbb{N}_W$.

(2) For every geometric point $\bar{w} \to \underline{W}$, the restriction of $\underline{f}$ to $\underline{C}_{\bar{w}}$ together with $\mathbf{p}$ is an ordinary stable map.

2.3.3. *Basic maps.* The crucial concept for defining moduli of stable logarithmic maps is the notion of *basic* stable logarithmic maps. To explain this in tropical terms, we begin by summarizing the discussion of [GS13], §1 where more details are available. The terminology used in [Che14, AC14] is *minimal* stable logarithmic maps.

2.3.4. *Induced maps of monoids.* Suppose given $(C/W, \mathbf{p}, f)$ a stable logarithmic map with $W = \mathrm{Spec}(Q' \to \Bbbk)$, with Q' an arbitrary sharp fs monoid and $\Bbbk$ an algebraically closed field. We will use the convention that a point denoted $p \in C$ is always a marked point, and a point denoted $q \in C$ is always a nodal point. Denoting $\underline{Q}' = \pi^{-1}Q'$, the morphism $\pi^\flat$ of logarithmic structures induces a homomorphism of sheaves of monoids $\psi : \underline{Q}' \to \overline{\mathcal{M}}_C$. Similarly $f^\flat$ induces $\varphi : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$.

2.3.5. *Structure of ψ.* The homomorphism ψ is an isomorphism when restricted to the complement of the special (nodal or marked) points of C. The sheaf $\overline{\mathcal{M}}_C$ has stalks $Q' \oplus \mathbb{N}$ and $Q' \oplus_{\mathbb{N}} \mathbb{N}^2$ at marked points and nodal points respectively. The latter fibred sum is determined by a map

$$\begin{aligned} \mathbb{N} &\longrightarrow Q' \\ 1 &\longmapsto \rho_q \end{aligned} \tag{2.3.2}$$

and the diagonal map $\mathbb{N} \to \mathbb{N}^2$ (see Def. 1.5 of [GS13]). The map ψ at these special points is given by the inclusion $Q' \to Q' \oplus \mathbb{N}$ and $Q' \to Q' \oplus_{\mathbb{N}} \mathbb{N}^2$ into the first component for marked and nodal points respectively.

2.3.6. *Structure of φ.* For $\bar{x} \in C$ a geometric point with underlying scheme-theoretic point x, the map φ induces maps $\varphi_{\bar{x}} : P_x \to \overline{\mathcal{M}}_{C,\bar{x}}$ for

$$P_x := \overline{\mathcal{M}}_{X,\underline{f}(\bar{x})}$$

(well-defined independently of the choice of $\bar{x} \to x$ since the logarithmic structure on X is Zariski). Following Discussion 1.8 of [GS13], we have the following behaviour at three types of points on C:

(i) $x = \eta$ is a generic point, giving a local homomorphism[2] of monoids
$$\varphi_{\bar{\eta}} : P_\eta \longrightarrow Q'.$$

(ii) $x = p$ is a marked point, giving u_p the composition
$$u_p : P_p \xrightarrow{\varphi_{\bar{p}}} Q' \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}.$$
The element $u_p \in P_p^\vee$ is called the *contact order* at p.

(iii) $x = q$ is a node contained in the closures of η_1, η_2. If $\chi_i : P_q \to P_{\eta_i}$ are the generization maps there exists a homomorphism
$$u_q : P_q \to \mathbb{Z},$$
called *contact order at* q, such that
$$\varphi_{\bar{\eta}_2}\big(\chi_2(m)\big) - \varphi_{\bar{\eta}_1}\big(\chi_1(m)\big) = u_q(m) \cdot \rho_q, \tag{2.3.3}$$
with $\rho_q \neq 0$ given in Equation (2.3.2), see [GS13], (1.8). This data completely determines the local homomorphism $\varphi_{\bar{q}} : P_q \to Q' \oplus_{\mathbb{N}} \mathbb{N}^2$.

The choice of ordering η_1, η_2 for the branches of C containing a node is called an *orientation* of the node. We note that reversing the orientation of a node q (by interchanging η_1 and η_2) results in reversing the sign of u_q.

2.3.7. *Dual graphs and combinatorial type.* As customary when studying nodal curves and their maps, a graph G will consist of a set of vertices $V(G)$, a set of edges $E(G)$ and a separate set of *legs* or *half-edges* $L(G)$, with appropriate incidence relations between vertices and edges, and between vertices and half-edges. In order to obtain the correct notion of automorphisms, we also implicitly use the convention that every edge $E \in E(G)$ of G is a pair of *orientations of* E or a pair of *half-edges of* E (disjoint from $L(G)$), so that the automorphism group of a graph with a single loop is $\mathbb{Z}/2\mathbb{Z}$.

Let G_C be the dual intersection graph of C. This is the graph which has a vertex v_η for each generic point η of C, an edge E_q joining v_{η_1}, v_{η_2} for each node q contained in the closure of η_1 and η_2, and where E_q is a loop if q is a double point in an irreducible component of C. Note that an ordering of the two branches of $\underline{C}$ at a node gives rise to an orientation on the corresponding edge. Finally, G_C has a leg L_p with endpoint v_η for each marked point p contained in the closure of η.

Definition 2.3.8. Let $(C/S, \mathbf{p}, f)$ be a stable logarithmic map. The *combinatorial type* of $(C/S, \mathbf{p}, f)$ consists of the following data:

(1) The dual graph G_C.
(2) The contact data u_p corresponding to marked points of C.

[2] A homomorphism of monoids $\varphi : P \to Q$ is *local* if $\varphi^{-1}(Q^\times) = P^\times$.

(3) The contact data u_q corresponding to oriented nodes of C.

2.3.9. *The basic monoid.* Given a combinatorial type of a logarithmic map $(C/S, \mathbf{p}, f)$ we define a monoid Q by first defining its dual

$$Q^\vee = \left\{ \big((V_\eta)_\eta, (e_q)_q\big) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \,\middle|\, \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}. \tag{2.3.4}$$

Here the sum is over generic points η of C and nodes q of C. We then set

$$Q := \mathrm{Hom}(Q^\vee, \mathbb{N}).$$

It is shown in [GS13], §1.5, that Q is a sharp monoid, necessarily fine and saturated by construction.

Given a stable logarithmic map $(C/W, \mathbf{p}, f)$ over $W = \mathrm{Spec}(Q' \to \Bbbk)$ of the given combinatorial type, we obtain a canonically defined map

$$Q \to Q' \tag{2.3.5}$$

which is most easily defined as the transpose of the map

$$(Q')^\vee \to Q^\vee \subseteq \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N},$$

given by

$$m \mapsto \big((\varphi_{\bar\eta}^t(m))_\eta, (m(\rho_q))_q\big).$$

where $\varphi_{\bar\eta}$ is defined in Section 2.3.6 and ρ_q is given in Equation 2.3.2.

Definition 2.3.10 (Basic maps)**.** Let $(C/W, \mathbf{p}, f)$ be a stable logarithmic map. If $W = \mathrm{Spec}(Q' \to \Bbbk)$ for some Q' and algebraically closed field $\Bbbk$, we say this stable logarithmic map is *basic* if the above map $Q \to Q'$ is an isomorphism. If W is an abitrary fs log scheme, we say the stable map is *basic* if $(C_{\bar w}/\bar w, \mathbf{p}, f|_{C_{\bar w}})$ is basic for all geometric points $\bar w \in W$.

2.3.11. *Degree data and class.* In what follows, we will need to make a choice of a notion of *degree data* for curves in X; we will write the group of degree data as $H_2(X)$. This could be 1-cycles on X modulo algebraic or numerical equivalence, or it could be $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$. If we work over $\mathbb{C}$, we can use ordinary singular homology $H_2(X, \mathbb{Z})$. In general, any family of stable maps $\underline{f} : \underline{C}/\underline{W} \to \underline{X}$ should induce a well-defined class $\underline{f}_*[C_{\bar w}] \in H_2(X)$ for $\bar w \in \underline{W}$ a geometric point; if $\underline{W}$ is connected, this class should be independent of the choice of $\bar w$.

Definition 2.3.12. A *class* β of stable logarithmic maps to X consists of the following:

(i) The data $\underline{\beta}$ of an underlying ordinary stable map, i.e., the genus g, the number of marked points k, and data $A \in H_2(X)$.

(ii) Integral elements $u_{p_1}, \ldots, u_{p_k} \in |\Sigma(X)|$. [3]

We say a stable logarithmic map $(C/W, \mathbf{p}, f)$ *is of class* β if two conditions are satisfied. First, the underlying ordinary stable map must be of type $\underline{\beta} = (g, k, A)$. Second, define the closed subset $\underline{Z}_i \subseteq \underline{X}$ to be the union of strata with generic points η such that u_{p_i} lies in the image of $\sigma_\eta \to |\Sigma(X)|$. Then for any i we have $\mathrm{im}(\underline{f} \circ p_i) \subset \underline{Z}_i$ and for any geometric point $\bar{w} \to \underline{W}$ such that $p_i(\bar{w})$ lies in the stratum of X with generic point η, the composed map

$$\overline{\mathcal{M}}_{X,\underline{f}(p_i(\overline{w}))} \xrightarrow{\chi} \overline{\mathcal{M}}_{X,\bar{\eta}} \xrightarrow{\overline{f}^\flat} \overline{\mathcal{M}}_{C,p_i(\bar{w})} = \overline{\mathcal{M}}_{W,\underline{w}} \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N},$$

thought of as an element of σ_η, maps to $u_{p_i} \in |\Sigma(X)|$ under the map $\sigma_\eta \to |\Sigma(X)|$. Here χ is the generization map. In particular, s_i specifies the contact order u_{p_i} at the marked point $p_i(\bar{w})$.

We emphasize that the class β does not specify the contact orders u_q at nodes.

Definition 2.3.13. Let $\mathscr{M}(X/B, \beta)$ denote the stack of basic stable logarithmic maps of class β. This is the category whose objects are basic stable logarithmic maps $(C/W, \mathbf{p}, f)$ of class β, and whose morphisms $(C/W, \mathbf{p}, f) \to (C'/W', \mathbf{p}', f')$ are commutative diagrams

$$\begin{array}{ccccc} C & \xrightarrow{g} & C' & \xrightarrow{f'} & X \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & W' & \longrightarrow & B \end{array}$$

with the left-hand square cartesian, $W \to W'$ strict, and $f = f' \circ g$, $\mathbf{p} = \mathbf{p}' \circ g$.

Theorem 2.3.14. *If $X \to B$ is proper, then $\mathscr{M}(X/B, \beta)$ is a proper Deligne-Mumford stack. If furthermore $X \to B$ is logarithmically smooth, then $\mathscr{M}(X/B, \beta)$ carries a perfect obstruction theory, defining a virtual fundamental class $[\mathscr{M}(X/B, \beta)]^{\mathrm{virt}}$ in the rational Chow group of the underlying stack $\underline{\mathscr{M}}(X/B, \beta)$.*

Proof. The stack $\mathscr{M}(X/B, \beta)$ is constructed in [GS13] in general for X a Zariski log scheme and in [Che14], [AC14] with the stronger assumption that there is a monoid P and a sheaf homomorphism $\underline{P} \to \overline{\mathcal{M}}_Y$ which locally lifts to a chart. Properness was proved in the latter two references in those cases, and in [GS13] with a certain hypothesis, combinatorial finiteness, see [GS13], Def. 3.3. Properness was shown in general in [ACMW17].

The existence of a perfect obstruction theory when $X \to B$ is logarithmically smooth was proved in [GS13], §5. ♠

[3] We remark that this definition of contact orders is different than that given in [GS13, Definition 3.1]. Indeed, the definition given there does not work when X is not monodromy free, and [GS13, Remark 3.2] is not correct in that case. However, [GS13, Definition 3.1] may be used in the monodromy free case.

We remark that the stack $\mathscr{M}(X/B,\beta)$ with its logarithmic structure also defines a stack in groupoids over the category of logarithmic schemes. As such it parametrizes all stable logarithmic maps, without requiring them to be basic.

2.4. **Stack of prestable logarithmic curves.** In [GS13], §5, one constructs a relative obstruction theory over $\mathfrak{M}_B$, the Artin stack of all prestable logarithmically smooth curves defined over B. We recall briefly how this moduli space is constructed, see [GS13], Appendix A for details.

First, working over a field $\Bbbk$, there is a moduli space $\mathbf{M}$ of prestable basic logarithmic curves over $\operatorname{Spec}\Bbbk$, essentially constructed by F. Kato in [Kat00]. Of course $\mathbf{M}$ is an algebraic log stack over $\operatorname{Spec}\Bbbk$.

If B is an arbitrary fs log scheme over $\operatorname{Spec}\Bbbk$, a morphism $\underline{W}\to\mathbf{M}\times_{\operatorname{Spec}\Bbbk}\underline{B}$ determines a logarithmic structure on $\underline{W}$ which has, as direct summand, the basic logarithmic structure for the family of curves $\underline{C}\to\underline{W}$ induced by $\underline{W}\to\mathbf{M}$. Thus $\mathbf{M}\times_{\operatorname{Spec}\Bbbk}\underline{B}$ acquires the structure of an algebraic log stack over the log scheme B.

We then set

$$\mathfrak{M}_B := \operatorname{Log}_{\mathbf{M}\times B},$$

Olsson's stack parametrizing logarithmic structures over $\mathbf{M}\times B$.[4]

2.5. **The tropical interpretation.** The precise form of the basic monoid Q came from a tropical interpretation, which will play an important role here. We review this in our general setting. Given a stable logarithmic map $(C/W,\mathbf{p},f)$, we obtain an associated diagram of cone complexes,

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ {\scriptstyle\Sigma(\pi)}\downarrow & & \downarrow \\ \Sigma(W) & \longrightarrow & \Sigma(B). \end{array} \tag{2.5.1}$$

This diagram can be viewed as giving a family of tropical curves mapping to $\Sigma(X)$, parameterized by the cone complex $\Sigma(W)$. Indeed, a fibre of $\Sigma(\pi)$ is a graph and the restriction of $\Sigma(f)$ to such a fibre can be viewed as a tropical curve mapping to $\Sigma(X)$. We make this precise.

First, we need to be a bit careful about the diagram giving a presentation of $\Sigma(X)$. To avoid difficulties in notation, we shall assume that in fact X is simple. This is not a restrictive assumption in this paper, as our results will only apply when X is log smooth over the trivial point $\operatorname{Spec}\Bbbk$, and as X is assumed to be Zariski in any event, it follows that X is simple.

We can then use the reduced presentation of $\Sigma(X)$ given by [ACP15, Proposition 2.6.2]: every face of a cone in the diagram presenting $\Sigma(X)$ is in the diagram, and every isomorphism is a self-map. But since X is in particular monodromy free, all isomorphisms are identities, and simplicity then implies that if

[4] In [GS13] one explicitly writes $\mathfrak{M}_B := \operatorname{Log}_B^{\bullet\to\bullet}\times_{\operatorname{Log}_B}(\mathbf{M}\times_{\operatorname{Spec}\Bbbk} B)$.

$\tau, \sigma \in \Sigma(X)$ with the image of τ in $|\Sigma(X)|$ a face of the image of σ, then there is a unique face map $\tau \to \sigma$ in the diagram.

Definition 2.5.1. Let G be a connected graph. We admit multiple edges, loops and legs (loose ends) and write $V(G)$, $E(G)$ and $L(G)$ for the sets of vertices, edges and legs, respectively. A *genus-weight function* is a function $\mathbf{g} : V(G) \to \mathbb{N}$. The pair $(G, \mathbf{g})$ is called *a genus-weighted graph*, and the value $\mathbf{g}(v)$ is called the genus associated to the vertex v.

Definition 2.5.2. A *tropical curve* $\Gamma = (G, \mathbf{g}, \ell)$ *of combinatorial type* $(G, \mathbf{g})$ is an assignment

$$\ell : E(G) \to \mathbb{R}_{>0}$$

of lengths to each edge of G.

There is an evident geometric realization of Γ as a metric space. We denote it $|\Gamma|$.

The genus of Γ is defined by the expression

$$g(\Gamma) = b_1(|\Gamma|) + \sum_{v \in V(G)} \mathbf{g}(v). \tag{2.5.2}$$

It depends only on the genus-weighted graph $(G, \mathbf{g})$.

Definition 2.5.3. A *tropical curve in* $\Sigma(X)$ consists of the following data:

(1) A tropical curve $\Gamma = (G, \mathbf{g}, \ell)$ as in Definition 2.5.2.
(2) A map
$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$$
(viewing $\Sigma(X)$ as a set of cones). This data must satisfy the condition that if E is either a leg or edge incident to a vertex v, then there is an inclusion of faces $\boldsymbol{\sigma}(v) \subseteq \boldsymbol{\sigma}(E)$ in the (reduced) presentation of $\Sigma(X)$.
(3) Leg ordering: writing $k = \#L(G)$, we are given a bijection between $L(G)$ and $\{1, \ldots, k\}$.
(4) Edge marking: for each edge $E_q \in E(G)$ with a choice of orientation, an element $u_q \in N_{\boldsymbol{\sigma}(E_q)}$; reversing the orientation of E_q results in replacing u_q by $-u_q$.
(5) Leg marking: for each leg $E_p \in L(G)$ an element $u_p \in N_{\boldsymbol{\sigma}(E_p)} \cap \boldsymbol{\sigma}(E_p)$.
(6) A map $f : |\Gamma| \to |\Sigma(X)|$ satisfying conditions
  (a) For $v \in V(G)$ we have $f(v) \in \mathrm{Int}(\boldsymbol{\sigma}(v))$.
  (b) Let $E_q \in E(G)$ be an edge with endpoints v_1 and v_2, oriented from v_1 to v_2. Then
    (i) $f(\mathrm{Int}(E_q)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_q))$,
    (ii) f maps E_q affine linearly to the line segment in $\boldsymbol{\sigma}(E_q)$ joining $f(v_1)$ and $f(v_2)$, with the points $f(v_1)$ and $f(v_2)$ viewed as elements of $\boldsymbol{\sigma}(E_q)$ via the unique inclusions of faces $\boldsymbol{\sigma}(v_i) \subseteq \boldsymbol{\sigma}(E_q)$.

(iii) we have [5]

$$f(v_2) - f(v_1) = \ell(E_q)u_q.$$

(c) For $E_p \in L(G)$ a leg with vertex v, it holds

$$f(\mathrm{Int}(E_p)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_p)),$$

and f maps E_p affine linearly to the ray

$$f(v) + \mathbb{R}_{\geq 0}u_p \subseteq \boldsymbol{\sigma}(E_p).$$

We often just write $f : \Gamma \to \Sigma(X)$ to refer to the data (1)–(6) of a tropical curve.

The *combinatorial type* $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ of a decorated tropical curve in $\Sigma(X)$ is the graph G along with data (2)–(5) above. Note that we are suppressing the leg numbering, viewing the set $L(G)$ as identical with $\{1, \ldots, k\}$.

2.5.4. *Tropical curves from logarithmically smooth curves.* Now suppose

$$W = \mathrm{Spec}(Q \to \Bbbk)$$

for some monoid Q. Then the diagram (2.5.1) is interpreted as follows. First,

$$\Sigma(W) = Q^\vee_{\mathbb{R}} := \mathrm{Hom}(Q, \mathbb{R}_{\geq 0}).$$

For any point $m \in Q^\vee_{\mathbb{R}}$, the inverse image $\Sigma(\pi)^{-1}(m)$ is a tropical curve. When m lies in the interior of $Q^\vee_{\mathbb{R}}$, then the combinatorial type of the curve is the dual intersection graph Γ_C of C. Explicitly:

(i) If η is a generic point of C, then $\sigma_\eta = Q^\vee_{\mathbb{R}}$ and $\Sigma(\pi)|_{\sigma_\eta}$ is the identity. Thus each fibre of $\Sigma(\pi)|_{\sigma_\eta}$ is a point v. We put the weight $\mathbf{g}(v) = g(C_\eta)$, the geometric genus of the component with generic point η.

(ii) If q is a node of C, then

$$\sigma_q = \mathrm{Hom}(Q \oplus_{\mathbb{N}} \mathbb{N}^2, \mathbb{R}_{\geq 0}) = Q^\vee_{\mathbb{R}} \times_{\mathbb{R}_{\geq 0}} \mathbb{R}^2_{\geq 0},$$

where the maps $Q^\vee_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ and $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ are given by $\rho_q \in Q \setminus \{0\}$ and $(a, b) \mapsto a + b$, respectively. Thus the fibre of $\Sigma(\pi)|_{\sigma_q}$ over $m \in Q^\vee_{\mathbb{R}}$ is an interval. This interval admits an integral affine isomorphism to an interval of affine length $m(\rho_q)$. Since $\rho_q \in Q \setminus \{0\}$, this length is non-zero if $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$. If the corresponding edge of $\Gamma = \Sigma(\pi)^{-1}(m)$ is called E_q, then we set $\ell(E_q) = m(\rho_q)$.

(iii) If $p \in C$ is a marked point, then $\sigma_p = Q^\vee_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, and $\Sigma(\pi)|_{\sigma_p}$ is the projection onto the first component. Thus a fibre of $\Sigma(\pi)|_{\sigma_p}$ is a ray we denote by E_p.

This analysis then makes clear the claim that Γ is the dual intersection graph Γ_C of C whenever $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$. However, the tropical structure of Γ_C, i.e., the lengths of the edges, depends on m.

If m lies in the boundary of $Q^\vee_{\mathbb{R}}$, then $\Sigma(\pi)^{-1}(m)$ is obtained from Γ_C by contracting the bounded edges E_q such that $m(\rho_q) = 0$. For example, $\Sigma(\pi)^{-1}(0)$

[5]Note that u_q and ℓ are required data for a tropical map, as they cannot in general be recovered from the image.

consists of a single vertex with an attached unbounded edge for each marked point of C. In general, if $m \in \mathrm{Int}(\tau)$, $m' \in \mathrm{Int}(\sigma)$ with $\tau \subseteq \sigma$ faces of $Q^\vee_{\mathbb{R}}$, then there is a continuous map with connected fibres

$$\xi : \Sigma(\pi)^{-1}(m') \to \Sigma(\pi)^{-1}(m) \tag{2.5.3}$$

which contracts precisely those edges of the first graph whose lengths go to zero over m. This is compatible with the weight $\mathbf{g}$ in the sense that for a vertex $v \in \Sigma(\pi)^{-1}(m)$ we have $\mathbf{g}(v) = g(\xi^{-1}(v))$, where $g(\xi^{-1}(v))$ is calculated using Equation (2.5.2).

2.5.5. *Tropical curves in $\Sigma(X)$ from stable logarithmic maps.* Continuing with $W = \mathrm{Spec}(Q \to \Bbbk)$, the map $\Sigma(f)$ encodes the map $\varphi : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ and defines a family of tropical curves in $\Sigma(X)$ in the sense of Definition 2.5.3. The data (1)–(6) are specified as follows, given $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$:

(1) Identify Γ_C with $\Sigma(\pi)^{-1}(m)$, which we have seen is a tropical curve.
(2) An element x of $V(G) \cup E(G) \cup L(G)$ corresponds to a point $\bar{x} \in C$ — either a generic point, a double point, or a marked point. Define

$$\boldsymbol{\sigma}(x) := (P_x)^\vee_{\mathbb{R}} = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\underline{f}(\bar{x})}, \mathbb{R}_{\geq 0}) \in \Sigma(X).$$

(3) If the marked points are $p_1, \ldots, p_k$, the bijection $L(G) \leftrightarrow \{1, \ldots, k\}$ is $E_{p_i} \leftrightarrow i$.
(4) The edge marking data u_q are the vectors associated to $C \to X$ defined in §2.3; note u_q depends on a choice of orientation of E_q and is replaced by $-u_q$ when the orientation is reversed.
(5) The leg markings u_p are defined in the same section.
(6) For v_η a vertex of Γ_C, we have by definition of $\Sigma(f)$ that with $\varphi_{\bar{\eta}} : P_\eta \to Q$,

$$\Sigma(f)(v_\eta) = \varphi^t_{\bar{\eta}}(m) \in \mathrm{Int}(\boldsymbol{\sigma}(v_\eta)),$$

the latter since $P_\eta \to Q$ is a local homomorphism. This shows item (6)(a) of Definition 2.5.3. Property (6)(b) follows from (2.3.3). The remaining properties of a tropical curve in $\Sigma(X)$ are easily checked.

2.5.6. *Basic maps and tropical universal families.* Basicness of the map f can then be recast as follows. If $(C/W, f)$ is basic, then the above family of tropical curves is universal. Indeed, the definition of the dual of the basic monoid $Q^\vee$ precisely encodes the data of a tropical curve in $\Sigma(X)$ with the combinatorial type described above. A tuple $((V_\eta)_\eta, (e_q)_q) \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$ specifies a unique tropical curve $\Sigma(f) : \Gamma_C \to \Sigma(X)$ of the given combinatorial type with $\Sigma(f)(v_\eta) = V_\eta$.

Given another stable logarithmic map $(C'/W', \mathbf{p}, f')$ over $W' = \mathrm{Spec}(Q' \to \Bbbk)$ coinciding with $(C/W, \mathbf{p}, f)$ at the scheme level, the canonically defined map (2.3.5) can be viewed at the tropical level as a map $(Q')^\vee_{\mathbb{R}} \to Q^\vee_{\mathbb{R}}$. This map takes a point $m \in \mathrm{Int}((Q')^\vee_{\mathbb{R}})$ to the data specifying the map $\Sigma(f')|_{\Sigma(\pi)^{-1}(m)} : \Gamma_C \to \Sigma(X)$.

Remark 2.5.7. Note that if W is not a log point, the diagram (2.5.1) still exists, but the fibres of $\Sigma(\pi)$ may not be the expected ones. In particular, if $\bar{w}$ is a geometric point of W, there is a functorial diagram

$$\begin{array}{ccc} \Sigma(C_{\bar{w}}) & \longrightarrow & \Sigma(C) \\ \downarrow & & \downarrow \\ \Sigma(\bar{w}) & \longrightarrow & \Sigma(W) \end{array}$$

but this diagram need not be Cartesian. This might reflect monodromy in the family W. For example, it is easy to imagine a situation where $C_{\bar{w}}$ has two irreducible components and two nodes for every geometric point $\bar{w}$, but the nodal locus of $C \to W$ is irreducible, as there is monodromy interchanging the two nodes. Then a fibre of $\Sigma(C) \to \Sigma(W)$ may consist of two vertices joined by a single edge, while a fibre of $\Sigma(C_{\bar{w}}) \to \Sigma(\bar{w})$ will have two vertices joined by two edges. Similarly, there may be monodromy interchanging irreducible components, hence a fibre of $\Sigma(C) \to \Sigma(W)$ may have fewer vertices than $C_{\bar{w}}$ has irreducible components.[6]

3. From toric decomposition to virtual decomposition

3.1. **The toric picture.** There is an underlying fact — a simple decomposition formula in toric varieties — which makes the decomposition formula possible. The first ingredient is the following. Let W be a toric variety and $\pi : W \to \mathbb{A}^1$ a toric morphism. We write Σ_W and $\Sigma_{\mathbb{A}^1}$ for the associated fans, noting that these can be abstractly identified with $\Sigma(W)$ and $\Sigma(\mathbb{A}^1)$ as cone complexes, although the latter do not naturally lie inside a vector space. Associated to π we have a morphism of fans $\Sigma_\pi : \Sigma_W \to \Sigma_{\mathbb{A}^1}$. Prime toric divisors in W correspond to rays in Σ_W. Let Ω_W be the set of these rays, and for each ray $\tau \in \Omega_W$ write D_τ for the corresponding toric divisor. Write M_W for the character lattice and N_W for the lattice of 1-parameter subgroups of the torus of W, and similarly write $M_{\mathbb{A}^1}$, $N_{\mathbb{A}^1}$. We write $N_\tau = (N \cap \tau)^{\mathrm{gp}}$ for the lattice of integral points tangent to τ. We denote by $\tau_{\mathbb{A}^1} \in \Sigma(\mathbb{A}^1)$ the unique one-dimensional cone. The toric decomposition formula is the following standard observation:

Proposition 3.1.1. *(1) For $\tau \in \Omega_W$, we have isomorphisms $\tau \cap N_\tau \simeq \mathbb{N}$ and $\tau_{\mathbb{A}^1} \cap N_{\mathbb{A}^1} \simeq \mathbb{N}$, and the map $\tau \cap N_\tau \to \tau_{\mathbb{A}^1} \cap N_{\mathbb{A}^1}$ between them is given by multiplication by a non-negative integer m_τ.*
(2) The multiplicity of the divisor $\pi^*(\{0\})$ along D_τ is m_τ.

In other words, we have an equality of Weil divisors

$$\pi^*(\{0\}) = \sum_\tau m_\tau D_\tau. \tag{3.1.1}$$

[6] In [CCUW17], Chan, Cavalieri, Ulirsch and Wise redefine moduli of tropical curves as stacks, so this issue is resolved.

Proof. (1) follows since τ is a rational ray. The map Σ_π is given by a linear function $m : N \to \mathbb{Z}$, and hence π is given by the regular monomial z^m. It is standard that the order of vanishing of z^m on the divisor D_τ is the value of m on the generator of $\tau \cap N_\tau$. But this value is precisely m_τ, giving the result. ♠

Remark 3.1.2. In fact, the data m_τ only depends on the map $\Sigma(\pi) : \Sigma(W) \to \Sigma(\mathbb{A}^1)$ as abstract cone complexes, as the lattice N_τ is the lattice giving the integral structure on cones $\tau \in \Sigma(W)$ corresponding to the codimension one strata of W.

3.2. **Decomposition in the toroidal case.** The proposition immediately applies to a finite type logarithmically smooth morphism $W \to B$, where B is a smooth curve with toroidal divisor $\{b_0\}$ and W a toroidal algebraic stack: Associated to $W \to B$ there is a morphism of generalized cone complexes $\Sigma(W) \to \Sigma(B)$. We have $\Sigma(B) \simeq \mathbb{R}_{\geq 0}$ with the lattice $N_B \simeq \mathbb{Z}$. There is also still a correspondence between rays $\tau \in \Sigma(W)$ and toroidal divisors $D_\tau \subset W$. For a ray τ with integral lattice N_τ, we have $\tau \cap N_\tau \simeq \mathbb{N}$, and the monoid homomorphism $\tau \cap N_\tau \to \tau_B \cap N_B$ is multiplication by an integer m_τ.

Corollary 3.2.1.

$$\pi^*(\{b_0\}) = \sum_\tau m_\tau D_\tau. \tag{3.2.1}$$

Proof. Choosing a covering in the smooth topology of W by a scheme we may assume W is a scheme. Fix a geometric point x on the open stratum of D_τ. We may assume that we have a toroidal chart, namely a commutative diagram

$$\begin{array}{ccccc} V_W & \longleftarrow & U_W & \longrightarrow & W \\ {\scriptstyle \pi_V}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ V_B & \longleftarrow & U_B & \longrightarrow & B \end{array}$$

where

(1) all the horizontal arrows are étale,
(2) $U_W \to W$ is an étale neighborhood of x,
(3) $\pi_V : V_W \to V_B$ is a toric morphism of affine toric varieties.

Since B is a curve, $V_B \simeq \mathbb{A}^1$. Write 0 for its origin. Replacing V_W and U_W by open sets we may assume V_W contains a unique toric divisor D_V. Then the multiplicity of $\pi_W^*(\{b_0\})$ along D_τ coincides with the multiplicity of $\pi_V^*(\{0\})$ along D_V. This is m_τ by Equation (3.1.1). ♠

3.3. **Decomposition in the stack of logarithmic structures.** In this paper we apply Equation (3.2.1) in the generality of Artin stacks which are not necessarily of finite type. We continue to work with (B, b_0) a pointed smooth curve. We have Log_B, Olsson's stack of log schemes over B, the objects of which are log morphisms $X \to B$.[7] There is a morphism $\mathrm{Log}_B \to B$, the forgetful map which forgets the logarithmic structures. Viewing $b_0 \in B$ as the standard log point, we similarly have a morphism $\mathrm{Log}_{b_0} \to b_0$.

Definition 3.3.1. Define the closed substack $\mathcal{D}_m$ of Log_B as follows. For $m \in \mathbb{N}$, let $m : \mathbb{N} \to \mathbb{N}$ denote the multiplication by m map, inducing $\mathcal{A}_{\mathbb{N}} \to \mathcal{A}_{\mathbb{N}} = \mathcal{A}$. This morphism of log stacks induces by projection a morphism of log stacks $\mathcal{A}_{\mathbb{N}} \times_{\mathcal{A}} B \to B$, hence a morphism of stacks $m : \mathcal{A}_{\mathbb{N}} \times_{\mathcal{A}} B \to \mathrm{Log}_B$. We take $\mathcal{D}_m$ to be the closure of the image of $[0/\mathbb{G}_m] \times_{\mathcal{A}} B$ under m with the reduced induced stack structure.

We observe:

Lemma 3.3.2. *(1)* $\mathrm{Log}_B \to B$ *is logarithmically étale.*

(2) $\mathrm{Log}_B \times_B b_0 \cong \mathrm{Log}_{b_0}$.

(3) For each $m \in \mathbb{N}$, $\mathcal{D}_m \subset \mathrm{Log}_B$ *is a generically reduced prime divisor. When* $m > 0$ *the divisor* $\mathcal{D}_m$ *is contained in* Log_{b_0}.

(4) For $m > 0$ *write* $i_m : \mathcal{D}_m \to \mathrm{Log}_{b_0}$ *for the embedding above. For any finite type open substack* $U \subset \mathrm{Log}_B$, *we have the following identity in the Chow group* $A_*(U)$ *of* U *introduced in* [Kre99]*:*

$$[\mathrm{Log}_{b_0} \cap U] = \sum_m m \cdot i_{m*}[\mathcal{D}_m \cap U], \tag{3.3.1}$$

For convenience, we denote the above identity formally as follows:

$$[\mathrm{Log}_{b_0}] = \sum_m m \cdot i_{m*}[\mathcal{D}_m] \tag{3.3.2}$$

without specifying the specific choice of U.

Remark 3.3.3. In [Kre99], the Chow groups are constructed for Artin stacks of finite type which admits finite sum of cycles. Note that the stack Log_B is not of finite type, and the summation in (3.3.2) has infinitely many non-zero terms. The equation (3.3.2) is not an identity of Chow cycles in the sense of [Kre99].

Proof. (1) This is generally true for any fine log scheme B, but to simplify notation and clarify the structure of Log_B, we just show it for the given B. It is sufficient to restrict to an étale cover of Log_B, which is described in [Ols03, Corollary 5.25]. One can cover Log_B by stacks indexed by morphisms of monoids $\mathbb{N} \to P$. We set $\mathcal{A} := \mathcal{A}_{\mathbb{N}} = [\mathbb{A}^1/\mathbb{G}_m]$ (following the notation of (2.2.1)). By [Cad07, Lemma 2.1.1], an object of the category $\mathcal{A}$ over a scheme $\underline{W}$ is a pair (L, s) where L is a line bundle on $\underline{W}$ and s is a section of L. In particular, taking a line bundle L on B corresponding to the divisor b_0, and taking a section s of L vanishing precisely

[7] We emphasize that, since we assume all our logarithmic structures are fine and saturated, this stack Log_B parametrizes only fine and saturated logarithmic structures, and is only an open substack of Olsson's full stack of logarithmic structures. Olsson denotes our stack Log_B by $\mathcal{T}or_B$.

once, at b_0, we obtain a strict morphism $B \to \mathcal{A}$. (In fact, $\mathcal{A}$ is the Artin fan of B and this is the canonical morphism).

Then a morphism $\mathbb{N} \to P$ gives the element $\mathcal{A}_P \times_{\mathcal{A}} B$ of the étale open cover of Log_B. Here the morphism $\mathcal{A}_P \to \mathcal{A}$ is functorially induced by $\mathbb{N} \to P$. The composition of $\mathcal{A}_P \times_{\mathcal{A}} B \to \mathrm{Log}_B$ with the forgetful morphism $\mathrm{Log}_B \to B$ is just the projection to B, the base-change of $\mathcal{A}_P \to \mathcal{A}$, which is étale by [Ols03, Corollary 5.23].

(2) is clear since giving a morphism $\underline{W} \to \mathrm{Log}_B \times_B b_0$ is the same thing as giving a log morphism $W \to B$ which factors through the inclusion $b_0 \hookrightarrow B$.

For (3), first note that the image of $[0/\mathbb{G}_m] \times_{\mathcal{A}} B$ under the projection to B is b_0 except in the case when $m = 0$, in which case the image is $B \smallsetminus \{b_0\}$. Thus $\mathcal{D}_m$ is contained in Log_{b_0} for $m \geq 1$.

The divisor $\mathcal{D}_m$ can be described in terms of the étale cover of Log_B given above. Fix $\varphi : \mathbb{N} \to P$ giving $\mathcal{A}_P \times_{\mathcal{A}} B \to \mathrm{Log}_B$ étale. For each toric divisor D_τ of $\mathrm{Spec}\, \Bbbk[P]$, there is a monoid homomorphism $P \to \mathbb{N}$ given by order of vanishing, i.e., $p \mapsto \mathrm{ord}_{D_\tau}(z^p)$. The composition $\mathbb{N} \to \mathbb{N}$ is multiplication by the integer m_τ, precisely the order of vanishing of $z^{\varphi(1)}$ along D_τ. Also write φ for the induced maps $\varphi : \mathrm{Spec}\, \Bbbk[P] \to \mathbb{A}^1$ and $\varphi : \mathcal{A}_P \to \mathcal{A}$. At the generic point ξ of D_τ, the map $\overline{\mathcal{M}}_{\mathbb{A}^1, \varphi(\bar\xi)} \to \overline{\mathcal{M}}_{\mathrm{Spec}\, \Bbbk[P], \bar\xi}$ coincides with $m_\tau : \mathbb{N} \to \mathbb{N}$ if $m_\tau > 0$; otherwise it is the map $0 \to \mathbb{N}$. Thus we see that the pull-back of $\mathcal{D}_m$ to $\mathcal{A}_P \times_{\mathcal{A}} B$ is $\sum_{D_\tau : m_\tau = m} [D_\tau / \mathrm{Spec}\, \Bbbk[P^{\mathrm{gp}}]] \times_{\mathcal{A}} B$. In particular, $\mathcal{D}_m$ is generically reduced, and by Proposition 3.1.1,(2) we can write as divisors

$$\varphi^*(b_0) = \sum_\tau m_\tau [D_\tau / \mathrm{Spec}\, \Bbbk[P^{\mathrm{gp}}]] \times_{\mathcal{A}} B. \tag{3.3.3}$$

under the induced map $\varphi : \mathcal{A}_P \times_{\mathcal{A}} B \to B$.

Consider a finite type open substack $U \subset \mathrm{Log}_B$. By [Ols03, Corollary 5.25], there is a finite set of monoid homomorphisms $\{\phi_i : \mathbb{N} \to P_i\}$ such that U is contained in the image of $\cup \phi_i : \cup_i \mathcal{A}_{P_i} \times_{\mathcal{A}} B \to \mathrm{Log}_B$. To prove (3.3.1), we may first apply the identity (3.3.3) to the image of $\cup \phi_i$, then restrict it to U. This implies (4).

♠

3.4. **Decomposition of the moduli space: first step.** We now fix a proper and logarithmically smooth morphism $X \to B$ with B a smooth curve with divisorial logarithmic structure given by $b_0 \in B$. Fix a class β of stable logarithmic map. The moduli space $\mathscr{M}(X/B, \beta)$ is neither a toric variety nor logarithmically smooth over B. Its saving grace is the fact that it has a perfect obstruction theory over $\mathfrak{M}_B$, see Section 2.4. For $b \in B$ an arbitrary closed point, $b \neq b_0$, we have the following diagram with all squares Cartesian, as is easily checked with the same argument as in Lemma 3.3.2, (2):

$$\begin{array}{ccccc}
\mathscr{M}(X_b/b,\beta) & \overset{j_b}{\hookrightarrow} & \mathscr{M}(X/B,\beta) & \overset{j_{b_0}}{\hookleftarrow} & \mathscr{M}(X_0/b_0,\beta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}_b & \hookrightarrow & \mathfrak{M}_B & \hookleftarrow & \mathfrak{M}_{b_0} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Log}_b & \hookrightarrow & \mathrm{Log}_B & \hookleftarrow & \mathrm{Log}_{b_0} \\
\downarrow & & \downarrow & & \downarrow \\
b & \hookrightarrow & B & \hookleftarrow & b_0
\end{array}$$

Consider the complex $\mathbf{E}^\bullet := \left(R\pi_* f^* T_{X/B}\right)^\vee$, where $T_{X/B}$ stands for the logarithmic tangent bundle and the dual is taken in the derived sense: $\mathbf{F}^\vee := R\mathrm{Hom}(\mathbf{F}, \mathcal{O}_W)$. This is a perfect 2-term complex supported in degrees 0 and -1 admitting a morphism to the cotangent complex $\mathbf{L}_{\mathscr{M}(X/B,\beta)\,/\,\mathfrak{M}_B}$. Since $\mathfrak{M}_B$ is pure-dimensional, this provides a well-defined virtual fundamental class $[\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$, as shown in [GS13, §5].

The obstruction theory for $\mathscr{M}(X/B,\beta)$ pulls back to the obstruction theory for $\mathscr{M}(X_b/b,\beta)$ and $\mathscr{M}(X_0/b_0,\beta)$, and hence by [BF97, Proposition 7.2], we have:

Proposition 3.4.1.

$$[\mathscr{M}(X_b/b,\beta)]^{\mathrm{virt}} = j_b^![\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$$

and

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = j_{b_0}^![\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}.$$

This allows us to focus on $\mathscr{M}(X_0/b_0,\beta)$

Write

$$\begin{aligned}
\mathfrak{M}_m &:= \mathfrak{M}_{b_0} \times_{\mathrm{Log}_{b_0}} \mathcal{D}_m = \mathfrak{M}_B \times_{\mathrm{Log}_B} \mathcal{D}_m \\
\mathscr{M}_m(X_0/b_0,\beta) &:= \mathscr{M}(X_0/b_0,\beta) \times_{\mathrm{Log}_{b_0}} \mathcal{D}_m,
\end{aligned} \tag{3.4.1}$$

which amounts to adding a column on the right of the diagram above

$$\begin{array}{ccccc}
\mathscr{M}(X/B,\beta) & \overset{j_{b_0}}{\hookleftarrow} & \mathscr{M}(X_0/b_0,\beta) & \hookleftarrow & \mathscr{M}_m(X_0/b_0,\beta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}_B & \hookleftarrow & \mathfrak{M}_{b_0} & \hookleftarrow & \mathfrak{M}_m \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Log}_B & \hookleftarrow & \mathrm{Log}_{b_0} & \hookleftarrow & \mathcal{D}_m \\
\downarrow & & \downarrow & & \\
B & \hookleftarrow & b_0 & &
\end{array}$$

Note $\mathscr{M}_m(X_0/b_0,\beta)$ has a natural perfect obstruction theory $\mathbb{E}_m$ over $\mathfrak{M}_m$, pulled back from an obstruction theory $\mathbb{E}$ of $\nu:\mathscr{M}(X_0/b_0,\beta)\to\mathfrak{M}_{b_0}$. There are natural maps we also denote $i_m:\mathfrak{M}_m\to\mathfrak{M}_{b_0}$ and $i_m:\mathscr{M}_m(X_0/b_0,\beta)\to\mathscr{M}(X_0/b_0,\beta)$, by abuse of notation.

Proposition 3.4.2.

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}}=\sum_m m\cdot i_{m*}[\mathscr{M}_m(X_0/b_0,\beta)]^{\mathrm{virt}} \tag{3.4.2}$$

Proof. We use the cartesian diagram of algebraic stacks

$$\begin{array}{ccc}
\coprod_m \mathscr{M}_m(X_0/b_0,\beta) & \xrightarrow{\sqcup i_m} & \mathscr{M}(X_0/b_0,\beta)\\
\Big\downarrow{\scriptstyle \sqcup\nu_m} & & \Big\downarrow{\scriptstyle \nu}\\
\coprod_m \mathfrak{M}_m & \xrightarrow{\sqcup i_m} & \mathfrak{M}_{b_0}.
\end{array}$$

Since $\mathfrak{M}_m\to\mathcal{D}_m$ and $\mathfrak{M}_{b_0}\to\mathrm{Log}_{b_0}$ are smooth, hence flat, Equation (3.3.2) gives $[\mathfrak{M}_{b_0}]=\sum_m m\cdot i_{m*}[\mathfrak{M}_m]$. Applying Manolache's refinement [Man12, Theorem 4.1 (3)(i)] of Costello's [Cos06, Theorem 5.0.1] we obtain that

$$\begin{aligned}
\sum_m m\cdot i_{m*}[\mathscr{M}_m(X_0/b_0,\beta)]^{\mathrm{virt}} &= (\sqcup i_m)_*\,(\sqcup\nu_m)^!_{\mathbb{E}_m}\Big(\sum m[\mathfrak{M}_m]\Big)\\
&= \nu^!_{\mathbb{E}}[\mathfrak{M}_{b_0}]\\
&= [\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}},
\end{aligned}$$

giving the required equality. ♠

As it stands, (3.4.2) is not easy to use: it says that a piece $\mathscr{M}_m(X_0/b_0,\beta)$ of the moduli space appears with multiplicity m. We need to describe in a natural, combinatorial way what $\mathscr{M}_m(X_0/b_0,\beta)$ is. The combinatorics of logarithmic structures provides an avenue to do this. A bridge to that combinatorial picture is provided by an unobstructed variant $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta)$ of $\mathscr{M}_m(X_0/b_0,\beta)$.

4. Tropical moduli spaces and the main theorem

We now show how to construct the desired set Ω in Theorem 1.1.2. This will be the set of isomorphism classes of tropical curves mapping to $\Sigma(X)$ which are rigid, in a sense we will define below. The integer m_τ will be read off immediately from such a tropical curve.

First we introduce a stack $\mathcal{X}$ and prestable logarithmic maps in $\mathcal{X}/B$ which serve as a bridge from geometry to combinatorics.

4.1. **Prestable logarithmic maps in $\mathcal{X}$.**

We have constructed in Proposition 2.2.6 a canonical morphism $\mathcal{A}_X\to\mathcal{A}_B$. Define $\mathcal{X}=\mathcal{A}_X\times_{\mathcal{A}_B}B$ and $\mathcal{X}_0=\mathcal{X}\times_B\{b_0\}=\mathcal{A}_X\times_{\mathcal{A}_B}\{b_0\}$. Associated to any cone $\sigma\in\Sigma(X)$ is a (locally closed) stratum $X_\sigma\subseteq X$, and we write X_σ^{cl} for the closure of this stratum. Similarly we write $\mathcal{X}_\sigma$ and $\mathcal{X}_\sigma^{\mathrm{cl}}$ for the corresponding strata of $\mathcal{X}$. In particular, the map

$X \to \mathcal{X}$ provides a one-to-one correspondence between logarithmic strata $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ of X and logarithmic strata $\mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ of $\mathcal{X}$. We note that $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} = \mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} \times_{\mathcal{X}} X$, and similarly for the underlying schemes.

We use the notation $\beta' = (g, u_{p_i})$ for discrete data in $\mathcal{X}/B$ or $\mathcal{X}_0/b_0$: these are the same as discrete data $\beta = (g, A, u_{p_i})$ in X or X_0 with the curve class A removed.

Denote by $\mathfrak{M}(\mathcal{X}/B, \beta')$ the stack of basic prestable logarithmic maps in $\mathcal{X}/B$, with its natural logarithmic structure. By [ACMW17] it is an algebraic stack provided with a morphism $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}(\underline{\mathcal{X}}/\underline{B}, \underline{\beta}')$. Restricting to b_0 we obtain a morphism of algebraic stacks $\mathfrak{M}(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}(\underline{\mathcal{X}}_0/\underline{b}_0, \underline{\beta}')$. Using notation of Section 3.4 we may further restrict and define

$$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta') = \mathfrak{M}(\mathcal{X}/B, \beta') \times_{\mathrm{Log}_B} \mathcal{D}_m = \mathfrak{M}(\mathcal{X}/B, \beta') \times_{\mathfrak{M}_B} \mathfrak{M}_m.$$

Since $X \to \mathcal{X}$ is strict we obtain natural strict morphisms

$$\mathscr{M}(X/B, \beta) \to \mathfrak{M}(\mathcal{X}/B, \beta')$$
$$\mathscr{M}(X_0/b_0, \beta) \to \mathfrak{M}(\mathcal{X}_0/b_0, \beta')$$
$$\mathscr{M}_m(X_0/b_0, \beta) \to \mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$$

The key fact is

Proposition 4.1.1. *The morphisms*

$$\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_B$$
$$\mathfrak{M}(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_{b_0}$$
$$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_m$$

are strict étale.

Proof. The morphisms are strict by definition.

Recall that $\mathcal{A}_X \to \mathcal{A}_B = \mathcal{A}$ is logarithmically étale. It follows that $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}} B$ is logarithmically étale over B and so $\mathcal{X}_0 \to b_0$ is logarithmically étale. The fact that $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_B$ is étale is equivalent to $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_{g,k} \times B$ being logarithmically étale, which is proven in [AW13, Proposition 3.1.2]. The result for $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ and $\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$ follows by pulling back. ♠

Since $\mathfrak{M}_B \to B$ is logarithmically smooth it follows that $\mathfrak{M}(\mathcal{X}/B, \beta') \to B$ is a toroidal morphism. Corollary 3.2.1 says that a meaningful decomposition result for $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ or $\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$ results from a meaningful description of rays in the polyhedral cone complex of $\mathfrak{M}(\mathcal{X}/B, \beta')$. This is where Section 2.5 becomes useful.

Let $W = \mathrm{Spec}(Q \to \Bbbk)$ and let $(C/W, \mathbf{p}, f : C \to \mathcal{X})$ be an object of $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$, namely a basic prestable logarithmic map to $\mathcal{X}/B$ lying over b_0. We recall that in Section 2.5.5 we introduced a family of tropical curves in $\Sigma(X)$ over the cone $\tau_Q := (Q^\vee_{\mathbb{R}}, Q^*)$, which is universal since $(C/W, \mathbf{p}, f)$ is basic, see Section 2.5.6. We denote by $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ the combinatorial type underlying this family. We write τ for the isomorphism class of the combinatorial type $\widetilde{\tau}$ under graph isomorphisms fixing $\mathbf{g}, \boldsymbol{\sigma}$ and u.

Proposition 4.1.2. *(1) The basic prestable map*

$$(C/W, \mathbf{p}, f : C \to \mathcal{X}) \in \mathfrak{M}(\mathcal{X}_0/b_0, \beta')\,(\Bbbk)$$

belongs to a codimension-1 stratum of $\mathfrak{M}(\mathcal{X}/B, \beta')$ if and only if $Q \simeq \mathbb{N}$; equivalently, the corresponding tropical moduli space is a ray.

(2) Assume $Q \simeq \mathbb{N}$ with generator v, let $g : Q^\vee \to \mathbb{N}$ be the map associated to $\tau_Q \to \Sigma(B)$ and $m_\tau = g(v)$. The map $(C/W, \mathbf{p}, f : C \to \mathcal{X})$ lies in $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ if and only if $m_\tau \neq 0$, in which case

$$(C/W, \mathbf{p}, f : C \to \mathcal{X}) \in \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta').$$

Proof. We may restrict $\mathfrak{M}(\mathcal{X}/B, \beta')$ to an neighbourhood of $(C/W, \mathbf{p})$ in the smooth topology. As in the proof of Lemma 3.3.2,(3) this reduces the statement to a toric morphism, treated in Proposition 3.1.1. ♠

Denote by $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')_\tau \subset \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta')$ the divisor corresponding to prestable maps whose corresponding tropical curve has discrete data τ. Denote also

$$\mathscr{M}(X_0/b_0, \beta)_\tau = \mathfrak{M}(\mathcal{X}_0/b_0, \beta')_\tau \times_{\mathfrak{M}(\mathcal{X}_0/b_0, \beta')} \mathscr{M}(X_0/b_0, \beta).$$

Corollary 3.2.1 gives a decomposition

$$[\mathfrak{M}(\mathcal{X}_0/b_0, \beta')] = \sum_\tau m_\tau [\mathfrak{M}(X_0/b_0, \beta')_\tau],$$

and an application of Costello's push-forward formula to Equation (3.4.2) gives

$$[\mathscr{M}(X_0/b_0, \beta)]^{\mathrm{virt}} = \sum_\tau m_\tau \cdot i_{m_\tau *}[\mathscr{M}(X_0/b_0, \beta)_\tau]^{\mathrm{virt}}. \tag{4.1.1}$$

The Main Theorem (Theorem 1.1.2) makes this decomposition more precise, in a way amenable to immediate computations and appropriate for our work on the degeneration formula.

We start by explicitly delineating the available combinatorial data.

4.2. **Decorated tropical curves.** We fix $X \to B$ logarithmically smooth over the one-dimensional base (B, b_0) as usual. As in the discussion in §2.5, we can use the fact that X is Zariski and logarithmically smooth over B to conclude that X is simple, and hence take the reduced presentation for $\Sigma(X)$. The cones of $\Sigma(X)$ are then indexed by the strata of X.

Let $\rho : \Sigma(X) \to \Sigma(B) = \mathbb{R}_{\geq 0}$ be the induced map on generalized cone complexes.[8] We will write $\Delta(X) = \rho^{-1}(1)$, which can be thought of as a generalized polyhedral complex.

Note that just as the cones of $\Sigma(X)$ are in one-to-one correspondence with strata of X, the polyhedra of $\Delta(X)$ are in one-to-one correspondence with the

[8] In fact these are cone complexes, since Zariski logarithmically smooth schemes have no monodromy.

strata of X contained in X_0. If $\sigma \in \Delta(X)$, we write X_σ for the corresponding (locally closed) stratum of X_0 and X_σ^{cl} for its closure.

Definition 4.2.1. A *decorated tropical curve* $(\Gamma \to \Delta(X), \mathbf{A})$ *in* $\Delta(X)$ consists of the following data:

(i) A tropical curve $\Gamma \to \Sigma(X)$ as in Definition 2.5.3 factoring through the inclusion $\Delta(X) \hookrightarrow \Sigma(X)$.

(ii) Decoration:

$$\mathbf{A} : V(G) \to \coprod_{\sigma \in \Delta(X)} H_2(X_\sigma^{\mathrm{cl}}),$$

with $\mathbf{A}(v) \in H_2(X_{\boldsymbol{\sigma}(v)}^{\mathrm{cl}})$.

The *decorated graph* (or *combinatorial type*) $(\widetilde{\tau}, \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$ is the combinatorial type $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ of a tropical curve (see Definition 2.5.3) in $\Sigma(X)$ with all u_x mapping to 0 under the map $N_{\boldsymbol{\sigma}(x)} \to \mathbb{Z} = N_{\Sigma(B)}$, along with the data $\mathbf{A}$ in (ii) above.

Recall that given a tropical curve, we have defined its genus as

$$g(\Gamma) = b_1(\Gamma) + \sum_v \mathbf{g}(v).$$

Given a decoration $\mathbf{A}$, we define its curve class as

$$A(\mathbf{A}) = \sum_v \mathbf{A}(v) \in H_2(X_0),$$

where $\mathbf{A}(v)$ is viewed as an element of $H_2(X_0)$ via the push-foward map $H_2(X_{\boldsymbol{\sigma}(v)}^{\mathrm{cl}}) \to H_2(X_0)$. When $A = A(\mathbf{A})$ we say that $\mathbf{A}$ is a partition of A and write $\mathbf{A} \vdash A$. Clearly both genus $g(\Gamma)$ and curve class $A(\mathbf{A})$ depend only on the combinatorial type $(\widetilde{\tau}, \mathbf{A})$.

An *isomorphism* ϕ between decorated tropical curves $(f_1 : \Gamma_1 \to \Delta(X), \mathbf{A}_1)$ and $(f_2 : \Gamma_2 \to \Delta(X), \mathbf{A}_2)$ in $\Delta(X)$ is an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ of tropical curves (necessarily preserving the genus decoration and ordering of legs), such that

(1) $f_1 = f_2 \circ \phi$,
(2) $\mathbf{A}_2(\phi(v)) = \mathbf{A}_1(v)$ for all $v \in V(G)$,
(3) $u_{\phi(q)} = u_q$ and $u_{\phi(p)} = u_p$ for all edges and legs.

This defines the *automorphism group* $\mathrm{Aut}(\Gamma \to \Delta(X), \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$. In the same manner we define isomorphisms of decorated graphs and the *automorphism group* $\mathrm{Aut}(\widetilde{\tau}, \mathbf{A})$ of a decorated graph $(\widetilde{\tau}, \mathbf{A})$, and similarly for $\mathrm{Aut}(\widetilde{\tau})$.

4.3. **Contractions and rigid curves.** Fix the combinatorial type $(\widetilde{\tau}, \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$. The space $M_{\widetilde{\tau},\mathbf{A}}^{\mathrm{trop}}(\Delta(X))$ of decorated tropical curves $(f : \Gamma \to \Delta(X), \mathbf{A})$ in $\Delta(X)$ with these data is the interior of a possibly unbounded polyhedron determined by the positions $f(v)$ of vertices of Γ and the lengths $\ell(E_q)$ of the compact edges for which $u_q = 0$. Note that we consider maps with the fixed graph G and do not identify maps differing by an automorphism in $\mathrm{Aut}(\Gamma \to \Delta(X), \mathbf{A})$. This polyhedron is rational, since $\Delta(X)$ is a complex of rationally defined polytopes and the equations in Definition 2.5.3(6)(b)(iii) have rational coefficients.

The interior of each face of $M^{\mathrm{trop}}_{\widetilde{\tau},\mathbf{A}}(\Delta(X))$ is naturally identified with $M^{\mathrm{trop}}_{\widetilde{\tau}',\mathbf{A}'}(\Delta(X))$, where $(\widetilde{\tau}', \mathbf{A}'), \widetilde{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', u')$ is a *contraction* of the decorated graph $(\widetilde{\tau}, \mathbf{A})$, namely:

(1) $\pi : G \to G'$ is a graph contraction, preserving the ordering of legs $L(G) = L(G')$.
(2) Whenever $\pi(v) = v'$ we have that $\boldsymbol{\sigma}'(v')$ is a face of $\boldsymbol{\sigma}(v)$.
(3) For $v' \in V(G')$ we have $\mathbf{g}'(v') = g(\pi^{-1}(v'))$ and $\mathbf{A}(v') = \sum_{\pi(v)=v'} \mathbf{A}(v)$. Here $\pi^{-1}(v')$ is the inverse image subgraph of G with the restriction of the genus function and $\mathbf{A}(v)$ is viewed as a curve class on the stratum $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v')}$, which contains $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ by the previous condition.
(4) Whenever an edge $E_q = E_{q'}$ is not contracted under π we have $\boldsymbol{\sigma}'(E_{q'})$ is a face of $\boldsymbol{\sigma}(E_q)$ and $u_{q'} = u_q$.
(5) For every leg $E_p = E_{p'}$ we have $\boldsymbol{\sigma}'(E_{p'})$ is a face of $\boldsymbol{\sigma}(E_p)$ and $u_{p'} = u_p$.

Definition 4.3.1. A decorated tropical curve in $\Delta(X)$ is *rigid* if it is not contained in a non-trivial family of decorated tropical curves of the same decorated graph. In other words, the relevant polyhedron $M^{\mathrm{trop}}_{\widetilde{\tau},\mathbf{A}}(\Delta(X))$ is a point. Note that this notion depends only on the decorated graph $(\widetilde{\tau}, \mathbf{A})$, or just $\widetilde{\tau}$, so it makes sense to say that $(\widetilde{\tau}, \mathbf{A})$ is rigid. Thus when $(\widetilde{\tau}, \mathbf{A})$ is rigid it uniquely determines a rigid decorated tropical curve in $\Delta(X)$, and we will sometimes refer to $(\widetilde{\tau}, \mathbf{A})$ as a rigid decorated tropical curve.

As we saw above, the tropical moduli space is rational, therefore a rigid tropical curve must be rationally defined, i.e., $f(v_\eta)$ are rational points of $\Delta(X)$ and $\ell(E_q) \in \mathbb{Q}$. We define the *multiplicity* of a rigid decorated tropical curve $m_{\widetilde{\tau}} \in \mathbb{Z}_{>0}$ to be the smallest positive integer $m_{\widetilde{\tau}}$ such that $m_{\widetilde{\tau}}\ell(E_q) \in \mathbb{Z}$ for all edges E_q and $m_{\widetilde{\tau}} f(v)$ is integral (i.e., lies in $N_{\boldsymbol{\sigma}(v)}$) for each vertex v of Γ.

We will see in Proposition 4.6.1 that $m_{\widetilde{\tau}}$ is compatible with the multiplicity m_τ of Proposition 4.1.2.

4.4. **Decorated logarithmic maps in X_0.** Fix a rigid decorated graph $(\widetilde{\tau}, \mathbf{A})$ in $\Delta(X)$. We explicitly write each edge E_q as a pair of half-edges named $E_{q,1}, E_{q,2}$; a leg is already considered a half-edge, having only one endpoint. We define the stratum function on half-edges by $\boldsymbol{\sigma}(E_{q,i}) := \boldsymbol{\sigma}(E_q)$. For each vertex v of Γ we write H_v for the set of half-edges (of the form $E_{q,i}$ or E_p) incident to v. We write $\beta_v = (\mathbf{g}(v), u|_{H_v}, \mathbf{A}(v))$.

Definition 4.4.1. A *stable logarithmic map in X_0 over a base scheme S of class β marked by* $(\widetilde{\tau}, \mathbf{A})$ is the following data:

(1) An object $f : C \to X_0$ of $\mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0, \beta) := \mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} \mathcal{D}_{m_{\widetilde{\tau}}}$ over the scheme S.

(2) For each vertex v of G a stable map $\underline{f}_v : \underline{C}_v \to \underline{X}^{\mathrm{cl}}_{\sigma(v)}$, an object of $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\sigma(v)}, \underline{\beta}_v)$ over S (with genus $\mathbf{g}(v)$, curve class $\mathbf{A}(v)$, and markings $s_E : S \to \coprod \underline{C}_v$ labelled by $E \in H_v$) with the marked point corresponding to the half-edge E landing in the stratum $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$.

(3) An isomorphism

$$\underline{C} \simeq \left(\coprod \underline{C}_v\right) / \langle s_{E_{q,1}} = s_{E_{q,2}} \rangle.$$

These data must satisfy

(i)

$$\underline{f}|_{\underline{C}_v} = \underline{f}_v,$$

and

(ii) the contact order of f at a node $q \in \underline{C}$ agrees with u_q from Definition 2.5.3, with orientation at q given by the gluing construction (3).

A *morphism* of stable logarithmic maps in X_0 over S marked by $(\widetilde{\tau}, \mathbf{A})$ is defined as a fibre square as usual.

We denote by $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$ the category of stable logarithmic maps in X_0 over S marked by $(\widetilde{\tau}, \mathbf{A})$ and define

$$\mathscr{M}_{\widetilde{\tau}}(X_0/b_0, \beta) := \coprod_{\mathbf{A} \vdash A} \mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta).$$

Proposition 4.4.2. *(1) The category $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$ of stable logarithmic maps in X_0 over S marked by $(\widetilde{\tau}, \mathbf{A})$ is a proper Deligne-Mumford stack.*

(2) The mapping $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta) \to \mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0, \beta)$ sending an object of $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$ to the underlying stable logarithmic map $f : C \to X_0$ is a morphism of algebraic stacks invariant under $\mathrm{Aut}(\widetilde{\tau}, \mathbf{A})$.

The proof is given below along with the proof of Proposition 4.5.2.

4.5. **Decorated logarithmic maps in $\mathcal{X}_0$.** To define a virtual fundamental class on $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$ we use prestable maps in $\mathcal{X}_0/b_0$. As for β' we use the notation $\beta'_v = (\mathbf{g}(v), u|_{H_v})$ for discrete data of maps in $\mathcal{X}_{\boldsymbol{\sigma}(v)}$: these are the same β_v with the curve classes $\mathbf{A}(v)$ removed.

Definition 4.5.1. A *logarithmic map in $\mathcal{X}_0$ over b_0 marked by $\widetilde{\tau}$* is the following data:

(1) A *prestable* logarithmic map $f : C \to \mathcal{X}_0$ which is an object of $\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0, \beta') = \mathfrak{M}(\mathcal{X}_0/b_0, \beta') \times_{\mathrm{Log}_{b_0}} \mathcal{D}_{m_{\widetilde{\tau}}}$.

(2) For each vertex v a prestable map $\underline{f}_v : \underline{C}_v \to \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ over $\underline{S}$, which is an object of $\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}, \underline{\beta}'_v)$ with genus $\mathbf{g}(v)$ and markings given by H_v. We impose the additional condition that the marking corresponding to a half-edge E lands in the stratum $\mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$.

(3) An isomorphism

$$\underline{C} \simeq \left(\coprod \underline{C}_v\right) / \langle s_{E_{q,1}} = s_{E_{q,2}} \rangle.$$

These data must satisfy the same conditions (i)–(ii) as in Definition 4.4.1.

We denote by $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ the category of stable logarithmic maps in $\mathcal{X}_0$ over S marked by $\widetilde{\tau}$.

Proposition 4.5.2. *(1) The category $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ of prestable logarithmic maps in X_0 over S marked by $\widetilde{\tau}$ is an algebraic stack.*

(2) The mapping $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')$ sending an object of $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ to the underlying prestable logarithmic map $f: C \to \mathcal{X}_0$ is a morphism of algebraic stacks invariant under $\operatorname{Aut}(\widetilde{\tau})$.

(3) The mapping $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta) \to \mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ composing maps of an object of $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ with the projection $X_0 \to \mathcal{X}_0$ is a morphism of algebraic stacks.

Proof of Propositions 4.4.2 and 4.5.2. Step 1: stacks parametrizing f and $\underline{f}_v$. By [GS13, AC14, ACMW17] the category $\mathscr{M}(X_0/b_0,\beta)$ is a proper Deligne–Mumford stack, see Section 2.3 above. It comes with a morphism $\mathscr{M}(X_0/b_0,\beta) \to \mathscr{M}(\underline{X}_0,\underline{\beta})$. Similarly, by [ACMW17], $\mathfrak{M}(\mathcal{X}_0/b_0,\beta')$ is an algebraic stack endowed with a morphism $\mathfrak{M}(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}(\underline{\mathcal{X}_0},\underline{\beta'})$.

Since $\mathscr{M}(X_0/b_0,\beta) \to \mathfrak{M}(\mathcal{X}_0/b_0,\beta')$ is a morphism of fibred categories, it is a morphism of algebraic stacks. Considering the fibred products we obtain a morphism $\mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0,\beta) \to \mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')$.

By [BM96, Theorem 3.14] the stack $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}_v)$ is a proper Deligne-Mumford stack. Similarly, by [Wis16b, Corollary 1.1.1] the stack $\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}'_v)$ is algebraic.

Again we obtain morphisms of algebraic stacks $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}_v) \to \mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}'_v)$.

Step 2: requiring marked points to land in the correct strata. For each half-edge $E \in H_v$ we have an evaluation map $e_E : \mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}_v) \to \underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$. Define $\mathscr{M}_v := \bigcap_E e_E^{-1}\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$. This is the proper Deligne–Mumford stack parametrizing maps where the marked point corresponding to E lands in the stratum $\boldsymbol{\sigma}(E)$. Replacing $\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ by $\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ we similarly obtain evaluations $\epsilon_E : \mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)},\underline{\beta}'_v) \to \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ and an algebraic stack $\mathfrak{M}_v := \bigcap_E \epsilon_E^{-1}\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$.

Step 3: requiring maps to glue at nodes. Let $\mathscr{M}^{\mathrm{prod}} = \prod_v \mathscr{M}_v$. The group $\operatorname{Aut}(\tilde{\tau})$ acts on this moduli stack. For each edge E_q of Γ we have two evaluation maps $e_{E_{q,i}} : \mathscr{M}^{\mathrm{prod}} \to X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E_{q_1})} = X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E_{q_2})}$. Define $\mathscr{M}_q = \mathscr{M}^{\mathrm{prod}} \times_{(X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E_{q_1})})^2} X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E_{q_1})}$, where the map on the left is $e_{E_{q,1}} \times e_{E_{q,2}}$ and the map on the right is the diagonal. It has a natural morphism $\mathscr{M}_q \to \mathscr{M}^{\mathrm{prod}}$. Let $\mathscr{M}^{\mathrm{glue}} = \mathscr{M}_{q_1} \times_{\mathscr{M}^{\mathrm{prod}}} \cdots \times_{\mathscr{M}^{\mathrm{prod}}} \mathscr{M}_{q_{|E(\Gamma)|}}$ be the fibred product of all these. It is a proper Deligne–Mumford stack parametrizing maps $\underline{f}_v$ with gluing data for a stable map. It therefore carries a family of glued stable maps $\underline{f} : \underline{C}^{\mathrm{glue}} \to \underline{X}_0$ by the universal property of pushouts. Hence we have a morphism $\mathscr{M}^{\mathrm{glue}} \to \mathscr{M}(\underline{X}_0,\underline{\beta})$. The action of $\operatorname{Aut}(\tilde{\tau})$ on $\mathscr{M}^{\mathrm{prod}}$ clearly lifts to $\mathscr{M}^{\mathrm{glue}}$ and the morphism $\mathscr{M}^{\mathrm{glue}} \to \mathscr{M}(\underline{X}_0,\underline{\beta})$ is invariant.

Replacing X_0 by $\mathcal{X}_0$ we obtain an algebraic stack $\mathfrak{M}^{\text{glue}}$, parametrizing maps $\underline{f}_v$ with gluing data. Hence $\mathfrak{M}^{\text{glue}}$ carries a family of glued maps $\underline{f} : \underline{C}^{\text{glue}} \to \underline{\mathcal{X}}_0$ providing an invariant morphism $\mathfrak{M}^{\text{glue}} \to \mathfrak{M}(\underline{\mathcal{X}}_0, \underline{\beta}')$.

The morphism $\mathscr{M}^{\text{glue}} \to \mathfrak{M}^{\text{glue}}$ is canonically $\operatorname{Aut}(\tilde\tau)$-equivariant.

Step 4: compatibility of maps. We have

$$\mathscr{M}_{\tilde\tau}(X_0/b_0, \beta) = \mathscr{M}^{\text{glue}} \times_{\mathscr{M}(\underline{X}_0, \underline{\beta})} \mathscr{M}_{m_{\tilde\tau}}(X_0/b_0, \beta),$$

hence a proper Deligne-Mumford stack, with morphism to $\mathscr{M}_{m_{\tilde\tau}}(X_0/b_0, \beta)$. The action of $\operatorname{Aut}(\tilde\tau)$ canonically lifts.

Replacing X_0 by $\mathcal{X}_0$ we similarly have an algebraic stack $\mathfrak{M}_{\tilde\tau}(\mathcal{X}_0/b_0) = \mathfrak{M}^{\text{glue}} \times_{\mathfrak{M}(\underline{\mathcal{X}}_0, \underline{\beta}')} \mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0, \beta')$ with invariant morphism to $\mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0, \beta')$. The resulting morphism $\mathscr{M}_{\tilde\tau}(X_0/b_0, \beta) \to \mathfrak{M}_{\tilde\tau}(\mathcal{X}_0/b_0, \beta')$ is canonically equivariant. ♠

4.6. **Costello's diagram.** We denote by τ the isomorphism class of $\widetilde{\tau}$, equivalently the unique isomorphism class of rigid tropical curves with combinatorial type $\widetilde{\tau}$ under graph isomorphisms fixing the decorations $\mathbf{g}, \boldsymbol{\sigma}$ and u. We also write $m_\tau := m_{\widetilde{\tau}}$.

We denote

$$\begin{aligned} \mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') &:= \left[\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0, \beta') \Big/ \operatorname{Aut}(\widetilde{\tau})\right], \\ \mathscr{M}_\tau(X_0/b_0, \beta) &:= \left[\mathscr{M}_{\widetilde{\tau}}(X_0/b_0, \beta) \Big/ \operatorname{Aut}(\widetilde{\tau})\right] \end{aligned}$$

and

$$\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta) := \left[\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta) \Big/ \operatorname{Aut}(\widetilde{\tau}, \mathbf{A})\right].$$

It follows that

$$\mathscr{M}_\tau(X_0/b_0, \beta) = \coprod_{\mathbf{A} \vdash A} \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta).$$

Proposition 4.6.1. *(1) We have a cartesian diagram*

$$\begin{array}{ccc} \mathscr{M}_\tau(X_0/b_0, \beta) & \longrightarrow & \mathscr{M}_{m_\tau}(X_0/b_0, \beta) \\ \downarrow & & \downarrow \\ \mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') & \longrightarrow & \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta') \end{array}$$

(2) The morphism

$$\Psi : \coprod_{\tau\, :\, m_\tau = m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$$

is of pure degree 1 in the sense of Costello [Cos06].

Proof. (1) Both the $\operatorname{Aut}(\tilde\tau)$-invariant composed morphisms

$$\mathscr{M}_{\tilde\tau}(X_0/b_0, \beta) \to \mathscr{M}_{m_{\tilde\tau}}(X_0/b_0, \beta) \to \mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0, \beta')$$

and

$$\mathscr{M}_{\widetilde{\tau}}(X_0/b_0,\beta) \to \mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')$$

send a decorated logarithmic map $C \to X$ to the composite morphism $C \to X \to \mathcal{X}$. Hence we obtain an $\mathrm{Aut}(\tilde{\tau})$-equivariant morphism

$$\mathscr{M}_{\widetilde{\tau}}(X_0/b_0,\beta) \to \mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta') \times_{\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')} \mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0,\beta).$$

An element of the fibred product consists of

$$\left(\left(f : C \to \mathcal{X}_0, \underline{f}_v : \underline{C}_v \to \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}\right), \tilde{f} : C \to X_0\right)$$

where the composite $C \xrightarrow{\tilde{f}} X_0 \to \mathcal{X}_0$ is f. Since $\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} = \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} \times_{\underline{\mathcal{X}}_0} \underline{X}_0$ we obtain morphisms $\underline{\tilde{f}}_v : \underline{C}_v \to \underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ which clearly glue together to the given $\underline{\tilde{f}} : \underline{C} \to \underline{X}_0$. These morphisms are stable since $\tilde{f}$ is, giving a morphism

$$\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta') \times_{\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')} \mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0,\beta) \to \mathscr{M}_{\widetilde{\tau}}(X_0/b_0,\beta).$$

The functorial nature of the two morphisms we have constructed shows they are inverse to each other, giving that the diagram is cartesian, as required in part (1).

(2) Consider the open, dense locus $\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ \subset \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')$ where

- the curves $\underline{C}_v$ are smooth, and
- the image $\underline{f}_v(\underline{C}_v)$ meets the interior $\underline{\mathcal{X}}_v$ of $\underline{\mathcal{X}}^{\mathrm{cl}}_v$.

It suffices to show:

Claim. The morphism $\coprod \mathfrak{M}_{\tau: m_{\widetilde{\tau}}=m}(\mathcal{X}_0/b_0,\beta')^\circ \to \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')$ induced by Ψ is an open embedding, whose image is the union of open strata $\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')^\circ$ of $\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')$.

Proposition 4.6.1(2) follows from the claim, since under these conditions, Ψ gives an isomorphism between the open set

$$\coprod_{\tau: m_{\widetilde{\tau}}=m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ = \Psi^{-1}\Psi\left(\coprod_{\tau: m_{\widetilde{\tau}}=m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ\right)$$

and its open dense image in $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')$.

To prove the claim, we first show that

$$\Psi\Big(\coprod_{\tau: m_{\widetilde{\tau}}=m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ\Big) \quad\subset\quad \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta').$$

In fact, the tropical curve of an object

$$(C/W,\mathbf{p},f) \in \Psi\left(\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ\right)(\Bbbk)$$

is the rigid tropical curve $\widetilde{\tau}$, and by Proposition 4.1.2 it lies in a codimension-1 stratum of $\mathfrak{M}(\mathcal{X}/B,\beta')$, namely in an open stratum of $\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')^\circ$.

We construct a map in the other direction. For any object $(C/W,\mathbf{p},f) \in \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ(\Bbbk)$ the isomorphism class τ of its tropical curve $\widetilde{\tau}$ is determined uniquely by the tropicalization process. Further, since $\mathfrak{M}(\mathcal{X}/B,\beta')$ is log smooth over B, Corollary 3.2.1 gives a one-to-one correspondence between rigid tropical curves and open strata of $\mathfrak{M}(\mathcal{X}_0/b_0,\beta')$. In particular, τ must be an isomorphism class of a *rigid* tropical curve. Lifting the decomposition of $(C/W,\mathbf{p},f)$ to an étale neighborhood $\mathfrak{U}\to\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ$ we obtain a morphism $\mathfrak{U}\to\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ$. This shows that τ does not depend on the choice of $\Bbbk$-point in the stratum, and provides a map $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ\to\coprod_\tau\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ$.

We claim that the multiplicity of $\widetilde{\tau}$ in this construction coincides with the multiplicity m, namely the index of $Q^\vee\to\mathbb{N}$, of the ray corresponding to the stratum in Proposition 4.1.2(2). Indeed, using the notation of that proposition, note that $Q^\vee_\mathbb{R}$ parameterizes a family of tropical curves with points of $Q^\vee$ corresponding to those curves whose assigned edge lengths are integral and whose vertices map to integral points of $\Sigma(X)$. Thus a primitive generator of $\tilde{\tau}$ corresponds precisely to the curve $\Psi(\tilde{\tau})$ rescaled by a factor of $m_{\tilde{\tau}}$.

This provides a morphism $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ\to\coprod_{\tau:m_\tau}\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ$. It is not difficult to show that this is an inverse of Ψ, as needed. ♠

4.7. Obstruction theories.

We note that $\mathcal{X}_0$ is logarithmically étale over b_0.

Proposition 4.7.1. *(1) The complex $R\pi_*f^*T_{X_0/\mathcal{X}_0}$ defines compatible perfect relative obstruction theories both on the morphism $\mathscr{M}_\tau(X_0/b_0,\beta)\to\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')$ and on the morphism $\mathscr{M}_m(X_0/b_0,\beta)\to\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')$. These perfect obstruction theories give rise to virtual fundamental classes*

$$[\mathscr{M}_\tau(X_0/b_0)]^{\mathrm{virt}}\qquad\text{and}\qquad[\mathscr{M}_m(X_0/b_0)]^{\mathrm{virt}}.$$

(2) The resulting virtual fundamental class on $\mathscr{M}_m(X_0/b_0,\beta)$ coincides with the virtual fundamental class defined relative to $\mathfrak{M}_m$ in (3.4.2).

Proof. (1) Since $\mathcal{X}\to\mathrm{Log}_B$ is étale we have $R\pi_*f^*T_{X/\mathcal{X}}=R\pi_*f^*T_{X/\mathrm{Log}_B}$, which is precisely the complex giving rise to the perfect relative obstruction theory for $\mathscr{M}(X/B,\beta)\to\mathfrak{M}_B$ introduced in [GS13]. Since $\mathfrak{M}(\mathcal{X}/B,\beta')\to\mathfrak{M}_B$ is étale this complex induces a perfect relative obstruction theory for $\mathscr{M}(X/B,\beta)\to\mathfrak{M}(\mathcal{X}/B,\beta')$. It is a general fact, see [BL00, Proposition A.1] or [Wis11, Proposition 6.2], that this induces a relative obstruction theory on pullbacks of $\mathscr{M}(X/B,\beta)\to\mathfrak{M}(\mathcal{X}/B,\beta')$ along $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')\to\mathfrak{M}(\mathcal{X}/B,\beta')$ or $\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')\to\mathfrak{M}(\mathcal{X}/B,\beta')$.

(2) follows since $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')\to\mathfrak{M}_m$ is strict étale. ♠

4.8. Proof of the main theorem.

By Costello and Manolache we get

$$\sum_{\tau\,:\,m_\tau=m}\Psi_*\left[\mathscr{M}_\tau(X_0/b_0,\beta)\right]^{\mathrm{virt}}=[\mathscr{M}_m(X_0/b_0)]^{\mathrm{virt}}.$$

Therefore Proposition 3.4.2 shows

$$\sum_\tau m_\tau\Psi_*\left[\mathscr{M}_\tau(X_0/b_0,\beta)\right]^{\mathrm{virt}}\quad=\quad[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}}$$

and similarly

$$\sum_{(\tau,A)\ :\ \mathbf{A}\vdash A} m_\tau \Psi_* \left[\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)\right]^{\mathrm{virt}} = \left[\mathscr{M}(X_0/b_0,\beta)\right]^{\mathrm{virt}}.$$

Writing $i_{\tau,\mathbf{A}}$ for Ψ restricted to $\mathscr{M}_\tau(X_0/b_0)$, we obtain Theorem 1.1.2.

Since $\mathscr{M}_{\tilde{\tau},A}(X_0/b_0,\beta) \to \mathscr{M}_{\tau,A}(X_0/b_0,\beta)$ has degree $|\operatorname{Aut}(\tilde{\tau},\mathbf{A})|$, we have the following alternative formulation.

Theorem 4.8.1.

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{(\tilde{\tau},\mathbf{A})\ :\ \mathbf{A}\vdash A} \frac{m_{\tilde{\tau}}}{|\operatorname{Aut}(\tilde{\tau},\mathbf{A})|} (i_{\tilde{\tau},\mathbf{A}})_* \left[\mathscr{M}_{\tilde{\tau},A}(X_0/b_0,\beta)\right]^{\mathrm{virt}}.$$

where $i_{\tilde{\tau},\mathbf{A}}$ is the composite of $i_{\tau,\mathbf{A}}$ with the quotient morphism.

Part 2. Practice

5. Logarithmic modifications and transversal maps

There is a general strategy which is often useful for constructing stable logarithmic maps. This is the most powerful tool we have at our disposal at the moment; eventually, the hope is that gluing technology will replace this construction. However, we expect it to be generally useful, especially in the examples in the next section.

Suppose we wish to construct a stable logarithmic map to X/B, as usual X logarithmically smooth with a Zariski logarithmic structure over one-dimensional B with logarithmic structure induced by $b_0 \in B$. Suppose further we wish the stable logarithmic map to map into the fibre X_0 over b_0. This construction is accomplished by a two step process.

5.1. **Logarithmic modifications.** First, we will choose a logarithmic modification $h : \tilde{X} \to X$, i.e., a morphism which is proper, birational, and log étale. The modification h is chosen to accommodate a situation at hand — in our applications the datum of a rigid tropical curve.

Given a modification h, [AW13] constructed a morphism $\mathscr{M}(h) : \mathscr{M}(\tilde{X}/B) \to \mathscr{M}(X/B)$ of moduli stacks of basic stable logarithmic maps, satisfying

$$\mathscr{M}(h)_*([\mathscr{M}(\tilde{X}/B)]^{\mathrm{virt}}) = [\mathscr{M}(X/B)]^{\mathrm{virt}}.$$

The construction of $\mathscr{M}(h)$ is as follows. Given a stable logarithmic map $\tilde{f} : \tilde{C}/W \to \tilde{X}/B$, one obtains on the level of schemes the stabilization of $h \circ \tilde{f}$, i.e., a factorization of $h \circ \tilde{f}$ given by

$$\underline{\tilde{C}}/\underline{W} \xrightarrow{g} \underline{C}/\underline{W} \longrightarrow \underline{X}$$

such that $\underline{C}/\underline{W} \to \underline{X}$ is a stable map. One gives $\underline{C}$ the logarithmic structure $\mathcal{M}_C := g_*\mathcal{M}_{\tilde{C}}$, and with this logarithmic structure one obtains a factorization of $h \circ \tilde{f}$ through C at the level of log schemes, giving $f : C/W \to X/B$. If $\tilde{f}$ was basic, there is no expectation that f is basic, but by [GS13], Proposition 1.22

or [AC14], Corollary 5.11 there is a unique basic map with the same underlying stable map of schemes such that the above constructed f is obained from pull-back from the basic map. This yields the map $\mathscr{M}(h)$.

5.2. **Transverse maps, logarithmic enhancements, and strata.** Second, if we have a stable map to $\underline{X}_0$ which interacts sufficiently well with the strata, we will compute in Theorem 5.3.3 the number of log enhancements of this curve. This generalizes a key argument of Nishinou and Siebert in [NS06]. There are two differences: our degeneration $X \to B$ is only logarithmically smooth and not necessarily toric; and the fibre X_0 is not required to be reduced. Not requiring X_0 to be reduced makes the situation more complex and perhaps explains why it was avoided in the past; we hope our treatment here will find further uses. The precise meaning of "interacting well with logarithmic strata" is as follows:

Definition 5.2.1 (Transverse maps and constrained points)**.** Let $X \to B$ be a logarithmically smooth morphism over B one-dimensional carrying the divisorial logarithmic structure $b_0 \in B$ as usual. Let $X_0^{[d]}$ denote the union of the (open) codimension d logarithmic strata of X_0. Suppose $\underline{f} : \underline{C}/\operatorname{Spec}\Bbbk \to \underline{X}_0$ is a stable map. We say that $\underline{f}$ is a *transverse map* if the image of $\underline{f}$ is contained in $X_0^{[0]} \cup X_0^{[1]}$, and $\underline{f}^{-1}(X_0^{[1]})$ is a finite set.

We call a node $q \in \underline{C}$ a *constrained node* if $\underline{f}(q) \in X_0^{[1]}$ and otherwise it is a *free node*. Similarly a marked point $x \in \underline{C}$ with $\underline{f}(x) \in X_0^{[1]}$ is a *constrained marking*, otherwise it is a *free marking*.

The term "transverse map" is shorthand for "a map meeting strata in a logarithmically transverse way".

Cones and strata in the transverse setting. For the rest of this section strata of higher codimension are irrelevant and we henceforth assume $X_0 = X_0^{[0]} \cup X_0^{[1]}$. Then $\Sigma(X_0)$ is a purely two-dimensional cone complex, with rays in bijection with the irreducible components of X_0. There are two types of two-dimensional cones: first, there is one cone for each component of the double locus $X_0^{[1]}$; Second, there is one cone for each other component of $X_0^{[1]}$, forming a divisor in the regular locus of X_0.

Logarithmic enhancement of a map. We codify what it means to take a stable map and endow it with a logarithmic structure:

Definition 5.2.2. Let $X \to B$ be as above and $\underline{f} : \underline{C} \to \underline{X}_0$ a stable map. A *logarithmic enhancement* $f : C \to X$ is a stable logarithmic map whose underlying map is $\underline{f}$. Two logarithmic enhancements f_1, f_2 are *isomorphic enhancements* if there is an isomorphism between f_1 and f_2 which is the identity on the underlying $\underline{f}$. Otherwise we say they are *non-isomorphic* or *distinct enhancements*.

Discrete invariants in the transverse case.

Notation 5.2.3. Let $\underline{f}: \underline{C}/\operatorname{Spec}\Bbbk \to \underline{X}_0$ be a transverse map and $x \in \underline{C}$ a closed point with $\underline{f}(x)$ contained in a stratum $S \subset X_0^{[1]}$ and let $\eta \in \underline{C}$ be a generic point with $x \in \operatorname{cl}(\eta)$. Then $P_x = \overline{\mathcal{M}}_{X,\underline{f}(x)}$ is a rank two toric monoid. Denote by $m_{\eta,x} \in P_x$ the generator of the kernel of the localization map $P_x \to \overline{\mathcal{M}}_{X,\underline{f}(\eta)} \simeq \mathbb{N}$ and by $m'_{\eta,x} \in P_x$ the generator of the other extremal ray. Denote by $n_{\eta,x}, n'_{\eta,x} \in P_x^\vee$ the dual generators of the extremal rays of $P_x^\vee$, satisfying $\langle n_{\eta,x}, m_{\eta,x}\rangle = 0$. A third distinguished element $\rho_x \in P_x$ is defined by pulling back the generator of $\Gamma(B, \overline{\mathcal{M}}_B) = \mathbb{N}$ under the log morphism $X \to B$.

For the following discussion denote by $\ell(m)$ the integral length of an element $m \in M \otimes_{\mathbb{Z}} \mathbb{Q}$, that is, for $m \neq 0$ the maximum of $\lambda \in \mathbb{Q}_{>0}$ with $\lambda^{-1} \cdot m \in M$, while $\ell(0) = 0$.

Definition 5.2.4. (1) The *index* of $x \in \underline{C}$ or of the stratum $S \subset X_0^{[1]}$ containing $\underline{f}(x)$ is the index of the sublattices in P_x^{gp} or in P_x^* generated by $m_{\eta,x}, m'_{\eta,x}$ and $n_{\eta,x}, n'_{\eta,x}$, respectively, that is,

$$\operatorname{Ind}(S) = \operatorname{Ind}_x = \langle n_{\eta,x}, m'_{\eta,x}\rangle = \langle n'_{\eta,x}, m_{\eta,x}\rangle.$$

For a constrained node q, the *length* $\lambda(q) = \lambda(S) \in \mathbb{Q}$ is the integral length of $\rho_q^{-1}(1)$ when viewing ρ_q as a map $P_q^* \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$.

(2) If $\eta \in \underline{C}$ is a generic point with $x \in \operatorname{cl}(\eta)$, denote by $w_{\eta,x} \in \mathbb{N} \setminus \{0\}$ the local intersection number of $\underline{f}|_{\operatorname{cl}(\eta)}$ at x with S inside the irreducible component of X_0 containing $\underline{f}(\eta)$.

When the choice of x and η is understood we write $m_1 = m_{\eta,x}$, $m_2 = m'_{\eta,x}$, $n_1 = n_{\eta,x}$, $n_2 = n'_{\eta,x}$, $\rho_x \in P_x$ and $w_1 = w_{\eta,x}$.

Relations between discrete invariants.

Lemma 5.2.5. *In the situation of Definition 5.2.4 denote by μ_1 the multiplicity of the irreducible component of $\underline{X}_0$ containing $\underline{f}(\eta)$. If the stratum $S \subset X_0^{[1]}$ is contained in two irreducible components of $\underline{X}_0$, denote by μ_2 the multiplicity of the other component and otherwise define $\mu_2 = 0$.*

$$(1)\quad \mu_i = \langle n_i, \rho_x\rangle. \quad (2)\quad \operatorname{Ind}_x \cdot \rho_x = \mu_2 m_1 + \mu_1 m_2. \quad (3)\quad \lambda(q) = \frac{\ell(\rho_q)\cdot \operatorname{Ind}_q}{\mu_1\mu_2}.$$

Proof. For (1) note that since $n_i \in P_x^\vee$ is a primitive vector with $\langle n_i, m_i\rangle = 0$, the pairing with n_i computes the integral distance from the face $\mathbb{N}\cdot m_i$ of P_x. Now ρ_x defines the local model at $\underline{f}(x)$ for the log smooth morphism $X \to B$, and hence the multiplicity μ_i equals the integral distance of ρ_x to $\mathbb{N} \cdot m_i$.

For (2), since the sublattice of P^{gp} generated by m_1, m_2 is of index Ind_x, there are $a_1, a_2 \in \mathbb{Z}$ with $\operatorname{Ind}_x \cdot \rho_x = a_1 m_1 + a_2 m_2$. Pairing with n_1 and using (1) and the definition of Ind_x yields

$$\operatorname{Ind}_x \cdot \mu_1 = \operatorname{Ind}_x \cdot \langle n_1, \rho_x\rangle = a_2\langle n_1, m_2\rangle = a_2 \cdot \operatorname{Ind}_x.$$

This shows $a_2 = \mu_1$, and similarly $a_1 = \mu_2$, yielding the claim.

To prove (3) note that (1) implies

$$\langle \mu_2 n_1, \rho_q \rangle = \mu_1\mu_2 = \langle \mu_1 n_2, \rho_q \rangle.$$

Hence $\mu_1\mu_2 \cdot \lambda(q)$ is the integral length of $\mu_2 n_1 - \mu_1 n_2$. Choosing an isomorphism of P_q with

$$\mathbb{Z}^2 \cap \big(\mathbb{R}_{\geq 0} \cdot (1,0) + \mathbb{R}_{\geq 0} \cdot (r,s)\big)$$

with $r, s > 0$ pairwise prime and ρ_q mapping to (a, c), then

$$m_1 = (1,0), \quad m_2 = (r,s), \quad \mu_1 = c, \quad \mu_2 = as - cr, \quad \mathrm{Ind}_q = s.$$

In the dual lattice $P_q^* \simeq \mathbb{Z}^2$ we have $n_1 = (0,1)$, $n_2 = (s, -r)$ and $\mu_2 n_1 - \mu_1 n_2 = s \cdot (-c, a)$ has integral length $\mathrm{Ind}_q\, \ell(\rho_q)$. Thus $\lambda(q) = \mathrm{Ind}_q\, \ell(\rho_q)/\mu_1\mu_2$ as claimed. ♠

Necessary conditions for enhancement. The data listed in Definition 5.2.4 determine the discrete invariant $u_x \in P_x^\vee$ at each special point $x \in \underline{C}$. Recall that Equation (2.3.3) characterizing u_q implies $\langle u_q, \rho_x \rangle = 0$. To fix the sign of u_q we use the convention that χ_1 in the defining equation is the generization map to η. Similarly, for each marked point p, it holds $\langle u_p, \rho_p \rangle = 0$ by definition of u_p. We now deduce a number of necessary conditions for a logarithmic enhancement of a transverse stable map to exist.

Proposition 5.2.6. *Let $f : C \to X$ be a logarithmic enhancement of a transverse stable map $\underline{f} : \underline{C} \to \underline{X}_0$. Let $\eta \in \underline{C}$ be a generic point and $x \in \mathrm{cl}(\eta)$. If $\underline{f}(x) \in X_0^{[1]}$ then following Definition 5.2.4 write $m_1 = m_{\eta,x}$, $m_2 = m'_{\eta,x}$, $n_1 = n_{\eta,x}$, $n_2 = n'_{\eta,x}$, $\rho_x \in P_x$ and $w_1 = w_{\eta,x}$.*

I) **(Node)** *If $x = q$ is a constrained nodal point of $\underline{C}$, then the second generic point η' of C with $x \in \mathrm{cl}(\eta')$ maps to a different irreducible component of X_0 than η. Moreover, with $w_2 = w_{\eta',x}$ the following holds:*

(1) $u_q = \dfrac{1}{\mathrm{Ind}_q} \cdot (w_1 n_2 - w_2 n_1)$.
(2) $u_q(m_1) = w_1$, $u_q(m_2) = -w_2$.
(3) $\mu_1 w_2 = \mu_2 w_1$.
(4) The integral length of u_q equals $\ell(u_q) = \dfrac{\mu_2 w_1 \lambda(q)}{\mathrm{Ind}_q} = \dfrac{w_1}{\mu_1}\ell(\rho_q)$.

If $x = q$ is a free node then $u_q = 0$.

II) **(Marked point)** *If x is a smooth point of $\underline{C}$, then $\underline{f}(x)$ is contained in only one irreducible component of X_0. Moreover, if $x = p$ is a marked point then $u_p = 0$ in the free case, while in the constrained case the following holds.*

(1) w_1 is a multiple of Ind_p.
(2) $u_p = \dfrac{w_1}{\mathrm{Ind}_p} n_2$.

Proof. Setup for (I). Let C be defined over the log point $W = \mathrm{Spec}(Q \to \Bbbk)$. For any generic point $\eta \in \underline{C}$, there is a commutative square

$$\begin{array}{ccc} \mathbb{N} \simeq P_\eta = \overline{\mathcal{M}}_{X_0, \underline{f}(\eta)} & \xrightarrow{\overline{f}_\eta^\flat} & \overline{\mathcal{M}}_{C,\eta} \\ \uparrow & & \uparrow \\ \mathbb{N} \simeq \overline{\mathcal{M}}_{B,b_0} & \longrightarrow & Q. \end{array}$$

Free node. In the case of a free node, both generic points $\eta, \eta' \in \underline{C}$ containing q in their closure map to the same irreducible component of X_0. Thus $u_q = 0$ by the defining equation (2.3.3).

Image components of constrained node. Let now $x = q$ be a constrained node. Since the generization map $\chi_\eta : P_q \to P_\eta$ is a localization of fine monoids there exists $m \in P_q \setminus \{0\}$ with $\chi_\eta(m) = 0$. Then also $\overline{f}_q^\flat(m)$ is a non-zero element in $\overline{\mathcal{M}}_{C,q}$ with vanishing generization at η. But $\overline{\mathcal{M}}_C$ has no local section with isolated support at q. Hence $\chi_{\eta'}(m) \neq 0$, which implies that the two branches of C at q map to different irreducible components of X_0.

Computations for a constrained node. (1) follows from (2) by pairing both sides with m_1, m_2 since these elements generate $P_q \otimes_{\mathbb{Z}} \mathbb{Q}$. We now prove (2). Since u_q is preserved under base-change, we may assume C is defined over the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$. Then $\overline{\mathcal{M}}_{C,q} \simeq S_e$ for some $e \in \mathbb{N} \setminus \{0\}$ with S_e the submonoid of $\mathbb{Z}^2$ generated by $(e, 0), (0, e), (1, 1)$, see e.g. [GS13], §1.3. The generator $1 \in \mathbb{N}$ of the standard log point maps to $(1, 1)$, while a chart at q maps $(e, 0)$ to a function restricting to a coordinate on one of the two branches of C, say on $\mathrm{cl}(\eta)$, while vanishing on the other. Similarly, $(0, e)$ restricts to a coordinate on $\mathrm{cl}(\eta')$. By transversality we conclude

$$\overline{f}_q^\flat(m_1) = w_1 \cdot (e, 0), \quad \overline{f}_q^\flat(m_2) = w_2 \cdot (0, e).$$

Equation (2.3.3) defining u_q says

$$\chi_2 \circ \overline{f}_q^\flat - \chi_1 \circ \overline{f}_q^\flat = u_q \cdot e, \tag{5.2.1}$$

with $\chi_i : S_e \to \mathbb{N}$ the generization maps. With our presentation, χ_1 and χ_2 are induced by the projections $S_e \subset \mathbb{Z}^2 \to \mathbb{Z}$ to the second and first factors, respectively. Hence

$$\begin{aligned} \big(\chi_2 \circ \overline{f}_q^\flat - \chi_1 \circ \overline{f}_q^\flat\big)(m_1) &= w_1 \cdot e \\ \big(\chi_2 \circ \overline{f}_q^\flat - \chi_1 \circ \overline{f}_q^\flat\big)(m_2) &= -w_2 \cdot e, \end{aligned}$$

showing (2).

(3) is obtained by evaluating (1) on ρ_q:

$$0 = \mathrm{Ind}_q \cdot \langle u_q, \rho_q \rangle = w_1 \langle n_2, \rho_q \rangle - w_2 \langle n_1, \rho_q \rangle = w_1 \mu_2 - w_2 \mu_1.$$

For (4) observe from (1) that $\mathrm{Ind}_q \cdot u_q$ is the vector connecting the extremal elements $w_2 n_1$ and $w_1 n_2$ of $P_q^\vee$. Thus $\mathrm{Ind}_q \cdot \ell(u_q)$ equals the integral length of

$\rho_q^{-1}(h)$ for $h = \langle w_1 n_2, \rho_q\rangle = \mu_2 w_1 = \mu_1 w_2 = \langle w_2 n_1, \rho_q\rangle$. This length equals $h\cdot\lambda(q)$, yielding the stated formula. This finishes the proof of (I).

Marked point. Turning to (II), let $x\in \underline{C}$ be a smooth point with $\underline{f}(x)\in X_0^{[1]}$ and again assume without restriction C is defined over the standard log point. If $s_u\in \mathcal{M}_{X,\underline{f}(x)}$ is a lift of m_1, then by transversality, $f_x^\flat(s_u)\in\mathcal{M}_{C,x}$ maps under the structure homomorphism $\mathcal{M}_{C,x}\to\mathcal{O}_{C,x}$ to z^{w_1}, with z a local coordinate of $\underline{C}$ at x. Thus $x=p$ is a marked point, $\overline{\mathcal{M}}_{C,x}=\mathbb{N}^2$ and

$$\overline{f}_p^\flat : P_p \longrightarrow \mathbb{N}^2$$

maps m_1 to $(0,w_1)$. Here we are taking the morphism $C\to \operatorname{Spec}(\mathbb{N}\to\Bbbk)$ to be defined by $\mathbb{N}\to\mathbb{N}^2$, $1\mapsto(1,0)$. Moreover, by compatibility of $f_p^\flat$ with the morphism of standard log points that C and X_0 are defined over, $\overline{f_p}^\flat(\rho_p)=(b,0)$ for some $b\in\mathbb{N}\smallsetminus\{0\}$. Thus by Lemma 5.2.5,(2), $\rho_p=\frac{\mu_1}{\operatorname{Ind}_p}m_2$ spans an extremal ray of P_p. In particular, $\underline{f}(p)$ is contained in only one irreducible component of X_0 and $u_p(m_2)=0$. Thus

$$u_p(m_1)=w_1=\frac{w_1}{\operatorname{Ind}_p}\langle n_2,m_1\rangle,\quad u_p(m_2)=0=\frac{w_1}{\operatorname{Ind}_p}\langle n_2,m_2\rangle.$$

This shows (2), which implies (1) since n_2 is a primitive vector.

Finally, at a free marked point $p\in\underline{C}$, commutativity over the standard log point again readily implies $u_p=0$. ♠

Transverse pre-logarithmic maps. Summarizing the necessary conditions of Proposition 5.2.6, we are led to the following definition.

Definition 5.2.7. Let $X\to B$ be as above, and let $\underline{f}:\underline{C}/\operatorname{Spec}\Bbbk\to\underline{X}_0$ be a transverse map. We say $\underline{f}$ is a *transverse pre-logarithmic map* if any $x\in\underline{C}$ with $\underline{f}(x)\in X_0^{[1]}$ is a special point and if in the notation of Proposition 5.2.6 the following holds.

(I) **(Constrained node)** If $x=q$ is a constrained node then the two branches of C at q map to different irreducible components of X_0. In additon, $\mu_1w_2=\mu_2w_1$ and the *reduced branching order*

$$\overline{w}_q := \frac{w_i}{\mu_i}\ell(\rho_q),\quad i=1,2 \tag{5.2.2}$$

is an integer.

(II) **(Constrained marking)** If $x=p$ is a constrained marking then $\underline{f}(x)$ is a smooth point of X_0 and $w_1/\operatorname{Ind}_p\in\mathbb{N}$.

Note that if a logarithmic enhancement of $\underline{f}$ exists, then by Proposition 5.2.6 the reduced branching order $\overline{w}_q$ agrees with $\ell(u_q)$.

Definition 5.2.8 (Base order)**.** For a transverse pre-logarithmic map $\underline{f}:\underline{C}/\operatorname{Spec}\Bbbk\to\underline{X}_0$ define its *base order* $b\in\mathbb{N}$ *to be the least common multiple of the following natural numbers: (1) all multiplicities of irreducible components of* X_0 *intersecting*

$\underline{f}(\underline{C})$ and (2) for each constrained node $q \in \underline{C}$ the quotient $\mu_1 w_2/\gcd(\mathrm{Ind}_q, \mu_1 w_2)$, notation as in Proposition 5.2.6.

Theorem 5.2.9. *Let $X \to B$ be as above, and let $\underline{f} : \underline{C}/\operatorname{Spec} \Bbbk \to \underline{X}_0$ be a transverse map. Suppose that there is an enhancement of $\underline{f}$ to a basic stable logarithmic map $f : C/W \to X/B$. Then*

(1) $\underline{f}$ is a transverse pre-logarithmic map.
(2) The combinatorial type of f is uniquely determined up to possibly a number of marked points p with $u_p = 0$, and the basic monoid Q is

$$Q = \mathbb{N} \oplus \bigoplus_{q \text{ a free node}} \mathbb{N}.$$

(3) The map $W = \operatorname{Spec}(Q \to \Bbbk) \to B$ induces the map $\overline{\mathcal{M}}_{b_0} = \mathbb{N} \to Q$ given by $1 \mapsto (b, 0, \ldots, 0)$, where the integer $b \in \mathbb{N}$ is the base order of $\underline{f}$.

Proof. (1) and (2) follow readily from Proposition 5.2.6. For (3), recall that the basic monoid Q is dual to the monoid $Q^\vee \subset Q^\vee_{\mathbb{R}}$, the latter being the moduli space of tropical curves $h : \Gamma \to \Sigma(X)$ of the given combinatorial type, and $Q^\vee$ consists of those tropical curves whose edge lengths are integral and whose vertices map to integral points of $\Sigma(X)$.

If η is a generic point of $\underline{C}$, denote by μ_η the multiplicity of the irreducible component of $(X_0)_{\mathrm{red}}$ in X_0 containing $\underline{f}(\eta)$. Thus the induced map $\mathbb{N} \to P_\eta \simeq \mathbb{N}$ coming from the structure map $X \to B$ is multiplication by μ_η. Write $\rho : \Sigma(X) \to \Sigma(B)$ for the tropicalization of $X \to B$. The restriction of ρ to the ray $\mathbb{R}_{\geq 0} P_\eta^\vee$ of $\Sigma(X)$ corresponding to the irreducible component of X_0 containing $\underline{f}(\eta)$ is multiplication by μ_η. Thus given a tropical curve $h : \Gamma \to \Sigma(X)$ with vertex v_η for $\eta \in \underline{C}$ and b the image of $\rho \circ h$ in $\Sigma(B)$, we see that $h(v_\eta)$ is integral if and only if $\mu_\eta | b$.

The edges of Γ corresponding to free nodes have arbitrary length independent of μ. But an edge corresponding to a constrained node q must have length

$$e_q = b\frac{\lambda(q)}{\ell(u_q)} = b\frac{\mathrm{Ind}_q}{\mu_1 w_2}. \tag{5.2.3}$$

This must also be integral for h to represent a point in $Q^\vee$. Thus the map $\Sigma(W) \to \Sigma(B)$ must be given by $(\alpha, (\alpha_q)_q) \mapsto b\alpha$ where b is as given in the statement of the theorem. Dually, we obtain the stated description of the map $W \to B$. ♠

5.3. **Existence and count of enhancements of transverse pre-logarithmic maps.** We now turn to count the number of logarithmic enhancements of a transverse stable map $\underline{f} : \underline{C} \to \underline{X}_0$. Denote by $\mathcal{M} := \underline{f}^* \mathcal{M}_{X_0}$ the pull-back log structure on $\underline{C}$ and by $\mathcal{M}^{\mathrm{Zar}}$ the corresponding sheaf of monoids in the Zariski topology, noting that the log structure on X_0 is assumed to be defined in the Zariski topology.

The torsor of roots. The count of logarithmic enhancements involves a torsor $\mathcal{F}$ under a sheaf of finite cyclic groups $\mathcal{G}$ on a finite topological space encoding compatible choices of roots of elements occurring in the construction of logarithmic enhancements. Given a transverse map $\underline{f}:\underline{C}/\operatorname{Spec}\Bbbk\to\underline{X}_0$, the finite topological space consists of the set of constrained nodes $q\in\underline{C}$ and generic points $\eta\in\underline{C}$. As basis for the topology we take the sets $U_\eta=\{\eta\}\cup\{q\in\operatorname{cl}(\eta)\}$ and $U_q=\{q\}$ (which is opposite to the topology as a subset of $\underline{C}$). Let $\rho\in\Gamma(\underline{C},\mathcal{M})$ be the preimage of a generator ρ_0 of $\mathcal{M}_{B,b_0}$, that we assume fixed in this subsection. The stalks at a constrained node $q\in\underline{C}$ and at a generic point $\eta\in\underline{C}$ are various roots of the germs ρ_x of ρ:

$$\begin{aligned}\mathcal{F}_q &= \big\{\sigma_q\in\mathcal{M}_q^{\mathrm{Zar}}\,\big|\,\sigma_q^{\ell(\rho_q)}=\rho_q\big\},\\ \mathcal{F}_\eta &= \big\{\sigma_\eta\in\mathcal{M}_\eta^{\mathrm{Zar}}\,\big|\,\sigma_\eta^{\mu_\eta}=\rho_\eta\big\}.\end{aligned}$$

We note that any of these sets may be empty, as Example 5.3.2 below shows. In such case we do not define a sheaf $\mathcal{F}$ and declare $|\Gamma(\mathcal{F})|=\emptyset$ in what follows. Otherwise we define the sheaf $\mathcal{F}$ as follows. For $q\in\operatorname{cl}(\eta)$, a choice σ_η with $\sigma_\eta^{\mu_\eta}=\rho_\eta$ determines a unique $\sigma_{\eta,q}\in\mathcal{F}_q$ with restriction to η equal to $\sigma_\eta^{\mu_\eta/\ell(\rho_q)}$. Note that by Lemma 5.2.5,(1) we have $\mu_\eta/\ell(\rho_q)\in\mathbb{N}$. We define the generization map $\mathcal{F}_\eta\to\mathcal{F}_q$ by mapping σ_η to $\sigma_{\eta,q}$. Observe that a different choice of ρ leads to an isomorphic sheaf $\mathcal{F}$.

Replacing the elements ρ_q and ρ_η in the definition of $\mathcal{F}$ by the element 1 we obtain a sheaf $\mathcal{G}$ of abelian groups, for which $\mathcal{F}$ is evidently a torsor.

Global sections of $\mathcal{G}$ and $\mathcal{F}$. General theory [Sta17, Tag 03AH], or direct computation, implies that the *set* of global sections $\Gamma(\mathcal{F})$ is a *pseudo-torsor* for the *group* $G:=\Gamma(\mathcal{G})$. Here G is computed as the kernel of the sheaf-axiom homomorphism

$$\partial:\prod_{\eta\in\underline{C}}\mathbb{Z}/\mu_\eta\longrightarrow\prod_{q\in\underline{C}}\mathbb{Z}/\ell(\rho_q),\quad \partial\big(\big(\zeta_\eta\big)_\eta\big):=\Big(\zeta_{\eta(q)}^{\mu_{\eta(q)}/\ell(\rho_q)}\cdot\zeta_{\eta'(q)}^{-\mu_{\eta'(q)}/\ell(\rho_q)}\Big)_q. \tag{5.3.1}$$

Here $\eta(q),\eta'(q)$ are the generic points of the two adjacent branches of a constrained node $q\in\underline{C}$, viewed in the étale topology. The notation implies a chosen order of branches. Multiplication of σ_q by ζ_q and of σ_η by ζ_η describes the natural action of $G=\Gamma(\mathcal{G})$ on $\Gamma(\mathcal{F})$.

Lemma 5.3.1. *If $\Gamma(\mathcal{F})\neq\emptyset$ the action of G on $\Gamma(\mathcal{F})$ is simply transitive. In particular, it then holds $\big|\Gamma(\mathcal{F})\big|=\big|G\big|$. If C is rational or if X_0 is reduced then $\Gamma(\mathcal{F})\neq\emptyset$.*

Proof. Simple transitivity is the fact that $\Gamma(\mathcal{F})$ is a pseudo-torsor for G.

If X_0 is reduced then $\mu_\eta=1$ for all η and $\Gamma(\mathcal{F})=\prod_q\mathcal{F}_q$ is non-empty. If C is rational we can construct a section by inductive extension over the irreducible components. Indeed, if $\sigma_q\in\mathcal{F}_q$ and η is the generic point of the next irreducible component, we can define σ_η as any $\mu_\eta/\ell(\rho_q)$-th root of the restriction of σ_q to η. By the definition of $\mathcal{F}$ this choice then also defines $\sigma_{q'}$ for all other $q'\in\operatorname{cl}(\eta)$. ♠

Example 5.3.2. Here is a simple example with $\Gamma(\mathcal{F}) = \emptyset$, in fact $\mathcal{F}_\eta = \emptyset$ for the unique point η in our space. Let $\underline{X} \to \underline{B} = \mathbb{A}^1$ be an elliptically fibred surface with $\underline{X}_0 \subset \underline{X}$ a b-fold multiple fibre with smooth reduction. Endow $\underline{X}$ and $\underline{B}$ with the divisorial log structures for the divisors $\underline{X}_0 \subset \underline{X}$ and $\{0\} \subset \underline{B}$. Then the generator $\overline{\rho}_0 \in \overline{\mathcal{M}}_{B,0}$ maps to b times the generator $\overline{\sigma} \in \Gamma(\underline{X}_0, \overline{\mathcal{M}}_{X_0}) = \mathbb{N}$. The preimage of $\overline{\sigma}$ under $\mathcal{M}_{X_0^{\rm red}} \to \overline{\mathcal{M}}_{X_0^{\rm red}}$ is the torsor with associated line bundle the conormal bundle $N^\vee_{X_0^{\rm red}|X}$. This conormal bundle is not trivial, but has order b in $\operatorname{Pic}(\underline{X}_0^{\rm red})$. Thus there exists no section σ_η with σ_η^b extending to a global section ρ of $\mathcal{M}_{X_0}$ lifting $\overline{\rho} = b \cdot \overline{\sigma}$.

The following statement generalizes and gives a more structural proof of [NS06], Proposition 7.1, which treated a special case with reduced central fibre.

Theorem 5.3.3. *Suppose given $X \to B$ as above, and let*

$$\underline{f} : (\underline{C}, p_1, \ldots, p_n)/\operatorname{Spec} \Bbbk \to \underline{X}_0$$

be a transverse pre-logarithmic map. Suppose further that the marked points $\{p_i\}$ include all points of $\underline{f}^{-1}(X_0^{[1]})$ mapping to non-singular points of $(X_0)_{\rm red}$.

Then there exists an enhancement of $\underline{f}$ to a basic stable logarithmic map if and only if $\Gamma(\mathcal{F}) \neq \emptyset$, in which case the number of pairwise non-isomorphic enhancements is

$$\frac{|G|}{b} \prod_q \overline{w}_q.$$

Here G is as in (5.3.1), the integer b is the base order (Definition 5.2.8), and the product is taken over the reduced branching orders (5.2.2) at constrained nodes.

Proof. Counting rigidified objects. We are going to count diagrams of the form

$$\begin{array}{ccc} C = (\underline{C}, \mathcal{M}_C) & \xrightarrow{f} & X_0 = (\underline{X}_0, \mathcal{M}_{X_0}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{Spec}(Q \to \Bbbk) & \xrightarrow[g]{} & B = (\underline{B}, \mathcal{M}_B), \end{array} \tag{5.3.2}$$

with p and $\underline{f}$ given by assumption and g determined by b as in Theorem 5.2.9,(3) uniquely up to isomorphism. For the final count we will divide out the $\mathbb{Z}/b$-action coming from the automorphisms of $\operatorname{Spec}(Q \to \Bbbk) \to B$.

Simplifying the base. By Theorem 5.2.9 we have $Q = \mathbb{N} \oplus \bigoplus_{\text{free nodes}} \mathbb{N}$ and the map $\mathbb{N} = \overline{\mathcal{M}}_{B,b_0} \to Q$ is the inclusion of the first factor multiplied by the base order b. Pulling back by any fixed sharp map $Q \to \mathbb{N}$ replaces the lower left corner by the standard log point $O^\dagger = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$. To be explicit, we take $Q \to \mathbb{N}$ to restrict to the identity on each summand. Since this map $Q \to \mathbb{N}$ is surjective we do not introduce automorphisms or ramification. The universal property of basic objects guarantees that the number of liftings is not changed.

The composition $\overline{\mathcal{M}}_{B,b_0} \to Q \to \mathbb{N}$ is then multiplication by b. We have now arrived at a counting problem over a standard log point. Note also that the given data already determines (5.3.2) at the level of ghost monoids, that is, the data determines the sheaf $\overline{\mathcal{M}}_C$, and maps $\overline{f}^\flat : \overline{\mathcal{M}} = \underline{f}^*\overline{\mathcal{M}}_{X_0} \to \overline{\mathcal{M}}_C$ and $\overline{\pi}^\flat : \overline{\mathcal{M}}_{O^\dagger} \to \overline{\mathcal{M}}_C$, uniquely.

Pulling back the target monoid. Pull-back yields the two log structures $\mathcal{M} = \underline{f}^*\mathcal{M}_{X_0}$ and $\pi^*\mathcal{M}_{O^\dagger}$ on $\underline{C}$. Recall our choice of generator ρ_0 of $\mathcal{M}_{B,b_0}$ and its pull-back $\rho \in \Gamma(\underline{C}, \mathcal{M})$, introduced earlier in Section 5.3. For later use let also $\tau_0 \in \mathcal{M}_{O^\dagger,0}$ be a generator with $g^\flat(\rho_0) = \tau_0^b$. Then for any log smooth structure $\mathcal{M}_C$ on $\underline{C}$ over $O^\dagger$ we have a distinguished section $\tau = \pi^\flat(\tau_0)$.

Simplifying the target monoid. Now define $\mathcal{M}'_C$ as the fine monoid sheaf given by pushout of these two monoid sheaves over $\underline{\pi}^*\underline{g}^*\mathcal{M}_B$:

$$\mathcal{M}'_C = \underline{f}^*\mathcal{M}_{X_0} \oplus^{\text{fine}}_{\underline{\pi}^*\underline{g}^*\mathcal{M}_B} \pi^*\mathcal{M}_{O^\dagger}.$$

Since $\overline{\mathcal{M}}_{B,b_0} = \mathbb{N}$, by [Kat89, (4.4)(ii)] $X_0 \to B$ is an integral morphism. Hence the pushout $\mathcal{M}'_C$ in the category of fine monoid sheaves agrees with the ordinary pushout. In particular, the structure morphisms of X_0 and $O^\dagger$ define a structure morphism $\alpha'_C : \mathcal{M}'_C \to \mathcal{O}_C$.

Restating the counting problem. Classifying diagrams (5.3.2) amounts to finding an fs log structure $\mathcal{M}_C$ on $\underline{C}$ together with a morphism of monoid sheaves

$$\phi : \mathcal{M}'_C \longrightarrow \mathcal{M}_C$$

compatible with $f^\sharp$ and such that the composition $\pi^*\mathcal{M}_{O^\dagger} \to \mathcal{M}_C$ of ϕ with $\pi^*\mathcal{M}_{O^\dagger} \to \mathcal{M}'_C$ is log smooth.

We will soon see that $\bar{\phi} : (\overline{\mathcal{M}}'_C)^{\text{gp}} \longrightarrow \overline{\mathcal{M}}_C^{\text{gp}}$ necessarily decomposes into the quotient by some finite torsion part and the inclusion of a finite index subgroup. Lifting the quotient morphism to $\mathcal{M}'_C$ leads to the factor $|G|$, while the finite extension of the resulting log structure to $\mathcal{M}_C$ receives a contribution by the reduced branching order $\overline{w}_q$ from each constrained node. Note that $\overline{w}_q = \ell(u_q)$ by Proposition 5.2.6,(I)(4); it is in this form that it appears in the proof.

The ghost kernel. To understand the torsion part to be divided out, note that $(\overline{\mathcal{M}'_{C,x}})^{\text{gp}}$ at $x \in \underline{C}$ equals $P_x^{\text{gp}} \oplus_{\mathbb{Z}} \mathbb{Z}$ with $1 \in \mathbb{Z}$ mapping to $\overline{\rho}_x \in P_x^{\text{gp}}$ and to the base order $b \in \mathbb{Z}$, respectively. Since $\ell(\overline{\rho}_x)$ divides the multiplicities of some irreducible components of $\underline{X}_0$, Theorem 5.2.9,(3) implies $b/\ell(\overline{\rho}_x) \in \mathbb{N}$. If $\overline{\rho}_x$ has integral length $\ell(\overline{\rho}_x) > 1$, then $\big(\overline{\rho}_x/\ell(\overline{\rho}_x), -b/\ell(\overline{\rho}_x)\big)$ is a generator of the torsion subgroup $((\overline{\mathcal{M}'_{C,x}})^{\text{gp}})_{\text{tor}}$, which has order $\ell(\overline{\rho}_x)$. This element has to be in the kernel of the map to the torsion-free monoid $\overline{\mathcal{M}}_C$.

The $|G|$ embodiments of the ghost image. The interesting fact is that the lift of $\big(\overline{\rho}_x/\ell(\overline{\rho}_x), -b/\ell(\overline{\rho}_x)\big)$ to $(\mathcal{M}'_{C,x})^{\text{gp}}$ is only unique up to an $\ell(\overline{\rho}_x)$-torsion element in $\mathcal{O}^\times_{\underline{C},x}$, that is, up to an $\ell(\overline{\rho}_x)$-th root of unity $\zeta_x \in \Bbbk^\times$. Explicitly, the lift is equivalent to a choice $\sigma_x \in \mathcal{M}_x$ with $\sigma_x^{\ell(\overline{\rho}_x)} = \rho_x$ by taking the torsion subsheaf in $(\mathcal{M}'_C)^{\text{gp}}$ generated by $\big(\sigma_x, \tau_x^{-b/\ell(\overline{\rho}_x)}\big)$. The quotient by this subsheaf means that we upgrade the relation $f^\flat(\rho_x) = \tau_x^b$ coming from the commutativity of (5.3.2) to $f^\flat(\sigma_x) = \tau_x^{b/\ell(\overline{\rho}_x)}$.

To define this quotient of the monoid $\mathcal{M}'_C$ globally amounts to choosing the roots σ_x of ρ_x compatibly with the generization maps, leading to a global section of the sheaf $\mathcal{F}$ introduced directly before the statement of the theorem. For this statement note that for $x = \eta$ a generic point, $\ell(\overline{\rho}_\eta)$ equals the multiplicity μ_η of the irreducible component of $\underline{X}_0$ containing $\underline{f}(\eta)$.

The quotient is a logarithmic structure. Assume now $\sigma \in \Gamma(\mathcal{F})$ has been chosen and denote by $\mathcal{M}''_C$ the quotient of $\mathcal{M}'_C$ by the corresponding torsion subgroup of $(\mathcal{M}'_C)^{\mathrm{gp}}$. Since $\alpha'_C(\sigma_x) = \alpha'_C(\tau_x) = 0$, the homomorphism α'_C descends to the quotient, thus defining a structure homomorphism $\alpha''_C : \mathcal{M}''_C \to \mathcal{O}_C$.

The log structure $\mathcal{M}_C$ is determined at smooth points. Note that the map $(\pi^*\mathcal{M}_{O^\dagger})_\eta \to \mathcal{M}''_{C,\eta}$ is an isomorphism and hence we must have $\mathcal{M}_{C,\eta} = \mathcal{M}''_{C,\eta}$. The log structure $\mathcal{M}_C$ is then also defined at each marked point $p \in \underline{C}$ by adding a generator of the maximal ideal in $\mathcal{O}_{\underline{C},p}$ as an additional generator to $\mathcal{M}_{C,p}$. It is also clear that $\mathcal{M}''_{C,p} \to \mathcal{M}_{C,p}$ exists and is determined by the corresponding map at η and by $f^\sharp$.

The log structure $\mathcal{M}_C$ is determined at free nodes. At a free node q we have $\underline{f}(q) \in (X_0)_{\mathrm{reg}}$ and hence there is a unique specialization $\sigma_q \in \mathcal{M}_q$ of σ_η for the two generic points $\eta \in \underline{C}$ with $q \in \mathrm{cl}(\eta)$. The log structure $\mathcal{M}_C$ on C is then determined by $f_\eta^\flat$ and by the universal log structure $\mathcal{M}_C^\circ$ of $\underline{C}$ as follows. Let $x, y \in \mathcal{O}_{C,q}$ be coordinates of the two branches of $\underline{C}$ at q in the étale topology. Then there exist unique lifts $s_x, s_y \in \mathcal{M}_{C,q}^\circ$ such that $s_x \cdot s_y$ is the pull-back of a generator ϵ_q of the q-th factor in the universal base log structure $\mathrm{Spec}\big(\bigoplus_{\text{nodes of } \underline{C}} \mathbb{N} \to \Bbbk\big)$. Our choice of pull-back $O^\dagger \to \mathrm{Spec}\,(Q \to \Bbbk)$ turns ϵ_q into $\lambda \tau_0^{e_q}$ for some $\lambda \in \Bbbk^\times$ and $e_q \in \mathbb{N}$ determined by basicness as in (5.2.3). Thus $\mathcal{M}_{C,q}$ is generated by s_x, s_y and τ_q with single relation $s_x \cdot s_y = \lambda \tau_q^{e_q}$ and mapping to x, y and 0 under the structure homomorphism, respectively. The morphism $f^\flat : \mathcal{M}_q \to \mathcal{M}_{C,q}$ factors over $\pi^*\mathcal{M}_{O^\dagger}$ and is therefore completely determined by $f_\eta^\flat(\sigma_q) = \tau_q^{b/\mu_\eta}$.

Constrained nodes: Study of the image log structure $\mathcal{M}''_C$. It remains to extend $\mathcal{M}''_C$ to the correct log structure at each constrained node $q \in \underline{C}$. On the level of ghost sheaves we have the following situation, where we include the above description of the kernel for completeness.

Proposition 5.3.4. *The homomorphism of abelian groups*

$$P_q^{\mathrm{gp}} \oplus \mathbb{Z} \longrightarrow \overline{\mathcal{M}}_{C,q}^{\mathrm{gp}}, \quad (m, k) \longmapsto \overline{f}_q^\flat(m) + k \cdot \overline{\tau}_q,$$

has kernel generated by $\big(\overline{\rho}_q/\ell(\overline{\rho}_q), -b/\ell(\overline{\rho}_q)\big)$ *and cokernel a cyclic group of order* $\ell(u_q)$.

Proof. The kernel is described in the discussion above. Indeed, if $(m,k)\in P_q^{\mathrm{gp}}\oplus\mathbb{Z}$ lies in the kernel then $\overline{f}_q^\flat(m)\in\mathbb{Z}\cdot\overline{\tau}_q$. Because $\overline{f}_q^\flat$ is injective and $\overline{f}_q^\flat(\overline{\rho}_q)=\overline{\tau}_q^b$ we conclude that (m,k) is proportional to $(\overline{\rho}_q,-b)$. The stated element is a primitive element of this one-dimensional subspace.

For the determination of the cokernel observe that the composition

$$P_q^{\mathrm{gp}}\xrightarrow{\overline{f}_q^\flat}\overline{\mathcal{M}}_{C,q}^{\mathrm{gp}}\longrightarrow\overline{\mathcal{M}}_{C,q}^{\mathrm{gp}}/\mathbb{Z}\overline{\tau}_q\simeq\mathbb{Z}$$

equals u_q up to sign. Indeed, the quotient by $\mathbb{Z}\overline{\tau}_q$ maps the two generators of extremal rays of $\overline{\mathcal{M}}_{C,q}$ to $\pm1\in\mathbb{Z}$. Hence $m_i\in P_q^{\mathrm{gp}}$ maps to $\pm w_i$, which by Proposition 5.2.6, I,(2) agrees with $\pm u_q(m_i)$. The order of the cokernel now agrees with the greatest common divisor of the components of u_q, that is, with $\ell(u_q)$. ♠

Once again we follow [GS13], §1.3 and denote by $S_e\subset\mathbb{Z}^2$ the submonoid generated by $(e,0),(1,1),(0,e)$, for $e\in\mathbb{N}\smallsetminus\{0\}$. Up to a choice of ordering of extremal rays there is a canonical isomorphism

$$\overline{\mathcal{M}}_{C,q}\xrightarrow{\simeq}S_{e_q}. \tag{5.3.3}$$

Using Proposition 5.3.4 we can now determine the saturation of $\overline{\mathcal{M}}''_{C,q}$.

Corollary 5.3.5. *Using the description (5.3.3), the saturation of $\overline{\mathcal{M}}''_{C,q}$ equals $S_{\ell(u_q)e_q}\subset S_{e_q}$.*

Proof. By construction of $\mathcal{M}''_C$, the image of the homomorphism in Proposition 5.3.4 equals $(\overline{\mathcal{M}}''_{C,q})^{\mathrm{gp}}$. In the notation of (5.3.3), the statement now follows from the fact that by the proposition, the image has index $\ell(u_q)$ in $\overline{\mathcal{M}}_{C,q}^{\mathrm{gp}}$ and $(e_q,0)\in S_{e_q}$. Hence $(\ell e_q,0)\in(\overline{\mathcal{M}}''_{C,q})^{\mathrm{gp}}$, which together with $(1,1)\in(\overline{\mathcal{M}}''_{C,q})^{\mathrm{gp}}$ generates $S_{\ell e_q,0}\subset S_{e_q}$.

The saturation is then computed by taking all integral points in the real cone in $\overline{\mathcal{M}}_{C,q}^{\mathrm{gp}}\otimes_{\mathbb{Z}}\mathbb{R}$ spanned by $\overline{\mathcal{M}}''_{C,q}$. ♠

Constrained nodes: Extending the log structure.

In this step we extend the log structure to the saturation of $\mathcal{M}''_C$, described in Corollary 5.3.5. For readability we write $\ell=\ell(u_q)$.

Lemma 5.3.6. *The log structure $\alpha'':\mathcal{M}''_C\to\mathcal{O}_C$ extends uniquely to the saturation $(\mathcal{M}''_C)^{\mathrm{sat}}$.*

Proof. The saturation can at most be non-trivial at a constrained node q. By Corollary 5.3.5 we have an isomorphism $(\overline{\mathcal{M}}''_{C,q})^{\mathrm{sat}}\simeq S_{\ell e_q}$. The definition of the weights w_i implies $(w_1e_q,0),(0,w_2e_q)\in\overline{\mathcal{M}}''_{C,q}$, for the appropriate ordering of the branches of C at q. As a sanity check, notice that w_i divides $\ell=\ell(u_q)$ by Proposition 5.2.6,(I)(2). Let $\beta_q:\overline{\mathcal{M}}''_{C,q}\to\mathcal{O}_{C,q}$ be the composition of a choice of splitting $\overline{\mathcal{M}}''_{C,q}\to\mathcal{M}''_{C,q}$ and the structure morphism $\mathcal{M}''_{C,q}\to\mathcal{O}_{C,q}$. Then $\beta_q\big((w_1e_q,0)\big)$ vanishes at q to order w_1 on one branch of C and $\beta_q\big((0,w_2e_q)\big)$

vanishes to order w_2 on the other branch. Thus, étale locally there exist generators $x, y \in \mathcal{O}_{C,q}$ for the maximal ideal at q with

$$\beta_q\big((w_1 e_q, 0)\big) = x^{w_1}, \quad \beta_q\big((0, w_2 e_q)\big) = y^{w_2}.$$

Thus any extension β_q^{sat} of β_q to a chart for $(\mathcal{M}''_{C,q})^{\mathrm{sat}}$ has to fulfill

$$\beta_q^{\mathrm{sat}}\big((\ell e_q, 0)\big) = \zeta_1 \cdot x^\ell, \quad \beta_q^{\mathrm{sat}}\big((0, \ell e_q)\big) = \zeta_2 \cdot y^\ell, \tag{5.3.4}$$

with $\zeta_i \in \Bbbk$, $\zeta_i^{w_1/\ell} = 1$. On the other hand, the ζ_i are uniquely determined by compatibility of β_q^{sat} with β at the generic points of the two branches of C at q since $(\ell e_q, 0), (0, \ell e_q) \in \overline{\mathcal{M}}_{C,q}$. Conversely, with this choice of the ζ_i, the equations (5.3.4) provide the requested extension of α''. ♠

Finally, we extend $\mathcal{M}''_C$ to a log structure of a log smooth curve over the standard log point. The situation is largely the same as with admissible covers, see, e.g., [Moc95, §3].

Lemma 5.3.7. *Up to isomorphism of log structures over the standard log point, there are $\ell = \ell(u_q)$ pairwise non-isomorphic extensions $\alpha_q : \mathcal{M}_{C,q} \to \mathcal{O}_{C,q}$ of the image log structure $\alpha''_q : \mathcal{M}''_{C,q} \to \mathcal{O}_{C,q}$ at the constrained node q to a log structure of a log smooth curve.*

Proof. Let

$$\beta_q^{\mathrm{sat}} : S_{\ell e_q} \longrightarrow \mathcal{O}_{C,q}$$

be a chart for the log structure $(\mathcal{M}''_C)^{\mathrm{sat}}$ at q. The task is to classify extensions to a chart $\tilde\beta_q : S_{e_q} \to \mathcal{O}_{C,q}$ up to isomorphisms of induced log structures. Similar to the reasoning in Lemma 5.3.6, in terms of coordinates $x, y \in \mathcal{O}_{C,q}$ with

$$\beta_q^{\mathrm{sat}}\big((\ell e_q, 0)\big) = x^\ell, \quad \beta_q^{\mathrm{sat}}\big((0, \ell e_q)\big) = y^\ell,$$

we have to define

$$\tilde\beta_q\big((e_q, 0)\big) = \zeta_1 \cdot x, \quad \tilde\beta_q\big((0, e_q)\big) = \zeta_2 \cdot y, \tag{5.3.5}$$

with $\zeta_i \in \Bbbk$, $\zeta_i^\ell = 1$. Dividing out isomorphisms amounts to working modulo $\varphi \in \mathrm{Hom}(S_{e_q}, \mathbb{Z}/\ell)$ with $\varphi\big((1,1)\big) = 1$. In other words, we can change ζ_1, ζ_2 by $\zeta\zeta_1, \zeta^{-1}\zeta_2$ for any ℓ-th root of unity ζ. This leaves us with ℓ pairwise non-isomorphic extensions of the log structure at q. ♠

Counting non-rigidified lifts. For the final count we need to divide out the action of $\mathbb{Z}/b$ by composition with automorphisms of $\mathrm{Spec}(Q \to \Bbbk)$ over B. The stated count follows once we prove that this action is free. The action changes τ_0 to $\zeta \cdot \tau_0$ for ζ a b-th root of unity. For this change to lead to an isomorphic log structure $\mathcal{M}_C$ requires $\zeta_1\zeta_2 \in \Bbbk^\times$ in (5.3.5) to be unchanged at any constrained node $q \in \underline{C}$. This shows $\zeta^{e_q} = 1$ for all q. Similarly, for the map $\mathcal{M}_{X_0, \underline{f}(\eta)} \to \mathcal{M}_{C,\eta}$ to stay unchanged relative $\mathcal{M}_{B,b_0} \to \mathcal{M}_{O^\dagger,0}$ requires $f_\eta^\flat(\sigma_\eta) = \tau_\eta^{b/\mu_\eta}$ to stay unchanged. Thus also $\zeta^{b/\mu_\eta} = 1$ for all generic points $\eta \in \underline{C}$. But by Theorem 5.2.9 the base order b is the smallest natural number with all $e_q = b \cdot \mathrm{Ind}_q / \mu_2 w_1$ and all b/μ_η integers. Thus the e_q and b/μ_η have no common factor. This shows that $\zeta^{e_q} = 1$ and $\zeta^{b/\mu_\eta} = 1$ for all q, η implies $\zeta = 1$. We conclude that the action of $\mathbb{Z}/b$ is free as claimed. ♠

Remark 5.3.8. The obstruction to the existence of a logarithmic enhancement in Theorem 5.3.3 can be interpreted geometrically as follows.

Let $\bar{\mu}$ be a positive integer and $\underline{\tilde{B}} \to \underline{B}$ be the degree d cyclic cover branched with ramification index d over b_0. Let $\underline{\bar{X}} = \underline{X} \times_{\underline{B}} \underline{\tilde{B}}$, and let $\underline{\tilde{X}} \to \underline{\bar{X}}$ be the normalization, giving a family $\underline{\tilde{X}} \to \underline{\tilde{B}}$. It is a standard computation that the inverse image of a multiplicity μ irreducible component of $\underline{X}_0$ in $\underline{\tilde{X}}$ is a union of irreducible components of $\underline{\tilde{X}}_0$, each with multiplicity $\mu/\gcd(\mu, d)$.

At the level of log schemes, in fact $\bar{X}$ carries a fine but not saturated logarithmic structure via the description $\bar{X} = X \times_B \tilde{B}$ in the category of fine log schemes, while $\tilde{X}$ carries an fs logarithmic structure via the description $\tilde{X} = X \times_B \tilde{B}$ in the category of fs logarithmic structures. Here $\tilde{B}$ carries the divisorial logarithmic structure given by $\tilde{b}_0 \in \tilde{B}$, the unique point mapping to b_0.

Similarly, the central fibres are related as follows. The map $\tilde{B} \to B$ induces a morphism on standard log points $\tilde{b}_0 \to b_0$ induced by $\mathbb{N} \to \mathbb{N}, 1 \mapsto d$ for some integer d. Then $\bar{X}_0 = X_0 \times_{b_0} \tilde{b}_0$ in the category of fine log schemes, and $\tilde{X}_0 = X_0 \times_{b_0} \tilde{b}_0$ in the category of fs log schemes.

Given a transverse pre-logarithmic map $\underline{f} : \underline{C}/\operatorname{Spec} \Bbbk \to \underline{X}$, take the integer d above to be the positive integer b given by Theorem 5.2.9,(3). Then one checks readily that $\underline{f}$ has a logarithmic enhancement if and only if there is a lift $\underline{\tilde{f}} : \underline{C} \to \underline{\tilde{X}}_0$ of $\underline{f}$. Indeed, if $\underline{f}$ has a logarithmic enhancement $f : C/W \to X_0$ with W carrying the basic log structure, the morphism $W \to b_0$ factors through $\tilde{b}_0$ by the description of Theorem 5.2.9. Thus the universal property of fibred product gives a morphism $\tilde{f} : C \to \tilde{X}_0$. Conversely, given a lift, it follows again from the definition of b in Theorem 5.2.9 that the multiplicity μ of any irreducible component of $\underline{X}_0$ meeting $\underline{f}(\underline{C})$ divides b. So the multiplicity of any component of $\underline{\tilde{X}}_0$ meeting $\underline{\tilde{f}}(\underline{C})$ is 1 and by shrinking $\underline{X}_0$ we can assume that $\underline{\tilde{X}}_0$ is reduced. One also checks that the reduced branching order $\overline{w}_q$ associated to a node q is the same for $\underline{f}$ and $\underline{\tilde{f}}$, and thus $\underline{\tilde{f}}$ is still transverse pre-logarithmic. By Lemma 5.3.1 and Theorem 5.3.3, $\underline{\tilde{f}}$ has a logarithmic enhancement, and then the composed morphism $C \xrightarrow{\tilde{f}} \tilde{X}_0 \longrightarrow X_0$ gives the desired logarithmic enhancement of $\underline{f}$.

6. Examples

6.1. **The classical case.** Suppose $X \to B$ is a simple normal crossings degeneration with $X_0 = Y_1 \cup Y_2$ a reduced union of two irreducible components, with $Y_1 \cap Y_2 = D$ a smooth divisor in both Y_1 and Y_2. In this case, $\Sigma(X) = (\mathbb{R}_{\geq 0})^2$ and the map $\Sigma(X) \to \Sigma(B)$ is given by $(x, y) \mapsto x + y$, so that the generalized cone complex $\Delta(X)$ from §4.2 admits an affine-linear isomorphism with the unit interval $[0, 1]$, see Figure 1.

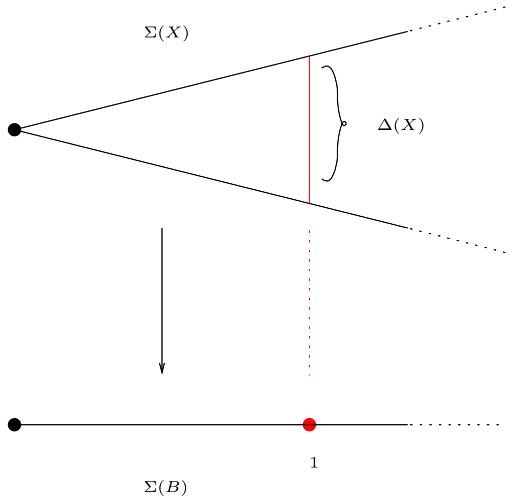

FIGURE 1. The cones $\Sigma(X)$ and $\Sigma(B)$ and the interval $\Delta(X)$

Proposition 6.1.1. *In the above situation, let $f : \Gamma \to \Delta(X)$ be a decorated tropical curve. Then f is rigid if and only if every vertex v of Γ maps to the endpoints of $\Delta(X)$ and every edge of Γ surjects onto $\Delta(X)$.*

Note that necessarily every leg of Γ is contracted, as $\Delta(X)$ is compact.

Proof. First note that if an edge E_q is contracted, then $u_q = 0$ and the length of the edge is arbitrary. By changing the length, one sees f is not rigid, see Figure 2 on the left.

Next, suppose v is a vertex with $f(v)$ lying in the interior of $\Delta(X)$. Identifying the latter with $[0, 1]$, we can view $u_q \in \mathbb{Z}$ for any q. Let $E_{q_1}, \ldots, E_{q_r}$ be the edges of Γ adjacent to v with lengths $\ell_1, \ldots, \ell_r$, oriented to point away from v. We can then write down a family f_t of tropical curves, t a real number close to 0, with $f = f_0$, $f_t(v') = f(v')$ for any vertex $v' \neq v$, and $f_t(v) = f(v) + t$. In doing so, we also need to modify the lengths of the edges E_{q_i}, as indicated in Figure 2 on the right. Any unbounded edge attached to v is contracted to $f_t(v)$. So f is not rigid. Thus if f is rigid, we see that all vertices of Γ map to endpoints of $\Delta(X)$, and any compact edge is not contracted, hence surjects onto $\Delta(X)$. The converse is clear.

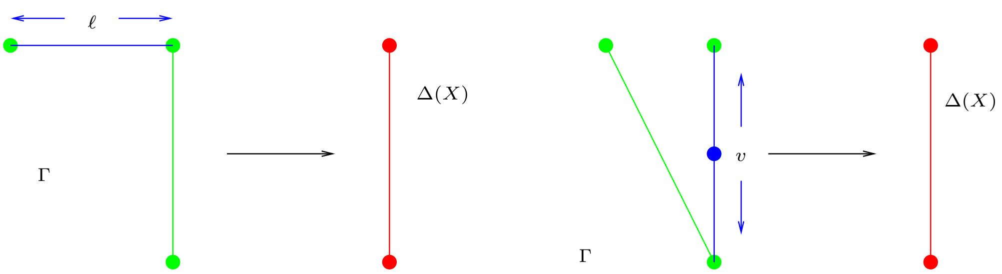

FIGURE 2. A graph with a contracted bounded edge or an interior vertex is not rigid.

A choice of decorated rigid tropical curve in this situation is then exactly what Jun Li terms an *admissible triple* in [Li02]. Indeed, by removing $f^{-1}(1/2)$ from Γ, one obtains two graphs (possibly disconnected) Γ_1, Γ_2 with legs and what Jun Li terms *roots* (the half-edges mapping non-trivially to $\Delta(X)$). The weights of a root, in Li's terminology, coincide with the absolute value of the corresponding u_q. The set I in the definition of admissible triple indicates which labels occur for unbounded edges mapping to, say, $0 \in \Delta(X)$. An illustration is given in Figure 3.

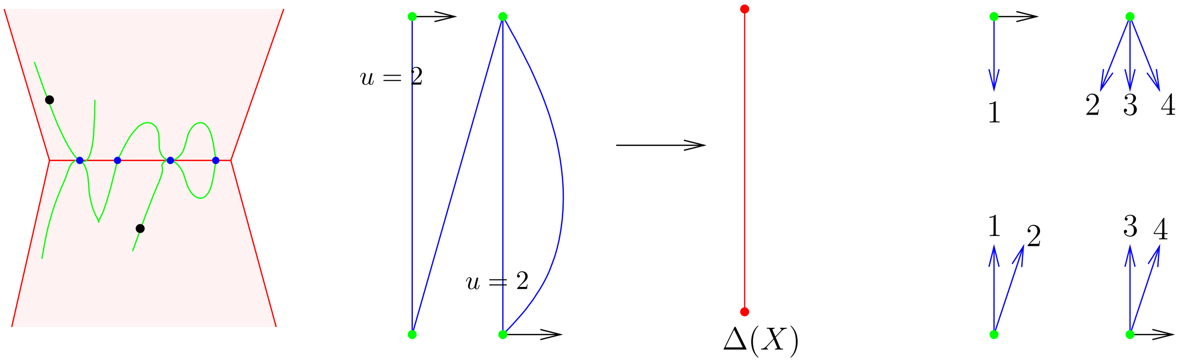

FIGURE 3. A rigid tropical curve is depicted with four edges and two legs, the latter corresponding to marked points with contact order 0. The corresponding admissible triple of Jun Li is depicted on the right, with roots corresponding to half-edges and legs corresponding to the legs of the original graph. The half-edges marked 1 and 3 have $u = 2$.

6.2. **Rational curves in a pencil of cubics.** It is well-known that if one fixes 8 general points in $\mathbb{P}^2$, the pencil of cubics passing through these 8 points contains precisely 12 nodal rational curves. Blowing up 6 of these 8 points, we get a cubic surface we denote $X_1' \subset \mathbb{P}^3$, and the enumeration of 12 nodal rational cubics translates to the enumeration of 12 nodal plane sections of X_1' passing through the remaining two points p_1, p_2.

We will give here a non-trivial demonstration of the decomposition formula by degenerating the cubic surface to a normal crossings union $H_1 \cup H_2 \cup H_3$ of three blown-up planes.

6.2.1. *Degenerating a cubic to three planes.* Using coordinates $x_0, \ldots, x_3$ on $\mathbb{P}^3$, consider a smooth cubic surface $X_1' \subset \mathbb{P}^3$ with equation

$$f_3(x_0, x_1, x_2, x_3) + x_1x_2x_3 = 0.$$

We then have a family $X' \to B = \mathbb{A}^1$ given by $X' \subseteq \mathbb{A}^1 \times \mathbb{P}^3$ defined by $tf_3 + x_1x_2x_3 = 0$. The fibre X_0' is the union of three planes $H_1' \cup H_2' \cup H_3'$. Pick two sections $p_1, p_2 : B \to X'$ such that $p_i(0) \in H_i'$. This can be achieved by choosing two appropriate points on the base locus $f_3(x_0, x_1, x_2, x_3) = x_1x_2x_3 = 0$.

6.2.2. *Resolving to obtain a normal crossings family.* The total space of X' is not a normal crossings family: it has 9 ordinary double points over $t = 0$, assuming f_3 is chosen generally: these are the points of intersection of the singular lines $H_i' \cap H_j'$ with $f_3 = 0$. One manifestation is the fact that H_i' are Weil divisors which are not Cartier. By blowing up H_1' followed by H_2', we resolve the ordinary double points. We obtain a family $X \to B$, which is normal crossings, hence logarithmically smooth, in a neighbourhood of $t = 0$, as depicted on the left in Figure 4. Denote by H_i the proper transform of H_i'.

We identify $\Sigma(X)$ with $(\mathbb{R}_{\geq 0})^3$, so that $\Delta(X)$ is identified with the standard simplex $\{(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0\}$, as depicted on the right in Figure 4.

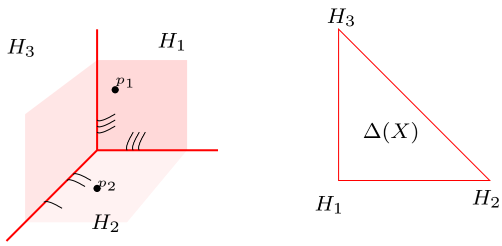

FIGURE 4. The left-hand picture depicts X_0 as a union of three copies of $\mathbb{P}^2$, blown up at 6, 3 or 0 points. The right-hand picture depicts $\Delta(X)$.

6.2.3. *Limiting curves: triangles.* Since the limit of plane curves on $X_t' = X_t$ should be a plane curve on X_0', limiting curves on X_0 would map to plane sections of X_0' through p_1, p_2. This greatly limits the possible limiting curves — in particular the image in each of H_i' is a line.

General triangles do not occur. It is easy to see that a plane section of X_0' passing through p_1, p_2 whose proper transform in X_0 is a triangle of lines cannot be the image of a stable logarithmic curve $C \to X_0$ of genus zero. Indeed, there would be a smooth point of $\underline{C}$ mapping to $(X_0)_{\text{sing}}$, contradicting Proposition 5.2.6,(II).

Triangles through double points. On the other hand, consider the total transform of a triangle in X_0' passing through p_1, p_2, and one of the 9 ordinary double points of X'. The resulting curve will be a cycle of 4 rational curves, one of the curves being part of the exceptional set of the blowup of H_1' and H_2'. We can partially normalize this curve at the node contained in the smooth part of X_0, getting a stable logarithmic curve of genus 0. See Figure 5 for one such case.

Tropical picture. We depict to the right the associated rigid tropical curve. Here the lengths of each edge are 1, and the contact data u_q take the values $(-1, 1, 0)$, $(0, -1, 1)$ and $(1, 0, -1)$. This accounts for 9 curves.

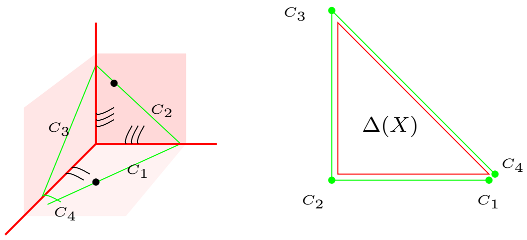

FIGURE 5. Proper transform of a triangle through a double point. The curve is normalized where C_1 and C_4 meet.

Logarithmic enhancement and logarithmic unobstructedness. Note that the above curves are transverse pre-logarithmic curves, and hence by Theorem 5.3.3, each of these curves has precisely one basic logarithmic enhancement. Since the curve is immersed it has no automorphisms. One can use a natural absolute, rather than relative, obstruction theory to define the virtual funamental class, which is governed by the logarithmic normal bundle. In this case each curve is unobstructed: since it is transverse with contact order 1, the logarithmmic normal bundle coincides with the usual normal bundle. The normal bundle restricts to $O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(1), O_{\mathbb{P}^1}(1)$, and $O_{\mathbb{P}^1}(-1)$ on the respective four components C_1, C_2, C_3 and C_4, hence it is non-special. We note that this does not account for the incidence condition that the marked points land at p_i. This can be arranged, for instance, using (6.3.1) in Section 6.3.2.

It follows that indeed each of these nine curves contributes precisely once to the desired Gromov-Witten invariant.

6.2.4. *Limiting curves: the plane section through the origin.* The far more interesting case is when the plane section of X_0' passes through the triple point. Then one has a stable map from a union of four projective lines, with the central component contracted to the triple point, see Figure 6 on the left.

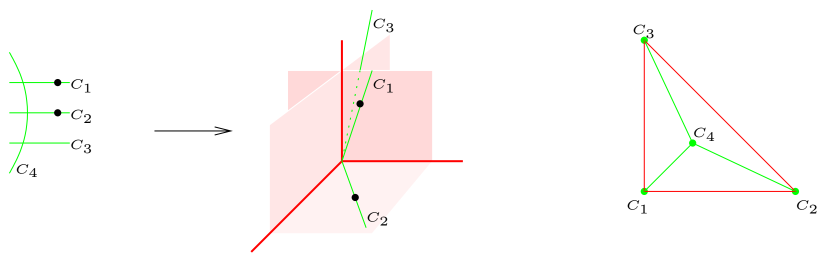

FIGURE 6. A curve mapping to a plane section through the origin, and its tropicalization.

There is in fact a one-parameter family $\underline{W}$ of such stable maps, as the line in H_3 is unconstrained and can be chosen to be any element in a pencil of lines. Only one member of this family lies in a plane, and we will see below that indeed only one member of the family admits a logarithmic enhancement.

Tropical picture. To understand the nature of such a logarithmic curve, we first analyze the corresponding tropical curve. The image of such a curve will be as depicted in Figure 6 on the right, with the central vertex corresponding to the contracted component landing somewhere in the interior of the triangle. However, the tropical balancing condition must hold at this central vertex, by [GS13], Proposition 1.14. ¿From this one determines that the only possibility is that the values of u_q of $(-2,1,1)$, $(1,1,-2)$ and $(1,-2,1)$, all lengths are $1/3$, and the central vertex is $(1/3,1/3,1/3)$. This rigid tropical curve Γ then has multiplicity $m_\Gamma = 3$ (Definition 4.3.1).

6.2.5. *Logarithmic enhancement using a logarithmic modification.* We now show that only one of the stable maps in the family $\underline{W}$ has a logarithmic enhancement. To do so, we use the techniques of §5, first refining $\Sigma(X)$ to obtain a logarithmic modification of X. The subdivision visible in Figure 6 gives a refinement of $\Sigma(X)$, the central star subdivision of $\Sigma(X)$. This corresponds to the ordinary blow-up $h : \tilde{X} \to X$ at the triple point of X_0. We may then identify logarithmic curves in $\tilde{X}$ and use the induced morphism $\mathscr{M}(\tilde{X}/B) \to \mathscr{M}(X/B)$.

Lifting the map to $\tilde{X}_0$. The central fibre $\tilde{X}_0$ is now as depicted in Figure 7. We then try to build a transverse pre-logarithmic curve in $\tilde{X}$ lifting one of the stable maps of Figure 6. Writing $C = C_1 \cup C_2 \cup C_3 \cup C_4$, with C_4 the central component, we map C_1 and C_2 to the lines L_1 and L_2 containing the preimages of p_1 and p_2, respectively, as depicted in Figure 7, while C_3 maps to some line L_3 in H_3. On the other hand, by (5.2.2) in the definition of transverse pre-logarithmic maps, C_4 must map to the exceptional $\mathbb{P}^2 = E$, which is of multiplicity 3, in such a way that it is triply tangent to ∂E precisely at the points of intersection with L_i, $i = 1,2,3$.

Uniqueness of liftable map. We claim that there is precisely one such map, necessarily with image containing a curve of degree 3 in the exceptional $\mathbb{P}^2$, with image as depicted in Figure 7. First, since $L_1 \cap E$, $L_2 \cap E$ are fixed, one can apply the tropical vertex [GPS10] to calculate the number of such maps as 1. One can also deduce this explicitly by considering linear series as follows. The three contact points on $C_4 \simeq \mathbb{P}^1$ can be taken to be $0, 1$ and ∞, and the map $C_4 \to \mathbb{P}^2$ corresponds, up to a choice of basis, to the unique linear system on $\mathbb{P}^1$ spanned by the divisors $3\{0\}$, $3\{1\}$ and $3\{\infty\}$. Since these points map to the coordinate lines, the choice of basis is limited to rescaling the defining sections. The choice of scaling of the defining sections results in fixing the images of 0 and 1, and the image point of ∞ is then uniquely determined.

This determines uniquely the point $L_3 \cap E$, in particular the line L_3 is determined. Thus we see that there is a unique transverse prelogarithmic map $\underline{f} : C \to \underline{\tilde{X}}_0$ such that $\underline{h} \circ \underline{f}$ lies in the family $\underline{W}$ of stable maps to X.

Logarithmic enhancement. Since the curve is rational, Theorem 5.3.3 assures the existence of a logarithmic enhancement. Only the exceptional component is non-reduced, of multiplicity $\mu = 3$ and for each node $q \in C$ we have $\mathrm{Ind}_q = 1$

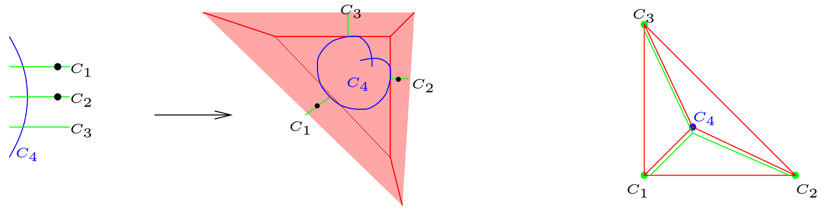

FIGURE 7. The lifted map. The middle figure is only a sketch: the nodal cubic curve C_4 meets each of the visible coordinate lines at one point with multiplicity 3. Moreover, these three points are collinear.

and $\overline{w}_q = 1/1 = 3/3 = 1$. Hence $b = 3$, $G = \mathbb{Z}/\mu = \mathbb{Z}/3$ and the count of Theorem 5.3.3 gives

$$\frac{|G|}{b}\prod_q \overline{w}_q = \frac{3}{3}\cdot 1^3 = 1$$

basic log enhancement of this transverse prelogarithmic curve. This gives one more basic stable logarithmic map $h\circ f$.

Unobstructedness. Once again we check that $h\circ f$ is unobstructed, if one makes use of an absolute obstruction theory: the logarithmic normal bundle has degree 0 on each line, hence degree 1 on C_4, and is non-special. Again the map has no automorphisms, which accounts for 1 curve, with multiplicity 3, because $m_\Gamma = 3$. Hence the final accounting is

$$9 + 3\times 1 \quad = \quad 12,$$

which is the desired result.

6.2.6. *Impossibility of other contributions.* Note our presentation has not been thorough in ruling out other possibilities for stable logarithmic maps, possibly obstructed, contributing to the total. For example, $\underline{W}$ includes curves where L_3 falls into the double point locus of X_0, but a more detailed analysis of the tropical possibilities rules out a possible log enhancement. We leave it to the reader to confirm that we have found all possibilities.

6.3. **Degeneration of point conditions.** We now consider a situation which is common in applications of tropical geometry; this includes tropical counting of curves on toric varieties [Mik05],[NS06]. We fix a pair (Y, D) where Y is a variety over a field $\Bbbk$ and D is a reduced Weil divisor such that the divisorial logarithmic structure on Y is logarithmically smooth over the trivial point $\operatorname{Spec}\Bbbk$. We then consider the trivial family

$$X = Y\times \mathbb{A}^1 \to \mathbb{A}^1 = B,$$

where now X is given the divisorial logarithmic structure with respect to the divisor $(D\times B)\cup(Y\times\{0\})$.

6.3.1. *Evaluation maps and moduli.* Fix a type β of stable logarithmic maps to X over B, getting a moduli space $\mathscr{M}(X/B,\beta)$. We assume now that the curves of type β have n marked points $p_1,\ldots,p_n$ with $u_{p_i}=0$ — and possibly some additional marked points $x_1,\ldots,x_m$ with non-trivial contact order with D. Given a stable map $(C/W,\mathbf{x},\mathbf{p},f)$, a priori for each i we have an evaluation map $\mathrm{ev}_i:(W,p_i^*\mathcal{M}_C)\to X$ obtained by restricting f to the section p_i. Noting that $u_{p_i}=0$, the map $\mathrm{ev}_i^\flat:(f\circ p_i)^{-1}\overline{\mathcal{M}}_X\to\overline{\mathcal{M}}_W\oplus\mathbb{N}$ factors through $\overline{\mathcal{M}}_W$, and thus we have a factorization $\mathrm{ev}_i:(W,p_i^*\mathcal{M}_C)\to W\to X$. In a slight abuse of notation we write ev_i for the morphism $W\to X$ also, and thus obtain a morphism

$$\mathrm{ev}:\mathscr{M}(X/B,\beta)\to X^n:=X\times_B X\times_B\times\cdots\times_B X.$$

If we choose sections $\sigma_1,\ldots,\sigma_n:B\to X$, we obtain a map

$$\sigma:=\prod_{i=1}^n\sigma_i:B\to X^n.$$

This allows us to define *the moduli space of curves passing through the given sections,*

$$\mathscr{M}(X/B,\beta,\sigma):=\mathscr{M}(X/B,\beta)\times_{X^n}B,$$

where the two maps are ev and σ.[9]

6.3.2. *Virtual fundamental class on $\mathscr{M}(X/B,\beta,\sigma)$.* We note that the moduli space $\mathscr{M}(X/B,\beta,\sigma)$ of curves passing through the given sections carries a virtual fundamental class. The perfect obstruction theory is defined by

$$E^\bullet=\big(R\pi_*[f^*\Theta_{X/B}\to\bigoplus_{i=1}^n(f^*\Theta_{X/B})|_{p_i(W)}]\big)^\vee,\tag{6.3.1}$$

for the stable map $(\pi:C\to W,\mathbf{x},\mathbf{p},f)$. Here the map of sheaves above is just restriction.

6.3.3. *Choice of sections and $\Delta(X)$.* We can now use the techniques of previous sections to produce a virtual decomposition of the fibre over $b_0=0$ of $\mathscr{M}(X/B,\beta,\sigma)\to B$. However, to be interesting, we should in general choose the sections to interact with D in a very degenerate way over b_0. In particular, restricting to b_0 (which is now the standard log point), we obtain maps

$$\sigma_i:b_0\to Y^\dagger,$$

where $Y^\dagger=Y\times O^\dagger$ is the product with the standard log point. Note that

$$\Sigma(Y^\dagger)=\Sigma(X)=\Sigma(Y)\times\mathbb{R}_{\geq0},$$

with $\Sigma(X)\to\Sigma(B)$ the projection to the second factor. So $\Delta(X)=\Sigma(Y)$ and $\Sigma(\sigma_i):\Sigma(B)\to\Sigma(X)$ is a section of $\Sigma(X)\to\Sigma(B)$ and hence is determined by a point $P_i\in\Delta(X)$, necessarily rationally defined.

[9]Recall that all fibred products are in the category of fs log schemes.

6.3.4. *Tropical fibred product.* We wish to understand the fibred product $\mathscr{M}(X/B,\beta,\sigma) := \mathscr{M}(X/B,\beta)\times_{X^n} B$ at a tropical level. We observe

Proposition 6.3.5. *Let X, Y and S be fs log schemes, with morphisms $f_1 : X \to S$, $f_2 : Y \to S$. Let $Z = X \times_S Y$ in the category of fs log schemes, p_1, p_2 the projections. Suppose $\bar z \in Z$ with $\bar x = p_1(\bar z)$, $\bar y = p_2(\bar z)$, and $\bar s = f_1(p_1(\bar z)) = f_2(p_2(\bar z))$. Then*

$$\mathrm{Hom}(\overline{\mathcal{M}}_{Z,\bar z},\mathbb{N}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar x},\mathbb{N}) \times_{\mathrm{Hom}(\overline{\mathcal{M}}_{S,\bar s},\mathbb{N})} \mathrm{Hom}(\overline{\mathcal{M}}_{Y,\bar y},\mathbb{N})$$

and

$$\mathrm{Hom}(\overline{\mathcal{M}}_{Z,\bar z},\mathbb{R}_{\geq 0}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar x},\mathbb{R}_{\geq 0}) \times_{\mathrm{Hom}(\overline{\mathcal{M}}_{S,\bar s},\mathbb{R}_{\geq 0})} \mathrm{Hom}(\overline{\mathcal{M}}_{Y,\bar y},\mathbb{R}_{\geq 0}).$$

Proof. The first statement follows immediately from the universal property of fibred product applied to maps $\bar z^\dagger \to Z$, where $\bar z^\dagger$ denotes the geometric point $\bar z$ with standard logarithmic structure. The second statement then follows from the first. ♠

6.3.6. *Tropical moduli space.* We now see a simple interpretation for the tropicalization of $W := \mathscr{M}(X/B,\beta,\sigma)$. If $\bar w \in W$ is a geometric point, let Q be the basic monoid associated with $\bar w$ as a stable logarithmic map to X. Then by Proposition 6.3.5, we have

$$\mathrm{Hom}(\overline{\mathcal{M}}_{W,\bar w},\mathbb{R}_{\geq 0}) = \mathrm{Hom}(Q,\mathbb{R}_{\geq 0}) \times_{\prod_i \mathrm{Hom}(P_{p_i},\mathbb{R}_{\geq 0})} \mathbb{R}_{\geq 0}.$$

Here as usual $P_{p_i} = \overline{\mathcal{M}}_{X,\underline{f}(p_i)}$. The maps defining the fibred product are as follows. The map $\mathrm{Hom}(Q,\mathbb{R}_{\geq 0}) \to \prod_i \mathrm{Hom}(P_{p_i},\mathbb{R}_{\geq 0})$ can be interpreted as taking a tropical curve $\Gamma \to \Sigma(X)$ to the point of $\mathrm{Hom}(P_{p_i},\mathbb{R}_{\geq 0})$ which is the image of the contracted edge corresponding to the marked point p_i. The map $\mathbb{R}_{\geq 0} \to \prod_i \mathrm{Hom}(P_{p_i},\mathbb{R}_{\geq 0})$ is $\prod_i \Sigma(\sigma_i)$ and hence takes 1 to $(P_1,\ldots,P_n)$.

This yields:

Proposition 6.3.7. *Let $m \in \Delta(W)$, and let $\Gamma_C = \Sigma(\pi)^{-1}(m)$. Then $\Sigma(f) : \Gamma_C \to \Delta(X)$ is a tropical curve with the unbounded edges E_{p_i} being mapped to the points P_i. Furthermore, as m varies within its cell of $\Delta(W)$, we obtain the universal family of tropical curves of the same combinatorial type mapping to $\Delta(X)$ and with the edges E_{p_i} being mapped to P_i.*

6.3.8. *Restatement of the decomposition formula.* Denote

$$\mathscr{M}(Y^\dagger/b_0,\beta,\sigma) := \mathscr{M}(X_0/b_0,\beta)\times_{X^n} B$$

and for $\mathbf{A} \vdash A$

$$\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0,\beta,\sigma) := \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)\times_{X^n} B.$$

Theorem 1.1.2 now translates to the following:

Theorem 6.3.9 (The logarithmic decomposition formula for point conditions)**.** *Suppose Y is logarithmically smooth. Then*

$$[\mathscr{M}(Y^\dagger/b_0,\beta,\sigma)]^{\mathrm{virt}} = \sum_{\tau\in\Omega} m_\tau \cdot \sum_{\mathbf{A}\vdash A} (i_{\tau,\mathbf{A}})_*[\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0,\beta,\sigma)]^{\mathrm{virt}}$$

Example 6.3.10. The above discussion allows a reformulation of the approach of [NS06] to tropical counts of curves in toric varieties. Take Y to be a toric variety with the toric logarithmic structure, and fix a homology class β of curve and a genus g. By fixing an appropriate number n of points in Y, one can assume that the moduli space of curves of genus g and class β passing through these points has expected dimension 0. Then after choosing suitable degenerating sections $\sigma_1,\ldots,\sigma_n$, one obtains points $P_1,\ldots,P_n\in\Sigma(Y)$, the fan for Y. Then the question of understanding $\mathscr{M}(X/B,\beta,\sigma)$ is reduced to an analysis for each rigid tropical curve in $\Sigma(Y)$ with the correct topology. In particular, the domain curve should have genus g (taking into account the genera assigned to each vertex) and should have $D_\rho\cdot\beta$ unbounded edges parallel to a ray $\rho\in\Sigma(Y)$, where $D_\rho\subseteq Y$ is the corresponding divisor. The argument of [NS06] essentially carries out an explicit analysis of possible logarithmic curves associated with each such rigid curve after a log blow-up $\tilde{Y}^\dagger\to Y^\dagger$.

6.4. **An example in $\mathbb{F}_2$.** We now consider a very specific case of the previous subsection. This example deliberately deviates slightly from the toric case mentioned above and exhibits new phenomena.

6.4.1. *A non-toric logarithmic structure on a Hirzebruch surface.* Let Y be the Hirzebruch surface $\mathbb{F}_2$. Viewed as a toric surface, it has 4 toric divisors, which we write as f_0, f_∞, C_0 and C_∞. Here f_0,f_∞ are the fibres of $\mathbb{F}_2\to\mathbb{P}^1$ over 0 and ∞, C_0 is the unique section with self-intersection -2, and C_∞ is a section disjoint from C_0, with C_∞ linearly equivalent to $f_0+f_\infty+C_0$.

We will give Y the (non-toric) divisorial logarithmic structure coming from the divisor $D=f_0+f_\infty+C_\infty$.

6.4.2. *The curves and their marked points.* We will consider rational curves representing the class C_∞ passing through 3 points y_1,y_2,y_3. Of course there should be precisely one such curve.

A general curve of class C_∞ will intersect D in four points, so we will set this up as a logarithmic Gromov-Witten problem by considering genus 0 stable logarithmic maps

$$f:(C,p_1,p_2,p_3,x_1,x_2,x_3,x_4)\to Y,$$

imposing the condition that $f(p_i)=y_i$, and f is constrained to be transversal to f_0,f_∞,C_∞ and C_∞ at x_i for $i=1,\ldots,4$ respectively. This transversality determines the vectors u_{x_i}, while we take the contact data $u_{p_i}=0$.

Since the maps have the points x_3 and x_4 ordered, we expect the final count to amount to 2 rather than 1.

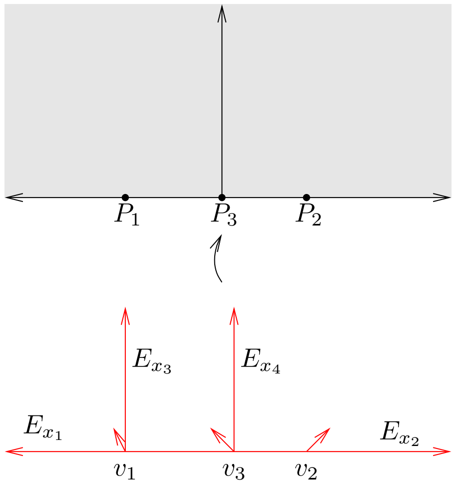

FIGURE 8. The polyhedral complex $\Delta(X) = \Sigma(Y)$ and a potential tropical map. The small arrows indicate E_{p_i} which are contracted to P_i. The suggested positions of E_{x_3} and E_{x_4} are shown below to contribute 0 to the virtual count.

6.4.3. *Choice of degeneration.* We will now see what happens when we degenerate the point conditions as in §6.3, by taking $X = Y \times \mathbb{A}^1$ and considering sections $\sigma_i : \mathbb{A}^1 \to X$, $1 \leq i \leq 3$. We choose these sections to be general subject to the condition that

$$\sigma_1(0) \in f_0, \quad \sigma_2(0) \in f_\infty, \quad \sigma_3(0) \in C_0.$$

Since $C_0 \cap C_\infty = \emptyset$, any curve in the linear system $|C_\infty|$ which passes through this special choice of 3 points must contain C_0, and hence be the curve $f_0 + f_\infty + C_0$.

6.4.4. *The complex $\Delta(X)$ and the tropical sections.* Note that $\Delta(X)$ is as depicted in Figure 8, an abstract gluing of two quadrants, not linearly embedded in the plane. The choice of sections σ_i determines points $P_i \in \Sigma(X)$ as explained in §6.3. For example, if, say, the section σ_1 is transversal to $f_0 \times \mathbb{A}^1$, then P_1 is the point at distance 1 from the origin along the ray corresponding to f_0. Since C_0 is not part of the divisor determining the logarithmic structure, P_3 is in fact the origin.

6.4.5. *The tropical curves.* One then considers rigid decorated tropical curves passing through these points.

- The curves must have 7 unbounded edges, E_{p_i}, E_{x_j}.
- The map contracts E_{p_i} to P_i.
- Each E_{x_j} is mapped to an unbounded ray going to infinity in the direction indicating which of the three irreducible components of D the point x_j is mapped to.

6.4.6. *Rigid tropical curves.* It is then easy to see that to be rigid, the tropical curve must have three vertices, v_1, v_2, v_3, with the edge E_{p_i} attached to v_i and v_i necessarily being mapped to P_i.

The location of the E_{x_i} is less clear. One can show using the balancing condition [GS13], Proposition 1.15, that E_{x_1} must be attached to v_1 and E_{x_2} must be attached to v_2. There remains, however, some choice about the location of E_{x_3} and E_{x_4}. Indeed, they may be attached to the vertices v_1, v_2 or v_3 in any manner. Figure 8 shows one such choice.

6.4.7. *Decorated rigid tropical curves.* We must however consider *decorated* rigid tropical curves, and in particular we need to assign curve classes $\beta(v)$ to each vertex v. Let n_i be the number of edges in $\{E_{x_3}, E_{x_4}\}$ attached to the vertex v_i. Since E_{x_3} and E_{x_4} indicate which "virtual" components of the domain curve have marked points mapping to C_∞, it then becomes clear that the class associated to v_1 and v_2 must be $n_1 f$ and $n_2 f$ respectively, while the class associated to v_3 must be $C_0 + n_3 f$, where $n_1 + n_2 + n_3 = 2$.

6.4.8. *The seeming contradiction.* In fact, as we shall see shortly, there are logarithmic curves whose tropicalization yields any one of the curves with $n_1 = n_2 = 1$, and there is no logarithmic curve *over the standard log point* whose tropicalization is the tropical curve with $n_3 = 2$. Suprisingly at first glance, in fact the only decorated rigid tropical curve which provides a non-trivial contribution to the Gromov-Witten invariant is the one which can not be realised, with $n_3 = 2$. We will also see that the case $n_1 = 2$ or $n_2 = 2$ plays no role.

6.4.9. *Curves with $n_1 = n_2 = 1$ contribute 0.* To explain this seemingly contradictory conclusion, first recall the standard fact that there is a flat family $\underline{\mathcal{X}} \to \underline{\mathbb{A}}^1$ such that $\underline{\mathcal{X}}_0 \cong \mathbb{F}_2$ and $\underline{\mathcal{X}}_t \cong \underline{\mathbb{P}}^1 \times \underline{\mathbb{P}}^1$ for $t \neq 0$. Furthemore, the divisor $f_0 \cup f_\infty \cup C_\infty$ extends to a normal crossings divisor on $\underline{\mathcal{X}}$ with three irreducible components: $\{0\} \times \mathbb{P}^1, \{\infty\} \times \mathbb{P}^1$, and a curve of type $(1,1)$. This endows $\underline{\mathcal{X}}$ with a divisorial logarithmic structure, logarithmically smooth over $\mathbb{A}^1$ with the trivial logarithmic structure. However, no curve of class C_0 or $C_0 + f$ in $\mathcal{X}_0$ deforms to $\mathcal{X}_t$ for $t \neq 0$. Hence no curve representing a point in the moduli space $\mathscr{M}_\tau$ for τ one of the decorated rigid tropical curves with $n_3 \leq 1$ deforms. The usual deformation invariance of Gromov-Witten invariants then implies that the contribution to the Gromov-Witten invariant from such a τ is zero.

The fact that this contribution is 0 can also be deduced from our formalism of the gluing formula using punctured maps in [ACGS17].

6.4.10. *Expansion and description of moduli space for $n_1 = n_2 = 1$.* To explore the existence of the relevant logarithmic curves, we again turn to §5. First let us construct a curve whose decorated tropical curve has $n_1 = n_2 = 1$. The image of this curve in $\Delta(X)$ yields a subdivision of $\Delta(X)$ which in turn yields a refinement of $\Sigma(X)$, and hence a log étale morphism $\tilde{X} \to X$. It is easy to see that this is just a weighted blow-up of $f_0 \times \{0\}$ and $f_\infty \times \{0\}$ in $X = Y \times \mathbb{A}^1$; the weights

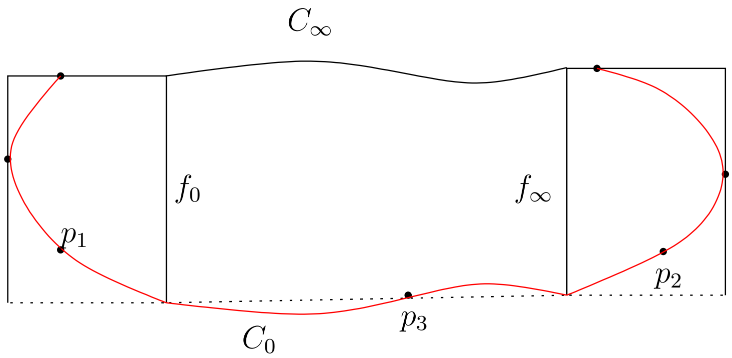

FIGURE 9.

depend on the precise location of P_1 and P_2, but if they are taken to have distance 1 from the origin, the subdivision will correspond to an ordinary blow-up. The central fibre is now as depicted in Figure 9, with the proper transforms of the sections meeting the central fibre at the points p_1, p_2, p_3 as depicted.

The logarithmic curve then has three irreducible components, one mapping to C_0 and the other two mapping to the two exceptional divisors, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. These latter two components each map isomorphically to a curve of class $(1,1)$ on the exceptional divisor, and is constrained to pass through p_i and the point where C_0 meets the exceptional divisor. There is in fact a pencil of such curves. We remark that all 7 marked points are visible in Figure 9, but the curves in the exceptional divisors meet the left-most and right-most curves transversally, and not tangent as it appears in the picture. By Theorem 5.3.3, any such stable map then has a log enhancement, and composing with the map $\tilde{X} \to X$ gives a stable logarithmic map over the standard log point whose tropicalization is one of the rigid curves with $n_1 = n_2 = 1$.

One can show that the relevant moduli space in $\tilde{X}_0$ has two components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, depending on which sides x_3 and x_4 lie. The virtual fundamental class of each component is in fact the top Chern class of the rank-2 trivial bundle, namely 0. This moduli space maps injectively to the moduli space of X_0.

6.4.11. *Curves with $n_1 = n_2 = 0$.* Now consider the case that $n_1 = n_2 = 0$ and $n_3 = 2$. This rigid tropical curve cannot be realised as the tropicalization of a stable logarithmic map over the standard log point. Indeed, to be realised, the curve must have an irreducible component of class $C_0 + 2f = C_\infty$, and we know there is no such curve passing through $\sigma_3(0)$, a general point on C_0. However, this tropical curve can in fact be realised as a degeneration of another tropical curve, as depicted in Figure 10.

To construct an actual logarithmic curve, we use refinements again. Assume for simplicity of the discussion that P_1 and P_2 have been taken to have distance 2 from the origin. Subdivide $\Delta(X)$ by introducing vertical rays with endpoints P_1 and P_2, and in addition introduce vertical rays which are the images E_{x_3} and

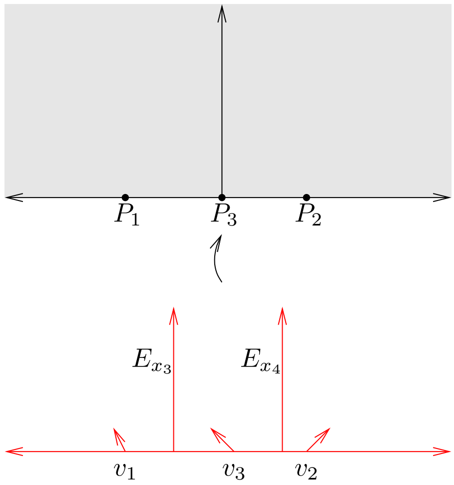

FIGURE 10.

E_{x_4}; again for simplicity of the discussion take the endpoints of these rays to be at distance 1 from the origin.

This corresponds to a blow-up $\tilde{X} \to X$ involving four exceptional components, and Figure 11 shows the central fibre of $\tilde{X} \to \mathbb{A}^1$, along with the image of a stable logarithmic map which tropicalizes appropriately (once again the curves on the second and fourth components of $\tilde{X}$ meet the first and fifth components with order 1, and no tangency). Composing this stable logarithmic map with $\tilde{X} \to X$ then gives a non-basic stable logarithmic map to X over the standard log point. It is not hard to see that the corresponding basic monoid Q has rank 3, parameterizing the image of the curve in $\Sigma(B)$ as well as the location of the edges E_{x_3} and E_{x_4}. The degenerate tropical curve where the edges E_{x_3} and E_{x_4} are attached to the vertex v_3 represents a one-dimensional face of $Q^\vee$, so the rigid tropical curve with $n_3 = 2$ does appear in the family $Q^\vee$, but only as a degeneration of a tropical curve which is realisable by an actual stable logarithmic curve over the standard log point.

One can again show that the relevant moduli space in $\tilde{X}_0$ has two components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This time the virtual fundamental class of each component is the top Chern class of $O(1) \boxplus O(1)$, which has degree 1. In contrast the corresponding moduli space $\mathscr{M}_{n_i=0}(X_0)$ is discrete.

6.4.12. *Curves with* $n_i = 2$. To complete the analysis, we end by noting that the case $n_1 = 2$ or $n_2 = 2$ cannot occur. Consider the case $n_1 = 2$. Any stable logarithmic curve over the standard log point with a tropicalization which degenerates to such a rigid tropical curve must have a decomposition into unions of irreducible components corresponding to the vertices v_1, v_2 and v_3, with the homology class of the image of the stable map restricted to each of these unions of irreducible components being $2[f_0]$, 0 and $[C_0]$ respectively. In particular, this

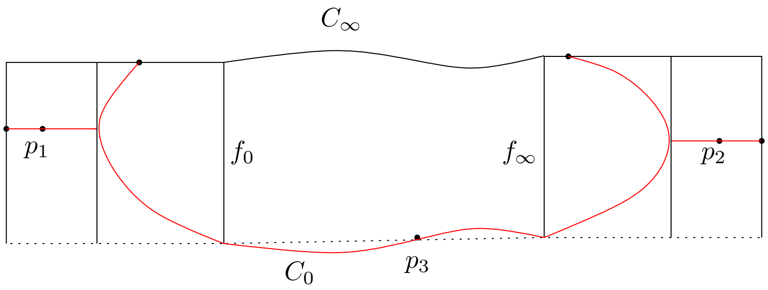

FIGURE 11.

will prevent the possibility of having any irreducible component whose image contains $\sigma_2(0)$. Thus this case does not occur.

6.4.13. *Deformation invariance, toric maps, and fundamental cycles.* Recall that logarithmic Gromov–Witten invariants are deformation invariant, therefore the number 2 of stable logarithmic maps is calculated equally well when p_i map to general points as when they specialize. When the images of p_i are in general position there are precisely two maps - with identical curves but the points x_3, x_4 reordered, which are necessarily unobstructed. The virtual fundamental class at b_0 is represented by the limiting cycle, which corresponds to two stable logarithmic maps on X. We claim that this cycle consists of two stable logarithmic maps with $n_1 = n_2 = 1$. One is bound to ask — how is this possible?

First, let us argue that indeed this is the case. Here the toric picture is helpful since it is unobstructed. Let Y^t be the Hirzebruch surface with its toric logarithmic structure coming from the divisor $f_0 + f_\infty + C_0 + C_\infty$, and let $Y^t \to Y$ be the natural logarithmic map where C_0 is left out. The two general stable logarithmic maps corresponding to p_i generic are disjoint from C_0, hence they lift to Y^t. Using the formalism of [NS06], one can choose the points P_1, P_2, P_3 so that the limiting curves on Y^t have precisely three components and compose to two maps having $n_1 = n_2 = 1$ on Y.

Next, let us explain why this does not constitute a contradiction within mathematics, and what we must learn from this example.

First, the moduli space $\mathscr{M}(Y^\dagger/b_0, \beta, \sigma)$ with $Y^\dagger = Y \times O^\dagger = X_0$ has four relevant components, two corresponding to curves with $n_1 = n_2 = 1$ and two corresponding to curves with $n_1 = n_2 = 0$. It has virtual fundamental class represented by the 0-cycle representing the two limiting stable logarithmic maps, which happens to lie on the components with $n_1 = n_2 = 1$. It is also the direct image of the class $[\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)]^{\mathrm{virt}}$ on the moduli space of curves $\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$ mapping to the locus of stable logarithmic maps with $n_1 = n_2 = 1$. But there is nothing here to guarantee that the limiting cycle is the image of any cycle coming from $\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$, and in fact this is not the case here. (One can obtain the limiting cycle as the image with a different choice of point constraints σ.) Figure 12 describes the situation.

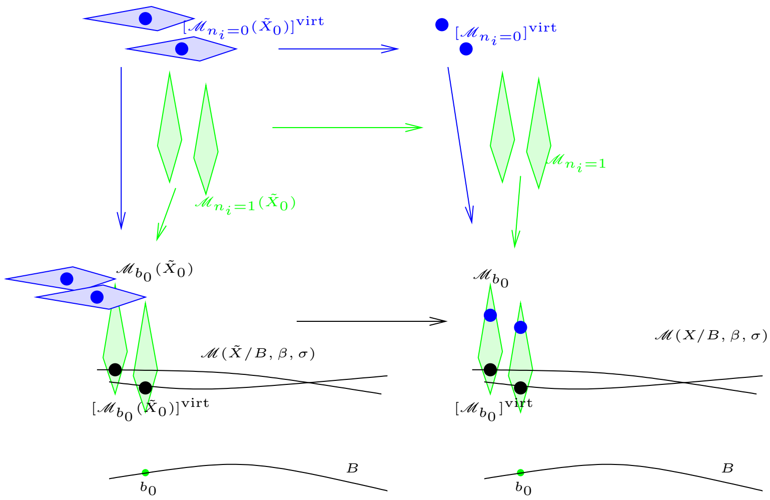

FIGURE 12. On the right: the fibre $\mathscr{M}(Y^\dagger/b_0,\beta,\sigma)$ of the moduli space $\mathscr{M}(X/\mathbb{A}^1,\beta,\sigma)$ and its virtual fundamental class. The class is represented by the limiting cycle marked by $[\mathscr{M}_{b_0}]^{\mathrm{virt}}$, but also by the image of the class $[\mathscr{M}_{n_i=0}]^{\mathrm{virt}}$. On the left: the corresponding picture for the expansion $\tilde{X}$.

Second, we point out that the virtual decomposition is not compatible with the map $Y^t \to Y$ from the toric logarithmic structure.

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