ON THE REAL LOCUS IN THE KATO-NAKAYAMA SPACE OF LOGARITHMIC SPACES WITH A VIEW TOWARD TORIC DEGENERATIONS

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Abstract. We study the real loci of toric degenerations of complex varieties with reducible central fibre, as introduced in the joint work of the second author with Mark Gross on mirror symmetry. The topology of such degenerations can be explicitly described via the Kato-Nakayama space of the central fibre as a log space. The paper provides generalities of real structures in log geometry and their lift to Kato-Nakayama spaces, the description of the Kato-Nakayama space of a toric degeneration and its real locus, as bundles determined by tropical data. Examples include real toric degenerations of K3-surfaces.

Contents

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Introduction.

We study real structures in the toric degenerations introduced by Gross and the second author in the context of mirror symmetry [GS1], [GS3]. A *toric degeneration* in this sense is a degeneration of algebraic varieties $\delta : \mathfrak{X} \to T = \operatorname{Spec} R$ with R a discrete valuation ring and with central fibre $X_0 = \delta^{-1}(0)$ a union of toric varieties, glued pairwise along toric divisors. Here $0 \in \operatorname{Spec} R$ is the closed point. We also require that δ is toroidal at the zero-dimensional toric strata, that is, étale locally near these points, δ is given by a monomial equation in an affine toric variety. For an introductory survey of toric degenerations see [GS4].

Probably the most remarkable aspect of toric degenerations is that they can be produced *canonically* from the central fibre X_0 and some residual information on the family $\mathfrak{X}$, captured by what is called a *log structure*. While the reconstruction is done by an inductive procedure involving a *wall structure* [GS3], and is typically impossible to carry through in practice, many features of the family are already contained in the log structure. A simple characterization of the nature of the log structure in the present situation has been given in [GS1], Theorem 3.27. It says that if at a general point of the singular locus of X_0 where two irreducible components meet, $\mathfrak{X}$ is given as $xy = f \cdot t^e$ with $t \in R$ generating the maximal ideal, the log structure captures $e \in \mathbb{N}$ and the restriction of f to $x = y = 0$.

An important feature of the log structure for the present paper is that the topology of the degeneration can be read off canonically. Indeed, just as for any logarithmic space over the complex numbers, to X_0 there is a canonically and functorially associated topological space X_0^{KN}, its *Kato-Nakayama space* or *Betti realization* [KN]. It comes with a continuous map to the analytic space X_0^{an} associated to X_0. Moreover, in the present case, there is a map $X_0^{\mathrm{KN}} \to S^1$, coming from functoriality and the fact that the closed point in T as a divisor also comes with a log structure, with S^1 its Kato-Nakayama space. Now it follows from the main result of [NO] that the map $X_0^{\mathrm{KN}} \to S^1$ is homeomorphic to the preimage under δ of a small circle about 0 in T. See the discussion at the beginning of §4.2 for details. In particular, by restricting to the fibre over, say $1 \in S^1$, we obtain a topological space $X_0^{\mathrm{KN}}(1)$ homeomorphic to a general fiber $\mathcal{X}_t$ of an analytic model $\mathcal{X}$ of the degeneration $\mathfrak{X}$.

Our primary interest in this paper are real structures in X_0 and their lift to X_0^{KN}. The main reason for being interested in real structures in this context is that the real locus produces natural Lagrangian submanifolds on any complex projective manifold defined over $\mathbb{R}$. Thus assuming the analytic model $\mathcal{X}$ is defined over $\mathbb{R}$, it comes with a natural family of degenerating Lagrangian submanifolds. Again we can study these Lagrangians by means of their analogues in X_0^{KN}. Note that if X_0 is defined over $\mathbb{R}$

and the functions f on the double locus defining the log structure are as well, then the canonical family $\mathfrak{X}$ is already defined over $\mathbb{R}$, see [GS3], Theorem 5.2.

Once we have a real Lagrangian $L \subset \mathcal{X}_t$, a holomorphic disc with boundary on L glues with its complex conjugate to a rational curve $C \subset \mathcal{X}_t$ with a real involution. Real rational curves are amenable to techniques of algebraic geometry and notably of log Gromov-Witten theory of the central fibre X_0. Thus real Lagrangians provide an algebraic-geometric path to open Gromov-Witten invariants and the Fukaya category. See [So],[PSW] and [FOOO] for previous work in this direction without degenerations.

In Section 1 we introduce the straightforward notion of a real structure on a log space along with basic properties. Our main example is the central fibre of a degeneration defined over $\mathbb{R}$, with its natural log structure. In Section 2 we recall the definition of the Kato-Nakayama space X^{KN} over a log scheme X as a topological space along with some properties needed in later sections. We then show that the real involution on a real log scheme lifts canonically to its Kato-Nakayama space (Section 3). Section 4 is devoted to the toric degeneration setup. We describe the Kato-Nakayama space as glued from standard pieces, torus bundles over the momentum polytopes of the irreducible components of the central fibre $X_0 \subset \mathfrak{X}$, and in terms of global monodromy data. Under the presence of a real structure we give a similar description for the real locus. For real structures inducing the standard real structure on each toric irreducible component of X_0, the real locus in the Kato-Nakayama space of X_0 is a branched cover of the union B of momentum polyhedra, the integral affine manifold of half the real dimension of a general fibre governing the inductive construction of $\mathfrak{X}$. For a concrete example we study the case of a toric degeneration of quartic $K3$ surfaces, reproducing a result of Castaño-Bernard and Matessi [CBM] on the topology of the real locus of an SYZ-fibration with compatible real involution in our setup.

Conventions. We work in the category of log schemes of finite type over $\mathbb{C}$ with log structures in the étale topology, but use the analytic topology from Section 2 on. Similar discussions are of course possible in the categories of algebraic log stacks over $\mathbb{C}$ or of complex analytic log spaces. Throughout this paper we assume basic familiarity with log geometry at the level of [Kf]. For more details we encourage the reader to also look at [Kk], [O]. The structure homomorphism of a log space $(X, \mathcal{M}_X)$ is denoted $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$, or just α if X is understood. The standard log point $(\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^\times)$ is denoted by $O^\dagger$.

For $a = re^{i\varphi} \in \mathbb{C} \setminus \{0\}$ we denote by $\arg(a) = \varphi \in \mathbb{R}/2\pi i\mathbb{Z}$ and by $\operatorname{Arg}(a) = e^{i\varphi} = a/|a|$.

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1. Real structures in log geometry

Recall that for a scheme $\bar{X}$ defined over $\mathbb{R}$ the Galois group $G(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ acts on the assoicated complex scheme $X = \bar{X} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ by means of the universal property of the cartesian product

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{R}. \end{array}$$

The generator of the Galois action thus acts on X as an involution of schemes over $\mathbb{R}$ making the following diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\operatorname{conj}} & \operatorname{Spec}\mathbb{C}. \end{array} \tag{1.1}$$

Here conj denotes the $\mathbb{R}$-linear automorphism of $\operatorname{Spec}\mathbb{C}$ defined by complex conjugation.

Conversely, a *real structure* on a complex scheme X is an involution $\iota : X \to X$ of schemes over $\mathbb{R}$ fitting into the commutative diagram (1.1). It is not hard to see that if X is separated then X is defined over $\mathbb{R}$ with ι the generator of the Galois action ([Hr], II Ex.4.7). A pair (X, ι) is called a *real scheme*. By abuse of notation we usually omit ι when talking about real schemes.

Definition 1.1. Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{C}$ with a real structure $\iota_X : X \to X$ on the underlying scheme. Then a *real structure* on $(X, \mathcal{M}_X)$ *(lifting ι_X)* is an involution

$$\tilde{\iota}_X = (\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \longrightarrow (X, \mathcal{M}_X)$$

of log schemes over $\mathbb{R}$ with underlying scheme-theoretic morphism ι_X. The data consisting of $(X, \mathcal{M}_X)$ and the involutions ι_X, $\iota_X^\flat$ is called a *real log scheme.*

In talking about real log schemes the involutions ι_X, $\iota_X^\flat$ are usually omitted from the notation. We also sometimes use the notation ι_X for the involution of the log space $(X, \mathcal{M}_X)$ and in this case write $\underline{\iota}_X$ if we want to emphasize we mean the underlying morphism of schemes.

Definition 1.2. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be real log schemes. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of real log schemes is called *real* if the following diagram is commutative.

$$\begin{tikzcd}
f^{-1}\iota_Y^{-1}\mathcal{M}_Y \arrow[r, "f^{-1}\iota_Y^\flat"] \arrow[d, "\iota_X^{-1}f^\flat"'] & f^{-1}\mathcal{M}_Y \arrow[d, "f^\flat"] \\
\iota_X^{-1}\mathcal{M}_X \arrow[r, "\iota_X^\flat"] & \mathcal{M}_X.
\end{tikzcd}$$

Here the left-hand vertical arrow uses the identification $\iota_Y \circ f = f \circ \iota_X$.

Remark 1.3. For a real morphism of real log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ the following diagram commutes.

$$\begin{tikzcd}
& f^{-1}\iota_Y^{-1}\mathcal{O}_Y \arrow[rr] \arrow[dd] & & f^{-1}\mathcal{O}_Y \arrow[dd] \\
f^{-1}\iota_Y^{-1}\mathcal{M}_Y \arrow[ur] \arrow[rr] \arrow[dd] & & f^{-1}\mathcal{M}_Y \arrow[ur] \arrow[dd] & \\
& \iota_X^{-1}\mathcal{O}_X \arrow[rr] & & \mathcal{O}_X \\
\iota_X^{-1}\mathcal{M}_X \arrow[ur] \arrow[rr] & & \mathcal{M}_X \arrow[ur] &
\end{tikzcd}$$

In fact, commutativity on the (1) bottom, (2) top, (3) right (4) left (5) back and (6) front faces follows from the assumptions that (1) $(X, \mathcal{M}_X)$ is a real log scheme, (2) $(Y, \mathcal{M}_Y)$ is a real log scheme, (3) f is a morphism of log schemes, (4) ι_X^{-1} applied to the right face plus the identity $f \circ \iota_X = \iota_Y \circ f$, (5) f induces a real morphism on the underlying schemes and (6) f is a real morphism of real log structures.

Given a real log scheme $(X, \mathcal{M}_X)$ with $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the structure homomorphism, for any geometric point $\bar{x} \to X$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{X,\bar{x}} \arrow[r, "\iota^\flat"] \arrow[d, "\alpha_{\bar{x}}"'] & \mathcal{M}_{X,\iota(\bar{x})} \arrow[r, "\iota^\flat"] \arrow[d, "\alpha_{\iota(\bar{x})}"'] & \mathcal{M}_{X,\bar{x}} \arrow[d, "\alpha_{\bar{x}}"'] \\
\mathcal{O}_{X,\bar{x}} \arrow[r, "\iota^\sharp"] & \mathcal{O}_{X,\iota(\bar{x})} \arrow[r, "\iota^\sharp"] & \mathcal{O}_{X,\bar{x}}.
\end{tikzcd}$$

The compositions of the maps in the two horizontal sequences are the identity on $\mathcal{M}_{X,\bar{x}}$ and on $\mathcal{O}_{X,\bar{x}}$, respectively. For the next result recall that if X is a pure-dimensional scheme and $D \subset X$ is a closed subset of codimension one, then the subsheaf $\mathcal{M}_{(X,D)} \subset \mathcal{O}_X$ of regular functions with zeros contained in D defines the *divisorial log structure* on X associated to D.

Proposition 1.4. *Let X be a pure-dimensional scheme, $D \subset X$ a closed subset of codimension one and let $\mathcal{M}_X = \mathcal{M}_{(X,D)}$ be the associated divisorial log structure. Then* *a real structure ι on X lifts to $\mathcal{M}_X$ iff $\iota(D) = D$. Moreover, in this case the lift $\iota^\flat$ is uniquely determined as the restriction of $\iota^\sharp$ to $\mathcal{M}_{(X,D)} \subset \mathcal{O}_X$.*

Proof. Let $\iota : X \to X$ be a real structure on X with $\iota(D) = D$. Then $\iota(X \setminus D) = X \setminus D$ and hence $\iota^\sharp$ restricts to an isomorphism $\varphi : \iota^{-1}\mathcal{O}^\times_{X\setminus D} \to \mathcal{O}^\times_{X\setminus D}$. By definition of $\mathcal{M}_{(X,D)}$, φ induces an isomorphism $\iota^\flat : \iota^{-1}\mathcal{M}_{(X,D)} \to \mathcal{M}_{(X,D)}$. Hence we get a real structure $(\iota, \iota^\flat) : (X, \mathcal{M}_{(X,D)}) \to (X, \mathcal{M}_{(X,D)})$ on $(X, \mathcal{M}_{(X,D)})$ lifting ι.

Conversely, let the real structure $\iota : X \to X$ lift to $(\iota, \iota^\flat) : (X, \mathcal{M}_{(X,D)}) \longrightarrow (X, \mathcal{M}_{(X,D)})$. In other words, there exists a morphism $\iota^\flat : \iota^{-1}\mathcal{M}_{(X,D)} \longrightarrow \mathcal{M}_{(X,D)}$ making the following diagram commute.

(1.2)
$$\begin{array}{ccc} \iota^{-1}\mathcal{M}_{(X,D)} & \xrightarrow{\iota^{-1}\alpha} & \iota^{-1}\mathcal{O}_X \\ {\scriptstyle \iota^\flat}\downarrow & & \downarrow{\scriptstyle \iota^\sharp} \\ \mathcal{M}_{(X,D)} & \xrightarrow{\alpha} & \mathcal{O}_X \end{array}$$

Let $D = \bigcup_\mu D_\mu$ be the decomposition into irreducible components. Since $\iota^2 = \mathrm{id}_X$ it suffices to show $\iota(D) \subset D$, or $\iota(D_\mu) \subset D$ for every μ. Fix μ and let $U \subset X$ be an affine open subscheme with $U \cap D_\mu \neq \emptyset$. Let $f \in \mathcal{O}_X(U) \setminus \{0\}$ be such that $D \subset V(f)$. Then $U \cap D_\mu \subset U \cap D \subset V(f)$. Write $V(f) = (D_\mu \cap U) \cup E$ with $E \subset V(f)$ the union of the irreducible components of $V(f)$ different from D_μ. Replacing U by $U \setminus E$ we may assume $V(f) = U \cap D_\mu$. Note that U may not be affine anymore, but this is not important from now on.

Taking sections of Diagram (1.2) over $\iota^{-1}(U)$ shows that $f \circ \iota = \iota^\sharp(f)$ lies in $\mathcal{M}_{(X,D)}\big(\iota^{-1}(U)\big) \subset \mathcal{O}_X\big(\iota^{-1}(U)\big)$. By the definition of $\mathcal{M}_{(X,D)}$ this implies $V(f \circ \iota) \subset D$. But also

$$V(f \circ \iota) = \iota^{-1}\big(V(f)\big) = \iota^{-1}(U \cap D_\mu) = \iota(U \cap D_\mu).$$

Taken together this shows that $\iota(U \cap D_\mu) \subset D$. Since U is open with $U \cap D_\mu \neq \emptyset$ we obtain the desired inclusion $\iota(D_\mu) \subset D$. □

Proposition 1.5. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be strict and assume that the morphism $\underline{f}$ of the underlying schemes is compatible with real structures ι_X on X and ι_Y on Y. Then for any real structure $\iota_Y^\flat$ on $\mathcal{M}_Y$ lifting ι_Y there exists a unique real structure $\iota_X^\flat$ on $\mathcal{M}_X$ lifting ι_X and compatible with f.*

Proof. By strictness we can assume the log structure $\mathcal{M}_X$ on X is the pull-back log structure $f^*\mathcal{M}_Y = f^{-1}\mathcal{M}_Y \oplus_{f^{-1}\mathcal{O}_Y^\times} \mathcal{O}_X^\times$. Hence,

$$\iota_X^{-1}\mathcal{M}_X = \iota_X^{-1} f^{-1}\mathcal{M}_Y \oplus_{\iota_X^{-1} f^{-1}\mathcal{O}_Y^\times} \iota_X^{-1}\mathcal{O}_X^\times = f^{-1}\iota_Y^{-1}\mathcal{M}_Y \oplus_{f^{-1}\iota_Y^{-1}\mathcal{O}_Y^\times} \iota_X^{-1}\mathcal{O}_X^\times.$$

Now for a lift $\iota_X^\flat : \iota_X^{-1}\mathcal{M}_X \to \mathcal{M}_X$ of $\iota_X^\sharp$ compatible with f, the composition

$$\varphi : f^{-1}\iota_Y^{-1}\mathcal{M}_Y \longrightarrow \iota_X^{-1}\mathcal{M}_X \xrightarrow{\iota_X^\flat} \mathcal{M}_X = f^*\mathcal{M}_Y$$

factors over $f^{-1}\iota_Y^\flat : f^{-1}\iota_Y^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{M}_Y$ and is hence determined by f and $\iota_Y^\flat$. Similarly, the composition

$$\psi : \iota_X^{-1}\mathcal{O}_X^\times \longrightarrow \iota_X^{-1}\mathcal{M}_X \xrightarrow{\iota_X^\flat} \mathcal{M}_X = f^*\mathcal{M}_Y$$

factors over $\iota_X^\sharp : \iota_X^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$ and thus is known by assumption. Since $\underline{f}$ is a real morphism of real schemes, φ and ψ agree on $f^{-1}\iota_Y^{-1}\mathcal{O}_Y^\times$. Hence the unique existence of $\iota_X^\flat$ with the requested properties follows from the universal property of the fibered sum. □

Explicit computations are most easily done in charts adapted to the real structure. For simplicity we provide the following statements for log structures in the Zariski topology, the case sufficient for our main application to toric degenerations. Analogous statements hold in the étale or analytic topology.

Definition 1.6. Let $(X, \mathcal{M}_X)$ be a log scheme with a real structure $(\iota_X, \iota_X^\flat)$. A chart $\beta : P \to \Gamma(U, \mathcal{M}_X)$ for $(X, \mathcal{M}_X)$ is called a *real chart* if (1) $\iota_X(U) = U$ and (2) there exists an involution $\iota_P : P \to P$ such that for all $p \in P$ it holds $\beta\big(\iota_P(p)\big) = \iota_X^\flat\big(\beta(p)\big)$.

Example 1.7. An involution ι_P of a toric monoid P induces an antiholomorphic involution on $\mathbb{C}[P]$ by mapping $\sum_p a_p z^p$ to $\sum_p \overline{a_p} z^{\iota_P(p)}$. The induced real structure on the toric variety $X_P = \operatorname{Spec}\mathbb{C}[P]$ permutes the irreducible components of the toric divisor $D_P \subset X_P$ and hence, by Proposition 1.4 induces a real structure on $(X_P, \mathcal{M}_{(X_P,D_P)})$. We claim the canonical toric chart

$$\beta : P \longrightarrow \Gamma(X_P, \mathcal{M}_{(X_P,D_P)}), \quad p \longmapsto z^p$$

is a real chart. Indeed, for any $p \in P$ we have $\beta(\iota_P(p)) = z^{\iota(p)} = \iota_{X_P}^\sharp(z^p) = \iota_{X_P}^\flat(z^p)$, the last equality due to Proposition 1.4.

Real charts may not exist, a necessary condition being that X has a cover by affine open sets that are invariant under the real involution ι_X. This is the only obstruction:

Lemma 1.8. *Let $(X, \mathcal{M}_X)$ be a real log scheme with involution ι_X. Let $U \subset X$ be a ι_X-invariant open set supporting a chart $\beta : P \to \Gamma(U, \mathcal{M}_X)$. Then there also exists a real chart $\beta' : P' \to \Gamma(U, \mathcal{M}_X)$ for $\mathcal{M}_X$ on U.*

Proof. We claim that

$$\tilde\beta : P \oplus P \longrightarrow \Gamma(U, X), \quad \tilde\beta(p, p') = \beta(p) \cdot \iota_X^\flat\big(\beta(p')\big)$$

is a real chart. Since $\tilde{\beta}$ restricts to β on the first summand of $\tilde{P}$, this is still a chart for $\mathcal{M}_X$ on U. For the involution on the monoid $\tilde{P} = P \oplus P$ we take $\iota_{\tilde{P}}(p, p') = (p', p)$. Then indeed for any $(p, p') \in \tilde{P}$ we have

$$\tilde{\beta}\big(\iota_{\tilde{P}}(p, p')\big) = \tilde{\beta}(p', p) = \beta(p') \cdot \iota_X^\flat(\beta(p)) = \iota_X^\flat\big(\iota_X^\flat(\beta(p')) \cdot \beta(p)\big) = \iota_X^\flat\big(\tilde{\beta}(p, p')\big),$$

verifying the condition for a real chart. □

Note that if X is a separated scheme, real charts always exist at any point x in the fixed locus of ι_X. In fact, take any chart defined in a neighbourhood U of X, restrict to $U \cap \iota_X(U)$, still an affine open set by separatedness, and apply Lemma 1.8.

Proposition 1.9. *Cartesian products exist in the category of real log schemes.*

Proof. Let $(X, \mathcal{M}_X)$, $(S, \mathcal{M}_S)$, $(T, \mathcal{M}_T)$ be real log schemes endowed with morphisms $f : (X, \mathcal{M}_X) \to (T, \mathcal{M}_T)$ and $g : (S, \mathcal{M}_S) \to (T, \mathcal{M}_T)$. Then the fibre product in the category of log schemes $(S \times_T X, \mathcal{M}_{S\times_T X})$ fits into the following cartesian diagram.

(1.3)
$$\begin{tikzcd}
(S \times_T X, \mathcal{M}_{S\times_T X}) \arrow[r, "p_X"] \arrow[d, "p_S"] & (X, \mathcal{M}_X) \arrow[d, "f"] \\
(S, \mathcal{M}_S) \arrow[r, "g"] & (T, \mathcal{M}_T)
\end{tikzcd}$$

The log structure on the fiber product $S \times_T X$ is given by $\mathcal{M}_{S\times_T X} = p_X^*\mathcal{M}_X \oplus_{p_T^*\mathcal{M}_T} p_S^*\mathcal{M}_S$. By the universal property of the fibered coproduct the existence of real structures on $(X, \mathcal{M}_X), (S, \mathcal{M}_S)$ and $(T, \mathcal{M}_T)$ ensures the existence of a real structure on $(S \times_T X, \mathcal{M}_{S\times_T X})$. □

Note that in general the amalgamated sum of fine log structures $p_X^*\mathcal{M}_X \oplus_{p_T^*\mathcal{M}_T} p_Y^*\mathcal{M}_Y$ is only coherent, but not even integral. To take the fibred product in the category of fine log schemes requires the further step of integralizing $(S\times_T X, \mathcal{M}_{S\times_T X})$. Given a monoid P with integralization P_{int} and a chart $U \to \operatorname{Spec} \mathbb{Z}[P]$ for a log scheme $(U, \mathcal{M}_U)$, the integralization of $(U, \mathcal{M}_U)$ is the closed subscheme $U \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[P^{\mathrm{int}}]$ of U with the log structure defined by the chart $U \to \operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}[P^{\mathrm{int}}]$. A similar additional step is needed for staying in the category of saturated log schemes. Fortunately, we are only interested in the case that g is strict, and in this case the fibre product in all categories agree. See [O], Ch.III, §2.2.1, for details.

Example 1.10. Let S be the spectrum of a discrete valuation ring with residue field $\mathbb{C}$ and $\delta : \mathfrak{X} \to S$ be a flat morphism. Let $0 \in S$ be the closed point, $X_0 = \delta^{-1}(0)$ and consider δ as a morphism of log schemes with divisorial log structures $\delta : (\mathfrak{X}, \mathcal{M}_{(\mathfrak{X},X_0)}) \to (S, \mathcal{M}_{(S,0)})$. If δ commutes with real structures on $\mathfrak{X}$ and S, then by Proposition 1.4, the morphism δ is naturally a real morphism of real log schemes. Taking the base change by the strict morphism $(\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^\times) \to (S, \mathcal{M}_{(S,0)})$, Proposition 1.9 leads to a real log scheme $(X_0, \mathcal{M}_{X_0})$ over the standard log point $O^\dagger = (\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^\times)$.

2. The Kato-Nakayama space of a log space

2.1. **Generalities on Kato-Nakayama spaces.** For the rest of the paper we work in the analytic topology. If R is a finitely generated $\mathbb{C}$-algebra we write $\operatorname{Specan} R$ for the analytic space associated to the complex scheme $\operatorname{Spec} R$.

To any log scheme $(X, \mathcal{M}_X)$ over $\mathbb{C}$, Kato and Nakayama in [KN] have introduced a topological space $(X, \mathcal{M}_X)^{\mathrm{KN}}$, its *Kato-Nakayama space* or *Betti-realization*. We review this definition and its basic properties first before discussing the additional properties coming from a real structure. Denote by $\Pi^\dagger = (\operatorname{Specan}\mathbb{C}, \mathcal{M}_\Pi)$ the *polar log point*, with log structure

$$\alpha_\Pi : \mathcal{M}_{\Pi,0} = \mathbb{R}_{\geq 0} \times U(1) \longrightarrow \mathbb{C}, \quad (r, e^{i\varphi}) \longmapsto r\cdot e^{i\varphi}.$$

There is an obvious map $\Pi^\dagger \to \operatorname{Specan}\mathbb{C}$ making $\Pi^\dagger$ into a log space over $\mathbb{C}$. Note $\overline{\mathcal{M}}_{\Pi,0} = U(1)$, so this log structure is not fine. As a set define

$$(X, \mathcal{M}_X)^{\mathrm{KN}} := \operatorname{Hom}\big(\Pi^\dagger, (X, \mathcal{M}_X)\big),$$

the set of morphisms of complex analytic log spaces $\Pi^\dagger \to (X, \mathcal{M}_X)$. Note that a log morphism $f : \Pi^\dagger \to (X, \mathcal{M}_X)$ is given by its set-theoretic image, a point $x = \varphi(0) \in X$, and a monoid homomorphism $f^\flat : \mathcal{M}_{X,x} \to \mathbb{R}_{\geq 0} \times U(1)$. Forgetting the monoid homomorphism thus defines a map

$$\pi : (X, \mathcal{M}_X)^{\mathrm{KN}} \longrightarrow X. \tag{2.1}$$

We endow $(X, \mathcal{M}_X)^{\mathrm{KN}}$ with the following topology. A local section $\sigma \in \Gamma(U, \mathcal{M}_X^{\mathrm{gp}})$, $U \subset X$ open, defines a map

$$\operatorname{ev}_\sigma : \pi^{-1}(U) \longrightarrow \mathbb{R}_{\geq 0} \times U(1), \quad f \longmapsto f^\flat \circ \sigma. \tag{2.2}$$

As a subbasis of open sets on $(X, \mathcal{M}_X)^{\mathrm{KN}}$ we take $\operatorname{ev}_\sigma^{-1}(V)$, for any $U \subset X$ open, $\sigma \in \Gamma(U, \mathcal{M}_X^{\mathrm{gp}})$ and $V \subset \mathbb{R}_{\geq 0} \times U(1)$ open. The forgetful map π is then clearly continuous.

If the log structure is understood, we sometimes write X^{KN} instead of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ for brevity.

Remark 2.1. The following more explicit set-theoretic description of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ is sometimes useful. A log morphism $f : \Pi^\dagger \to (X, \mathcal{M}_X)$ with $f(0) = x$ is equivalent to a choice of monoid homomorphism $f^\flat$ fitting into the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{X,x} \arrow[r, "f^\flat=(\rho,\theta)"] \arrow[d, "\alpha_{X,x}"'] & \mathbb{R}_{\geq 0}\times U(1) \arrow[d, "\alpha_\Pi"] \\
\mathcal{O}_{X,x} \arrow[r, "\mathrm{ev}_x"'] & \mathbb{C}
\end{tikzcd}$$

Here $\alpha_{X,x}$ is the stalk of the structure morphism $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ of X and ev_x takes the value of a function at the point x. This diagram implies that the first component ρ of $f^\flat$ is determined by x and the structure homomorphism by

$$\rho(\sigma) = \big|\big(\alpha_{X,x}(\sigma)\big)(x)\big|.$$

Thus giving f is equivalent to selecting the point $x \in X$ and a homomorphism $\theta : \mathcal{M}_{X,x} \to U(1)$ with the property that for any $\sigma \in \mathcal{M}_{X,x}$ it holds

$$(\alpha_{X,x}(\sigma))(x) = \big|(\alpha_{X,x}(\sigma))(x)\big| \cdot \theta(\sigma).$$

Since both sides vanish unless $\sigma \in \mathcal{O}^\times_{X,x} \subset \mathcal{M}_{X,x}$, this last property needs to be checked only on invertible elements. Note also that a homomorphism $\mathcal{M}_{X,x} \to U(1)$ extends to $\mathcal{M}^{\mathrm{gp}}_{X,x}$ since $U(1)$ is an abelian group. Summarizing, we have a canonical identification

$$(2.3)\qquad (X,\mathcal{M}_X)^{\mathrm{KN}} = \Big\{(x,\theta) \in \textstyle\prod_x \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X,x}, U(1)) \,\Big|\, \forall h \in \mathcal{O}^\times_{X,x} : \theta(h) = \frac{h(x)}{|h(x)|}\Big\}$$

In this description we adopt the occasional abuse of notation of viewing $\mathcal{O}^\times_{X,x}$ as a submonoid of $\mathcal{M}_{X,x}$ by means of the structure homomorphism $\mathcal{M}_{X,x} \to \mathcal{O}_{X,x}$. From (2.3), for $s \in \mathcal{M}_{X,x}$ any point $f \in X^{\mathrm{KN}}$ over $x \in X$ defines an element $\theta(s) \in U(1)$. We refer to this element of $U(1)$ as the *phase* of s at f. If $s \in \mathcal{O}^\times_{X,x}$ then the phase of any point of X^{KN} over x agrees with $\mathrm{Arg}(s) = e^{i\arg(s)}$.

Next we give an explicit description of $(X,\mathcal{M}_X)^{\mathrm{KN}}$ assuming the log structure has a chart with a fine monoid. For a fine monoid P, we have $P^{\mathrm{gp}} \simeq T \oplus \mathbb{Z}^r$ with T finite. Thus the set $\mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ is in bijection with $|T|$ copies of the real torus $U(1)^r$ by means of choosing generators. This identification is compatible with the topology on $\mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ defined by the subbasis of topology consisting of the sets

$$(2.4)\qquad V_p := \big\{\varphi \in \mathrm{Hom}(P^{\mathrm{gp}}, U(1)) \,\big|\, \varphi(p) \in V\big\},$$

for all $V \subset U(1)$ open and $p \in P^{\mathrm{gp}}$.

Proposition 2.2. *Let P be a fine monoid and let X be an analytic space endowed with the log structure defined by a holomorphic map $g : X \to \mathrm{Specan}\,\mathbb{C}[P]$. Then there is a canonical closed embedding of $(X,\mathcal{M}_X)^{\mathrm{KN}}$ into $X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ with image*

$$\Big\{(x,\lambda) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1)) \,\Big|\, \forall p \in P :\ g^\sharp(z^p)_x \in \mathcal{O}^\times_{X,x} \Rightarrow \lambda(p) = \mathrm{Arg}(g^\sharp(z^p)(x))\Big\}.$$

Proof. Denote by $\beta : P \to \Gamma(X, \mathcal{M}_X)$ the chart given by g and by $\beta_x : P \to \mathcal{M}_{X,x}$ the induced map to the stalk at $x \in X$. Recall the description (2.3) of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ by pairs (x, θ) with $x \in X$ and $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ a group homomorphism extending $h \mapsto h(x)/|h(x)|$ for $h \in \mathcal{O}_{X,x}^\times$. With this description, the canonical map in the statement is

$$\Psi : (X, \mathcal{M}_X)^{\mathrm{KN}} \longrightarrow X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1)), \quad (x, \theta) \longmapsto \big(x, \theta \circ \beta_x^{\mathrm{gp}}\big).$$

Here $\beta_x^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathcal{M}_{X,x}^{\mathrm{gp}}$ is the map induced by β_x on the associated groups.

To prove continuity of Ψ, let $p \in P^{\mathrm{gp}}$ and $V \subset U(1)$ be open. Then Ψ^{-1} of $X \times V_p$ with $V_p \subset \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ the basic open set from (2.4), equals $\mathrm{ev}_{\beta^{\mathrm{gp}}(p)}^{-1}(\mathbb{R}_{\geq 0} \times V)$, with ev_σ defined in (2.2). By the definition of the topology, $\Psi^{-1}(X \times V_p) \subset (X, \mathcal{M}_X)^{\mathrm{KN}}$ is thus open. Continuity of the first factor π of Ψ being trivial, this shows that Ψ is continuous.

We next check that $\mathrm{im}(\Psi)$ is contained in the closed subset of $X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ stated in the assertion. Let $(x, \theta) \in (X, \mathcal{M}_X)^{\mathrm{KN}}$. For $p \in P$ the required equation $g^\sharp(z^p)(x) = \lambda(p) \cdot \big|g^\sharp(z^p)(x)\big|$ for $\lambda = \theta \circ \beta_x^{\mathrm{gp}}$ is non-trivial only if $h := g^\sharp(z^p) \in \mathcal{O}_{X,x}^\times$. In this case, $\beta_x(p)$ maps to h under the structure homomorphism $\mathcal{M}_{X,x} \to \mathcal{O}_{X,x}$ and hence

$$\big(\theta \circ \beta_x(p)\big)(x) = \frac{h(x)}{\big|h(x)\big|} = \frac{g^\sharp(z^p)(x)}{\big|g^\sharp(z^p)(x)\big|},$$

verifying the required equality.

Conversely, assume $(x, \lambda) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ fulfills

$$g^\sharp(z^p)(x) = \lambda(p) \cdot \big|g^\sharp(z^p)(x)\big|. \tag{2.5}$$

Denote by $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the structure homomorphism. Then $\mathcal{M}_{X,x}$ fits into the cocartesian diagram of monoids

$$\begin{tikzcd} \beta_x^{-1}(\mathcal{O}_{X,x}^\times) \arrow[r] \arrow[d, "\alpha_x \circ \beta_x"'] & P \arrow[d, "\beta_x"] \\ \mathcal{O}_{X,x}^\times \arrow[r] & \mathcal{M}_{X,x} \end{tikzcd} .$$

Consider the pair of homomorphisms $\mathrm{Arg} \circ \mathrm{ev}_x : \mathcal{O}_{X,x}^\times \to U(1)$, and $\lambda : P \to U(1)$, with ev_x evaluation at x. In view of $(\alpha \circ \beta)(p) = g^\sharp(z^p)$, Equation (2.5) says precisely that the compositions of these two maps with the maps from $\beta_x^{-1}(\mathcal{O}_{X,x}^\times)$ agree. By the universal property of fibred sums we thus obtain a homorphism $\mathcal{M}_{X,x} \to U(1)$. Define $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ as the induced map on associated groups. For $h \in \mathcal{O}_{X,x}^\times$ it holds $\theta(h) = h(x)/|h(x)|$ and hence $(x, \theta) \in (X, \mathcal{M}_X)^{\mathrm{KN}}$. It is now not hard to see that the map $(x, \lambda) \mapsto (x, \theta)$ is inverse to Ψ and continuous as well. $\square$

2.2. **Examples of Kato-Nakayama spaces.** We next discuss a few examples of Kato-Nakayama spaces, geared towards toric degenerations. Unless otherwise stated, N denotes a finitely generated free abelian group, $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual and $N_\mathbb{R}$, $M_\mathbb{R}$ are the associated real vector spaces. If $\sigma \subset N_\mathbb{R}$ is a cone then the set of monoid homomorphisms $\sigma^\vee = \mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0}) \subset M_\mathbb{R}$ denotes its dual cone. A *lattice polyhedron* is the intersection of rational half-spaces in $M_\mathbb{R}$ with an integral point on each minimal face.

The basic example is a canonical description of the Kato-Nakayama space of a toric variety defined by a momentum polytope. We use a rather liberal definition of a momentum map, not making any reference to a symplectic structure. Let $\Xi \subset M_\mathbb{R}$ be a full-dimensional, convex lattice polyhedron. Let X be the associated complex toric variety. A basic fact of toric geometry states that the fan of X agrees with the normal fan Σ_Ξ of Ξ. From this description, X is covered by affine toric varieties $\mathrm{Specan}\, \mathbb{C}[\sigma^\vee \cap M]$, for $\sigma \subset \Sigma_\Xi$. Since the patching is monomial, it preserves the real structure of each affine patch. Hence the real locus $\mathrm{Hom}(\sigma^\vee, \mathbb{R}) \subset \mathrm{Hom}(\sigma^\vee, \mathbb{C})$ of each affine patch glues to the real locus $X_\mathbb{R} \subset X$. Unlike in the definition of $\sigma^\vee$, here $\mathbb{R}$ and $\mathbb{C}$ are multiplicative monoids. Moreover, inside the real locus of each affine patch there is the distinguished subset

$$\sigma = \mathrm{Hom}(\sigma^\vee, \mathbb{R}_{\geq 0}) \subset \mathrm{Hom}(\sigma^\vee, \mathbb{R}),$$

with "Hom" referring to homomorphisms of monoid. These also patch via monomial maps to give the *positive real locus* $X_{\geq 0} \subset X_\mathbb{R}$.

Having introduced the positive real locus $X_{\geq 0} \subset X_\mathbb{R}$ we are in position to define abstract momentum maps.

Definition 2.3. Let X be the complex toric variety defined by a full-dimensional lattice polyhedron $\Xi \subset M_\mathbb{R}$. Then a continuous map

$$\mu : X \longrightarrow \Xi$$

is called an *(abstract) momentum map* if the following holds.

(1) μ is invariant under the action of $\mathrm{Hom}(M, U(1))$ on X.
(2) The restriction of μ maps $X_{\geq 0}$ homeomorphically to Ξ, thus defining a section $s_0 : \Xi \to X$ of μ with image $X_{\geq 0}$.
(3) The map

$$\mathrm{Hom}(M, U(1)) \times \Xi \longrightarrow X, \quad (\lambda, x) \longmapsto \lambda \cdot s_0(x) \tag{2.6}$$

induces a homeomorphism $\mathrm{Hom}(M, U(1)) \times \mathrm{Int}(\Xi) \simeq X \setminus D$, where $D \subset X$ is the toric boundary divisor.

Projective toric varieties have a momentum map, see e.g. [Fu], §4.2. For an affine toric variety $\operatorname{Specan}\mathbb{C}[P]$, momentum maps also exist. One natural construction discussed in detail in [NO], §1, is a simple formula in terms of generators of the toric monoid P ([NO], Definition 1.2). Some work is however needed to show that if $P = \sigma^\vee \cap M$, then the image of this momentum map is the cone $\sigma^\vee$ spanned by P. We give here another, easier but somewhat ad hoc construction of a momentum map.

Proposition 2.4. *An affine toric variety $X = \operatorname{Specan}\mathbb{C}[\sigma^\vee \cap M]$ has a momentum map with image the defining rational polyhedral cone $\sigma^\vee \subset M_\mathbb{R}$.*

Proof. If the minimal toric stratum $Z \subset X$ is of dimension $r > 0$, we can decompose $\sigma^\vee \simeq C + \mathbb{R}^r$ and acordingly $X \simeq \overline{X} \times (\mathbb{C}^*)^r$ with $\overline{X}$ a toric variety with a zero-dimensional toric stratum. The product of a momentum map $\overline{X} \to C$ with the momentum map

$$(\mathbb{C}^*)^r \longrightarrow \mathbb{R}^r, \quad (z_1, \ldots, z_r) \longmapsto \big(\log|z_1|, \ldots, \log|z_r|\big)$$

is then a momentum map for X. We may therefore assume that X has a zero-dimensional toric stratum, or equivalently that $\sigma^\vee$ is strictly convex.

Now embed X into a projective toric variety $\tilde{X}$ and let $\mu : \tilde{X} \to \Xi$ be a momentum map mapping the zero-dimensional toric stratum of X to the origin. Then the cone in $M_\mathbb{R}$ spanned by Ξ equals $\sigma^\vee$. By replacing Ξ with its intersection with an appropriate affine hyperplane we may assume that Ξ is the convex hull of 0 and a disjoint facet $\omega \subset \Xi$. Then $X = \mu^{-1}(\Xi \setminus \omega)$. To construct a momentum map for X with image $\sigma^\vee$ let $q : M_\mathbb{R} \to \mathbb{R}$ be the quotient by T_ω. Then $q(\Xi)$ is an interval $[0, a]$ with $a > 0$. Now $f(x) = x/(a - x)$ maps the half-open interval $[0, a)$ to $\mathbb{R}_{\geq 0}$. A momentum map for X with image $\sigma^\vee$ is then defined by

$$z \longmapsto (f \circ q)\big(\mu(z)\big) \cdot \mu(z).$$

□

Our next result concerns the announced canonical description of the Kato-Nakayama space of a toric variety with a momentum map.

Proposition 2.5. *Let X be a complex toric variety with a momentum map $\mu : X \to \Xi \subset M_\mathbb{R}$ and let $\mathcal{M}_X$ be the toric log structure on X. Then the map* (2.6) *factors through a canonical homeomorphism*

$$\Xi \times \operatorname{Hom}(M, U(1)) \longrightarrow (X, \mathcal{M}_X)^{\mathrm{KN}}.$$

Proof. The toric variety X is covered by open affine sets of the form $\operatorname{Specan}\mathbb{C}[P]$ with $P^{\mathrm{gp}} = M$, and these are charts for the log structure. Thus the local description of X^{KN} in Proposition 2.2 as a closed subset globalizes to define a closed embedding

$$\iota : X^{\mathrm{KN}} \longrightarrow X \times \operatorname{Hom}(M, U(1)).$$

With $s_0 : \Xi \to X$ the section of μ with image $X_{\geq 0} \subset X$, consider the continuous map

$$\Phi : \Xi \times \operatorname{Hom}(M, U(1)) \longrightarrow X \times \operatorname{Hom}(M, U(1)), \quad (a, \lambda) \longmapsto \big(\lambda \cdot s_0(a), \lambda\big).$$

Here $\lambda \in \operatorname{Hom}(M, U(1))$ acts on X as an element of the algebraic torus $\operatorname{Hom}(M, \mathbb{C}^\times)$. The map Φ has the continuous left-inverse $\pi \times \mathrm{id}$. Thus to finish the proof it remains to show $\operatorname{im}(\Phi) = \operatorname{im}(\iota)$.

Indeed, according to Proposition 2.2, $(x, \lambda) \in X \times \operatorname{Hom}(M, U(1))$ lies in $\iota(X^{\mathrm{KN}})$ iff for all $m \in M$ with z^m defined at x it holds $z^m(x) = \lambda(m) \cdot |z^m(x)|$. But this equation holds if and only if $x = \lambda \cdot \sigma(a)$ for $a = \mu(x)$ since $\sigma(a) \in X_{\geq 0}$ implies

$$z^m(\lambda \cdot \sigma(a)) = \lambda(m) \cdot z^m(\sigma(a)) = \lambda(m) \cdot |z^m(x)|.$$

Thus $(x, \lambda) \in X^{\mathrm{KN}}$ iff $(x, \lambda) = (\lambda \cdot \sigma(\mu(x)), \lambda)$, that is, iff $(x, \lambda) \in \operatorname{im}(\Phi)$. □

Remark 2.6. The left-hand side in the statement of Proposition 2.5 can also be written $T^*_\Xi/\check{\Lambda}$ where $\check{\Lambda} \subset T_\Xi$ is the local system of integral cotangent vectors. Indeed, for any $y \in \Xi$ we have the sequence of canonical isomorphisms

$$T^*_{\Xi,y}/\check{\Lambda}_y \longrightarrow \operatorname{Hom}(M, \mathbb{R})/\operatorname{Hom}(M, \mathbb{Z}) = \operatorname{Hom}(M, \mathbb{R}/\mathbb{Z}) = \operatorname{Hom}(M, U(1)).$$

Example 2.7. Let $X = \mathbb{A}^1 = \operatorname{Specan}\mathbb{C}[\mathbb{N}]$ be endowed with the divisorial log structure $\mathcal{M}_{(X,\{0\})}$. By Proposition 2.4 there exists a momentum map $\mu : X \to \mathbb{R}_{\geq 0}$. Explicitly, in the present case one may simply take $\mu(z) = |z|$ where z is the toric coordinate. According to Proposition 2.5, $X^{\mathrm{KN}} \cong \mathbb{R}_{\geq 0} \times S^1$ canonically. The map $\pi : X^{\mathrm{KN}} \to X$ is a homeomorphism onto the image over $\mathbb{A}^1 \setminus \{0\}$ and has fibre $S^1 = \operatorname{Hom}(\mathcal{M}_{(X,\{0\})}, U(1)) = \operatorname{Hom}(\mathbb{N}, U(1))$ over 0. Thus X^{KN} is homeomorphic to the *oriented real blow up* of $\mathbb{A}^1$ at 0.

Example 2.8. More generally, Let $(X, \mathcal{M}_{(X,D)})$ be the divisorial log structure on a complex scheme X with a normal crossings divisor $D \subset X$. Then the Kato-Nakayama space X^{KN} of X can be identified with the oriented real blow up of X along D. At a point $x \in X$ the map $X^{\mathrm{KN}} \to X$ has fibre $(S^1)^k$ with k the number of irreducible components of D containing x.

Example 2.9. Let $X = \mathbb{P}^2$ with the toric log structure. There exists a momentum map $\mu : \mathbb{P}^2 \to \Xi$ with $\Xi = \operatorname{conv}\{(0,0), (1,0), (0,1)\} \subset M_{\mathbb{R}}$ the 2-simplex and $M = \mathbb{Z}^2$. The momentum map exhibits the algebraic torus $(\mathbb{C}^\times)^2 \subset \mathbb{P}^2$ as a trivial $(S^1)^2$-bundle over $\operatorname{Int}\Xi$. Intrinsically, the 2-torus fibres of μ over $\operatorname{Int}(\Xi)$ are $\operatorname{Hom}(M, U(1))$. Over a face $\tau \subset \Xi$, the 2-torus fibre collapses via the quotient map given by restriction,

$$\operatorname{Hom}(M, U(1)) \longrightarrow \operatorname{Hom}(M \cap T_\tau, U(1)),$$

where $T_\tau \subset M_\mathbb{R}$ is the tangent space of τ. The quotient yields an S^1 over the interior of an edge of Ξ and a point over a vertex.

Now going over to the Kato-Nakayama space simply restores the collapsed directions, thus yielding the trivial product $\Xi \times (S^1)^2$. The fibre of $X^{\mathrm{KN}} \to X$ over the interior of a toric stratum given by the face $\tau \subset \Xi$ are the fibres of $\operatorname{Hom}(M, U(1)) \to \operatorname{Hom}(M \cap T_\tau, U(1))$.

An analogous discussion holds for all toric varieties with a momentum map.

We finish this section with an instructive non-toric example that features a non-fine log structure. It discusses the most simple non-toric example of a toric degeneration, the subject of Section 4.

Example 2.10. Let $X = \operatorname{Specan} \mathbb{C}[x, y, w^{\pm 1}, t]/(xy - t(w+1))$, considered as a holomorphic family of complex surfaces $\delta : X \to \mathbb{C}$ via projection by t. For fixed $t \neq 0$ we can eliminate w to arrive at $\delta^{-1}(t) \simeq \mathbb{C}^2$. For $t = 0$ we have $\pi^{-1}(0) = \mathbb{C}^2 \amalg_\mathbb{C} \mathbb{C}^2$, two copies of the affine plane with coordinates x, w and y, w, respectively, glued semi-normally along the line $x = y = 0$. Denote $X_0 = \delta^{-1}(0)$, let $\mathcal{M}_X = \mathcal{M}_{(X,X_0)}$ be the log structure defined by the family and $\mathcal{M}_{X_0}$ its restriction to the fibre over 0. Then $(X_0, \mathcal{M}_{X_0})$ comes with a log morphism f to the standard log point $O^\dagger = (\mathrm{pt}, \mathbb{C}^\times \oplus \mathbb{N})$. We want to discuss the Kato-Nakayama space of $(X_0, \mathcal{M}_{X_0})$ together with the map to S^1, the Kato-Nakayama space of $O^\dagger$.

First note that X has an A_1-singularity at the point p_0 with coordinates $x = y = t = 0$, $w = -1$. Any Cartier divisor at p_0 with support contained in X_0 is defined by a power of t. Hence $\overline{\mathcal{M}}_{X_0,p_0} = \mathbb{N}$, while at a general point p of the double locus $(X_0)_{\mathrm{sing}} \simeq \mathbb{C}$, the central fibre is a normal crossings divisor in a smooth space and $\overline{\mathcal{M}}_{X_0,p} = \mathbb{N}^2$. In particular, $\overline{\mathcal{M}}_{X_0}$ is not a fine sheaf at p_0. On the other hand $(X_0, \mathcal{M}_{X_0})$ is a typical example of Ogus' notion of *relative coherence*. In this category, the main result of [NO] still says that X^{KN} is homeomorphic relative $(\mathbb{C}, \mathcal{M}_\mathbb{C})^{\mathrm{KN}} = S^1 \times \mathbb{R}_{\geq 0}$ to $X_0^{\mathrm{KN}} \times \mathbb{R}_{\geq 0}$. In particular, the fibre of $f^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to S^1 = (O^\dagger)^{\mathrm{KN}}$ over $e^{i\phi} \in S^1$ is homeomorphic to $\mathbb{C}^2$. We want to verify this statement explicitly.

As a matter of notation we write s_x, s_y, s_t for the sections of $\mathcal{M}_X$ or of $\mathcal{M}_{X_0}$ defined by the monomial functions indicated in the subscripts. We also use s_t to denote the generator of the log structure $\mathcal{M}_{O^\dagger}$ of $O^\dagger$.

Since s_t generates $\mathcal{M}_{X_0,p_0}$ as a log structure, according to (2.3) the fibre of $\pi : X_0^{\mathrm{KN}} \to X_0$ over p_0 is a copy of $U(1)$, by mapping $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,p_0}, U(1))$ to its value on s_t. The projection to $S^1 = (O^\dagger)^{\mathrm{KN}}$ can then be viewed as the identity.

On the complement of p_0 the log structure is fine, but there is no global chart. We rather need two charts, defined on the open sets

$$\begin{aligned} U &= X_0 \setminus (x = y = 0) = \mathrm{Specan}\left(\mathbb{C}[x^{\pm 1}, w^{\pm 1}] \times \mathbb{C}[y^{\pm 1}, w^{\pm 1}]\right) \\ V &= X_0 \setminus (w = -1) = \mathrm{Specan}\, \mathbb{C}[x, y, w^{\pm 1}]_{w+1}, \end{aligned}$$

respectively. The charts are as follows:

$$\begin{aligned} \varphi : \mathbb{N} &\longrightarrow \Gamma(U, \mathcal{M}_{X_0}), \quad \varphi(1) = s_t. \\ \psi : \mathbb{N}^2 &\longrightarrow \Gamma(V, \mathcal{M}_{X_0}), \quad \varphi(a, b) = s_x^a \cdot s_y^b. \end{aligned}$$

Proposition 2.2 now exhibits U^{KN}, V^{KN} as closed subsets of $U \times U(1)$ and $V \times U(1)^2$, respectively. In each case, the projections to the $U(1)$-factors are defined by evaluation of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on monomials. We write these $U(1)$-valued functions defined on open subsets of X_0^{KN} by $\theta_t, \theta_x, \theta_y, \theta_w$ according to the corresponding monomial. Since $f : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ is strict over U, we have $U^{\mathrm{KN}} = U \times U(1)$ with $f^{\mathrm{KN}} = \theta_t$ the projection to $U(1)$. For V^{KN}, over the double locus $x = y = 0$ the fibre of the projection $V^{\mathrm{KN}} \to V$ is all of $U(1)^2$, while for $x \neq 0$ the value of θ_x is determined by $\arg x$. An analogous statement holds for $y \neq 0$.

To patch the descriptions of X_0^{KN} over the two charts amounts to understanding the map $V^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$, the image telling the value of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on s_t. Over $V = X_0 \setminus (w = -1)$ we have the equation $s_t = (w+1)^{-1} s_x s_y$. Thus, say over $x \neq 0$, we had the description of V^{KN} by the value of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on s_y. Then

$$\theta_t = \frac{\mathrm{Arg}(x)}{\mathrm{Arg}(w+1)} \cdot \theta_y. \tag{2.7}$$

Thus the identification with U^{KN} is twisted both by the phases of x and of $w + 1$. A similar description holds for $y \neq 0$.

For $t = \tau e^{i\phi} \neq 0$ denote by $X_0^{\mathrm{KN}}(e^{i\phi})$ the fibre over $e^{i\phi} \in U(1) = (O^\dagger)^{\mathrm{KN}}$ and similarly $U^{\mathrm{KN}}(e^{i\phi})$, $V^{\mathrm{KN}}(e^{i\phi})$. It is now not hard to construct a homeomorphism between $U^{\mathrm{KN}}(e^{i\phi}) \cup V^{\mathrm{KN}}(e^{i\phi})$ and $\delta^{-1}(t) \simeq \mathbb{C}^2 \setminus \{0\}$. For example, there exists a unique such homeomorphism that on $(x = 0) \subset U^{\mathrm{KN}}(e^{i\phi})$ restricts to

$$(\mathbb{C}^*)^2 \ni (y = se^{i\psi}, w) \longmapsto \left(\frac{(w+1) \cdot \tau e^{i(\phi - \psi)}}{s + |(w+1)\tau|^{1/2}}, (s + |(w+1)\tau|^{1/2}) e^{i\psi} \right) \in \mathbb{C}^2,$$

and to a similar map with the roles of x and y swapped on $(y = 0) \subset U^{\mathrm{KN}}(e^{i\phi})$. This form of the homeomorphism comes from considering the degeneration $xy = (w+1)t$

as a family of normal crossing degenerations of curves parametrized by $w = \text{const}$. Details are left to the reader.

Example 2.11. An alternative and possibly more useful way to discuss the Kato-Nakayama space of the degeneration $xy = (w+1)t$ in Example 2.10, is in terms of closed strata and the momentum maps of the irreducible components $Y_1 = (y = 0)$, $Y_2 = (x = 0)$ of X_0 and of their intersection $Z = Y_1 \cap Y_2$. Endow Y_1, Y_2, Z with the log structures making the inclusions into X_0 strict. Away from p_0 we then have global charts defined by s_t, s_x for Y_1, by s_t, s_y for Y_2 and by s_x, s_y for Z. By functoriality, the fibre of $\pi : X_0^{\mathrm{KN}} \to X_0$ over these closed strata Y_1, Y_2, Z agrees with Y_1^{KN}, Y_2^{KN}, Z^{KN}, respectively. Therefore, we can compute X_0^{KN} as the fibred sum

$$X_0^{\mathrm{KN}} = Y_1^{\mathrm{KN}} \amalg_{Z^{\mathrm{KN}}} Y_2^{\mathrm{KN}}.$$

Away from the singular point $p_0 \in X_0$ of the log structure, Y_1^{KN} is the Kato-Nakayama space of Y_1 as a toric variety times an additional S^1-factor coming from s_t, and similarly for Y_2. Since each Y_i has a momentum map μ_i with image the half-plane $\mathbb{R}_{\geq 0} \times \mathbb{R}$, Proposition 2.5 gives a description of Y_i^{KN} as $\mathbb{R}_{\geq 0} \times \mathbb{R} \times U(1)^3/\sim$ with the $U(1)$-factors telling the phases of w, s_t and of s_x (for $i = 1$) or of s_y (for $i = 2$), respectively. We assume that the momentum map maps p_0 to $(0,0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Note that $w \neq 0$, so the phase of w is already determined uniquely at any point of X_0. The indicated quotient takes care of the special point p_0 by collapsing a $U(1)$ over $(x, y, w) = (0, 0, -1)$ as follows. Restricting the projection

$$\mathbb{R}_{\geq 0} \times \mathbb{R} \times U(1)^3 \longrightarrow Y_1$$

to $x = 0$ yields a $U(1)^2$-bundle over $\mathbb{C}^*$, the w-plane. The two $U(1)$-factors record the phases of s_t and s_x, respectively. Now the quotient collapses the second $U(1)$-factor over $w = -1$, reflecting the fact that only s_t survives in $\mathcal{M}_{X_0,p_0}$.

Again by functoriality, the restriction of either Y_i^{KN} to $\{0\} \times \mathbb{R}$ yields Z^{KN}. Using s_x, s_y as generators for $\mathcal{M}_Z$ over $Z \setminus \{p_0\} = \mathbb{C}^* \setminus \{-1\}$ we see

$$Z^{\mathrm{KN}} \ = \ \mathbb{R} \times U(1)^3/\sim \ = \ \mathbb{C}^* \times U(1)^2/\sim$$

Now the three $U(1)$-factors tell the phases θ_w, θ_x, θ_y of w, s_x, s_y. The equivalence relation collapses the $U(1)$-subgroup

$$\big\{(\theta_x, \theta_y) \in U(1)^2 \,\big|\, \theta_x \cdot \theta_y = 1\big\}$$

over $-1 \in \mathbb{C}^*$. Thus over the circle $|w| = a$ inside the double locus $x = y = 0$, X_0^{KN} is a trivial $U(1)^2$-bundle as long as $a \neq 1$, hence a 3-torus. This 3-torus fibres as a trivial bundle of 2-tori over $(O^\dagger)^{\mathrm{KN}} = S^1$. If $a = 1$, one of the $U(1)$-factors collapses to a point over $w = -1$, leading to a trivial family of pinched 2-tori over S^1.

A nontrival torus fibration arises if we consider a neighbourhood of the double locus. This is most easily understood by viewing X_0^{KN} as a torus fibration over $\mathbb{R}^2$ by taking the union of the momentum maps

$$\mu : X_0 \longrightarrow \mathbb{R}^2, \quad \mu|_{Y_1} = \mu_1, \quad \mu|_{Y_2} = \kappa \circ \mu_2,$$

with $\kappa(a,b) = (-a,b)$. Denote by $X_0^{\mathrm{KN}}(e^{i\phi})$ the fibre of $X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = S^1$ over $e^{i\phi} \in S^1$. Write $\mu^{\mathrm{KN}} = \pi \circ \mu : X_0^{\mathrm{KN}} \to \mathbb{R}^2$ and $\mu^{\mathrm{KN}}(e^{i\phi})$ for the restriction to $X_0^{\mathrm{KN}}(e^{i\phi})$. For any $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ the fibre $(\mu^{\mathrm{KN}})^{-1}(a,b)$ is a 3-torus trivially fibred by 2-tori over $(O^\dagger)^{\mathrm{KN}} = S^1$. We also have trivial torus bundles over the half-spaces $(\mathbb{R}_{\geq 0} \times \mathbb{R}) \setminus \{(0,0)\}$ and $(\mathbb{R}_{\leq 0} \times \mathbb{R}) \setminus \{(0,0)\}$ as well as over $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$. However, the torus bundle is non-trivial over any loop about $(0,0) \in \mathbb{R}^2$. The reason is that the equation $xy = t(w+1)$ gives the identification of torus fibrations over the two half planes via

$$\theta_y = \theta_x^{-1} \cdot \mathrm{Arg}(w+1) \cdot \theta_t.$$

Now $\mathrm{Arg}(w+1)$ restricted to the circle $|w| = a$ with $a < 1$ is homotopic to a constant map, while for $a > 1$ this restriction has winding number 1. This means that for $e^{i\phi} \in S^1$, the topological monodromy of the 2-torus fibration $\mu^{\mathrm{KN}}(e^{i\phi}) : X_0^{\mathrm{KN}}(e^{i\phi}) \to \mathbb{R}^2$ along a counterclockwise loop about $(0,0) \in \mathbb{R}^2$ is a (negative) Dehn-twist. Thus $\mu^{\mathrm{KN}}(e^{i\phi})$ is homeomorphic to a neighbourhood of an I_1-singular fibre (a nodal elliptic curve) of an elliptic fibration of complex surfaces.

3. The Kato-Nakayama space of a real log space

Let us now combine the topics of Sections 1 and 2 and consider the additional structure on the Kato-Nakayama space of a log space induced by a real structure. Throughout this section we identify $\mathbb{Z}/2\mathbb{Z}$ with the multiplicative group with two elements ± 1.

The conjugation involution on $\mathbb{C}$ lifts to the log structure of the polar log point $\mathcal{M}_\Pi = \mathbb{R}_{\geq 0} \times U(1)$ by putting $\iota_\Pi^\flat(r, e^{i\varphi}) = (r, e^{-i\varphi})$.

Definition 3.1. Let $(X, \mathcal{M}_X)$ be a real log space with $(\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \to (X, \mathcal{M}_X)$ its real involution. We call the map

$$\iota_X^{\mathrm{KN}} : X^{\mathrm{KN}} \longrightarrow X^{\mathrm{KN}}, \quad \big(f : \Pi^\dagger \to (X, \mathcal{M}_X)\big) \longrightarrow \iota_X^\flat \circ f \circ \iota_\Pi^\flat.$$

the *lifted real involution*.

Proposition 3.2. *The lifted real involution ι_X^{KN} is continuous and is compatible with the underlying real involution ι_X of X under the projection $\pi : X^{\mathrm{KN}} \to X$.*

Proof. Both statements are immediate from the definitions. □

By the definition and proposition we thus see that the real locus of X has a canonical lift to X^{KN}.

Definition 3.3. Let $(X, \mathcal{M}_X)$ be a real log space and $\iota_X^{\mathrm{KN}} : X^{\mathrm{KN}} \to X^{\mathrm{KN}}$ the lifted real involution. We call the fixed point set of ι_X^{KN} the *real locus of* X^{KN}, denoted $X_{\mathbb{R}}^{\mathrm{KN}} \subset X^{\mathrm{KN}}$.

To describe the real locus in toric degenerations, one main interest in this paper is the study of the real locus $X_{\mathbb{R}}^{\mathrm{KN}} \subset X^{\mathrm{KN}}$. We first discuss the fibres of the restriction $\pi_{\mathbb{R}} : X_{\mathbb{R}}^{\mathrm{KN}} \to X_{\mathbb{R}}$ of the projection $\pi : X^{\mathrm{KN}} \to X$. If $x \in X_{\mathbb{R}}$ then $\iota_X^\flat$ induces an involution on $\mathcal{M}_{X,x}$ and on the quotient $\overline{\mathcal{M}}_{X,x} = \mathcal{M}_{X,x}/O_{X,x}^\times$. If the involution on $\overline{\mathcal{M}}_{X,x}$ is trivial, $\pi_{\mathbb{R}}^{-1}(x)$ is easy to describe. Recall from (2.3) that $\pi^{-1}(x)$ can be identified with the set of homomorphisms $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ given on invertible functions $h \in \mathcal{O}_{X,x}^\times$ by $\theta(h) = h(x)/|h(x)|$.

Proposition 3.4. *Let $(X, \mathcal{M}_X)$ be a real log space and $x \in X_{\mathbb{R}}$.*

(1) *In the description (2.3) of $\pi^{-1}(x)$, an element $\theta \in \mathrm{Hom}(\mathcal{M}_{X,x}^{\mathrm{gp}}, U(1))$ lies in $X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $\theta \circ \iota_{X,x}^\flat = \overline{\theta}$, the complex conjugation of θ.*

(2) *If $\iota_X^\flat$ induces a trivial action on $\overline{\mathcal{M}}_{X,x}$, then $\pi_{\mathbb{R}}^{-1}(x)$ is canonically a torsor for the group $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$.*

Proof. 1) Let $\tilde{x} \in \pi^{-1}(x)$, given by a log morphism $f : \Pi^\dagger \to (X, \mathcal{M}_X)$ with image x. Then $\tilde{x} \in X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $f \circ \iota_\Pi = \iota_X \circ f$. Now writing $\tilde{x} = (x, \theta)$ as in (2.3), we have $f \circ \iota_\Pi = (x, \overline{\theta})$ and $\iota_X \circ f = (x, \theta \circ \iota_{X,x}^\flat)$. Comparing the two equations yields the statement.
2) Denote by $\kappa : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ the quotient homomorphism. We define the action of $\sigma \in \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$ on $\pi^{-1}(x)$ by

$$\big(\sigma \cdot \theta\big)(s) = \sigma(\kappa_x(s)) \cdot \theta(s)$$

for $\theta \in \mathrm{Hom}(\mathcal{M}_{X,x}^{\mathrm{gp}}, U(1))$. In this definition we take $\sigma(\kappa_x(s)) \in \mathcal{O}_{X,x}^\times$ by means of the identifcation $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$. Now by (1) together with the additional hypotheses, θ defines a point in $X_{\mathbb{R}}^{\mathrm{KN}}$ iff $\theta = \overline{\theta}$. This is the case iff θ takes values ± 1. This condition is preserved by the action of $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$. Conversely, if θ_1, θ_2 define elements in $\pi_{\mathbb{R}}^{-1}(x) \subset X_{\mathbb{R}}^{\mathrm{KN}}$ then they both take values in $\mathbb{Z}/2\mathbb{Z} \subset U(1)$ and in any case they agree on $\mathcal{O}_{X,x}^\times$. Thus $\theta_1 \circ \theta_2^{-1}$ factors over the quotient map $\kappa_x : \mathcal{M}_{X,x}^{\mathrm{gp}} \to \overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}$ to define a homomorphism $\sigma : \overline{\mathcal{M}}_{X,x}^{\mathrm{gp}} \to \mathbb{Z}/2\mathbb{Z}$. Then $\theta_1 = \sigma \cdot \theta_2$, showing that the action is simply transitive. □

Concretely, in the fine saturated case, Proposition 3.4,(2) says that if the stalk of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ at $x \in X_{\mathbb{R}}$ has rank r, and $\iota_x^\flat$ induces a trivial action on $\overline{\mathcal{M}}_{X,x}$, then $\pi^{-1}(x)$

consists of 2^r points. This seems to contradict the expected smoothness of $X_{\mathbb{R}}^{\mathrm{KN}}$ in log smooth situations, but we will see in the toric situation how this process can sometimes merely separate sheets of a branched cover. The reason is that the real picture interacts nicely with the momentum map description of X^{KN}.

Proposition 3.5. *Let $(X, \mathcal{M}_X)$ be a toric variety with its toric log structure and $\mu : X \to \Xi \subset M_{\mathbb{R}}$ a momentum map. Let ι_X be the unique real structure on $(X, \mathcal{M}_X)$ lifting the standard real structure according to Proposition 1.4. Then there is a canonical decomposition*

$$X_{\mathbb{R}}^{\mathrm{KN}} \simeq \Xi \times \mathrm{Hom}(M, \mathbb{Z}/2\mathbb{Z}),$$

with the projection to Ξ giving the composition $\mu \circ \pi_{\mathbb{R}} : X_{\mathbb{R}}^{\mathrm{KN}} \to \Xi$.

Proof. Recall the section $\sigma : \Xi \to X$ of the momentum map with image $X_{\geq 0} \subset X_{\mathbb{R}}$. For $x \in \Xi$, Proposition 2.5 identifies $\pi^{-1}(\mu^{-1})(x) \subset X^{\mathrm{KN}}$ with pairs $(\lambda \cdot \sigma(x), \lambda) \in X \times \mathrm{Hom}(M, U(1))$. The action of ι_X^{KN} on this fibre is

$$(\lambda \cdot \sigma(x), \lambda) \longmapsto (\overline{\lambda} \cdot \sigma(x), \overline{\lambda}).$$

Thus $(\lambda \cdot \sigma(x), \lambda)$ gives a point in $X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $\lambda = \overline{\lambda}$. This is the case iff λ takes values in $\mathbb{R} \cap U(1) = \{\pm 1\}$, giving the result. $\square$

Without the assumption of a trivial action on the ghost sheaf $\overline{\mathcal{M}}_{X,x}$, the fibre of $X_{\mathbb{R}}^{\mathrm{KN}} \to X_{\mathbb{R}}$ can be non-discrete.

Example 3.6. Let X be a complex variety with a real structure $\underline{\iota}_X$ and a $\underline{\iota}_X$-invariant simple normal crossings divisor D with two irreducible components D_1, D_2. Assume there is a real point $x \in D_1 \cap D_2$ and $\underline{\iota}_X$ exchanges the two branches of D at x. Denote by $\iota_X^\flat$ the induced real structure on $\mathcal{M}_X = \mathcal{M}_{(X,D)}$ according to Proposition 1.4. Then $P = \overline{\mathcal{M}}_{X,x} = \mathbb{N}^2$ and $\iota_{X,x}^\flat(a, b) = (b, a)$. The action extends to an involution ι_M of $M = P^{\mathrm{gp}} = \mathbb{Z}^2$. In the present case there is a subspace $M' \subset M$ with $M' \oplus \iota_M(M') = M$, e.g. $M' = \mathbb{Z} \cdot (1, 0)$. Then $\theta : M \to U(1)$ can be prescribed arbitrarily on M' and extended uniquely to M by enforcing $\theta \circ \iota_M = \overline{\theta}$. Thus in the present case $\pi_{\mathbb{R}}^{-1}(x) = \mathrm{Hom}(\mathbb{Z}, U(1)) = S^1$.

In the general case, say with $\overline{\mathcal{M}}_{X,x}$ a fine monoid, we can write $\mathcal{M}_{X,x}^{\mathrm{gp}} = M \oplus \mathcal{O}_{X,x}^\times$ with M a finitely generated abelian group and such that $\iota_{X,x}^\flat$ acts by an involution ι_M on M and by $\iota_{X,x}^\sharp$ on $\mathcal{O}_{X,x}^\times$. Then $\pi^{-1}(x) = \mathrm{Hom}(M, U(1))$ is a disjoint union of tori, one copy of $\mathrm{Hom}(M/T, U(1))$ for each element of the torsion subgroup $T \subset M$. The fibres $\pi_{\mathbb{R}}^{-1}(x)$ for $x \in X_{\mathbb{R}}$ are the preimage of the diagonal torus of the map

$$\mathrm{Hom}(M, U(1)) \longrightarrow \mathrm{Hom}(M, U(1)) \times \mathrm{Hom}(M, U(1)), \quad \theta \longmapsto (\theta \circ \iota_M, \overline{\theta}).$$

4. The case of toric degenerations

4.1. **Toric degenerations and their intersection complex.** We now focus attention to toric degenerations, as first introduced in [GS1], Definition 4.1. As already stated in the introduction, a *toric degeneration* in this sense is a proper flat map of normal connected schemes $\delta : \mathfrak{X} \to \operatorname{Spec} R$ with R a discrete valuation ring and such that the central fibre X_0 is a reduced union of toric varieties; the toric irreducible components of X_0 are glued pairwise along toric strata in such a way that the dual intersecting complex is a closed topological manifold, of the same dimension n as the fibres of δ. In particular, the notion of toric strata of X_0 makes sense. It is then also required that near each zero-dimensional toric stratum of X_0, étale locally δ is isomorphic to a monomial map of toric varieties. Since R is a discrete valuation ring this amounts to describing $\mathfrak{X}$ étale locally as $\operatorname{Spec} \mathbb{C}[P]$ with P a toric monoid and f by one monomial $t = z^{\rho_P}$, $\rho_P \in P$. This last formulation then holds locally outside a closed subset $Z \subset X_0$ of codimension 2 and not containing any zero-dimensional toric strata. For the precise list of conditions we refer to [GS1], Definition 4.1. Under these conditions it turns out that the generic fibre $\mathfrak{X}_\eta$ is a Calabi-Yau variety.

We refer to [GS3], §1 for a more thorough review of toric degenerations as described here. Various generalizations of toric degenerations have also been considered, notably including dual intersection complexes that are non-compact or have non-empty boundary [CPS], higher dimensional base spaces [GHK],[GHKS] and log singular loci containing zero-dimensional toric strata [GHK]. While much of the following discussion holds in these more general setups, to keep the presentation simple we restrict ourselves to the original Calabi-Yau case.

Asuming that X_0 is projective, let $\mathscr{P} = \{\sigma\}$ be the set of momentum polytopes of the toric strata and $\mathscr{P}_{\max} \subset \mathscr{P}$ the maximal elements under inclusion. For $\tau \in \mathscr{P}$ we denote by $X_\tau \subset X_0$ the correspoinding toric stratum. View $B = \bigcup_{\sigma \in \mathscr{P}_{\max}} \sigma$ as a cell complex with attaching maps defined by the intersection patterns of the toric strata. The barycentric subdivision of $(B, \mathscr{P})$ is then canonically isomorphic to the barycentric subdivision of the dual intersection complex of X_0, as simplicial complexes. Thus B is a topological manifold. There is a generalized momentum map $\mu : X_0 \to B$ that restricts to the toric momentum map $X_\tau \to \tau$ on each toric stratum of X_0 ([RS], Proposition 3.1). Unlike in [GS1], for simplicity of notation we assume that no irreducible component of X_0 self-intersects. On the level of the cell complex $(B, \mathscr{P})$ this means that for any $\tau \in \mathscr{P}$ the map $\tau \to B$ is injective. We call $(B, \mathscr{P})$ the *intersection complex* or *cone picture* of the polarized central fibre X_0.

The log structure $\mathcal{M}_{X_0}$ on X_0 induced from the degeneration can be conveniently described as follows. At a general point of $(X_0)_{\mathrm{sing}}$, exactly two irreducible components

X_σ, $X_{\sigma'} \subset X_0$ intersect. At such a point there is a local description of $\mathfrak{X}$ of the form

$$uv = f(z_1, \ldots, z_{n-1}) \cdot t^\kappa, \tag{4.1}$$

with t a generator of the maximal ideal of R, $z_1, \ldots, z_{n-1}$ toric coordinates for the maximal torus of $X_\sigma \cap X_{\sigma'}$ and u, v restricting either to 0 or to a monomial on X_σ, $X_{\sigma'}$. One of the main results of [GS1] is the statement that the restriction of the function f is well-defined after choosing u, v and that this restriction classifies $\mathcal{M}_{X_0}$. The zero locus of f in X_ρ specifies the locus where the log structure $\mathcal{M}_{X_0}$ is not fine. Thus $\mathcal{M}_{X_0}$ is fine outside a closed subset $Z \subset (X_0)_{\mathrm{sing}}$ of codimension two, a union of hypersurfaces on the irreducible components X_ρ of X_0, $\dim \rho = n-1$. Conversely, there is a sheaf on X_0 with support on $(X_0)_{\mathrm{sing}}$ which is an invertibe $\mathcal{O}_{X_0}$-module on the open dense subset where X_0 is normal crossings, with sections classifiying log structures arising from a local embedding into a toric degeneration (see [GS1], Theorem 3.22 and Definition 4.21). Thus the moduli space of log structures on X_0 that look like coming from a toric degeneration can be explicitly described by an open subset of $\Gamma((X_0)_{\mathrm{sing}}, \mathcal{F})$ for some coherent sheaf $\mathcal{F}$ on $(X_0)_{\mathrm{sing}}$.

The ghost sheaf $\overline{\mathcal{M}}_{X_0}$ can be read off from a multivalued piecewise affine function φ on $\mathscr{P}$. This function is uniquely described by one integer κ_ρ on each codimension one cell $\rho \in \mathscr{P}$. If $uv = f \cdot t^\kappa$ is the local description of $\mathfrak{X}$ at a general point of X_ρ, then $\kappa_\rho = \kappa$. Each codimension two cell τ imposes a linear condition on the κ_ρ for all $\rho \supset \tau$ assuring the existence of a local single-valued representative of φ in a neighbourhood of τ (see [GHKS], Example 1.11). Note that the local representative φ is only defined up to a linear function. Thus globally φ can be viewed as a *multi-valued piecewise linear function*, a section of the sheaf of pieceweise linear functions modulo linear functions. We write $(B, \mathscr{P}, \varphi)$ for the complete tuple of discrete data associated to a toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$, still refereed to as the intersection complex (now *polarized* by φ).

The interpretation of the cells of $\mathscr{P}$ as momentum polyhedra endows B with the structure of an *integral affine manifold* on the interiors of the maximal cells, that is, a manifold with coordinate changes in $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$. On such manifolds it makes sense to talk about integral points as the preimage of $\mathbb{Z}^n$ under any chart, and they come with a local system Λ of integral tangent vectors. An important insight is that the log structure on X_0 provides a canonical extension of this affine structure over the complement in B of the amoeba image $\mathcal{A} := \mu(Z)$ of the log singular locus $Z \subset (X_0)_{\mathrm{sing}}$. In the local description at a codimension one cell $\rho = \sigma \cap \sigma'$, the affine structure of the adjacent maximal cells σ, σ' already agree on their common face ρ. So the extension at $x \in \mathrm{Int}\, \rho \setminus \mathcal{A}$ only requires the identification of $\xi \in \Lambda_{\sigma,x}$ with

$\xi' \in \Lambda_{\sigma',x}$, each complementary to $\Lambda_{\rho,x}$. In a local description $uv = f \cdot t^{\kappa_\rho}$ we have $u|_{X_\sigma} = z^m$, $v|_{X_{\sigma'}} = z^{m'}$ by the assumption on u, v to be monomial on one of the adjacent components X_σ, $X_{\sigma'}$. One then takes $\xi = m$, $\xi' = -m' + m_x$. Here $m_x \in \Lambda_\rho$ is defined by the homotopy class of $f|_{\mu^{-1}(x)}$, see [RS], Construction 2.2 for details. This defines the integral affine structure on $B \setminus \mathcal{A}$ away from codimension two cells.

Lemma 4.1. *The integral affine structure on the interiors of the maximal cells $\sigma \in \mathscr{P}$ and on* $\operatorname{Int}\rho \setminus \mathcal{A}$ *for all codimension one cells ρ extends uniquely to $B \setminus \mathcal{A}$.*

Proof. Uniqueness is clear because the extension is already given on an open and dense subset.

At a vertex $v \in B$ we have $\mu^{-1}(v) = X_v$, a zero-dimensional toric stratum. Let $U \to \operatorname{Specan}\mathbb{C}[P]$ with $P = K \cap \mathbb{Z}^{n+1}$ and $t = z^{\rho_P}$, $\rho_P \in P$, be a toric chart for $\delta : \mathfrak{X} \to \operatorname{Specan} R$ at X_v. Here K is an $(n+1)$-dimensional rational polyhedral cone, not denoted $\sigma^\vee$ to avoid confusion with the cells of B. There is then a local identification of μ with the composition

$$\mu_v : \operatorname{Specan}\mathbb{C}[P] \xrightarrow{\mu_P} K \longrightarrow \mathbb{R}^{n+1}/\mathbb{R} \cdot \rho_P$$

of the momentum map for $\operatorname{Specan}\mathbb{C}[P]$ with the projection from the cone K along the line through ρ_P. Since $\rho_P \in \operatorname{Int} K$, this map projects ∂K to a complete fan Σ_v in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \rho_P$. The irreducible components of X_0 containing X_v have affine toric charts given by the facets of K. Thus this fan describes X_0 at X_v as a gluing of affine toric varieties. Now any momentum map μ of a toric variety provides an integral affine structure on the image with $R^1\mu_*\underline{\mathbb{Z}}$ the sheaf of integral tangent vectors on the interior. In the present case, this argument shows first that the restriction of $R^1\mu_{v*}\underline{\mathbb{Z}}$ to the interior of each maximal cone $K' \in \Sigma_v$ can be canonically identified with the sheaf of integral tangent vectors Λ on the interiors of maximal cells of B. Second, the argument shows that $R^1\mu_{v*}\underline{\mathbb{Z}}$ restricted to $\operatorname{Int} K'$ can be identified with the (trivial) local system coming from the integral affine structure provided by $\mathbb{Z}^{n+1}/\mathbb{Z} \cdot \rho_P$. The fan thus provides an extension of the sheaf Λ over a neighbourhood of σ and hence also of the integral affine structure. A possible translational part in the local monodromy does not arise by the given gluing along lower dimensional cells.

For any $\tau \in \mathscr{P}$, the extension at the vertices of $\mathscr{P}$ provides also the extension on any connected component of $\tau \setminus \mathcal{A}$ containing a vertex. If $\mathcal{A} \cap \tau$ has connected components not containing a vertex, one can in any case show the existence of a toric model with fan $\partial K/\mathbb{R} \cdot \rho_P$ of not necessarily strictly convex rational polyhedral cones. The argument given at a vertex then works analogously. $\square$

4.2. **The Kato-Nakayama space of a toric degeneration.** Throughout the following discussion we fix $(X_0, \mathcal{M}_{X_0})$ the central fibre of a toric degeneration with log singular locus $Z \subset (X_0)_{\mathrm{sing}}$, $(B, \mathscr{P})$ its intersection complex and $\mu : X_0 \to B$ a momentum map. The main result of the section gives a canonical description of the Kato-Nakayama space of $(X_0, \mathcal{M}_{X_0})$ as a torus bundle over B, away from the amoeba image $\mathcal{A} = \mu(Z) \subset B$. We denote by

$$\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \xrightarrow{\pi} X_0 \xrightarrow{\mu} B$$

the composition of the projection of the Kato-Nakayama space with the momentum map and write $Z^{\mathrm{KN}} = \pi^{-1}(Z) \subset X_0^{\mathrm{KN}}$. We also fix once and for all a generator t of the maximal ideal of R and accordingly identify the closed point in $\operatorname{Spec} R$ with the induced log structure with the standard log point $O^\dagger$. Thus we have a log morphism $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$, inducing a continuous map $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$.

Our interest in X_0^{KN} comes from the fact that it captures the topology of an analytic family inducing the given log structure on X_0, for a large class of spaces. This statement is based on a result of Nakayama and Ogus, which involves the following generalization of the notion of a fine log structure. A log space $(X, \mathcal{M}_X)$ is *relatively coherent* if locally in X the log structure $\mathcal{M}_X$ is isomorphic to a sheaf of faces of a fine log structure.

Theorem 4.2. *(*[NO]*, Theorem 5.1) Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a proper, separated, exact and relatively smooth morphism of log analytic spaces, with $(Y, \mathcal{M}_Y)$ fine and $(X, \mathcal{M}_X)$ relatively coherent. Then $f^{\mathrm{KN}} : (X, \mathcal{M}_X)^{\mathrm{KN}} \to (Y, \mathcal{M}_Y)^{\mathrm{KN}}$ is a topological fibre bundle with fibres oriented manifolds with boundary.*

Being a topological fibre bundle says that f^{KN} is locally in Y homeomorphic to the projection from a product. The technical heart of the proof is a local product decomposition for maps of real cones induced by exact homomorphisms of fine monoids ([NO], Theorem 0.2). From this result it follows easily that f^{KN} is a topological submersion, that is, locally in X a projection of a product ([NO], Theorem 3.7). In a final step one applies a result of Siebenmann ([Si], Corollary 6.14) to conclude the fibre bundle property.

We can verify the hypothesis of Theorem 4.2 for analytic smoothings of $(X_0, \mathcal{M}_{X_0})$ for the case of *simple singularities*. The notion of simple singularities has been introduced in [GS1] as an indecomposability condition on the local affine monodromy around the singular locus $\Delta \subset B$ of the affine structure on the dual intersection complex of $(X_0, \mathcal{M}_{X_0})$. It implies local rigidity of the singular locus of the log structure as needed in the smoothing algorithm ([GS3], Definition 1.26), but unlike local rigidity, being simple imposes conditions in all codimensions.

Proposition 4.3. *Let $(X_0, \mathcal{M}_{X_0})$ be the central fibre of a troic degeneration with simple singularities. Then $(X_0, \mathcal{M}_{X_0}) \to \operatorname{Spec} O^\dagger$ as well as any analytic family $\mathcal{X} \to D = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ with X_0 as central fibre and inducing the given log structure $\mathcal{M}_{X_0}$, fulfills the conditions of Theorem 4.2. In particular, $\delta^{\mathrm{KN}} : (X_0, \mathcal{M}_{X_0}) \to U(1)$ and $(\mathcal{X}, \mathcal{M}_{\mathcal{X},X_0})^{\mathrm{KN}} \to (D, \mathcal{M}_{D,0})^{\mathrm{KN}}$ are topological fibre bundles with fibres closed manifolds.*

Proof. Under the assumptions of simple singularities, [GS2], Theorem 2.11 and Corollary 2.17 show that $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ as well as any analytic family inducing the given log structure on X_0 away from codimension three are relatively coherent. The log structure on the parameter space being generated by one element, exactness is trivial. Moreover, δ is *vertical* as a log morphism (the image of $\delta^\flat$ is not contained in any proper face), and hence the fibres have no boundary according to [NO], Theorem 5.1. □

The preceding discussion motivates the study of $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to U(1)$. First we show that the log singular locus can be dealt with by taking closures, even stratawise. For $\tau \in \mathscr{P}$ denote by $X_\tau^{\mathrm{KN}} = \pi^{-1}(X_\tau) \subset X_0^{\mathrm{KN}}$ and by $Z_\tau^{\mathrm{KN}} = Z^{\mathrm{KN}} \cap X_\tau^{\mathrm{KN}}$.

Lemma 4.4. *On each toric stratum $X_\tau \subset X_0$, the preimage $Z_\tau^{\mathrm{KN}} \subset X_\tau^{\mathrm{KN}}$ of the log singular locus is a nowhere dense closed subset.*

Proof. It suffices to prove the statement over an irreducible component $X_\sigma \subset X_0$, $\sigma \subset B$ a maximal cell. Let $x \in Z \cap X_\sigma$ and $X_\tau \subset X_\sigma$ the minimal toric stratum containing x. Since Z does not contain zero-dimensional toric strata, x is not the generic point $\eta \in X_\tau$. We claim that the generization map $\chi_{\eta x} : \overline{\mathcal{M}}_{X_0,x} \to \overline{\mathcal{M}}_{X_0,\eta}$ is injective. In fact, $\mathcal{M}_{X_0}$ is locally the divisorial log structure for a toric degeneration. Hence the stalks of $\overline{\mathcal{M}}_{X_0}$ are canonically a submonoid of $\mathbb{N}^r$ with r the number of irreducible components of X_0 containing X_τ, say $X_{\sigma_1}, \ldots, X_{\sigma_r}$. An element $a \in \mathbb{N}^r$ lies in $\overline{\mathcal{M}}_{X_0,x}$ iff $\sum a_i X_{\sigma_i}$ is locally at x a Cartier divisor, in a local description as the central fibre of a toric degeneration. In any case, both $\overline{\mathcal{M}}_{X_0,x}$ and $\overline{\mathcal{M}}_{X_0,\eta}$ are submonoids of the same $\mathbb{N}^r$, showing the claimed injectivity of $\chi_{\eta x}$.

Now take a chart $\overline{\mathcal{M}}_{X_0,\eta} \to \Gamma(U, \mathcal{M}_{X_0})$ with U a Zariski-open neighbourhood of η in $X_0 \setminus Z$. Then Proposition 2.2 yields a canonical homeomorphism $\pi^{-1}(U) = U \times \operatorname{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1))$. By the definition of the topology on X_0^{KN}, the composition

$$U \times \operatorname{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1)) \longrightarrow \operatorname{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1)) \longrightarrow \operatorname{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,x}, U(1))$$

of the projection with pull-back by the generization map $\chi^*_{\eta x}$ is continuous. By injectivity of $\chi_{\eta x}$ this composition is surjective. Since $x \in \operatorname{cl}(U)$ we conclude that $\pi^{-1}(x)$ is contained in the closure of $\pi^{-1}(U)$, showing the desired density. □

For $\sigma \in \mathscr{P}_{\max}$ denote by $\mathcal{M}_{X_\sigma}$ the toric log structure for the irreducible component $X_\sigma \subset X_0$ and by X_σ^{KN} its Kato-Nakayama space. By [GS1], Lemma 5.13, there is a canonical isomorphism

$$\mathcal{M}_{X_0}^{\mathrm{gp}}|_{X_\sigma \setminus Z} \simeq \mathcal{M}_{X_\sigma}^{\mathrm{gp}}|_{X_\sigma \setminus Z} \oplus \mathbb{Z}, \tag{4.2}$$

the $\mathbb{Z}$-factor generated by the generator t of $\mathfrak{m}_R$ chosen above.

Lemma 4.5. *For $\sigma \in \mathscr{P}_{\max}$ denote by $\Lambda_\sigma = \Gamma(\mathrm{Int}\,\sigma, \Lambda)$ the group of integral tangent vector fields on σ. Then there is a canonical continuous surjection*

$$\Phi_\sigma : \sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) \longrightarrow (\mu^{\mathrm{KN}})^{-1}(\sigma) \subset X_0^{\mathrm{KN}},$$

which is a homeomorphism onto the image over the complement of the log singular locus $Z \subset X_0$.

With respect to the product decomposition

$$\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) = X_\sigma^{\mathrm{KN}} \times U(1),$$

of the domain of Φ, the restrictions of $\pi : X_0^{\mathrm{KN}} \to X_0$ and $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to U(1)$ to $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ are given by the projection to X_σ^{KN} followed by $X_\sigma^{\mathrm{KN}} \to X_\sigma$ and by the projection to $U(1)$, respectively.

Proof. By Lemma 4.4 we may establish the result away from Z and then extend Φ_σ by continuity. For any $x \in X_\sigma \setminus Z$, the isomorphism (4.2) establishes a canonical bijection

$$\mathrm{Hom}(\mathcal{M}_{X_0,x}^{\mathrm{gp}}, U(1)) \longrightarrow \mathrm{Hom}(\mathcal{M}_{X_\sigma,x}^{\mathrm{gp}}, U(1)) \times \mathrm{Hom}(\mathbb{Z}, U(1)).$$

This bijection is compatible with the fibrewise description of the Kato-Nakayama spaces of X_0 and of X_σ in (2.3), respectively, as well as with the definition of the topology. Now varying $x \in X_\sigma \setminus Z$, Proposition 2.5 turns the first factor into the complement of a closed, nowhere dense subset (to become $\Phi_\sigma^{-1}(Z^{\mathrm{KN}})$) in $\sigma \times \mathrm{Hom}(\Lambda_\sigma, U(1))$. The inverse of this description of $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ over the complement $X_\sigma^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$ defines the map Φ_σ over $X_0^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$.

The statements in the second paragraph are immediate from the definitions. □

Proposition 4.6. *Away from the amoeba image $\mathcal{A} \subset B$ of the log singular locus $Z \subset (X_0)_{\mathrm{sing}}$, the projection $\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to B$ is a bundle of real $(n+1)$-tori. Similarly, over $B \setminus \mathcal{A}$ the restriction of μ^{KN} to a fibre of $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$ is a bundle of n-tori.*

Proof. For $\sigma \in \mathscr{P}_{\max}$ denote by $T_\sigma = \mathrm{Hom}(\Lambda_\sigma, U(1)) \times U(1)$ the $(n+1)$-torus fibre of μ^{KN} over σ in the description of Lemma 4.5. For $x \in B \setminus \mathcal{A}$ let $\tau \in \mathscr{P}$ be the unique cell with $x \in \mathrm{Int}\,\tau$. Let $n = \dim B$ and $k = \dim \tau$. Then in an open contractible neighbourhood $U \subset B \setminus \mathcal{A}$ of x, the polyhedral decomposition $\mathscr{P}$ looks like the product of $\Lambda_\tau \otimes_{\mathbb{Z}} \mathbb{R}$ with an $n-k$-dimensional complete fan Σ_τ in the vector space with lattice $\Lambda_x/\Lambda_{\tau,x}$. Over each maximal cell σ containing τ, we have the canonical homeomorphism of $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ with $\sigma \times \mathrm{Hom}(\Lambda_\sigma, U(1)) \times U(1)$ provided by Lemma 4.5. Thus for any pair of maximal cells $\sigma, \sigma' \supset \tau$ we obtain a homeomorphism of torus bundles

$$\Phi_{\sigma'\sigma} : (U \cap \sigma \cap \sigma') \times T_\sigma \longrightarrow (U \cap \sigma \cap \sigma') \times T_{\sigma'}.$$

We only claim a fibre-preserving homeomorphism of total spaces here, $\Phi_{\sigma,\sigma'}$ does in general not preserve the torus actions. In any case, these homeomorphisms are compatible over triple intersections, hence provide homeomorphisms of torus bundes also for maximal cells intersecting in higher codimension. This way we have described $\pi^{-1}(U)$ as the gluing of trivial torus bundles over a decomposition of U into closed subsets, a clutching construction.

To prove local triviality from this description of $\pi^{-1}(U)$, replace U by a smaller neighbourhood of x that is star-like with respect to a point $y \in \mathrm{Int}(\sigma)$ for some maximal cell $\sigma \ni x$. By perturbing y slightly, we may assume that the rays emanating from y intersect each codimension one cell ρ with $\rho \cap U \neq \emptyset$ transversaly. To obtain a fibre-preserving homeomorphism $\pi^{-1}(U) \simeq U \times T_\sigma$, connect any other point $y \in U$ with x by a straight line segment γ. Then γ passes through finitely many maximal cells σ'. At each change of maximal cell apply the relevant $\Phi_{\sigma'\sigma''}$ to obtain the identification of the fibre over y with T_σ. □

Remark 4.7. Let us describe explicitly the homeomorphism of torus bundles $\Phi_{\sigma'\sigma}$ in the proof of Proposition 4.6, locally around some $x \in B \setminus \mathcal{A}$. We restrict to the basic case $\sigma\cap\sigma' = \rho$ of codimension one. Let f_ρ be the function defining the log structure along X_ρ according to (4.1). Then there is first a strata-preserving isomorphism of $X_\rho \subset X_\sigma$ with $X_\rho \subset X_{\sigma'}$. This isomorphism is given by *(closed) gluing data*, see [GS1], Definition 2.3 and Definition 2.10. In the present case, gluing data are homomorphisms $\Lambda_\rho \to \mathbb{C}^\times$ fulfilling a cocyle condition in codimension two. The Kato-Nakayama space has an additional $U(1)$-factor coming from the deformation parameter t. This additional factor gets contracted in X_0 along X_ρ, but not in X_0^{KN}. Thus over X_ρ, the Kato-Nakayama space is a $U(1)^2$-fibration. One factor captures the phase of the deformation parameter t, the other the phase of the monomial u (or v) describing X_ρ as a divisor in X_σ and $X_{\sigma'}$, respectively. In these coordinates for X_0^{KN} over σ and σ', the gluing $\Phi_{\sigma'\sigma}$ is determined by taking the argument of (4.1):

$$\arg(u) + \arg(v) = \kappa_\rho \cdot \arg(t) + \arg(f). \tag{4.3}$$

Unless f is monomial, this equation is not compatible with the fibrewise action of $\mathrm{Hom}\big(\Lambda_x, U(1)\big) \times U(1)$ suggested by Lemma 4.5. In the case of nontrivial *(open)* *gluing data*, this definition of $\Phi_{\sigma'\sigma}$ has to be corrected by the scaling factors in $\mathbb{C}^\times$ by which u, v differ from toric monomials in X_σ, $X_{\sigma'}$, respectively.

By Proposition 4.6 we may now view the subset $(\mu^{\mathrm{KN}})^{-1}(B \setminus \mathcal{A}) \subset X_0^{\mathrm{KN}}$ as a torus bundle over $B \setminus \mathcal{A}$. For $U \subset B \setminus A$ any subset, we write $X_0|_U = (\mu^{\mathrm{KN}})^{-1}(U)$, viewed as a topological torus bundle over U. Generally, topological r-torus bundles are fibre bundles with structure group $\mathrm{Homeo}(T^r)$, the group of homeomorphisms of the r-torus. There is the obvious subgroup $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$ of homeomorphisms that lift to affine transformations on the universal covering $\mathbb{R}^{r+1} \to T^r = \mathbb{R}^r/\mathbb{Z}^r$. In higher dimensions (certainly for $r \geq 5$), there exist exotic homeomorphisms that are not isotopic to a linear one ([Ht], Theorem 4.1). However, in the present situation such exotic transition maps do not occur, and we can even find a system of local trivializations with transition maps induced by locally constant affine transformations.

Lemma 4.8. *The torus bundle $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ has a distinguished atlas of local trivializations with transition maps in $U(1)^{n+1} \rtimes \mathrm{GL}(n+1, \mathbb{Z})$.*

Proof. It suffices to consider the attaching maps between the trivial pieces $(\mu^{\mathrm{KN}})^{-1}(\sigma) = \sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$ of Lemma 4.5 for maximal cells σ, σ' with $\rho = \sigma \cap \sigma'$ of codimension one. Let $V \subset \rho$ be a connected component of $\rho \setminus \mathcal{A}$. In Remark 4.7 we saw that the transition maps over V are given by the equation $\arg(v) = -\arg(u) + \arg(f) + \kappa_\rho \cdot \arg(t)$. Now $\mu^{\mathrm{KN}}|_V$ factors over the Kato-Nakayama space of X_ρ, which can be trivialized as $V \times \mathrm{Hom}(\Lambda_\rho, U(1)) \times U(1)^2$. The last factor is given by $(\mathrm{Arg}(u), \mathrm{Arg}(t))$, say, and the transition map transforms this trivialization into the description with $(\mathrm{Arg}(v), \mathrm{Arg}(t))$. Thus this transition is the identity on the first $n-1$ coordinates given by Λ_ρ and on $\mathrm{Arg}(t)$, while on the last coordinate it is given by $\mathrm{Arg}(u)^{-1}$ times the phase of the algebraic function $f \cdot t^{\kappa_\rho}$. The homotopy class of this map is given by the winding numbers of a generating set of closed loops in $\pi_1(T^{n-1} \times T^1) = \mathbb{Z}^n$. These winding numbers define a monomial function z^m on $V \times \mathrm{Hom}(\Lambda_\rho \oplus \mathbb{Z}, U(1))$ with $z^{-m} \cdot f$ homotopic to a constant map. The transition function is therefore isotopic to $(\mathrm{id}_{T^{n-1}}, \mathrm{Arg}(z^m \cdot t^{\kappa_\rho} \cdot u^{-1}), \mathrm{id}_{U(1)})$, fibrewise a linear transformation of $T^{n+1} = T^n \times U(1)$ with coordinates $\mathrm{Arg}(z)$ for T^n and $\mathrm{Arg}(u)$ for $U(1)$, respectively.

The translational factor of $U(1)^{n+1}$ arises because non-trivial gluing data change the meaning of monomials on the maximal cells by constants. See the discussion after Corollary 4.8 below for some comments on gluing data. $\square$

The topological classification of torus bundles with transition functions taking values in $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$ works in analogy with the Lagrangian fibration case discussed in [Du]. Let $\mu : X \to B$ be such a torus bundle of relative dimension r. Then $\Lambda = R^1\mu_*\underline{\mathbb{Z}}$ is a local system with fibres $\Lambda_x = H^1(\mu^{-1}(x), \mathbb{Z}) \simeq \mathbb{Z}^r$. The torus $\mathrm{Hom}(\Lambda_x, U(1)) \simeq U(1)^r$

acts fibrewise by translation on any trivialization $\mu^{-1}(U) \simeq U \times T^r$, and this action does not depend on the trivialization. Hence $X \to B$ can be viewed as a $\mathcal{H}om(\Lambda, \underline{U}(1))$-torsor. In particular, if $\mu : X \to B$ has a section, then $X \simeq \mathcal{H}om(\Lambda, \underline{U}(1)) \simeq \check{\Lambda} \otimes U(1)$ is isomorphic to the trivial torsor. Here we write $\check{\Lambda} = \mathcal{H}om(\Lambda, \underline{\mathbb{Z}})$ and view the locally constant sheaf $\mathcal{H}om(\Lambda, \underline{U}(1))$ as a topological space via its espace étalé. The cohomology group $H^1(X, \check{\Lambda} \otimes U(1))$ classifies isomorphism classes of $\check{\Lambda} \otimes U(1)$-torsors by the usual Čech description. Moreover, from the exact sequence

$$0 \longrightarrow \check{\Lambda} \longrightarrow \check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \check{\Lambda} \otimes_{\mathbb{Z}} U(1) \longrightarrow 0,$$

exhibiting $\check{\Lambda} \otimes_{\mathbb{Z}} U(1)$ fibrewise as a quotient of a vector space modulo a lattice, we obtain the long exact cohomology sequence

$$\ldots \longrightarrow H^1(B, \check{\Lambda}) \longrightarrow H^1(B, \check{\Lambda} \otimes \mathbb{R}) \longrightarrow H^1(B, \check{\Lambda} \otimes U(1)) \xrightarrow{\delta} H^2(B, \check{\Lambda}) \longrightarrow \ldots$$

The image of the class $[X] \in H^1(B, \check{\Lambda} \otimes U(1))$ of $X \to B$ as a $\check{\Lambda} \otimes U(1)$-torsor under the connecting homomorphism, defines the obstruction $\delta[X] \in H^2(B, \check{\Lambda})$ to the existence of a continuous section of $\mu : X \to B$. The proof works as in the symplectic case discussed in [Du], p.696f.

For a local system Λ on a topological space B with fibres $\mathbb{Z}^r$ we write $\mathrm{Hom}(\Lambda, U(1))$ for the associated torus bundle. As a set, $\mathrm{Hom}(\Lambda, U(1)) = \coprod_{x \in B} \mathrm{Hom}(\Lambda_x, U(1))$, and the topology is generated by sets, for $m \in \Gamma(U, \Lambda)$ with $U \subset X$ open and $V \subset U(1)$ open,

$$V_m = \big\{(x, \varphi) \,\big|\, x \in U,\, \varphi \in \mathrm{Hom}(\Lambda_x, U(1)),\, \varphi(m) \in V\big\}.$$

With this notation, we summarize the general discussion on torus bundles with locally constant transition functions in the following proposition.

Proposition 4.9. *Let B be a topological manifold and $\mu : X \to B$ a fibre bundle with locally constant transition functions with values in $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$. Then up to isomorphism, $X \to B$ is given uniquely by the local system $\Lambda = R^1\mu_*\underline{\mathbb{Z}}$ with fibres $\mathbb{Z}^r$ and a class $[X] \in H^1(B, \check{\Lambda} \otimes U(1))$.*

Moreover, a continuous section of μ exists if and only if $\delta([X]) \in H^2(B, \check{\Lambda})$ vanishes. In this case, $X \simeq \mathrm{Hom}(\Lambda, U(1))$, as a torus bundle with locally constant transition functions in $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$. □

Remark 4.10. A torus bundle $X \to B$ with locally constant transition functions in $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$ as in the preceding proposition can be explicitly reconstructed from its class $[X] \in H^1(B, \check{\Lambda} \otimes U(1))$ as follows. Since $H^1(B, \check{\Lambda} \otimes U(1)) = \mathrm{Ext}^1(\Lambda, U(1))$, any such class defines an extension

$$1 \xrightarrow{i} U(1) \longrightarrow \mathcal{E} \xrightarrow{q} \Lambda \longrightarrow 0$$

of Λ by $U(1)$. Then the set $\mathrm{Hom}^0(\mathcal{E}, U(1))$ of homomorphisms $\varphi: \mathcal{E}_x \to U(1)$, $x \in B$ with $\varphi \circ i = 1$ is naturally a $\check\Lambda \otimes U(1)$-torsor by letting $\lambda: \check\Lambda_x \to U(1)$ act by $\lambda \otimes \varphi \mapsto (\lambda \circ q) \cdot \varphi$. By the description of gluing trivial bundles via a Čech 1-cocycle it is then not hard to construct an isomorphism of $\mathrm{Hom}^0(\mathcal{E}, U(1))$ with $X \to B$ as $\check\Lambda \otimes U(1)$-torsors.

For the Kato-Nakayama space $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$, the governing bundle $R^1\mu_*\underline{\mathbb{Z}}$ is identified as follows. Recall that the multivalued piecewise affine function φ encoded in the $\kappa_\rho \in \mathbb{N}$ defines an integral affine manifold $\mathbb{B}_\varphi$ with an integral affine action by $(\mathbb{R}, +)$, making $\mathbb{B}_\varphi$ a torsor over $B = \mathbb{B}_\varphi/\mathbb{R}$ ([GHKS], Construction 1.14). This torsor comes with a canonical piecewise affine section locally representing φ. The pull-back of $\Lambda_{\mathbb{B}_\varphi}$ under this section defines an extension

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{P} \longrightarrow \Lambda \longrightarrow 0 \tag{4.4}$$

of Λ by the constant sheaf $\underline{\mathbb{Z}}$ on $B \setminus \mathcal{A}$. The extension class of this sequence equals $c_1(\varphi) \in \mathrm{Ext}^1(\Lambda, \underline{\mathbb{Z}}) = H^1(B \setminus \mathcal{A}, \check\Lambda)$, called the first Chern class of φ from its mirror dual interpretation (see [GHKS], Equation (1.5)).

For each point $x \in B$ there is a chart for the log structure on X_0 with monoid $\mathbb{C}[\mathcal{P}_x^+]$, where $\mathcal{P}_x^+ \subset \mathcal{P}_x$ is a certain submonoid of positive elements with $(\mathcal{P}_x^+)^{\mathrm{gp}} = \mathcal{P}_x$ ([GHKS], §2.2 and [GS3], Construction 2.7). Hence from the local description of $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ in Lemma 4.5 and in Proposition 4.6, the following result is immediate:

Lemma 4.11. *Writing $\breve\mu$ for the restriction of $\mu^{\mathrm{KN}}: X_0^{\mathrm{KN}} \to B$ to $B \setminus \mathcal{A}$, there exists a canonical isomorphisms of local systems $R^1\breve\mu_*\underline{\mathbb{Z}} = \mathcal{P}$.* □

Remark 4.12. Much as in the discussion of the radiance obstruction in [GS1], §1.1, the first Chern classs $c_1(\varphi)$ can be interpreted as an element in group cohomology $H^1(\pi_1(B \setminus \mathcal{A}, x), \check\Lambda_x)$. Here $\check\Lambda_x \simeq \mathbb{Z}^n$ is a $\pi_1(B \setminus \mathcal{A}, x)$-module by means of parallel transport in $\check\Lambda$ along closed loops based at some fixed $x \in B \setminus \mathcal{A}$. As an element in group cohomology, $c_1(\varphi)$ is given by a twisted homomorphism $\lambda: \pi_1(B \setminus \mathcal{A}, x) \to \check\Lambda_x$, $\gamma \mapsto \lambda_\gamma$, determining the monodromy of $\mathcal{P}$ around a closed loop γ based at x as follows:

$$\Lambda_x \oplus \mathbb{Z} \longrightarrow \Lambda_x \oplus \mathbb{Z}, \quad (v, a) \longmapsto \big(T_\gamma \cdot v, \lambda_\gamma \cdot v + a\big).$$

Here $T_\gamma \in \mathrm{GL}(\Lambda_x)$ is the monodromy of Λ along γ and we have chosen an isomorphism $\mathcal{P}_x \simeq \Lambda_x \oplus \mathbb{Z}$. Being a twisted homomorphism means that for a composition $\gamma_1\gamma_2$ of two loops based at x,

$$\lambda_{\gamma_1\gamma_2} = \lambda_{\gamma_2} \circ T_{\gamma_1} + \lambda_{\gamma_1}.$$

Here we use the convention that $\gamma_1\gamma_2$ runs through γ_2 first, and hence $T_{\gamma_1\gamma_2} = T_{\gamma_2} \circ T_{\gamma_1}$. This interpretation is also compatible with the fact that under discrete Legendre duality, the roles of $c_1(\varphi)$ and the radiance obstruction swap ([GS1], Proposition 1.50,3).

In view of Lemma 4.11, an immediate corollary from Proposition 4.9 is a complete topological description of $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ over large open sets.

Corollary 4.13. *Denote by $\tilde{\mathcal{A}}\subset B$ a closed subset containing $\mathcal{A}$ and such that $B\setminus\tilde{\mathcal{A}}$ retracts to a one-dimensional cell complex. Then as a topological torus bundle, $X_0^{\mathrm{KN}}|_{B\setminus\tilde{\mathcal{A}}}$ is isomorphic to* $\mathrm{Hom}(\mathcal{P},\underline{U}(1))$.

Proof. By Lemma 4.8 we can treat $X_0^{\mathrm{KN}}|_{B\setminus\tilde{\mathcal{A}}}$ as a torus bundles with locally constant transition functions in $U(1)^{n+1}\rtimes\mathrm{GL}(n+1,\mathbb{Z})$. By Proposition 4.9 the obstruction to the existence of a continuous section then lies in $H^2(B\setminus\tilde{\mathcal{A}},\mathcal{P}^\vee)$. This cohomology group vanishes by the assumption on the existence of a retraction. □

We should emphasize that in this corollary, we have first used Lemma 4.8 to reduce to the case of locally constant transition functions. As discussed in Remark 4.7, the transition functions for $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}\to B\setminus\mathcal{A}$ between the canonical charts coming from toric geometry are not locally constant, and hence Corollary 4.8 makes a purely topological statement.

The remainder of this subsection derives a more canonical description of X_0^{KN} over a somewhat smaller set by controlling the gluing data. The various charts for $(X_0,\mathcal{M}_{X_0})$ are related by parallel transport inside $\mathcal{P}$, but the monomials may be rescaled due to non-trivial *gluing data*. Gluing data have already been introduced in [GS1], but §1.2 in [GS3] or §5.2 in [GHKS] may contain more palatable accounts. Multiples of the deformation parameter t are well-defined on all charts, hence define a constant subsheaf with fibres $\mathbb{Z}\oplus\mathbb{C}^\times$. Monomials therefore define a refinement of (4.4):

$$0\longrightarrow\underline{\mathbb{Z}}\oplus\underline{\mathbb{C}}^\times\longrightarrow\tilde{\mathcal{P}}\longrightarrow\Lambda\longrightarrow 0. \tag{4.5}$$

The extension class

$$\big(c_1(\varphi),s\big)\in\mathrm{Ext}^1(\Lambda,\underline{\mathbb{Z}}\oplus\underline{\mathbb{C}}^\times)=H^1(B\setminus\mathcal{A},\check{\Lambda})\oplus H^1(B\setminus\mathcal{A},\check{\Lambda}\otimes\mathbb{C}^\times)$$

has as second component (the restriction to $B\setminus\mathcal{A}$ of) the gluing data s, as discussed in [GHKS], Remark 5.16.[1] Furthermore, dividing out $\mathbb{R}_{>0}\subset\mathbb{C}^\times$ defines an extension $\widehat{\mathcal{P}}$ of Λ by $\underline{\mathbb{Z}}\oplus\underline{U}(1)$ with class

$$\big(c_1(\varphi),\mathrm{Arg}(s)\big)\in\mathrm{Ext}^1(\Lambda,\underline{\mathbb{Z}}\oplus\underline{U}(1))=H^1(B\setminus\mathcal{A},\check{\Lambda})\oplus H^1(B\setminus\mathcal{A},\check{\Lambda}\otimes U(1)).$$

Taking this latter extension and the extension of Λ by $\underline{\mathbb{Z}}$ from (4.4) as two columns, we obtain the following commutative diagram with exact rows and columns:

[1]The discussion in [GHKS] is on the complement of a part $\tilde{\Delta}\subset B$ of the codimension two skeleton of the barycentric subdivision. There is a retraction of $\mathcal{A}$ to a subset of $\tilde{\Delta}$. However, the discussion on monomials works on any cell $\tau\in\mathscr{P}$ not contained in $\mathcal{A}$ and with $\mathfrak{X}\to\operatorname{Spec}R$ locally toroidal at some point of X_τ.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & \underline{U}(1) \arrow[r] \arrow[d, "="'] & \underline{\mathbb{Z}} \oplus \underline{U}(1) \arrow[r] \arrow[d] & \underline{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \underline{U}(1) \arrow[r] & \widehat{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & 0 \\
 & & \Lambda \arrow[r, "="] \arrow[d] & \Lambda \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{4.6}
$$

Note that the extension of $\underline{\mathbb{Z}}$ by $\underline{U}(1)$ in the second row is trivial by construction. The middle row now defines an extension of $\mathcal{P}$ by $\underline{U}(1)$.

We can use the extension $\widehat{\mathcal{P}}$ to give a *canonical* description of X_0^{KN} on a large subset of $B \setminus \mathcal{A}$, assuming the open gluing data *normalizes* the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ ([GS1], Definition 4.23). Being normalized means that if $f_{\rho,v}$ is the slab function describing the log structure near a zero-dimensional toric stratum $x \in X_0$ along a codimension one stratum $X_\rho \subset X_0$ with $x \in X_\rho$, then $f_{\rho,v}(x) = 1$. By the discussion after Definition 4.23 in [GS1], there always exist open gluing data normalizing a given toric log Calabi-Yau space, so this assumption imposes no restriction. Note that while previously we had viewed slab functions only as functions on (analytically) open subsets of the big torus of X_ρ, hence as Laurent polynomials, in the setup of [GS1] or [GS3] they extend as regular functions to the zero-dimensional toric stratum they take reference to.

Proposition 4.14. *Denote by $B' = \bigcup_{\sigma\in\mathscr{P}^{\max}} \operatorname{Int}\sigma \cup \big\{v \in \mathscr{P}^{[0]}\big\}$ the subset of B covered by the interiors of maximal cells and the vertices of $\mathscr{P}$. Assume that the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ is normalized with respect to open gluing data s. Denote by $\widehat{\mathcal{P}}$ the extension of Λ by $\mathbb{Z} \oplus U(1)$ in (4.6). Write $\mathrm{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ \subset \mathrm{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))$ for the space of fibrewise homomorphisms restricting to the identity on $\underline{U}(1) \subset \widehat{\mathcal{P}}$. Then there is a canonical homeomorphism*

$$\mathrm{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ|_{B'} \xrightarrow{\simeq} X_0^{\mathrm{KN}}\big|_{B'}$$

of topological fibre bundles over B'.

Moreover, the class in $H^1(B', \mathcal{P}^\vee \otimes U(1))$ *defining* $X_0^{\mathrm{KN}}|_{B'}$ *as a topological torus bundle with locally constant transition functions in* $U(1)^{n+1} \rtimes \mathrm{GL}(n+1, \mathbb{Z})$ *according to Lemma 4.8, agrees with the class of the extension* $\widehat{\mathcal{P}}$ *of* $\mathcal{P}$ *by* $\underline{U}(1)$ *in* (4.6).

Proof. From its origin in the bundle $\tilde{\mathcal{P}}$ of monomials, we obtain a canonical description of $\widehat{\mathcal{P}}$ over B' as follows. Over a maximal cell σ, we have a canonical isomorphism of $\widehat{\mathcal{P}}|_\sigma$ with the trivial bundle with fibre $\Lambda_\sigma \oplus \mathbb{Z} \oplus U(1)$. Then if $\sigma, \sigma' \in \mathscr{P}_{\max}$ and $v \in \sigma \cap \sigma'$ is a vertex, the open gluing data s define a multiplicative function $s_{v,\sigma} : \Lambda_\sigma \to \mathbb{C}^\times$, and similarly for σ'. Now glue the trivial bundles on σ, σ' by means of $\mathrm{Arg}\left(s_{v,\sigma'} \cdot s_{v,\sigma}^{-1}\right)$ on the $U(1)$-factor. The gluing on the discrete part $\Lambda_\sigma \oplus \mathbb{Z}$ is governed by a local representative of the MPL function φ, to yield $\mathcal{P}$. Accordingly, we obtain a description of $\mathrm{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ$ by gluing trivial pieces $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$.

Now the point is that if f is normalized, the gluing of X_0^{KN} from the same canonical trivial pieces in Lemma 4.5 is given by exactly the same procedure over vertices. Indeed, given a codimenson one cell ρ and a vertex $v \in \rho$, in the formula $\arg(u) + \arg(v) = \arg(f_{\rho,v}) + \kappa_\rho \cdot \arg(t)$ the term involving $f_{\rho,v}$ disappears due to the normalization condition.

The statement on the extension class is immediate from Remark 4.10. □

To describe the fibres of $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$, we need another extension. For $\phi \in U(1)$ denote by $\Psi_\phi : \mathbb{Z} \oplus U(1) \to U(1)$ the homomorphism mapping $(1,1)$ to ϕ and inducing the identity on $U(1)$. We have a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} \oplus \underline{U}(1) & \longrightarrow & \widehat{\mathcal{P}} & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi_\phi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \underline{U}(1) & \longrightarrow & \widehat{\mathcal{P}}_\phi & \longrightarrow & \Lambda & \longrightarrow & 0, \end{array}$$

with the lower row having extension class $\Psi_\phi(s) \in \mathrm{Ext}^1(\Lambda, \underline{U}(1)) = H^1(B \setminus \mathcal{A}, \check{\Lambda} \otimes \underline{U}(1))$.

Corollary 4.15. *With the same assumptions as in Proposition 4.14, the fibre of* δ^{KN} : $X_0^{\mathrm{KN}}|_{B'} \to (\mathrm{Spec}\, O^\dagger)^{\mathrm{KN}} = U(1)$ *over* $\phi \in U(1)$ *is isomorphic to the* n*-torus bundle with local system* Λ *and with extension class* $\Psi_\phi(s) \in H^1(B', \check{\Lambda} \otimes \underline{U}(1))$.

Proof. By Lemma 4.5, the restriction of δ^{KN} to the canonical piece $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$ is given by composing with the inclusion $\mathbb{Z} \to \Lambda_\sigma \oplus \mathbb{Z}$. The statement follows by tracing through the gluing descriptions of $\widehat{\mathcal{P}}_\phi$ and of the extension class defined by $(\delta^{\mathrm{KN}})^{-1}(\phi)$. □

Remark 4.16. Let us emphasize the role of the normalization condition here. The canonical description over maximal cells in Lemma 4.5 is based on toric monomials. To extend this description over a point $x \in B \setminus \mathcal{A}$ in a codimension one cell ρ, we need the gluing equation (4.3) to be monomial along the fibres of the momentum map. This condition means that the restriction of f to a fibre of the momentum map $X_\rho \to \rho$ is monomial. This is a strong condition that in case the Newton polyhedron of f is full-dimensional fails everywhere except at the zero-dimensional toric strata of X_ρ. The normalization condition then says that the non-trivial gluing of the torus fibres over the vertices only comes from the gluing data, hence is entirely determined by the extension class of $\widehat{\mathcal{P}}$.

4.3. **Study of the real locus.** We now turn to toric degenerations with a real structure, or rather to the corresponding toric log Calabi-Yau space ([GS1], Definition 4.3) that arise as central fibre $(X_0, \mathcal{M}_{X_0})$ of a toric degeneration, as discussed in §4.1. We call a toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ *standard real* if it has a real structure inducing the standard real structure on its toric irreducible components and which is compatible with the standard real structure on the standard log point. Since the morphism $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ is strict at the generic points of the irreducible components of X_0, and since any section of $\mathcal{M}_{X_0}$ that is supported on higher codimensional strata is trivial (constant 1), there is at most one such real structure on $(X_0, \mathcal{M}_{X_0})$. This real structure is called the *standard real structure*.

Standard real structures appear to be the only class of real structures on toric log-Calabi-Yau spaces that exist in great generality. While other real structures, for example those lifting an involution on B, should be extremely interesting in more specific situations, we therefore restrict the following discussion to standard real structures.

Proposition 4.17. *Let $(X_0, \mathcal{M}_{X_0})$ be a polarized toric log Calabi-Yau space (*[GS1]*, Definition 4.3) with intersection complex $(B, \mathscr{P})$. Then there is a standard real structure on $(X_0, \mathcal{M}_{X_0})$ if and only if the following hold:*

1. *There exist open gluing data $s = (s_{\omega\tau})$ with $X_0 \simeq X_0(B, \mathscr{P}, s)$ such that $s_{\omega\tau}$ takes values in $\mathbb{R}^\times$ rather than in $\mathbb{C}^\times$.*
2. *The slab functions $f_{\rho,v} \in \mathbb{C}[\Lambda_\rho]$ describing the log structure $\mathcal{M}_{X_0}$ for gluing data as in (1) are defined over $\mathbb{R}$, that is $f_{\rho,v} \in \mathbb{R}[\Lambda_\rho]$ for any $\rho \in \mathscr{P}$ of codimension one and $v \in \rho$ a vertex.*

Proof. The proof is by revision of the arguments in [GS1].

1) If $s = (s_{\omega\tau})_{\omega,\tau}$ are open gluing data taking values in $\mathbb{R}^\times \subset \mathbb{C}^\times$, the construction of $X_0(B, \mathscr{P}, s)$ by gluing affine toric varieties in [GS1], Definition 2.28 readily shows that the real structures on the irreducible components induce a real structure on $X_0(B, \mathscr{P}, s)$.

Conversely, given $(X_0, \mathcal{M}_{X_0})$ with a standard real structure, Theorem 4.14 in [GS1] constructs open gluing data s and an isomorphism $X_0 \simeq X_0(B, \mathscr{P}, s)$. The construction

has two steps. First, X_0 being glued from toric varieties, there exist *closed* gluing data $\overline{s}$ inducing this gluing. If X_0 admits a standard real structure, $\overline{s}$ automatically takes real values. In a second step one shows that the closed gluing data are the image of open gluing data as in [GS1], Lemma 2.29 and Proposition 2.32,2. This step uses a chart for the log structure at a zero-dimensional toric stratum $x \in X_0$. In view of the given real structure on $(X_0, \mathcal{M}_{X_0})$, this chart can be taken real (Lemma 1.8). With this choice of chart, the construction of open gluing data in the proof of [GS1], Theorem 4.14, indeed produces real open gluing data.
2) The relation between the slab functions $f_{\rho,v}$ and charts for the log structure is given in [GS1], Theorem 3.22. At a zero-dimensional toric stratum $x \in X_0$ the description in terms of open gluing data yields an isomorphism of an open affine neighbourhood in $\operatorname{Spec} X_0$ with $\operatorname{Spec} \mathbb{C}[P]/(z^{\rho_P})$, with $P = \overline{\mathcal{M}}_{X,x}$ and $\rho_P \in P$ corresponding to the deformation parameter t. The facets of P are in one-to-one correspondence with the irreducible components of X_0 containing x. Now charts for the log structure on this open subset are of the form

$$P \longrightarrow \mathbb{C}[P]/(z^{\rho_P}), \quad p \longmapsto h_p \cdot z^p$$

with h_p an invertible function on $V(p)$, the closure of the open subset $(z^p \neq 0) \subset \operatorname{Spec} \mathbb{C}[P]$. The equation describing this chart in terms of functions on codimension one strata writes the slab function as a quotient of piecewise multiplicative functions g_v defined in terms of h_p. This equation, with the slab function written $\xi_\omega(h)$ in [GS1], shows that describing a real chart via real open gluing data yields real slab functions $f_{\rho,v}$.

Conversely, given real open gluing data and real slab functions, the real structure on $\operatorname{Spec} \mathbb{C}[P]$ induces the involution $\iota^\flat_{X_0}$ defining a standard real structure on $(X_0, \mathcal{M}_{X_0})$. □

In the case of positive and simple singularities, a polarized toric log Calabi-Yau space with given intersection complex $(B, \mathscr{P}, \varphi)$ is defined uniquely up to isomorphism by so-called *lifted gluing data* $s \in H^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^\times)$ ([GS1], Theorem 5.4).[2] Lifted gluing data both contain moduli of open gluing data and moduli of the log structure given by the slab functions. In terms of lifted gluing data the existence of a standard real structure has a simple cohomological formulation.

Corollary 4.18. *Assuming $(B, \mathscr{P})$ positive and simple, then the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ defined by lifted gluing data $s \in H^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^\times)$ is standard real if*

[2]The theorem makes also the converse statement using the dual intersection complexes; working polarized as we do here, imposes a codimension one constraint, see the definition of $A_{\mathbb{P}}$ in [GHKS], §A.2.

and only if s lies in the image of

$$H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{R}^\times) \longrightarrow H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^\times).$$

Proof. This follows again by inspection of the corresponding results in [GS1], here Theorems 5.2 and 5.4. □

Remark 4.19. It is worthwhile pointing out that real structures on $(X_0, \mathcal{M}_{X_0})$ are compatible with the smoothing algorithm of [GS3] in the following way. Assume that $(X_0, \mathcal{M}_{X_0})$ is a toric log Calabi-Yau space for which the smoothing algorithm of [GS3] works, for example with associated intersection complex $(B, \mathscr{P})$ positive and simple. Assume that $(X_0, \mathcal{M}_{X_0})$ has a real structure, not necessarily standard. The real involution then induces a possibly non-trivial involution on the intersection complex $(B, \mathscr{P})$. But in any case, $(X_0, \mathcal{M}_{X_0})$ has a description by open gluing data $s = (s_{\omega\tau})$ and slab functions $f_{\rho,v}$ with the real involution lifting to an action on these data. By the strong uniqueness of the smoothing algorithm it is then not hard to see that the real involution extends to the constructed family $\mathfrak{X} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$.

Note also that by Proposition 4.17, the locally rigid case with standard real structure is already covered in [GS3], Theorem 5.2.

Let us now assume we have a standard real structure on $(X_0, \mathcal{M}_{X_0})$. We wish to understand the topology of the real locus $X_{0,\mathbb{R}}^{\mathrm{KN}} \subset X_0^{\mathrm{KN}}$, the fixed locus of the lifted real involution of X_0^{KN} from Definition 3.1. First, since $(O^\dagger)_{\mathbb{R}}^{\mathrm{KN}} = \{\pm 1\}$, the real locus of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ decomposes into two parts, the preimages of ± 1 under $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$. Denote by $X_{0,\mathbb{R}}^{\mathrm{KN}}(\pm 1)$ these two fibres.

Proposition 4.20. *The restriction of $\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to B$ to the real locus exhibits $X_{0,\mathbb{R}}^{\mathrm{KN}}$ as a surjection with finite fibres. Over $B \setminus \mathcal{A}$, this map is a topological covering map with fibres of cardinality 2^{n+1}.*

Proof. Let $\sigma \in \mathscr{P}$ be a maximal cell. In the canonical identification Φ_σ of Lemma 4.5, the standard real involution on $(\mu^{\mathrm{KN}})^{-1}(\sigma) \subset X_0^{\mathrm{KN}}$ lifts to the involution of $\sigma \times \operatorname{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$ that acts by the identity on σ and by multiplication by -1 in $\Lambda_\sigma \oplus \mathbb{Z}$. The fixed point set of this involution over each point in σ is the set of two-torsion points $(\pm 1, \ldots, \pm 1)$ of $U(1)^{n+1}$. In particular, away from $\mathcal{A} \subset B$, the projection $X_{0,\mathbb{R}}^{\mathrm{KN}} \to B$ is a 2^{n+1}-fold unbranched cover.

In any case, $\Phi_\sigma\big(\sigma \times \operatorname{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})\big)$ is a closed subset in X_0^{KN} containing $X_{\sigma,\mathbb{R}} \setminus Z$ and projecting with fibres of cardinality at most 2^{n+1} to σ. The statement on finiteness of all fibres then follows if Z is nowhere dense in $X_{0,\mathbb{R}}^{\mathrm{KN}}$. This statement follows as in Lemma 4.4 noting that the generization maps between stalks of $\mathcal{M}_{X_0}$ at real points are compatible with the real involution. □

We thus see that $X_{0,\mathbb{R}}^{\mathrm{KN}}$ can be understood by studying (a) the unbranched covering over $B \setminus \mathcal{A}$ and (b) the behaviour near the log singular locus by means of the canonical uniformization map Φ_σ of Lemma 4.5. Sometimes, e.g. in dimension two, the unbranched cover together with the fact that $X_{0,\mathbb{R}}^{\mathrm{KN}}$ is a topological manifold, determines $X_{0,\mathbb{R}}^{\mathrm{KN}}$ completely.

For the unbranched cover, Lemma 4.5 together with the gluing equation (4.3) in Remark 4.7 provide a full description of $X_{0,\mathbb{R}}^{\mathrm{KN}}$. Note also that the gluing equation involves the term $\kappa_\rho \cdot \arg(t)$, which for κ_ρ odd and $\mathrm{Arg}(t) = -1$ leads to a difference in the identification of branches over neighboring maximal cells.

We formulate this discussion as a proposition, omitting the obvious proof.

Proposition 4.21. *Let $(X_0, \mathcal{M}_{X_0})$ be endowed with a standard real structure, described by real open gluing data and real slab functions, following Proposition 4.17. Then the description of X_0^{KN} in Remark 4.7 as glued from trivial pieces $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) \setminus \Phi_\sigma^{-1}(Z^{\mathrm{KN}})$ via Equation (4.3), exhibits the real locus $X_{0,\mathbb{R}}^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$ as glued from $\big(\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})\big) \setminus \Phi_\sigma^{-1}(Z^{\mathrm{KN}})$. In particular, the sign of the function f describing the gluing over a connected component of $\rho \setminus \mathcal{A}$, $\dim \rho = n-1$, influences the identification of branches suggested by the identification of integral tangent vectors through affine parallel transport.* □

As emphasized, in general the specific choice of slab functions changes the topology of $X_{0,\mathbb{R}}^{\mathrm{KN}}$, and hence has to be studied case by case. Assuming without loss of generality that the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ is normalized by the open gluing data, we can however give a neat global description over the large subset $B' \subset B$ considered in Proposition 4.14. In the simple case, there is a retraction of $B \setminus \mathcal{A}$ to B' and this result is strong enough to understand the unbranched cover over $B \setminus \mathcal{A}$ completely. In the general case, this result can be complemented by separate studies along the interior of codimension one cells to gain a complete understanding of the part of the real locus covering $B \setminus \mathcal{A}$.

As a preparation, we need to discuss the effect of the real involution on Diagram (4.6), and in particular on the middle vertical part, the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \oplus \underline{U}(1) \longrightarrow \widehat{\mathcal{P}} \longrightarrow \Lambda \longrightarrow 0.$$

The action on the discrete part $\underline{\mathbb{Z}}$ and Λ is induced by the action on the cohomology of the torus fibres, which is multiplication by -1. Similarly, we can act by multiplication with -1 on each entry of the sequence defining $\mathcal{P}$, forming the next to rightmost column in (4.6). For the extension by $U(1)$, however, taking the pushout with complex conjugation on $U(1)$, maps the extension class $s \in \mathrm{Ext}^1(\Lambda, \underline{U}(1))$ to its complex conjugate $\overline{s}$. Thus only if this class is real, reflected in a real choice of open gluing data (Proposition 4.17), there is an involution on $\widehat{\mathcal{P}}$ inducing multiplication by -1 on $\underline{\mathbb{Z}}$ and Λ and the conjugation on $U(1)$. Note also that the extension class is real if and only if it lies in the image of $\mathrm{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \{\pm 1\})$ under the inclusion $\{\pm 1\} \to U(1)$. In this case, the extension of Λ by $\underline{\mathbb{Z}} \oplus U(1)$ is obtained by pushout from an extension by $\underline{\mathbb{Z}} \oplus \{\pm 1\}$. We now assume such an involution $\iota_{\widehat{\mathcal{P}}}$ of $\widehat{\mathcal{P}}$ exists.

Proposition 4.22. *Assume that the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ is given by real open gluing data and real normalized slab functions. Then in the canonical description of X_0^{KN} over $B' \subset B$ given in Proposition 4.14, the real locus is given by*

$$\mathrm{Hom}(\widehat{\mathcal{P}}, \{\pm 1\})^0|_{B'} \subset \mathrm{Hom}(\widehat{\mathcal{P}}, U(1))^0|_{B'}.$$

Proof. Recall the trivialization with fibres $\Lambda_\sigma \oplus \mathbb{Z} \oplus U(1)$ of $\widehat{\mathcal{P}}$ over the interior of a maximal $\sigma \in \mathscr{P}$ used in the proof of Proposition 4.14. In this trivialization, the involution $\iota_{\widehat{\mathcal{P}}}$ acts by -1 on $\Lambda_\sigma \oplus \mathbb{Z}$ and by conjugation on $U(1)$. Taking homomorphisms to $U(1)$ and restricting to those homomorphisms inducing the identity on the $U(1)$-factor, identifies the fibres of X_0^{KN} over $\mathrm{Int}\,\sigma$ with $\mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$. The fixed point locus of the induced action of $\iota_{\widehat{\mathcal{P}}}$ is then the set of homomorphisms to the two-torsion points of $U(1)$, that is, $\mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})$, as claimed. $\square$

Remark 4.23. The topology of the 2^{n+1}-fold cover of B' can also be described in terms of the monodromy representation as follows. Analogously to the discussion for $\mathcal{P}$ in Remark 4.12, the monodromy representation of $\widehat{\mathcal{P}}$ is given by viewing $(c_1(\varphi), s) \in \mathrm{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \underline{U}(1))$ as a pair of twisted homomorphisms,

$$(\lambda, \theta) : \pi_1(B', x) \longrightarrow \mathrm{Hom}(\Lambda_x, \mathbb{Z} \oplus U(1)).$$

Explicitly, for a closed loop γ at x, the action of $(\lambda, \theta)(\gamma) = (\lambda_\gamma, \theta_\gamma)$ on the fibre of $\widehat{\mathcal{P}}_x \simeq \Lambda_x \oplus \mathbb{Z} \oplus U(1)$ is

$$\Lambda_x \oplus \mathbb{Z} \oplus U(1) \ni (v, a, \beta) \longmapsto (T_\gamma \cdot v, \lambda_\gamma \cdot v + a, \theta_\gamma(v) \cdot \beta).$$

Here $T_\gamma \in \mathrm{GL}(\Lambda_x)$ is from parallel transport in Λ. If the open gluing data s are real, θ takes values in $\{\pm 1\} \subset U(1)$. Thus in the real case, (λ, θ) is a twisted homomorphism with values in $\mathrm{Hom}(\Lambda_x, \mathbb{Z} \oplus \{\pm 1\})$.

In view of Proposition 4.22, the monodromy representation of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ over B' is given by the induced action on $\mathrm{Hom}\big(\Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}, \{\pm 1\}\big)^\circ = \mathrm{Hom}(\Lambda_x, \{\pm 1\}) \oplus \mathrm{Hom}(\mathbb{Z}, \{\pm 1\})$. Note that the last summand in $\Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}$ does not contribute to the right-hand side, since we restricted to those homomorphisms inducing the identity on $0 \oplus 0 \oplus \{\pm 1\}$. The action of a closed loop γ on $\mathrm{Hom}(\Lambda_x, \{\pm 1\}) \oplus \mathrm{Hom}(\mathbb{Z}, \{\pm 1\})$ is now readily computed as

$$(\varphi, \mu) \longmapsto \big(\varphi \circ T_\gamma + \mu \circ \lambda_\gamma + \theta_\gamma, \mu\big), \tag{4.7}$$

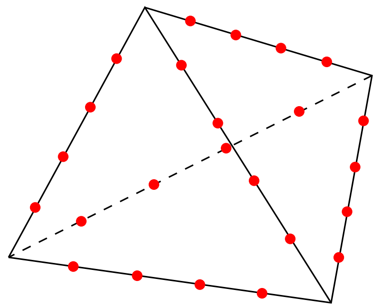

FIGURE 5.1. $(B, \mathscr{P})$ for a quartic K3 surface

Here we wrote the group structure on $\{\pm 1\}$ additively. This formula gives an explicit description of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ over B' in terms of a permutation representation of $\pi_1(B', x)$ on the set $\check{\Lambda}_x/2\check{\Lambda}_x \oplus \{\pm 1\}$ of cardinality 2^{n+1}.

In this description, the map to the real part $(O^\dagger)_{\mathbb{R}}^{\mathrm{KN}} = \{\pm 1\}$ of the Kato-Nakayama space $U(1)$ of the standard log point, is induced by the inclusion $\mathbb{Z} \oplus \{\pm 1\} \to \Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}$. Thus to describe the fibres over $\{\pm 1\} \subset (O^\dagger)^{\mathrm{KN}}$ in $X_{0,\mathbb{R}}^{\mathrm{KN}}$ simply amounts to restricting to $\mu = \pm 1$ in (4.7). In particular, $c_1(\varphi)$ only becomes relevant for the fibre over -1. This fact can also be seen from the gluing description of (4.3), where κ_ρ is the only place for $c_1(\varphi)$ to enter.

5. EXAMPLES

5.1. **A toric degeneration of quartic K3 surfaces.** As a first application of the general results in this paper, we look at an example of a toric degeneration of real quartic K3 surfaces.

Consider $(B, \mathscr{P})$ the polyhedral affine manifold that as an integral cell complex is the boundary of a 3-simplex, with four focus-focus singularities on each edge and with the complete fan at each vertex the fan of $\mathbb{P}^2$ (Figure 5.1). There are four maximal cells, each isomorphic to the standard simplex in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. The edges have integral length 1 and are identified pairwise to yield the boundary of a tetrahedron. On each of the six edges there are four singular points of the afffine structure, with monodromy conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. We use standard ("vanilla") gluing data, that is, $s_{\omega\tau} = 1$ for all $\omega, \tau \in \mathscr{P}$, $\omega \subset \tau$. Then X_0 is isomorphic as a scheme to $Z_0Z_1Z_2Z_3 = 0$ in $\mathbb{P}^3$, a union of four copies of $\mathbb{P}^2$. As the MPL-function defining the ghost sheaf $\overline{\mathcal{M}}_{X_0}$ we take the function with kink $\kappa_\rho = 1$ on each of the edges. The moduli space of toric log Calabi-Yau structures on X_0 with the given $\overline{\mathcal{M}}_{X_0}$ is described by the space of global sections of an invertible sheaf $\mathcal{LS}$ on the double locus $(X_0)_{\mathrm{sing}}$. This line bundle has degree 4 on each of the six $\mathbb{P}^1$-components. The section is explicitly described by the 12 slab functions $f_{\rho,v}$. For each edge $\rho \in \mathscr{P}$ there are two slab functions, related by the equation $f_{\rho,v}(x) = x^4 f_{\rho,v'}(x^{-1})$ for x the toric coordinate on $\mathbb{P}^1$ (see e.g. [GS3], Equation 1.11). Explicitly, restricting to the normalized case, we have $f_{\rho,v} = 1 + a_1x + a_2x^2 + a_3x^3 + x^4$, the highest and lowest coefficients being 1 due to the normalization condition at the two zero-dimensional toric strata of the projective line $X_\rho \subset X_0$. The other coefficients $a_i \in \mathbb{C}$ are free, to give a total of $6 \cdot 3 = 18$ parameters. Taking into account the additional deformation parameter t, this number is in agreement with the 19 dimensions of projective smoothings of X_0. See the appendix of [GHKS] for a discussion of projectivity in this context.

This example does not have simple singularities, but it is locally rigid in the sense of [GS3], Definition 1.26. Thus the smoothing algorithm of [GS3] works, yielding a one-parameter smoothing of X_0, one for each choice of slab functions. According to Proposition 4.17 this smoothing is real if and only if all slab functions are real, that is, all coefficients $a_i \in \mathbb{R}$. To obtain 4 focus-focus singularities on each edge of $\mathscr{P}$ as drawn in Figure 5.1, we need to choose the a_i in such a way that the 4 zeroes of $f_{\rho,v} = 1 + a_1x + a_2x^2 + a_3x^3 + x^4$ have pairwise different absolute values. These are then also all real. This condition is open in the Euclidean topology, but the closure is a proper subset of $\mathbb{R}^3$, the space of tuples (a_1, a_2, a_3). The precise choice does not matter for the following discussion and we assume such a choice has been made on each edge.

Proposition 5.1. *Let $(X_0, \mathcal{M}_{X_0})$ be the union of four copies of $\mathbb{P}^2$ with the real log structure as described. Denote by $\mathcal{A} \subset B$ the pairwise different images of the 24 singular points of the log structure (the zero loci of the slab functons).*

Then the fibre $X_{0,\mathbb{R}}^{\mathrm{KN}}(1)$ of $\delta_{\mathbb{R}}^{\mathrm{KN}} : X_{0,\mathbb{R}}^{\mathrm{KN}} \to \{\pm 1\} \subset U(1)$ has two connected components, one mapping homeomorphically to B, the other a branched covering of degree 3, unbranched over $B \setminus \mathcal{A}$ and with a simple branch point over each point of $\mathcal{A}$. In particular, the latter component is a closed orientable surface of genus 10.

Proof. In the present case of vanilla gluing data, the positive real sections $\sigma \times \{1\} \subset \sigma \times \mathrm{Hom}(\Lambda_\sigma, \{\pm 1\})$ of the pieces over maximal cells (Lemma 4.5) are compatible to yield a section of $X_{0,\mathbb{R}}^{\mathrm{KN}}(1) \to B$. At each point of $\mathcal{A}$ there are local analytic coordinates x, y, w, defined over $\mathbb{R}$, with $(X_0, \mathcal{M}_{X_0})$ isomorphic to the central fibre of the degeneration $xy = t(w + 1)$ discussed in Example 2.11. In this example, the real locus of the Kato-Nakayama space has three connected components, with two being sections and one a two-fold branched cover with one branch point.

To finish the proof we have to study the global monodromy representation $\pi_1(B \setminus \mathcal{A}) \to S_4$ into the permutations of a fibre and show it has at most two irreducible subrepresentations. We compute a part of the affine monodromy representation and

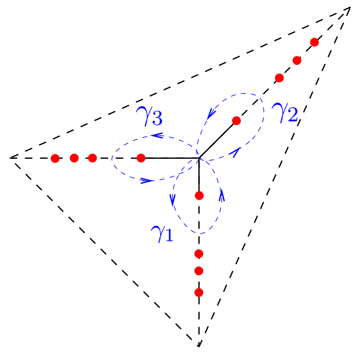

FIGURE 5.2. Chart at a vertex of $(B, \mathscr{P})$ from Figure 5.1

then use Remark 4.23 and notably Equation (4.7) to obtain the induced monodromy representation in S_4.

Figure 5.2 depicts a chart at a vertex with its three adjacent maximal cells. The chart gives the affine coordinates in the union of the three triangles minus the dotted lines. The locations of the 12 singular points on the outer dotted lines are irrelevant in this chart and are hence omitted. We look at the part of the fundamental group spanned by the three loops γ_1, γ_2, γ_3. Each encircles one focus-focus singularities on an edge containing the vertex and hence the affine monodromy is conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. In particular, the translational part vanishes. Concretely, in standard coordinates of $\mathbb{R}^2$, the monodromy matrices T_i along γ_i are

$$T_1 = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad T_2 = \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}, \quad T_3 = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}. \tag{5.1}$$

Now while the γ_i are not loops inside B' as treated in Remark 4.23, it is not hard to see that (4.7) still applies in the present case. We have $\mu = 1$ since we look at $X_{0,\mathbb{R}}^{\mathrm{KN}}(1)$ and $\theta_{\gamma_i} = 1$ also for the translational parts. Thus (4.7) says that the branches transform according to the linear part of the affine monodromy. Now indeed a slab function $f_{\rho,v}$ changes signs locally along the real locus over an edge whenever crossing a focus-focus singularity. For $X_{0,\mathbb{R}}^{\mathrm{KN}}(1)$ this means that the two branches given by $\mathrm{Hom}(\Lambda_\rho, \{\pm 1\}) \subset \mathrm{Hom}(\Lambda_\sigma, \{\pm 1\})$ have trivial monodromy around any focus-focus singularity on ρ, while the two other branches swap.

It thus remains to compute the action of T_i on the two-torsion points of $\mathbb{Z}^2/2\mathbb{Z}^2$. These are the four vectors

$$v_0 = \begin{pmatrix}0\\0\end{pmatrix}, \quad v_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\1\end{pmatrix}.$$

Here v_0 is the point in the positive real locus, yielding the section of $X_{0,\mathbb{R}}^{\mathrm{KN}}(1) \to B$. The permutation of the indices of the other three vectors yield the three transpositions (12), (13), (23) for T_1, T_2 and T_3, respectively. These transpositions act transitively on $\{1, 2, 3\}$, showing connectedness of the cover of degree 3. This component is a genus 10 surface by the Riemann Hurwitz formula. $\square$

5.2. **Toric degenerations of K3 surfaces for simple singularities.** As a second, related family of examples we consider toric degenerations of K3 surfaces such that the associated intersection complex $(B, \mathscr{P})$ has simple singularities. In this case the possible topologies of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ are determined by Proposition 4.22. Interestingly, for the fibre over $1 \in (O^\dagger)^{\mathrm{KN}} = U(1)$, the question becomes a purely group-theoretic one. In fact, according to (4.7), for $\mu = 1$ the translational part λ of the affine monodromy representation does not enter in the computation. Moreover, by a classical result of Livné and Moishezon, the linear part of the monodromy representation for an affine structure on S^2 with 24 focus-focus singularities is unique up to equivalence [Mo], p.179. The result says that there exists a set of standard generators $\gamma_1, \ldots, \gamma_{24}$ of $\pi_1(S^2 \setminus 24 \text{ points}, x)$, closed loops pairwise only intersecting at x and with composition $\gamma_1 \cdot \ldots \cdot \gamma_{24}$ homotopic to the constant loop, such that the monodromy representation takes the form

$$T_{\gamma_i} = \begin{cases} T_3, & i \text{ odd} \\ T_1, & i \text{ even}, \end{cases}$$

with T_1, T_3 as in (5.1). As in §5.1, the corresponding monodromy of the four elements in $\check{\Lambda}_x / 2\check{\Lambda}_x \simeq \mathbb{Z}^2/2\mathbb{Z}^2$ are (12) and (23), respectively. Thus the computation only depends on the choice of the twisted homomorphism $\theta \in H^1(B', \check{\Lambda} \otimes \{\pm 1\})$. Now each θ_γ acts by translation on the fibre $\mathbb{Z}^2/2\mathbb{Z}^2$. If θ_γ is nontrivial, the permutation is a double transposition. But any double transposition together with (12) and (23) acts transitively on the 4-element set. Thus $X_{0,\mathbb{R}}^{\mathrm{KN}}$ is connected as soon as $\theta \neq 0$; otherwise we have two connected components as in Proposition 5.1.

Proposition 5.2. *Let $(X_0, \mathcal{M}_{X_0})$ be a toric log K3 surface with intersection complex $(B, \mathscr{P})$ having simple singularities and endowed with a standard real structure. Denote by $\theta \in H^1(B, \check{\Lambda} \otimes \{\pm 1\})$ be the argument of the associated lifted real gluing data according to Corollary 4.18. Then $X_{0,\mathbb{R}}^{\mathrm{KN}}(1)$ has a connected component mapping homeomorphically to $B \simeq S^2$ if and only if $\theta = 0$, and is otherwise connected.* $\square$

For $X_{0,\mathbb{R}}^{\mathrm{KN}}(-1)$, the translational part of the affine monodromy enters in (4.7). The action is also by translation, hence lead to a double transposition if non-trivial. A similar analysis then shows that if $X_{0,\mathbb{R}}^{\mathrm{KN}}(-1)$ is not connected, one connected component maps homeomorphically to $B \simeq S^2$.

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