Proceedings of Symposia in Pure Mathematics

Intrinsic mirror symmetry and punctured Gromov-Witten invariants

Mark Gross and Bernd Siebert

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Introduction.

The very first occurences of mirror symmetry in the string-theoretic literature involved constructions of mirror pairs. The work of Greene and Plesser in [**19**] produced the mirror to the quintic three-fold via an orbifold construction, and the work of Candelas, Lynker and Schimmrigk [**12**] found a symmetry in Hodge numbers in the classification of Calabi-Yau hypersurfaces in weighted projective spaces. While mirror symmetry has developed in many directions, initially spurred by the genus zero calculations of [**13**], one of the essential open questions in the field has remained: how broadly do mirrors exist?

The first broad mathematical constructions of mirror pairs were the Batyrev [**10**] and Batyrev–Borisov [**11**] constructions respectively for Calabi-Yau hypersurfaces and complete intersections in toric varieties. There have been many other proposed constructions of mirrors to Calabi-Yau varieties, but such constructions tend to yield a smaller number of examples and are rarely completely orthogonal to the Batyrev and Batyrev–Borisov constructions. In particular, these two constructions remain the most practical and widely used.

In 1994, Givental [**18**] proposed extending mirror symmetry to the case of Fano manifolds, in which case the mirror is expected to be a Landau-Ginzburg model, i.e., a variety equipped with a regular function. This has resulted in a significant expansion of the realm of relevance of mirror symmetry. Analogously to the Batyrev–Borisov case, Givental and Hori–Vafa [**34**] provided constructions of

This work was partially supported by NSF grant 1262531 and a Royal Society Wolfson Research Merit Award.

mirrors of Fano complete intersections in toric varieties, later generalized to the case where the anti-canonical divisor is nef [**37**]. Even more recently, aspects of mirror symmetry have been observed much more generally for complete intersections in toric varieties with nef but non-zero canonical class in [**38**], [**27**].

This might suggest that mirror constructions are largely toric phenomena, leaving one to wonder to what extent one can find mirror constructions which apply more broadly. If mirror symmetry were to be a general phenomenon for Calabi-Yau varieties, one should have a construction which goes beyond toric geometry. Indeed, while it is frequently difficult to show that any given Calabi-Yau variety is *not* a complete intersection in a toric variety, the expectation is that a vast majority of Calabi-Yau varieties are not complete intersections. Thus one desires more intrinsic constructions which do not depend on embeddings in toric varieties.

The first sign of an intrinsic geometry of mirror symmetry was the 1996 proposal of Strominger, Yau and Zaslow (SYZ) [**56**]. They proposed a conjectural picture in the realm of differential geometry, suggesting that a mirror pair X, $\check{X}$ of Calabi-Yau manifolds should carry dual special Lagrangian torus fibrations $f : X \to B$, $\check{f} : \check{X} \to B$ over some base B. Although recently there has been significant progress in constructing special Lagrangian submanifolds and currents (see [**33**] and [**16**]), special Lagrangian fibrations remain elusive. The reader may consult the first author's contribution to the proceeedings of the 2005 edition of the AMS Algebraic Geometry Symposium, [**21**] for more discussion of the conjecture.

Nevertheless, as explained in [**21**], the SYZ conjecture led directly to our joint project which now appears to be colloquially referred to as the *Gross–Siebert program*. This project reinterpreted the SYZ conjecture inside of algebraic geometry. In particular, the base of the SYZ fibration B carries additional structure: it is not just a topological space but an *affine manifold with singularities*. One forgets the fibration, and hopes to work purely with the base B with this additional structure. As also suggested by Kontsevich and Soibelman in the context of [**46**], one should view B as a dual intersection complex of a degeneration of Calabi-Yau manifolds.

An initial achievement of this program was our 2007 result [**29**], where we showed that such an idea lead to a theoretically powerful mirror construction. This construction works in the context of *toric degenerations*. A toric degeneration, roughly, is a degenerating family of Calabi-Yau manifolds $f : \mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ whose central fibre is a union of toric varieties glued along toric strata, and such that in neighbourhoods of zero-dimensional strata of the central fibre, the morphism f is, étale locally, a monomial morphism between toric varieties. Here $\Bbbk$ is an algebraically closed field of characteristic zero.

The main result of [**29**] then produces, given a sufficiently "nice" toric degeneration (somewhat akin to the notion of large complex structure limit), a mirror toric degeneration. This construction, generalizing a non-Archimedean version carried out for K3 surfaces by Kontsevich and Soibelman in [**47**], which in turn was inspired by the speculative work of Fukaya in [**17**], proceeds by producing inductively a kind of combinatorial, tropical *structure* (sometimes called a *scattering diagram*) on B which encodes "instanton corrections" to gluing standard toric smoothings. The construction of the mirror, in this case, can be viewed as giving a tropical hint as to why mirror symmetry has something to do with counting curves: indeed, the tropical curves on B contributing to the structure describing the mirror degeneration $\check{\mathcal{X}} \to \operatorname{Spec} \Bbbk[\![t]\!]$ can be viewed as tropicalizations of "holomorphic disks on the generic fibre of $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ with boundary on a fibre of the SYZ fibration." This statement should, of course, be taken with a grain of salt, but a basic goal is to make such a statement sensible inside algebraic geometry. The main result of [**28**] provided moral support for the enumerative interpretation of structures.

The first author's paper [**20**] in fact implies that this construction is at least as strong as the Batyrev–Borisov construction, and in particular replicates the Batyrev–Borisov construction when one starts with natural degenerations of complete intersections in toric varieties to a union of toric strata. While it is easy to show the construction of [**29**] applies to more cases than the Batyrev–Borisov construction, e.g., by considering certain quotients of complete intersection Calabi-Yau manifolds, it is not clear how general the construction is. Nevertheless, if one would like a duality between degenerations which is an involution on the class of degenerations considered, one is led naturally to restrict to the case of toric degenerations.

After [**29**], several different threads converged to suggest the existence of canonical bases of sections of line bundles on the constructed families $\check{\mathcal{X}} \to \operatorname{Spec} \Bbbk[\![t]\!]$. First, discussions between Mohammed Abouzaid and ourselves led to the notion of *tropical Morse trees*, discussed in the simple case of elliptic curves in [**9**]. These yield a tropical analogue of the Floer theory of a "general fibre" of $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$, capturing the Floer homology between certain Lagrangian sections of the putative SYZ fibration. These sections would be mirror to powers of a canonically defined relatively ample line bundle $\mathcal{L}$ on $\check{\mathcal{X}}$. Under this correspondence, one anticipates a canonical basis of global sections of $\mathcal{L}^{\otimes d}$ indexed by points of $B(\frac{1}{d}\mathbb{Z})$, the set of points on B with coordinates lying in $\frac{1}{d}\mathbb{Z}$. Furthermore, multiplication of sections should be described, in analogy with Floer multiplication, as a sum over trees with two inputs and one output. This predicts that the homogeneous coordinate ring of $\check{\mathcal{X}}$ can be described directly in terms of tropical objects on B. The motivation for this point of view is explained in some detail in the expository paper [**31**].

On the other hand, while seeking to understand Landau-Ginzburg mirror symmetry for $\mathbb{P}^2$ in [**22**], the first author developed the notion of *broken line*, which was then used by the second author, working with Carl and Pumperla [**14**], to describe regular functions on (non-proper) families $\check{\mathcal{X}} \to \operatorname{Spec} \Bbbk[\![t]\!]$ in the context of [**29**], and in particular to construct Landau-Ginzburg mirrors to varieties with effective anti-canonical bundle.

Using a combination of the above ideas, the first author, working with Paul Hacking and Sean Keel, gave in [**24**] a general construction for mirrors to log Calabi-Yau surfaces with maximal boundary. Specifically, one considers pairs (Y, D) with Y a rational surface and D an anti-canonical divisor consisting of a cycle of n copies of $\mathbb{P}^1$. The idea was to use [**28**] to construct directly from the pair (Y, D) a structure on B (an affine manifold with singularities homeomorphic to $\mathbb{R}^2$ being a kind of "fan" for the pair (Y, D), see §2.1 for the construction of B as a cone complex and §2.4 for the affine structure) which governs the construction of the mirror. However, B carries one singularity, at the origin, and is a singularity of a quite different nature than that which appeared in [**29**]. As a consequence, the mirror family, which should be a family with central fibre the *n-vertex*, isomorphic to the affine cone over D, cannot be constructed directly as there is no local model for the smoothing at the vertex of the cone. This was dealt with as follows. For A an Artin local ring of the form $\Bbbk[Q]/I$, where Q is a suitably chosen toric monoid and I a monomial ideal, one constructs a deformation $\mathcal{U}$ over $\operatorname{Spec} A$ of an open subset of the n-vertex. One then constructs theta functions, the functions defined using broken lines, on $\mathcal{U}$. The theta functions can then be used to embed $\mathcal{U}$ into $\mathbb{A}^n_A$, where one then takes the scheme-theoretic closure to get a scheme $\mathcal{X}$ affine over $\operatorname{Spec} A$. Taking the limit, one obtains the mirror family as a formal family of affine schemes over the completion of $\Bbbk[Q]$ with respect to its maximal monomial ideal. So theta functions play a key role in the construction. Furthermore, [**24**] gave a formula for multiplication of theta functions in terms of trees of broken lines on B analogous to the formula using tropical Morse trees suggested in the discussions with Abouzaid.

In forthcoming work [**26**], a similar construction is carried out for K3 surfaces: here one starts with a maximally unipotent, normal crossings, relatively minimal family $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ of K3 surfaces and uses it to construct, using a combination of the ideas of [**29**] and [**28**], a structure on an affine manifold with singularities B, homeomorphic to the two-sphere. This dictates a deformation $\mathcal{U}$ over $\operatorname{Spec} A$ of an open subset of a singular union of $\mathbb{P}^2$'s. Theta functions, now sections of an ample line bundle on $\mathcal{U}$, are used to embed $\mathcal{U}$ into $\mathbb{P}^N_A$ for some N, and one compactifies by taking the scheme-theoretic closure. Here, a multiplication rule for theta functions can be described precisely by the one originally suggested in conversations with Abouzaid. This multiplication rule then describes the homogeneous coordinate ring of the mirror.

Suppose one would like to use similar ideas to construct mirrors in general. Following these ideas, if one starts with a suitable log Calabi-Yau pair (X, D) or a suitable degeneration $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ of Calabi-Yau manifolds, the above discussion suggests three steps:

(1) Construct a structure on B, the dual intersection complex of (X, D) or $\mathcal{X}_0$ (see §2.1), controlling the deformation theory of an open subset of a singular scheme combinatorially determined by B. This structure is expected to be definable in terms of enumerative geometry of X or $\mathcal{X}$; in the case of a log Calabi-Yau pair (X, D), the structure should be determined by suitable counts of $\mathbb{A}^1$-*curves*: morally these are rational curves meeting the boundary at one point, and defined rigorously via logarithmic Gromov-Witten invariants [**30**], [**15**], [**1**].
(2) Using broken lines, construct theta functions on the above open subset. Use these theta functions to extend the smoothing to an affine or projective family in the two cases.
(3) While the second step constructs the mirror, the multiplication rule on theta functions is also determined by trees on B, and hence the affine or projective coordinate ring of the mirror family is completely determined by the structure on B.

Given (1) and (2), step (3) is straightforward, and is carried out in general in [**31**], §3.5. However, both broken lines appearing in step (2) and trees appearing in step (3) are tropical objects. In particular, the relevant counts of tropical objects should reflect some kind of holomorphic curve, or more precisely, a form of logarithmic stable map.

In this paper, we will explain that the above philosophy can be used to construct a mirror in very great generality. In particular, the logarithmic invariants necessary are generalizations of those already defined in [**30**], [**15**], [**1**], called *punctured invariants*. These are currently being developed in [**4**] jointly with Abramovich and

Chen. In this announcement, we will in fact first skip directly to step (3), in §§2.1–2.3, and explain how the affine or homogeneous coordinate rings can be constructed directly from punctured invariants without the intervention of steps (1) and (2). However, the first two steps do give a far more detailed description of the mirror, and in §2.4 we sketch how the first two steps will also be carried out.

In particular, once the correct invariants are defined, one can then easily write down a description of the coordinate ring to the mirror of a log Calabi-Yau manifold (X, D) or a maximally unipotent normal crossings degeneration of Calabi-Yau manifolds. Determining the mirror in practice remains a difficult task in general, however.

In fact, the construction discussed here should be viewed as the construction of a piece of a quantum cohmology ring associated with any (log smooth) pair (X, D). Following discussions with Daniel Pomerleano, it appears that the construction we have given in §2.1 should give an algebro-geometric version of $SH^0(X \setminus D)$ (symplectic cohomology) in the case that (X, D) is log Calabi-Yau, and ongoing work of Ganatra and Pomerleano further suggests there may be a version of SH^*, the quantum cohomology of the pair (X, D). However, the construction given here only uses the degree 0 part of this hypothetical ring. In any event, this fits well with the conjectures of §0.5 of the first preprint version of [**24**] concerning the relationship between symplectic homology and the mirror construction given there. For toric degenerations of Calabi-Yau varieties the relation between broken lines, punctured log invariants, symplectic cohomology and the homogeneous coordinate ring of the mirror was suggested by the second author following discussions with Mohammed Abouzaid. The case of elliptic curves and their relation to tropical geometry has been studied in detail by Hülya Argüz [**7**].

The idea of using punctured log curves to assign algebro-geometric enumerative meaning to broken lines originates in 2012, after initial discussions with Abramovich and Chen suggested the existence of such invariants (although it took some time to iron out the details of punctured curves).

In an alternative approach to these ideas, Tony Yu has been developing the theory of non-Archimedean Gromov-Witten invariants, and has used these to interpret broken lines in the case that (X, D) is a Looijenga surface pair with D supporting an ample divisor, see [**59**] and references therein. The full development of non-Archimedean Gromov-Witten theory should allow the replacement, in the construction described here, of punctured invariants with non-Archimedean invariants. The advantage of the latter is that they are manifestly independent of the choice of compactification $X \setminus D \subseteq X$ (or of the birational model of a degeneration $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$). The advantage of punctured invariants is that they are technically much easier to define. They also relate to traditional algebraic geometry more directly and hence should be more amenable to explicit computations.

Turning to the structure of the paper, in §1, we will give a brief overview of logarithmic Gromov-Witten invariants as defined in [**30**], [**15**], [**1**] and the punctured invariants of [**4**]. Logarithmic Gromov-Witten invariants provide a natural way to both talk about Gromov-Witten theory of degenerations of varieties as well as relative Gromov-Witten theory with tangency conditions with respect to more complicated divisors than the usual theory of relative invariants allows [**48**]. Furthermore, punctured log maps can be viewed as a slight further generalization which allows marked points with a "negative order of tangency" with a divisor. In §2, we then use these to describe the actual construction. The only detail of the construction not given here is the proof of associativity of the multiplication operation, which is quite technical and will be presented in [**32**].

Acknowledgements: This work would have been impossible without the long-term collaboration with Paul Hacking and Sean Keel. In particular, Sean Keel has been lobbying for a long time for a Gromov-Witten theoretic interpretation for broken lines, which was the starting point for the work described here. We also thank Dan Abramovich and Qile Chen, who are collaborators in the work on punctured Gromov-Witten invariants described here. In addition, a conversation of the first author with Daniel Pomerleano led to the realisation that one can most efficiently describe the mirror construction directly via step (3) above, leading to the exposition given here. The second author is grateful to Mohammed Abouzaid for discussions in 2012 shaping his view on the role of symplectic cohomology in mirror symmetry and the relation to punctured invariants. The first author would also like to thank the organizing committee for the invitation to speak at the AMS Summer Institute in Algebraic Geometry in Salt Lake City, where these results were announced.

1. Punctured invariants

1.1. A short review of logarithmic geometry. Punctured invariants are a generalization of logarithmic Gromov-Witten invariants. They are both based on abstract logarithmic geometry as introduced by Illusie and K. Kato [**36**], [**43**]. While for a more comprehensive survey of this theory we have to point to other sources such as [**23**], Chapter 3 or [**2**], we would like to recall the gist of it.

Log geometry provides a powerful abstraction of pairs consisting of a scheme X and a divisor D with mild singularities, say normal crossings. In such a situation one is interested in the behaviour of functions having zeroes exclusively inside D. Such functions can be multiplied without losing this property, but they cannot be added. Thus the corresponding subsheaf $\mathcal{M}_{(X,D)}$ of $\mathcal{O}_X$, sloppily written as $\mathcal{O}^\times_{X\setminus D}\cap\mathcal{O}_X$, is a sheaf of multiplicative monoids containing $\mathcal{O}_X^\times$ as a subsheaf. The inclusion defines a homomorphism of monoid sheaves

$$\alpha_X : \mathcal{M}_{(X,D)} \longrightarrow \mathcal{O}_X$$

with the property that it induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$. With only this structure it is possible to define the sheaf of differential forms with logarithmic poles $\Omega_X(\log D)$, the $\mathcal{O}_X$-module locally generated by $\mathrm{d}f/f$ for f defining D locally.

Now quite generally, a *log structure* on a scheme X is a sheaf of (commutative) monoids $\mathcal{M}_X$ together with a homomorphism of sheaves of multiplicative monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ inducing an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$. There is an obvious notion of morphism of log schemes

$$f : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y),$$

which apart from an ordinary morphism $f : X \to Y$ of schemes provides a homomorphism $f^\flat : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ compatible with $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ via the structure morphisms $f^{-1}\alpha_Y$ and α_X.

Now as long as we have a pair (X, D), all abstract constructions can of course be written directly in terms of functions on X. The point is, however, that log structures have excellent functorial properties making it possible, for example, to work on a space with normal crossings as if it were embedded as a divisor in a smooth space, or even as the central fibre of a semi-stable degeneration, neither of which may exist.

To explain how this works, we first remark that given a log space $(Y, \mathcal{M}_Y)$ and a morphism $f : X \to Y$, one can define the *pull-back log structure* $f^*\mathcal{M}_Y$ on X as the fibred sum $\mathcal{O}_X^\times \oplus_{f^{-1}\mathcal{O}_Y^\times} f^{-1}\mathcal{M}_Y$. The structure map $f^*\mathcal{M}_Y \to \mathcal{O}_X$ is induced by the inclusion $\mathcal{O}_X^\times \to \mathcal{O}_X$ and by $f^\sharp \circ f^{-1}\alpha_Y : f^{-1}\mathcal{M}_Y \to \mathcal{O}_X$.

Now let $\mathcal{X} \to \operatorname{Spec} R$ be a flat morphism of schemes with R a discrete valuation ring with residue field $\Bbbk$. Then the preimage of the closed point $0 \in \operatorname{Spec} R$, the central fibre $X_0 \subseteq \mathcal{X}$, is a divisor. We thus obtain a morphism of log schemes

$$\big(\mathcal{X}, \mathcal{M}_{(\mathcal{X}, X_0)}\big) \longrightarrow \big(\operatorname{Spec} R, \mathcal{M}_{(\operatorname{Spec} R, 0)}\big),$$

and by functoriality of pull-back, its restriction to the central fibre

$$\big(X_0, \mathcal{M}_{X_0}\big) \longrightarrow (\operatorname{Spec} \Bbbk, \mathcal{M}_{(\operatorname{Spec} \Bbbk, 0)}\big).$$

The log structure on $\operatorname{Spec} \Bbbk$ can be described explicitly by observing that if $t \in R$ is a generator of the maximal ideal, then any element of $R \setminus \{0\}$ can be written uniquely as $h \cdot t^n$ with $n \geq 0$ and $h \in R^\times$. Thus the choice of t induces an isomorphism $\mathcal{M}_{(\operatorname{Spec} R, 0), 0} \simeq \mathbb{N} \oplus R^\times$, and the restriction to 0 yields the *standard log point* $\operatorname{Spec} \Bbbk^\dagger := (\operatorname{Spec} \Bbbk, \mathbb{N} \oplus \Bbbk^\times)$ with structure morphism

$$\mathbb{N} \oplus \Bbbk^\times \longrightarrow \Bbbk, \quad (n, h) \longmapsto \begin{cases} h, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Observe that all but the copy of $\Bbbk^\times$ in $\mathcal{M}_{\Bbbk^\dagger}$ maps to $0 \in \Bbbk$, reflecting the fact that $h \cdot t^n$ vanishes at 0 for $n > 0$.

We thus see that a log scheme arising as the central fibre of a degeneration comes with a morphism to the standard log point. Note such a morphism to the standard log point is uniquely determined by the pull-back of the generator $(1, 1) \in \mathbb{N} \oplus \Bbbk^\times$. This pull-back provides a global section $s_0 \in \Gamma(X_0, \mathcal{M}_{X_0})$, which provides logarithmic information about the deformation $\mathcal{X} \to \operatorname{Spec} R$ of X_0. For example, if X_0 is a reduced normal crossings divisor locally given inside $\mathcal{X}$ by $x_0 \cdot \ldots \cdot x_k = f \cdot t^e$ with f non-vanishing, then the log structure records $e \in \mathbb{N}$ and the restriction of f to the singular locus of X_0. Thus a log structure $\mathcal{M}_X$ carries both discrete information, given by the monoid quotient sheaf $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^\times$, and algebro-geometric information from the extension

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{M}_X^{\mathrm{gp}} \xrightarrow{\kappa} \overline{\mathcal{M}}_X^{\mathrm{gp}} \longrightarrow 0.$$

of the associated abelian sheaves. For example, for each section $\overline{m}$ of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ over an open set $U \subseteq X$, one obtains an $\mathcal{O}_U^\times$-torsor $\kappa^{-1}(\overline{m})$, or equivalently, the associated line bundle. In general we write the torsor as $\mathcal{L}_{\overline{m}}^\times \subseteq \mathcal{M}_X^{\mathrm{gp}}$ and the associated line bundle as $\mathcal{L}_{\overline{m}}$. Since $\overline{\mathcal{M}}_X$ has a sometimes subtle influence on the possibilities of the behaviour of the log structure, we like to call $\overline{\mathcal{M}}_X$ the *ghost sheaf* of $\mathcal{M}_X$.[1]

Without further restrictions log structures can be very pathological, and one usually restricts to so-called fine log structures defined as follows. If P is a finitely generated submonoid of a free monoid (a *fine monoid*) then $Y_P = \operatorname{Spec} \mathbb{Z}[P]$ is a (generalized) toric variety, which comes with its distinguished divisor D_P with

[1]More common seems to be the usage of *characteristic*, but we feel this word is already used too often in mathematics.

ideal generated by all elements of P not contained in a facet of P. We write such finitely generated monoids and monoid sheaves additively, thinking of its elements as exponents in $\mathbb{Z}[P]$. A log structure is called *fine* if it is locally obtained by pull-back of $\mathcal{M}_{(Y_P,D_P)}$. Given a log space $(X,\mathcal{M}_X)$, a morphism $f:U\to\operatorname{Spec}\mathbb{Z}[P]$ from an open subspace $U\subseteq X$ together with an isomorphism $f^*\mathcal{M}_{(Y_P,D_P)}\simeq\mathcal{M}_X|_U$ is called a *chart*. Explicit computations with fine log structures are all done in charts, hence their importance. A chart is uniquely determined by the homomorphism of monoids $P\to\Gamma(U,\mathcal{M}_X)$.

For example, if X is smooth over a field and $D\subseteq X$ is a divisor with simple normal crossings, with D on $U\subseteq X$ given by $x_1\cdot\ldots\cdot x_k=0$, then

$$\mathbb{N}^k\longrightarrow\Gamma(U,\mathcal{M}_{(X,D)}),\quad (a_1,\ldots,a_k)\longmapsto x_1^{a_1}\cdot\ldots\cdot x_k^{a_k} \tag{1.1}$$

is a chart for $(X,\mathcal{M}_{(X,D)})$ on U.

Caution has to be exercised with the topology, for the Zariski topology is often too coarse for applications, and one rather works at least in the étale topology. For example, to include in (1.1) normal crossing divisors with self-intersections already requires the étale topology.

Note that $\mathbb{Z}[P]$ is integrally closed only if P contains all $p\in P^{\mathrm{gp}}$ with $n\cdot p\in P$ for some $n>0$. Such monoids and corresponding log structures are called *saturated*. For fine log structures the ghost sheaf $\overline{\mathcal{M}}_X$ is a subsheaf of a constructible sheaf, its associated sheaf of abelian groups $\overline{\mathcal{M}}_X^{\mathrm{gp}}$. The log structure is called saturated if $\overline{\mathcal{M}}_X$ is a sheaf of saturated monoids. We typically work in the category of fine and saturated (fs) log structures: this is most important in considerations of fibre products, which are dependent on the particular subcategory of the category of log schemes being used.

A last concept in log geometry before we can turn to log Gromov-Witten invariants is *log smoothness*. By definition, a morphism of log schemes $f:(X,\mathcal{M}_X)\to(Y,\mathcal{M}_Y)$ is smooth if it fulfills the logarithmic analogue of formal smoothness in scheme theory (infinitesimal lifting criterion). Under mild assumptions, this statement is equivalent to saying that a chart $Q\to\Gamma(V,\mathcal{M}_Y)$ of Y can locally in X be lifted to a chart $P\to\Gamma(U,\mathcal{M}_X)$ in such a way that f is induced by a monoid homomorphism $Q\to P$ and $X\to Y\times_{\operatorname{Spec}\mathbb{Z}[Q]}\operatorname{Spec}\mathbb{Z}[P]$ is a smooth morphism of schemes ([**43**], Theorem 3.5 and [**41**], Theorem 4.1).

Thus a log smooth morphism is the abstraction of a toric morphism. Remarkably, the underlying morphism need not even be flat, a toric blowing up being the typical example. An instructional case is also smoothness of a fine saturated $(X,\mathcal{M}_X)$ over a point $\operatorname{Spec}\Bbbk$ with trivial log structure. Such a *log smooth variety* is nothing but a *toroidal pair* (X,D) with the divisor $D\subseteq X$ the support of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$.

1.2. Logarithmic Gromov-Witten invariants. Since deformation theory in log geometry is quite analogous to ordinary deformation theory, it is natural to try to extend Gromov-Witten theory with target X a smooth variety to the case with target $(X,\mathcal{M}_X)$ a log smooth variety, or more generally, work over a base log scheme $(S,\mathcal{M}_S)$ with $(X,\mathcal{M}_X)\to(S,\mathcal{M}_S)$ a log smooth morphism. Such an extension is in fact of great interest, as the logarithmic category naturally captures the tangency conditions that first appeared in relative Gromov-Witten theory as defined in [**49**], [**39**], [**40**] and [**48**]. In fact, the second author of this paper suggested in [**55**] that logarithmic Gromov-Witten theory was the natural context for thinking about relative invariants. Since then a full theory has been developed by ourselves

[**30**] and also, building on [**55**], by Abramovich and Chen [**1**], [**15**]. A theory which serves many of the same purposes but is more suitable for the symplectic category has also been developed independently by Brett Parker in [**53**], [**54**].

To define logarithmic Gromov-Witten theory, one simply adds the prefix *log-* to all spaces and morphisms in the definitions and hopes this works. This is almost the case, but it is a little more subtle and interesting because an ordinary stable map may support many log structures that are trivially related by base change. The way out is to pick a universal one. The universal choice is detected purely on the level of ghost sheaves, and this correspondence turns out to be closely related to tropical geometry. We now explain this story in more detail.

Gromov-Witten theory is based on stable maps, whose domains are nodal curves. The logarithmic geometry of nodal curves and their moduli spaces indeed already contains a number of crucial aspects of the theory [**42**]. We thus start discussing these first. Given a nodal curve C over a separably closed field $\Bbbk$, the log smooth enhancements $f:(\operatorname{Spec} C,\mathcal{M}_C)\to(\operatorname{Spec}\Bbbk,\mathcal{M}_{\operatorname{Spec}\Bbbk})$ can be classified easily. First, $\mathcal{M}_{\operatorname{Spec}\Bbbk}=Q\oplus\Bbbk^\times$ with Q a fine monoid with $Q^\times=\{0\}$, turning $\operatorname{Spec}\Bbbk$ into a *log point*. Log smoothness implies that at a generic point of C the morphism $f^*\mathcal{M}_{\operatorname{Spec}\Bbbk}\to\mathcal{M}_C$ is an isomorphism. This property hence fails at finitely many points, necessarily at all singular points $q\in C$ and possibly also at some smooth points $p\in C$.

At a smooth special point $p\in C$ the only possibility for a smooth morphism of log structures is given by the morphism of charts $Q\to Q\oplus\mathbb{N}$ with the chart $Q\oplus\mathbb{N}\to\Gamma(U,\mathcal{M}_C)$ defined by $f^\flat$ on Q and mapping $1\in\mathbb{N}$ to some local section σ of $\mathcal{M}_C$ with $\alpha_C(\sigma)=z$ a generator of $\mathfrak{m}_p\subseteq\mathcal{O}_{C,p}$. Note there is no freedom of $\mathcal{M}_C$ coming from the choice of chart because the second summand simply generates $\alpha_C(\mathcal{M}_C)\subseteq\mathcal{O}_C$. Thus p is nothing but a marked point of C, unlabelled for now.

At a node $q\in C$ smoothness implies a chart for f of the interesting form

$$Q\longrightarrow Q\oplus_{\mathbb{N}}\mathbb{N}^2,\quad q\longmapsto(q,0),$$

where the map $\mathbb{N}\to\mathbb{N}^2$ is the diagonal morphism $1\mapsto(1,1)$ and $\mathbb{N}\to Q$ defines an element $\rho_q\in Q\setminus\{0\}$. Thus $Q\oplus_{\mathbb{N}}\mathbb{N}^2$ is generated by Q and by two more elements $e_1=\big(0,(1,0)\big)$, $e_2=\big(0,(0,1)\big)$ with single relation $\rho_q=e_1+e_2$. The images of e_1, e_2 in $\mathcal{M}_{C,q}$ map to the defining equations $x,y\in\mathcal{O}_{C,q}$ of the two branches of C at q under α, and this property determines the corresponding sections $\sigma_x,\sigma_y\in\mathcal{M}_{C,p}$ up to invertible functions. The meaning of this chart is that the node $q\in C$ looks as if it is embedded in the deformation $\operatorname{Spec}\Bbbk[Q][x,y]/(xy-t^{\rho_q})\to\operatorname{Spec}\Bbbk[Q]$. Thus each node determines an element $\rho_q\in Q$ which records something like a virtual speed of smoothing of the node under variations of the base point. Note that if q is a self-intersection point of an irreducible component of C, the chart has to be understood in the étale topology.

It is then straightforward to generalize the notion of log smooth curve to families, arriving at a stack $\tilde{\mathscr{M}}$. In contrast to the classical case, fixing the degree, numbers of marked points and imposing stability (finiteness of automorphism group) is not enough to yield an open substack of finite type. The reason is that any homomorphism of monoids $\varphi:Q\to Q'$ with the property $\varphi^{-1}(0)=0$ defines a morphism of log points $(\operatorname{Spec}\Bbbk,Q'\oplus\Bbbk^\times)\to(\operatorname{Spec}\Bbbk,Q\oplus\Bbbk^\times)$. Base change of a log smooth curve $(C,\mathcal{M}_C)\to(\operatorname{Spec}\Bbbk,Q\oplus\Bbbk^\times)$ then defines another smooth structure $(C,\mathcal{M}'_C)\to(\operatorname{Spec}\Bbbk,Q'\oplus\Bbbk^\times)$.

Given a nodal curve C over $\Bbbk$ there is, however, always a universal (*minimal* or *basic*) log structure by taking $Q = \mathbb{N}^e$ with e the number of nodes and $\rho_q = e_q$ the corresponding generator. Any other log smooth enhancement of $C \to \operatorname{Spec}\Bbbk$ is then obtained by a unique pull-back, see [**42**], Proposition 2.1. Restricting to families with all geometric fibres carrying the universal log structure defines an open substack $\mathscr{M} \subseteq \tilde{\mathscr{M}}$. Once stability is imposed, the proper connected components can be identified with the smooth Deligne-Mumford stacks of ordinary stable curves $\mathbf{M}_{g,k}$. From this perspective we can also understand the universal log structure quite easily. Singular curves define a divisor with normal crossings $\mathbf{D}_{g,k} \subseteq \mathbf{M}_{g,k}$. The universal log structure on a family of stable curves is the pull-back by the morphism to $\mathscr{M}_{g,k}$ of $\mathcal{M}_{(\mathbf{M}_{g,k},\mathbf{D}_{g,k})}$.

Now that we have a fair understanding of the domains, we can turn to stable log maps. Given a log space $(X, \mathcal{M}_X)$, we define a *stable log map* over a scheme W and with target $(X, \mathcal{M}_X)$, to be a log smooth curve $(C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ for some log structure $\mathcal{M}_W$ on W together with a morphism of log spaces

$$(C, \mathcal{M}_C) \longrightarrow (X, \mathcal{M}_X).$$

Similarly to the case of curves, we now obtain an algebraic stack $\tilde{\mathscr{M}}(X, \mathcal{M}_X)$ of stable log maps. This stack is too big for the same reason that $\tilde{\mathscr{M}}$ is and we rather have to find an open substack of stable log maps with a minimality property, which we call basicness. We follow the exposition of [**30**], but basicness has been studied in much greater generality more recently in [**58**]. There are a few essential insights regarding the issue of basicness.

First, since basicness should be an open property, it is enough to define it on geometric points, that is, for a stable log map over an algebraically closed field.

Second, a stable map $(f : C \to X, \mathbf{x})$ with $\mathbf{x}$ the tuple of marked points, defines a log structure $f^*\mathcal{M}_X$ on C that typically is not the log structure of a log smooth curve. For example, the rank of $f^*\overline{\mathcal{M}}_X$ may jump on the smooth locus of C, contradicting strictness of $(C, \mathcal{M}_C) \to (\operatorname{Spec}\Bbbk, Q \oplus \Bbbk^\times)$ on this locus. Then the question of basicness boils down to finding a log structure $\mathcal{M}_C$ on C such that (i) it has a log smooth morphism $(C, \mathcal{M}_C) \to (\operatorname{Spec}\Bbbk, Q \oplus \Bbbk^\times)$ for some Q, (ii) there is a morphism of log structures $f^*\mathcal{M}_X \to \mathcal{M}_C$ and (iii) $\mathcal{M}_C$ is minimal with the properties (i) and (ii). Of course, given $f^*\mathcal{M}_X$, no such $\mathcal{M}_C$ may exist, which just means that $(f : C \to X, \mathbf{x})$ is not in the image of the forgetful morphism $\tilde{\mathscr{M}}(X, \mathcal{M}_X) \to \mathbf{M}(X)$ to the stack of ordinary stable maps.

The third insight is that basicness can be checked at the level of ghost sheaves. The reason is that given a log structure $\mathcal{M}_C$ on the domain curve C and a morphism $\overline{\mathcal{M}} \to \overline{\mathcal{M}}_C$ of sheaves of fine monoids, then $\mathcal{M} := \overline{\mathcal{M}} \times_{\overline{\mathcal{M}}_C} \mathcal{M}_C$ is again a log structure on C. Now if there exists any log enhancement $(C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ of a given stable map $(C \to X, \mathbf{x})$ and $\overline{\mathcal{M}}$ is the universal ghost sheaf for such log morphisms, then $(C, \mathcal{M} = \overline{\mathcal{M}} \times_{\overline{\mathcal{M}}_C} \mathcal{M}_C) \to (X, \mathcal{M}_X)$ is the desired basic log enhancement.

Just as for the domain we have three types of charts which readily describe the maps on the level of ghost sheaves. Let $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ be a stable log map over the log point $(\operatorname{Spec}\Bbbk, Q \oplus \Bbbk^\times)$ with $\Bbbk$ an algebraically closed field. For a scheme-theoretic point $y \in C$ denote $P_y = \overline{\mathcal{M}}_{X,f(y)}$.

(I) At a *generic point* $\eta \in C$ we have $\overline{\mathcal{M}}_{C,\eta} = Q$ and hence a homomorphism

$$V_\eta : P_\eta \longrightarrow Q.$$

(II) At a *marked point* $p \in C$ we have $\overline{\mathcal{M}}_{C,p} = Q \oplus \mathbb{N}$ with the projection to the first factor the generization map $\overline{\mathcal{M}}_{C,p} \to \overline{\mathcal{M}}_{C,\eta}$ to the generic point η with p in its closure. Thus the composition $P_p \to \overline{\mathcal{M}}_{C,p}$ with the projection to Q agrees with the generization map $P_p \to P_\eta$ on X composed with V_η. The additional data at p is the composition with the projection to $\mathbb{N}$, defining a homomorphism

$$u_p : P_p \longrightarrow \mathbb{N}.$$

(III) The most interesting data is at a *node* $q \in C$. Then $\overline{\mathcal{M}}_{C,q} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$ with the fibred sum defined by the relation $(\rho_q, (0,0)) = (0, (1,1))$ in $Q \oplus \mathbb{N}^2$. The generization maps $\overline{\mathcal{M}}_{C,q} \to \overline{\mathcal{M}}_{C,\eta_i}$ to the generic points η_1, η_2 of the two branches of C at q are given by embedding $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ into $Q \oplus Q$ via

$$\big(m, (a,b)\big) \longmapsto (m + a\rho_q, m + b\rho_q),$$

and projecting to one of the factors. Thus

$$P_q \longrightarrow Q \oplus_{\mathbb{N}} \mathbb{N}^2 \subseteq Q \oplus Q$$

equals the pair of compositions of the generization maps $\chi_i : P_q \to P_{\eta_i}$ with V_{η_i}, $i = 1, 2$. Since $(m_1, m_2) \in Q \oplus Q$ lies in the image of $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ iff $m_1 - m_2 \in \mathbb{Z}\rho_q$, we derive the existence of a homomorphism

$$u_q : P_q \longrightarrow \mathbb{Z}$$

fulfilling the important equation

$$V_{\eta_1} \circ \chi_1 - V_{\eta_2} \circ \chi_2 = u_q \cdot \rho_q. \tag{1.2}$$

Note that the sign of u_q depends on an ordering of the branches of C at q. Moreover, u_q is determined uniquely by this equation since $\rho_q \neq 0$.

From this description it is not hard to see that for any stable log map over a (separably closed) field, there is a universal choice of monoid Q, $\overline{\mathcal{M}}_C$ and morphism $f^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ of the desired form. The base monoid Q can be defined as an explicit quotient of $\prod_{\eta \in C} P_\eta \times \prod_{q \in C} \mathbb{N}$, see [**30**], Equation (1.14). We skip the formula and rather give an interpretation in terms of tropical geometry ([**30**], Remark 1.18) in §1.3 below.

In any case, with the characterization of basicness via the notion of stable log maps with basic log structure, we arrive at an open substack $\mathscr{M}(X, \mathcal{M}_X) \subseteq \tilde{\mathscr{M}}(X, \mathcal{M}_X)$, which, if X is proper over the base field, is a proper Deligne-Mumford stack over the base field once the degree and numbers of marked points are bounded ([**30**], Theorem 0.2, [**6**], Theorem 1.1.1). If in addition $(X, \mathcal{M}_X)$ is (log-) smooth over the base field then $\mathscr{M}(X, \mathcal{M}_X)$ comes with a virtual fundamental class with the expected properties. The corresponding intersection theoretic numbers are our log Gromov-Witten invariants.

A straightforward generalization works relative a fixed log scheme $(S, \mathcal{M}_S)$. Then for $(X, \mathcal{M}_X)$ smooth and proper over $(S, \mathcal{M}_S)$ one obtains a Deligne-Mumford stack $\tilde{\mathscr{M}}((X, \mathcal{M}_X)/(S, \mathcal{M}_S))$ of stable log maps over $(S, \mathcal{M}_S)$ that is proper over S and hence the corresponding virtual fundamental class and log Gromov-Witten invariants.

An important aspect of the theory is the discrete logarithmic data u_p at the marked points. Unlike the other discrete data V_η, u_q, ρ_q determining the basic monoid and map of ghost sheaves, u_p can be fixed once the marked points are labelled. To understand the meaning of u_p consider the situation of a toroidal pair (X, D) with $\mathcal{M}_X = \mathcal{M}_{(X,D)}$ the divisorial log structure and the component of C containing p not mapping into D. If $f(p)$ lies in only one irreducible component of D then D is Cartier at $f(p)$, say defined by $h = 0$. Then h generates $P_p = \mathbb{N}$, and the map $f^\flat : \mathcal{M}_{X,f(p)} \to \mathcal{M}_{C,p}$ is determined by $f^\sharp : \mathcal{O}_{X,f(p)} \to \mathcal{O}_{C,p}$, $h \mapsto g\cdot z^{u_p(1)}$, $g \in \mathcal{O}^\times_{C,p}$, z a local uniformizer at p. Thus u_p records the *contact order* of f with D at p. It is one merit of log geometry that this contact order still makes sense for marked points on components mapping into D.

If D has several irreducible components D_μ containing p, they may not be individually Cartier, but u_p records the contact order with respect to any linear combination $\sum_\mu a_\mu D_\mu$, $a_\mu \ge 0$, that is Cartier at p.

To set up a log Gromov-Witten counting problem, one needs to specify both the contact orders described above and degree data. We specify contact orders at each marked point by selecting a stratum $Z \subseteq X$ and compatible choices of homomorphisms $P_x \to \mathbb{N}$ for any $x \in Z$. In other words, we take a section $s \in \Gamma(Z, (\overline{\mathcal{M}}^{\mathrm{gp}}_X)^*)$ to fix u_p. We insist this choice of contact order is maximal by requiring that s does not extend to any larger closed subset of X.[2] We call such data *maximal contact data*. In particular, we can now define a *class* β of stable log map. Such a class consists of data the curve class $\underline{\beta} \in H_2(X, \mathbb{Z})$, the genus g of the curve, a number n of marked points, and a choice of maximal contact data as above for each marked point. Then the substack $\mathscr{M}_\beta(X, \mathcal{M}_X)$ of $\mathscr{M}(X, \mathcal{M}_X)$ consisting of stable log maps realising the homology class $\underline{\beta}$, of the given genus and number of marked points, and contact order at the marked points given by the sections, is in fact an open and closed substack of $\mathscr{M}(X, \mathcal{M}_X)$.

Example 1.1. As a simple example, let $X = \mathbb{P}^2$ with its toric log structure defined by $D = D_0 \cup D_1 \cup D_2$, the union of coordinate lines $D_\mu = (Z_\mu = 0)$. Consider the log Gromov-Witten count for genus $g = 0$ and three marked points p_1, p_2, p_3 mapping to the strata $D_1 \cap D_2$, $\mathbb{P}^2$ and D_0, respectively. One has $\Gamma(D_1 \cap D_2, (\overline{\mathcal{M}}^{\mathrm{gp}}_X)^*) = \mathbb{Z}^2$, $\Gamma(\mathbb{P}^2, (\overline{\mathcal{M}}^{\mathrm{gp}}_X)^*) = 0$ and $\Gamma(D_0, (\overline{\mathcal{M}}^{\mathrm{gp}}_X)^*) = \mathbb{Z}$; we take $u_{p_1} = (1, 1)$, $u_{p_2} = 0$, $u_{p_3} = 1$ and look at curves of degree one. If β symbolizes the given choice of discrete data, the corresponding moduli space $\mathscr{M}_\beta(X, \mathcal{M}_X)$ of stable log maps is isomorphic to the blowing up $\mathrm{Bl}_x \mathbb{P}^2$ of $\mathbb{P}^2$ at $x = D_1 \cap D_2$. The map $\mathscr{M}_\beta(X, \mathcal{M}_X) \to \mathbb{P}^2$ is given by evaluation at p_2.

Indeed, for $p_2 \notin D_1 \cap D_2$ there is a unique line $\ell \simeq \mathbb{P}^1$ through p_2 and $D_1 \cap D_2$, and the pull-back $f^\sharp$ of functions readily defines $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_C$. This is a curve without nodes and trivial monoid $P_\eta = 0$ at the generic point. Hence the minimal log structure has $Q = 0$, so it is a stable log map defined over the trivial log point $\operatorname{Spec} \Bbbk$. Now fixing the image ℓ and letting p_2 approach $D_1 \cap D_2$, the limit is a stable log map with two irreducible components C_1, C_2. The universal base monoid turns out to be $\mathbb{N}$, so this is a stable log map over the standard log point. The first component C_1 maps isomorphically to ℓ and has one marked point p_3 mapping to D_0, while C_2 is contracted to the intersection point $x = D_1 \cap D_2$

[2] We assume the log structure on X to be defined in the Zariski topology to avoid subtle points related to monodromy in $\overline{\mathcal{M}}$. However, [**58**] removes this condition.

and has two marked points p_1, p_2. Local equations $z_1 = 0$, $z_2 = 0$ of D_1 and D_2 at x can be viewed as generators of $\mathcal{M}_{\mathbb{P}^2,x} \subseteq \mathcal{O}_{\mathbb{P}^2,x}$. The choice of u_{p_2} says that $f^\flat(z_i)$ both map to $(1,0) \in \overline{\mathcal{M}}_{C,p_2} = Q \oplus \mathbb{N}$. Hence $f^\flat_{p_2}(z_1) = h \cdot f^\flat_{p_2}(z_2)$ for some $h \in \mathcal{O}^\times_{C_2,p_2}$. The value $h(p_2)$ then tells the direction that ℓ passes through x, that is, the point in the fibre of $\mathrm{Bl}_x \mathbb{P}^2$ over x. A complete analysis also has to address the special situations of $\ell \subseteq D_1 \cup D_2$, but this situation does not lead to any additional refinement, other than increasing the size of the base monoid.

1.3. The tropical interpretation. For any logarithmic point $(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\times)$, the moduli space of log morphisms

$$(\operatorname{Spec} \Bbbk, \mathbb{N} \oplus \Bbbk^\times) \longrightarrow (\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\times)$$

from the standard log point is non-empty. Its connected components are labelled by monoid homomorphisms $\psi : Q \to \mathbb{N}$ with $\psi^{-1}(0) = 0$, and each is a torsor under the multiplicative action of $\operatorname{Hom}(Q, \mathbb{G}_m)$. Thus we can probe the moduli space of stable log maps $\mathscr{M}(X, \mathcal{M}_X)$ by considering stable log maps over the standard log point. This point of view also provides the link to tropical geometry.

First, to any fine log space $(X, \mathcal{M}_X)$ in the Zariski topology one can functorially associate a polyhedral complex $\operatorname{Trop}(X, \mathcal{M}_X)$. This was carried out in [**30**], Appendix B, with a refinement given by Ulirsch in [**57**], §6, to handle correctly the tropicalization of a general fine log scheme or stack as a generalized cone complex.

Here we give the construction by identifying rational polyhedral cones along common faces as follows. For each scheme-theoretic point $x \in X$ define the rational polyhedral cone $C_x = \overline{\mathcal{M}}^\vee_{X,x}$, where for a monoid Q we write $Q^\vee = \operatorname{Hom}(Q, \mathbb{R}_{\geq 0})$. Then if $x \in \operatorname{cl}(y)$, the generization map $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,y}$ identifies C_y with a face of C_x. Define $\operatorname{Trop}(X) = \varinjlim C_x$, the colimit taken over all scheme-theoretic points in X partially ordered by specialization. Note that if the log structure is only defined in the étale topology, monodromy issues may lead to self-identifications of C_x and it is better to think of $\operatorname{Trop}(X)$ as a diagram in the category of polyhedral cones with arrows being inclusions of faces.

Example 1.2. Let X be a toric variety defined by a fan $\Sigma = \{\sigma\}$ in $N \simeq \mathbb{Z}^n$ and endowed with the log structure $\mathcal{M}_X$ defined by its toric divisor $D \subseteq X$. If $x \in X$ lies in the interior of the closed toric stratum defined by $\sigma \in \Sigma$, then $\overline{\mathcal{M}}_{X,x} = \operatorname{Hom}(\sigma \cap N, \mathbb{N})$. Hence $\operatorname{Trop}(X) = \varinjlim_{\sigma \in \Sigma} \sigma = \Sigma$.

Note however that $\operatorname{Trop}(X)$ does not contain information on the embedding of its cones into the fixed vector space $N_\mathbb{R}$. This embedding reflects the torus action on X, which is irrelevant in the construction of $\operatorname{Trop}(X)$.

By functoriality, the tropicalization of a stable log map $(C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ over $(W, \mathcal{M}_W)$ yields two maps of cone complexes

$$\operatorname{Trop}(C, \mathcal{M}_C) \longrightarrow \operatorname{Trop}(X, \mathcal{M}_X), \quad \operatorname{Trop}(C, \mathcal{M}_C) \longrightarrow \operatorname{Trop}(W, \mathcal{M}_W).$$

Each pull-back to a standard log point $(\operatorname{Spec} \Bbbk, \Bbbk^\times \oplus \mathbb{N}) \to (W, \mathcal{M}_W)$ leads to restriction of these maps to the fibre over the image of $\operatorname{Trop}(\operatorname{Spec} \Bbbk, \Bbbk^\times \oplus \mathbb{N}) = \mathbb{R}_{\geq 0}$. Of course, this situation is entirely described by the fibre over $1 \in \mathbb{R}_{\geq 0}$, which is a complex of polyhedra rather than cones.

Example 1.3. (*Traditional tropical curves from the toric case*) To make the contact of this point of view to traditional tropical geometry, see e.g. [**50**], let

$(X, \mathcal{M}_X)$ be a toric variety defined by a fan Σ in $N_{\mathbb{R}}$ as in Example 1.2. Let $\tilde{X} = X \times \mathbb{A}^1$ be the trivial degeneration with its toric log structure and viewed as a log space over $(\mathbb{A}^1, \mathcal{M}_{(\mathbb{A}^1,0)})$. Note that $\operatorname{Trop}(\tilde{X}, \mathcal{M}_{\tilde{X}}) = N_{\mathbb{R}} \times \mathbb{R}_{\ge 0}$ with the cell decomposition given by $\sigma \times \mathbb{R}_{\ge 0}$ and $\sigma \times \{0\}$, $\sigma \in \Sigma$. Now consider a stable log map over the standard log point to $(\tilde{X}, \mathcal{M}_{\tilde{X}})$ over $(\mathbb{A}^1, \mathcal{M}_{(\mathbb{A}^1,0)})$, i.e., a commutative diagram

$$\begin{tikzcd}
(C, \mathcal{M}_C) \arrow[r, "f"] \arrow[d] & (\tilde{X}, \mathcal{M}_{\tilde{X}}) \arrow[d] \\
(\operatorname{Spec} \Bbbk, \mathbb{N} \oplus \Bbbk^\times) \arrow[r] & (\mathbb{A}^1, \mathcal{M}_{(\mathbb{A}^1,0)})
\end{tikzcd} \tag{1.3}$$

The map $(\operatorname{Spec} \Bbbk, \mathbb{N} \oplus \Bbbk^\times) \to (\mathbb{A}^1, \mathcal{M}_{(\mathbb{A}^1,0)})$ is given by mapping $\operatorname{Spec} \Bbbk$ to $0 \in \mathbb{A}^1$ and mapping the toric coordinate t on $\mathbb{A}^1$ to $(b, 1) \in \mathbb{N} \oplus \Bbbk^\times$ for some $b > 0$. Tropicalizing this diagram, one can view the result as describing the cone over a traditional tropical curve in $N_{\mathbb{R}}$. Indeed, for each generic point $\eta \in C$ the tropicalization now is nothing but multiplication by V_η:

$$\begin{aligned}
h_\eta : \mathbb{R}_{\ge 0} = \overline{\mathcal{M}}^\vee_{C,\eta} &\longrightarrow N_{\mathbb{R}} \times \mathbb{R}_{\ge 0}, \\
\lambda &\longmapsto \lambda \cdot V_\eta \in \overline{\mathcal{M}}^\vee_{\tilde{X}, f(\eta)} = \sigma \times \mathbb{R}_{\ge 0} \subseteq N_{\mathbb{R}} \times \mathbb{R}_{\ge 0}.
\end{aligned}$$

Here $\sigma \in \Sigma$ labels the smallest stratum of X containing $f(\eta)$. The composition of h_η with the projection $N_{\mathbb{R}} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is multiplication by b.

For a double point $q \in C$ we have $\rho_q \in \mathbb{N}$ and hence $\overline{\mathcal{M}}^\vee_{C,q} \simeq \mathbb{R}_{\ge 0} \cdot ([0, \rho_q] \times \{1\})$, a cone over an interval of length ρ_q. Denoting this cone K_{ρ_q} and letting $\sigma \in \Sigma$ label the smallest stratum of X containing $f(q)$, the tropicalization of f at q defines the map of cones

$$h_q : K_{\rho_q} \longrightarrow \overline{\mathcal{M}}^\vee_{\tilde{X}, f(q)} = \sigma \times \mathbb{R}_{\ge 0} \subseteq N_{\mathbb{R}} \times \mathbb{R}_{\ge 0}.$$

The restriction of h_q to the two rays forming ∂K_{ρ_q} is h_{η_1}, h_{η_2} for η_i the generic points of the two branches of C at q.

Finally, a marked point $p \in C$ with tangency condition $u_p \in N \oplus \mathbb{N}$ yields the map

$$\begin{aligned}
h_p : \mathbb{R}^2_{\ge 0} = \overline{\mathcal{M}}^\vee_{C,p} &\longrightarrow \sigma \times \mathbb{R}_{\ge 0} \subseteq N_{\mathbb{R}} \times \mathbb{R}_{\ge 0}, \\
(\lambda, \mu) &\longmapsto \lambda \cdot V_\eta + \mu \cdot u_p.
\end{aligned}$$

Again $\sigma \in \Sigma$ labels the smallest stratum containing $f(p)$. Note that a non-logarithmic (traditional) incidence condition at p leads to $u_p = 0$ and makes h_p contract the submonoid $\{0\} \times \mathbb{R}_{\ge 0} \subseteq \mathbb{R}^2_{\ge 0}$.

Now assume that the tangency conditions u_p lie in $N \subseteq N \oplus \mathbb{N}$, that is, they are pulled back from X. Then the restriction of $\operatorname{Trop}(f)$ to the fibre over $1 \in \mathbb{R}_{\ge 0}$ as described before is a traditional tropical curve in $N_{\mathbb{R}}$. The vertices are in bijection with the irreducible components of C, the bounded edges correspond to double points and unbounded edges are given by marked points. The balancing condition at a vertex $\eta \in C$ comes from triviality of certain $\mathcal{O}_C^\times$-torsors related to the log morphism and is special to the toric case, see [**30**], Proposition 1.15 and Example 7.5, as well as earlier work [**52**].

In the non-toric situation of a log scheme $(X, \mathcal{M}_X)$ over the standard log point, the tropicalization of a stable log map can still be viewed as a tropical curve, but the

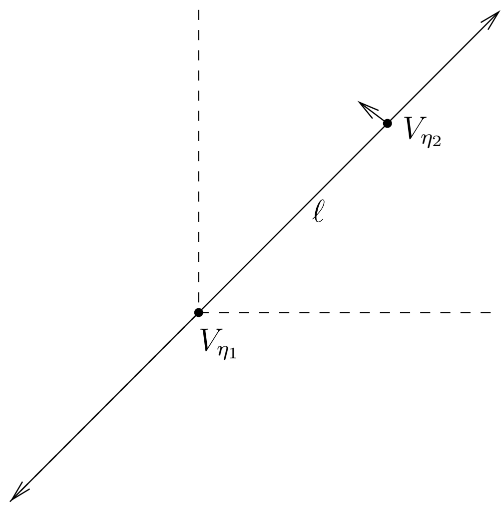

FIGURE 1. The tropical curve associated to the stable log map of Example 1.4. Here the dotted lines are rays in the fan for $\mathbb{P}^2$, while the diagonal line in the direction $(-1,-1)$ is both a ray in the fan for $\mathbb{P}^2$ and part of the tropical curve. The domain tropical curve has three unbounded edges corresponding to p_1, p_2 and p_3, but the edge associated to p_2 is contracted, depicted here by the short arrow.

balancing condition at a vertex only contains information along the given stratum and is inhomogeneous with a correction term determined by the underlying map of schemes $C \to X$, see [**30**], Proposition 1.15.

EXAMPLE 1.4. Returning to Example 1.1, consider the curve C with two irreducible components, $C = C_1 \cup C_2$, described there, with three marked points p_1, p_2, p_3, and $p_3 \in C_1$, $p_2, p_3 \in C_2$. The stable log map described in Example 1.1 to $\mathbb{P}^2$ also describes, after taking the product of $(C, \mathcal{M}_C)$ with the standard log point and replacing $\mathbb{P}^2$ with $\mathbb{P}^2 \times \mathbb{A}^1$, a stable log map with target space $\mathbb{P}^2 \times \mathbb{A}^1$. In particular, the curve $(C, \mathcal{M}_C)$ is defined over $(W, \mathcal{M}_W) = (\operatorname{Spec} \Bbbk, \mathbb{N}^2 \oplus \Bbbk^\times)$, where the first factor in $\mathbb{N}^2$ comes from the standard log point factor and the second comes from the base monoid $\mathbb{N}$ appearing in Example 1.1. One can make a base-change $(\operatorname{Spec} \Bbbk, \mathbb{N} \oplus \Bbbk^\times) \to (\operatorname{Spec} \Bbbk, \mathbb{N}^2 \oplus \Bbbk^\times)$ given by the monoid homomorphism $\mathbb{N}^2 \to \mathbb{N}, (a_1, a_2) \mapsto a_1\ell_1 + a_2\ell_2$ for some positive integers ℓ_1, ℓ_2. In this way one obtains a diagram (1.3) with $\tilde{X} = \mathbb{P}^2 \times \mathbb{A}^1$ and such that the restriction of $\operatorname{Trop}(f)$ to the fibre over $1 \in \operatorname{Trop}(\operatorname{Spec} \Bbbk, \mathbb{N} \times \Bbbk^\times)$ is a tropical curve in $\mathbb{R}^2$, depicted in Figure 1. Here $\ell = \ell_2/\ell_1$ determines the length of the edge corresponding to the node of C.

We can now give a tropical interpretation of the basic monoid Q. Namely, $Q^\vee$ can be identified with the moduli space of tropical curves with fixed combinatorial data. In this definition we admit real variations of the vertices inside the given cones of $\operatorname{Trop}(X)$, but fix the directions of the edges. In particular, the balancing condition does not play a role in the definition of $Q^\vee$ once *one* tropical curve in the deformation class arises as the tropicalization of a stable log map over the standard log point. Note that only those tropical curves respecting the integral structure actually arise in this fashion, the others arise from $\mathrm{Hom}(Q,\mathbb{R}_{\geq 0})$ and have a direct geometric interpretation only in non-archimedean geometry.

1.4. Punctured invariants. As a motivation for punctured Gromov-Witten invariants and the process of "puncturing", consider a log smooth curve over the standard log point $\tilde\pi : (\tilde C,\mathcal{M}_{\tilde C}) \to (\operatorname{Spec}\Bbbk, \mathbb{N}\oplus\Bbbk^\times)$ with two irreducible components $\tilde C = C\cup C'$ intersecting in one node $q\in\tilde C$ and with $\rho_q = \ell$. Thus there are $\zeta,\omega,\tau\in\mathcal{M}_{\tilde C,q}$ with $\tau = \tilde\pi^\flat(1,1)$ and single relation $\zeta\omega = \tau^\ell$ and such that any $\sigma\in\mathcal{M}_{\tilde C,q}$ can be written uniquely as $\sigma = h\cdot\zeta^a\omega^b\tau^c$ with $h\in\mathcal{O}^\times_{\tilde C,q}$ and $0\le c<\ell$. Moreover, we can arrange $z=\alpha_{\tilde C}(\zeta)$, $w=\alpha_{\tilde C}(\omega)$ to be local uniformizers of C,C' at q, respectively.

Now consider the log structure $\mathcal{M}_C$ on C over the standard log point defined by restriction of $\mathcal{M}_{\tilde C}$ to C. At a point $x\in C\setminus\{q\}$ the composition $(C,\mathcal{M}_C)\to(\tilde C,\mathcal{M}_{\tilde C})\to(\operatorname{Spec}\Bbbk,\mathbb{N}\oplus\Bbbk^\times)$ is a strict log morphism as before. At $x=q$ denote the restrictions of ζ,ω,τ to C by the same symbol. Then we can still write any $\sigma\in\mathcal{M}_{C,q}$ uniquely as $\sigma = h\cdot\zeta^a\omega^b\tau^c$ with $h\in\mathcal{O}^\times_{C,q}$ and $0\le c<\ell$. But now ω maps to $0\in\mathcal{O}_{C,q}$ because $w|_C=0$.

In $\mathcal{M}^{\mathrm{gp}}_{C,q}$ we can nevertheless write $\omega=\zeta^{-1}\tau^\ell$. Eliminating ω we arrive at a description of q analogous to a marked point as follows. Write $\mathbb{N}\oplus_{\mathbb{N}}\mathbb{N}^2$ defined by $\rho_q=\ell\in\mathbb{N}\setminus\{0\}$ and $(1,1)\in\mathbb{N}^2$, as the submonoid

$$S_\ell = \big(\mathbb{R}_{\geq 0}\cdot(-1,\ell)+\mathbb{R}_{\geq 0}\cdot(1,0)\big)\cap\mathbb{Z}^2,$$

of $\mathbb{Z}^2$ with generators $(-1,\ell),(0,1),(1,0)$. Note that $S_\ell^\vee = K_\ell$ from Example 1.3. The log structure of $(C,\mathcal{M}_C)$ at q is then defined by

$$S_\ell \longrightarrow \mathcal{O}_{C,q},\quad (a,b)\longmapsto \begin{cases} z^a, & b=0\\ 0, & b>0.\end{cases}$$

This description does not depend on anything but a marked point on C (here, q) and $\ell\in\mathbb{N}\setminus\{0\}$.

Conversely, starting from a marked point p on a log smooth curve $(C,\mathcal{M}_C)$ over a standard log point, for any $\ell\in\mathbb{N}$ there is a log structure $\mathcal{M}_C^\ell$ with $\mathcal{M}_C\subseteq\mathcal{M}_C^\ell\subseteq\mathcal{M}_C^{\mathrm{gp}}$ as follows. Take $\zeta,\tau\in\mathcal{M}_{C,p}$ to be generators up to $\mathcal{O}^\times_{C,p}$ as before. Then define $\mathcal{M}_C^\ell$ to agree with $\mathcal{M}_C$ away from p, while $\mathcal{M}^\ell_{C,p}$ is generated by $\zeta^a\tau^c$ with $c\ge 0$, $\ell a+c\ge 0$ and with structure homomorphism

$$\alpha^\ell_C(\zeta^a\tau^c) = \begin{cases} z^a, & c=0\\ 0, & c>0.\end{cases}$$

For a log smooth curve over the standard log point we have thus defined a notion of *puncture* at any marked point $p\in C$, depending on the choice of $\ell\in\mathbb{N}\setminus\{0\}$. It is designed to admit a log morphism $(C,\mathcal{M}_C^\ell)\to(\tilde C,\tilde{\mathcal{M}}_C)$, a typical example of a *punctured stable map*.

More generally, for any $r,s\in\mathbb{N}_{>0}$ we can embed the nodal curve $xy=0$ as boundary divisor into the two-dimensional affine toric variety $\operatorname{Spec}\mathbb{C}[P_{r,s}]$ with $P_{r,s}=\big(\mathbb{R}_{\geq 0}\cdot(-r,s)+\mathbb{R}_{\geq 0}\cdot(1,0)\big)\cap\mathbb{Z}^2$. Restriction to the component with coordinate defined by $(1,0)\in P_{r,s}$ then produces a log structure on $\mathbb{A}^1$ with a morphism to the standard log point defined by $(0,1)\in P_{r,s}$, which is strict except at one special point $p=0$ with monoid $P_{r,s}$. A universal choice is obtained by taking the direct limit of $P_{r,s}$ over all (r,s), ordered by inclusion, that is, by the slope r/s. The governing monoid at p is then

$$\varinjlim_{(r,s)} P_{r,s} = \big\{(a,b) \in \mathbb{N} \oplus \mathbb{Z} \,\big|\, b = 0 \Rightarrow a \geq 0\big\}.$$

For the puncturing of a curve at a smooth point p, this definition generalizes to arbitrary base monoids Q giving

$$\overline{\mathcal{M}}^{\circ}_{C,p} = \big\{(a,b) \in Q \oplus \mathbb{Z} \,\big|\, b = 0 \Rightarrow a \geq 0\big\}$$

A slight reason for discomfort with the universal punctured log structure $\mathcal{M}^{\circ}_C$ is that $\overline{\mathcal{M}}^{\circ}_{C,p}$ is not a finitely generated monoid. In the application to punctured stable maps, this in fact never matters, for we can work with a smaller fine and saturated log structure, see Remark 1.6.

The general definition for the puncturing $(C, \mathcal{M}^{\circ}_C)$ of a log smooth curve $(C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ at a section $p : W \to C$ with image disjoint from any special points has to treat non-reduced base schemes properly. Denote by $\alpha_P : \mathcal{P} \to \mathcal{O}_C$ the divisorial log structure defined by the Cartier divisor $\mathrm{im}(p) \subseteq C$. Now define the *puncturing of* $(C, \mathcal{M}_C)$ *along* p by the subsheaf $\mathcal{M}^{\circ}_C \subseteq \mathcal{M}_C \oplus_{\mathcal{O}^{\times}_C} \mathcal{P}^{\mathrm{gp}}$ agreeing with $\mathcal{M}_C$ away from p and generated by pairs $(\sigma, \zeta) \in \mathcal{M}_{C,p} \oplus \mathcal{P}^{\mathrm{gp}}_p$ with the property

$$\alpha(\sigma) \neq 0 \quad \Longrightarrow \quad \zeta \in \mathcal{P}.$$

Thus $\mathcal{M}^{\circ}_C$ is the largest subsheaf of $\mathcal{M}_C \oplus_{\mathcal{O}^{\times}_C} \mathcal{P}^{\mathrm{gp}}$ to which the sum of structure homomorphisms $\mathcal{M}_C \to \mathcal{O}_C$ and $\mathcal{P} \to \mathcal{O}_C$ extends.

We now see how punctured curves can allow negative contact orders. Suppose given a puncturing $(C, \mathcal{M}^{\circ}_C)$ of a log smooth curve $\pi : (C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ along a section p, and suppose given a log morphism $f : (C, \mathcal{M}^{\circ}_C) \to (X, \mathcal{M}_X)$. Then we obtain a composed map

$$u_p : \overline{\mathcal{M}}_{X,f(p)} = P_p \xrightarrow{\bar{f}^\flat} \overline{\mathcal{M}}^{\circ}_C \subseteq Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$$

where $Q = \overline{\mathcal{M}}_{W,\pi(p)}$. This is clearly analogous to the contact order u_p in the non-punctured case, but now it is possible that the image of u_p does not lie in $\mathbb{N}$.

Example 1.5. Let X be a non-singular surface containing a non-singular curve $D \cong \mathbb{P}^1$ with $D^2 = -1$. Consider the target space $(X, \mathcal{M}_{(X,D)})$. Take as domain curve $C = D$, defined over the standard log point $\operatorname{Spec} \Bbbk^{\dagger}$. Choose a point $p \in C$ as a puncture. Thus if η is the generic point of C, then $\overline{\mathcal{M}}^{\circ}_{\eta} = \mathbb{N}$ but $\overline{\mathcal{M}}^{\circ}_{p}$ is a non-finitely generated monoid contained in $\mathbb{N} \oplus \mathbb{Z}$. We can define a log morphism $f : (C, \mathcal{M}^{\circ}_C) \to (X, \mathcal{M}_{(X,D)})$ which is the identification of C with D as an ordinary morphism. Noting that $\overline{\mathcal{M}}_{(X,D)} = \mathbb{N}_D$, the constant sheaf on D with stalk $\mathbb{N}$, the map $\bar{f}^\flat : \overline{\mathcal{M}}_{(X,D)} \to \overline{\mathcal{M}}^{\circ}_C$ is given by $1 \mapsto (1,-1) \in \mathbb{N}_C \oplus \mathbb{Z}_p$. Note that $(1,-1) \in \overline{\mathcal{M}}^{\circ}_{C,p}$.

To see that this choice of $\bar{f}^\flat$ lifts to an actual log morphism, it is enough to map the torsor $\mathcal{L}^{\times}_1|_D$ associated to $1 \in \mathbb{N}_D$ to the torsor $\mathcal{L}^{\times}_{(1,-1)}$ associated to $(1,-1) \in \mathbb{N}_C \oplus \mathbb{Z}_p$. But the line bundle associated to the former torsor $\mathcal{L}^{\times}_1|_D$ is the conormal bundle of D in X, i.e., $\mathcal{O}_D(1)$. Since a section of the form $(a,0)$ of $\overline{\mathcal{M}}^{\circ}_C$ is pulled-back from the standard log point, the torsor $\mathcal{L}^{\times}_{(1,0)}$ is trivial, and the line bundle associated to the torsor $\mathcal{L}^{\times}_{(0,1)}$ is the ideal sheaf of p in C, i.e., $\mathcal{O}_C(-1)$. Thus the line bundle associated to the torsor $\mathcal{L}^{\times}_{(1,-1)}$ is $\mathcal{O}_C(1)$. We choose an isomorphism between $\mathcal{O}_C(1)$ and $f^*\mathcal{O}_D(1)$ to define the log morphism f.

Thus we have constructed a punctured curve which can be viewed as being "tangent to D to order -1."

With the definition of a punctured curve at hand, we can now define a punctured stable map with a number k of marked points and a number k' of punctures. The construction of the corresponding stack $\tilde{\mathscr{M}}_{k,k'}(X,\mathcal{M}_X)$ is indeed completely straightforward. Moreover, since the basicness condition does not involve the marked points, the same definition also works for punctured stable maps.

What is a little more difficult is the right version of obstruction theory for the construction of the virtual fundamental class. The difficulty arises essentially because punctured log structures do not behave well under base-change, and in particular even the ghost sheaf $\overline{\mathcal{M}}^{\circ}_C$ does not pull back under base change, but may get bigger. In particular, in a deformation theory situation, where one considers $\bar{W} \subseteq W$ a closed subscheme defined by a square zero ideal, and given a log smooth family $C \to W$ restricting to $\bar{C} = C \times_W \bar{W} \to \bar{W}$, with a choice of puncturing section $p : W \to C$, one will have $\overline{\mathcal{M}}^{\circ}_C \subseteq \overline{\mathcal{M}}^{\circ}_{\bar{C}}$ (as sheaves on the same underlying toplogical space), but equality often fails. As a result, given a morphism $\bar{f} : (\bar{C}, \mathcal{M}^{\circ}_{\bar{C}}) \to (X, \mathcal{M}_X)$, it may be possible that the image of $\bar{f}^{\flat}$ does not lie in the smaller sheaf $\overline{\mathcal{M}}^{\circ}_C$, and hence there is an essentially local, combinatorial obstruction to lifting the morphism $\bar{f}$ to a morphism $f : (C, \mathcal{M}^{\circ}_C) \to (X, \mathcal{M}_X)$. Such an obstruction cannot be encoded in a cohomology group.

The solution to this problem is roughly as follows. Set $W = \mathscr{M}_{k,k'}(X,\mathcal{M}_X)$ for now. In [**5**], a construction of the *Artin fan* $\mathcal{A}_W$ of a log stack W is given. Without going into detail, this is a zero-dimensional Artin stack which captures the combinatorial content of the log structure on W, and has an étale open cover by toric stacks of the form $[\operatorname{Spec} \Bbbk[Q]/\operatorname{Spec}\Bbbk[Q^{\mathrm{gp}}]]$ with Q ranging over stalks of $\overline{\mathcal{M}}_W$. In [**4**], a closed substack $\mathcal{T}_W$ of $\mathcal{A}_W$ is constructed, such that the morphism $W \to \mathcal{A}_W$ factors through $\mathcal{T}_W$. Roughly, one can define a relative perfect obstruction theory over $\mathcal{T}_W$.

This stack $\mathcal{T}_W$ has one immediate disadvantage, which is that depending on the combinatorics of the situation, it may not be equi-dimensional. Thus there is no virtual fundamental class in general, and one needs to take care in various contexts to extract numbers. Some examples of where one can extract useful numbers appear in §2.

Remark 1.6. From several points of view, punctured log structures are not particularly well-behaved, e.g., they don't behave well under pull-back and the stalks of the ghost sheaves at punctures are not in general finitely generated. There is however a natural choice of a fine saturated sub-log structure on C associated to any punctured log map. Given $\pi : (C,\mathcal{M}_C) \to (W,\mathcal{M}_W)$ and $f : (C,\mathcal{M}^{\circ}_C) \to (X,\mathcal{M}_X)$, there is a unique smallest sub-log structure $\mathcal{M}^{\mathrm{fs}}_C \subseteq \mathcal{M}^{\circ}_C$ which is fine and saturated, contains the image of $f^*\mathcal{M}_X$ under $f^{\flat}$, contains the image of $\pi^*\mathcal{M}_W$ under $\pi^{\flat}$, and contains the log structure $\mathcal{M}_{(C,p)}$. We note that $(C,\mathcal{M}^{\mathrm{fs}}_C) \to (W,\mathcal{M}_W)$ is not log smooth, and the log structure $\mathcal{M}^{\mathrm{fs}}_C$ depends on f. However, it has a pleasant tropical interpretation. Suppose $(W,\mathcal{M}_W)$ is the standard log point. Then the fibre over 1 of $\operatorname{Trop}(C,\mathcal{M}^{\mathrm{fs}}_C) \to \operatorname{Trop}(W,\mathcal{M}_W) = \mathbb{R}_{\geq 0}$ has an edge corresponding to a puncture, which may either be bounded or unbounded, and the restriction of

$\mathrm{Trop}(f)$ to this edge maps the edge to the longest possible line segment or ray in $\overline{\mathcal{M}}^{\vee}_{X,f(p)}$ with one end-point given by $V_\eta \in \overline{\mathcal{M}}^{\vee}_{X,f(\eta)}$, where η is the generic point of the irreducible component of C containing p. If this edge is unbounded, then in fact $u_p \in \overline{\mathcal{M}}^{\vee}_{X,f(p)}$ only takes non-negative values, and it is not necessary to puncture the curve.

2. The construction of mirrors

2.1. Algebras associated to pairs. We begin with a simple normal crossings pair (X, D): X is a smooth projective variety and D is a reduced simple normal crossings divisor. We also assume that for any collection $\{D_i\}$ of irreducible components of D, $\bigcap_i D_i$ is also irreducible if non-empty. We obtain from this pair a log scheme $(X, \mathcal{M}_{(X,D)})$. We shall write short-hand $\mathcal{M}_X := \mathcal{M}_{(X,D)}$, and usually just write X instead of $(X, \mathcal{M}_{(X,D)})$; it should be clear from context when we are talking about the log scheme. More generally, for any log scheme $(W, \mathcal{M}_W)$, we shall leave off the $\mathcal{M}_W$ in the notation.

We let $B = \mathrm{Trop}(X)$ be the tropicalization of X. Explicitly, let $\mathrm{Div}_D(X) \subseteq \mathrm{Div}(X)$ be the subspace of divisors supported on D, $\mathrm{Div}_D(X)_{\mathbb{R}} = \mathrm{Div}_D(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that $\mathrm{Div}_D(X) = \Gamma(X, \overline{\mathcal{M}}^{\mathrm{gp}}_X)$. We can write $\mathrm{Trop}(X)$ as a polyhedral cone complex in the dual space $\mathrm{Div}_D(X)^*_{\mathbb{R}}$ as follows. Let $D = \bigcup_i D_i$ be the decomposition of D into irreducible components, and write $\{D_i^*\}$ for the dual basis of $\mathrm{Div}_D(X)^*_{\mathbb{R}}$. Then define $\mathscr{P}$ to be the collection of cones

$$\mathscr{P} := \left\{ \sum_{i \in I} \mathbb{R}_{\geq 0} D_i^* \,|\, I \subseteq \{1, \ldots, m\} \text{ such that } \textstyle\bigcap_{i \in I} D_i \neq \emptyset \right\},$$

and define

$$B := \bigcup_{\tau \in \mathscr{P}} \tau.$$

Writing $\mathrm{Div}_D(X)^* = \mathrm{Hom}(\mathrm{Div}_D(X), \mathbb{Z})$, we define

$$B(\mathbb{Z}) = B \cap \mathrm{Div}_D(X)^*.$$

Given $p \in B(\mathbb{Z})$, we have a stratum

$$Z_p := \bigcap_{i : \langle p, D_i \rangle > 0} D_i$$

and the open stratum $Z_p^\circ \subseteq Z_p$ obtained by deleting all deeper strata. Note Z_p, Z_p° only depend on the minimal cone of $\mathrm{Trop}(X)$ containing p.

We fix a finitely generated, saturated submonoid $P \subset H_2(X, \mathbb{Z})$ containing all effective curve classes and such that $P^\times := P \cap (-P) \subseteq H_2(X, \mathbb{Z})_{\mathrm{tors}}$.[3] Let $\mathfrak{m} \subseteq \Bbbk[P]$ be the monomial ideal generated by monomials in $P \setminus P^\times$, and fix an ideal $I \subseteq \Bbbk[P]$ with $\sqrt{I} = \mathfrak{m}$. We write $A_I := \Bbbk[P]/I$, and set

$$R_I := \bigoplus_{p \in B(\mathbb{Z})} A_I \vartheta_p \tag{2.1}$$

a free A_I-module. Our immediate goal is to define structure constants for an A_I-algebra structure on R_I. In complete generality, this algebra structure will not be

[3] Note that if $H_2(X, \mathbb{Z})$ has torsion, then $\mathrm{Spec}\, \Bbbk[P]$ has a number of connected components, and the mirror family we build will thus have a number of connected components. This fits with the expectation in [**8**].

associative, but will be associative under some hypotheses on the pair (X, D), see e.g., Theorem 2.2.

We do this by defining structure constants:

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \vartheta_r$$

with $\alpha_{pqr} \in A_I$. We are going to write monomials in A_I as $t^{\underline{\beta}}$, $\underline{\beta} \in P$, to emphasize the character of A_I as the base ring of a deformation. We then write

$$\alpha_{pqr} = \sum_{\underline{\beta} \in P \setminus I} N^{\underline{\beta}}_{pqr} t^{\underline{\beta}} \tag{2.2}$$

where $N^{\underline{\beta}}_{pqr} \in \mathbb{Q}$ are defined as follows.

The data $\underline{\beta}, p, q$ and r determine a class β of punctured curve on X, as follows. The associated homology class is $\underline{\beta}$. We consider curves of genus zero with two marked points, x_1 and x_2, and one puncture, x_3. Now let $Z_1 := Z_p$, $Z_2 := Z_q$, $Z_3 := Z_r$. Then p, q, r determine sections s_i of $\Gamma(Z_i, (\overline{\mathcal{M}}_X)|^*_{Z_i})$. Indeed, to define s_1, we define a map $(\overline{\mathcal{M}}_X)|_{Z_1} \to \underline{\mathbb{N}}$ as follows. One can identify the stalk $\overline{\mathcal{M}}_{X,\eta_1}$ at the generic point η_1 of Z_1 with $\bigoplus_{i:\langle p, D_i\rangle > 0} \mathbb{N} D_i$. Then s_1 is defined on an open set $U \subseteq Z_1$ as the composition

$$(\overline{\mathcal{M}}_X)|_{Z_1}(U) \to \overline{\mathcal{M}}_{Z_1,\eta_1} \xrightarrow{p} \mathbb{N}, \tag{2.3}$$

where the first map takes a section to its germ at η_1. Put another way, we are imposing the condition that the curve should be tangent to D_i at the point x_1 to order $\langle p, D_i\rangle$.

We define s_2 similarly using q, whereas to define s_3, we use $-r$ instead of r, and in particular, $-r$ defines a map $(\overline{\mathcal{M}}_X)|_{Z_3} \to \mathbb{Z}$, and unless $r = 0$, x_3 must be viewed as a punctured rather than a marked point.

We now obtain a moduli space $\mathscr{M}_\beta(X)$ of punctured maps to X of class β. The next step is to impose a point constraint on the point x_3 by selecting a point $z \in Z_3^\circ$ and constraining the punctured point x_3 to map to z. This is a slightly delicate condition to impose in the log category. Indeed, there is of course an evaluation map at the level of underlying stacks $\underline{ev} : \mathscr{M}_\beta(X) \to Z_3$, so the first thought would be to define this moduli space to be the fibre product in the category of stacks $\mathscr{M}_\beta(X) \times_{Z_3} z$. However, this is ignoring the log structures. There is no log extension of the evaluation map since the log structure on the moduli space is smaller than the log structure at the punctured point. This is a general feature of imposing constraints for stable log maps, and the solution is the *evaluation space* introduced in [**3**], or rather a punctured version of it. There is an Artin stack $\mathscr{P}(X, r)$, the *evaluation space of punctures of type* r, along with an evaluation map $\mathrm{ev} : \mathscr{M}_\beta(X) \to \mathscr{P}(X, r)$. The Artin stack $\mathscr{P}(X, r)$ is a $B\mathbb{G}_m$-gerbe over Z_3. The choice of $z \in Z_3$ gives a stack morphism $B\mathbb{G}_m \to \mathscr{P}(X, r)$. In fact, r determines a canonical logarithmic extension of this map, with $B\mathbb{G}_m$ carrying a universal log structure induced by the divisorial log structure $B\mathbb{G}_m \subseteq [\mathbb{A}^1/\mathbb{G}_m]$. With this log structure, $\overline{\mathcal{M}}_{B\mathbb{G}_m} = \mathbb{N}$, the stalk of $\overline{\mathcal{M}}_{\mathscr{P}(X,r)}$ at the point corresponding to z is $\overline{\mathcal{M}}_{X,z}$, and the log stack morphism $B\mathbb{G}_m \to \mathscr{P}(X, r)$ is given by $r : \overline{\mathcal{M}}_{X,z} \to \mathbb{N}$ at the level of ghost sheaves. Once this morphism is defined, we can define $\mathscr{M}_{\beta,z}(X) = \mathscr{M}_\beta(X) \times_{\mathscr{P}(X,r)} B\mathbb{G}_m$. We have:

LEMMA 2.1. *$\mathscr{M}_{\beta,z}(X)$ is a proper Deligne-Mumford stack carrying a virtual fundamental class with virtual dimension $-\underline{\beta}\cdot(K_X+D)$.*

When this expected dimension is zero, we set

$$N^{\underline{\beta}}_{pqr} = \int_{[\mathscr{M}_{\beta,z}(X)]^{\mathrm{virt}}} 1,$$

and otherwise set

$$N^{\underline{\beta}}_{pqr} = 0.$$

THEOREM 2.2. *If either K_X+D or $-(K_X+D)$ is nef, then the product given by the structure constants α_{pqr} is associative.*

SKETCH OF PROOF. A complete proof will be given in [**32**]. The idea is standard. We would like to show the coefficient of ϑ_r is the same in the two products $(\vartheta_{p_1}\cdot\vartheta_{p_2})\cdot\vartheta_{p_3}$ and $\vartheta_{p_1}\cdot(\vartheta_{p_2}\cdot\vartheta_{p_3})$. Expanding in terms of the classes $\underline{\beta}$, we need to show that for each $\underline{\beta}\in P\setminus I$, $r\in B(\mathbb{Z})$, we have

$$\sum_{\substack{\underline{\beta_1},\underline{\beta_2},s\\ \underline{\beta_1}+\underline{\beta_2}=\underline{\beta}}} N^{\underline{\beta_1}}_{p_1p_2s}N^{\underline{\beta_2}}_{sp_3r} = \sum_{\substack{\underline{\beta_1},\underline{\beta_2},s\\ \underline{\beta_1}+\underline{\beta_2}=\underline{\beta}}} N^{\underline{\beta_1}}_{p_2p_3s}N^{\underline{\beta_2}}_{p_1sr} \tag{2.4}$$

where the sums are over all splittings of $\underline{\beta}$ in $P\setminus I$ and all $s\in B(\mathbb{Z})$. To show this equality, one fixes a point $z\in Z_r^\circ$ and considers the moduli space $\mathscr{M}_{\beta,z}(X)$ of genus 0 four-pointed punctured curves, with tangency conditions at the four marked points given by p_1,p_2,p_3 and $-r$, defined exactly as in the definition of $\mathscr{M}_{\beta,z}(X)$ in the three-pointed case. The virtual dimension of this moduli space is $-\underline{\beta}\cdot(K_X+D)+1$.

Note that by construction the numbers $N^{\underline{\beta_i}}$ are all zero unless $\underline{\beta_i}\cdot(K_X+D)=0$, so we may assume that $\underline{\beta}\cdot(K_X+D)=0$. Thus the virtual dimension of $\mathscr{M}_{\beta,z}(X)$ is 1, and there is a "virtually finite" morphism $\psi:\mathscr{M}_{\beta,z}(X)\to\overline{\mathcal{M}}_{0,4}$, with a suitable notion of "logarithmic virtual degree". One needs to show that either side of (2.4) coincides with the virtual degree of this morphism. This is done by looking at the (logarithmic) fibre of this morphism over different boundary points of $\overline{\mathcal{M}}_{0,4}$, and showing that certain logarithmic fibres represent curves which can be decomposed into curves contributing, say, to $N^{\underline{\beta_1}}_{p_1p_2s}$ and $N^{\underline{\beta_2}}_{sp_3r}$ respectively. The condition on $\pm(K_X+D)$ being nef guarantees via dimension counting arguments that all contributions to the "virtual degree" of ψ arise in this manner. Since the virtual degree is then independent of the choice of boundary point, we obtain (2.4).

The main technical difficulty involves the fact that one has to glue logarithmic curves, and doing this at the level of the virtual cycles is still technologically difficult. General gluing techniques are currently under development with D. Abramovich and Q. Chen. □

REMARK 2.3. The hypotheses of the above theorem (unfortunately omitted in an earlier version of this paper) reflect the fact that there really should exist a full analogue of quantum cohomology, a "relative quantum cohomology ring," an algebro-geometric analogue of symplectic cohomology. We have described the degree 0 part of the ring, and the product we have defined is only the projection of the product to the degree 0 part of the ring. Such a projection in general would not be expected to preserve associativity. We are working with D. Pomerleano to define this relative quantum cohomology ring.

The following will be useful for analyzing a number of situations:

PROPOSITION 2.4. *Let β be a class of punctured curve with $n+m$ tangency conditions, with $x_1,\ldots,x_n$ being marked points with tangency condition specified by $p_1,\ldots,p_n \in B(\mathbb{Z})$ and $x_{n+1},\ldots,x_{n+m}$ being punctured points with tangency condition specified by $-p_{n+1},\ldots,-p_{n+m}$ with $p_{n+1},\ldots,p_{n+m} \in B(\mathbb{Z})$. Then in order for $\mathscr{M}_\beta(X)$ to be non-empty, we must have for any $D' \in \mathrm{Div}_D(X)$,*

$$\underline{\beta}\cdot D' = \sum_{i=1}^{n}\langle p_i, D'\rangle - \sum_{i=n+1}^{n+m}\langle p_i, D'\rangle.$$

PROOF. Note that $\Gamma(X,\overline{\mathcal{M}}_X)$ can be naturally identified with the submonoid $\bigoplus \mathbb{N} D_i \subseteq \mathrm{Div}_D(X)$. For any $\overline{m} \in \Gamma(X,\overline{\mathcal{M}}_X)$, we have the associated line bundle $\mathcal{L}_{\overline{m}}$. Then $\mathcal{L}_{D_i} = \mathcal{O}_X(-D_i)$. If we have a punctured curve representing the type β, say $f : C \to X$, with C defined over the standard log point, then $\underline{f}^*\mathcal{O}_X(-D_i)$ must be the line bundle $\mathcal{L}_i$ associated to the torsor corresponding to $\bar{f}^\flat(D_i)$, where $\bar{f}^\flat : \Gamma(X,\overline{\mathcal{M}}_X) \to \Gamma(C,\overline{\mathcal{M}}_C)$ is induced by $f^\flat : \underline{f}^{-1}\mathcal{M}_X \to \mathcal{M}_C$.

Now the value of the total degree of $\mathcal{L}_i$ can be calculated using [**30**], Lemma 1.14 in the case there are no punctures, and the same result continues to hold in the punctured case [**4**]. In particular, the total degree of $\mathcal{L}_i$ is

$$-\sum_{j=1}^{n}\langle p_j, D_i\rangle + \sum_{j=n+1}^{n+m}\langle p_j, D_i\rangle.$$

This degree must coincide with the degree of $\underline{f}^*\mathcal{O}_X(-D_i)$, yielding the desired formula. $\square$

EXAMPLE 2.5. Consider the case that $I = \mathfrak{m}$. In this case, the only curve classes which may contribute to (2.2) are elements $\underline{\beta}$ of $H_2(X,\mathbb{Z})_{\mathrm{tors}}$. Such a class can only be represented by a constant map, and hence $\underline{\beta} = 0$. In particular, any punctured log map $f : (C, x_1, x_2, x_3) \to X$ representing a point in $\mathscr{M}_{\beta,z}(X)$ must be the constant map with image z. In fact, $\mathscr{M}_{\beta,z}(X)$ consists of a single point, with no automorphisms, and $N_{pqr}^{\beta} = 1$. In addition, p, q and r must all lie in the same cone σ of $\mathrm{Trop}(X)$. Then by Proposition 2.4, we have the equality $p + q = r$ in σ. We then obtain

$$R_{\mathfrak{m}} = A_{\mathfrak{m}}[B] := \bigoplus_{p\in B(\mathbb{Z})} A_{\mathfrak{m}}\vartheta_p,$$

with the multiplication rule given by

$$\vartheta_p\cdot\vartheta_q = \begin{cases}\vartheta_{p+q} & p,q \text{ in a common cone of } \mathrm{Trop}(X)\\ 0 & \text{otherwise,}\end{cases}$$

see [**25**], §2.1.

It is not difficult to show along the lines of [**25**], Prop. 3.17, that $\mathrm{Spec}\, R_I \to \mathrm{Spec}\, A_I$ is a flat deformation of $\mathrm{Spec}\, R_{\mathfrak{m}} \to \mathrm{Spec}\, A_{\mathfrak{m}}$. Note that $\mathrm{Spec}\, A_{\mathfrak{m}}$ consists of a finite number of points, and is a single point if $H_2(X,\mathbb{Z})_{\mathrm{tors}} = 0$.

REMARK 2.6. The construction of the multiplication law given here can be viewed as a generalization of the Frobenius structure conjecture given in §0.4 of the first arXiv version of [**24**]. In particular, the coefficient of ϑ_0 in $\vartheta_p \cdot \vartheta_q$ is precisely as described in §0.4.

2.2. The log Calabi-Yau case. Consider a simple normal crossings pair (X, D) with $U = X \setminus D$. We say (X, D) is *log Calabi-Yau* if for all $m > 0$ the space $H^0(X, \omega_X(D)^{\otimes m}) \subseteq H^0(U, \omega_U^{\otimes m})$ is one-dimensional, generated by $\Omega^{\otimes m}$ for Ω a nowhere-vanishing form $\Omega \in H^0(U, \omega_U)$. A result of Iitaka [**35**] yields that this subspace is independent of the compactification of U, so this is really an intrinsic property of U. In particular, $K_X + D$ is effective and K_X is supported on D. Thus we can write $K_X = \sum (a_i - 1) D_i$ with $a_i \geq 0$.

We have $\mathrm{Trop}(X) \subseteq \mathrm{Div}_D(X)^*_{\mathbb{R}}$ as before. We define $\mathrm{Div}'_D(X)^*_{\mathbb{R}}$ to be the subspace of $\mathrm{Div}_D(X)^*_{\mathbb{R}}$ spanned by those D_i^* with $a_i = 0$. Set

$$B := \mathrm{Trop}(X) \cap \mathrm{Div}'_D(X)^*_{\mathbb{R}}.$$

So if $K_X + D = 0$, then $B = \mathrm{Trop}(X)$.

Definition 2.7. We say (X, D) is a *maximal log Calabi-Yau pair* if B is pure-dimensional of dimension $\dim_{\mathbb{R}} B = \dim X$.

For the remainder of this subsection we assume that (X, D) is a maximal log Calabi-Yau pair.

We thus obtain an A_I-module R_I given by (2.1), with a not necessarily associative algebra structure given by (2.2). Note that unless (X, D) is a minimal model, i.e., $K_X + D = 0$, associativity does not follow from Theorem 2.2. We define, however, a sub-A_I-module $S_I \subseteq R_I$ defined by

$$S_I = \bigoplus_{p \in B(\mathbb{Z})} A_I \vartheta_p.$$

Proposition 2.8. *S_I is closed under the non-associative algebra structure on R_I, turning S_I into an associative A_I-algebra.*

Sketch of proof. We need to show (1) S_I is closed under the multiplication law

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in \mathrm{Trop}(X)(\mathbb{Z})} \alpha_{pqr} \vartheta_r;$$

(2) this multiplication law is associative.

(1) is straightforward. Fixing

$$p, q \in B(\mathbb{Z}) \subset \mathrm{Trop}(X)(\mathbb{Z}),$$

consider $r \in \mathrm{Trop}(X)(\mathbb{Z})$. We want to show $\alpha_{pqr} = 0$ if $r \notin B(\mathbb{Z})$. In order for a curve class $\underline{\beta}$ to contribute to N^{β}_{pqr}, we need $-\underline{\beta} \cdot (K_X + D) = 0$ by Lemma 2.1. In fact we claim that $\underline{\beta} \cdot (K_X + D) < 0$. Indeed, $K_X + D = \sum_i a_i D_i$, so it is sufficient to show that $N^{\beta}_{pqr} \neq 0$ implies $\underline{\beta} \cdot D_i \leq 0$ for all i with inequality for at least one i with $a_i > 0$. By Proposition 2.4, if $\mathcal{M}_\beta(X) \neq \emptyset$ then $\underline{\beta} \cdot D_i = \langle p, D_i \rangle + \langle q, D_i \rangle - \langle r, D_i \rangle$ for all i. Moreover, if $a_i > 0$, then $\langle p, D_i \rangle = \langle q, D_i \rangle = 0$ by assumption that $p, q \in B$, so $\underline{\beta} \cdot D_i = -\langle r, D_i \rangle$. Since $\langle r, D_i \rangle > 0$ for at least one i with $a_i > 0$, as otherwise $r \in B$, we get the claim.

For (2), one follows the same argument of associativity as given in Theorem 2.2. One again needs to check that there is no contribution to the logarithmic virtual degree of $\psi : \mathscr{M}_{\beta,z}(X) \to \overline{\mathcal{M}}_{0,4}$ coming from a decomposition $\underline{\beta} = \underline{\beta_1} + \underline{\beta_2}$ with $\underline{\beta_i} \cdot (K_X + D) \neq 0$. A similar argument using Proposition 2.4 as given above can be used to show that if $\underline{\beta_2} \cdot (K_X + D) \neq 0$, then in fact $-\underline{\beta_2} \cdot (K_X + D) < 0$ and thus the moduli space defining $N^{\beta_2}_{sp_3 r}$ is of negative virtual dimension. Then such a splitting cannot contribute to the logarithmic virtual degree. $\square$

CONSTRUCTION 2.9. The mirror family to the log Calabi-Yau (X,D) is the formal scheme $\check{X} := \operatorname{Spf}\widehat{S} \to \operatorname{Spf}\widehat{\Bbbk[P]}$, where $\widehat{\Bbbk[P]}$ is the completion of $\Bbbk[P]$ at the monomial ideal $\mathfrak{m}$, and $\widehat{S} = \varprojlim S_I$, where the limit is over all monomial ideals I with $\sqrt{I} = \mathfrak{m}$.[4]

Note that as in Example 2.5, $\check{X}$ is flat over $\operatorname{Spf}\widehat{\Bbbk[P]}$, and the condition that (X,D) is a maximal pair implies that the relative dimension of $\check{X}$ over $\operatorname{Spf}\widehat{\Bbbk[P]}$ coincides with the dimension of X.

EXAMPLE 2.10. Let $\bar{X} = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\bar{D} \subseteq \bar{X}$ be its toric boundary, so that $\bar{D} = \bar{D}_1 + \cdots + \bar{D}_4$, in some chosen cyclic ordering. Choose a point $p \in \bar{D}_1^\circ$, blow up this point to obtain $\pi : X \to \bar{X}$, and let D be the proper transform of $\bar{D}$. Then (X,D) is a log Calabi-Yau pair with $K_X + D = 0$. Noting that $H_2(X,\mathbb{Z})$ is generated by the classes of $C_1 = D_1, C_2 = D_2, C_3 = E$ where E is the exceptional curve of the blowup, and these classes also generate the cone of effective curves, we can take $P = \bigoplus_{i=1}^3 \mathbb{N}C_i$.

B, as an abstract cone complex, can be identified with $\operatorname{Trop}(\bar{X})$, but this identification is only piecewise linear. Let $p_i = D_i^* \in B(\mathbb{Z})$. The contributions to $\vartheta_{p_i} \cdot \vartheta_{p_{i+1}}$ (the index i taken modulo 4) only come from constant maps, so that, as in Example 2.5, we have the monomial relations

$$\vartheta_{p_i} \cdot \vartheta_{p_{i+1}} = \vartheta_{p_i + p_{i+1}}.$$

On the other hand, consider $\vartheta_{p_1} \cdot \vartheta_{p_3}$. Any curve class $\underline{\beta}$ contributing to the coefficient of ϑ_r in this product must satisfy, with $r = \sum_{i=1}^4 r_i D_i^*$,

$$\underline{\beta} \cdot D_i = \langle p_1, D_i\rangle + \langle p_3, D_i\rangle - \sum_j \langle r_j D_j^*, D_i\rangle = \begin{cases} 1 - r_i & i = 1,3 \\ -r_i & i = 2,4 \end{cases}$$

by Proposition 2.4. If $\underline{\beta} = \sum_{i=1}^3 \beta_i C_i$ with $\beta_i \geq 0$, then we obtain from the above constraints that

$$\begin{aligned} -\beta_1 + \beta_2 + \beta_3 &= 1 - r_1 \\ \beta_1 &= -r_2 \\ \beta_2 &= 1 - r_3 \\ \beta_1 &= -r_4 \end{aligned}$$

Thus in particular $r_2 = r_4$, and since necessarily $r_i = r_{i+1} = 0$ for some i (indices taken modulo 4), we must have $r_2 = r_4 = 0$ and at most one of r_1, r_3 non-zero. In particular, $\beta_1 = 0$. By non-negativity of the β_i, if $r_3 \neq 0$, then $r_3 = 1$ and $\beta_2 = 0$, $\beta_3 = 1$, so $\underline{\beta} = E$. But no curve of class E intersects D_3, so this possibility does not occur. If $r_1 \neq 0$, then non-negativity of the β_i rules out a solution. Thus the only choice of $\underline{\beta}$ satisfying the above constraints is $r = 0$, $\underline{\beta} = C_2$. Since $Z_r = X$, we fix

[4] In some cases, notably if D supports an ample divisor, $\hat{S}$ contains a natural subring S' finitely generated as a $\widehat{\Bbbk[P]}$-algebra and $\check{X}$ is the completion at the closed fibre of $\operatorname{Spec} S' \to \operatorname{Spec}\widehat{\Bbbk[P]}$. In general, this is not the case and the mirror presently can only be constructed as a formal scheme.

a general point $z \in X$, and there is a unique line in the class C_2 passing through z and intersecting D_1 and D_3 transversally. Thus we obtain

$$\vartheta_{p_1} \cdot \vartheta_{p_3} = t^{C_2}\vartheta_0.$$

Similarly, consider $\vartheta_{p_2} \cdot \vartheta_{p_4}$. Any curve class $\underline{\beta}$ contributing to the coefficient of ϑ_r in this product must satisfy

$$\underline{\beta} \cdot D_i = \begin{cases} -r_i & i = 1, 3 \\ 1 - r_i & i = 2, 4. \end{cases}$$

A similar analysis shows the only possible classes are $\underline{\beta} = C_1 + C_3 \sim D_3$ or $\underline{\beta} = C_1$. The first has $r = 0$ and the second $r = p_1$. In the first case, after choosing $z \in X$ general, the only curve in $\mathscr{M}_{\beta,z}(X)$ is a line of class D_3 passing through z, transversal to D_2 and D_4. In the second case, one chooses $z \in Z_r^\circ = D_1^\circ$, and the only curve in $\mathscr{M}_{\beta,z}(X)$ is the curve D_1 itself, with the points x_1 and x_2 mapping to D_2 and D_4 respectively, and x_3 a point of tangency order -1 with D_1, much as in Example 1.5. Thus we obtain

$$\vartheta_{p_2} \cdot \vartheta_{p_4} = t^{C_1+C_3}\vartheta_0 + t^{C_1}\vartheta_{p_1}.$$

In particular, with ϑ_0 the unit in the ring, we have

$$\check{X} := \operatorname{Spf} \widehat{\Bbbk[P]}[\vartheta_{p_1}, \ldots, \vartheta_{p_4}]/(\vartheta_{p_1}\vartheta_{p_3} - t^{C_2}, \vartheta_{p_2}\vartheta_{p_4} - t^{C_1+C_3} - t^{C_1}\vartheta_{p_1}).$$

This coincides with the mirror of the pair (X, D) defined in [**24**]. Note that as in [**24**], Corollary 6.11, this mirror is defined over $\operatorname{Spec} \Bbbk[P]$, which, in the surface case, is the case whenever D supports an ample divisor.

2.3. The Calabi-Yau case. Consider now the situation that we are given a simple normal crossings degeneration $\mathcal{X} \to T$, where T is the spectrum of a discrete valuation ring and whose generic fibre $\mathcal{X}_\eta$ is a non-singular Calabi-Yau variety, i.e., $K_{\mathcal{X}_\eta} = 0$. Let $0 \in T$ be the closed point. We view $(\mathcal{X}, \mathcal{X}_0)$ as a log Calabi-Yau pair of dimension $\dim \mathcal{X}_\eta + 1$, and thus can apply the construction of the previous sub-section, with some minor alterations. For convenience here, we shall assume that $\mathcal{X}_0$ is reduced, and while this can always be achieved via stable reduction, in fact this is unnecessary, and only maximally unipotent monodromy is needed to get a sensible result out of the construction we give here.

Let $A_1(\mathcal{X}/T)$ denote the group of algebraic equivalence classes of complete curves in $\mathcal{X}$ contracted by the map to T. This group contains the cone of effective curve classes, and we choose a finitely generated monoid $P \subseteq A_1(\mathcal{X}/T)$ such that $P \cap (-P) \subseteq A_1(\mathcal{X}/T)_{\text{tors}}$ and P contains every effective curve class. As in §2.2, we obtain $\operatorname{Trop}(\mathcal{X}) \subseteq \operatorname{Div}_{\mathcal{X}_0}(\mathcal{X})^*_{\mathbb{R}}$ and a sub-complex which we shall write as $\mathbf{C}B$ rather than as B, and then define

$$B := \{p \in \mathbf{C}B \,|\, \langle \mathcal{X}_0, p \rangle = 1\}.$$

Here we interpret $\mathcal{X}_0 \in \operatorname{Div}_{\mathcal{X}_0}(\mathcal{X})$. We assume that $\dim_{\mathbb{R}} B = \dim \mathcal{X}_\eta$. This is equivalent to maximal unipotency of the degeneration $\mathcal{X} \to T$.

We note here that B in fact coincides with the Kontsevich-Soibelman skeleton of $\mathcal{X} \to T$, a canonically defined topological subspace of the Berkovich analytic space of $\mathcal{X}_\eta$, introduced in [**46**] and studied in more detail in [**51**]. The latter reference in particular shows that B is a closed pseudo-manifold.

Note that the divisor $\mathcal{X}_0$ defines an $\mathbb{N}$-grading on $\mathbf{C}B(\mathbb{Z})$, with $\deg p = \langle \mathcal{X}_0, p\rangle$. In particular, with $A_I = \Bbbk[P]/I$ as before for a choice of monomial ideal I with $\sqrt{I} = \mathfrak{m}$, the A_I-module

$$S_I = \bigoplus_{p\in \mathbf{C}B(\mathbb{Z})} A_I \vartheta_p$$

is $\mathbb{N}$-graded. The multiplication rule on S_I is then defined using (2.2). The structure coefficients only count punctured curves mapping to the central fibre $\mathcal{X}_0$, so only depend on $\mathcal{X}_0$ as a log scheme, rather than on more refined information carried by $\mathcal{X}$. Noting that $\langle \mathcal{X}_0, \underline{\beta}\rangle = 0$ for any class $\underline{\beta} \in P$, it follows from Proposition 2.4 that if $N^{\beta}_{pqr} \neq 0$, then $\langle \mathcal{X}_0, p\rangle + \langle \mathcal{X}_0, q\rangle = \langle \mathcal{X}_0, r\rangle$, i.e., $\deg p + \deg q = \deg r$. Thus the multiplication law respects the grading and S_I is a graded ring. This gives a finite order deformation $\check{X}_I := \operatorname{Proj} S_I \to \operatorname{Spec} \Bbbk[P]/I$.

At this point one can take the limit in two different ways. First, one can take the limit of the $\check{X}_I$ over all I and obtain a formal scheme $\check{\mathfrak{X}}$ projective over $\operatorname{Spf} \widehat{\Bbbk[P]}$. Grothendieck existence then yields a projective family $\check{\mathcal{X}} \to \operatorname{Spec} \widehat{\Bbbk[P]}$. More directly, unlike in the general affine case, one can define

$$\widehat{S} := \bigoplus_{p\in \mathbf{C}B(\mathbb{Z})} \widehat{\Bbbk[P]} \vartheta_p,$$

and use the same structure constants α_{pqr} for defining the product $\vartheta_p \cdot \vartheta_q$. Since B is compact, the degree d part of $\widehat{S}$ is a finitely generated free module over $\widehat{\Bbbk[P]}$, and in particular $\vartheta_p \cdot \vartheta_q$ is a sum over only a finite number of ϑ_r (with formal power series coefficients). Then we have $\check{\mathcal{X}} = \operatorname{Proj} \widehat{S}$.

Note that points of $\mathbf{C}B(\mathbb{Z})$ of degree d are canonically identified, by dividing by d, with the points of $B(\frac{1}{d}\mathbb{Z})$, giving the indexing of sections of the line bundle $\mathcal{O}_{\operatorname{Proj} \widehat{S}}(d)$ on $\operatorname{Proj} \widehat{S}$ mentioned in the introduction.

Construction 2.11. The mirror family to the degeneration $\mathcal{X} \to T$ of Calabi-Yau varieties is the flat family $\check{\mathcal{X}} := \operatorname{Proj} \widehat{S} \to \operatorname{Spec} \widehat{\Bbbk[P]}$.

There are of course a host of questions associated with such a construction, the most immediate being:

Question 2.12. *Show that the above construction coincides with previously known constructions.*

See Remark 2.15 for a bit of discussion on this.

2.4. Scattering diagrams and broken lines. In this subsection we will give a somewhat rougher outline explaining the more detailed general picture suggested in the introduction. To avoid some complexities, we will make some simplifying assumptions and work with a maximal log Calabi-Yau pair (X, D) with $K_X + D = 0$, i.e., a *minimal model* of a log Calabi-Yau variety. We continue to assume that D is simple normal crossings; this is not a sufficient degree of generality, as one would expect log Calabi-Yau pairs to have dlt minimal models. Somewhat more generally, one may assume that (X, D) is log smooth, so that the pair has toroidal singularities, but the class of toroidal singularities are orthogonal to the class of dlt singularities. Indeed, if (X, D) has dlt singularities, the divisor D is generically normal crossings on each stratum of D, whereas this is not true for toroidal singularities.

In [**25**], Construction 1.1, we define the notion of a *polyhedral affine manifold*. In the case at hand, this will be the pair $(B, \mathscr{P})$. Such a polyhedral (in this case cone) complex is asked to satisfy five properties: (1) Each $\tau \in \mathscr{P}$ injects into B: this of course comes from the simple normal crossings assumption, and is manifest in the description of $B \subseteq \mathrm{Div}_D(X)^*_{\mathbb{R}}$. (2) The intersection of two cones in $\mathscr{P}$ is a cone in $\mathscr{P}$: this is again manifest from the description of B, and follows from the assumption that $\bigcap_{i\in I} D_i$ is connected when non-empty. (3) B is pure dimension n, where $n = \dim X$. Indeed, maximality implies $\dim B = n$. Further, by [**45**], Theorem 2, (i), B is pure dimension. (4) Every codimension one cone of $\mathscr{P}$ is contained in precisely two top-dimensional cones. We see this as follows. If Z is any stratum of X, we denote by D_Z the union of all closed substrata of X contained properly in Z. By repeated use of adjunction, (Z, D_Z) is a log Calabi-Yau pair. The fact that B has the same dimension at every point implies (Z, D_Z) is also maximal for any stratum Z. Now if $\dim Z = 1$, then this implies $Z \cong \mathbb{P}^1$ and D_Z consists of two points. (5) *The S_2 condition.* We do not repeat this condition here, but it follows from the observation that if Z is a stratum with $\dim Z \geq 2$, then D_Z is connected, see [**45**], §2, or [**44**], 4.37. This condition guarantees that the constructed zeroth order mirror as described in Example 2.5 satisfies Serre's S_2 condition.

Thus $(B, \mathscr{P})$ satisfies all five hypotheses of that Construction. We can now give $(B, \mathscr{P})$ the structure of a polyhedral affine manifold in the sense of [**25**], Construction 1.1, generalizing the one given in [**24**]. Indeed, let $\Delta \subseteq B$ be the union of codimension ≥ 2 cones of B. We shall describe an integral affine structure on $B_0 := B \setminus \Delta$. Each cone τ in B carries a canonical integral affine structure. Denote by Λ_τ the lattice $\mathbb{R}\tau \cap \mathrm{Div}_D(X)^*$ of integral tangent vectors to τ. To construct an affine structure on B_0 compatible with the affine structures on codimension zero and one cones, it suffices, as in [**25**], §1, to give for every codimension one cone ρ contained in two maximal cones σ_1, σ_2 an identification of Λ_{σ_1} with Λ_{σ_2} preserving $\Lambda_\rho \subseteq \Lambda_{\sigma_i}$. To do this, if $u_1 \in \Lambda_{\sigma_1}$ is such that $\Lambda_\rho + \mathbb{Z}u_1 = \Lambda_{\sigma_1}$, we need to provide $u_2 \in \Lambda_{\sigma_2}$ such that $\Lambda_\rho + \mathbb{Z}u_2 = \Lambda_{\sigma_2}$. We then identify Λ_{σ_1} with Λ_{σ_2} by taking u_1 to $\pm u_2$ with the sign adjusting for the local orientation.

Let $Z_\rho \cong \mathbb{P}^1$ be the stratum of X corresponding to ρ. Suppose ρ is generated by $D^*_{i_2}, \ldots, D^*_{i_n}$, $\rho \subseteq \sigma_1, \sigma_2$ with additional generators of σ_1 and σ_2 being $D^*_{i_1}$ and $D^*_{i'_1}$ respectively. We can take $u_1 = D^*_{i_1}$, in which case we take

$$u_2 = -D^*_{i'_1} - \sum_{j=2}^{n} (D_{i_j} \cdot Z_\rho) D^*_{i_j}.$$

This formula can be viewed in terms of punctured invariants: given any suitable choice of u_1, there is a unique choice of u_2 such that one can construct a punctured log map with underlying domain $C = Z_\rho$, with two punctures p_1, p_2 mapping to Z_{σ_1}, Z_{σ_2} respectively, and with $u_{p_1} = u_1$, $u_{p_2} = -u_2$. Considerations similar to those of Proposition 2.4 yield the above formula from this.

This completes the description of $(B, \mathscr{P})$ as a polyhedral affine manifold in the sense of [**25**], Construction 1.1.

We next turn to the construction of a wall structure on B (sometimes referred to as a scattering diagram, e.g., in [**24**]). The definition of a wall structure on a polyhedral affine manifold, generalizing the original source [**29**], is given in [**25**], Definition 2.11. We give an abbreviated version of the definition of a (conical) wall structure here, leaving off some conditions which are unnecessary for the discussion at hand. For full details, we refer the reader to [**25**]. In the discussion that follows, fix a monomial ideal $I \subseteq P$ with radical $\mathfrak{m}$.

Definition 2.13. 1) A *wall* on B is a codimension one rational polyhedral cone $\mathfrak{p}$ contained in some maximal cone σ of B, along with an element

$$f_{\mathfrak{p}} = \sum_{m \in \Lambda_{\mathfrak{p}},\, \underline{\beta} \in P} c_{m,\underline{\beta}} t^{\underline{\beta}} z^m \in A_I \otimes_{\Bbbk} \Bbbk[\Lambda_{\mathfrak{p}}].$$

Here $\Lambda_{\mathfrak{p}}$ is the lattice of integral tangent vectors to $\mathfrak{p}$.

2) A *structure* $\mathscr{S}$ is a finite set of walls.

We now explain how to construct the *canonical structure* on B using the pair (X, D). This generalizes the canonical scattering diagram of [**24**], Definition 3.3. Fix $\tau \in \mathscr{P}$, a class $\underline{\beta} \in P \setminus I$, and a vector $u_p \in \Lambda_\tau$. Writing $u_p = \sum_i a_i D_i^*$, assume that $a_i \neq 0$ whenever $D_i^* \in \tau$, so that u_p is not tangent to any proper face of τ. Then u_p determines maximal contact data consisting of the pair Z_τ and $u_p \in \Gamma(Z_\tau, (\overline{\mathcal{M}}_X|_{Z_\tau})^*)$, defined using u_p instead of p as in (2.3). In particular, $\underline{\beta}$ along with this maximal contact data at the one punctured point determines a type β of punctured curve, yielding a moduli space $W := \mathscr{M}_\beta(X)$, with universal family of punctured maps $(\pi : C \to W, p, f)$. The virtual dimension of W over the stack $\mathcal{T}_W$, the substack of the Artin fan $\mathcal{A}_W$ described at the end of §1.4, is $n-2$, where $n = \dim X$. Let C^{fs} be the auxilliary fine saturated log structure on C, as described in Remark 1.6, and let W^{fs} be the pull-back of this log structure to W via the section p. We then have a diagram

$$\begin{tikzcd} W^{\mathrm{fs}} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ W \end{tikzcd}$$

yielding a tropicalized diagram

$$\begin{tikzcd} \mathrm{Trop}(W^{\mathrm{fs}}) \arrow[r, "\mathrm{Trop}(f)"] \arrow[d, "\mathrm{Trop}(\pi)"'] & \mathrm{Trop}(X) = B \\ \mathrm{Trop}(W) \end{tikzcd}$$

The fibres of $\mathrm{Trop}(\pi)$ are either line segments or rays.

We might expect $\mathrm{Trop}(W)$ to be $(n-2)$-dimensional, as W is of virtual dimension $n-2$ over $\mathcal{T}_W$, but this would not be the case in general. However, the $(n-2)$-dimensional skeleton of $\mathrm{Trop}(W)$ can be viewed as a "virtual" complex of the correct dimension. Explicitly, let $\sigma \in \mathrm{Trop}(W)$ be an $(n-2)$-dimensional cone with the property that $\mathfrak{p}_\sigma := \mathrm{Trop}(f)(\mathrm{Trop}(\pi)^{-1}(\sigma))$ is an $(n-1)$-dimensional cone. We will associate a number N_σ to this data as follows, which can be thought of as the pull-back of the virtual fundamental class on W to the (virtually) codimension $n-2$ stratum of W indexed by σ. Recall the discussion of the obstruction theory for punctured curves in §1.4. The cone σ yields an étale morphism $\mathcal{A}_\sigma := \big[\operatorname{Spec} \Bbbk[\sigma^\vee \cap \Lambda_\sigma^*] / \operatorname{Spec} \Bbbk[\Lambda_\sigma^*]\big] \to \mathcal{A}_W$, where Λ_σ is the group of integral tangent vectors to σ. Let $y \in \mathcal{A}_\sigma$ be the closed point, with stabilizer $\Lambda_\sigma \otimes \mathbb{G}_m$. Then one can show the image of y lies in $\mathcal{T}_W$. Denote by $\mathrm{cl}(y)$ the closure of y in

$\mathcal{T}_W$ with the reduced induced stack structure. Then as a stack $\mathrm{cl}(y)$ has dimension $-n+2$. We obtain a Cartesian diagram

$$\begin{tikzcd} W_{\mathrm{cl}(y)} \arrow[r] \arrow[d] & W \arrow[d] \\ \mathrm{cl}(y) \arrow[r] & \mathcal{T}_W \end{tikzcd}$$

Note that possibly unlike $\mathcal{T}_W$, $\mathrm{cl}(y)$ is pure-dimensional. Pulling back the fundamental class of the stack $W = \mathscr{M}_\beta(X)$ to $W_{\mathrm{cl}(y)}$ gives a virtual fundamental class $[W_{\mathrm{cl}(y)}]^{\mathrm{virt}}$. Set

$$N_\sigma := \int_{[W_{\mathrm{cl}(y)}]^{\mathrm{virt}}} 1.$$

Using this, we can define a wall $(\mathfrak{p}_\sigma, f_{\mathfrak{p}_\sigma})$ with

$$f_{\mathfrak{p}_\sigma} = \exp\left(k_\sigma N_\sigma t^{\underline{\beta}} z^{-u_p}\right),$$

where k_σ is the index of the image of the lattice of integral tangent vectors to $\mathrm{Trop}(\pi)^{-1}(\sigma)$ in $\Lambda_{\mathfrak{p}}$. We then define $\mathscr{S}$ to be the collection of all such walls, running over all choices of $u_p \neq 0$ and all choices of $\underline{\beta} \in P \setminus I$, and all choices of cones σ as described above.

This gives the generalization of the canonical scattering diagram defined in [**24**].

[**25**] develops the theory of broken lines in §3.1, and then defines the notion of *consistency* in §3.2. Roughly put, given a structure, one can use it to glue together some standard open charts (see [**25**], §2.4) to obtain a family $\check{\mathfrak{X}}^\circ$ over $\mathrm{Spec}\, A_I$ in any event. Sums over broken lines define regular functions on these charts, and the consistency of the structure is equivalent to these functions being compatible under the gluing maps, yielding global functions on $\check{\mathfrak{X}}^\circ$, the theta functions. One then defines $\check{\mathfrak{X}} = \mathrm{Spec}\,\Gamma(\check{\mathfrak{X}}^\circ, \mathcal{O}_{\check{\mathfrak{X}}^\circ})$. The fact that $\check{\mathfrak{X}}$ is flat over $\mathrm{Spec}\, A_I$ then follows from the existence of theta functions, see [**25**], Proposition 3.21.

THEOREM 2.14. *The canonical structure $\mathscr{S}$ described above is consistent.*

The proof of this theorem, to be given in future work, requires a punctured interpretation of broken lines. We only summarize this here as we do not wish to review the definition of broken line. This interpretation can be accomplished in a similar way to the description of the multiplication law: one needs to consider curves which have a marked point with an ordinary tangency condition specified by $p \in B(\mathbb{Z})$, and a punctured point specifying the final direction of the broken line. An additional point is required to fix the endpoint of the broken line. The details will appear elsewhere. We merely remark here that Tony Yu has provided an interpretation for broken lines for log Calabi-Yau surfaces in [**59**].

REMARK 2.15. Here we sketch a possible approach to Question 2.12. An obvious approach to comparing the present mirror construction with the construction of [**29**] (which is known to agree with the Batyrev-Borisov construction when applied to a natural choice of toric degeneration of complete intersection Calabi-Yau varieties in a toric variety, see [**20**]) runs as follows. By the strong uniqueness properties of the inductive construction of the wall structure in [**29**] it is enough to (a) derive the initial wall structure by a local computation of punctured invariants near the log singular locus and (b) show that the inductive insertion of walls leads to consistency in codimension two. This consistency should follow from showing independence of the variation of tropical end points of the counting of punctured curves around a codimension two locus, set up similarly to the interpretation of broken lines.

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DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK

E-mail address: mgross@dpmms.cam.ac.uk

FB Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany

E-mail address: siebert@math.uni-hamburg.de