Local mirror symmetry in the tropics

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Abstract. We discuss how the reconstruction theorem of [GrSi3] applies to local mirror symmetry [CKYZ]. This theorem associates to certain combinatorial data a degeneration of (log) Calabi-Yau varieties. While in this case most of the subtleties of the construction are absent, an important normalization condition already introduces rich geometry. This condition guarantees the parameters of the construction are canonical coordinates in the sense of mirror symmetry. The normalization condition is also related to a count of holomorphic disks and cylinders, as conjectured in [GrSi3] and partially proved in [CLL],[CLT],[CCLT]. We sketch a possible alternative proof of these counts via logarithmic Gromov-Witten theory.

There is also a surprisingly simple interpretation via rooted trees marked by monomials, which points to an underlying rich algebraic structure both in the relevant period integrals and the counting of holomorphic disks.

1. Introduction

In [GrSi1], [GrSi3], we proposed a mirror construction as follows. We begin with a polarized degenerating flat family $\mathcal{X} \to T = \operatorname{Spec} R$ of n-dimensional Calabi-Yau varieties where R is a complete local ring. We consider only degenerations of a special sort which we term *toric degenerations*, see [GrSi1], Def. 4.1. Roughly, these are degenerations for which the central fibre is a union of toric varieties glued along toric strata, and such that the map $\mathcal{X} \to T$ is locally given by a monomial near the zero-dimensional strata of the central fibre X_0. Associated to this degeneration we construct the *dual intersection complex* $(B, \mathscr{P}, \varphi)$, where

(a) B is an n-dimensional integral affine manifold with singularities (possibly with boundary). In other words, B is a topological manifold with an open subset B_0 with $\Delta := B \setminus B_0$ of codimension ≥ 2, such that B_0 has an atlas of coordinate charts whose transition maps lie in $\mathrm{Aff}(\mathbb{Z}^n)$, the group of integral affine transformations.

(b) $\mathscr{P}$ is a decomposition of B into convex lattice polyhedra (possibly unbounded). The singular locus Δ is typically the union of codimension two cells of the first barycentric subdivision of $\mathscr{P}$ not intersecting the interior of a maximal

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cell of $\mathscr{P}$ nor containing a vertex of $\mathscr{P}$. There is a one-to-one inclusion reversing correspondence between elements of $\mathscr{P}$ and toric strata of X_0. The local structure of $\mathscr{P}$ near a vertex is determined by the fan defining the corresponding irreducible component. The maximal cells of $\mathscr{P}$ are determined by the toric structure of the map $\mathcal{X} \to T$ near the corresponding zero-dimensional strata of X_0.

(c) φ is a *multi-valued piecewise affine function*. This is a collection $\{(U_i, \varphi_i)\}$ of $\mathbb{R}$-valued functions φ_i on an open cover $\{U_i\}$ of B, with each φ_i piecewise affine linear with respect to the polyhedral decomposition $\mathscr{P}$, and $\varphi_i - \varphi_j$ being affine linear on $U_i \cap U_j$. We assume the slopes of the φ_i on cells of $\mathscr{P}$ to be integral. In this case, φ is determined by the polarization on $\mathcal{X}$, with local representatives near vertices given by a piecewise linear function defined by restricting the polarization to the corresponding irreducible component.

Given this data, we obtain the mirror to the degeneration $\mathcal{X} \to T$ by reinterpreting $(B, \mathscr{P}, \varphi)$ as the *intersection complex* of another polarized toric degeneration $\mathcal{Y} \to \operatorname{Spec} \Bbbk[\![t]\!]$ (in the projective case). This time, there is a one-to-one inclusion preserving correspondence between cells of $\mathscr{P}$ and toric strata of X_0, the central fibre of this new degeneration. The cells of $\mathscr{P}$ are the Newton polytopes for the polarization restricted to the various strata of X_0, and φ is determined by the local toric structure of the map near zero-dimensional strata.

The prime difficulty in the program lies in reconstructing $\mathcal{Y} \to \operatorname{Spec} \Bbbk[\![t]\!]$ from the data $(B, \mathscr{P}, \varphi)$. The main result of [GrSi3] gives an algorithm for constructing a *structure* $\mathscr{S}$ of walls which tell us how to construct the degeneration.

More recently [GHKS] has considered families constructed using the technology of [GrSi3] over higher dimensional base schemes. This represents a modification of the above procedure. In the typical example, instead of choosing a fixed polarization on $\mathcal{Y}$, one chooses a monoid P of polarizations. Let $Q = \operatorname{Hom}(P, \mathbb{N})$ be the dual monoid. Then this data determines a multi-valued piecewise linear function φ taking values in $Q^{\mathrm{gp}}_{\mathbb{R}} := Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. If $\mathfrak{m}$ is the maximal monomial ideal of $\Bbbk[Q]$, and $\widehat{\Bbbk[Q]}$ denotes the completion of $\Bbbk[Q]$ with respect to this ideal, then the construction gives a family $\mathcal{Y} \to \operatorname{Spec} \widehat{\Bbbk[Q]}$.

The history of the problem of associating a geometric object (complex manifold, non-Archimedean space, toric degeneration...) to an integral affine manifold with singularities began with work of Fukaya [Fuk]. Fukaya gave a heuristic suggesting that one should be able to construct the mirror to a K3 surface using objects that look like structures in two dimensions (in two dimensions, we can think of a structure as just consisting of a possibly infinite number of unbounded rays). Fukaya observed that holomorphic disks with boundary on fibres of an SYZ fibration ([SYZ]) gave similar pictures of structures on the mirror side. In

2004, Kontsevich and Soibelman in [KS] gave the first construction of a structure, showing how given a two-dimensional affine sphere with singularities one could construct a consistent structure and from this structure a non-Archimedean K3 surface. We combined the picture of toric degenerations we had been developing independently of the above-mentioned authors with some ideas from [KS], allowing us to construct degenerations from structures in all dimensions in [GrSi3].

In the first two sections of this paper, we shall illustrate the program by carrying it out completely for toric Calabi-Yau manifolds, a case usually referred to as local mirror symmetry [CKYZ]. This particular case can be viewed as being complementary to the case that the ideas of [KS] was able to handle. In the remaining sections, we shall analyze enumerative meaning and a tropical interpretation of this construction.

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2. Degenerations of toric Calabi-Yau varieties

Our running example is the construction of the mirror of what is called "local $\mathbb{P}^2$", the total space X of the canonical bundle $K_{\mathbb{P}^2}$ over $\mathbb{P}^2$. Since X itself is a toric variety, its anti-canonical divisor $-K_X$ is linearly equivalent to the sum of toric divisors. There are four toric divisors, the zero section $S \subset X$, which is the maximal compact subvariety of X, and the preimages F_0, F_1, F_2 of the three coordinate lines in $\mathbb{P}^2$ under the bundle projection $X \to \mathbb{P}^2$. Toric methods show that $S + F_0 + F_1 + F_2 \sim 0$ and hence X is a non-compact Calabi-Yau threefold. The normal bundle $N_{S|X} = \mathcal{O}_{\mathbb{P}^2}(-3)$ is determined by the adjunction formula from the Calabi-Yau condition and it is the dual of an ample line bundle. Hence, by a result of Grauert [Gt], any embedded $\mathbb{P}^2$ in a Calabi-Yau threefold has an analytic neighbourhood biholomorphic to an analytic neighbourhood of S in X.

For the general description, fix throughout $M = \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.[1] Let $\sigma \subseteq M_{\mathbb{R}}$ be a compact lattice polytope, and assume $0 \in \sigma$. Define

$$C(\sigma) = \{(rm, r) \,|\, m \in \sigma, r \in \mathbb{R}_{\geq 0}\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

The cone $C(\sigma)$ viewed as a fan defines an affine toric variety X_σ. A polyhedral decomposition $\overline{\mathscr{P}}$ of σ into standard simplices leads to a fan $\Sigma = \{C(\tau) \,|\, \tau \in \overline{\mathscr{P}}\}$

[1] Since M, N will eventually be treated as data for the mirror side our conventions in this section are opposite to the usual ones in toric geometry.

which is a refinement of $C(\sigma)$. This yields a toric resolution of singularities $X_\Sigma \to X_\sigma$. Assume also that the fan Σ supports at least one strictly convex piecewise linear function.

For the case of local $\mathbb{P}^2$ take $n=2$ and $\sigma = \mathrm{Conv}\{(1,0),(0,1),(-1,-1)\}$, where $\mathrm{Conv}(S)$ denotes the convex hull of the set S. Then the dual cone $C(\sigma)^\vee$ is $C(\sigma^*)$, the cone over the polar polytope σ^* with vertices $(-1,-1)$, $(2,-1)$, $(-1,2)$. It turns out that $X_\sigma = \mathrm{Spec}(\mathbb{C}[C(\sigma)^\vee \cap \mathbb{Z}^3])$ is the cyclic quotient $\mathbb{A}^2/\mathbb{Z}_3$ with $\mathbb{Z}_3$ acting diagonally on the coordinates by multiplication with third roots of unity. Taking the polyhedral decomposition as shown in Figure 2.1 yields for X_Σ the blowing up of the origin of X_σ. One can show that X_Σ is the total space of $K_{\mathbb{P}^2}$ and the map to X_σ is the contraction of the zero section. Note also that the projection $C(\sigma) \to M_\mathbb{R}$ defines a map from Σ to the fan of $\mathbb{P}^2$, which indeed corresponds to the bundle projection $X_\Sigma \to \mathbb{P}^2$.

In general, the map $X_\Sigma \to X_\sigma$ has a reducible exceptional locus, with one component for each vertex of $\overline{\mathscr{P}}$ that is not a vertex of σ, and the explicit description of the geometry is more complicated.

It turns out that constructing a mirror to X_Σ does not fit well with our program. The reason is that X_Σ does not seem to possess a fibration by Lagrangian tori of the kind expected by mirror symmetry [Gr1]. Rather, such a fibration will exist only after removal of a hypersurface in X_Σ that is disjoint from the exceptional fibre of $X_\Sigma \to X_\sigma$. To run our program we could give an ad hoc construction of an affine manifold with singularities derived from the fan Σ or write down a toric degeneration of X_Σ. The local $\mathbb{P}^2$ case has been discussed from the former point of view in [GrSi4], Examples 5.1 and 5.2. Since it can be done easily in the present case we follow the latter method here. This method is motivated by the construction of toric degenerations of hypersurfaces in toric varieties in [Gr2].

To exhibit X_Σ as an anticanonical hypersurface in a toric variety we embed the fan Σ in $M_\mathbb{R} \oplus \mathbb{R}$ as a subfan of a fan $\tilde{\Sigma}$ in $M_\mathbb{R} \oplus \mathbb{R}^2$. For each maximal cone $C \in \Sigma$ the fan $\tilde{\Sigma}$ has two maximal cones

$$C_1 = C \times 0 + \mathbb{R}_{\geq 0} \cdot (0,1,-1), \quad C_2 = C \times 0 + \mathbb{R}_{\geq 0} \cdot (0,0,1).$$

Then Σ is the subfan of $\tilde{\Sigma}$ consisting of cones lying in the hyperplane $M_\mathbb{R} \oplus \mathbb{R} \oplus 0 \subset M_\mathbb{R} \oplus \mathbb{R}^2$. The fan $\tilde{\Sigma}$ only has two rays not contained in Σ, with generators $(0,0,1)$ and $(0,1,-1)$. The inclusion $M_\mathbb{R} \oplus \mathbb{R} \oplus 0 \subset M_\mathbb{R} \oplus \mathbb{R}^2$ induces a map of fans from Σ to $\tilde{\Sigma}$, hence an embedding $j : X_\Sigma \hookrightarrow X_{\tilde{\Sigma}}$ identifying X_Σ with the closure of the orbit of the subtorus defined by this inclusion through the distinguished point (the unit of the toric variety). Note that the projection to $\mathbb{R}^2$ maps $\tilde{\Sigma}$ to the fan $\Sigma_{\widehat{\mathbb{A}}^2}$ of the toric blowing up $\widehat{\mathbb{A}}^2$ of $\mathbb{A}^2$, with rays generated by $(0,1), (1,0), (1,-1)$.

Under this map the subfan $\Sigma \subset \tilde{\Sigma}$ maps to the interior ray $\mathbb{R}_{\geq 0} \cdot (1,0)$. Viewing this interior ray as giving a map of fans, from the one-dimensional fan defining $\mathbb{A}^1$ to the two-dimensional fan defining $\widehat{\mathbb{A}}^2$, we obtain an embedding $i : \mathbb{A}^1 \hookrightarrow \widehat{\mathbb{A}}^2$.

We thus obtain a cartesian diagram of toric morphisms

$$\begin{tikzcd} X_\Sigma \arrow[r, "j"] \arrow[d, "p"'] & X_{\tilde{\Sigma}} \arrow[d, "q"] \\ \mathbb{A}^1 \arrow[r, "i"] & \widehat{\mathbb{A}}^2 \end{tikzcd}$$

The left vertical arrow is induced by the projection $M_{\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$, hence is given by the pull-back to X_Σ of the distinguished monomial x on X_σ defining the toric boundary as a reduced subscheme.

Explicitly, write x, y for the toric coordinates on $\mathbb{A}^2$ and $\widehat{\mathbb{A}}^2 = (xu - yv = 0) \subset \mathbb{A}^2 \times \mathbb{P}^1$ for the blowing up. Then $\mathrm{im}(i)$ is the strict transform of the diagonal $x = y$. Dehomogenizing $u = 1$ or $v = 1$ we obtain the usual two coordinate patches with coordinates y, v and x, u respectively with the transitions $v = u^{-1}$ and $x = yv$ or $y = xu$. We use the same notation for the pull-back of x, y, u, v to the corresponding two types of affine patches with $u \neq 0$ or $v \neq 0$ of $X_{\tilde{\Sigma}}$.

To describe $X_{\tilde{\Sigma}}$ let $C \in \Sigma$ be a maximal cone. Then if $(m, a) \in N \oplus \mathbb{Z}$ defines a facet $C' \subset C$, that is, (m, a) generates an extremal ray of $C^\vee$, the element $(m, a, a) \in N \oplus \mathbb{Z}^2$ defines the facet $C' + \mathbb{R}_{\geq 0}(0, 1, -1)$ of C_1. There is only one more facet of C_1, namely C itself, defined by $(0, 0, -1)$, and hence

$$C_1^\vee = \{(m, a, a) \,|\, (m, a) \in C^\vee\} + \mathbb{R}_{\geq 0} \cdot (0, 0, -1).$$

The rays of $C_2^\vee$ are generated by $(m, a, 0)$ for (m, a) an extremal ray of $C^\vee$, and by $(0, 0, 1)$, so $C_2^\vee = C^\vee \times 0 + \mathbb{R}_{\geq 0}(0, 0, 1)$. In either case, we have an identification

$$\operatorname{Spec} \Bbbk[C_i^\vee \cap (N \oplus \mathbb{Z}^2)] = \operatorname{Spec} \Bbbk[C^\vee \cap N] \times \mathbb{A}^1 \subset X_\Sigma \times \mathbb{A}^1.$$

The toric coordinate for $\mathbb{A}^1$ is $v = z^{(0,0,-1)}$ for C_1 and $u = z^{(0,0,1)}$ for C_2. From this description it is clear that the embedding of X_Σ in $X_{\tilde{\Sigma}}$ is given by $u = 1$ in affine patches with $v \neq 0$ and by $v = 1$ in the affine patches with $u \neq 0$.

To write down a degeneration of X_Σ to the toric boundary $\partial X_{\tilde{\Sigma}} \subset X_{\tilde{\Sigma}}$ view u, v as sections of the line bundle $q^*\mathcal{O}(-E)$ where $E \subset \widehat{\mathbb{A}}^2$ is the exceptional curve. Then X_Σ is the zero locus of $s := u - v$. On the other hand, $xu = yv$ defines a section s_0 of $q^*\mathcal{O}(-E)$ with zero locus $\partial X_{\tilde{\Sigma}}$. Thus the hypersurface $\mathcal{X} \subset X_{\tilde{\Sigma}} \times \mathbb{A}^1$ with equation

$$ts + s_0 = 0$$

defines a pencil in $X_{\tilde{\Sigma}}$ with members X_Σ at $t=\infty$ and with the toric boundary $\partial X_{\tilde{\Sigma}}$ at $t=0$. Note this pencil is the preimage of the pencil on $\widehat{\mathbb{A}}^2$ defined by the same equations. In particular, by direct computation $\mathcal{X}_t$ is completely contained in either type of coordinate patch for $t\neq 0$. Working in a patch with $v\neq 0$ we have $s=u-1$, $s_0=xu$ and the equation

$$0=ts+s_0=t(u-1)+xu=u(t+x)-t$$

shows $u(t+x)=t\neq 0$. Thus $t+x\neq 0$ and u can be eliminated. In other words, $\mathcal{X}_t\simeq X_\Sigma\setminus Z_t$ with $Z_t\subset X_\Sigma$ the hypersurface $x=-t$. Note also that our notation is consistent in that x indeed descends to the defining equation of the toric boundary of X_σ.

It is not difficult to show that $\mathcal{X}\to\mathbb{A}^1$ is a toric degeneration. Indeed, we have already checked that $\mathcal{X}_0$ is the toric boundary of $X_{\tilde{\Sigma}}$. Some harder work shows that locally near the zero-dimensional strata of X_0, the projection $\mathcal{X}\to\mathbb{A}^1$ is toric. We omit the details, but this can be done similarly to arguments given in [Gr2].

The dual intersection complex is then easily described along the lines given in [Gr2], where, for a Calabi-Yau hypersurface in a toric variety, B was described as the boundary of a reflexive polytope, with the cones over the faces of the polytope yielding the fan defining the ambient toric variety. Topologically, we can write $B\subseteq M_{\mathbb{R}}\oplus\mathbb{R}^2$ as

$$B=\tilde{\sigma}_1\cup\tilde{\sigma}_2$$

where

$$\begin{aligned}\tilde{\sigma}_1 &= \operatorname{Conv}\big((0,1,-1)\cup(\sigma\times\{(1,0)\})\big),\\ \tilde{\sigma}_2 &= \operatorname{Conv}\big((0,0,1)\cup(\sigma\times\{(1,0)\})\big).\end{aligned}$$

Note that the support of the fan $\tilde{\Sigma}$ above is the cone over $\tilde{\sigma}_1\cup\tilde{\sigma}_2$. We then take $\mathscr{P}=\{C\cap B\,|\,C\in\tilde{\Sigma}\}$.

Finally, the affine structure on B is defined as follows. Identify σ with $\sigma\times\{(1,0)\}\subseteq B$, and take the discriminant locus Δ to be the union of cells of the first barycentric subdivision of $\overline{\mathscr{P}}$ not containing vertices of $\overline{\mathscr{P}}$, see Figure 2.1. We then define affine charts as follows. First, we define affine charts $\iota_i:\tilde{\sigma}_i\setminus\sigma\hookrightarrow\mathbb{A}_i$ as the inclusions, where $\mathbb{A}_i$ denotes the affine hyperplane in $M_{\mathbb{R}}\oplus\mathbb{R}^2$ spanned by $\tilde{\sigma}_i$. Second, for each vertex $v\in\overline{\mathscr{P}}$, choose a neighbourhood U_v of $(v,1,0)\in B$. These neighbourhoods can be chosen so that $U_v\cap U_{v'}=\emptyset$ if $v\neq v'$ and the two sets $\tilde{\sigma}_i\setminus\sigma$ along with the open sets U_v cover $B\setminus\Delta$. Define a chart $\iota_v:U_v\to(M_{\mathbb{R}}\oplus\mathbb{R}^2)/\mathbb{R}(v,1,0)$ via the inclusion followed by the projection. It is easy to check that these charts give an integral affine structure. This again precisely follows the

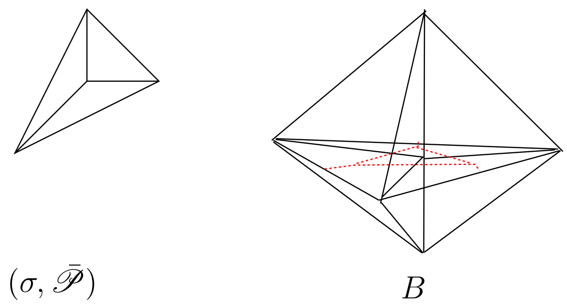

Figure 2.1. On the left is the initial polytope with its decomposition. We take the vertices of σ to be $(1,0)$, $(0,1)$ and $(-1,-1)$. On the right is the resulting B with its discriminant locus Δ, indicated by the dotted line.

procedure for Calabi-Yau hypersurfaces in toric varieties considered in [Gr2]. This gives rise to the pair $(B, \mathscr{P})$.

In general, a pair $(B, \mathscr{P})$ can be described by specifying the lattice polytopes in $\mathscr{P}$ and specifying a *fan structure* at each vertex v, that is, the identification of a neighbourhood of each vertex with the neighbourhood of 0 in a fan Σ_v. This identification gives a one-to-one inclusion preserving correspondence between cells of $\mathscr{P}$ containing v and cones of Σ_v, along with integral affine identifications of the tangent wedges of each cell $\tau \in \mathscr{P}$ containing v with the corresponding cone of Σ_v. These identifications patch together to give an affine chart in a neighbourhood of the vertex v.

In our example, it is worth describing the fan structure at a vertex $v \in \sigma$. Since the fan structure at a vertex must be the fan yielding the corresponding irreducible component of $\mathscr{X}_0$, toric geometry tells us this fan structure must be given as the quotient fan obtained from $\tilde{\Sigma}$ by dividing out by the ray generated by this vertex. Explicitly, we use the chart

$$\iota_v : U_v \to (M_{\mathbb{R}} \oplus \mathbb{R}^2)/\mathbb{R} \cdot (v,1,0) \cong M_{\mathbb{R}} \oplus \mathbb{R}, \tag{2.1}$$

the latter isomorphism given by $(m, r_1, r_2) \mapsto (m - r_1 v, r_2)$. The fan Σ_v can then be described as the fan of tangent wedges to images of cells $\tau \in \mathscr{P}$ containing v. The set of maximal cones of this fan, described as subsets of $M_{\mathbb{R}} \oplus \mathbb{R}$, is

$$\{T_v\tau + \mathbb{R}_{\geq 0}(-v,-1) \,|\, v \in \tau \in \overline{\mathscr{P}}_{\max}\} \cup \{T_v\tau + \mathbb{R}_{\geq 0}(0,1) \,|\, v \in \tau \in \overline{\mathscr{P}}_{\max}\}, \tag{2.2}$$

where $T_v\tau$ denotes the tangent wedge to $v \in \tau$ in $M_{\mathbb{R}} \oplus 0$. Figure 2.2 shows some of the fan structures when σ is an interval $[-1,1]$ of length two.

One can understand the nature of the singularities of B by studying the local system Λ of integral vector fields on B_0. Given integral affine coordinates

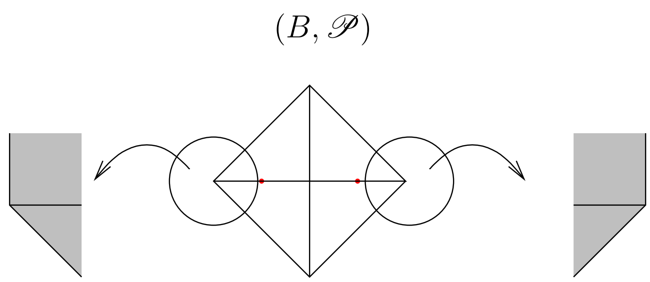

Figure 2.2.

$y_1, \ldots, y_n$, Λ is locally the family of lattices in the tangent bundle of B_0 generated by $\partial/\partial y_1, \ldots, \partial/\partial y_n$. If $v, v' \in \overline{\mathscr{P}}$ are adjacent vertices, consider a path γ passing from v through $\tilde\sigma_1$ to v' and then through $\tilde\sigma_2$ back to v. To identify Λ_v, we can use the chart (2.1), which gives an identification of Λ_v with $M \oplus \mathbb{Z}$. It is an easy exercise to check that parallel transport in Λ around γ yields a *monodromy transformation*

$$\begin{aligned} T_{vv'} : \Lambda_v &\to \Lambda_v \\ (m, r) &\mapsto (m + r(v - v'), r) \end{aligned} \tag{2.3}$$

The final piece of data for the dual intersection complex $(B, \mathscr{P}, \varphi)$ of $\mathcal{X} \to \mathbb{A}^1$ is a multi-valued piecewise linear function φ describing aspects of the Kähler geometry of the situation.

In the next section, we will build the mirror family over some base scheme. The natural choice for this base is related to the Kähler cone of X_Σ, or the Picard group. By toric geometry, $\mathrm{Pic}(X_\Sigma)$ equals piecewise linear functions on Σ modulo linear functions. It will thus be convenient to normalize the piecewise linear functions as follows. Choose a maximal cell $\tau \in \overline{\mathscr{P}}_{\max}$ which has 0 as a vertex. Let P be the monoid of integral convex piecewise linear functions on the fan Σ which take the value 0 on the cone $C(\tau)$. Note that $P^{\mathrm{gp}} \cong \mathrm{Pic}\, X_\Sigma$. Setting $Q := \mathrm{Hom}(P, \mathbb{N})$, there is a universal piecewise linear function $\psi : |\Sigma| \to Q^{\mathrm{gp}}_{\mathbb{R}}$, with

$$\psi(x) = (P \ni \varphi \mapsto \varphi(x)).$$

This function is strictly convex in the sense of [GHK], Definition 1.12. In the local $\mathbb{P}^2$ case normalized piecewise linear functions are determined by the value at the one remaining vertex of σ not contained in τ and hence $Q = \mathbb{N}$.

The multi-valued piecewise linear function φ comes from the universal piecewise linear function ψ on $|\Sigma|$ by descent to a quotient fan, or rather from a choice of extension of this function to $\widetilde{\Sigma}$. This choice can be made by choosing an element

$q \in Q \setminus \{0\}$. While the choice of q affects the family of polarizations on $X_{\tilde{\Sigma}}$, it does not affect the family after restriction to $\mathcal{X}_t$ for $t \neq 0$. However, it does affect the polarization on $\mathcal{X}$, and hence will play some role in the mirror, seen explicitly in (3.7). We take $\widetilde{\psi}$ to be the $Q^{\mathrm{gp}}_{\mathbb{R}}$-valued piecewise linear extension of ψ which takes the value 0 at $(0,1,-1)$ and the value q at $(0,0,1)$. One can check that this function is strictly convex in the sense of [GHK], Definition 1.12.

We can then construct φ from $\widetilde{\psi}$ as follows. For each $C \in \tilde{\Sigma}$, let $\tau = C \cap B$ be the corresponding cell of $\mathscr{P}$. The function $\widetilde{\psi}$ induces a function on the quotient fan of $\widetilde{\Sigma}$ along C (this quotient fan determining the fan structure of B along τ) as follows. Let $\widetilde{\psi}_\tau \in \mathrm{Hom}(M \oplus \mathbb{Z}^2, Q)$ be a linear extension of $\widetilde{\psi}|_C$. Then $\widetilde{\psi} - \widetilde{\psi}_\tau$ is a piecewise affine function on $\tilde{\Sigma}$ vanishing on C, hence descending to the quotient fan of $\tilde{\Sigma}$ along C. We take $(\widetilde{\psi} - \widetilde{\psi}_\tau)|_B$ as a representative of φ on a small open neighbourhood of $\mathrm{Int}(\tau)$ in B; this is clearly the pull-back of the corresponding function on the quotient fan of $\tilde{\Sigma}$ along C under the projection to $(M_{\mathbb{R}} \oplus \mathbb{R}^2)/\mathbb{R}C$.

3. The mirror degeneration and slab functions

Having described $(B, \mathscr{P}, \varphi)$ in our example as the dual intersection complex of a degeneration of the local Calabi-Yau X_Σ, we turn to the construction of the mirror, which shall be a family $\mathcal{Y} \to \mathrm{Spec}\,\widehat{\Bbbk[Q]}$ over a generally higher-dimensional base.

This family is constructed by constructing families $\mathcal{Y}_k \to \mathrm{Spec}\,\Bbbk[Q]/\mathfrak{m}^{k+1}$ to each order k, giving rise to a formal scheme $\widehat{\mathcal{Y}} \to \mathrm{Spf}\,\widehat{\Bbbk[Q]}$. As the case at hand will be projective, the Grothendieck existence theorem gives rise to the desired family. Alternatively, $\mathcal{Y}$ can be constructed using a graded ring of theta functions, following [GHKS].

Here is a brief summary of the construction. The central fibre Y_0 can be described as

$$Y_0 = \bigcup_{\sigma \in \mathscr{P}_{\max}} \mathbb{P}_\sigma$$

where $\mathscr{P}_{\max}$ denotes the maximal cells of $\mathscr{P}$ and $\mathbb{P}_\sigma$ is the toric variety (projective if σ is compact) determined by the polyhedron σ. These toric varieties are glued together in a manner reflecting the combinatorics of $\mathscr{P}$: if $\sigma_1 \cap \sigma_2 = \tau$, then the strata $\mathbb{P}_\tau \subseteq \mathbb{P}_{\sigma_1}$, $\mathbb{P}_\tau \subseteq \mathbb{P}_{\sigma_2}$ are identified.

Local models for the k^{th} order deformation of Y_0 are determined by the function φ. A key point of the construction involves an invariant description for the local models, which we explain here. The function φ, defined on an open cover $\{U_i\}$ by single-valued functions $\varphi_i : U_i \to Q^{\mathrm{gp}}_{\mathbb{R}}$, determines an extension of Λ by $\underline{Q}^{\mathrm{gp}}$, the constant sheaf with coefficients in Q^{gp}. Indeed, on $U_i \cap B_0$, this extension will split as $\Lambda|_{U_i} \oplus \underline{Q}^{\mathrm{gp}}$, and on the overlap, (m,r) as a section of $\Lambda|_{U_i} \oplus \underline{Q}^{\mathrm{gp}}$ is identified on $U_i \cap U_j$ with $(m, r + d(\varphi_j - \varphi_i)(m))$ as a section of $\Lambda|_{U_j} \oplus \underline{Q}^{\mathrm{gp}}$, interpreting $d(\varphi_j - \varphi_i) \in \mathrm{Hom}(\Lambda|_{U_i}, Q^{\mathrm{gp}})$. We then have an exact sequence

$$0 \to \underline{Q}^{\mathrm{gp}} \to \mathcal{P} \to \Lambda \to 0 \tag{3.1}$$

on B_0. We write the map $\mathcal{P} \to \Lambda$ as $m \mapsto \bar{m}$. After choosing a representative φ_i of φ in a neighbourhood of a point $x \in B_0$, the stalk $\mathcal{P}_x$ is identified with $\Lambda_x \oplus Q^{\mathrm{gp}}$. There is a fan $\Sigma_x = \{T_x\sigma \,|\, x \in \sigma \in \mathscr{P}\}$ (of not-necessarily strictly convex cones), where $T_x\sigma$ denotes the tangent wedge to σ at x. This allows us to define a convex PL function $\varphi_x : |\Sigma_x| \to Q^{\mathrm{gp}}_{\mathbb{R}}$ whose slope on $T_x\sigma$ coincides with the slope of $\varphi_i|_\sigma$. We then set

$$P_x := \{(m,q) \,|\, m \in \Lambda_x \cap |\Sigma_x|, q \in Q^{\mathrm{gp}}, q - \varphi_x(m) \in Q\} \subseteq \mathcal{P}_x \tag{3.2}$$

While this definition as described inside of $\Lambda_x \oplus Q$ depends on the choice of representative, in fact it is independent of this choice when viewed as a submonoid of $\mathcal{P}_x$.

Note that Q acts naturally on P_x, giving $\Bbbk[P_x]$ a $\Bbbk[Q]$-algebra structure. For a vertex v, we can now view $\mathrm{Spec}\,\Bbbk[P_v]/\mathfrak{m}^{k+1}$ as a local model for the k^{th} order deformation of Y_0 in a neighbourhood of the stratum of Y_0 corresponding to v. In addition, the local system $\mathcal{P}$ gives a method of defining parallel transport of monomials.

Let us describe certain aspects of this construction for our local mirror symmetry example. Using the fan structure given by (2.2), we can describe the monoid $P_v \subseteq \mathcal{P}_v$ as $\{(m,r,s) \,|\, s - \varphi_v(m,r) \in Q\} \subseteq M \oplus \mathbb{Z} \oplus Q^{\mathrm{gp}}$ using the identifications

$$\mathcal{P}_v \cong \Lambda_v \oplus Q^{\mathrm{gp}} \cong M \oplus \mathbb{Z} \oplus Q^{\mathrm{gp}} \tag{3.3}$$

induced first by the representative φ_v at v and second by the affine coordinate chart on U_v. In particular, for the purposes of the discussion below, we can describe the most relevant part of P_v as follows. First, we choose the representative φ_v by choosing the linear function $\widetilde{\psi}_v$ to be $(0, \bar{\psi}(v), 0) \in (N \oplus \mathbb{Z}^2) \otimes_{\mathbb{Z}} Q^{\mathrm{gp}}$, with $\bar{\psi} = \psi|_{\sigma \times \{1\}}$. Let $\bar{P}_v \subseteq P_v$ be the submonoid consisting of $m \in P_v$ with $\bar{m}$ tangent to σ. Then $\bar{P}_v$ is naturally described in terms of $\bar{\psi}$. Indeed, consider the convex hull of the graph of $\bar{\psi}$,

$$\Xi_{\bar{\psi}} := \{(m, 0, s) \,|\, m \in \sigma, s - \bar{\psi}(m) \in Q\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R} \oplus Q^{\mathrm{gp}}_{\mathbb{R}},$$

an unbounded polyhedron with vertices mapping to vertices of $\overline{\mathscr{P}}$ under the projection $M_{\mathbb{R}} \oplus \mathbb{R} \oplus Q^{\mathrm{gp}}_{\mathbb{R}} \to M_{\mathbb{R}}$. Then we can identify $\bar{P}_v$ with the integral points in the tangent wedge of $\Xi_{\bar{\psi}}$ at $(v, 0, \bar{\psi}(v))$.

We also note that under the identification (3.3) of $\mathcal{P}_v$, the monodromy of Λ described in (2.3) lifts to a monodromy transformation of $\mathcal{P}_v$ given by

$$\begin{aligned} &T_{vv'} : \mathcal{P}_v \to \mathcal{P}_v \\ &(m, r, q) \to (m + r(v - v'), r, q + r(\bar{\psi}(v) - \bar{\psi}(v'))) \end{aligned} \tag{3.4}$$

The key additional (and usually most complex) ingredient for constructing $\mathcal{Y}_k$ is a *structure* $\mathscr{S}$. A structure encodes data about how certain forms of these local models are glued together. We will explain this structure in our example, but not go into too much detail. A more detailed explanation for how this works is given in the expository paper [GrSi4].

The structure takes a particularly simple form here. In general, a structure is a collection of walls, polyhedral cells in B of codimension one each contained in a cell of $\mathscr{P}$ carrying the additional data of certain formal power series. In [GrSi3] we distinguish a special sort of wall, namely those contained in codimension one cells, and call them *slabs*. They tend to have a different behaviour. In the case at hand, only slabs appear, and these cover σ. The functions attached to the slabs are determined from the monodromy around the discriminant locus Δ.

In this example, the slabs are the sets $\tau \times \{(1,0)\}$ for $\tau \in \overline{\mathscr{P}}_{\max}$. For a slab $\mathfrak{b}$, associated to any point $x \in \mathfrak{b} \setminus \Delta$ is a formal power series $f_{\mathfrak{b},x} = \sum_{m \in P_x} c_m z^m$. This should only depend on the connected component of $\mathfrak{b} \setminus \Delta$ containing x, so there is in fact one such expression for each vertex v of τ, and we can write $f_{\mathfrak{b},v} = \sum_{m \in P_v} c_m z^m$. Furthermore, $c_m \neq 0$ implies $\bar{m}$ is tangent to $\mathfrak{b}$, so in fact the sum is over $m \in \bar{P}_v$.

The series $f_{\mathfrak{b},v}$ are completely determined by a number of simple properties. This follows in the case under consideration from having chosen $\overline{\mathscr{P}}$ to consist of *standard* simplices. In what follows we will want to compare $f_{\mathfrak{b},v}$ with $f_{\mathfrak{b},v'}$ for different vertices v, v' of $\mathfrak{b}$. To do so, we use parallel transport in $\mathcal{P}$ from v to v'. Given the identification $\mathcal{P}_v$ with $M \oplus \mathbb{Z} \oplus Q^{\mathrm{gp}}$ used to give the formula (3.4) and noting that only monomials of the form $z^{(m,0,p)}$ can appear in $f_{\mathfrak{b},v}$, we see that the particular path chosen between v and v' is irrelevant.

We can now state the conditions determining the $f_{\mathfrak{b},v}$:

1. The constant term of each $f_{\mathfrak{b},v}$ is 1.

2. If v and v' are adjacent vertices of $\mathfrak{b}$, then the corresponding slab functions are related by
$$f_{\mathfrak{b},v'} = z^{(v-v',0,\bar{\psi}(v)-\bar{\psi}(v'))} f_{\mathfrak{b},v}. \tag{3.5}$$

   Here the equality makes sense after parallel transport of the exponents from

v to v' in the local system $\mathcal{P}$, and $(v-v', 0, \bar\psi(v)-\bar\psi(v')) \in P_{v'}$ using the identification of $\mathcal{P}_{v'}$ given by (3.3).

3. $\log f_v$ contains no terms of the form z^q for $q \in Q \setminus \{0\}$. Here we view $Q \subseteq P_v$ via the natural inclusion $Q^{\mathrm{gp}} \subseteq \mathcal{P}_v$.

4. If v lies in slabs $\mathfrak{b}, \mathfrak{b}'$, then $f_{\mathfrak{b},v} = f_{\mathfrak{b}',v}$.

Item 1 is a normalization which originated in [GrSi1], Def. 4.23. However, we shall see its enumerative importance in §4. Item 2 is the crucial point of slabs: they allow us to define parallel transport of monomials through slabs in a way which cancels the effects of monodromy. We shall say more about this shortly. The condition 3 is interpreted by writing $f_v = 1 + \cdots$ and using the Taylor expansion for $\log(1+x) = \sum_{i=1}^\infty (-1)^{i+1} x^i/i$. This can be interpreted inside some suitably completed ring. After expanding out each expression $(\cdots)^i$, one demands that no monomials of the form z^q appear for any $q \in Q \setminus \{0\}$. Finally, 4 tells us how expressions propagate across $\sigma \times \{1\}$.

To see the significance of the second condition, let $w \in \mathrm{Int}(\tilde\sigma_1)$, $w' \in \mathrm{Int}(\tilde\sigma_2)$. Suppose we want to compare monomials defined at w (that is, monomials with exponent in P_w) with monomials defined at w' (that is, monomials with exponent in $P_{w'}$). If we parallel transport from $\mathcal{P}_w$ to $\mathcal{P}_{w'}$, the result depends on the path. For example, let v, v' be adjacent vertices of $\tau \in \overline{\mathscr{P}}_{\max}$. Let T_v, $T_{v'}$ denote parallel transport in Λ from w to w' via the vertices v and v' respectively. Then from (3.4), it follows that for $(m,r,q) \in \mathcal{P}_w = M \oplus \mathbb{Z} \oplus Q$,

$$T_{v'}(m,r,q) - T_v(m,r,q) = \big(r(v-v'), 0, r(\bar\psi(v)-\bar\psi(v'))\big).$$

For convenience, we can identify $\mathcal{P}_w$ and $\mathcal{P}_{w'}$ with $\mathcal{P}_v$ so that T_v is the identity. This difference between T_v and $T_{v'}$ creates problems for comparing the rings $\Bbbk[P_w]$ and $\Bbbk[P_{w'}]$. However, we can follow the rule that if we wish to transport a monomial $z^{(m,r,q)}$ along a path between w and w' which crosses a slab $\mathfrak{b}$ in a connected component of $\mathfrak{b} \setminus \Delta$ containing a vertex v, we apply an automorphism

$$z^{(m,r,q)} \mapsto z^{(m,r,q)} f_{\mathfrak{b},v}^{-r}. \tag{3.6}$$

Here r represents the result of projecting $\overline{(m,r,q)} = (m,r)$ via the projection $\pi : \Lambda_v \to \mathbb{Z}$ obtained by dividing out by the tangent space to the slab. If instead we pass through the slab $\mathfrak{b}$ via the connected component of $\mathfrak{b} \setminus \Delta$ containing v', we get

$$z^{(m,r,q)} \longmapsto z^{(m+r(v-v'), r, q+r(\bar\psi(v)-\bar\psi(v')))} f_{v'}^{-r} = z^{(m,r,q)} f_v^{-r},$$

coinciding with (3.6). Here we use the above expression for $T_v - T_{v'}$ and (3.5). Hence we see that the ambiguity produced by monodromy is resolved by the slab functions.

Examples 3.1. In the following examples, we express the various functions $f_{\mathfrak{b},v}$ as formal power series with exponents appearing in $\bar{P}_v$, using the representation of $\bar{P}_v$ as the integral points of the tangent wedge of $\Xi_{\bar{\psi}}$ at $(v, 0, \bar{\psi}(v))$.

(1) Take σ to be the interval $[-1, 1]$ as in Figure 2.2, with $\mathscr{P}$ as given there. The monoid of convex piecewise linear functions on Σ is generated by the function which takes the values 0, 0 and 1 respectively at $(-1, 1)$, $(0, 1)$ and $(1, 1)$. Thus we have $Q = \mathbb{N}$, and the universal piecewise linear function ψ coincides with the above generator. For a vertex v, with $\bar{P}_v \subseteq M \oplus 0 \oplus Q^{\mathrm{gp}}$, write $x = z^{(1,0,0)}$, $t = z^{(0,0,1)}$, t being the generator of $\Bbbk[Q]$. Then we have

$$\begin{aligned} f_{[-1,0],-1} &= 1 + x + x^2 t + xt, \\ f_{[-1,0],0} = f_{[0,1],0} &= 1 + x^{-1} + xt + t, \\ f_{[0,1],1} &= 1 + x^{-1}t^{-1} + x^{-2}t^{-1} + x^{-1}. \end{aligned}$$

Note that $\log f_{[-1,0],-1}$, $\log f_{[0,1],1}$ are clearly devoid of pure powers of t as any power, say, of $x + x^2t + xt$ clearly produces only terms with positive powers of x. On the other hand, $f_{[-1,0],0} = (1 + x^{-1})(1 + xt)$, and taking logs we get $\log(1 + x^{-1}) + \log(1 + xt)$ which will again involve no pure t power. The t term in $f_{[0,1],0}$ was necessary to achieve this.

(2) Take σ to be as in Figure 2.1. Again, the monoid of convex piecewise linear functions on the fan Σ is generated by, say, the function taking the values 0 at $(0, 0, 1)$, $(1, 0, 1)$ and $(0, 1, 1)$ and the value 1 at $(-1, -1, 1)$. So again $Q = \mathbb{N}$, with the universal function ψ agreeing with this generator. Writing $x = z^{(1,0,0,0)}$, $y = z^{(0,1,0,0)}$, $t = z^{(0,0,0,1)}$, it is easy to see that the terms of the slab function $f_{\mathfrak{b},(0,0)}$ (independent of $\mathfrak{b}$ by the fourth condition) required by conditions 1 and 2 are $1 + x + y + tx^{-1}y^{-1}$. The normalization condition forces us to add some additional terms:

$$f_{\mathfrak{b},(0,0)} = 1 + x + y + tx^{-1}y^{-1} + \sum_{k \geq 1} a_k t^k,$$

where the a_k are uniquely determined by the requirement that

$$\sum_{i=1}^{\infty} (-1)^{i+1} \frac{(x + y + tx^{-1}y^{-1} + \sum_{k\geq 1} a_k t^k)^i}{i}$$

contains no pure powers of t. This series in t begins as

$$-2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 + \cdots .$$

(3) Let σ be the convex hull of the points $(\pm 1, 0)$, $(0, \pm 1)$ and take $\mathscr{P}$ to be the star subdivision at the origin. Now the monoid P of convex piecewise linear functions which are 0 on $(0,0,1)$, $(1,0,1)$ and $(0,1,1)$ is isomorphic to $\mathbb{N}^2$, determined by the values α_1, α_2 of the function at generators of the other two rays. Thus we can write $Q = \mathbb{N}^2$, $t_1 = z^{(1,0)} \in \Bbbk[Q]$, $t_2 = z^{(0,1)} \in \Bbbk[Q]$. Using x, y as defined in the previous example, one can check that for any slab $\mathfrak{b}$,

$$f_{\mathfrak{b},0} := 1 + x + y + t_1 x^{-1} + t_2 y^{-1} + t_1 + t_2 + 3t_1t_2 + 5t_1^2 t_2 + 5t_1 t_2^2 + \cdots .$$

The additional terms represented by $\cdots$ give a power series in t_1, t_2.

We now describe the degeneration $\mathcal{Y} \to \operatorname{Spec} \widehat{\Bbbk[Q]}$ produced by the above data. In fact, it is not difficult to do this in terms of equations, as follows. First, define

$$C(\Xi_{\bar\psi}) := \overline{\{((um, 0, uq, u) \,|\, (m, 0, q) \in \Xi_{\bar\psi},\ u \in \mathbb{R}_{\geq 0}\}} \subseteq M_{\mathbb{R}} \oplus \mathbb{R} \oplus Q^{\mathrm{gp}}_{\mathbb{R}} \oplus \mathbb{R}.$$

Here the closure is necessary because $\Xi_{\bar\psi}$ is unbounded. We then obtain a graded ring

$$S_{\bar\psi} := \Bbbk[C(\Xi_{\bar\psi}) \cap (M \oplus \mathbb{Z} \oplus Q^{\mathrm{gp}} \oplus \mathbb{Z})]$$

where the grading is given by the projection from $M \oplus \mathbb{Z} \oplus Q^{\mathrm{gp}} \oplus \mathbb{Z}$ onto the last copy of $\mathbb{Z}$. Note the closure in the definition of cone adds the cone $\{0\} \times \{0\} \times \mathbb{R}_{\geq 0} Q \times \{0\}$ to the set, so we see the degree 0 part of $S_{\bar\psi}$ is $\Bbbk[Q]$. We can then complete, with

$$\widehat{S}_{\bar\psi} := S_{\bar\psi} \otimes_{\Bbbk[Q]} \widehat{\Bbbk[Q]}.$$

It is then natural to think of the slab functions as being given by a single degree 1 element of $\widehat{S}_{\bar\psi}$. Indeed, given a vertex $v \in \overline{\mathscr{P}}$, we obtain from $f_{\mathfrak{b},v}$ an element of degree 1 by multiplying all monomials of $f_{\mathfrak{b},v}$ by $z^{(v,0,\bar\psi(v),1)}$. It follows from (3.5) that this is independent of the choice of v and gives an element $F \in \widehat{S}_{\bar\psi}$ of degree 1. One can then show that

$$\mathcal{Y} = \operatorname{Proj} \widehat{S}_{\bar\psi}[U, W]/(UW - z^q V_0 F). \tag{3.7}$$

Here U, W are of degree 1, $V_0 \in \widehat{S}_{\bar\psi}$ is the element corresponding to $(0,0,0,1)$ (which lies in $\Xi_{\bar\psi}$ by the assumption that $0 \in \sigma$ and ψ has been chosen so that $\bar\psi(0) = 0$). The element $q \in Q$ is the element chosen in the definition of $\widetilde{\psi}$ at the end of §2. This can be shown in much the way the special case discussed in [GrSi4], Example 5.2, being the case of Examples 3.1, (2). Note that after localizing at z^q, this family does not depend on the choice of q up to isomorphism, just as the choice of q did not affect the polarization on the general fibres of $\mathcal{X} \to \mathbb{A}^1$.

The homogeneous coordinate ring of $\mathcal{Y}$ is generated in degree 1 by *theta functions*, as explored in [GHKS]. Each point of $B(\mathbb{Z})$ (the set of points of B with integral coordinates) corresponds to a generator of this ring as a $\widehat{\Bbbk[Q]}$-algebra. Explicitly, the integral points in this example are the integral points of σ and the apexes of the pyramids $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. If v is an integral point of σ, then $z^{(v,0,\bar{\psi}(v),1)} \in \widehat{S}_{\bar{\psi}}$ is the corresponding theta function. On the other hand, the monomials U and W correspond to the two apexes.

This description of $\mathcal{Y}$ can be related to the more traditional mirror to X_Σ as described in [CKYZ]. Here $\mathcal{Y}$ can be decompactified by setting $V_0 = 1$, obtaining an open subset $\mathcal{Y}^o$. Passing to the generic fibre $\mathcal{Y}^o_\eta$ of $\mathcal{Y}^o \to \operatorname{Spec} \widehat{\Bbbk[Q]}$, we obtain a variety defined over the field of fractions K of $\widehat{\Bbbk[Q]}$. We can describe $\mathcal{Y}^o$ as a subvariety of $\mathbb{A}^2 \times (N \otimes_{\mathbb{Z}} \mathbb{G}_m)$ over the field K given by the equation

$$uw = z^q f_{\mathfrak{b},0}, \tag{3.8}$$

where $\mathfrak{b}$ is any slab containing $0 \in \sigma$. Here u, w are coordinates on $\mathbb{A}^2$. Without the normalization condition, we could take $f_{\mathfrak{b},0} = \sum_{m\in\sigma\cap M} z^{(m,\bar{\psi}(m))}$, which would lead to the mirror of X_Σ being precisely that given in [CKYZ].

Remark 3.2. The crucial feature of the mirror family we have just described, as opposed to the one given in [CKYZ], is that the monomial coordinates on the base $\operatorname{Spec} \widehat{\Bbbk[Q]}$ are canonical in the sense of mirror symmetry. To describe this briefly, we work over the field $\Bbbk = \mathbb{C}$, and assume that the power series $f := f_{\mathfrak{b},0}$ is convergent in some analytic neighbourhood U of the zero-dimensional stratum in $\operatorname{Spec} \mathbb{C}[Q]$. Let $U^* = U \setminus \partial \operatorname{Spec} \mathbb{C}[Q]$, the complement of the union of toric divisors. Thus we can view $\mathcal{Y}^o$ as giving an analytic family $\mathcal{Y}^o \to U^*$. We write $\mathcal{Y}^o_t$ for the fibre over $t \in U^*$. On such a fibre, one has the holomorphic volume form on the fibres of $\mathcal{Y}^o \to \operatorname{Spec} \widehat{\Bbbk[Q]}$ given by

$$\Omega = (2\pi i)^{-n-1} d\log u \wedge d\log x_1 \wedge \cdots \wedge d\log x_n$$

One then finds that there is a monodromy invariant cycle $\alpha_0 \in H_{n+1}(\mathcal{Y}^o_t, \mathbb{Z})$ such that $\int_{\alpha_0} \Omega = 1$, so that Ω is a normalized holomorphic form in the sense of mirror symmetry. Further, if $q_1, \ldots, q_r \in Q^{\mathrm{gp}}$ are a basis for Q^{gp}, one can find (multi-valued) flat families of $(n+1)$-cycles $\alpha_1, \ldots, \alpha_r$ with $\int_{\alpha_i} \Omega = \log z^{q_i}$. The key point of this calculation is to take the logarithmic derivative of these period integrals and reduce the resulting integral to an integral on the hypersurface $f_{\mathfrak{b},v} = 0$ in $N \otimes_{\mathbb{Z}} \mathbb{G}_m$. Via residues, this is translated into an integral of the derivative of $\log f_{\mathfrak{b},0}$ over various tori in $N \otimes_{\mathbb{Z}} \mathbb{G}_m$. The fact that these integrals are then constant follows precisely from the normalization condition on $f_{\mathfrak{b},0}$.

4. Enumerative predictions

So far we have seen two interpretations of the slab functions and the normalization condition. The first came from the desire to write down a correction to the patching of the naive toric models for the mirror degeneration $\mathcal{Y} \to \operatorname{Spec} \widehat{\Bbbk[Q]}$ in a way consistent with local monodromy of the affine structure on B. We discussed in §2 how this condition along with the normalization condition determines the slab functions uniquely. Then in Remark 3.2 we saw that the normalization condition is responsible for making our families canonically parametrized in the sense of mirror symmetry. Both of these arguments concern the *complex geometry* of the mirror degeneration $\mathcal{Y} \to \operatorname{Spec} \widehat{\Bbbk[Q]}$.

In the following two sections we will give two related interpretations of normalized slab functions related to the *symplectic geometry* of the degeneration $\mathcal{X} \to \mathbb{A}^1$ of the local Calabi-Yau variety X_Σ we started with. The interpretation supports the view that the degenerations constructed by structures are indeed the ones expected from homological mirror symmetry and open-closed string theory.

Since the completion of [GrSi3], a clearer idea emerged as to the precise meaning of structures. This picture has arisen from several converging points of view: (1) The heuristic correspondence between tropical Morse trees and Floer homology emerging in discussions between us and Mohammed Abouzaid. Some of these ideas were discussed in [Clay] and [GrSi5]. (2) Auroux's work [Au] on T-duality on complements of anti-canonical divisors, describing the complex structure on the SYZ dual of a Lagrangian fibration using counts of Maslov index two disks. This has inspired quite a bit of work, which is realising Fukaya's original dream of correcting the complex structure of the mirror via counts of holomorphic disks. (3) [GPS] made explicit the enumerative content of the key part of the algorithm of [KS] (or the two-dimensional version of [GrSi3]). In particular, this established an enumerative meaning for functions attached to walls of a structure.

Heuristically, one expects the following interpretation in the SYZ picture of mirror symmetry. Suppose given a (special) Lagrangian fibration $f : X \to B$ from a Calabi-Yau X, with the general fibre being a torus. Consider Maslov index zero holomorphic disks with boundary a fibre of f. For dimensional reasons the expectation is that the set of points in $x \in B$ such that $f^{-1}(x)$ bounds a Maslov index zero holomorphic disk is real codimension one in B, forming a collection of walls. These walls should determine the structure necessary to build the mirror to X, but one needs to attach functions to the walls. Again, heuristically, these functions are expected to take the shape, at a point $x \in B$ with $L = f^{-1}(x)$,

$$\exp\left(\sum_{\substack{\beta\in\pi_2(X,L)\\ \partial\beta\neq 0}} k_\beta n_\beta z^\beta\right). \tag{4.1}$$

Here the sum is over all relative homotopy classes β such that $\partial\beta \in \pi_1(L)$ is non-zero, k_β is the index of $\partial\beta \in \pi_1(L)$ and n_β is some count of Maslov index zero disks with boundary on L. This series should be defined as a formal power series in some suitable ring. One can note that as $x \in B$ varies, the groups $\pi_2(X,L)$ vary forming a local system on B_0 (where $B_0 = \{x \in B \mid f^{-1}(x) \text{ is non-singular}\}$). This local system is analogous to the sheaf $\mathcal{P}$ of §2, with the exact sequence of homotopy groups

$$\pi_2(L) = 0 \to \pi_2(X) \to \pi_2(X,L) \to \pi_1(L) \to \pi_1(X)$$

being analogous to the exact sequence (3.1).

It is difficult to give exact definitions for the numbers n_β. There have been several approaches to dealing with this. For example, Auroux [Au] pioneered, in the case of an effective anti-canonical divisor, the use of counts of Maslov index two disks to define holomorphic coordinates which are then transformed by wall-crossing automorphisms as we cross walls in B over which Maslov index zero disks live.

A different approach, using log geometry, originates in [GPS]. There, working with Pandharipande, we used relative Gromov-Witten invariants to make sense of the formula (4.1). The situation there was effectively that of a rational surface with an anti-canonical divisor D, and the n_β of (4.1) are replaced with counts of curves meeting the divisor D in one point. This was used for a general mirror symmetry construction for such surfaces in [GHK].

It is interesting to note how these two points of view apply to the case of local mirror symmetry considered in this paper. Auroux's point of view was used effectively in a sequence of papers [CLL], [CLT], [CCLT] to study the same local mirror symmetry situation as discussed in this paper. Wall-crossing formulas for counts of Maslov index two disks are used to obtain what should be the same slab functions as discussed in this paper. The count of Maslov index two disks is reduced to a closed Gromov-Witten invariant on a toric variety, which can then be calculated via known mirror symmetry results. This allows one to show show that the slab functions defined using their counts give rise to canonical coordinates just as our slab functions do.

On the other hand, generalising the idea of replacing holomorphic disks with relative curves, one should be able to work with a certain kind of logarithmic curve called a *punctured curve*, the theory of which is currently being developed in a joint project with Abramovich and Chen [ACGS]. These curves will live in the central fibre of the toric degeneration $\mathcal{Y} \to \mathbb{A}^1$ constructed in §2, and can be viewed as a substitute for holomorphic curves with boundary in an algebro-geometric context. Then (4.1) can be used to define slab functions, where now n_β is a count of genus 0 logarithmic curves with one puncture.

We do not propose calculating the slab functions in this way. Rather, we should be able to show that the slab functions defined in this way satisfy the same determining properties that the slab functions of §2 did. This is done by probing slabs by broken lines (see [CPS], [GHK], [GHKS]) and interpreting these enumeratively using a different type of punctured curve, roughly corresponding to cylinders. These punctured curves play the same role that Maslov index two disks play in the analysis of slab functions of [CLL], [CLT], [CCLT]. Crucially, we need to use the gluing formula of [ACGS] to relate broken lines and punctured curves.

While the details of this approach will be given elsewhere, let us demonstrate this using the simple example from Examples 3.1, (1). We depict in Figure 4.1 the central fibre of the degeneration $\mathcal{X} \to \mathbb{A}^1$ constructed in §2 in this case. The total space $\mathcal{X}$ has two ordinary double points, situated on the singular locus of $\mathcal{X}_0$, where the map $\mathcal{X} \to \mathbb{A}^1$ is not normal crossings. The inclusion $\mathcal{X}_0 \subseteq \mathcal{X}$ induces a log structure on $\mathcal{X}_0$, but the log structure is not well-behaved at the two points (not *fine* in the sense of log geometry). In particular, the theory of log Gromov-Witten invariants as developed in [GrSi6], [AC], [Ch] cannot be used directly. While a theory of invariants which can deal directly with this poorly behaved log structure is under development, for the moment we will deal with it via a small resolution of the ordinary double points. There are four choices of such resolutions, one of which is shown on the right in Figure 4.1. These choices can be thought of in terms of the affine geometry of the dual intersection complex B, with the resolutions corresponding to sliding the two singularities of the affine structure along σ to various choices of vertices.

We have different slab functions $f_{\mathfrak{b},v}$ for the vertices $v = -1, 0, 1$. To identify the slab function at a vertex v as a generating function, we choose a small resolution $\widetilde{\mathcal{X}} \to \mathcal{X}$ so that the irreducible component indexed by v remains toric. This effectively slides the singularities away from the vertex. The resolution in Figure 4.1 is used for the vertex $v = 0$. The slab function is given by (4.1) where n_β is a count of log curves of genus 0 with one puncture mapping to the boundary of the component X_v indexed by v. In Figure 4.2 we show the two obvious such curves for $v = 0$. However, multiple covers of these curves totally ramified at the puncture points are also possible, and a d-fold cover will contribute with multiplicity

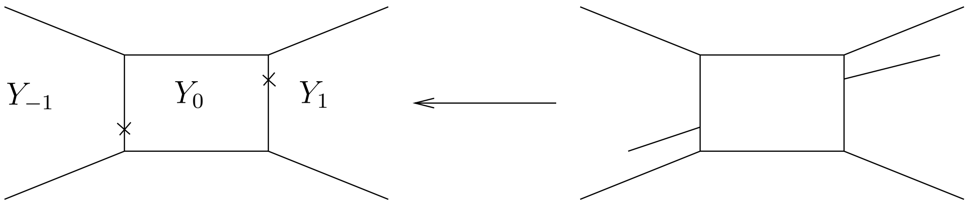

Figure 4.1. The left-hand figure shows the five irreducible components of $\mathcal{X}_0$, with the three labelled components indexed by the vertices of σ. Here $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and X_{-1}, X_1 are isomorphic to the blow-up of $\mathbb{A}^2$ at a point.

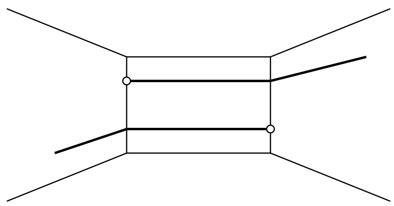

Figure 4.2. The two punctured curves corresponding to holomorphic disks. The curves include the exceptional divisors of the small resolution, and the punctures are represented by the white circles.

$(-1)^{d+1}/d^2$. The slab function is then, following (4.1),

$$(1+x^{-1})(1+xt) = \exp\left(\sum_{d=1}^{\infty} d \cdot \frac{(-1)^{d+1}}{d^2} x^{-d} + \sum_{d=1}^{\infty} d \cdot \frac{(-1)^{d+1}}{d^2} (tx)^d\right),$$

with the monomials x^{-1} and tx and their powers playing the role of z^β.

To prove this formula without a direct calculation, we show the slab functions defined by these counts satisfy conditions 1-4 of §3. Conditions 1 and 3 are obvious from (4.1), as the statement that only monomials z^β with $\partial\beta \neq 0$ appear is analogous to the statement that no terms of the form z^q for $q \in Q \setminus \{0\}$ appear inside the exponential. Condition 4 is automatic because in this situation the slab function only depends on the vertex. It remains to show condition 2, and we use broken lines for this, which can be reviewed in [GrSi5]. A broken line is a piecewise linear path with monomials $c_L z^{m_L}$ attached to each domain of linearity, and the derivative of the line in the domain L is $-\bar{m}_L$. When the broken line crosses a wall, we may change the monomial by applying the wall-crossing automorphism (3.6) to the monomial and choosing a new monomial being one of the terms in the expression obtained after applying this automorphism. In Figure 4.3, we consider

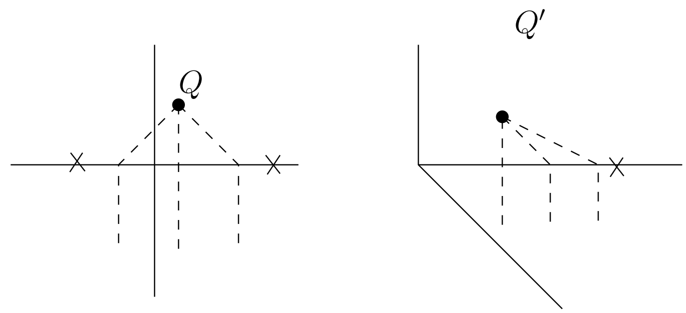

Figure 4.3. There are four broken lines with endpoints Q on the left, two of which don't bend. All have initial monomial z^m with $\bar{m} = (0, -1)$. Once the broken line crosses the slab, there are four possible attached monomials: z^m, tz^m, $x^{-1}z^m$, xtz^m. The right-hand picture shows a different choice of basepoint Q', and there are again four broken lines.

germs of broken lines which come vertically from below with initial monomial z^m with $\bar{m} = (0, -1) \in M_{\mathbb{R}} \oplus \mathbb{R}$. We define $\mathrm{Lift}_Q(m)$ to be the sum over all broken lines ending at a basepoint Q of the final attached monomials. Note that if Q is near a vertex v of σ, then in fact $\mathrm{Lift}_Q(m) = z^m f_{\mathfrak{b},v}$. It is then not difficult to show that (3.5) holds for all pairs of adjacent vertices if and only if $\mathrm{Lift}_Q(m)$ is independent of Q chosen above the slabs as in Figure 4.3.

Broken lines can be viewed as a purely combinatorial (tropical) way to count holomorphic cylinders. But we can actually count logarithmic curves of genus 0 with two punctures to emulate cylinders, and there will be a correspondence between such twice-punctured logarithmic curves and broken lines. Varying the basepoint can be achieved by varying a point constraint for one of the punctures. The key point is that various ways of degenerating the point constraint can lead to different broken lines with different endpoints. However, the count of these punctured curves will be independent of the constraint.

To see this explicitly, let's look at the example of the straight line in the left-hand diagram in Figure 4.3 with attached monomial z^m. To understand what happens when we move this broken line through the singularity, it is helpful to move the singularity to the vertex 0 by using the small resolution depicted in Figure 4.4. Consider the family of twice-punctured curves given by the vertical fibres of X_0, the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. Any curve in this family has a tropicalization (see [GrSi6], §3). The tropicalization of a general curve in this family is just the vertical line through the singularity on the right-hand side of Figure 4.4. Combinatorially, this just indicates that the curve intersects the upper and lower boundary divisors of X_0.

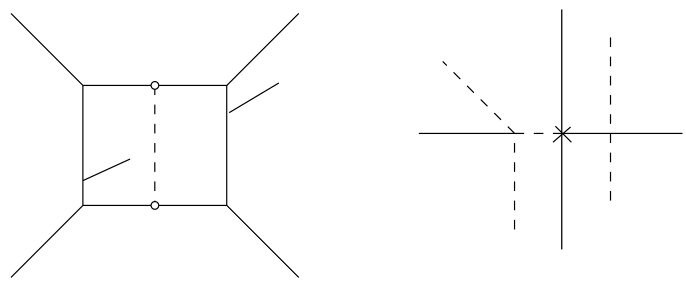

Figure 4.4.

However, the family has two special members which are degenerate with respect to the log structure on the central fibre. The tropicalization of the punctured curve when it falls into $X_0 \cap X_1$ is the straight broken line depicted in Figure 4.4 to the right of the vertex. If on the other hand we move the punctured curve to the left, it becomes reducible, the union of $X_{-1} \cap X_0$ and the exceptional curve of the small resolution. This curve tropicalizes to the tropical curve depicted on the left, now with two vertices corresponding to the two components. The bend is a consequence of gluing the once-punctured curve with support the exceptional curve to the thrice-punctured curve with support $X_0 \cap X_{-1}$. The broken line is then a subset of this tropical curve.

The point is that the two broken lines make the same contribution to $\mathrm{Lift}_Q(m)$ as Q varies because they can both be viewed as counting the number of curves in the one-parameter family described passing through some point in X_0. The point can degenerate into $X_{-1} \cap X_0$ or $X_0 \cap X_1$, giving the two types of broken line behaviour. Thus the invariance of $\mathrm{Lift}_Q(m)$ can be viewed as the fact that these lifts are generating functions for counts of certain types of punctured curves.

The key point for the argument is then to prove that broken lines really calculate Gromov-Witten invariants of punctured curves. This shall be shown using a general gluing formula [ACGS].

Note so far we have not actually calculated the Gromov-Witten invariants of X_Σ. These should be extracted in the B-model from some additional period integrals past the ones discussed in Remark 3.2. A significant challenge remaining is to give a tropical description for these period integrals and Gromov-Witten invariants.

5. Tropical disks and slab functions

The picture of counting holomorphic disks and cylinders from §4 suggests an interpretation of the slab functions in terms of tropical curves. In this section we give a surprisingly simple interpretation of this sort. The arguments are by algebraic manipulations of the slab functions. We are thus lead to the challenge of interpreting the tropical counts in terms of the counting of holomorphic disks on X_Σ.

We study the collection of slab functions at a vertex $v \in \overline{\mathscr{P}}$ with $v \in \operatorname{Int}\sigma$. By Condition (4) of slab functions all the $f_{\mathfrak{b},v}$ for slabs $\mathfrak{b}$ containing v agree. Dehomogenizing (3.7) at v we are thus left with the local model $uw - ft = 0$ for the mirror degeneration for some $f \in \widehat{\Bbbk[P]}$. Here $P = \bar{P}_v$ is a toric submonoid of $M \oplus Q^{\mathrm{gp}}$ with $P^\times = \{0\}$ and the completion is with respect to $P \setminus \{0\}$. Recall also the projection

$$M \oplus Q^{\mathrm{gp}} \longrightarrow M, \quad m \longmapsto \bar{m}.$$

For example, for the mirror of local $\mathbb{P}^2$ we had $Q = \mathbb{N}$, $P \subset \mathbb{N}^3$ generated by $(1,0,0)$, $(0,1,0)$, $(-1,-1,1)$, hence $\widehat{\Bbbk[P]} = \Bbbk[x,y,z][\![t]\!]/(xyz-t)$, and

$$f = 1 + x + y + z - 2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 + \cdots.$$

In general we assume $f = 1 + \sum_{i=1}^r z^{m_i} + g$ with $\bar{m}_i \neq 0$ for all i and $g = \sum_q b_q \cdot z^q \in \widehat{\Bbbk[Q]}$ taking care of the normalization condition.[2] Under this assumption we are going to give an infinite product expansion

$$f = \prod_{\{m \mid \bar{m} \neq 0\}} (1 + a_m z^m)$$

in $\widehat{\Bbbk[P]}$, with each a_m having an interpretation in terms of tropical disks in $M_\mathbb{R}$ with root weight m. Moreover, each coefficient b_q of g has an interpretation in terms of pointed tropical curves of genus zero.[3]

To this end consider the following definition of the *type of a tropical disk*. A *rooted tree* is a partially ordered finite set with a unique maximal element, called the *root vertex*, which is connected and without cycles when viewed as a graph. The *predecessors* of a vertex v are the adjacent vertices that are smaller than v. The minimal elements of a tree are called its *leaves*, so these are the elements without

[2] Note that by the universal nature of Q the sum over z^{m_i} implicitly comprises a universal choice of coefficients.

[3] If σ has several interior integral points the change of vertex formula (3.4) provides a non-trivial identity between expressions labelled by different sets of tropical trees. It would be interesting to give an interpretation of this formula within the following discussion.

predecessors. We require that there are no elements with only one predecessor. In graph theory language this means that the interior vertices are at least trivalent and the leaves are the unique univalent vertices.

We now define types of tropical trees weighted by elements of P. Note that Q can be identified with the submonoid $\{m \in P \,|\, \bar{m} = 0\}$ of P.

Definition 5.1. The *type of a P-labelled tropical disk* is a rooted tree Γ with sets V_Γ of vertices and E_Γ of edges along with a *vertex-labeling* map

$$w : V_\Gamma \longrightarrow P \setminus Q, \quad v \longmapsto m_v$$

fulfilling the following conditions:

1. For any non-leaf vertex $v \in V_\Gamma$ with predecessors $v_1, \ldots, v_\ell$ the balancing condition
$$m_v = m_{v_1} + \cdots + m_{v_\ell}$$
holds.
2. For any vertex v the weights $m_1, \ldots, m_\ell$ of the adjacent predecessor vertices are pairwise distinct.

By abuse of notation we suppress the labelling function in the notation and write just Γ for the type of a tropical disk. The set of non-leaf vertices is denoted $\hat{V}_\Gamma$.

If we take the weight m_Γ of the root vertex in Q rather than in $P \setminus Q$ and otherwise leave the definition unchanged we arrive at the notion of *type of P-labelled pointed rational tropical curve*.

Each type of tropical disk or rational tropical curve determines an isotopy class of traditional tropical curves in $M_\mathbb{R}$ with edges labelled by lifts of the direction vector (an element of M) to P, the labelling of the predecessor vertex. In the disk case one may add another edge to force the balancing condition at the root vertex.

The balancing condition for a tropical disk implies that the labelling function is uniquely determined by its values on the leaf vertices $v_1, \ldots, v_\ell$. In particular, for the weight of the root vertex we have

$$m_\Gamma = m_{v_1} + \cdots + m_{v_\ell}.$$

Let now $S = \{m_1, \ldots, m_r\}$ be the set of exponents m occuring in f with $\bar{m} \neq 0$. For $m \in P$ with $\bar{m} \neq 0$ denote by $\mathscr{T}_m(S)$ the set of types of P-labelled tropical disks Γ with $m_\Gamma = m$ and with leaf labels in S. Similarly $\mathscr{R}_q(S)$ denotes the set of types of P-labelled pointed rational tropical curves with leaf labels in S and $m_\Gamma = q$.

Proposition 5.2. *For $f = 1+\sum_{m\in S} z^m + g$ with $g = \sum_{q\neq 0} b_q z^q$ as above it holds*

$$f = \prod_{\{m\,|\,\bar m\neq 0\}} (1+a_m z^m) \tag{5.1}$$

with

$$a_m = \sum_{\Gamma\in\mathscr{T}_m(S)} (-1)^{|\hat V_\Gamma|} \quad \textit{and} \quad b_q = \sum_{\tilde\Gamma\in\mathscr{R}_q(S)} (-1)^{|V_{\tilde\Gamma}|-1}.$$

Proof. Expanding the infinite product in the statement and gathering according to monomials yields

$$\prod_{\{m\,|\,\bar m\neq 0\}} (1+a_m z^m) = \sum_{m\in P} \left(\sum_{\ell=1}^{\infty} \sum_{\substack{m=m_1+\cdots+m_\ell \\ \Gamma_i\in\mathscr{T}_{m_i}(S)}} (-1)^{|\hat V_{\Gamma_1}|}\cdots(-1)^{|\hat V_{\Gamma_\ell}|} \right) z^m. \tag{5.2}$$

In this expansion ℓ is the number of a_m-terms in the infinite product to be multiplied. Thus the third sum on the right-hand side is over all decompositions $m = m_1+\cdots+m_\ell$ of m into ℓ *pairwise distinct* summands in P. Recall that $\hat V_\Gamma$ is the set of non-leaf vertices. Fix m with $\bar m\neq 0$ now and consider the coefficient of z^m. Then for $\ell\geq 2$ any collection $\Gamma_1,\ldots,\Gamma_\ell$ of types of tropical disks with $m = m_{\Gamma_1}+\cdots+m_{\Gamma_\ell}$ can be merged into a new type of tropical disk $\Gamma\in\mathscr{T}_m(S)$ by connecting the root vertex of each Γ_i by one edge to the root vertex $v_0\in V_\Gamma$. Thus the root vertex of Γ is ℓ-valent with adjacent predecessor trees $\Gamma_1,\ldots,\Gamma_\ell$. Now this merged tree Γ contributes to the coefficient of z^m as one term for $\ell=1$. Since the vertices of Γ other than the root vertex are in bijection with the vertices of $\Gamma_1,\ldots,\Gamma_\ell$ it holds

$$(-1)^{|\hat V_\Gamma|} = -(-1)^{|\hat V_{\Gamma_1}|}\cdots(-1)^{|\hat V_{\Gamma_\ell}|}.$$

Thus each term with $\ell\geq 2$ in the sum of the right-hand side of (5.2) cancels with one term for $\ell=1$. Conversely, if the root vertex of the type of a tropical disk Γ has valency $\ell\geq 2$ then Γ is obtained by this merging procedure. On the right-hand side of (5.2) we are thus left only with those m with $\bar m = 0$ and in addition with those trees with only one vertex. The latter condition means that the root vertex is also a leaf vertex. These terms yield the sum $\sum_{m\in S} z^m$. The terms with $\bar m = 0$ define a power series $1+h\in\widehat{\Bbbk[Q]}$. We have thus shown

$$\prod_{\{m\,|\,\bar m\neq 0\}} (1+a_m z^m) = 1+\sum_{i=1}^{r} z^{m_i} + h,$$

with $h\in\widehat{\Bbbk[Q]}$. Since the left-hand side of this equation is clearly normalized we see that $h=g$. Tropically, the coefficient of z^q in g is the weighted sum of types

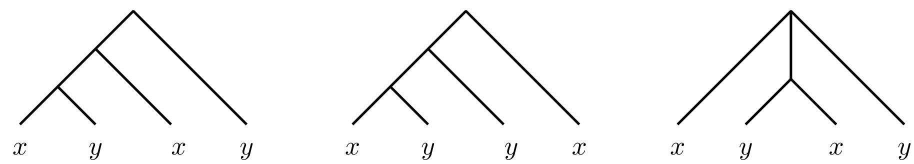

Figure 5.1.

of P-labelled pointed rational tropical curves, with the marked point (of valency $\ell \geq 2$) the merging point of ℓ tropical trees $\Gamma_1, \ldots, \Gamma_\ell$. The balancing condition of an underlying tropical curve at the marked point is the statement

$$\bar{m}_{\Gamma_1} + \cdots + \bar{m}_{\Gamma_\ell} = \bar{m} = 0.$$

□

For $f = 1 + x + y + z + g$ the expansion up to order 4 is

$$\begin{aligned} &(1+x)(1+y)(1+z)(1-xy)(1-yz)(1-xz)(1+x^2y) \\ &\cdot(1+xy^2)\ldots(1+yz^2)(1-x^2y^2)(1-y^2z^2)(1-x^2z^2) \\ &\cdot(1-x^2yz)(1-xy^2z)(1-xyz^2)(1-xz^3)\ldots(1-yz^3) \end{aligned}$$

Figure 5.1 shows the tropical trees contributing to the coefficient $-1 = (-1)^3 + (-1)^3 + (-1)^2$ of x^2y^2. Note that many labelled trees with four leaves are ruled out because of the third condition in Definition 5.1 that no two predecessor subtrees at some vertex be isomorphic.

We finish this section with two remarks on a possible enumerative interpretation of the expansion in terms of tropical disks and trees. First, according to (4.1) we should write the product expansion (5.1) in exponential form. Indeed, we can also write

$$f = \exp\left(\sum_{\{m \mid \bar{m} \neq 0\}} \sum_{\Gamma \in \tilde{\mathscr{T}}_m(S)} \frac{(-1)^{|\hat{V}_\Gamma|}}{|\operatorname{Aut}(\Gamma)|} z^m \right).$$

Here the sum is over the space $\tilde{\mathscr{T}}_m(S)$ of tropical disks with the stability condition Definition 5.1,2 dropped. Expanding exp in a Taylor series the proof is largely the same as the one given, with extra care taken concerning automorphisms.

Second, in log Gromov-Witten theory the log structure on the moduli space only depends on the type of tropical curve associated to a stable log map [GrSi6]. It is tempting to believe in a formulation of the counting problem by a symmetric obstruction theory [BeFa] on a moduli space with a log structure stratified by types of tropical disks, with each stratum contributing $(-1)^{|\hat{V}_\Gamma|}/|\operatorname{Aut}(\Gamma)|$.

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