

Inverse Limits and Topologies of Infinite Graphs ^{*†}

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Abstract

Two of the natural topologies for infinite graphs with edge-ends are ETOP and ITOP . In this paper, we study and characterize them. We show that ITOP can be constructed by inverse limits of inverse systems of graphs with finitely many vertices. Furthermore, as an application of the inverse limit approach, we construct a topological spanning tree in ITOP .

1 Introduction

Studying graphs as topological spaces has a vast number benefits, see [2, 6, 14, 17, 20]. This view allows us to compactify graphs. For instance, considering infinite graphs as compact spaces enables us to define infinite cycles, see [2]. Compactifying infinite graphs is one of the controversial problems in infinite graph theory, see [6].

In 1931, Freudenthal [11] introduced ends of locally compact, connected, locally connected, σ -compact, Hausdorff topological spaces X as points at infinity for compactification purposes. Essentially, Freudenthal's ends are defined as descending sequences $U_1 \supseteq U_2 \supseteq \dots$ of connected open sets with compact boundaries in a such way that $\bigcap \overline{U_i} = \emptyset$. Adding these ends with new appropriate open sets around them to X leads to a new space which is compact. This new compact space is called the *Freudenthal compactification* of X . In 1963, Halin [13], introduced graph-theoretical vertex-ends as equivalence classes of rays independently. Those ends are, in general, distinct from Freudenthal's ends. In 2004, Diestel and Kühn [8] showed that these two kind of ends coincide for locally finite graphs. More precisely, let G be a locally finite graph. Then the geometric realization of G is one-dimensional complex and we compactify G with the Freudenthal compactification and so some topological ends are obtained. Topological ends of G correspond to vertex-ends introduced by Halin.

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It turns out that the Freudenthal compactification for locally finite graphs and the definition of infinite cycles as homeomorphic images of the unit circle S^1 in the Freudenthal compactification of a graph are good approaches to extend extremal finite graph theory for infinite graphs, see [6].

The most commonly used topologies on infinite graphs are TOP , ETOP , ITOP , $\ell\text{-TOP}$, MTOP , VTOP , for comprehensive details see [3, 4]. It is worth mentioning that for a locally finite graph all those topologies coincide. So the importance of studying of them is when a given graph has a vertex of infinite degree. The topology ETOP was first defined by Diestel [7] though it appeared first in [19] by Schulz. Among of all non-trivial topologies for infinite graphs ETOP is the coarsest. The ends considered in ETOP are edge-ends rather than the usual vertex ends. Note that ETOP is always a compact space, see [19, Satz 2.1]. The topology ETOP is not always Hausdorff. By identifying any two points that have the same open neighborhoods and use the quotient topology on ETOP , the topological space ITOP is obtained.¹ In this paper, we reconstruct the topologies ETOP and ITOP with different methods and show that all of them are homeomorphic. First, we introduce a new topology for infinite graphs, namely FCTOP , with respect to edge-ends, which turns out to be equivalent to the topology ETOP . Then we define two families of inverse systems whose inverse limits are homeomorphic to ITOP . Furthermore, as an application of our approach, we will construct topological spanning tree for an infinite graphs with ITOP in Section 5.

2 Preliminaries

We refer readers to Diestel [5] and Munkres [15] for the standard terminologies and notations of graph theory and topology, respectively.

2.1 Graphs

Throughout this paper, graphs are infinite and connected and G will be reserved for graphs with the vertex set $V(G)$ and the edge set $E(G)$. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its *tails*. The union of a ray with infinitely many disjoint finite paths having precisely their first vertex is a *comb* and the last vertices of these paths are *teeth*. Two rays in a graph G are *edge equivalent* if for any finite set F of edges, R_1 and R_2 have a tail in the same component of G without inner points of edges of F . The corresponding edge equivalence classes of rays are the *edge-ends* of G and for a ray R and we show the corresponding edge-end by $[R]$. We denote the set of all edge-end of G by $\Omega'(G)$. It is important to notice that by replacing edge by vertex in the definition of edge-end, we obtain the *vertex-end*, however in this paper we are only concerned with edge-ends. For distinguishing between vertex and edge ends see [12]. For instance, let G be a graph as depicted on Figure 2.1. Then G has exactly one edge-end and the vertex v dominates it. Note that we are not able to separate them by a finite cut.

¹Other names in the literature for ETOP and ITOP are ETOP' and ETOP , respectively, see [6].

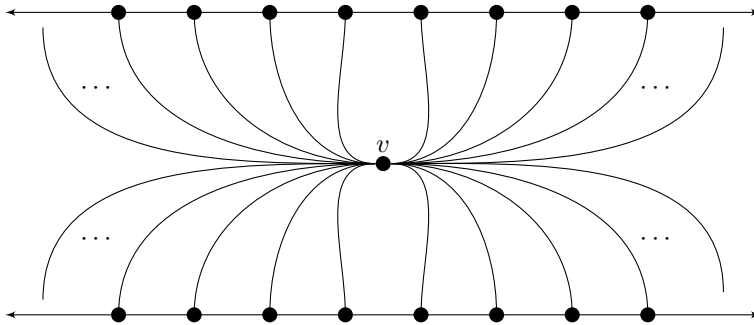


Figure 2.1 a graph with only one edge-end

For a given subset A of vertices of G , we denote the induced subgraph with vertices of A by $G[A]$. Suppose that a pair (A, B) is a partition of the vertices of a graph into two disjoint subsets such that the number of edges between two sides is finite. The set of these edges between A and B is called a *finite cut*. So we represent every finite cut C by a pair (A, B) where A and B are subsets of $V(G)$ such that $A \cap B = \emptyset$ and $V(G) = A \cup B$. Thus with the above notation C is the set of edges which joins $G[A]$ to $G[B]$. We note that the set of all finite cuts with empty forms a vector space over \mathbb{Z}_2 . We denote *finite cut space* by $\mathcal{B}_{\text{fin}}(G)$. Let R be a ray. Then we say that a vertex v *dominates* R if for any finite set F of the set of edges, there is $v - R'$ path in G without inner points of edges of F where R' is a tail of R . So a vertex dominates an end if it dominates the corresponding ray of this end. An edge-end ω *lives* in a component C of G if $V[C]$ contains one ray belonging to C or equivalently each ray. Let F be a subset of $E(G)$. Then by $\overset{\circ}{F}$, we mean all inner points of edges of F .

2.2 Topology

By a basic closed set, we mean the complement of a basic open set in a topological space. For a set X , we denote the power set of X by $\mathcal{P}(X)$. Let X be a space that is the union of the subspaces X_α , for $\alpha \in I$. The topology of X is said to be *coherent* with the subspaces X_α provided a subset C of X is closed in X if $C \cap X_\alpha$ is closed in X_α for each $\alpha \in I$. An equivalent condition is that a set be open in X if its intersection with each X_α is open in X_α . Now we move to topologies of graphs. First the geometric realization of graphs is the one dimensional complex.² We denote the geometric realization without considering its topology of G by $\|G\|$. So we are able to regard inner points of edges of a graph G as points of $\|G\|$. For defining ETOP on $\|G\| \cup \Omega'(G)$, we describe open sets. For each $e \in E(G)$, $\overset{\circ}{e}$ inherits the topology of open interval $(0, 1)$. For any finite set F of edges of G , we remove a finite set X of inner points of edges of F . Suppose that C is a component of $\|G\| \setminus X$. Then $C \cup \{\omega \in \Omega'(G) \mid \omega \text{ lives on } C\} \cup L$ forms an open set, where L is the set of all partial edges like $[a, b)$ and b is the inner point which is picked up from the edge $ac \in F$ with a lying in C . With a similar method, we can define when c lies in C with replacing $(b, c]$ with $[a, b)$. We denote the open set around an end ω with respect to a finite set X of $E(G)$, by $\mathcal{O}_X(\omega)$. The topology generated

²For the definition of the geometric realization, see [16].

by these open sets is called ETOP. For a locally finite graph G , ETOP and the Freudenthal compactification [11] of the 1-complex of the graph G are the same, see [8].

It is worth noting that $(\|G\| \cup \Omega'(G), \text{ETOP})$ is not Hausdorff. The solution for obtaining a Hausdorff space is identifying any two points that have the same open neighborhoods. In other words, we define an equivalence relation between points i.e. for two points $x, y \in \|G\| \cup \Omega'(G)$, we define $x \sim y$ if and only if we cannot separate x and y with a finite subset of edges. For instance, every dominating vertex is equivalent with the corresponding edge ends, see Figure 2.1. Then we use the quotient topology and obtained a new topological space \tilde{G} . We denote this topology by ITOP. Strictly speaking, ITOP is not a topology for an infinite graph, as we are identifying some points.

For defining our topologies, we need inverse systems and inverse limits. Since these terminologies are one of central notations of this paper, let us review here. Let (I, \preceq) denote a directed poset, that is, a set with a binary relation \preceq satisfying reflexivity, antisymmetry, transitivity and moreover if $i, j \in I$ there exists some $k \in I$ such that $i, j \preceq k$. An *inverse system* of topological spaces over I consists of a collection $\{X_i \mid i \in I\}$ of topological spaces indexed by I and a collection of continuous maps $f_{ij}: X_i \rightarrow X_j$ defined whenever $i \preceq j$ such that the diagrams of the form

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ij}} & X_j \\ & \searrow f_{ik} & \swarrow f_{jk} \\ & & X_k \end{array}$$

Figure 2.2

commute whenever they are defined, i.e., whenever $i, j, k \in I$ and $i \preceq j \preceq k$. In addition we assume that f_{ii} is the identity mapping id_{X_i} on X_i . We denote this inverse system over I by (X_i, f_{ij}, I) .

Now, assume that Y be a topological space and $g_i: Y \rightarrow X_i$ is a continuous map for each $i \in I$. The maps g_i s are called compatible if $f_{ij} \circ g_i = g_j$ for every $i, j \in I$. A topological space X with compatible continuous map $f_i: X \rightarrow X_i$ for $i \in I$ is called an *inverse limit* of the inverse system (X_i, f_{ij}, I) , if there is a unique continuous map $f: Y \rightarrow X$ satisfying $f_i \circ f = g_i$.

For comprehensive detail about the inverse limit of topological spaces, see [18]. The following lemma plays a vital role in this paper. In fact this is an immediate corollary of [18, Lemma 1.1.2].

Lemma 1. *If (X_i, f_{ij}, I) is an inverse system of compact Hausdorff topological spaces, then $\varprojlim X_i$ is compact.*

3 New Topologies

In this section, we define a new topology for infinite graphs with edge-ends. To define this topology, we use finite cuts and instead of defining basic open sets for each point of $\|G\| \cup \Omega'(G)$, we introduce basic closed set for them. Then we

introduce two new topological spaces. In order to introduce these new topological spaces, we define two families of auxiliary graphs with two different methods and we show that they constitute inverse systems. We start with the definition of a new topology for infinite graphs.

First for any edge e of G , \mathring{e} is endowed by the open interval $(0, 1)$. For any finite cut $C = (A, B)$ of G , we remove \mathring{C} of G . We now define every component of $G \setminus \mathring{C}$ as basic closed set with respect to C . We need to define a basic closed set for a given end ω . A basic closed around an end ω is $\mathcal{C}_C(\omega) = F \cup \{\omega \in \Omega'(G) \mid \omega \text{ lives on } C\}$ where F is the unique component which ω lives in it. We call the above topology FCTOP . It is worth mentioning that after removing \mathring{C} , we will have a finite number of components.

Recall that for defining ITOP , we identified any two points that have the same open neighborhoods in ETOP . Equivalently, we used an equivalence relation between vertices so that for two vertices we have $x \sim y$ if and only if we cannot separate x and y with a finite cut. Also if we have an end which is dominated by a vertex, see Figure 2.1, then we identify them. Now let us get back to our definition. We need to get a Hausdorff space, but there might be some vertices which do not have any separation by finite cuts and the same problem like ETOP for dominating vertices by some edge ends. We identify these points by defining an equivalence relation on $\|G\| \cup \Omega'(G)$. Now we use the quotient topology on this quotient space. We denote this new space obtained by taking quotient of $\|G\| \cup \Omega'(G)$ by the equivalence relation and the quotient topology on it with \tilde{G} and IFCTOP , respectively.

To show that FCTOP is compact, we need the following famous lemma namely star-comb lemma.

Lemma 2. [5, Lemma 8.2.2] *Let U be an infinite set of vertices of a connected graph G . Then G contains either a comb with all teeth in U or a subdivision of an infinite star with all leaves in U .*

Theorem 3. *If G is a countable graph. Then (G, FCTOP) is a compact space.*

Proof. In order to show the compactness of (G, FCTOP) , we take any collection of basic closed sets $\{C_i\}_{i \in \mathbb{N} \cup \{0\}}$ with the finite intersection property and then we show that the intersection of this collection is not empty. We note that since G is countable, there are countably many basic closed sets. Let $x_0 \in C_0$. Then we can find a point $x_i \in C_0 \cap \dots \cap C_i$. Let U be the collection of all x_i 's with the above property. It follows from Lemma 2 that G contains either a comb with all teeth in U or a subdivision of an infinite star with all leaves in U . First suppose that we have a ray R with all teeth in U . We claim that the end $[R]$ is included $\bigcap C_i$. If every C_i contains a tail of a ray in $[R]$, then we are done. So assume to contrary that C_k has no tail of a ray in $[R]$. Then there are infinitely many vertices of R outside of C_k . We denote them by Y . It follows from the choice of x_i that there is an infinite subset Λ of $\mathbb{N} \cup \{0\}$ such that $x_i \in C_k$ for any $i \in \Lambda$. Let $X := \{x_i\}_{i \in \Lambda}$. It is not hard to see that there are infinitely many disjoint X - Y paths. On the other hand, C_i is a basic closed set which is separated by finitely many edges. It yields a contradiction with infinitely many disjoint X - Y paths from the outside of C_k to the inside of C_k . So the claim is proved.

Now suppose that G contains a subdivision of an infinite star with all leaves in U . Let v be the center of this infinite star. We show that v belongs to $\bigcap C_i$. Again there is C_k such that it does not contain v . There are infinitely many $i \in \mathbb{N} \cup \{0\}$ such that $x_i \in C_k$. Hence v has infinitely many leaves in C_k and it contradicts with being basic closed of C_k . Thus $\bigcap C_i$ is not empty and we deduce that (G, FCTOP) is compact, as desired. \square

A graph G is said *finitely separable* if every two vertices can be separated by some finite set of edges.

Corollary 4. *Let G be a finitely separable 2-connected. Then (G, FCTOP) is compact.*

Lemma 5. [15, Theorem 34.1](Urysohn metrization theorem) *Every regular space with a countable basis is metrizable.*

Note that every Hausdorff compact space is normal and so it is regular. Now let G be a countable graph. Theorem 3 implies that $(\tilde{G}, \text{IFCTOP})$ is a regular space and by theorem 5, we have the following theorem.

Theorem 6. *Let G be a countable graph. Then $(\tilde{G}, \text{IFCTOP})$ is metrizable.*

In the following, we introduce the first family of inverse systems. First, we define a family of auxiliary graphs with finitely many vertices. Next we study these auxiliary graphs and their connection with the primary graph. The following auxiliary graphs were defined for the first time in [14] for extending and generalizing flow theory of finite graphs to infinite graphs.

We can imagine our auxiliary graphs in the following way:

We consider a partition $\{V_1, \dots, V_t\}$ of G such that there are only finitely many cross-edges between these V_i 's. Then we contract all vertices of each partition to a single vertex, but we keep the edges. In other words, every partition with the above property gives a multi-graph with finitely many vertices. Next we define these partitions more precisely.

Definition 7. *Let $M = \{C_1, \dots, C_t\}$ be a finite subset of the space $\mathcal{B}_{\text{fin}}(G)$, where $C_i = (A_i, B_i)$. Define*

$$V(\mathcal{G}_M) = \left\{ X_1 \cdots X_t \mid X_i \in \{A_i, B_i\} \text{ and } \bigcap_{i=1}^t X_i \neq \emptyset \right\}.$$

We add edges between $U_1 = X_1 \cdots X_t$ and $U_2 = X'_1 \cdots X'_t$ for every edge between $\bigcap_{i=1}^t X_i$ and $\bigcap_{i=1}^t X'_i$ in the original graph G .

Every vertex $X_1 \cdots X_t \in V(\mathcal{G}_M)$ defines with $\bigcap_{i=1}^t X_i$ a subset of $V(G)$ and $V(\mathcal{G}_M)$ admits a partition of $V(G)$ and there are finitely many cross-edges between these partition, as we said before the definition.

For simplicity, we define $\Phi(U) = \bigcap_{i=1}^t X_i$, for every vertex $U = X_1 \cdots X_t$ of \mathcal{G}_M .

For a given finite subset M of $\mathcal{B}_{\text{fin}}(G)$, we define a natural map ϕ_M from $\|G\| \cup \Omega'(G)$ to $\|\mathcal{G}_M\|$. First, we define ϕ_M on the set of vertices of G . For every vertex $u \in V$, we associate a unique vertex U of \mathcal{G}_M . Consider a finite cut $C = (A, B)$ in M . By the definition of C , either A or B should contain u and we do this for every finite cut in M . Let X_i be the suitable part containing u

for $i = 1, \dots, t$. We define $\phi_M(u) = X_1 \cdots X_t$. For edge-ends, we can do the same. For a given end ω , one of A or B of a cut (A, B) should contain a tail of a ray corresponding to the end ω and with using an analogous argument, we can build up the unique word containing ω . Now, it is clear how we have to define the set of edges. If $uv \in E(G)$, then $\phi_M(uv) = UV$, where U and V are the corresponding vertices to u and v respectively. Thus we have the following lemma.

Lemma 8. *With the above notations the following holds:*

- (i) *The map ϕ_M is surjective.*
- (ii) *The restriction of ϕ_M from $E(G) \cup \hat{E}(G)$ to $E(\mathcal{G}_M) \cup \hat{E}(\mathcal{G}_M)$ is identity.*

It is worthy to mention that $\phi_M^{-1}(V) \cap \phi_M^{-1}(U) = \emptyset$ for vertices $U \neq V$ of \mathcal{G}_M . Therefore we get a partition for vertices of G .

In the following, we topologize $\|\mathcal{G}_M\|$. First we discuss the case when G is a countable graph. In order to put our topology on $\|\mathcal{G}_M\|$, we need to define topologies for three special subgraphs. The first one is well-known as the Hawaiian earring or infinite earring see [15, Example 1 of page 436], the second one is the finite version of the Hawaiian earring and the last graph is K_2 . The Hawaiian Earring is defined to be a subspace of \mathbb{R}^2 consisting of the union of planar circles c_i of radius $1/i$, tangent to the x -axis at the origin for $i \in \mathbb{N}$. This space is well-known to have interesting properties, see [1]. Note that the Hawaiian earring is a closed and bounded subset of \mathbb{R}^2 and so it is compact; but its fundamental group is uncountable. The finite version of the Hawaiian earring is a subspace of \mathbb{R}^2 consisting of the union of planar circles c_i of radius $1/i$, tangent to the x -axis at the origin for $i \in N$ where N is a finite subset of \mathbb{N} . We consider the same topology for the finite version of the Hawaiian earring. Moreover, every K_2 is endowed with the topology of the closed interval $[0, 1]$. We denote the subgraphs which are homeomorphic to the Hawaiian earring and a finite version of Hawaiian earring by $\mathcal{H}_{\mathbb{N}_0}$ and \mathcal{H}_N , respectively.

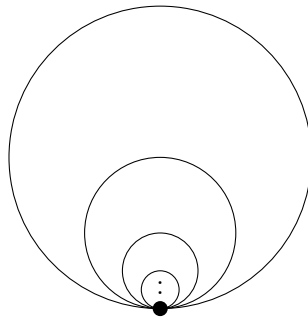


Figure 3.1 The Hawaiian earring

Now, we are ready to topologize $\|\mathcal{G}_M\|$ for a given finite subset M of $\mathcal{B}_{\text{fin}}(G)$. Since $\|\mathcal{G}_M\|$ is a union of finitely many of copies of the subgraphs $\mathcal{H}_{\mathbb{N}_0}$, \mathcal{H}_N and K_2 , we will not have any problem to use the coherent topology here, i.e. U is an open (closed) set in $\|\mathcal{G}_M\|$ if and only if the intersection of U with each of

these subspaces is an open (closed) set. Therefore we obtain a compact space, as $\|\mathcal{G}_M\|$ is a union of finitely many compact spaces. Note that $\|\mathcal{G}_M\|$ is a Hausdorff space as well. We denote this topology by τ_M , as $\|\mathcal{G}_M\|$ is constructed by the finite set of cuts M .

Now, let G be an uncountable graph. Then we cannot embed the Hawaiian earring in \mathbb{R}^2 , i.e. we are not able to embed uncountable many loops in \mathbb{R}^2 and use the induced topology of \mathbb{R}^2 . Now we define a new topological space for the uncountable version of the Hawaiian earring.³ Suppose that $I = [0, 1]$. Fix a point of S^1 say x_0 . Consider the quotient space $(S^1 \times I)/(\{x_0\} \times I)$. So we have an injective map ι from this space to the product of S^1 , I times. Let X_i be homeomorphic to S^1 for every $i \in I$. Then we have the following map.

$$\begin{aligned} \iota: \frac{S^1 \times I}{\{x_0\} \times I} &\longrightarrow \prod_{i \in I} X_i \\ (x, i) &\longmapsto \left(j \rightarrow \begin{cases} x_0 & \text{for } i \neq j \\ x & \text{for } i = j \end{cases} \right) \end{aligned}$$

We claim that the image ι i.e. $\{(x_j)_{j \in I} \mid \exists \text{ at most one } j \in I \text{ with } x_j \neq x_0\}$ is a closed set in $\prod_{i \in I} X_i$. Assume that $(x_\alpha)_{\alpha \in I}$ is an element of the complement of $\text{Im}(\iota)$. So there are indices α_0 and α_1 in I such that both of x_{α_0} and x_{α_1} are not x_0 . We can find open sets O_{α_0} and O_{α_1} containing x_{α_0} and x_{α_1} without x_0 , respectively. Now put $O := \prod O_i$ where $O_i = X_i$ for any $i \neq \alpha_0, \alpha_1$. It is not hard to see that O is an open set of the complement of $\text{Im}(\iota)$ containing $(x_\alpha)_{\alpha \in I}$ and the claim is proved. Since $\text{Im}(\iota)$ is a closed subset of a compact space, it is compact. We now define the $\text{Im}(\iota)$ as the uncountable Hawaiian earring and we denote it by $\mathcal{H}_{2^{\aleph_0}}$. With an analogous method we define \mathcal{H}_κ , for an arbitrary cardinal κ .

Let us get back to our objective which is defining a topology for \mathcal{G}_M . Whenever G is uncountable, we benefit from \mathcal{H}_κ for a suitable κ and like the above case we obtain a compact Hausdorff topology for $\|\mathcal{G}_M\|$.

Now we summarize all the above discussion on the following theorem.

Theorem 9. *Let M be a finite subset of $\mathcal{B}_{\text{fin}}(G)$. Then $(\|\mathcal{G}_M\|, \tau_M)$, is a compact Hausdorff space.*

Recall that we defined the map ϕ_M from $(\|G\| \cup \Omega'(G), \text{FCTOP})$ to $(\|\mathcal{G}_M\|, \tau_M)$ for a finite subset M of $\mathcal{B}_{\text{fin}}(G)$.

The next theorem reveals the relationship between the space $(\|G\| \cup \Omega'(G), \text{FCTOP})$ with the space $(\|\mathcal{G}_M\|, \tau_M)$.

Theorem 10. *The map $\phi_M: (\|G\| \cup \Omega'(G), \text{FCTOP}) \rightarrow (\|\mathcal{G}_M\|, \tau_M)$ is continuous.*

Proof. First, assume that I is an open set around an inner point x of an edge e . Without loss of generality, we can assume that I is an open interval and $I \subset e$. Since ϕ_M is the identity map from $E(G) \cup \hat{E}(G)$ to $E(\mathcal{G}_M) \cup \hat{E}(\mathcal{G}_M)$, the preimage of I is an open set in $\|G\|$. So $\phi_M^{-1}(I)$ is open in $\|G\| \cup \Omega'(G)$. Now

³For the sake of simplicity, we use a closed interval whose the cardinal is 2^{\aleph_0} , however the method works for any compact space with an arbitrary cardinal κ .

let O be a basic open set containing a vertex v of \mathcal{G}_M . If the degree of v is finite, then the preimage is a union of some vertices and some parts of edges $\{e_1, \dots, e_{m-1}\} \cup \{e'_1, \dots, e'_t\}$ and entire edges $\{e_m, \dots, e_n\}$, where the boundary of $\phi_M^{-1}(O)$ touches each e_i twice and each e'_j once for $i = 1, \dots, m-1$ and $j = 1, \dots, t$, see Figure 3.2. We may suppose that the set $\{e'_1, \dots, e'_t\}$ forms a finite cut which can be presented by $C = (A, B)$. In addition we can assume that the preimage of O is contained in $G[A] \cup C$. We show that the complement of $\phi_M^{-1}(O)$ is a closed set. Note that $G[B]$ is a union of finitely many components. For obtaining the complement of O , we need to add some partial edge to the closure of $G[B]$ which are some parts of edges e'_1, \dots, e'_t and we might need to add some parts of the rest of edges. But all of these partial edges are closed with FCTOP , see Figure 3.2.

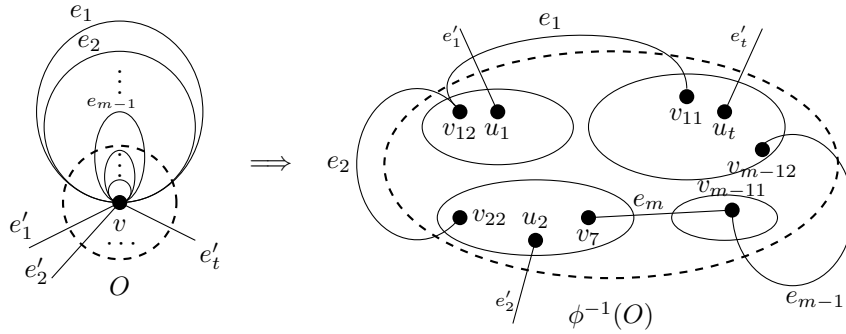


Figure 3.2

More precisely, each of e'_i can be regarded as the union of $(u_i, w_i) \cup [w_i, z_i)$ so that (u_i, w_i) is included in O . In addition, every e_i for $i = 1, \dots, m-1$ is divided by three intervals (v_{i1}, w_{i1}) , $[w_{i1}, w_{i2}]$ and (w_{i2}, v_{i2}) such that we have $(v_{i1}, w_{i1}) \cup (w_{i2}, v_{i2}) \subseteq O$ and also $\{e_{m+1}, \dots, e_n\}$ is included in O . Thus the complement of $\phi^{-1}(O)$ is

$$D \cup \bigcup_{i=1}^{m-1} [w_{i1}, w_{i2}] \cup \bigcup_{i=1}^t [w_i, z_i),$$

where D is the union of $G[B]$ with all ends which live in B . Let $B = C_1 \cup \dots \cup C_l$ where C_i is a component of $G \setminus \mathring{C}$. Let an end ω live in C_i . Note that the union C_i with all ends which live in C_i is $\mathcal{C}_C(\omega)$ that is closed by definition.

Now suppose that the degree of v is infinite. So there is an infinite Hawaiian earring which occurs at v . It is important to notice that there are only finitely many edges incident to v meeting O . In this case there are infinitely many edges inside of $\phi_M^{-1}(O)$ in spite of the last case. Hence with using a similar method which we used in the preceding case, we can show that the complement of $\phi_M^{-1}(O)$ is a closed set in $\|G\| \cup \Omega'(G)$. \square

Lemma 11. *The map $\phi_M: (\|G\| \cup \Omega'(G), \text{FCTOP}) \rightarrow (\|\mathcal{G}_M\|, \tau_M)$ is closed.*

Proof. Let K be a basic closed set around a vertex v of G . The image of K by ϕ_M contains finitely many vertices. More precisely, $\phi_M(K)$ is a union of finitely many Hawaiian earrings, finite Hawaiian earrings and finitely many copies of K_2 .

Note that it does not contain any partial edges and so it is closed. Let K be a closed set around an inner point x which is included in an edge e . Then since ϕ_M is identity on $E(G) \cup \mathring{E}(G)$, the set $\phi_M(K)$ is closed. Now let ω be an edge-end and let $\mathcal{C}_C(\omega)$ be an arbitrary basic closed set around ω with respect to the finite cut C in M . We show that $\phi_M(\mathcal{C}_C(\omega))$ is closed in $(\|\mathcal{G}_M\|, \tau_M)$. The image of $\mathcal{C}_C(\omega)$ contains finitely many vertices. Again $\phi_M(\mathcal{C}_C(\omega))$ is a union of finitely many Hawaiian earrings, finite Hawaiian earrings and some copies of K_2 . Note that it does not contain any partial edges. Thus the image of the basic closed set $\mathcal{C}_C(\omega)$ is closed, as desired. \square

We accomplished to study the connection between topological spaces $(\|G\| \cup \Omega'(G), \text{FCTOP})$ and $(\|\mathcal{G}_M\|, \tau_M)$. Next we investigate the graph \mathcal{G}_M for different finite subsets M of $\mathcal{B}_{\text{fin}}(G)$.

Lemma 12. *Let $M \subseteq M'$ be two finite subsets of $\mathcal{B}_{\text{fin}}(G)$. Then there exists a continuous map $\psi_{M'M}: (\|\mathcal{G}_{M'}\|, \tau_{M'}) \rightarrow (\|\mathcal{G}_M\|, \tau_M)$.*

Proof. Let $V' \in V(\mathcal{G}_{M'})$. Then $\Phi(V')^4$ is a subset of $V(G)$ which is obtained by elements of M' . On the other hand, every element of M is an element of M' . Hence we can find a word W in \mathcal{G}_M such that $\Phi(V')$ is included in $\Phi(W)$. Thus we assign the vertex V' of $\mathcal{G}_{M'}$ to the vertex W of \mathcal{G}_M . Note that every \mathcal{G}_M contains all edges of G . So we have a map $\psi_{M'M}: \mathcal{G}_{M'} \rightarrow \mathcal{G}_M$, where ψ carries V' to W and every edge e to e . We now show that the map $\psi_{M'M}$ is continuous. Since M is subset of M' , the partition of $\Phi(V(\mathcal{G}_{M'}))$ is a finer partition of $V(G)$ then $\Phi(V(\mathcal{G}_M))$. In fact we contract it to smaller pieces and the contraction is a continuous map such that the new partitions are contained in the old partitions. More precisely, let K be a closed set in $(\|\mathcal{G}_M\|, \tau_M)$. Since ϕ_M is continuous, $\phi_M^{-1}(K)$ is closed in $(\|G\| \cup \Omega'(G), \text{FCTOP})$ and it follows from Lemma 11 that $\phi_{M'}^{-1}(\phi_M^{-1}(K))$ is closed in $(\|\mathcal{G}_{M'}\|, \tau_{M'})$. So it is enough to show that $\psi_{M'M} \circ \phi_{M'} = \phi_M$. Since all maps are the identity on edges, it is enough to show equality for vertices and ends. By definition of $\phi_{M'}$, every vertex v of G maps to the unique vertex U of \mathcal{G}_M and it follows from definition of $\psi_{M'M}$ that the image of U by $\psi_{M'M}$ is exactly the same as the image v by ϕ_M . Now let ω be an end of G . Then with regarding to the cuts of the set M' , we can build up the unique word U which is a vertex of $\mathcal{G}_{M'}$.⁵ Again by the definition of $\psi_{M'M}$ the image of U by $\psi_{M'M}$ is equal to the image of ω by ϕ_M . Hence $\psi_{M'M}$ is continuous, as desired. \square

As a consequence of Lemma 1 and the previous lemma, we have the following theorem.

Theorem 13. *The system $(\|\mathcal{G}_M\|, \psi_{MM'}, \Gamma)$ is an inverse system, where Γ is the set of all finite subset of $\mathcal{B}_{\text{fin}}(G)$ and $M, M' \in \Gamma$ and moreover the space $\varprojlim \|\mathcal{G}_M\|$ is a compact Hausdorff space.*

Proof. In Lemma 12, we show that $\psi_{MM'}: \|\mathcal{G}_M\| \rightarrow \|\mathcal{G}_{M'}\|$ for $M' \subseteq M$ is continuous. Let $M_1 \subseteq M_2 \subseteq M_3$ be three finite subsets of $\mathcal{B}_{\text{fin}}(G)$. We have to show that $\psi_{M_1M_3} = \psi_{M_2M_3} \circ \psi_{M_1M_2}$. Note that every vertex of \mathcal{G}_{M_1} admits a partition of the set of vertices of G . In fact we contract this component to this

⁴For the definition of Φ , see after Definition 7.

⁵For building up the word, see the proof of Lemma 8.

vertex. Since $M_1 \subseteq M_2 \subseteq M_3$, we can deduce that $\Phi(V(\mathcal{G}_{M_i}))$ is a finer partition than $\Phi(V(\mathcal{G}_{M_{i-1}}))$, for $i = 2, 3$. Suppose that $\{V_1, \dots, V_\ell\} = V(\mathcal{G}_{M_2})$. Let $\{V_{i_1}, \dots, V_{i_\ell}\}$ be all vertices of \mathcal{G}_{M_2} in such a way that each $\Phi(V_{i_j})$ is contained in $\Phi(V)$ for a vertex $U \in V(\mathcal{G}_{M_1})$ for $j = 1, \dots, \ell$. We now contract all vertices $\{V_{i_1}, \dots, V_{i_\ell}\}$ to obtain U . With a similar method, we are able to contract the finer partition corresponding $\Phi(V(\mathcal{G}_{M_1}))$ to get the partition corresponding $\Phi(V(\mathcal{G}_{M_2}))$ and again contract to get the partition corresponding $\Phi(V(\mathcal{G}_{M_3}))$ or independently we can contract the partition corresponding $\Phi(V(\mathcal{G}_{M_1}))$ to get the partition corresponding $\Phi(V(\mathcal{G}_{M_3}))$. This shows that the above diagram is commutative. Now Lemma 1 completes the proof. \square

Now we introduce the other family of inverse system. First, we define our auxiliary graphs.

Definition 14. *Let $E \in \mathcal{P}(E(G))$ be a finite set. Then we remove E and we contract all vertices and edges of each component to a vertex.⁶ Now for every edge in E , we join the corresponding vertices in the new graph. We denote this new finite graph by $\mathcal{G}.E$.*

Now we are ready to topologize $\|\mathcal{G}.E\|$ for the auxiliary graph $\mathcal{G}.E$. Every edge of $\mathcal{G}.E$ is endowed by the topology of the closed unit interval $[0, 1]$. The topology on $\|\mathcal{G}.E\|$ is the coherent topology with all edges which is exactly the same as one complex topology here. Now suppose that $E' \subseteq E$ are two finite subsets of $E(G)$.

The definition of $\mathcal{G}.E$ leads to a map $f_{EE'}: \|\mathcal{G}.E\| \rightarrow \|\mathcal{G}.E'\|$. Every vertex of $\mathcal{G}.E$ corresponds a component of $G \setminus E$. Thus this component is contained a component of $G \setminus E'$ and so it defines $f_{EE'}$ on vertices of the graph $\mathcal{G}.E$. So we only need to define $f_{EE'}$ on $E \setminus E'$. Each of these edges has to be a component of $G \setminus E'$ and so $f_{EE'}$ carries this edge to the corresponding vertex of its component.

Note that each $\|\mathcal{G}.E\|$ is a compact Hausdorff space and it is not hard to see that $f_{EE'}$ is continuous. An analogous argument of Theorem 13 yields the following theorem.

Theorem 15. *The system $(\|\mathcal{G}.E\|, f_{EE'}, \Gamma)$ is an inverse system, where Γ is the set of all finite sets of edges and $E, E' \in \Gamma$ and moreover the space $\varprojlim \|\mathcal{G}.E\|$ is compact.*

4 Reconstruction of Topologies

In this section, we study connections between the following topological spaces: $(\|G\| \cup \Omega'(G), \text{ETOP})$, $(\|G\| \cup \Omega'(G), \text{FCTOP})$, $(\tilde{G}, \text{IFCTOP})$, (\tilde{G}, ITOP) , $\varprojlim \|\mathcal{G}.E\|$ and $\varprojlim \|\mathcal{G}_M\|$. In particular we show that the four last topological spaces are homeomorphic.

We start with the following theorem.

Theorem 16. *Let G be an arbitrary graph. Then topologies FCTOP and ETOP coincide.*

⁶So all ends which live on that component are corresponded to a vertex.

Proof. In order to show that they coincide, we have to prove that every basic closed set in ETOP is closed in FCTOP and vice versa. First let $\tilde{G} \setminus \mathcal{O}_X(\omega)$ be a basic closed set in ETOP, where $\mathcal{O}_X(\omega) = F \cup \{\omega \in \Omega'(G) \mid \omega \text{ lives on } C\} \cup F'$ and $X = \{x_1, \dots, x_n\}$ are inner points of $\{e_1, \dots, e_n\}$ of edges such that $x_i \in \mathring{e}_i$ for $i = 1, \dots, n$ and F is a component of $\tilde{G} \setminus \{e_1, \dots, e_n\}$ and F' is a finite set of partial edges as we defined in Section 2.2. So we can suppose that $\tilde{G} \setminus \mathcal{O}_X(\omega) = F_1 \cup \dots \cup F_t$, where each F_i is the topological components of $\tilde{G} \setminus X$ except $\mathcal{O}_X(\omega)$ and the corresponding inner points of X . Without loss of generality, we can assume $\{x_1, \dots, x_t\}$ are inner points separating F_i 's from $\mathcal{O}_X(\omega)$. Now consider the edges containing inner points $\{x_1, \dots, x_r\}$, say $C = \{e_1, \dots, e_r\}$. Hence C forms a finite cut and each F_i is a topological component, for $i = 1, \dots, t$. Thus every F_i is a closed set in FCTOP, as desired.

Now suppose that $C = (A, B)$ is an arbitrary finite cut and F_1, \dots, F_s are components after removing \mathring{C} . For a given $\omega \in \Omega'(G)$, let F_1 be the component which contains a tail of a ray of ω and let $\mathcal{C}_C(\omega)$ be a basic closed set around ω . Assume that $X = \{x_1, \dots, x_s\}$ are arbitrary inner points of edges of C and let $\mathcal{O}_X(\omega)$ be the corresponding basic open set containing F_1 . On the other hand, the union of all the other components of $G \setminus X$ except $\mathcal{O}_X(\omega)$ is an open set in ETOP. Thus the union of $(G \setminus \mathcal{O}_X(\omega))$ with \mathring{C} is an open set in ETOP. Therefore $\mathcal{C}_C(\omega)$ is closed in ETOP, as it is the complement of $(G \setminus \mathcal{O}_X(\omega)) \cup \mathring{C}$. \square

Now we have the following lemma.

Lemma 17. [19, Satz 2.1] *Let G be an infinite graph. Then $(\|G\| \cup \Omega'(G), \text{ITOP})$ is a compact Hausdorff space.*

In Theorem 10, we proved that $\phi_M: (\|G\| \cup \Omega'(G), \text{FCTOP}) \rightarrow (\|\mathcal{G}_M\|, \tau_M)$ is continuous. And since $(\tilde{G}, \text{IFCTOP})$ is a quotient space of $(\|G\| \cup \Omega'(G), \text{FCTOP})$, there is a continuous map from $(\tilde{G}, \text{IFCTOP})$ to $(\|\mathcal{G}_M\|, \tau_M)$. For the sake of simplicity, we call this map also ϕ_M .

Theorem 18. *Let G be an arbitrary infinite graph. Then the following topological spaces are homeomorphic.*

- (i) (\tilde{G}, ITOP)
- (ii) $(\tilde{G}, \text{IFCTOP})$
- (iii) $\varprojlim \|\mathcal{G}_M\|$
- (iv) $\varprojlim \|\mathcal{G}.E\|$

Proof. We prove it according to the following diagram:

$$(i) \iff (ii) \iff (iii) \iff (iv)$$

We show that topologies ITOP and IFCTOP are equivalent on \tilde{G} . We consider quotient maps $\pi_1: (\|G\| \cup \Omega'(G), \text{ETOP}) \rightarrow (\tilde{G}, \text{ITOP})$ and $\pi_2: (\|G\| \cup \Omega'(G), \text{FCTOP}) \rightarrow (\tilde{G}, \text{IFCTOP})$ and so Theorem 16 completes this part.

Now we show that (ii) \iff (iii). We use the universal property of the inverse limit of topological spaces. Let M_1 and M_2 be two finite subsets of $\mathcal{B}_{\text{fin}}(G)$ such that $M_2 \subseteq M_1$. Note that it follows from Lemma 8 and Theorem 10 that

we have the continuous surjective maps ϕ_{M_1} and ϕ_{M_2} from \tilde{G} to \mathcal{G}_{M_1} and \mathcal{G}_{M_2} respectively, as in the following commutative digram.

$$\begin{array}{ccc}
 & \tilde{G} & \\
 \phi_{M_1} \swarrow & \downarrow h & \searrow \phi_{M_2} \\
 & \varprojlim \|\mathcal{G}_M\| & \\
 \swarrow & & \searrow \\
 \|\mathcal{G}_{M_1}\| & \xrightarrow{\psi_{M_1 M_2}} & \|\mathcal{G}_{M_2}\|
 \end{array}$$

Figure 4.1

By the universal property of the inverse limit, there is a unique continuous map $h: \tilde{G} \rightarrow \varprojlim \|\mathcal{G}_M\|$ and moreover since each ϕ_M is surjective for finite subset $M \subseteq \mathcal{B}_{\text{fin}}(G)$, it follows from [18, Corollary 1.1.6] that the map h is surjective. Next we show that h is injective as well. Assume that there are two distinct points x and y belonging to \tilde{G} such that $h([x]) = h([y])$. There is a finite subset $M \subseteq \mathcal{B}_{\text{fin}}(G)$ such that the images x and y in \mathcal{G}_M are distinct and we deduce that the images $[x]$ and $[y]$ by h are different, as the above diagram is commutative. Note that Theorem 3 implies that \tilde{G} with FCTOP is a compact space and any continuous bijection from a compact space to a Hausdorff space is a homeomorphism map. So \tilde{G} is homeomorphic to $\varprojlim \|\mathcal{G}_M\|$, as desired.

In order to show (iii) \iff (iv), we find a continuous bijection map between spaces $\varprojlim \|\mathcal{G}_M\|$ and $\varprojlim \|\mathcal{G}.M\|$. Since this map is a continuous map from a compact space to a Hausdorff space, this would complete the prove. Suppose that Σ and Γ are sets of finite subsets of $E(G)$ and $\mathcal{B}_{\text{fin}}(G)$, respectively. Let $E \in \Sigma$. Then assume that M is the set $\{C \in \mathcal{B}_{\text{fin}}(G) \mid C \subseteq E\}$. Let V be a vertex of \mathcal{G}_M . Then every vertex of G in $\Phi(V)$ belongs to a component of $G \setminus E$. So we have a map $g: \mathcal{G}_M \rightarrow \mathcal{G}.E$ where g carries each vertex V to the contraction of the corresponding component and g contracts every edge of \mathcal{G}_M out side of M to the vertex of $\mathcal{G}.M$ corresponding with a suitable component of $G \setminus E$. It is not hard to see that the extension of $g: \|\mathcal{G}_M\| \rightarrow \|\mathcal{G}.E\|$ is a continuous map. Similarly, we define $\|\mathcal{G}_{\tilde{M}}\|$ and \tilde{g} for $\tilde{E} \subseteq E$. Thus we have the following commutative diagram:

$$\begin{array}{ccc}
 \|\mathcal{G}_M\| & \xrightarrow{g} & \|\mathcal{G}.E\| \\
 \psi_{M\tilde{M}} \downarrow & & \downarrow f_{E\tilde{E}} \\
 \|\mathcal{G}_{\tilde{M}}\| & \xrightarrow{\tilde{g}} & \|\mathcal{G}.\tilde{E}\|
 \end{array}$$

Figure 4.2

Each $g: \|\mathcal{G}_M\| \rightarrow \|\mathcal{G}.E\|$ induces a compatible continuous surjective map from $\varprojlim \|\mathcal{G}_M\|$ to $\|\mathcal{G}.E\|$. It follows from [18, Corollary 1.1.6] that the corresponding map which induces the mapping $\theta: \varprojlim \|\mathcal{G}_M\| \rightarrow \varprojlim \|\mathcal{G}.M\|$ is a surjective continuous map. Now we show that θ is injective. Assume that $x = (x_\alpha)_{\alpha \in I}$ and $y = (y_\alpha)_{\alpha \in I}$ are distinct in $\varprojlim \|\mathcal{G}_M\|$ such that $\theta((x_\alpha)_{\alpha \in I}) = \theta((y_\alpha)_{\alpha \in I})$. Since x and y are different, there is $\alpha_0 \in I$ such that x_{α_0} and y_{α_0} are different points in $\|\mathcal{G}_{M_{\alpha_0}}\|$. Then there is a finite cut C in M_{α_0} which separates x and y . Thus the images x and y in $\|\mathcal{G}.C\|$ are different and so $\theta(x) \neq \theta(y)$. The other cases are similar. Thus we found a continuous bijection between $\varprojlim \|\mathcal{G}_M\|$ and $\varprojlim \|\mathcal{G}.M\|$. \square

5 Topological Spanning Trees in ITOP

The aim of this section is to show how the auxiliaries graphs defined in the third section can be utilized to investigate topological spanning trees in (\tilde{G}, ITOP) . We first review some notations and definitions regarding topological spanning trees in (\tilde{G}, ITOP) .

An *arc* and a *circle* in the space (\tilde{G}, ITOP) is a subspace homeomorphic to the closed interval $[0, 1]$ and the unit circle S^1 , respectively. A subspace H of \tilde{G} is said a *standard subspace* if it is the closure of some subgraph of G .

Definition 19. *A topological spanning tree of \tilde{G} is an arc-connected standard subspace T of \tilde{G} that contains every vertex of \tilde{G} but contains no circle.*

We note that ITOP is obtained by taking quotient of ETOP and so ITOP is compact. since a topological spanning tree contains the class of every vertex of \tilde{G} , it should have every end as well.

We now need another terminology. A *continuum* is a compact connected Hausdorff space.

Lemma 20. [10, Problem 6.3.11] *Every locally connected metric continuum is arc-connected.*

Now, suppose that G is a countable graph and H is a connected standard subspace of (\tilde{G}, ITOP) . It follows from Theorem 3 that H is compact. Then by Theorem 6, H is metrizable. Thus H is connected metric continuum and Lemma 20 implies that H is arc-connected. If we summarize the above discussion, then we have the following theorem.

Theorem 21. *If G is countable, then every connected standard subspace of (\tilde{G}, ITOP) is arc-connected.*

The following well-known lemma is important. It can be found in [5] with a different formulation.

Lemma 22. *A standard subspace H of \tilde{G} is arc-connected if and only if H contains an edge from every finite cut of G of which it meets both sides.*

Proof. First let H be arc-connected. Then assume to the contrary that G has a finite cut $C = (A, B)$ which both A and B meet H such that H has no edge in C . Thus one can see that $H \subseteq \tilde{G} \setminus \overset{\circ}{C}$. On the other hand, we know that $\tilde{G} \setminus \overset{\circ}{C}$

is equal to $\overline{G[A]} \cup \overline{G[B]}$. Now we claim that this union is disjoint. Otherwise there is an element $x \in \overline{G[A]} \cap \overline{G[B]}$. Therefore the class of x is the class of an end ω in \tilde{G} , as $G[A] \cap G[B] = \emptyset$. Pick up an arbitrary inner point from each edge in the finite cut C . Let $\mathcal{O}(\omega)$ be a basic open neighbourhood of ω in ETOP containing $G[B]$ with respect to these inner points. Hence $\mathcal{O}(\omega)$ has no intersection with $G[A]$ and so the claim is proved. Since \tilde{G} is a disjoint union of closed sets, \tilde{G} is not connected and so is not arc-connected and it yields a contradiction.

Conversely, suppose that $H = \overline{(X, D)}$, where $X \subseteq V(G)$ and $D \subseteq E(G)$. Assume to the contrary that H is not arc-connected and equivalently by Theorem 21, we can assume that H is not connected. Let H be the disjoint union of open sets O_1 and O_2 and set $X_i := O_i \cap X$. Let C_1 be a component of X_1 and let P be a maximal edge-disjoint C_1 - X_2 paths. If there is a component of X_2 such that there are only finitely paths of P between this component and C_1 , then we have a finite cut between this component and C_1 . By the assumption, H has to meet this finite cut and we get a contradiction with $H = O_1 \cup O_2$. Otherwise there are infinitely many paths between C_1 and each component of X_2 . Choose from each path a vertex. So we have infinity many vertices. It follows from Lemma 2 that C_1 contains either an end ω or a vertex v with an infinite degree. If C_1 has a vertex v of the infinite degree, then we are not able to separate v from each vertex of any component of X_2 and a contradiction is obtained. So we can assume that there is an end which lives in C_1 . With a similar argument, we can show that any component of X_2 has an end. Therefore there is an end belonging to O_1 and O_2 and it yields a contradiction. \square

A strategy for finding a topological spanning tree in \tilde{G} is investigating spanning trees in each \mathcal{G}_M for every finite set M of $\mathcal{B}_{\text{fin}}(G)$ and extending this spanning tree to a topological spanning tree in \tilde{G} .

Theorem 23. *Let G be a countable graph. Then (\tilde{G}, ITOP) contains a topological spanning tree.*

Proof. We are going to construct trees in our inverse system inductively and we show that the limit of these trees is our required topological spanning tree. Let M be a finite subset of $\mathcal{B}_{\text{fin}}(G)$ and $C \notin M$ be a finite cut. We set $M' = M \cup \{C\}$. Then we show that there exists a spanning tree T_M of \mathcal{G}_M such that $E(T_{M'}) \cap E(\mathcal{G}_M) = E(T_M)$. Suppose that V_1, \dots, V_t are vertices of \mathcal{G}_M . Thus the set of $\{\Phi(V_i) \mid i = 1, \dots, t\}$ is a partition of the vertex set of G and so we have $\bigcup_{i=1}^t \Phi(V_i) = V(G)$. Adding the cut C refines the partition $\{\Phi(V_1), \dots, \Phi(V_t)\}$. We notice that $E(\mathcal{G}_M) = E(\mathcal{G}_{M'})$. Thus we are able to find the edges of T_M in $\mathcal{G}_{M'}$. We now add some edges of $\mathcal{G}_{M'}$ to T_M to assure that we have a tree. Let us denote the new tree with T'_M . We set $N_j = \{C_i \mid i \leq j\}$. Define

$$T := \overline{\bigcup_{i \in \mathbb{N}} E(T_{N_i})}.$$

We claim that T is a topological spanning tree of \tilde{G} . In order to show that T is arc-connected, we invoke Lemma 22. We have to show that every finite cut of G contains an edge from T . By definition of the graphs \mathcal{G}_M and T_M , we picked an edge up from each finite cut of M . Next we show that T contains no circle. Assume to contrary that T contains a circle C . Let u, v be two vertices of C .

Then there exists a finite cut F separating u and v . So we can choose M large enough that M contains F . Suppose that H is a fundamental cut with respect to T_M of \mathcal{G}_M separating u from v . It is important to notice that H gives us a finite cut of G . Since C meets H , it follows from Lemma 22 that C should contain an edge from H . Let $e \in C \cap H$. Then since $C \setminus e$ is still arc-connected, Lemma 22 implies that $C \setminus e$ meets H . Thus we can conclude that T has two edges in the finite cut H . Therefore we have a contradiction, as by definition one can see that $E(T_M) = E(T) \cap E(\mathcal{G}_M)$ and we picked only one edge up from each finite cut of \mathcal{G}_M . \square

We finish our paper with the following final remark.

Remark 24. *We have defined ITOP as the quotient topology of ETOP and in the above we constructed a topological spanning in \tilde{G} as a limit of spanning trees. In our proof, we benefit so much from the properties of ETOP and we cannot replace it with the others topologies. For instance, if we apply the quotient topology on TOP, not necessarily there is a topological spanning tree on \tilde{G} . Diestel and Kühn have discovered a counterexample that shows that the quotient topology of TOP does not contain any topological spanning tree, see [9, Corollary 3.5].*

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