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Worldsheet operator product expansions and p-point functions in $\mathrm{AdS}_3/\mathrm{CFT}_2$

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Abstract

We construct the operator product expansions (OPE) of the chiral primary operators in the worldsheet theory for strings on $AdS_3 \times S^3 \times T^4$. As an interesting application, we will use the worldsheet OPEs to derive a recursion relation for a particular class of extremal p-point correlators on the sphere. We compare our result with the corresponding recursion relation previously found in the symmetric orbifold theory on the boundary of AdS_3.

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Contents

1 Introduction

In the recent years, much progress has been made in matching correlation functions in the AdS_3/CFT_2 correspondence [1]. In the symmetric product orbifold theory on the boundary of the AdS_3 space, two- and three-point functions of single-cycle twist operators were computed in [2, 3]. In [4], this analysis was extended to some simple four-point functions, and recursion relations were found for some extremal p-point correlators. In the dual worldsheet theory for string theory on $AdS_3 \times S^3 \times T^4$, the two- and three-point correlators of chiral primary operators were derived in [5, 6] (see also [7, 8]) and intriguing agreement with the dual boundary correlators was found (Later, agreement with supergravity was achieved in [9], see also [10, 11, 12] for earlier work). Recently, in [13], (the one-particle contributions of) some extremal four-point correlators have been computed on the worldsheet using a general method for $SL(2)$ correlation functions developed in [14]. Again, agreement was found with the corresponding boundary result of [4]. Even though the string theory/supergravity and field theory correlators are computed at different points in the moduli space, they must and do agree as predicted by the non-renormalization theorem of [15]. This theorem states that all three-point functions as well as all extremal p-point functions ($p > 3$) of chiral primary operators are protected along the moduli space [15].

In this paper we extend the analysis of [13] by deriving a recursion relation for higher p-point correlators in the worldsheet theory. Such p-point functions may be factorized by means of *worldsheet* operator product expansions (OPE), which should not be mixed up with their dual spacetime OPEs. Their general properties were discussed in [16] for string theory on a general $AdS_{d+1} \times W$ background. Here we specialize to $AdS_3 \times S^3 \times T^4$ and compute the worldsheet operator product expansions of chiral primary operators in the associated $H_3^+ \times SU(2)$ Wess-Zumino-Witten (WZW) model. The chiral primaries are composite operators of the bosonic H_3^+ and $SU(2)$ primaries, usually dressed with some free fermions and ghosts, and their OPEs are obtained by combining the OPEs of the individual fields. Comparing the thus obtained (unintegrated) worldsheet OPEs with the corresponding spacetime OPEs, we find, not surprisingly, a one-to-one realization of the fusion rules of the chiral ring. We will also discuss some structural differences between both kinds of OPEs.

To find the recursion relation, we insert the worldsheet OPEs into a particular class of extremal p-point functions of chiral primary operators (In an extremal p-point function the spacetime scaling of the p-th operator is the sum of the spacetime scalings of the other $p-1$ operators). After performing the integrals over a (single) worldsheet coordinate and the $SL(2)$ representation label h, in a similar fashion as in [13], the p-point function factorizes into the product of a $p-1$-point function and a three-point function. In this way we find a recursion relation which, up to an overall factor F, is in agreement with the recursion relation of the dual boundary correlator previously found in [4]. We will comment on F in the conclusions.

2 Worldsheet operator product expansions in AdS_3

In the following we derive the operator product expansions of the chiral primary operators in the worldsheet theory for string theory on $AdS_3 \times S^3 \times T^4$. In the next section, we will use the resulting OPEs to find a recursion relation for a particular class of p-point functions.

2.1 Chiral primary operators

We begin by summarizing the worldsheet chiral primary operators [17, 18, 6]. Our conventions are as in [13]. In particular, it is understood that all operators depend on the complex worldsheet coordinate z, even though we often omit this dependence in the arguments of the operators.

The worldsheet theory is the product of an $\mathcal{N}=1$ WZW model on H_3^+, an $\mathcal{N}=1$ WZW model on $S^3 \simeq SU(2)$ and an $\mathcal{N}=1$ $U(1)^4$ free superconformal field theory. We emphasize here that, following [14], we consider an $H_3^+ = SL(2,\mathbb{C})/SU(2)$ sigma model whose target space is a *Euclidean* AdS_3. Likewise, the dual CFT_2 on the boundary is unitary and its time variable can be analytically continued to Euclidean time. In this way we avoid problems which arise in the definition of operator product expansions in the (Lorentzian) $SL(2,\mathbb{R})$ WZW model [19]–[22].

The above WZW model has the affine world-sheet symmetry $\widehat{sl}(2)_k \times \widehat{su}(2)_{k'} \times u(1)^4$. Criticality of the fermionic string on $AdS_3 \times S^3$ requires the identification of the levels k and k' [23], $k = k'$. The label k denotes the supersymmetric level of the affine Lie algebras and is identified with the bosonic levels k_b and k_b' as $k = k_b - 2 = k_b' + 2$. The bosonic currents are J^a for $SL(2)$ and K^a for $SU(2)$. The free fermions of $SL(2)$ are denoted by ψ^a, those of $SU(2)$ by χ^a ($a = (+,0,-)$ in either case). It is convenient to split the bosonic currents as

$$J^a = j^a + \hat{\jmath}^a \,, \qquad \hat{\jmath}^a = -\frac{i}{k}\varepsilon^a{}_{bc}\psi^a\psi^b \,, \tag{2.1}$$

and similarly K^a. Finally the $u(1)^4$ symmetry is described in terms of free bosons as $i\partial Y^i$, and the corresponding free fermions are λ_i $(i = 1, 2, 3, 4)$.

The chiral operators are constructed from the dimension zero operators

$$\mathcal{O}_j(x, y) = \Phi_h(x)\Phi'_j(y) \qquad \text{with} \qquad h = j + 1 \,, \quad j = 0, \tfrac{1}{2}, ..., \tfrac{k-2}{2} \,, \tag{2.2}$$

where $\Phi_h(x)$ and $\Phi'_j(y)$ are the primaries of the bosonic H_3^+ and $SU(2)$ WZW models with dimensions

$$\Delta(h) = -\frac{h(h-1)}{k_b - 2} \,, \qquad \Delta'(j) = \frac{j(j+1)}{k'_b + 2} \,, \tag{2.3}$$

respectively.[1] The labels x and y correspond to the $SL(2)$ and $SU(2)$ representation labels m and m', respectively. Our conventions for these models can be found in appendix A of [13]. Since $h = j + 1$, the operators $\mathcal{O}_j(x, y)$ have vanishing conformal dimensions, $\Delta(h) + \Delta'(j) = 0$.

Neveu-Schwarz sector

In the *Neveu-Schwarz* sector there are two families of chiral primaries. In the -1 picture they are[2]

$$\mathcal{O}_j^{(0)}(x, y) = e^{-\phi}\psi(x)\mathcal{O}_j(x, y) \,, \tag{2.4}$$

$$\mathcal{O}_j^{(2)}(x, y) = e^{-\phi}\chi(y)\mathcal{O}_j(x, y) \,, \tag{2.5}$$

where the fields $\psi(x)$ and $\chi(y)$ are given by

$$\begin{aligned} \psi(x) &= -\psi^+ + 2x\psi^3 - x^2\psi^- \,, \\ \chi(y) &= -\chi^+ + 2y\chi^3 + y^2\chi^- \,. \end{aligned} \tag{2.6}$$

The bosonized superghost field $e^{-\phi}$ ensures that the operators have ghost number -1.

Sometimes we will also need the corresponding ghost number 0 operators, which are obtained from (2.4) by acting with the picture changing operator Γ_{+1}. These operators will be needed to get the correct ghost number in the correlators. The ghost number 0 operators are [6, 5]

$$\tilde{\mathcal{O}}_j^{(0)}(x, y) = \left((1 - h)\hat{\jmath}(x) + j(x) + \tfrac{2}{k}\psi(x)\chi_a P_y^a\right)\mathcal{O}_j(x, y) \,, \tag{2.7}$$

$$\tilde{\mathcal{O}}_j^{(2)}(x, y) = \left(h\hat{k}(y) + k(y) + \tfrac{2}{k}\chi(y)\psi_A D_x^A\right)\mathcal{O}_j(x, y) \,, \tag{2.8}$$

[1] As mentioned above, we drop the dependence on the worldsheet coordinate z. For instance, the holomorphic H_3^+ operator is simply denoted by $\Phi_h(x)$ instead of $\Phi_h(x, z)$.

[2] In [17], these operators are denoted by $\mathcal{W}_j^-$ and $\mathcal{X}_j^+$, respectively.

where the operators D_x^A and P_y^a are

$$D_x^- = \partial_x \,, \quad D_x^3 = x\partial_x + h \,, \quad D_x^+ = x^2\partial_x + 2hx \,,$$
$$P_y^- = -\partial_y \,, \quad P_y^3 = y\partial_y - j \,, \quad P_y^+ = y^2\partial_y - 2jy \,. \tag{2.9}$$

Here we used again the compact notation

$$\hat{\jmath}(x) = -\hat{\jmath}^+ + 2x\hat{\jmath}^3 - x^2\hat{\jmath}^- \,,$$
$$\hat{k}(y) = -\hat{k}^+ + 2y\hat{k}^3 + y^2\hat{k}^- \,, \quad etc. \tag{2.10}$$

Ramond sector

In the *Ramond sector* there are also two families of chiral primaries, $\mathcal{O}_j^{(a)}(x,y)$ with $a = \pm 1$. For their construction we need the spin operators

$$S_{[\varepsilon_1,\varepsilon_2,\varepsilon_3]} = e^{\frac{i}{2}(\varepsilon_1\hat{H}_1+\varepsilon_2\hat{H}_2+\varepsilon_3\hat{H}_3)} \,, \tag{2.11}$$

where $\varepsilon_I = \pm 1$ and $\hat{H}_i$ $(i = 1,2,3)$ are bosonized fermions related to ψ^a and χ^a $(a = \pm, 0)$, as in [6] (Similarly, $\hat{H}_{4,5}$ are related to the fermions on the T^4, λ^i $(i = 1,2,3,4)$ [6]). Then, in the $-1/2$ and $-3/2$ picture the chiral primaries are given by[3]

$$\mathcal{O}_j^{(a)}(x,y) = e^{-\frac{\phi}{2}} s_-^{1,2}(x,y)\mathcal{O}_j(x,y) \qquad (a = \pm 1) \,, \tag{2.12}$$

and

$$\tilde{\mathcal{O}}_j^{(a)}(x,y) = -\sqrt{k}(2h-1)^{-1}e^{-\frac{3\phi}{2}} s_+^{1,2}(x,y)\mathcal{O}_j(x,y) \,, \tag{2.13}$$

respectively, where

$$s_\pm^1(x,y) = S_\pm(x,y)e^{+\frac{i}{2}(\hat{H}_4-\hat{H}_5)} \,, \qquad s_\pm^2(x,y) = S_\pm(x,y)e^{-\frac{i}{2}(\hat{H}_4-\hat{H}_5)} \tag{2.14}$$

and

$$S_\pm(x,y) = \mp xyiS_{[--\pm]} \mp xS_{[-+\mp]} + yiS_{[+-\mp]} + S_{[++\pm]} \,. \tag{2.15}$$

Full chiral primary operators

The full chiral primary operators are given by the product of a holomorphic with an anti-holomorphic operator,

$$\mathcal{O}_j^{(A,\bar{A})}(x,\bar{x},y,\bar{y}) \equiv \mathcal{O}_j^{(A)}(x,y)\bar{\mathcal{O}}_j^{(\bar{A})}(\bar{x},\bar{y}) \,, \tag{2.16}$$

where $A = 0, a, 2$ and $\bar{A} = \bar{0}, \bar{a}, \bar{2}$. When integrated over the worldsheet, these operators are dual to the chiral primary operators $O_n^{(A,\bar{A})}$ (n-cycle twist operators with $n = 2j+1$) in the symmetric orbifold theory on the boundary of AdS_3, defined *e.g.* in [2, 6, 4].

[3] $\mathcal{O}_j^{(+1)}$ and $\mathcal{O}_j^{(-1)}$ contain s_-^1 and s_-^2, respectively. In [17], these operators are denoted by $\mathcal{Y}_j^\pm$.

2.2 Worldsheet operator product expansions

The general structure of worldsheet operator product expansions for strings on $AdS_{d+1} \times W$ was studied in [16]. The vertex operators of this theory $\mathbb{O}_{h,j}$ are usually labeled by the spacetime scaling dimension h associated with the spacetime conformal group $SO(d+1,1)$ and a collective label j denoting some internal quantum numbers. Let us restrict to $d = 2$. As exemplified in section 2.1, for the special case of $AdS_3 \times S^3 \times T^4$, the vertex operators are products of the primaries $\Phi_h(z,x)$ and $\Phi'_j(z,y)$ of the bosonic H_3^+ and $SU(2)$ WZW models, dressed by a polynomial in the bosonic and fermionic worldsheet fields and their derivatives [17, 18, 23]. These operators depend on both the worldsheet coordinate z as well as the $SL(2)$ and $SU(2)$ representation labels x and y. As argued in [24], the label x can be identified with the coordinate on the boundary. Moreover, the $SL(2)$ current algebra on the string worldsheet induces a Virasoro algebra in spacetime conformal field theory. In addition to the usual worldsheet conformal weight $\Delta = \Delta(h,j)$ the vertex operators therefore also have a spacetime scaling dimension related to h.[4] Physical vertex operators have worldsheet dimension $\Delta(h,j) = 1$.

The Hilbert space of the worldsheet theory contains only the normalizable vertex operators with $h = \frac{1}{2} + is$ ($s \in \mathbb{R}$). For such operators, the most general form of an AdS_3 worldsheet OPE is, in the limit $z \to 0$, [16]:

$$\begin{aligned}\mathbb{O}_1(0)\mathbb{O}_2(x,\bar{x},z,\bar{z}) = \sum_j \int_{\mathcal{C}} dh \int d^2x' \, \frac{|z|^{2(\Delta(h,j)-\Delta(1)-\Delta(2))}}{|x|^{\alpha}|x'|^{\beta}|x'-x|^{\gamma}} \mathcal{F}(j_i,j,h_i,h)\mathbb{O}_{h,j}(x',\bar{x}',0,0) \\ + \text{descendants}\,, \qquad (2.17)\end{aligned}$$

where $\mathcal{F}$ is related to the 2-point and 3-point functions on the worldsheet. The parameters α, β and γ are functions of the spacetime conformal weights of the operators $\mathbb{O}_i \equiv \mathbb{O}_{h_i,j_i}$ ($i = 1,2$) and $\mathbb{O}_{h,j}$, respectively. $\Delta(1)$, $\Delta(2)$ and $\Delta(h,j)$ denote the corresponding worldsheet conformal weights. The dependence on z and x is completely determined by conformal invariance. The OPE contains an integral over the contour $h = \frac{1}{2}+is$, which is denoted by $\mathcal{C}$. In the following, we ignore contributions coming from the worldsheet descendants.

The above OPE is not directly applicable to worldsheet operators which are dual to spacetime operators. Such operators are non-normalizable and therefore not part of the Hilbert space. Instead they have spacetime scalings related to h located on the real axis of the complex h-plane. The OPE of such non-normalizable operators is obtained by careful analytic continuation in h. As shown in [16], this amounts to the inclusion of additional discrete contributions from the poles of $\mathcal{F}$. Otherwise, the form of (2.17) is preserved.

[4] The exact spacetime scaling depends on the actual form of the operator, *e.g.* $h[\mathcal{O}_j^{(0)}] = h[\mathcal{O}_j] + h[\psi] = h - 1$ for the operator $\mathcal{O}_j^{(0)}$ defined in (2.4).

2.3 Worldsheet operator product expansions of chiral primary operators

We now compute the OPE (2.17) for the case that the worldsheet operators are chiral primary. We begin by constructing the OPE of the dimension-zero operators

$$\mathcal{O}_j(x,\bar{x},y,\bar{y}) = \Phi_h(x,\bar{x})\Phi'_j(y,\bar{y}) \qquad (h=j+1)\,, \tag{2.18}$$

which form an essential part of the chiral primaries, as discussed after (2.2). The OPE is obtained from the OPEs of the H_3^+ and $SU(2)$ fields $\Phi_h(x,\bar{x})$ and $\Phi'_j(y,\bar{y})$.

The OPE of two H_3^+ primaries was found in [25]. As shown in Appendix A, it can be written as

$$\Phi_{h_2}(x_2,\bar{x}_2)\Phi_{h_1}(x_1,\bar{x}_1) = \int_{\mathcal{C}^+} dh \frac{C(h_1,h_2,h)|z_{12}|^{-2\Delta_{12}}|x_{12}|^{-2h_{12}}}{B(h)}\Phi_h(x_1,\bar{x}_1)\,, \tag{2.19}$$

with $h_{12}=h_1+h_2-h$ and $\Delta_{12}=\Delta_1+\Delta_2-\Delta$ ($\mathcal{C}^+ = 1/2+i\mathbb{R}^+$). $C(h_1,h_2,h_3)$ and $B(h)$ are the $SL(2)$ structure constants and the scaling of the $SL(2)$ two-point function, respectively. Similarly, the OPE of two $SU(2)$ primaries is given by [26, 27]

$$\Phi'_{j_2}(y_2,\bar{y}_2)\Phi'_{j_1}(y_1,\bar{y}_1) = \sum_j C'(j_1,j_2,j)|z_{12}|^{-2\Delta'_{12}}|y_{12}|^{2j_{12}}\Phi'_j(y_1,\bar{y}_1)\,, \tag{2.20}$$

with $j_{12}=j_1+j_2-j$ and $\Delta_{12}=\Delta'_1+\Delta'_2-\Delta'$. In both OPEs we ignored the contribution from current algebra descendants. Combining both OPEs yields the operator product expansion

$$\begin{aligned}
&\mathcal{O}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&\quad= \sum_j \int_{\mathcal{C}^+} dh \frac{C'C|z_{12}|^{-2(\Delta_{12}+\Delta'_{12})}|y_{12}|^{2j_{12}}}{B(h)|x_{12}|^{2h_{12}}}\Phi_h(x_1,\bar{x}_1,z_1,\bar{z}_1)\Phi'_j(y_1,\bar{y}_1,z_1,\bar{z}_1)\\
&\quad= \sum_j \int_{\mathcal{C}^+} dh \frac{C'C|z_{12}|^{2(\Delta(h)+\Delta'(j))}|y_{12}|^{2j_{12}}}{B(h)|x_{12}|^{2h_{12}}}\mathcal{O}_{j,h}(x_1,\bar{x}_1,y_1,\bar{y}_1)\,.
\end{aligned} \tag{2.21}$$

In the last line we defined the more general operators $\mathcal{O}_{j,h}\equiv\Phi_h\Phi'_j$, for which the labels h and j are not related in any way. Recall that the resulting operator $\mathcal{O}_{j,h}$ need not be physical.

Let us now construct the OPE of the operator $\mathcal{O}_j^{(0,0)}$ in the -1 picture and $\tilde{\mathcal{O}}_j^{(0,0)}$ in the 0 picture, which are defined by (2.4) and (2.7), respectively. We start from the expression

$$\begin{aligned}
&\tilde{\mathcal{O}}^{(0,0)}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}^{(0,0)}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&\quad= \big((1-h_2)\hat{\jmath}(x_2)+j(x_2)+\tfrac{2}{k}\psi(x_2)\chi_a P^a_{y_2}\big)e^{-\phi}\psi(x_1)\\
&\quad\times \big((1-h_2)\bar{\hat{\jmath}}(\bar{x}_2)+\bar{j}(\bar{x}_2)+\tfrac{2}{k}\bar{\psi}(\bar{x}_2)\bar{\chi}_a P^a_{\bar{y}_2}\big)e^{-\bar{\phi}}\bar{\psi}(\bar{x}_1)\mathcal{O}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\,.
\end{aligned} \tag{2.22}$$

Using the OPEs (B.2)-(B.6) in appendix B and the identity

$$2\chi_a P^a_y = \chi(y)\partial_y - j\partial_y\chi(y)\,, \tag{2.23}$$

this can also be written as

$$
\begin{aligned}
&\tilde{\mathcal{O}}^{(0,0)}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}^{(0,0)}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&= \left|(1-h_2)\left(\mathcal{D}^{(-1)}_{21}\psi(x_1)\right)+\psi(x_1)\mathcal{D}^{(h_1)}_{21}+\frac{x_{21}^2}{z_{21}}\left(\chi(y_2)\partial_{y_2}-j_2\partial_{y_2}\chi(y_2)\right)\right|^2\\
&\quad\times e^{-\phi-\bar{\phi}}\mathcal{O}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&= \left|\frac{x_{21}}{z_{21}}\psi(x_1)\left((1-h_2)2+x_{21}\partial_{x_1}-2h_1\right)+\frac{x_{21}^2}{z_{21}}\chi(y_2)\partial_{y_2}+...\right|^2 e^{-\phi-\bar{\phi}}\mathcal{O}_{j_2}\mathcal{O}_{j_1}\,. \qquad (2.24)
\end{aligned}
$$

The ellipses denote further terms involving derivatives of the type $\partial\psi$ and $\partial\chi$. In this analysis we neglect descendants and therefore ignore such terms. In the following we will also need to Taylor expand $\chi(y_2)=\chi(y_1)+y_{12}\partial\chi(y_1)+...$ and again drop derivatives of χ. $|...|^2$ indicates that there is the same factor in anti-holomorphic variables.

Substituting (2.21) into (2.24), we evaluate the derivatives on $\mathcal{O}_j$ such that $x_{21}\partial_{x_1}\to h_{12}$ and $y_{21}\partial_{y_2}\to j_{12}$ under the integral. We obtain

$$
\begin{aligned}
&\tilde{\mathcal{O}}^{(0,0)}_{j_2}(x_2,\bar{x}_2,y_2,\bar{y}_2)\mathcal{O}^{(0,0)}_{j_1}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&=\sum_j\int_{\mathcal{C}+}dh\frac{C'C|z_{12}|^{2(\Delta(h)+\Delta'(j)-1)}|y_{12}|^{2j_{12}}}{B(h)|x_{12}|^{2(h_{12}-1)}}\Big((h_1+h_2+h-2)^2\mathcal{O}^{(0,0)}_{j,h}(x_1,\bar{x}_1,y_1,\bar{y}_1)\\
&\qquad\qquad+(j_{12})^2\frac{|x_{21}|^2}{|y_{21}|^2}\mathcal{O}^{(2,2)}_{j,h}(x_1,\bar{x}_1,y_1,\bar{y}_1)+...\Big)\,, \qquad (2.25)
\end{aligned}
$$

where we ignored possible terms involving 'mixed' operators of the type $\mathcal{O}^{(0,2)}_{j,h}$ and $\mathcal{O}^{(2,0)}_{j,h}$.

Before we continue, let us recall how the fusion rules in the conformal field theory on the boundary can be reproduced from the worldsheet description [6]. The operators in the OPE must obey $U(1)$ charge conservation (as measured by the $SU(2)$ generator K^3_0, see [6]). Chiral (anti-chiral) operators in the boundary CFT are mapped to highest (lowest) weight states of $SU(2)$ in the worldsheet theory, *i.e.* $M=J$ ($M=-J$). $U(1)$ charge conservation in the fusion of two worldsheet operators, symbolically

$$\mathcal{O}^{(*)}_{j_1}\times\mathcal{O}^{(*)}_{j_2}=[\mathcal{O}^{(*)}_{j_3}]\,, \qquad (2.26)$$

therefore requires [6]

$$J=J_1+J_2\,, \qquad (2.27)$$

where $J_i=j_i+a_i$ and $a_i=0,1/2,1$ for the holomorphic operators $\mathcal{O}^{(0)},\mathcal{O}^{(a)},\mathcal{O}^{(2)}$, respectively. The fusion of two $SU(2)$ primary states requires $j_3\le j_1+j_2$ and therefore (2.27) implies

$$a_3\ge a_1+a_2\,. \qquad (2.28)$$

Clearly, the fusion rules must also obey the spin-statistics relations NS $\times$ NS $\to$ NS, NS $\times$ R $\to$ R, R $\times$ NS $\to$ R, and R $\times$ R $\to$ NS, where NS and R refer to the operators in the Neveu-Schwarz sector $(\mathcal{O}^{(0)}, \mathcal{O}^{(2)})$ and Ramond sector $(\mathcal{O}^{(a)})$, respectively. This allows for the following fusion rules in the holomorphic sector:

$$
\begin{aligned}
(0) \times (0) &= (0) + (2)\,, \\
(0) \times (2) &= (2)\,, \\
(0) \times (a) &= (a)\,, \\
(a) \times (a) &= (2)\,.
\end{aligned} \tag{2.29}
$$

Similar fusion rules hold in the anti-holomorphic sector. The four cases (2.29) can be freely combined between holomorphic and anti-holomorphic operators. Note however that in the fusion $(0,0) \times (0,0) \to (0,0) + (2,2)$ the resulting operator must be the same in the holomorphic and anti-holomorphic sector, *i.e.* the combinations $(0,2)$ and $(2,0)$ do not appear [6]. In (2.25) the fusion rules therefore only allow for terms involving the operators

$$
\begin{aligned}
\mathcal{O}^{(0,0)}_{j,h}: &\qquad j = j_1 + j_2 \equiv \tilde{j}\,, \\
\mathcal{O}^{(2,2)}_{j,h}: &\qquad j = j_1 + j_2 - 1 \equiv \tilde{j} - 1\,,
\end{aligned} \tag{2.30}
$$

where the j-values have been determined using (2.27). Terms proportional to $\mathcal{O}^{(0,2)}_{j,h}$ and $\mathcal{O}^{(2,0)}_{j,h}$ are forbidden by the worldsheet fusion rules.

In order to compare the worldsheet OPE with the corresponding boundary OPE, we need to rescale the operators as in [13] such that their (integrated) two-point functions scale as unity. For instance, the operators $\mathcal{O}^{(0,\bar{0})}_j(x,\bar{x})$ will be rescaled as

$$
\mathbb{O}^{(0,0)}_j(x,\bar{x}) = \frac{\sqrt{2\pi^2}}{\sqrt{k\,B(h)(2h-1)}} g_s\,\mathcal{O}^{(0,\bar{0})}_j(x,\bar{x})\,. \tag{2.31}
$$

Then, as shown in detail in appendix C, the OPE of the rescaled operators $\mathbb{O}^{(0,\bar{0})}_j$ following from (2.25) is

$$
\begin{aligned}
&\tilde{\mathbb{O}}^{(0,\bar{0})}_{j_2}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}^{(0,\bar{0})}_{j_1}(x_1,\bar{x}_1;y_1,\bar{y}_1) \\
&= \int_{\mathcal{C}^+} dh\, \frac{2h-1}{2\pi^2 k} \frac{|z_{21}|^{2(\Delta(h)-1)}}{|x_{21}|^{2(h_{21}-1)}} \Big(|z_{21}|^{2\Delta'(\tilde{j})}\, \mathbb{G}^{(000)}_3(j_1,j_2,\tilde{j},h)\mathbb{O}^{(0,\bar{0})}_{\tilde{j},h}(x_1,\bar{x}_1;y_1,\bar{y}_1) \\
&\qquad + |x_{21}|^2 |z_{21}|^{2\Delta'(\tilde{j}-1)}\, \mathbb{G}^{(002)}_3(j_1,j_2,\tilde{j}-1,h)\mathbb{O}^{(2,2)}_{\tilde{j}-1,h}(x_1,\bar{x}_1;y_1,\bar{y}_1) \Big)\,,
\end{aligned} \tag{2.32}
$$

where in the last line we defined the coefficients

$$
\begin{aligned}
\mathbb{G}^{(000)}_3(j_1,j_2,j_3,h_3) &\equiv P(j_1,j_2,j_3,h_3)\, \frac{g_s}{k} \frac{(h_1+h_2+h_3-2)^2}{\prod_i (2h_i-1)^{\frac{1}{2}}}\,, \\
\mathbb{G}^{(002)}_3(j_1,j_2,j_3,h_3) &\equiv P(j_1,j_2,j_3,h_3)\, \frac{g_s}{k} \frac{(j_1+j_2-j_3)^2}{\prod_i (2h_i-1)^{\frac{1}{2}}}\,,
\end{aligned} \tag{2.33}
$$

and

$$P(j_1,j_2,j_3,h_3) \equiv \frac{CC'\,2\pi}{\sqrt{B(h_1)B(h_2)B(h_3)\;c_\nu}} \qquad (c_\nu = 1/(2\pi^4 k^3))\,. \tag{2.34}$$

The factor $P(j_1,j_2,j_3,h_3)$ reflects the fact that h_3 is not related to j_3 in the third operator. This factor would be just one, $P(j_1,j_2,j_3,h_3)=1$, if h_3 were related to j_3 by $h_3=j_3+1$.[5] In that case, and if j_3 is related to j_1+j_2 as in (2.30), the coefficients reduce to the extremal three-point correlators

$$\begin{aligned}
\mathbb{G}_3^{(000)}(j_1,j_2,j_3,h_3)|_{h_3=j_3+1} &= \left\langle \mathbb{O}_{j_1}^{(0,0)}(\infty)\mathbb{O}_{j_2}^{(0,0)}(1)\tilde{\mathbb{O}}_{j_3}^{(0,0)}(0)\right\rangle\,,\\
\mathbb{G}_3^{(002)}(j_1,j_2,j_3,h_3)|_{h_3=j_3+1} &= \left\langle \mathbb{O}_{j_1}^{(0,0)}(\infty)\tilde{\mathbb{O}}_{j_2}^{(0,0)}(1)\mathbb{O}_{j_3}^{(2,2)}(0)\right\rangle\,,
\end{aligned} \tag{2.35}$$

found in [6, 5]. Note, for instance, that the $U(1)$ charge conservation $j_3=j_1+j_2$ is equivalent to $h_3=h_1+h_2-1$, if $h_3=j_3+1$. However, we stress that we do *not* assume any relation between h and $\tilde{j}$ at this stage, *i.e.* the operators on the right-hand-side of (2.32) need not be physical.

The other OPEs allowed by the fusion rules are computed in a similar way. We find

$$\begin{aligned}
&\tilde{\mathbb{O}}_{j_2}^{(0,\bar{0})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}_{j_1}^{(2,\bar{2})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&\quad = \int_{\mathcal{C}^+} dh\,\frac{2h-1}{2\pi^2 k}\frac{|z_{21}|^{2(\Delta(h)+\Delta'(\tilde{j})-1)}}{|x_{21}|^{2(h_{21}-1)}}\,\mathbb{G}_3^{(022)}(j_1,j_2,\tilde{j},h)\mathbb{O}_{\tilde{j},h}^{(2,\bar{2})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\,, && (2.36)\\
&\tilde{\mathbb{O}}_{j_2}^{(0,\bar{0})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}_{j_1}^{(a,\bar{a})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&\quad = \int_{\mathcal{C}^+} dh\,\frac{2h-1}{2\pi^2 k}\frac{|z_{21}|^{2(\Delta(h)+\Delta'(\tilde{j})-1)}}{|x_{21}|^{2(h_{21}-1)}}\,\mathbb{G}_3^{(0aa)}(j_1,j_2,\tilde{j},h)\mathbb{O}_{\tilde{j},h}^{(a,\bar{a})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\,, && (2.37)\\
&\mathbb{O}_{j_2}^{(a,\bar{a})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}_{j_1}^{(b,\bar{b})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&\quad = \int_{\mathcal{C}^+} dh\,\frac{2h-1}{2\pi^2 k}\frac{|z_{21}|^{2(\Delta(h)+\Delta'(\tilde{j})-1)}}{|x_{21}|^{2(h_{21}-1)}}\,\mathbb{G}_3^{(ab2)}(j_1,j_2,\tilde{j},h)\mathbb{O}_{\tilde{j},h}^{(2,\bar{2})}(x_1,\bar{x}_1;y_1,\bar{y}_1)\,, && (2.38)
\end{aligned}$$

with

$$\mathbb{G}_3^{(022)}(j_1,j_2,j_3,h_3) \equiv P(j_1,j_2,j_3,h_3)\frac{g_s}{k}\frac{(-h_1+h_2+h_3)^2}{\prod_i(2h_i-1)^{\frac{1}{2}}}\,, \tag{2.39}$$

$$\mathbb{G}_3^{(0aa)}(j_1,j_2,j_3,h_3) \equiv P(j_1,j_2,j_3,h_3)\frac{g_s}{k}\frac{(h_1+h_2+h_3-2)^2}{(2h_3-1)^2}\frac{(2h_1-1)^{\frac{1}{2}}(2h_3-1)^{\frac{1}{2}}}{(2h_2-1)^{\frac{1}{2}}}\,, \tag{2.40}$$

$$\mathbb{G}_3^{(ab2)}(j_1,j_2,j_3,h_3) \equiv P(j_1,j_2,j_3,h_3)\frac{g_s}{k}\frac{(2h_1-1)^{\frac{1}{2}}(2h_2-1)^{\frac{1}{2}}}{(2h_3-1)^{\frac{1}{2}}}\delta^{ab}\,. \tag{2.41}$$

The correlators (2.39)–(2.41) reduce again to the extremal three-point functions computed in [6], if $h_3=j_3+1$. Note that the total ghost number is preserved in the OPEs.

[5] This can be seen by using the identity (4.29) in [13]. This identity has first been found in [5, 6].

2.4 Discussion and comparison with boundary operator product expansions

Some comments on the worldsheet operator product expansions (2.32) are in order. Similar statements will hold for the OPEs (2.36)–(2.38).

First, let us first compare (2.32) with the general form (2.17). Defining $\Delta(h,j) \equiv \Delta(h) + \Delta'(j) + 1$, which is the worldsheet conformal dimension of $\mathbb{O}^{(0,0)}_{j,h}$ (and $\mathbb{O}^{(2,2)}_{j,h}$), we find that at small z and small x (2.32) agrees with the general form (2.17), since the chiral primaries have conformal dimension $\Delta(1) = \Delta(2) = 1$ and $|z_{21}|^{2(\Delta(h,j)-\Delta(1)-\Delta(2))} = |z_{21}|^{2(\Delta(h)+\Delta'(j)-1)}$. Recall also that $\mathbb{O}^{(0,0)}_{j,h}$ and $\mathbb{O}^{(2,2)}_{j,h}$ scale differently in x, $h^{(0)} = h - 1$ and $h^{(2)} = h$ [13]. The total x-dependence should be $|x_{21}|^{2(h^{(A)}-h_1^{(0)}-h_2^{(0)})}$ with $A = 0, 2$ in the first and second term of (2.32), respectively. Therefore there is an additional factor $|x_{21}|^2$ in the second term of (2.32). Consequently, we find that the OPE has the correct scaling in both x and z. (In (2.32) we have already used $U(1)$ charge conservation such that there is no sum over j anymore).

Second, another peculiar feature of (2.32) is the appearance of the factor

$$\frac{2h-1}{2\pi^2 k}\,. \tag{2.42}$$

As we will see later, when we use the OPE inside a general correlator, this factor will cancel against the residue of the h-integral, which is proportional to the inverse of the derivative of the $SL(2)$ conformal weight, $(\partial_h \Delta)^{-1} \propto k/(2h-1)$.

Third, it is also interesting to compare the worldsheet OPE (2.32) with the corresponding spacetime OPE of n-cycle twist operators of the type $O^{(0,0)}_n$ which are dual to the worldsheet operators $\mathbb{O}^{(0,0)}_j$. This OPE is given by [4][6]

$$O^{(0,0)}_{n_2} O^{(0,0)}_{n_1} = C_3 O^{(0,0)}_{\tilde{n}} + C_3' O^{(2,2)}_{\tilde{n}-2} + \ldots\,, \tag{2.43}$$

with $\tilde{n} = n_1 + n_2 - 1$ and structure constants C_3 and C_3'. The ellipses indicate terms coming from multi-cycle operators. Given that the cycle lengths n_i are related to j_i by $n_i = 2j_i + 1$ (and $\tilde{n} = 2\tilde{j} + 1$), we observe a structural resemblance between the worldsheet and the spacetime OPE, cf. (2.32) with (2.43). In particular, both OPEs satisfy the fusion relation $(0,0) \times (0,0) \to (0,0) + (2,2)$ of the chiral-chiral ring. More general, we find that the worldsheet OPEs (2.32), (2.36)-(2.38) mimic the fusion rules of the (c,c) ring in the spacetime conformal field theory,

$$\begin{aligned}
(0,0) \times (0,0) &= (0,0) + (2,2)\,,\\
(0,0) \times (2,2) &= (2,2)\,,\\
(0,0) \times (a,a) &= (a,a)\,,\\
(a,a) \times (a,a) &= (2,2)\,.
\end{aligned} \tag{2.44}$$

[6] See [4] for a precise definition of the operators $O^{(0,0)}_n$ and the corresponding OPE.

In fact, upon integration over the worldsheet coordinates, the worldsheet OPE (2.32) becomes identical to the spacetime OPE (2.43) (multi-cycle contributions ignored).[7]

Fourth, one might worry that (2.32) still depends on the spacetime coordinates x, while the spacetime OPE (2.43) has no singularities. We will see however in the next section that, when the OPE is employed inside an extremal p-point correlator of chiral primary operators, the x-dependence will drop out (Basically the integration over h will yield a relation between h and $\tilde{j}$ which eliminates the x-dependence in both terms in (2.32).).

3 Recursion relation for worldsheet p-point functions

In this section we derive a recursion relation for a particular extremal worldsheet p-point function and compare it with the corresponding relation for the dual boundary correlator previously computed in [4].

A simple worldsheet p-point function on the sphere is given by the product of p (rescaled) operators $\mathbb{O}_j \equiv \mathbb{O}_j^{(0,0)}$,

$$\mathbb{G}_p \equiv \mathbb{G}_p^{j_1,\dots,j_p} = g_s^{-2} \left\langle \tilde{\mathbb{O}}_{j_p}(\infty)\mathbb{O}_{j_{p-1}}(1) \left(\prod_{i=2}^{p-2} \int d^2 z_i \, \tilde{\mathbb{O}}_{j_i}(x_i,\bar{x}_i;z_i,\bar{z}_i) \right) \mathbb{O}_{j_1}(0) \right\rangle , \tag{3.1}$$

with the extremality condition

$$j_p = \sum_{i=1}^{p-1} j_i \,. \tag{3.2}$$

Modular invariance has been used to fix three of the p worldsheet points as $z_{1,p-1,p} = 0, 1, \infty$. Similarly, the continuous $SL(2)$ representation labels are chosen as $x_{1,p-1,p} = 0, 1, \infty$. The x labels will later be identified with the complex coordinates in the spacetime conformal field theory [24]. The correlator $\mathbb{G}_p$ involves $p-2$ ghost number zero and 2 ghost number -1 operators, $\tilde{\mathbb{O}}_j^{(0,0)}$ and $\mathbb{O}_j^{(0,0)}$, respectively. Recall that the total ghost number of a correlator on a genus-g surface must be $-\chi = -(2-2g)$, which is -2 on the sphere.

We now show that the p-point functions $\mathbb{G}_p$ satisfy the recursion relation

$$\mathbb{G}_p \simeq \left\langle \mathbb{O}_{\tilde{j}}^{(0,0)}(\infty)\tilde{\mathbb{O}}_{j_2}^{(0,0)}(1)\mathbb{O}_{j_1}^{(0,0)}(0) \right\rangle \mathbb{G}_{p-1} \tag{3.3}$$

with $\tilde{j} = j_1 + j_2$. The symbol $\simeq$ indicates that (3.3) is true up to a factor F which currently cannot be reproduced on the worldsheet. This factor is coming from two-particle contributions in the intermediate channel, which are nonlocal on the worldsheet.

[7] This can be seen by setting $z_1 = 0$ and $z = z_2$ and performing the integral over z and h as described in section 3 below. After the integration over h, $\mathbb{G}_3^{(000)}$ and $\mathbb{G}_3^{(002)}$ have reduced to the extremal correlators (2.35) which are identical to the coefficients C_3 and C_3' appearing in (2.43) [6, 5].

The factor F has however been determined in the dual symmetric orbifold theory. The recursion relation for the dual boundary correlators C_p is given by

$$C_p = \frac{n_p}{\tilde{n}} \left\langle O_{\tilde{n}}^{(0,0)\dagger}(\infty) O_{n_2}^{(0,0)}(1) O_{n_1}^{(0,0)}(0) \right\rangle C_{p-1} \tag{3.4}$$

with $\tilde{n} = n_1 + n_2 - 1$ [4]. The non-renormalization theorem of [15] predicts the equivalence of both recursion relations such that F can be identified as $F = \frac{n_p}{\tilde{n}} = \frac{2j_p+1}{2\tilde{j}+1}$.

Proof of (3.3): Substituting the worldsheet OPE (2.32) into $\mathbb{G}_p$, we obtain[8]

$$\begin{aligned}
\mathbb{G}_p &= g_s^{-2} \int d^2 z_2 \int_{\mathcal{C}} dh \left\langle \tilde{\mathbb{O}}_{j_p}(\infty) \mathbb{O}_{j_{p-1}}(1) \left(\prod_{i=3}^{p-2} \int d^2 z_i \, \tilde{\mathbb{O}}_{j_i}(x_i, \bar{x}_i; z_i, \bar{z}_i) \right) \mathbb{O}_{\tilde{j},h}(0) \right\rangle \\
&\quad \times \frac{2h-1}{2\pi^2 k} \frac{|z_2|^{2(\Delta(h)+\Delta'(\tilde{j})-1)}}{|x_2|^{2(h_{21}-1)}} \mathbb{G}_3^{(000)}(j_1, j_2, \tilde{j}, h) + \ldots \\
&= \int d^2 z \int_{\mathcal{C}} dh \frac{2h-1}{2\pi^2 k} \frac{|z|^{2(\Delta(h)+\Delta'(\tilde{j})-1)}}{|x|^{2(h_{21}-1)}} \mathbb{G}_{p-1} \, \mathbb{G}_3^{(000)}(j_1, j_2, \tilde{j}, h) + \ldots ,
\end{aligned} \tag{3.5}$$

where we set $z = z_2$ $(x = x_2)$ and introduced the short hand notation $\mathbb{G}_{p-1}$ for $\mathbb{G}_{p-1}^{j, j_3, \ldots, j_p}$. The ellipses indicate that there is in principle a second contribution from the operator $\mathbb{O}_{\tilde{j}-1,h}^{(2,2)}$ in the OPE (2.32). This contribution is zero, as will be shown below.

The integrals over z and h can be done as in the case of four-point functions [13, 14]. As in [13], we need to do the z-integral before the h-integral. In that case we have to be careful about the occurrence of divergencies and regularize the z-integral by introducing a cutoff ε [14]. Later, after the integrations, we will eventually take the limit $\varepsilon \to 0$. In general it is not known how to compute the z-integral over the whole range of z, but it can be computed in the limit of small $|z| < \varepsilon$. In this region, the z-integral can be performed by elementary methods,

$$\int_{|z|<\varepsilon} d^2 z \, |z|^{2(\lambda(h)-1)} = \frac{\pi}{\lambda(h)} \varepsilon^{2\lambda(h)} \tag{3.6}$$

with $\lambda(h) = \Delta(h) + \Delta'(\tilde{j})$. As discussed in [16, 14], the integral only over $|z| < \varepsilon$ captures the single-cycle (or, in higher dimensions, single-trace) terms in the spacetime OPE. By performing the integral only over $|z| < \varepsilon$, we omit nonlocal contributions from the large z region, which are expected to give the double-cycle terms in the spacetime OPE [14, 16]. This limitation prevents us from deriving the overall factor F, which is known to arise from double-cycle operators in the spacetime OPE [4].

We now turn to the integration over h. In general, after the z integration, there are additional discrete contributions coming from poles in the integrand of (3.5) [14, 16]. Such contributions arise when the poles cross the integration contour during

i) the analytic continuation in j_1 and j_2 (or $h_{1,2} = j_{1,2} + 1$), and

[8] Within a p-point function the integration over the half-axis $\mathcal{C}^+ = 1/2 + i\mathbb{R}^+$ can be extended to an integration over the full axis $\mathcal{C} = 1/2 + i\mathbb{R}$ [25].

ii) the shift of the contour from $h = 1/2 + is$ to $h = h_0 + is$ ($s \in \mathbb{R}$), where h_0 is defined by $\lambda(h_0) = 0$.[9]

There are altogether four types of poles [14, 16]:

$$
\begin{aligned}
\text{type I:} &\quad \lambda = 0\,, \\
\text{type II:} &\quad h = h_1 + h_2 + n\,, \\
\text{type III:} &\quad h = k - h_1 - h_2 + n\,, \\
\text{type IV:} &\quad h = |h_1 - h_2| - n\,, \qquad n \in \{0, 1, 2, ...\}\,.
\end{aligned}
$$

The poles of type II-IV are poles in the structure constants $C(h, h_1, h_2)$. As discussed extensively in [16], none of these poles contributes to the integral, at least if the preceding z integration is restricted to the regime $|z| < \varepsilon$. Even though naively one might interpret the contributions from the poles of type II as "double-cycle" operators in the spacetime CFT, such contributions go to zero in the $\varepsilon \to 0$ limit [16] (This is in agreement with the general expectation [16] that contributions from double-cycle operators arise non-locally, *i.e.* at large z and not in the $|z| < \varepsilon$ region). Type III poles do not appear if one assumes $h_1 + h_2 < \frac{k+1}{2}$ [14]. Other than the poles of type II, the type IV poles may contribute both during the analytic continuation and the additional shift in the contour. It was found in [16] that the contribution coming from crossing the contour during the analytic continuation is exactly the opposite of that during the subsequent shift of the contour. In effect, the poles of type IV do not modify the final result.

We are left with poles of type I, $\lambda(h) = 0$, corresponding to $h = h_0 \equiv \tilde{j} + 1$. The residue of this pole is

$$
\mathrm{Res}(f; h_0) = \frac{\pi \varepsilon^{2\lambda(h_0)}}{\lambda'(h_0)} \frac{2h_0 - 1}{2\pi^2 k} \mathbb{G}_{p-1}\, \mathbb{G}_3^{(000)}\,, \tag{3.7}
$$

where f is the integrand of (3.5) and $' \equiv \partial_h$. Remarkably, the first and second factor on the right-hand side cancel each other (up to 2π), since $\lambda'(h_0) = \partial_h \Delta(h_0)$. Moreover, the x-dependence drops out since $h_{21} - 1 = h_2 + h_1 - h_0 - 1 = j_2 + j_1 - \tilde{j} = 0$. Applying the residue theorem, we thus obtain

$$
\begin{aligned}
\mathbb{G}_p &= \mathbb{G}_{p-1}\, \mathbb{G}_3^{(000)}(j_1, j_2, \tilde{j}, h = \tilde{j} + 1) \\
&= \left\langle \mathbb{O}_{\tilde{j}}^{(0,0)}(\infty) \tilde{\mathbb{O}}_{j_2}^{(0,0)}(1) \mathbb{O}_{j_1}^{(0,0)}(0) \right\rangle \mathbb{G}_{p-1}\,,
\end{aligned} \tag{3.8}
$$

which is nothing but (3.3).

We still have to show that in (3.5) there are no contributions from the operator $\mathbb{O}^{(2,2)}_{\tilde{j}-1,h}$. The additional term in the integrand of (3.5) is proportional to

$$
\frac{|z|^{2(\Delta(h) + \Delta'(\tilde{j}-1) - 1)}}{|x|^{2(h_{21} - 2)}} \mathbb{G}_3^{(002)}(j_1, j_2, \tilde{j} - 1, h) \tag{3.9}
$$

[9] It is convenient to shift the contour in this way since, as we will see, most of the pole contributions vanish during the shift.

and has a pole at $h = \tilde{j}$. After applying the residue theorem, the x-dependence drops out, since $|x|^{2(h_2+h_1-h-2)} = |x|^{2((j_2+1)+(j_1+1)-\tilde{j}-2)} = 1$ and we get the additional contribution

$$\left\langle \mathbb{O}^{(2,2)}_{\tilde{j}-1}(\infty)\tilde{\mathbb{O}}^{(0,0)}_{j_2}(1)\mathbb{O}^{(0,0)}_{j_1}(0)\right\rangle \mathbb{G}'_{p-1}\,, \tag{3.10}$$

where $\mathbb{G}'_{p-1}$ is defined by

$$\mathbb{G}'^{\,\tilde{j}-1,j_3,\ldots,j_p}_{p-1} = g_s^{-2}\left\langle \mathbb{O}^{(0,0)}_{j_4}(\infty)\tilde{\mathbb{O}}^{(0,0)}_{j_3}(1)X\mathbb{O}^{(2,2)}_{\tilde{j}-1}(0)\right\rangle \tag{3.11}$$

and X denotes the product of $p-4$ $\tilde{\mathbb{O}}^{(0,0)}_{j}$ operators.

Clearly, for $p = 4$, the three-point correlator $\mathbb{G}'^{\,\tilde{j}-1,j_3,j_4}_{3}$ is zero, as can be seen as follows. The extremality condition (3.2) for $G_4^{j_1,j_2,j_3,j_4}$ can be written as

$$j_4 = j_1 + j_2 + j_3 = \tilde{j} + j_3\,, \tag{3.12}$$

which is formally the $U(1)$ charge conservation for the fusion of $\mathbb{O}^{(2,2)}_{\tilde{j}-1}$ and $\tilde{\mathbb{O}}^{(0,0)}_{j_3}$. However, the fusion rules require $a_4 \geq \tilde{a} + a_3$ (cf. with (2.28)), which is violated since $a_4 = 0$ and $\tilde{a} + a_3 = 1 + 0 = 1$, implying $\mathbb{G}'^{\,\tilde{j}-1,j_3,j_4}_{3} = 0$. A similar argument holds for $p > 4$. Thus, the term (3.10) vanishes identically.

4 Conclusions

In this paper we studied the worldsheet realization of the chiral ring structure of the $N = (4,4)$ symmetric orbifold theory on the boundary of $AdS_3 \times S^3 \times T^4$. Our main results are the (unintegrated) *worldsheet* operator product expansions (2.32) and (2.36)–(2.38), which nicely reflect the fusion rules of the chiral ring. Despite the similarity to the dual spacetime OPEs, there are also some structural differences which we discussed at length in section 2.4. In particular, the worldsheet OPEs are not simply given by the (extremal) worldsheet three-point functions of chiral primary operators [5, 6], as one might naively expect. In fact, the operators $\mathbb{O}_{h,j}$ appearing on the right hand side of the worldsheet OPEs need not even be physical, *i.e.* there is a priori no relation between the $SL(2)$ and $SU(2)$ labels h and j, whereas $h = j + 1$ for chiral primaries. In this respect, the OPEs are more general than the three-point functions in [5, 6]. However, when the worldsheet OPEs are integrated over the worldsheet coordinates, the h integral turns out to have a pole at $h = j + 1$, and the worldsheet OPEs become identical to those of the spacetime CFT.

As an interesting application, we used the worldsheet OPEs to derive a recursion relation for a particular class of extremal p-point correlators on the worldsheet. Our result (3.3) for the correlator (3.1) agrees with the recursion relation for the dual boundary p-point function [4], up to a simple overall factor $F = n_p/\tilde{n}$. In the spacetime OPE the factor F comes from two-cycle operators, whose contributions are not suppressed in extremal correlators at large N. Unfortunately, these contributions arise nonlocally on the worldsheet and are presently not very well understood [16]. It would be highly desirable to understand in more detail how multi-cycle (or, in general, multi-trace) operators are treated in worldsheet OPEs.

In this paper (and its precursors [5]–[8],[13]) worldsheet p-point functions on $AdS_3 \times S^3$ (with NSNS fluxes) are computed on the *full quantum* level. This may be compared to the *semi-classical* treatment of worldsheet p-point functions for string theory on $AdS_5 \times S^5$ (with RR fluxes), see e.g. [28]–[33]. To gain more insight into the latter approach, it would be interesting to repeat such semi-classical computations on $AdS_3 \times S^3$ and compare the results with the already known quantum correlators. It may also be of interest to attempt a full quantum computation on AdS_3 backgrounds with Ramond-Ramond fluxes, perhaps using techniques suggested in [34].

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Appendix

A OPE of H_3^+ primaries

In the following we derive the worldsheet operator product expansion of chiral primary operators in the H_3^+ model. – Important note: Other than in the rest of the paper, we use the conventions of Teschner [25] in this appendix, *i.e.* we use j to label the H_3^+ states.

The worldsheet OPE of two H_3^+ primaries is [25][10]

$$\begin{aligned}&\Phi_{j_1}(x_1,\bar{x}_1,z_1,\bar{z}_1)\Phi_{j_2}(x_2,\bar{x}_2,z_2,\bar{z}_2)\\&\quad=\int_{\mathcal{C}^+} dj_3\, C(j_1,j_2,j_3)|z_{12}|^{-2\Delta_{12}}(\mathcal{J}_{12}(j_3)\Phi_{-j_3-1})(z_2,\bar{z}_2)\,,\end{aligned} \tag{A.1}$$

where

$$(\mathcal{J}_{12}(j_3)\Phi_{-j_3-1})(z_2,\bar{z}_2) \equiv \int_{\mathbb{C}} d^2x_3\, |x_{12}|^{2j_{12}}|x_{23}|^{2j_{23}}|x_{31}|^{2j_{31}}\Phi_{-j_3-1}(x_3,\bar{x}_3,z_2,\bar{z}_2)\,. \tag{A.2}$$

Here $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$, $j_{12} = j_1 + j_2 - j_3$, etc. We prefer to express the OPE in terms of Φ_{j_3} rather than Φ_{-j_3-1}. We therefore substitute the expression

$$(\mathcal{J}_{12}(j_3)\Phi_{-j_3-1})(z_2,\bar{z}_2) = \frac{\gamma(-2j_3)}{(-\pi)\gamma(-j_{23})\gamma(-j_{31})}\frac{1}{B(j_3)}(\mathcal{J}_{12}(-j_3-1)\Phi_{j_3})(z_2,\bar{z}_2) \tag{A.3}$$

[10] We interchange the labels $1 \leftrightarrow 2$. In the following we ignore the contribution from descendants.

into (A.1) and obtain

$$\begin{aligned}&\Phi_{j_1}(x_1,\bar{x}_1,z_1,\bar{z}_1)\Phi_{j_2}(x_2,\bar{x}_2,z_2,\bar{z}_2) \\ &= \int_{\mathcal{C}+} dj_3\, C(j_1,j_2,j_3)|z_{12}|^{-2\Delta_{12}} \frac{\gamma(-2j_3)}{(-\pi)\gamma(-j_{23})\gamma(-j_{31})}\frac{1}{B(j_3)} \\ &\quad\times \int_{\mathbb{C}} d^2x_3\, |x_{12}|^{-2(-j_1-j_2-j_3-1)}|x_{23}|^{-2(1+j_{31})}|x_{31}|^{-2(1+j_{23})}\Phi_{j_3}(x_3,\bar{x}_3,z_2,\bar{z}_2)\,.\end{aligned} \tag{A.4}$$

We now simplify the expression by computing the x_3-integral

$$I = \int_{\mathbb{C}} d^2t'\, |t|^{-2(-j_1-j_2-j_3-1)}|t'|^{-2(1+j_{31})}|t-t'|^{-2(1+j_{23})}\Phi_{j_3}(x_2-t',\bar{x}_2-\bar{t}',z_2,\bar{z}_2)\,, \tag{A.5}$$

where we have defined $t = x_{12}$ and $t' = x_{23}$. Denoting $t = |t|\hat{t}$ and defining $y = t'/|t|$, we get

$$\begin{aligned}I &= |t|^{-2(-j_1-j_2+j_3+1)}\int_{\mathbb{C}} d^2t'\, (|t'|/|t|)^{-2(1+j_{31})}|(t'/|t|-\hat{t})|^{-2(1+j_{23})}\Phi_{j_3}(x_2-t',\bar{x}_2-\bar{t}',z_2,\bar{z}_2) \\ &= |t|^{2j_{12}}\int_{\mathbb{C}} d^2y\, |y|^{-2(1+j_{31})}|y-\hat{t}|^{-2(1+j_{23})}\Phi_{j_3}(x_2-y|t|,\bar{x}_2-\bar{y}|t|,z_2,\bar{z}_2)\,.\end{aligned} \tag{A.6}$$

In the OPE, x_1 and x_2 are assumed to be close to each other such that $|t|$ is small. We also ignore the subleading contributions from space-time descendants. We may then Taylor expanded the operator $\Phi_{j_3}(x_2-y|t|,\bar{x}_2-\bar{y}|t|,z_2,\bar{z}_2)$ around x_2 and obtain[11]

$$I \approx |t|^{2j_{12}}\Phi_{j_3}(x_2,\bar{x}_2,z_2,\bar{z}_2)\int_{\mathbb{C}} d^2y\, |y|^{-2(1+j_{31})}|y-\hat{t}|^{-2(1+j_{23})}\,. \tag{A.7}$$

Using the identity

$$\int_{\mathbb{C}} d^2y\, |y|^{2a}|1-y|^{2b} = -\pi\frac{\gamma(-1-a-b)}{\gamma(-a)\gamma(-b)}\,, \tag{A.8}$$

the integral I becomes

$$I = (-\pi)|x_{12}|^{2j_{12}}\frac{\gamma(1+2j_3)}{\gamma(1+j_{31})\gamma(1+j_{23})}\Phi_{j_3}(x_2,\bar{x}_2,z_2,\bar{z}_2)\,. \tag{A.9}$$

Thus,

$$\begin{aligned}&\Phi_{j_1}(x_1,\bar{x}_1,z_1,\bar{z}_1)\Phi_{j_2}(x_2,\bar{x}_2,z_2,\bar{z}_2) \\ &= \int_{\mathcal{C}+} dj_3\, C(j_1,j_2,j_3)|z_{12}|^{-2\Delta_{12}}\frac{1}{B(j_3)}|x_{12}|^{2j_{12}}\Phi_{j_3}(x_2,\bar{x}_2,z_2,\bar{z}_2)\,.\end{aligned} \tag{A.10}$$

Replacing $j \to -h$ ($\Phi_j \to \Phi_h$), we get (2.19).

[11] An almost identical expansion was done in Eq. (2.10) in [16].

B Some correlators and operator product expansions

In this appendix we list some worldsheet operator product expansions used in section 2. It is convenient to express these OPEs in terms of the operator

$$\mathcal{D}_{ki}^{(h_i)} = \frac{1}{z_{ki}} \left(x_{ki}^2 \partial_{x_i} - 2h_i x_{ki} \right) , \tag{B.1}$$

where h_i denotes the spacetime scaling of the operator it acts on. Some important worldsheet operator product expansions are [6, 13]:

$$j(x_k)\Phi_{h_i}(x_i) \sim \mathcal{D}_{ki}^{(h_i)} \Phi_{h_i}(x_i) \,, \tag{B.2}$$

$$j(x_1)j(x_2) \sim (k+2)\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}_{12}^{(-1)} j(x_2) \,, \tag{B.3}$$

$$\hat{\jmath}(x_1)\hat{\jmath}(x_2) \sim -2\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}_{12}^{(-1)} \hat{\jmath}(x_2) \,, \tag{B.4}$$

$$\hat{\jmath}(x_1)\psi(x_2) \sim \mathcal{D}_{12}^{(-1)} \psi(x_2) \,, \tag{B.5}$$

$$\psi(x_1)\psi(x_2) \sim k\frac{x_{12}^2}{z_{12}} \,. \tag{B.6}$$

C Rescaling the operators in the OPE

In this appendix we compute the rescaled OPE (2.32). For comparison with the boundary theory, it is useful to rescale the operators such that, when integrated over z, their two-point functions are just one (integration over $z_{1,2}$). The rescaled operators are [13]

$$\begin{aligned}
\mathbb{O}_j^{(0,0)}(x,\bar{x}) &= \frac{\sqrt{2\pi^2}}{\sqrt{k\,B(h)(2h-1)}} g_s\, \mathcal{O}_j^{(0,\bar{0})}(x,\bar{x}) \,, \\
\mathbb{O}_j^{(a,\bar{a})}(x,\bar{x}) &= \sqrt{\frac{2\pi^2(2h-1)}{B(h)}} g_s \mathcal{O}_j^{(a,\bar{a})}(x,\bar{x}) \,.
\end{aligned} \tag{C.1}$$

The operator $\mathcal{O}_j^{(2,2)}(x,\bar{x})$ is rescaled as $\mathcal{O}_j^{(0,0)}(x,\bar{x})$ (Tilded operators are rescaled as their untilded partners). Then, substituting the OPE (2.25) into

$$\begin{aligned}
&\tilde{\mathbb{O}}_{j_2}^{(0,\bar{0})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}_{j_1}^{(0,\bar{0})}(x_1,\bar{x}_1;y_1,\bar{y}_1) \\
&\quad = \frac{2\pi^2 g_s^2}{k\sqrt{B(h_1)(2h_1-1)B(h_2)(2h_2-1)}} \tilde{\mathcal{O}}_{j_2}^{(0,\bar{0})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathcal{O}_{j_1}^{(0,\bar{0})}(x_1,\bar{x}_1;y_1,\bar{y}_1) \,,
\end{aligned} \tag{C.2}$$

we get

$$\begin{aligned}
&\tilde{\mathbb{O}}_{j_2}^{(0,\bar{0})}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}_{j_1}^{(0,\bar{0})}(x_1,\bar{x}_1;y_1,\bar{y}_1) \\
&\quad = \sum_j \int_{\mathcal{C}} dh \frac{|z_{12}|^{2(\Delta(h)+\Delta'(j)-1)}|y_{12}|^{2j_{12}}}{|x_{12}|^{2(h_{12}-1)}} \frac{(2h-1)}{\sqrt{(2h-1)(2h_2-1)(2h_1-1)}} \frac{g_s\sqrt{2\pi^2}\; C'C}{\sqrt{k\;B(h_1)B(h_2)B(h)}} \\
&\quad \times \left((h_1+h_2+h-2)^2 \mathbb{O}_{j,h}^{(0,\bar{0})}(x_1,\bar{x}_1;y_1,\bar{y}_1) + (j_{12})^2 \frac{|x_{21}|^2}{|y_{21}|^2} \mathbb{O}_{j,h}^{(2,\bar{2})}(x_1,\bar{x}_1;y_1,\bar{y}_1) \right) \,,
\end{aligned} \tag{C.3}$$

which can be written as (2.32).

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