Definable Maximal Independent Families

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Abstract

We study maximal independent families for sets in the projective hierarchy. Our main result shows that in the Cohen model, there are no maximal independent families. We also consider a new cardinal invariant related to the question of destroying and preserving maximal independent families.

1 Introduction

In descriptive set theory, one often looks at objects defined in a con-constructive way, such as ultrafilters, Bernstein-type sets, maximal almost disjoint families etc., and asks the question "how low in the projective hierarchy do such objects first appear"? In this paper, we look at maximal independent families, a close relative of the maximal almost disjoint families studied in this way by the same authors in [1].

Definition 1.1. A family $\mathcal{I} \subseteq [\omega]^\omega$ is called *independent* if whenever we choose finite disjoint $F, G \subseteq \mathcal{I}$, we get

$$\sigma(F;G) := \left(\bigcap_{A \in F} A \right) \cap \left(\bigcap_{B \in G} (\omega \setminus B) \right) \text{ is infinite.}$$

A family $\mathcal{I} \subseteq [\omega]^\omega$ is called a *maximal independent family* (m.i.f.) if it is independent and maximal with regard to this property.

Note that maximality of $\mathcal{I}$ is equivalent to:

$$\forall X \in [\omega]^\omega \; \exists F \in [\mathcal{I}]^{<\omega} \; \exists G \in [\mathcal{I} \setminus F]^{<\omega} \; (\sigma(F;G) \subseteq^* X \; \vee \; \sigma(F;G) \cap X =^* \varnothing).$$

By identifying the space $[\omega]^\omega$ with 2^ω via characteristic functions, one can consider independent families as subsets of the reals and study their complexity in the projective hieararchy.

Remark 1.2. If $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. then it is $\mathbf{\Delta}^1_n$.

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Proof. Suppose $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. Then $\forall X \in [\omega]^\omega$:

$$X \notin \mathcal{I} \iff \exists F \in [\mathcal{I}]^{<\omega}\, \exists G \in [\mathcal{I} \setminus F]^{<\omega}$$
$$(X \notin F \wedge X \notin G \wedge (\sigma(F;G) \subseteq^* X \vee \sigma(F;G) \cap X =^* \varnothing)).$$

The last statement is easily seen to be $\mathbf{\Sigma}^1_n$. □

Theorem 1.3 (Miller; [2]). *There is no analytic m.i.f.*

An analysis of Miller's proof shows the following stronger result: $\mathbf{\Sigma}^1_n(\mathbb{C}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f., for all n, where we use "$\mathbf{\Sigma}^1_n(\mathbb{C})$" to denote the statement "all $\mathbf{\Sigma}^1_n$ sets have the Baire property". In particular, it follows that in the Cohen model there is no $\mathbf{\Sigma}^1_2$ m.i.f., that in the Solovay and the Shelah model (for projective Baire Property without inaccessible) there is no m.i.f. at all, and that $\mathsf{AD} \Rightarrow$ there is no m.i.f.

In this paper, we prove a much stronger result, namely, that in the Cohen model there is no projective m.i.f. Since $\mathbf{\Sigma}^1_2(\mathbb{C})$ is false in the Cohen model, this will show that the above implication cannot be reversed in general.

On the other hand, it is easy to construct a m.i.f. by induction using a wellorder of the reals. In particular, it is easy to see that in L, there exists a $\mathbf{\Sigma}^1_2$ m.i.f. In [2], Miller used sophisticated coding techniques to show that, in fact, there is a $\mathbf{\Pi}^1_1$ m.i.f. in L. Building on an idea due to Asger Törnquist [4], we will show that in fact this proof is unneccessary, since one can show directly in ZFC that if there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.

The paper is structured as follows: in Section 2, we prove the implication mentioned above. In Section 3 we present a break-down of Miller's original proof necessary for further development. In Section 4 we prove the main theorem about projective m.i.f.'s in the Cohen model, and in Section 5 we study a cardinal invariant related to the question of preserving or destroying a m.i.f.

2 $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$ m.i.f's

Theorem 2.1. *If there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.*

Proof. Suppose $\mathcal{I}_0$ is a $\mathbf{\Sigma}^1_2$ maximal independent family. Let $F_0 \subseteq ([\omega]^\omega)^2$ be a $\mathbf{\Pi}^1_1$ set such that $\mathcal{I}_0$ is the projection of F_0. Consider the space $\omega \,\dot\cup\, 2^{<\omega}$ as a disjoint union, and consider the mapping

$$g: \begin{array}{l} ([\omega]^\omega)^2 \longrightarrow \mathscr{P}(\omega \,\dot\cup\, 2^{<\omega}) \\ (x,y) \longmapsto x \cup \{\chi_y{\restriction}n \mid n < \omega\} \end{array}$$

where χ_y is the characteristic function of y. It is not hard to see that g is a continuous function (in the sense of the space $\mathscr{P}(\omega \,\dot\cup\, 2^{<\omega})$).

By $\mathbf{\Pi}^1_1$-uniformization, there exists a $\mathbf{\Pi}^1_1$ set $F \subseteq F_0$ which is the graph of a function, i.e., $\forall x \in \mathcal{I}_0\, \exists! y\, ((x,y) \in F)$. We let $\mathcal{I} := g[F]$ and claim that $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ m.i.f.

To see that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, note that for $z \in [\omega \,\dot\cup\, 2^{<\omega}]^\omega$, there is an explicit way to recover x and y such that $g(x,y) = z$, if such x and y exist. More precisely: for $B \subseteq 2^{<\omega}$, let $\lim(B) := \{y \in 2^\omega \mid \forall n\, (y{\restriction}n \in B)\}$. Note that if B is infinite then $\lim(B) \neq \varnothing$. Then we can say the following: $z \in \mathcal{I}$ if and only if

1. $\forall y, y' \ (y \in \lim(z \cap 2^{<\omega}) \wedge y' \in \lim(z \cap 2^{<\omega}) \ \rightarrow \ y = y')$, and
2. $\forall y \ (y \in \lim(z \cap 2^{<\omega}) \ \rightarrow \ (z \cap \omega, y) \in F)$.

This gives a $\mathbf{\Pi}^1_1$ definition of $\mathcal{I}$.

To see that $\mathcal{I}$ is independent, suppose we have $z_1, \dots z_n$ and $w_1, \dots w_\ell \in \mathcal{I}$, the z's being different from the w's. Write $a_i := z_i \cap \omega$ and $b_j := w_j \cap \omega$. Then all a_i and b_j are in $\mathrm{dom}(F) = \mathcal{I}_0$, and moreover, since F is a function, the a_i's are different from the b_j's. But then we have that $\sigma(z_1, \dots, z_n; w_1, \dots, w_\ell) \supseteq \sigma(a_1, \dots, a_n; b_1, \dots, b_\ell)$ is infinite, since the latter set is infinite by the independence of $\mathcal{I}_0$.

To show maximality of $\mathcal{I}$, suppose $W \in [\omega \dot\cup 2^{<\omega}]^\omega$ and $W \notin \mathcal{I}$. Let $A := W \cap \omega$. By maximality of $\mathcal{I}_0$, there are $a_1, \dots, a_n$ and different $b_1, \dots, b_\ell$ such that $\sigma(a_1, \dots, a_n, A; \ b_1, \dots b_\ell)$ is finite or $\sigma(a_1, \dots, a_n; \ b_1, \dots b_\ell \cup \{A\})$ is finite, w.l.o.g. the former. Then there are $z_1, \dots, z_n$ and different $w_1, \dots, w_\ell$ such that $a_i = z_i \cap \omega$ and $b_j = w_j \cap \omega$. To make sure that the "$2^{<\omega}$-part" of the z_i's and the w_j's does not make the intersection infinite, we pick two additional $t_0 \neq t_1 \in \mathcal{I}$, different from the z_i's and the w_j's. Let $t_0 = g(x_0, y_0)$ and $t_1 = g(x_1, y_1)$. If $y_0 = y_1$, then $(t_0 \setminus t_1) \cap 2^{<\omega} = \varnothing$, hence $\sigma(x_1, \dots, x_n, W, t_0; \ b_1, \dots b_\ell, t_1)$ is finite. If, on the other hand, $y_0 \neq y_1$, then the sets $\{\chi_{y_0} \restriction n \mid n < \omega\}$ and $\{\chi_{y_1} \restriction n \mid n < \omega\}$ are almost disjoint, so $(t_0 \cap t_1) \cap 2^{<\omega}$ is finite. In that case, $\sigma(x_1, \dots, x_n, W, t_0, t_1; \ b_1, \dots b_\ell)$ is finite. So in any case, $\mathcal{I} \cap \{W\}$ is not independent, completing the proof. $\square$

3 Perfect almost disjoint and almost covering sets

Next, we turn our attention to Miller's original proof of the non-existence of analytic m.i.f.'s., using it to prove a stronger result and breaking it down a bit, using the following definition.

Definition 3.1. A tree $T \subseteq 2^{<\omega}$ is called *perfect almost disjoint* (*perfect a.d.*) if it is a perfect tree and $\forall x, y \in [T] \ \{n \mid x(n) = y(n) = 1\}$ is finite. A tree $S \subseteq 2^{<\omega}$ is called *perfect almost covering* (*perfect a.c.*) if it is a perfect tree and $\forall x, y \in [T] \ \{n \mid x(n) = y(n) = 0\}$ is finite.

Definition 3.2.

1. A set $A \subseteq 2^\omega$ satisfies the *perfect-a.d.-a.c. property*, abbreviated by $\mathbb{S}_{ad\text{-}ac}$, if there exists a perfect a.d. tree T with $[T] \subseteq A$, or there exists a perfect a.c. tree S with $[S] \cap A = \varnothing$.
2. A set $A \subseteq 2^\omega$ satisfies the *perfect-a.c.-a.d. property*, abbreviated by $\mathbb{S}_{ac\text{-}ad}$, if there exists a perfect a.c. tree S with $[S] \subseteq A$, or there exists a perfect a.d. tree T with $[T] \cap A = \varnothing$.

Question 3.3. *Do the statements $\mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac})$ or $\mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$ have any interesting characterisations and/or has anything like this ever been studied previously, for example for $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ or $\mathbf{\Gamma} = \mathbf{\Sigma}^1_2$?*

Lemma 3.4. $\mathbf{\Gamma}(\mathbb{C}) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac}) \wedge \mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$ *for any projective pointclass* $\mathbf{\Gamma}$.

Proof. Let $A \subseteq 2^\omega$ be in $\mathbf{\Gamma}$. If A has the Baire property, in particular there is a basic open set $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$ (here $\subseteq^*$ and $=^*$ denote modulo meager). If we assume the former, we will find *both* a perfect a.d. tree T and a perfect a.c. tree S such that

$[T] \subseteq A$ and $[S] \subseteq A$. Analogously, if we assume the latter, we will find *both* a perfect a.d. tree T and a perfect a.c. tree S such that $[T] \cap A = \varnothing$ and $[S] \cap A = \varnothing$. Thus, it is clear that in effect we prove both $\mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac})$ and $\mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$. We only show how to construct the perfect almost disjoint tree T in the former case; the other cases are similar.

So, assume $[s] \subseteq^* A$ and let B_n be nowhere dense so that $[s] \setminus A \subseteq \bigcup_n B_n$.

- Let $s_\varnothing$ be an extension of s with at least one (new) non-zero digit, and such that $[s_\varnothing] \cap B_0 = \varnothing$. Let $k_0 := |s_\varnothing|$.
- Let $s_{\langle 0 \rangle}$ be an extension of $s_\varnothing$, with at least one new non-zero digit, and such that $[s_{\langle 0 \rangle}] \cap B_1 = \varnothing$. Let $k_1 := |s_{\langle 0 \rangle}|$.
- Let $s_{\langle 1 \rangle}$ be an extension of $s_\varnothing$ consisting only of 0's on the interval $[k_0, k_1)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 1 \rangle}] \cap B_1 = \varnothing$. Let $k_2 := |s_{\langle 1 \rangle}|$.
- Let $s_{\langle 0,0 \rangle}$ be an extension of $s_{\langle 0 \rangle}$, consisting only of 0's on the interval $[k_1, k_2)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 0,0 \rangle}] \cap B_2 = \varnothing$. Let $k_3 := |s_{\langle 0,0 \rangle}|$.
- Let $s_{\langle 0,1 \rangle}$ be an extension of $s_{\langle 0 \rangle}$, consisting only of 0's on the interval $[k_1, k_3)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 0,1 \rangle}] \cap B_2 = \varnothing$. Let $k_4 := |s_{\langle 0,0 \rangle}|$.
- Continue in the same way: $s_{\sigma ^\frown \langle i \rangle}$ extends s_σ with only 0's until the largest k_j which has been defined, followed by an arbitrary extension with at least one non-zero digit, such that $[s_{\sigma ^\frown \langle i \rangle}] \cap B_{|\sigma|+1} = \varnothing$.

Finally let T be the tree generated by $\{s_\sigma \mid \sigma \in 2^{<\omega}\}$. This is a perfect tree (because of the "new non-zero digit"), and clearly $[T] \subseteq [s] \cap A$. The construction clearly guarantees that $[T]$ is an almost disjoint tree.

To construct the perfect almost covering tree S in A, proceed analogously replacing "0" by "1" in the proof above. □

Remark 3.5. An equivalent formulation of the above lemma is: "for every countable model M there exists a perfect almost disjoint set and perfect almost covering set of Cohen reals over M".

Lemma 3.6. $\mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$*-m.i.f. and* $\mathbf{\Sigma}^1_n(\mathbb{S}_{ac\text{-}ad}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$*-m.i.f..*

Proof. We prove both statements simultaneously. Let $\mathcal{I}$ be $\mathbf{\Sigma}^1_n$, and assume, towards contradiction, that $\mathcal{I}$ is a m.i.f. Let

$$H := \{X \mid \exists F \in [\mathcal{I}]^{<\omega}\ \exists G \in [\mathcal{I} \setminus F]^{<\omega}\ (\sigma(F;G) \subseteq^* X)\}$$
$$K := \{X \mid \exists F \in [\mathcal{I}]^{<\omega}\ \exists G \in [\mathcal{I} \setminus F]^{<\omega}\ (\sigma(F;G) \cap X =^* \varnothing)\}$$

Then both H and K are $\mathbf{\Sigma}^1_n$ sets. Moreover, by maximality of $\mathcal{I}$, $[\omega]^\omega = H \cup K$.

Assume that $\mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac})$ was true. Then, applying this property to H, we either obtain a perfect almost disjoint tree T with $[T] \subseteq H$, or a perfect almost coverting tree S with $[S] \cap H = \varnothing$, hence $[S] \subseteq K$ (note that here, and in the rest of the proof, we identify subsets of ω with their characteristic function).

Alternatively, assume that $\mathbf{\Sigma}^1_n(\mathbb{S}_{ac\text{-}ad})$ was true. Then, applying this property to K, we either obtain a perfect almost covering tree S with $[S] \subseteq K$, or a perfect almost disjoint tree T with $[T] \cap K = \varnothing$, and therefore $[T] \subseteq H$.

In both cases, the proof proceeds analogously.

First assume there is a perfect almost disjoint T with $[T] \subseteq H$. For each $X \in [T]$ let F_X, G_X witness the fact that $X \in H$, and apply the Δ-systems Lemma to find distinct $X, Y \in [T]$ such that $(F_X \cup F_Y) \cap (G_X \cup G_Y) = \varnothing$. Then $\sigma(F_X \cup F_Y; G_X \cup G_Y) \subseteq^* X \cap Y =^* \varnothing$, contradicting the independence of $\mathcal{I}$.

Similarly, assume there is a perfect almost covering S with $[S] \subseteq K$, and proceed analogously. Then we obtain $\sigma(F_X \cup F_Y; G_X \cup G_Y) \cap (X \cup Y) =^* \varnothing$. But by assumption $(X \cup Y) =^* \omega$ so this implies that $\sigma(F_X \cup F_Y; G_X \cup G_Y) =^* \varnothing$, again contradicting the independence of $\mathcal{I}$. $\square$

Corollary 3.7. *$\mathbf{\Sigma}^1_n(\mathbb{C}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$ m.i.f.*

Remark 3.8. A curious aspect of this corollary is that the proof can proceed either via $\mathbb{S}_{ad\text{-}ac}$ or via $\mathbb{S}_{ac\text{-}ad}$; in fact, considering just any one of these dichotomy properties would be sufficient (see Figure 1), and in the proof of Lemma 3.6 it would be sufficient for just H or just K to be $\mathbf{\Sigma}^1_2$.

$$\begin{tikzcd}
\mathbf{\Gamma}(\mathbb{C}) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac}) & \mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac}) \arrow[dr, Rightarrow] & \\
& \mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad}) & \mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac}) \arrow[r, Rightarrow] & \nexists \mathbf{\Sigma}^1_n \text{ m.i.f.}
\end{tikzcd}$$

Figure 1: Impications in ZFC.

Question 3.9. *Can we strengthen Lemma 3.6 to $\mathbf{\Delta}^1_n(\mathbb{S}_{ad\text{-}ac}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f.? (Note that $\mathbf{\Delta}^1_n(\mathbb{S}_{ad\text{-}ac})$ and $\mathbf{\Delta}^1_n(\mathbb{S}_{ac\text{-}ad})$ are equivalent).*

4 Projective m.i.f's

The general question is: in which models do m.i.f.'s of complexity $\mathbf{\Gamma}$ exist? A recent abstract result of Schrittesser [3] shows:

Fact 4.1 (Schrittesser 2016). *In the iterated Sacks model (of any length) starting from L, there exists a (lightface) Δ^1_2 m.i.f.*

Theorem 4.2. *In the Cohen model there are no projective m.i.f.'s*

What we actually show is that in the Cohen model all projective sets (and even all sets in $L(\mathbb{R})$) satisfy $\mathbb{S}_{ad\text{-}ac}$ and $\mathbb{S}_{ac\text{-}ad}$. The main point is the following Lemma, closely related to Lemma 3.4:

Lemma 4.3. *If $c \in [s]$ is Cohen over V, then in $V[c]$ there exists a perfect almost disjoint set and a perfect almost covering set of Cohen reals over V, contained in $[s]$.*

Proof. Let $\mathbb{P}$ denote the partial order consisting of finite trees $T \subseteq 2^{<\omega}$ with the following property: $\exists k_0 < k_1 < \cdots < k_\ell$ such that $T \subseteq 2^{\leq k_\ell}$, and for every $i < \ell$, there is at most one $t \in T$ where $t \upharpoonright [k_i, k_{i+1})$ is not constantly 0 (notice that the tree constructed in the proof of Lemma 3.4 has this property). The trees are ordered by end-extension.

Notice that $\mathbb{P}$ generically adds a perfect tree T_G, defined as the limit of the trees in G. Moreover, using an analogous idea to Lemma 3.4, we can see that T is almost disjoint and that every $x \in [T_G]$ is Cohen-generic over the ground model.

Since $\mathbb{P}$ is countable, it is isomorphic to Cohen forcing. Therefore, if $V[c]$ is a Cohen extension of V, it is also a $\mathbb{P}$-generic extension of V, so there exists a perfect almost disjoint set $[T_G]$ of Cohen reals. W.l.o.g. T_G can be assumed to be within $[s]$.

To obtain a perfect almost covering set of Cohen reals in $[s]$, apply the same argument with "0" replaced by "1". □

Proof of Theorem 4.2. Let $W := V^{\mathbb{C}_\kappa}$ (for any κ), and let A be a set in W defined by a formula $\phi(x)$ with real or ordinal parameters, w.l.o.g. all of which are in V (so we can forget about them). In W, let c be Cohen over V, and assume w.l.o.g. that $\phi(c)$. Then $V[c] \models$ "$p \Vdash_{\mathbb{Q}} \phi(\check{c})$", where $\mathbb{Q}$ is the remainder forcing leading from $V[c]$ to W and p is some $\mathbb{Q}$-condition. However, since $\mathbb{C}_\kappa$ is the product forcing, $\mathbb{Q}$ is isomorphic to $\mathbb{C}_\kappa$. Moreover, since $\mathbb{C}_\kappa$ is homogeneous we can assume that p is the trivial condition, hence we really have:

$$V[c] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \phi(\check{c})\text{”}$$

Let $[s]$ be a Cohen condition with $c \in [s]$ forcing this statement in V. By Lemma 4.3, first we find a perfect a.d. tree T with $T \in V[c]$, $[T] \subseteq [s]$ and such that all $x \in [T]$ are Cohen over V. Note that this fact remains true in W, since "being a perfect set of Cohen reals" is upwards absolute. Now, for any such $x \in [T]$ (in W), we have that $x \in [s]$, and therefore $V[x]$ satisfies whatever $[s]$ forces, in particular

$$V[x] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \phi(\check{x})\text{”}$$

But, again, the remainder forcing leading from $V[x]$ to W is isomorphic to $\mathbb{C}_\kappa$, and it follows that $W \models \phi(x)$.

Similarly, we also find a perfect a.c. tree S with exactly the same properties. Thus A satisfies both $\mathbb{S}_{\textit{ad-ac}}$ and $\mathbb{S}_{\textit{ac-ad}}$, and the rest follows by Lemma 3.6. □

5 $\aleph_1$-Borel and $\aleph_1$-closed m.i.f's

The question of definable m.i.f's is closely related to questions concerning certain cardinal invariants (compare with [1]).

Definition 5.1.

1. $\mathfrak{i}$ is the least size of a m.i.f.
2. $\mathfrak{i}_{\mathrm{cl}}$ is the least κ such that there exists a collection $\{C_\alpha \mid \alpha < \kappa\}$, where each C_α is a **closed** independent family, and $\bigcup_{\alpha<\kappa} C_\alpha$ is a m.i.f.
3. $\mathfrak{i}_B$ is the least κ such that there exists a collection $\{B_\alpha \mid \alpha < \kappa\}$, where each B_α is a **Borel** independent family, and $\bigcup_{\alpha<\kappa} B_\alpha$ is a m.i.f.

It is clear that $\mathfrak{i}_B \leq \mathfrak{i}_{\text{cl}} \leq \mathfrak{i}$. It is also known that $\mathfrak{r} \leq \mathfrak{i}$ and $\mathfrak{d} \leq \mathfrak{i}$, where $\mathfrak{d}$ and $\mathfrak{r}$ denote the dominating and reaping numbers, respectively. Notice that if $\mathfrak{i}_B > \aleph_1$, then there are no $\mathbf{\Sigma}^1_2$ m.i.f.'s (since $\mathbf{\Sigma}^1_2$-sets are $\aleph_1$-unions of Borel sets).

Theorem 5.2. $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_B$.

Proof. Let $\kappa < \text{cov}(\mathcal{M})$ and let $\{B_\alpha \mid \alpha < \kappa\}$ be a collection of Borel independent families. We need to show that $\mathcal{I} := \bigcup_{\alpha<\kappa} B_\alpha$ is not maximal.

Suppose otherwise, and for every finite $E \subseteq \kappa$ define

$$H_E := \{X \mid \exists F \in [\textstyle\bigcup_{\alpha\in E} B_\alpha]^{<\omega} \; \exists G \in [\bigcup_{\alpha\in E} B_\alpha \setminus F]^{<\omega} \; (\sigma(F;G) \subseteq^* X)\}$$

$$K_E := \{X \mid \exists F \in [\textstyle\bigcup_{\alpha\in E} B_\alpha]^{<\omega} \; \exists G \in [\bigcup_{\alpha\in E} B_\alpha \setminus F]^{<\omega} \; (\sigma(F;G) \cap X =^* \varnothing)\}$$

Notice that by maximality of $\mathcal{I} = \bigcup_{\alpha<\kappa} B_\alpha$, we have

$$\bigcup \{H_E \cup K_E \mid E \in [\kappa]^{<\omega}\} = [\omega]^\omega.$$

Since $\kappa < \mathfrak{d} = \text{cov}(K_\sigma)$, there must exist a finite $E \subseteq \kappa$ such that $H_E \cup K_E \notin \mathcal{M}$. Suppose $H_E \notin \mathcal{M}$: since H_E is analytic, there exists a basic open $[s]$ with $[s] \subseteq^* H_E$. By the argument from Lemma 3.4, there exists a perfect a.d. tree T with $[T] \subseteq H_E$. But then, by the argument from Lemma 3.6, it follows that $\bigcup_{\alpha\in E} B_\alpha$ is not independent, contrary to the assumption. Likewise, if $K_E \notin \mathcal{M}$ then using the argument from Lemma 3.4, there exists a perfect a.c. tree S with $[S] \subseteq H_K$, and the rest is the same. $\square$

We end this section with the following open questions:

Question 5.3.

1. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{d}$ *or* $\mathfrak{i}_B < \mathfrak{d}$*?*
2. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{r}$ *or* $\mathfrak{i}_B < \mathfrak{r}$*?*
3. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{i}$ *or* $\mathfrak{i}_B < \mathfrak{i}$*?*
4. *Can we have* $\mathfrak{d} > \aleph_1$ *or* $\mathfrak{r} > \aleph_1$ *together with a* $\mathbf{\Sigma}^1_2$ *m.i.f.?*
5. *Does the existence of a* $\mathbf{\Sigma}^1_{n+1}$ *m.i.f. imply the existence of a* $\mathbf{\Pi}^1_n$ *m.i.f. for* $n > 2$*?*

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