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HITCHIN AND CALABI-YAU INTEGRABLE SYSTEMS VIA VARIATIONS OF HODGE STRUCTURES

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Abstract. A complex integrable system determines a family of complex tori over a Zariski-open and dense subset in its base. This family in turn yields an integral variation of Hodge structures of weight ± 1. In this paper, we study the converse of this procedure. Starting from an integral variation of Hodge structures of weight ± 1, we give a criterion for when its associated family of complex tori carries a Lagrangian structure, i.e. for when it can be given the structure of an integrable system. This sheaf-theoretic approach to (the smooth parts of) complex integrable systems enables us to apply powerful tools from Hodge and sheaf theory to study complex integrable systems. We exemplify the usefulness of this viewpoint by proving that the degree zero component of the Hitchin system for any simple adjoint or simply-connected complex Lie group G is isomorphic to a non-compact Calabi-Yau integrable system over a Zariski-open and dense subset in the corresponding Hitchin base. In particular, we recover previously known results for the case where G has a Dynkin diagram of type ADE and extend them to the remaining Dynkin types B_k, C_k, F_4 and G_2.

Contents

1. Introduction

A *complex integrable system* is classically defined by the existence of the maximal possible amount of independent Poisson-commuting holomorphic functions $f_1, \ldots, f_n : M \to \mathbb{C}$ on a holomorphic symplectic manifold (M, ω) of $\dim_{\mathbb{C}} M = 2n$. In examples coming from mathematical physics (e.g. the simple pendulum or spinning tops [AvMV04], [Aud08]), these functions can be thought of as complexified constants of motion. By the complex version of the Arnold-Liouville theorem, the generic fibers of

$$F := (f_1, \ldots, f_n) : M \to \mathbb{C}^n$$

are torsors for complex tori if we assume F to be proper[1]. Abstracting this property, a complex integrable system can be regarded as a proper morphism $\pi : (M, \omega) \to B$ between a holomorphic symplectic manifold (M, ω) and a complex manifold B such that the fibers over a Zariski-open and dense $B^\circ \subset B$ are torsors for complex tori (see Section 2 for a precise definition). In this paper we go one step further and study the Hodge-theoretic aspects of a complex integrable system. More precisely, π determines an integral variation of Hodge structures ($\mathbb{Z}$-VHS) of weight 1 over B° with underlying local system $R^1\pi_*\mathbb{Z}_{|B^\circ}$, i.e. the first integral cohomology of the fibers over B°. Conversely, any $\mathbb{Z}$-VHS V of weight 1 determines a torus fibration

$$\pi : \mathcal{J}(\mathsf{V}) \to B^\circ.$$

As we show in Proposition 2, the existence of a special section of V together with a non-degenerate polarization guarantees the existence of a Lagrangian structure on the torus fibration $\pi : \mathcal{J}(\mathsf{V}) \to B^\circ$, i.e. of a holomorphic symplectic form on the total space $\mathcal{J}(\mathsf{V})$ such that the fibers of π become Lagrangian. Since such a section has properties as the Seiberg-Witten differential of (generalized) Hitchin systems ([Don97], [HHP10]), we call it an *abstract Seiberg-Witten differential.*

The advantage of our approach is that it encodes the data of the smooth locus of a complex integrable system (with section) purely in terms of Hodge and sheaf theory. In this way powerful Hodge- and sheaf-theoretic tools can be applied to study complex integrable systems. We exemplify this approach by relating Hitchin systems to Calabi-Yau integrable systems building on and extending work by Diaconescu, Donagi and Pantev (DDP), see [DDP07]. In fact, this application was the initial motivation for our considerations and comprises the main part of this paper. We plan to explore further applications in subsequent work.

[1] This is already an idealization of complex integrable systems that are classically known. For these the generic fibers are typically non-canonically isomorphic to the complement of a divisor in a complex torus. However, they can be relatively compactified over the base (at least locally) so that we are in the described situation (see [AvMV04] for more details).

Hitchin systems and Calabi-Yau integrable systems form large classes of complex integrable systems that are constructed from Lie-theoretic and geometric data. *Hitchin systems* were originally discovered by Hitchin in [Hit87a], [Hit87b]. They are defined on the degree zero or neutral component $\mathbf{Higgs}_1(\Sigma, G) \subset \mathbf{Higgs}(\Sigma, G)$ of the moduli space of semistable G-Higgs bundles of degree zero on a compact Riemann surface Σ of genus larger than 2 and any simple complex Lie group G. In this case the generic fibers of the Hitchin map

$$\boldsymbol{h} : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}(\Sigma, G)$$

are even abelian varieties which can be described in terms of branched coverings of Σ ([Hit87a], [Don95], [DG02]). As it turns out, many classical complex integrable systems can be formulated as (generalized) Hitchin systems ([DM96b]). It is therefore tempting to believe that many explicitly describable complex integrable systems are (generalized) Hitchin systems.

Calabi-Yau integrable systems were first constructed by Donagi and Markman ([DM96a]). In contrast to Hitchin systems, which are a priori Lie-theoretic in nature, the construction of Calabi-Yau integrable systems is Hodge-theoretic. Given a (complete) family $\pi : \mathcal{X} \to B$ of compact Calabi-Yau threefolds, the intermediate Jacobians

$$J^2(X_b) = H^3(X_b, \mathbb{C})/(F^2H^3(X_b, \mathbb{C}) + H^3(X_b, \mathbb{Z}))$$

fit together to form a holomorphic family $\mathcal{J}^2(\mathcal{X}/B) \to B$ of complex tori. After the base change $\rho : \tilde{B} \to B$, where $\tilde{B}$ is the total space of the $\mathbb{C}^*$-bundle associated to the pushforward π_*K_π of the relative canonical bundle K_π, the family $\mathcal{J}^2(\rho^*\mathcal{X}/\tilde{B}) \to \tilde{B}$ of complex tori has a Lagrangian structure. In other words, the total space carries a holomorphic symplectic form making the fibers Lagrangian. The Lagrangian structure is governed by the *Yukawa cubic* which plays an important role in mirror symmetry of compact Calabi-Yau threefolds ([CK99], [DM96a]). In contrast to Hitchin systems, the fibers are almost[2] never abelian varieties but only non-degenerate complex tori (see e.g. [BL99]). Moreover, their geometry is in general difficult to describe.

Despite their different origins and properties, we study the relation between Hitchin systems and Calabi-Yau integrable systems in this paper. The fact that such a relation is possible goes back to DDP and requires to look at quasi-projective Calabi-Yau threefolds instead of compact ones. It is still possible to define intermediate Jacobians for quasi-projective Calabi-Yau threefolds ([Car80]). However, the construction method of [DM96a] does not immediately apply since it requires properties of *compact* Calabi-Yau threefolds. In [DDP07] first examples of Calabi-Yau integrable systems were constructed via the intermediate Jacobians of certain *quasi-projective* Calabi-Yau threefolds. We refer to such complex integrable systems as *non-compact Calabi-Yau integrable systems*. DDP then showed that certain Hitchin systems are isomorphic to non-compact Calabi-Yau integrable systems away from the discriminant.
A more systematic approach to non-compact Calabi-Yau integrable systems was given

[2]The only exception are the cases of rigid compact Calabi-Yau threefolds but then the resulting integrable system is trivial.

by Kontsevich and Soibelman ([KS14]) via deformation theory. However, it is in general difficult to make these non-compact Calabi-Yau integrable systems more explicit. In loc. cit. the question was raised, inspired by [DDP07], whether all Hitchin systems are non-compact Calabi-Yau integrable systems (at least away from the discriminant in the base). We can partially answer this question by extending [DDP07]:

Theorem 1 (= Corollary 5)**.** *Let G be a simple adjoint complex Lie group with Dynkin diagram Δ. Further let $(\Delta_h, \mathbf{C})$ be the unique pair consisting of an* ADE*-Dynkin diagram such that $\Delta = \Delta_h^{\mathbf{C}}$ for a subgroup $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$. Then there exists a family $\pi\colon \mathcal{X} \to \mathbf{B}(\Sigma, G)$ of quasi-projective Gorenstein threefolds* endowed with a $\mathbf{C}$-action and $\mathbf{C}$-trivial canonical classes *satisfying the following: Over a Zariski-open and dense subset $\mathbf{B}^\circ \subset \mathbf{B}(\Sigma, G)$ there is an isomorphism*

$$\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \xrightarrow{\cong} \mathbf{Higgs}_1^\circ(\Sigma, G) \tag{1}$$

of integrable systems over $\mathbf{B}^\circ$. Here $\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \subset \mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ)$ is determined by the $\mathbf{C}$-invariants in cohomology.

The procedure to go from an ADE-Dynkin diagram Δ_h to a not necessarily simply-laced Dynkin diagram $\Delta = \Delta_h^{\mathbf{C}}$ is known as *folding* in the literature on Lie theory ([Spr09], [Slo80b]), see Section 4.1 for more details. It can be depicted as follows:

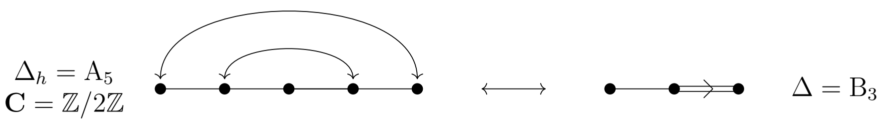

FIGURE 1. Folding of $\Delta_h = \mathrm{A}_5$ to $\Delta_h^{\mathbf{C}} = \Delta = \mathrm{B}_3$.

The ADE-case, i.e. $\mathbf{C} = 1$, appeared in [DDP07] which already alluded to the BCFG-*case*, i.e. where Δ is of type B_k, C_k, F_4 or G_2. Our construction of the Calabi-Yau threefolds, which is based on Slodowy's beautiful account of simple singularities ([Slo80b]), work for all cases at once. In fact, a closer analysis shows that our construction of the quasi-projective Calabi-Yau threefolds (without the $\mathbf{C}$-action) is in fact a special case of the one by DDP. We do not treat these aspects any further here and refer the reader to [Bec17], [Bec].

Our methods are general enough to deal with simple *simply-connected* complex Lie groups G as well by using compactly supported cohomology (Section 8). Altogether our results indicate that (non-)compact Calabi-Yau integrable systems are a much larger class than Hitchin systems. Moreover, the simply-connected cases further allow us to reobtain a simple instance of Langlands duality for Hitchin systems ([DP12, HT03]) of simple adjoint/simply-connected complex Lie groups as fiberwise-Poincaré duality (or Verdier duality) for the family $\mathcal{X}^\circ \to \mathbf{B}^\circ$, see Remark 29. Hence the geometry of the *single* family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ encodes properties of Hitchin systems of this kind over $\mathbf{B}^\circ$.

There are three main difficulties to overcome in our work:

a) Constructing the families as in Theorem 1. The main novelty in comparison with [DDP07] is the incorporation of graph automorphisms $\mathbf{C}$, in particular the construction of a $\mathbf{C}$-invariant volume form.
b) Showing that the integral variations of (mixed) Hodge structures ($\mathbb{Z}$-V(M)HS) V^H and V^{CY} over $\mathbf{B}^\circ$ determined by the Hitchin system and the quasi-projective Calabi-Yau threefolds respectively are isomorphic.
c) Deducing from b) the isomorphism of integrable systems.

In this paper, we focus on b) and c) but give a brief outline of the constructions from a) which is sufficient to understand this paper. For more details on a) we refer the reader to [Bec] and [Bec17]. The latter work will give a more geometric treatment of Theorem 1 by studying *integral* equivariant cohomolgy of the Calabi-Yau threefolds. This is subtle because integral equivariant cohomology for a finite group does not coincide with the invariants in cohomology in general. For example, this is the case for ADE-singularities under the action of graph automorphisms, i.e. for BCFG-*singularities* (Section 4.1). The forthcoming work [Bec17] also deals with *monodromy along the fibers* which is another natural possible approach to BCFG-Dynkin diagrams (as originally proposed in [DDP07]). But we will show that it yields different kinds of Hitchin systems.

Let us now give an outline of this paper's content. In **Section** 2 we give the sheaf-theoretic approach to integrable systems outlined above, in particular we introduce abstract Seiberg-Witten differentials. It abstracts some aspects of the work by Donagi-Markman ([DM96a]) and we relate it to their cubic condition. This approach is applied in **Section** 3 to the Hitchin system for a simple complex adjoint or simply-connected Lie group. It requires a detailed study of the induced $\mathbb{Z}$-VHS for which we make use of the important works [DG02] and [DP12]. The main result is an alternative description of the Hodge filtration using results by Zucker. It makes the link to M. Saito's (mixed) Hodge modules that allows us to deal with step b) above.
In **Section** 4 we summarize the construction of quasi-projective Calabi-Yau threefolds with $\mathbf{C}$-action and $\mathbf{C}$-trivial canonical bundle over the corresponding Hitchin bases following [Bec, Bec17].
In **Section** 5 we show how these families give rise to non-compact Calabi-Yau integrable systems. Since we work with quasi-projective Calabi-Yau threefolds, we cannot directly apply the results from [DM96a] for compact Calabi-Yau threefolds.
We then begin proving Theorem 1 in **Section** 6 by establishing item c) first. More concretely, we deduce Theorem 1 by showing that the Hitchin system $\mathbf{Higgs}_1^\circ(\Sigma, G) \to \mathbf{B}^\circ$ and the appropriate family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ determine isomorphic $\mathbb{Z}$-V(M)HS over $\mathbf{B}^\circ$. This requires some work because these $\mathbb{Z}$-V(M)HS are constructed from maps with mildly singular fibers as we explain in this section. From how we set things up, we can use powerful tools from Hodge and sheaf theory to deal with these complications in **Section** 7. Namely, we employ M. Saito's theory of (mixed) Hodge modules ([Sai88], [Sai90]) to show the isomorphism of $\mathbb{Z}$-V(M)HS. It allows us to show that the respective (non-trivial) Hodge filtrations, which consist of holomorphic bundles over $\mathbf{B}^\circ$, are isomorphic. In doing so, we give an example of how (mixed) Hodge modules can be

applied to problems that can be purely formulated in terms of V(M)HS but which seem to be out of reach (at least to the author) with the methods of V(M)HS.
In the final **Section** 8 we treat the Langlands dual version of Theorem 1, i.e. for simple simply-connected complex Lie groups, and briefly discuss how the geometry of the family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ relates to Langlands duality for Hitchin systems ([HT03, DP12]).

Notation. We fix some notation that is used throughout the text:
$\mathfrak{g}(\Delta)$: Simple complex Lie algebra with irreducible Dynkin diagram Δ.
$\mathfrak{t}(\Delta)$: Cartan subalgebra $\mathfrak{t}(\Delta) \subset \mathfrak{g}(\Delta)$ of type Δ with Weyl group $W = W(\Delta)$ and root space $R(\Delta)$.
$\mathfrak{t}_\alpha \subset \mathfrak{t}$: Fixed point set of the reflection s_α corresponding to $\alpha \in R(\Delta)$.
$G_{ad} = G_{ad}(\Delta)$, $G_{sc} = G_{sc}(\Delta)$: Simple complex Lie group of type Δ which is of adjoint type and simply-connected respectively.
$\mathbf{\Lambda}(G)$: cocharacter lattice $\mathbf{\Lambda}(G) = \mathrm{Hom}(\mathbb{C}^*, T)$ of G where $T \subset G$ is a(ny) maximal torus.
K_π: relative canonical sheaf for a Gorenstein morphism $\pi : X \to B$; if $B = pt$ is a point we simply write K_X.

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2. Approach to integrable systems via VHS

We fix our notation for variations of Hodge structures (VHS) by recalling the correspondence between families of complex tori and VHS of weight ± 1. Subsequently, we give a criterion when a family of complex tori gives rise to an integrable system in terms of the corresponding VHS. This is closely related to the considerations by Donagi-Markman in [DM96a] and we discuss the relation to their cubic condition. Before we start our considerations, we fix the class of integrable systems that we work with. We call these *complex integrable systems of index k*. They are not the most general class of integrable systems but are best suited for our purposes.

Let (M, ω) be a holomorphic symplectic manifold, i.e. M is a complex manifold and $\omega \in \Omega^{2,0}(M)$ is a holomorphic symplectic form. A *Lagrangian torus fibration* is a proper

Lagrangian submersion $\pi : M \to B$ with *connected* fibers. It follows as in the C^∞-case (e.g. [GS90]) that the fibers of π are torsors for complex tori. A *relative polarization of index* k, $k \in \mathbb{N}_0$, on the Lagrangian torus fibration $\pi : M \to B$ is a global section ρ of $R^2\pi_*\mathbb{Z}$ such that each pair (M_b, ρ_b), $b \in B$, is isomorphic to a non-degenerate complex torus of index k ([BL99]). More precisely, π is a torsor for the family $\hat{\pi} : T^*B/\Gamma \to B$ of non-degenerate complex tori of index k[3]. Here $\Gamma \subset T^*B$ is a relative lattice in T^*B which is determined by π (cf. [Fre99]). We call a Lagrangian torus fibration *polarizable of index* k if it admits a relative polarization of index k.

Definition 1. Let $(\mathbf{M}, \omega)$ be a holomorphic symplectic manifold. A *complex integrable system of index* $k \in \mathbb{N}_0$ is a holomorphic map $\pi : (\mathbf{M}, \omega) \to \mathbf{B}$ with the following property: There is a Zariski-open dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ such that the restriction

$$\pi^\circ := \pi_{|\mathbf{B}^\circ} : \mathbf{M}^\circ \to \mathbf{B}^\circ, \quad \mathbf{M}^\circ = \pi^{-1}(\mathbf{B}^\circ),$$

is a polarizable Lagrangian torus fibration of index k. If $k = 0$, then we simply speak of an *algebraic integrable system*[4].

Remark 2. As already mentioned, this is not the most general definition of an integrable system (cf. [DM96a]). For example, integrable systems that occur in nature often have disconnected fibers, as for example Hitchin systems. And it might happen that the index of the non-degenerate complex tori depend on the connected components of $\mathbf{B}^\circ$. However, all the integrable systems that we encounter are complex integrable systems of some fixed index k. Moreover, the relation to variation of Hodge structures of weight ± 1 becomes more direct by restricting to connecting fibers.

For the moment we concentrate on algebraic integrable systems, i.e. complex integrable systems of index 0. All of what follows easily generalizes to the more general case, cf. Remark 4.

Let $\pi : \mathbf{M} \to \mathbf{B}$ be an algebraic integrable system. By definition, there is a Zariski-open and dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ such that $\pi^\circ : \mathbf{M}^\circ \to \mathbf{B}^\circ$ is a torsor for a family of abelian varieties. Therefore the fiberwise integral cohomology and homology,

$$\mathsf{V}_\mathbb{Z} = R^1\pi^\circ_*\mathbb{Z}, \quad \mathsf{W}_\mathbb{Z} = (R^1\pi^\circ_*\mathbb{Z})^\vee,$$

underlie polarizable $\mathbb{Z}$-VHS of weight 1 and -1 respectively. In this section, we discuss the inverse process. In particular, we give a criterion when a polarizable $\mathbb{Z}$-VHS of weight ± 1 gives rise to an algebraic integrable system.

Let $\mathrm{VHS}^p_\mathbb{Z}(B, \pm 1)$ be the category of polarizable $\mathbb{Z}$-VHS on the complex manifold B and $\mathrm{AVF}^p(B)$ the category of families of abelian varieties over B (in particular, they have a global section and admit a global polarization, i.e. are polarizable). Then we have duality functors

$$(.)^\vee : \mathrm{VHS}^p_\mathbb{Z}(B, \pm 1) \to \mathrm{VHS}^p_\mathbb{Z}(B, \mp 1), \quad \mathsf{V} \mapsto \mathsf{V}^\vee := \mathrm{Hom}_{\mathrm{VHS}}(\mathsf{V}, \mathbb{Z}_B(0)),$$
$$\widehat{(.)} : \mathrm{AVF}^p(B) \to \mathrm{AVF}^p(B), \quad (\pi : M \to B) \mapsto (\hat{\pi} := \mathrm{Jac}(\pi) \to B).$$

[3]A family of complex tori is a torus fibration with a *global section.*

[4]Here we follow Freed's terminology ([Fre99]). These are in particular algebraically completely integrable systems.

Here $\mathbb{Z}_B(0)$ denotes the constant $\mathbb{Z}$-VHS of weight 0 and $\mathrm{Jac}(\pi) \to B$ is the relative Jacobian torus fibration associated with π. There are several ways to relate the two categories $\mathrm{VHS}^p_{\mathbb{Z}}(B, \pm 1)$ and $\mathrm{AVF}^p(B)$. To go from the latter to the former, we define the functors

$$\begin{aligned} \mathsf{VHS} : \mathrm{AVF}^p \to \mathrm{VHS}^p_{\mathbb{Z}}(B,1), &\quad \mathsf{VHS}(\pi) := (R^1\pi_*\mathbb{Z}, \mathcal{F}^\bullet R^1\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B), \\ \mathsf{VHS}^\vee : \mathrm{AVF}^p \to \mathrm{VHS}^p_{\mathbb{Z}}(B,-1), &\quad \mathsf{VHS}^\vee(\pi) := (.)^\vee \circ \mathsf{VHS}(\pi). \end{aligned}$$

Each polarization on π clearly induces one on $\mathsf{VHS}(\pi)$ and $\mathsf{VHS}^\vee(\pi)$ so that VHS and $\mathsf{VHS}^\vee$ are well-defined. We can also go the other way round by defining the functors

$$\begin{aligned} \mathcal{J} : \mathrm{VHS}^p_{\mathbb{Z}}(B,1) \to \mathrm{AVF}^p(B), &\quad \mathcal{J}(\mathsf{V}) = \mathsf{V}_{\mathcal{O}}/(\mathcal{F}^1\mathsf{V}_{\mathcal{O}} + \mathsf{V}_{\mathbb{Z}}), \\ \mathcal{A} : \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \to \mathrm{AVF}^p(B), &\quad \mathcal{A}(\mathsf{W}) := \mathsf{W}_{\mathcal{O}}/(\mathcal{F}^{-1}\mathsf{W}_{\mathcal{O}} + \mathsf{W}_{\mathbb{Z}}). \end{aligned}$$

The relations between these functors are summarized in the following

Proposition 1. *Let B be a complex manifold. Then the following diagram is commutative*

$$\begin{tikzcd} \mathrm{VHS}^p_{\mathbb{Z}}(B,1) \arrow[r, "\mathcal{J}"] \arrow[d, "(.)^\vee"'] & \mathrm{AVF}^p(B) \arrow[d, "\widehat{(.)}"] \\ \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \arrow[r, "\mathcal{A}"] & \mathrm{AVF}^p(B). \end{tikzcd} \tag{2}$$

Moreover, $\mathcal{A}$ and $\mathsf{V}^\vee$ yield an equivalence between $\mathrm{VHS}^p_{\mathbb{Z}}(B,-1)$ and $\mathrm{AVF}^p(B)$ whereas

$$\mathcal{J} \circ \mathsf{VHS} \simeq \widehat{(.)}, \quad \mathsf{VHS} \circ \mathcal{J} \simeq (-1) \circ (.)^\vee. \tag{3}$$

Here $(-1) : \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \to \mathrm{VHS}^p_{\mathbb{Z}}(B,1)$ is the Tate twist.

Proof. The diagram (2) is commutative because it commutes fiberwise by definition of the dual torus. To see the claimed equivalence, observe that $\mathcal{A} \circ \mathsf{VHS}^\vee(\pi) = \mathrm{Alb}(\pi)$, the Albanese fibration associated with π. But $\mathrm{Alb}(\pi) \to B$ is naturally isomorphic to $\pi \in \mathrm{AVF}^p(B)$ so that $\mathcal{A} \circ \mathsf{VHS}^\vee \simeq \mathrm{id}_{\mathrm{AVF}^p}$. Conversely, the dual of the VHS of weight 1 associated to $\mathcal{A}(\mathsf{W})$ is isomorphic to W itself. Hence we also have $\mathsf{VHS}^\vee \circ \mathcal{A} \simeq \mathrm{id}_{\mathrm{VHS}^p}$. The first relation in (3) follows by definition of the dual torus fibration. For the second relation denote $\pi : \mathcal{J}(\mathsf{V}) \to B$. Of course, if $A_b = \pi^{-1}(b)$ for $b \in B$, then $H^1(A_b, \mathbb{Z}) = \mathsf{V}^\vee_{\mathbb{Z},b}$. This implies

$$\mathsf{VHS}(\pi) = (R^1\pi_*\mathbb{Z}, \mathcal{F}^\bullet R^1\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B) \cong (\mathsf{V}^\vee_{\mathbb{Z}}, \mathcal{F}^\bullet \mathsf{V}^*_{\mathcal{O}})(-1) = \mathsf{V}^\vee(-1)$$

and therefore $\mathsf{V} \circ \mathcal{J} \simeq (-1) \circ (.)^\vee$. $\square$

Although VHS of weight -1 behave better in relation with families of abelian varieties/non-degenerate complex tori, we often work with VHS of weight 1 and the functor $\mathcal{J}$. The reason being that many of the families of abelian varieties/non-degenerate complex tori, that we consider, are induced from other families of varieties. And for the latter it is more natural to consider the induced VHS of positive weights. There is an immediate analogue for isogenous abelian varieties on the side of VHS.

Definition 2. Let V, V' be two $\mathbb{Z}$-VHS of weight $k \in \mathbb{Z}$ on a complex manifold B. We say that V and V' are *isogenous* to each other, $\mathsf{V} \simeq \mathsf{V}'$, if there is an isomorphism $\mathsf{V} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathsf{V}' \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-VHS.

Observe that two isogenous $\mathbb{Z}$-VHS V, V' have isomorphic associated filtered holomorphic bundles $(\mathsf{V} \otimes \mathcal{O}_B, \mathcal{F}^\bullet) \cong (\mathsf{V}' \otimes \mathcal{O}_B, \mathcal{F}'^\bullet)$. In particular, all statements that are independent of the underlying integer lattices $\mathsf{V}_{\mathbb{Z}}$, $\mathsf{V}'_{\mathbb{Z}}$ hold true for whole isogeny classes of VHS. Further specializing to $\mathbb{Z}$-VHS of weight ± 1, we see that isogenous $\mathbb{Z}$-VHS of weight ± 1 give rise to isogenous families of abelian varieties and vice versa.

Example 1. Let (V, Q) be a polarized $\mathbb{Z}$-VHS of weight 1 over B. Its (Tate-twisted) dual $\mathsf{V}^\vee(-1)$ is a $\mathbb{Z}$-VHS of weight 1 and the polarization $Q : \mathsf{V} \to \mathsf{V}^\vee(-1)$ is an isogeny. Under the functors $\mathcal{A}$ and $\mathcal{J}$ this corresponds to the fact that a family of abelian varieties is isogenous to its dual. Since there exist abelian varieties A such that its dual $\hat{A}$ is not isomorphic to itself, the polarization $Q : \mathsf{V} \to \mathsf{V}^\vee(-1)$ is in general not an isomorphism. For example, this is the case for (the neutral component of) Hitchin systems ([DP12]).

The next result gives a criterion when a $\mathbb{Z}$-VHS yields an algebraic integrable system. It is closely related to [DM96a].

Proposition 2. *Let V be a $\mathbb{Z}$-VHS of weight 1 over B, Q a polarization on $\mathsf{V}_R = \mathsf{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ for $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ and ∇ the Gauß-Manin connection on $\mathsf{V}_{\mathcal{O}}$. Assume there is a global section $\lambda \in \Gamma(B, \mathsf{V}_{\mathcal{O}})$ such that*

$$\phi_\lambda : TB \to \mathcal{F}^1\mathsf{V}, \quad X \mapsto \nabla_X \lambda, \tag{4}$$

is an isomorphism. Further let $\iota : \mathsf{V}_{\mathcal{O}}/\mathcal{F}^1 \to T^*B$ be the isomorphism induced by (4) and the polarization Q. Then the family

$$\pi : \mathcal{J}(\mathsf{V}) = \mathsf{V}_{\mathcal{O}}/(\mathcal{F}^1 + \mathsf{V}_{\mathbb{Z}}) \to B.$$

of abelian varieties carries a unique Lagrangian structure which makes the zero section Lagrangian and induces ι. It is independent of Q up to symplectomorphisms. Moreover, the same results hold true for $\mathcal{J}(\mathsf{V}') \to B$ where V' is any VHS in the isogeny class of V, in particular for $\mathsf{V}' = \mathsf{V}^\vee(-1)$.

Proof. We begin by recalling how $\iota : \mathsf{V}_{\mathcal{O}}/\mathcal{F}^1 \to T^*B$ is constructed. To this end, observe that the polarization Q induces an isomorphism

$$\phi_Q : \mathsf{V}_{\mathcal{O}}/\mathcal{F}^1 \to (\mathcal{F}^1)^*.$$

Then ι is simply the composition $\iota = \phi_\lambda^\vee \circ \phi_Q$. These isomorphisms further induce isomorphisms (denoted by the same symbols)[5]

$$\mathcal{J}(\mathsf{V}) \xrightarrow{\phi_Q} (\mathcal{F}^1)^*/\phi_Q(\mathsf{V}_{\mathbb{Z}}) \xrightarrow{\phi_\lambda^\vee} T^*B/\Gamma, \quad \Gamma := \phi_\lambda^*(\phi_Q(\mathsf{V}_{\mathbb{Z}})).$$

If we can show that $\Gamma \subset T^*B$ is Lagrangian, then the canonical symplectic structure η on T^*B descends to a symplectic structure $\hat{\eta}$ on T^*B/Γ. The induced symplectic

[5] If Q is not defined over $\mathbb{Z}$, then $\phi_Q(\mathsf{V}_{\mathbb{Z}})$ is *not* contained in $\mathsf{V}_{\mathbb{Z}}^\vee = Hom(\mathsf{V}_{\mathbb{Z}}, \mathbb{Z})$. In any case, $\phi_Q(\mathsf{V}_{\mathbb{Z}})$ is a local system of lattices.

structure on $\mathcal{J}(\mathsf{V})$ will satisfy all the claimed properties, in particular the zero section will be Lagrangian.

To show that $\Gamma \subset T^*B$ is Lagrangian, we have to prove that the image of $\mathsf{V}_{\mathbb{Z}}$ in T^*B under ι consists of closed (local) 1-forms. If γ is a local section of $\phi_Q(\mathsf{V}_{\mathbb{Z}}) \subset (\mathcal{F}^1)^*$, then its image is the local 1-from

$$\phi_\lambda^\vee(\gamma)(X) = \langle \gamma, \nabla_X \lambda \rangle, \quad X \in TB,$$

where the brackets are the duality pairing between $(\mathcal{F}^1)^*$ and $\mathcal{F}^1$. Its closedness can be shown similarly as in the proof of Theorem 3 in [DM96a]: Since $\mathsf{V}_{\mathbb{Z}}^\vee$ is a local system and ∇ is flat, we can represent γ around $b \in U \subset B$ as some fixed element $\gamma_0 \in \phi_\lambda^\vee(\mathsf{V}_{\mathbb{Z}})_b$, ∇ as d and λ as a map $f : U \to \mathsf{V}_b$. In particular, $v = \nabla_X \lambda \in \mathsf{V}$ is represented by $df(X)$ where $X \in TU$. It then follows that $g : U \to \mathbb{C}$, $g(b) = \langle \gamma_0, f(b) \rangle$, satisfies

$$\begin{aligned} dg(X) =& \frac{d}{dt}_{|t=0} g(\alpha(t)) \\ =& \langle \gamma_0, df(X) \rangle \\ =& \phi_\lambda^\vee(\gamma)(v). \end{aligned}$$

Here α is a curve representing the tangent vector X. Hence $\phi_\lambda^\vee(\gamma)$ is locally exact and therefore closed.
Now let Q' be another polarization. Then the previous construction can be performed for Q' as well and we denote by ω and ω' the corresponding Lagrangian structures. Morever, it follows that there is an automorphism $\psi : T^*B \to T^*B$ such that $\psi(\Gamma) = \Gamma' = \phi_\lambda^\vee \circ \phi_{Q'}(\mathsf{V}_{\mathbb{Z}})$. It induces a symplectomorphism $\psi : (T^*B/\Gamma, \hat{\eta}) \to (T^*B/\Gamma', \hat{\eta}')$. Since ω and ω' on $\mathcal{J}(\mathsf{V})$ are pull backs of $\hat{\eta}$ and $\hat{\eta}'$ respectively, it follows that ω and ω' are symplectomorphic to each other.
The last statement is immediate because if $\mathsf{V} \simeq \mathsf{V}'$ are isogenous, then V' admits a section $\lambda \in \Gamma(B, \mathsf{V}'_{\mathcal{O}})$ with the same properties as well. □

Definition 3. A section $\lambda \in \Gamma(B, \mathsf{V}_{\mathcal{O}})$, such that

$$TB \to \mathcal{F}^1\mathsf{V}, \quad X \mapsto \nabla_X \lambda,$$

is an isomorphism as above, will be called an *abstract Seiberg-Witten differential.*

Remark 3.

a) This definition is motivated by Seiberg-Witten differentials[6] of Hitchin systems, cf. Corollary 3 in the next section. It seems likely that integrable systems with an abstract Seiberg-Witten differential are in fact exact. At least this is true for Hitchin and Calabi-Yau integrable systems, which admit (abstract) Seiberg-Witten differentials.

[6]Seiberg-Witten differentials are often considered for *meromorphic* Hitchin systems, because these naturally occur in physics ([Don97]). In these cases they are meromorphic differentials. The Seiberg-Witten differentials, that we consider, are always holomorphic and are the analogues of the meromorphic ones for holomorphic Hitchin systems (cf. [HHP10]).

b) In [KS14] Kontsevich and Soibelman gave a similar approach that seems to be closely related to ours. We plan to investigate the relation between these two approaches in a future work.

Proposition 2 shows that the existence of an abstract Seiberg-Witten differential is a strong restriction on a VHS of weight 1. To illustrate this from another viewpoint, we check directly that the cubic condition of Donagi and Markman ([DM96a]) for the (local) period map $\mathcal{P}$ of $\mathcal{J}(\mathsf{V}) \to B$ is satisfied if V admits an abstract Seiberg-Witten differential. To this end, recall that we can write $d\mathcal{P}_b(v)(\alpha, \beta) = Q(\alpha, \nabla_v \beta)$ for $\alpha, \beta \in \mathcal{F}^1$. Since $Q(\mathcal{F}^1, \mathcal{F}^1) = 0$ and $\nabla Q = 0$, we conclude

$$Q(\alpha, \nabla_v \beta) = Q(\beta, \nabla_v \alpha). \tag{5}$$

By the property that $X \mapsto \nabla_X \lambda$ is an isomorphism, this can be written as

$$d\mathcal{P}(X, Y, Z) = Q(\nabla_X \lambda, \nabla_Y \nabla_Z \lambda) = d\mathcal{P}(Z, Y, X)$$

with $\alpha = \nabla_X \lambda$, $\beta = \nabla_Z \lambda$, $v = Y$. For the last equality we have employed (5). The symmetry in Y and Z can be seen by using flatness of ∇ together with $Q(\mathcal{F}^1, \mathcal{F}^1) = 0$:

$$Q(\nabla_X \lambda, \nabla_Y \nabla_Z \lambda) - Q(\nabla_X \lambda, \nabla_Z \nabla_Y \lambda) = Q(\nabla_X \lambda, \nabla_{[Y,Z]} \lambda) = 0.$$

Hence the cubic condition is satisfied.

Remark 4. There is a straightforward way to generalize Proposition 1 and Proposition 2 to families of nondegenerate complex tori and complex integrable systems of index $k > 0$. The only difference is that one has to weaken the notion of a polarization on a VHS of weight ± 1 to a non-degenerate pairing of index k.

As an application of this greater generality, we give the following example. It first appeared in [DM96a] and fits nicely into our framework.

Example 2 (Calabi-Yau integrable system)**.** Let $\pi : \mathcal{X} \to B$ be a *complete* family of compact Calabi-Yau 3-folds, i.e. the Kodaira-Spencer map at each point $b \in B$ is an isomorphism. Denote by V the $\mathbb{Z}$-VHS of weight 3 with underlying $\mathbb{Z}$-local system $\mathsf{V}_{\mathbb{Z}} = R^3 \pi_* \mathbb{Z}$. We denote by $\rho : \tilde{B} \to B$ the $\mathbb{C}^*$-bundle corresponding to the pushforward of the relative canonical bundle $F^3 \mathsf{V}_{\mathcal{O}} = \pi_* K_\pi$. Finally, let $\tilde{\pi} : \tilde{\mathcal{X}} = \rho^* \mathcal{X} \to \tilde{B}$ be the base change. Then the tautological section $\underline{s} : \tilde{B} \to \rho^* F^2 \mathsf{V}_{\mathcal{O}}$ is an abstract Seiberg-Witten differential. Indeed, the Gauß-Manin connection ∇ on V induces an isomorphism of bundles

$$T\tilde{B} \xrightarrow{\cong} \rho^* F^2 \mathsf{V}_{\mathcal{O}}, \quad v \mapsto (\rho^* \nabla)_v \underline{s},$$

see the proof of Theorem 3 in [DM96a]. The resulting complex integrable system is of index 1. We call it *compact Calabi-Yau integrable system* to emphasize that it is constructed from a family of compact Calabi-Yau threefolds.

3. VHS of Hitchin systems

We apply the viewpoint of the previous section to a well-known and intensively studied class of algebraic integrable systems, namely G-Hitchin systems ([Hit87a], [Hit87b], [Fal93], [Don95]). More precisely, we describe the $\mathbb{Z}$-VHS of weight 1 corresponding to the smooth part of a G-Hitchin system and describe the Lagrangian structure in terms of an abstract Seiberg-Witten differential. One crucial result is the description of the Hodge filtration via methods developed by Zucker ([Zuc79]). It establishes the link to M. Saito's (mixed) Hodge modules that allow us to establish the global isomorphism b) (cf. the introduction). We focus on simple complex Lie groups G which are either of *adjoint type* ($G = G_{ad}$) or *simply connected* ($G = G_{sc}$). These are the relevant cases for our later purposes. However, the methods of this section are expected to generalize to other types of G.

Fix a maximal torus $T \subset G$ with corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g} = \mathrm{Lie}(G)$, root system R and Weyl group W. Moreover, let Σ be a Riemann surface of genus $g(\Sigma) \geq 2$ and consider the bundles

$$\begin{aligned} u : \boldsymbol{U} &:= K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W \to \Sigma, \\ \tilde{u} : \tilde{\boldsymbol{U}} &:= K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t} \to \Sigma \end{aligned}$$

associated to the canonical bundle K_Σ of Σ. Here we let $\mathbb{C}^*$ act on $\mathfrak{t}$ and $\mathfrak{t}/W$ in the standard way.
The moduli space $\mathbf{Higgs}(\Sigma, G)$ of semistable G-Higgs bundle over Σ admits the Hitchin map

$$\boldsymbol{h} : \mathbf{Higgs}(\Sigma, G) \to \mathbf{B} := \mathbf{B}(\Sigma, G) := H^0(\Sigma, \boldsymbol{U})$$

which is an integrable system with disconnected fibers ([DG02], [DP12]). For our purposes we restrict attention to its restriction

$$\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}$$

to the degree zero or neutral component $\mathbf{Higgs}_1(\Sigma, G) \subset \mathbf{Higgs}(\Sigma, G)$, i.e. the isomorphism classes of G-Higgs bundles with degree zero. This is a complex integrable system in the sense of Definition 1. In particular, it has connected fibers ([DP12]).
To investigate the $\mathbb{Z}$-VHS that $\boldsymbol{h}_1$ determines over an open dense subset of $\mathbf{B}$, we need further background. First of all, we observe that each root α of $\mathfrak{t}$ yields a section

$$\boldsymbol{\alpha} : \tilde{\boldsymbol{U}} \to \tilde{u}^* K_\Sigma$$

and hence a reflection $s_{\boldsymbol{\alpha}} : \tilde{\boldsymbol{U}} \to \tilde{\boldsymbol{U}}$ with fixed point set $\tilde{\boldsymbol{U}}_\alpha = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}_\alpha$. In particular, the W-action on $\mathfrak{t}$ glues to give a W-action on $\tilde{\boldsymbol{U}}$. Since the quotient map $q : \mathfrak{t} \to \mathfrak{t}/W$ is $\mathbb{C}^*$-equivariant, it gives the morphism $\boldsymbol{q} : \tilde{\boldsymbol{U}} \to \boldsymbol{U}$ which coincides with the quotient map $\tilde{\boldsymbol{U}} \to \tilde{\boldsymbol{U}}/W = \boldsymbol{U}$ by construction. Finally, the element $\prod_{\alpha \in R} \alpha \in \mathbb{C}[\mathfrak{t}]^W$ induces the section

$$s_{\boldsymbol{br}} : \boldsymbol{U} \to u^* K_{|R|}. \tag{6}$$

With these notions at hand we can define the universal cameral curve via the cartesian square

$$\begin{tikzcd} \tilde{\boldsymbol{\Sigma}} \arrow[r] \arrow[d, "\boldsymbol{p}_1"] \arrow[dd, bend right=60, "\boldsymbol{p}"'] & \tilde{\boldsymbol{U}} \arrow[d, "\boldsymbol{q}"] \\ \Sigma \times \mathbf{B} \arrow[r, "ev"] \arrow[d, "\boldsymbol{p}_2=pr"] & \boldsymbol{U} \\ \mathbf{B}. & \end{tikzcd} \tag{7}$$

By construction $\tilde{\boldsymbol{\Sigma}}$ inherits a W-action. The pullback $\tilde{\Sigma}_b := i_b^*\tilde{\boldsymbol{\Sigma}}$ via the inclusion $i_b : \Sigma \to \{b\} \times \Sigma$ is the cameral curve $\tilde{\Sigma}_b \hookrightarrow \tilde{\boldsymbol{U}}$ corresponding to $b \in \mathbf{B}$ and we denote by

$$p_b := \boldsymbol{p}_{1,b} : \tilde{\Sigma}_b \to \Sigma$$

the induced map. These curves can be singular but for generic $b \in \mathbf{B}$ they are non-singular and $p_b : \tilde{\Sigma}_b \to \Sigma$ is a simply ramified Galois covering. More precisely, let

$$\mathbf{B}^\circ := \{b \in \mathbf{B} \mid b \text{ transversal to } \mathrm{discr}(\boldsymbol{q})^{sm}\} \tag{8}$$

where $\mathrm{discr}(\boldsymbol{q})^{sm}$ denotes the smooth locus of the discriminant $\mathrm{discr}(\boldsymbol{q})$ of $\boldsymbol{q}$. It can be shown that $\mathbf{B}^\circ$ is Zariski-open and dense in $\mathbf{B}$ ([Sco98]) and that it is precisely the locus of smooth cameral curves with simple Galois ramification. Moreover, $\mathbf{B}^\circ \subset \mathbf{B}$ is contained in the smooth locus of $\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}$.

3.1. **Stratifications.** In the following we will mainly work with $\mathbf{B}^\circ$ rather than the full Hitchin base $\mathbf{B}$. To get a clearer picture and for later purposes, we introduce a stratification on open subsets $\boldsymbol{U}^1 \subset \boldsymbol{U}$ and $\tilde{\boldsymbol{U}}^1 \subset \tilde{\boldsymbol{U}}$. They are defined as follows: Consider the open subsets

$$\mathfrak{t}^1 = \mathfrak{t} - \bigcup_{\alpha \neq \beta} \mathfrak{t}_\alpha \cap \mathfrak{t}_\beta \subseteq \mathfrak{t},$$
$$\mathfrak{t}^1/W := q(\mathfrak{t}^1) \subseteq \mathfrak{t}/W.$$

They are stratified via

$$\mathfrak{t}^1 = \mathfrak{t}^\circ \cup D := \mathfrak{t}^\circ \cup \left(\bigcup_{\alpha \in R} \mathfrak{t}_\alpha - \cup_{\beta \neq \gamma} \mathfrak{t}_\beta \cap \mathfrak{t}_\gamma \right),$$
$$\mathfrak{t}^1/W = \mathfrak{t}^\circ/W \cup (D/W)$$

where $\mathfrak{t}^\circ \subset \mathfrak{t}$ are the regular elements. Note that $D = \mathfrak{t}^1 - \mathfrak{t}^\circ$ decomposes into $D = D^s \cup D^l$ where D^s and D^l corresponds to short and long roots respectively. Moreover,

$$q : \mathfrak{t}^1 \to \mathfrak{t}^1/W$$

is a stratified map and $\mathrm{discr}(q)^{sm} = D$. Of course, $\mathfrak{t}^1$ and $\mathfrak{t}^1/W$ together with their stratifications can be glued to give

$$\tilde{\boldsymbol{U}}^1 = \tilde{\boldsymbol{U}}^\circ \cup \boldsymbol{D} = \tilde{\boldsymbol{U}}^\circ \cup \boldsymbol{D}^s \cup \boldsymbol{D}^l, \tag{9}$$

$$\boldsymbol{U}^1 = \boldsymbol{U}^\circ \cup (\boldsymbol{D}/W) = \boldsymbol{U}^\circ \cup (\boldsymbol{D}^s/W) \cup (\boldsymbol{D}^l/W) \tag{10}$$

and $\mathrm{discr}(\boldsymbol{q})^{sm} = \boldsymbol{D}$ as before. Hence $\mathbf{B}^\circ$ are precisely the sections $b : \Sigma \to \boldsymbol{U}^1$ that intersect $\boldsymbol{D}$ transversally.
Let us apply this stratification to the family $\boldsymbol{p}^\circ : \tilde{\boldsymbol{\Sigma}}^\circ \to \mathbf{B}^\circ$ of non-singular cameral curves. By construction, it factorizes as $\boldsymbol{p} = \boldsymbol{p}_2^\circ \circ \boldsymbol{p}_1^1$ where $\boldsymbol{p}_2^\circ = pr : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ is the projection and $\boldsymbol{p}_1^1 : \tilde{\boldsymbol{\Sigma}}^\circ \to \Sigma \times \mathbf{B}^\circ$ is obtained by the base change

$$\begin{tikzcd} \tilde{\boldsymbol{\Sigma}}^\circ \arrow[r] \arrow[d, "\boldsymbol{p}_1^1"'] & \tilde{\boldsymbol{U}}^1 \arrow[d] \\ \Sigma \times \mathbf{B}^\circ \arrow[r] & \boldsymbol{U}^1. \end{tikzcd}$$

The branch locus $Br \subset \Sigma \times \mathbf{B}^\circ$ of $\boldsymbol{p}_1^1$ is by definition of $\mathbf{B}$ a reduced divisor. Moreover, it can be described as

$$Br = ev^*\mathrm{discr}(\boldsymbol{q})^{sm} = V(ev^* s_{\boldsymbol{br}|\boldsymbol{U}^1})$$

where the latter is the vanishing locus of $ev^* s_{\boldsymbol{br}}$, cf. (6).

Lemma 1. *The (algebraic) divisor $Br \cap (\Sigma \times \mathbf{B}^\circ) \subset \Sigma \times \mathbf{B}^\circ$ is smooth.*

Proof. This is intuitively clear because the branch points of the cameral curves do not collide when we move in the Zariski-open $\mathbf{B}^\circ$. To make this precise, let $b_0 \in \mathbf{B}^\circ$ and choose (the germ of) a neighborhood $T \subset \mathbf{B}^\circ$ of b_0. The preimage $\boldsymbol{p}_2^{-1}(b_0)$ consists of $|R| \cdot \deg K_\Sigma$ points and we fix one of them, say $x_0 \in Br_{b_0}$. Around $(x_0, b_0) \in \Sigma \times \mathbf{B}^\circ$ the divisor $Br \cap (\Sigma \times \mathbf{B}^\circ)$ is given by

$$\{(x, b) \mid (ev^* s_{\boldsymbol{br}})(x, b) = 0\} \subset B \times T$$

where B is (the germ of) a neighborhood of x_0 that does not contain any other branch point of $b \in T$. This is possible because we work within $\mathbf{B}^\circ$. Now use local trivializations $\boldsymbol{U}_{|B} = u^{-1}(B) \cong B \times \mathfrak{t}/W$ and $(u^* K_\Sigma)_{|u^{-1}(B)} \cong B \times \mathfrak{t}/W \times \mathbb{C}$. In these terms $ev^* s_{\boldsymbol{br}}$ can be expressed as

$$(x, f_b) \mapsto (x, f_b(x), s_{br}(f_b(x))).$$

Here $f_b : B \to \mathfrak{t}/W$ corresponds to a (global) section $b \in \mathbf{B}^\circ$ in the local trivialization. Since f_b and s_{br} are transversal to each other by definition of $\mathbf{B}^\circ$, it follows that $ev^* s_{\boldsymbol{br}}$ is transversal to the zero section at (x_0, b_0). In other words, the divisor is smooth. $\square$

This lemma will become important in Section 7 and is false if we worked outside of $\Sigma \times \mathbf{B}^\circ$.

3.2. **Generic Hitchin fibers.** We next describe generic Hitchin fibers[7], more precisely generic fibers of $\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}$, following [DG02], [DP12]. Here we concentrate on the locus $\mathbf{B}^\circ \subset \mathbf{B}$ since the fibers $\boldsymbol{h}_1^{-1}(b)$, $b \in \mathbf{B}^\circ$, can be described via smooth cameral curves. To do so, consider the following sheaves for a fixed $b \in \mathbf{B}^\circ$

$$\overline{\mathcal{T}}(b) := p_{b,*}^W(\boldsymbol{\Lambda} \otimes \mathcal{O}_{\tilde{\Sigma}_b}^*),$$
$$\mathcal{T}^\circ(b) := p_{b,*}^W(\boldsymbol{\Lambda}) \otimes \mathcal{O}_\Sigma^*.$$

[7]The discussion in this subsection applies for any simple complex Lie group G, not necessarily of adjoint type or simply connected.

for the cocharacter lattice $\mathbf{\Lambda} = \mathbf{\Lambda}(G)$ of G. Here we denote by

$$p_{b,*}^W = (.)^W \circ p_{b,*}$$

the equivariant direct image for the diagonal W-action, i.e. the composition of taking W-invariants with the direct image functor. The actual sheaf $\mathcal{T}(b)$ of interest is

$$\mathcal{T}(b)(U) := \{t \in \overline{\mathcal{T}}(b)(U) \mid \alpha(t)_{|D^\alpha} = +1 \ \forall \alpha \in R(T)\}$$

where we consider a root $\alpha \in R$ as a morphism $\alpha : T \to \mathbb{C}^*$. The importance of $\mathcal{T}(b)$ stems from the following

Theorem 5 ([DG02]). *The generic Hitchin fiber $\boldsymbol{h}^{-1}(b)$, $b \in \mathbf{B}^\circ$, is a torsor for $H^1(\Sigma, \mathcal{T}(b))$.*

Remark 6. Hence the generic Hitchin fiber $\boldsymbol{h}^{-1}(b)$ is non-canonically isomorphic to $H^1(\Sigma, \mathcal{T}(b))$. Note that we do consider $\boldsymbol{h} : \mathbf{Higgs}(\Sigma, G) \to \mathbf{B}$ here and not its restrictions $\boldsymbol{h}_1$ to the neutral component.

To get hold of $H^1(\Sigma, \mathcal{T}(b))$, $b \in \mathbf{B}^\circ$, we will make use of $\overline{\mathcal{T}}(b)$ and $\mathcal{T}^\circ(b)$ as well. Observe that by definition,

$$\mathcal{T}^\circ(b) \subset \mathcal{T}(b) \subset \overline{\mathcal{T}}(b).$$

The next lemma shows that they all yield abelian varieties reflecting the fact that $\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}$ is an integrable system. It already appeared in [DP12] (Claim 3.5 (i)). We elaborate on their proof in order to obtain a useful corollary.

Lemma 2 ([DP12]). *The connected components $P^\circ(b)$, $P(b)$, $\overline{P}(b)$ of $H^1(\Sigma, \mathcal{T}^\circ(b))$, $H^1(\Sigma, \mathcal{T}(b))$, $H^1(\Sigma, \overline{\mathcal{T}}(b))$ are abelian varieties for $b \in \mathbf{B}^\circ$. All of them are isogenous to each other. In particular, the connected component $\boldsymbol{h}_1^{-1}(b) \subset \boldsymbol{h}^{-1}(b)$ is a torsor for $P(b)$.*

Proof. Consider the Grothendieck spectral sequence

$$R^p a_* R^q p_*^W \mathcal{F} \Rightarrow R^{p+q} \tilde{a}_*^W \mathcal{F}$$

for the composition $a_* \circ p_*^W = \tilde{a}_*^W$ where $a : \Sigma \to pt$ and $\tilde{a} : \tilde{\Sigma} = \tilde{\Sigma}_b \to pt$ are the constant maps. Note that $\tilde{a}_*^W(\mathcal{F}) = (.)^W \circ \tilde{a}_*(\mathcal{F}) = H^0(\tilde{\Sigma}, \mathcal{F})^W$ for any W-sheaf $\mathcal{F}$ on $\tilde{\Sigma}$. The corresponding five-term exact sequence of this spectral sequence reads as

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Sigma, p_*^W\mathcal{F}) \arrow[r, "\gamma"] & H^1(\tilde{\Sigma}, \mathcal{F})^W \arrow[r] & H^0(\Sigma, R^1 p_*^W \mathcal{F}) \arrow[dll, rounded corners] \\ & H^2(\Sigma, p_*^W\mathcal{F}) \arrow[r] & H^2(\tilde{\Sigma}, \mathcal{F})^W. \end{tikzcd} \tag{11}$$

Since p is a finite map, it follows that $R^1 p_*^W \mathcal{F} \cong \mathcal{H}^1(W, p_*\mathcal{F})$ (see [Gro57]). The latter sheaf has stalks $H^1(W, (p_*\mathcal{F})_x)$ which is finite because $H^k(W, M)$ is finite for $k \geq 1$ and any W-module M, cf. [Wei94]. As $\mathcal{H}^1(W, p_*\mathcal{F})$ is a local system on $\Sigma^\circ = \Sigma - Br_b$, it follows that $H^0(\Sigma, R^1 p_*^W \mathcal{F})$ is finite. This can be used to see that

$$\overline{P} = H^1(\Sigma, p_*^W \mathcal{F})^\circ, \quad \mathcal{F} = \mathbf{\Lambda} \otimes \mathcal{O}_{\tilde{\Sigma}},$$

is an abelian variety. Indeed, it is classical that the connected component $\tilde{P}$ of $H^1(\tilde{\Sigma}, \mathbf{\Lambda} \otimes \mathcal{O}^*_{\tilde{\Sigma}})^W$ is an abelian variety. Now restricting γ to the connected components in (11) shows that $\gamma^\circ : \overline{P} \to \tilde{P}$ is injective with finite cokernel, i.e. is an isogeny. In particular, $\overline{P}$ carries the structure of an abelian variety as well.
To prove the statement for P° and P, we observe that there are short exact sequences

$$0 \longrightarrow \mathcal{T}^\circ \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{T}^\circ \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{T} \longrightarrow \overline{\mathcal{T}} \longrightarrow \overline{\mathcal{T}}/\mathcal{T} \longrightarrow 0$$

by construction (cf. [DP12]). Note that the quotients are supported on the branch locus of $\tilde{\Sigma} \to \Sigma$, i.e. they are (sums of) skyscrapers. The corresponding long exact sequences show that each of the natural maps $H^1(\Sigma, \mathcal{T}^\circ) \to H^1(\Sigma, \mathcal{T}) \to H^1(\Sigma, \overline{\mathcal{T}})$ is surjective with finite kernel. Hence the restrictions $P^\circ \to P \to \overline{P}$ are isogenies. It follows that P and P° are abelian varieties as well. $\square$

The proof in particular determines the complex structure on P°, P and $\overline{P}$.

Corollary 1. *The $\mathbb{Z}$-Hodge structure $H_1(Q, \mathbb{Z})$ (for $Q = P^\circ, P, \overline{P}$) of weight -1 become isomorphic to $H^1(\tilde{\Sigma}, \mathbf{\Lambda} \otimes_{\mathbb{Z}} \mathbb{Q})^W(1)$ after tensoring with $\mathbb{Q}$. Hence the complex structures on these complex tori are determined by the (Tate twisted) Hodge filtration $F^\bullet H^1(\tilde{\Sigma}, \mathfrak{t})^W(1)$.*

It turns out that for our purposes (where G is either of adjoint type or simply connected), we only need to work with $\mathcal{T}^\circ$. To see this, it is helpful to introduce real versions $\mathcal{T}^\circ_{\mathbb{R}}$, $\mathcal{T}_{\mathbb{R}}$ and $\overline{\mathcal{T}}_{\mathbb{R}}$ ([DDP07], [DP12]) of the sheaves $\mathcal{T}^\circ$, $\mathcal{T}$ and $\overline{\mathcal{T}}$. These are defined by replacing $\mathcal{O}^*_{\tilde{\Sigma}}$ with the constant sheaf $\mathbb{S}^1_{\tilde{\Sigma}}$ for the circle group $\mathbb{S}^1$. If $d \in \Sigma$ is a branch point corresponding to a W-orbit $W \cdot \alpha$, $\alpha \in R$, then the stalks are given by (writing $\mathbf{\Lambda} = \mathbf{\Lambda}(G)$ multiplicatively)

$$\begin{aligned}
\overline{\mathcal{T}}_{\mathbb{R},d} &= \{\Pi_j \lambda_j \otimes z_j \in \mathbf{\Lambda} \otimes \mathbb{S}^1 \mid \alpha^\vee(\Pi_j z_j^{\langle \alpha, \lambda_j \rangle}) = 1 \in \mathbb{C}^*\}, \\
\mathcal{T}_{\mathbb{R},d} &= \{\Pi_j \lambda_j \otimes z_j \in \mathbf{\Lambda} \otimes \mathbb{S}^1 \mid \Pi_j z_j^{\langle \alpha, \lambda_j \rangle} = 1 \in \mathbb{S}^1\}, \\
\mathcal{T}^\circ_{\mathbb{R},d} &= \{\Pi_j \lambda_j \otimes z_j \in \mathbf{\Lambda} \otimes \mathbb{S}^1 \mid \Sigma_j \langle \alpha_j, \lambda_j \rangle = 0 \in \mathbb{Z}\},
\end{aligned} \tag{12}$$

cf. [DP12]. It is important to note that this description is actually independent of the chosen root in the W-orbit. A very useful observation (which first appeared in [DDP07]) is that the real versions give the same first cohomology groups.

Lemma 3 ([DP12])**.** *Let $b \in \mathbf{B}^\circ$ and let $\mathcal{F}$ be one of the sheaves $\mathcal{T}^\circ(b)$, $\mathcal{T}(b)$ or $\overline{\mathcal{T}}(b)$, $b \in \mathbf{B}^\circ$, and denote by $\mathcal{F}_{\mathbb{R}}$ the corresponding real version. Then the natural inclusion $\mathcal{F}_{\mathbb{R}} \hookrightarrow \mathcal{F}$ induces an isomorphism of abelian groups*

$$H^1(\Sigma, \mathcal{F}_{\mathbb{R}}) \cong H^1(\Sigma, \mathcal{F}).$$

3.3. **VHS in the adjoint case.** We now specialize to the case where $G = G_{ad}$ is a simple complex Lie group of adjoint type. For the reader's convenience we summarize relevant results from [DP12] which deals with the general case.

Proposition 3 ([DP12])**.** *Let G be a simple* adjoint *complex Lie group and $\mathbf{B} = \mathbf{B}(\Sigma, G)$ its Hitchin base. Then the inclusion*

$$\mathcal{T}_{\mathbb{R}}^{\circ}(b) \hookrightarrow \mathcal{T}_{\mathbb{R}}(b)$$

is in fact an equality for any $b \in \mathbf{B}^{\circ}$. Moreover, the cocharacter lattice $\operatorname{cochar}(P_b)$ *of the abelian variety $P_b^{\circ} = P_b \subset H^1(\Sigma, \mathcal{T}(b))$ is given by*

$$\operatorname{cochar}(P_b) = H^1(\Sigma, p_{b,*}^W \mathbf{\Lambda}). \tag{13}$$

Proof. Let $d \in \Sigma$ be a branch point corresponding to a W-orbit $W \cdot \alpha$ for a root $\alpha \in R$. According to (12) we have to show

$$z^{\langle \alpha, \lambda \rangle} = 1 \in \mathbb{S}^1 \quad \Longrightarrow \quad \langle \alpha, \lambda \rangle = 0 \in \mathbb{Z} \tag{14}$$

for $\lambda \otimes z \in \mathbf{\Lambda} \otimes \mathbb{S}^1$. Let $\epsilon_\alpha \in \mathbb{Z}_+$ be the positive generator of the image of $\langle \alpha, \bullet \rangle : \mathbf{\Lambda} \to \mathbb{Z}$. Then (14) follows if[8] $\epsilon_\alpha = 1$. But G is adjoint so that $\mathbf{\Lambda} = \operatorname{coweights}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ of G, i.e. we can find $\lambda \in \mathbf{\Lambda}$ such that $\langle \alpha, \lambda \rangle = 1 = \epsilon_\alpha$. Altogether we obtain $\mathcal{T}_{\mathbb{R}}^{\circ} = \mathcal{T}_{\mathbb{R}}$.
For the second statement we refer to Claim 3.6 (i) in [DP12]) which states that $\operatorname{cochar}(P^{\circ}) = H^1(\Sigma, (p_{b,*}\mathbf{\Lambda})^W)_{\mathrm{tf}}$. However, $H^1(\Sigma, p_{b,*}^W \mathbf{\Lambda})$ is actually torsion-free, see Remark 9 below, so that (13) follows. $\square$

Together with Corollary 1, this gives a complete description of the polarizable $\mathbb{Z}$-Hodge structure of weight 1 corresponding to $P_b = P_b(G_{ad})$, $b \in \mathbf{B}^{\circ}$, namely it is given by

$$(H^1(\Sigma, p_{b,*}^W \mathbf{\Lambda}), F^{\bullet} H^1(\tilde{\Sigma}_b, \mathfrak{t})^W).$$

This globalizes over $\mathbf{B}^{\circ}$ to yield the VHS in the adjoint case.

Corollary 2. *Let G_{ad} be a simple adjoint complex Lie group and*

$$\boldsymbol{h}_1^{\circ} = \boldsymbol{h}_{1,ad}^{\circ} : \mathbf{Higgs}_1(\Sigma, G_{ad})^{\circ} \to \mathbf{B}^{\circ}$$

the restriction of the Hitchin map to the neutral component and away from singular fibers. Then $\boldsymbol{h}_1^{\circ}$ is isomorphic as a family of abelian varieties to the family $\mathcal{J}(\mathsf{V}_{ad}^H) \to \mathbf{B}^{\circ}$ determined by the polarizable $\mathbb{Z}$-VHS

$$\mathsf{V}_{ad}^H := \left(R^1 \boldsymbol{p}_{2,*}(\boldsymbol{p}_{1,*}^W \mathbf{\Lambda}_{ad}), \mathcal{F}^{\bullet}(R^1 \boldsymbol{p}_*^W \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}})\right)_{|\mathbf{B}^{\circ}} \cong \mathsf{VHS}^{\vee}(\boldsymbol{h}_{1,ad}^{\circ})(-1)$$

of weight 1 over $\mathbf{B}^{\circ}$*, where $\mathbf{\Lambda}_{ad} = \mathbf{\Lambda}(G_{ad})$ is the cocharacter lattice of G_{ad}.*

Proof. It is not difficult to see that $R^1 \boldsymbol{p}_{2,*}(\boldsymbol{p}_{1,*}^W \mathbf{\Lambda})$ is a local system (cf. proof of Theorem 16), so the statement makes sense. It is known ([DG02], [DP12]) that $\boldsymbol{h}_1^{\circ} : \mathbf{Higgs}_1(\Sigma, G_{ad})^{\circ} \to \mathbf{B}^{\circ}$ is a torsor for $\mathcal{J}(\mathsf{V}_{ad}^H) \to \mathbf{B}^{\circ}$. But the former has sections, e.g. Hitchin sections, so that the claim follows. $\square$

Remark 7. The $\mathbb{Z}$-VHS V_{ad}^H differs from another VHS that can be found in the literature (e.g. [Bal06], [HHP10]), namely

$$\left(R^1 \boldsymbol{p}_*^W \mathbf{\Lambda}, \mathcal{F}^{\bullet} R^1 \boldsymbol{p}_*^W \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}}\right)_{|\mathbf{B}^{\circ}}$$

[8]Note that $\lambda \otimes 1 = 0 \in \mathbf{\Lambda} \otimes_{\mathbb{Z}} \mathbb{S}^1$ - one of the dangers when forming the tensor product of a multiplicative with an additive abelian group.

over $\mathbf{B}^\circ$ with fibers $(H^1(\tilde{\Sigma}_b, \mathbf{\Lambda})^W, H^1(\tilde{\Sigma}_b, \mathfrak{t})^W)$. As we have seen in the fiberwise case, this is in general only an isogenous VHS. Due to its simpler description, this VHS is particularly useful when the underlying integral structure is not important. This is for example the case, when one wants to compute the Donagi-Markman cubic of the Hitchin system as it is done in [Bal06], [HHP10]. However, since the integral structure is important for us, we work with V^H_{ad} when necessary.

We next describe the Lagrangian structure on $\boldsymbol{h}_1^\circ$ in terms of $\mathsf{V}^H = \mathsf{V}^H_{ad}$ by giving an abstract Seiberg-Witten differential. Of course, it is defined via the $\mathfrak{t}$-valued (holomorphic) Seiberg-Witten differentials $\lambda_b \in H^0(\tilde{\Sigma}_b, K_{\tilde{\Sigma}} \otimes \mathfrak{t})^W$, $b \in \mathbf{B}^\circ$ (e.g. [HHP10]). By construction, they are defined via the tautological section $\boldsymbol{\tau} : \tilde{\boldsymbol{U}} \to \tilde{u}^*\tilde{\boldsymbol{U}}$ and hence give a section

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_{SW} : \mathbf{B}^\circ \to \mathcal{F}^1\mathsf{V}^H_{\mathcal{O}}, \tag{15}$$

which we call Seiberg-Witten differential as well. We can now strengthen Corollary 2.

Corollary 3. *The section $\boldsymbol{\lambda} \in H^0(\mathbf{B}^\circ, \mathsf{V}^H_{\mathcal{O}})$ is an abstract Seiberg-Witten differential. It defines a Lagrangian structure on $\mathcal{J}(\mathsf{V}^H_{\mathcal{O}}) \to \mathbf{B}^\circ$ such that it becomes isomorphic as an integrable system to the Hitchin system $\boldsymbol{h}_1^\circ : \mathbf{Higgs}_1^\circ(\Sigma, G_{ad}) \to \mathbf{B}^\circ$ over $\mathbf{B}^\circ$.*

Proof. It is proven in Proposition 8.2. of [HHP10] that $\boldsymbol{\lambda}$ is an abstract Seiberg-Witten differential, i.e.

$$\phi_{\boldsymbol{\lambda}} : T\mathbf{B}^\circ \to \mathcal{F}^1\mathsf{V}^H_{\mathcal{O}}, \quad X \mapsto \nabla_X \boldsymbol{\lambda},$$

is an isomorphism. Hence $\mathcal{J}(\mathsf{V}^H) \to \mathbf{B}^\circ$ carries a Lagrangian structure $\omega_{\boldsymbol{\lambda}}$ by Proposition 2 where we use the natural polarization on $\mathsf{V}^H = \mathsf{V}^H_{ad}$. By construction of $\omega_{\boldsymbol{\lambda}}$, $\phi_{\boldsymbol{\lambda}}$ induces a symplectomorphism

$$(T^*\mathbf{B}^\circ/\Gamma, \hat{\eta}) \cong (\mathcal{J}(\mathsf{V}^H), \omega_{\boldsymbol{\lambda}}).$$

where $\Gamma \subset T^*\mathbf{B}^\circ$ is the corresponding bundle of lattices. Any choice of a Lagrangian section $s : \mathbf{B}^\circ \to \mathbf{Higgs}_1^\circ(\Sigma, G)$, say a Hitchin section, in turn yields a symplectomorphism $T^*\mathbf{B}^\circ/\Gamma \cong \mathbf{Higgs}_1^\circ(\Sigma, G)$ over $\mathbf{B}^\circ$. Altogether this yields the claim.

□

Before turning to the simply-connected case, let us outline another but equivalent way to endow $P_b = P_b^\circ$ with the structure of an abelian variety. The point is that there is another approach to describe the Hodge filtration on

$$H^1(\Sigma, p_*^W \mathbf{\Lambda})) \otimes \mathbb{C}$$

without appealing to the previous arguments. This viewpoint is crucial in Section 7 to make the relation to M. Saito's mixed Hodge modules. It uses the following theorem of Zucker:

Theorem 8 ([Zuc79])**.** *Let Σ be a compact Riemann surface and $j : \Sigma^\circ = \Sigma - S \hookrightarrow \Sigma$ the complement of a finite subset $S \subset \Sigma$. Further let V be a polarized $\mathbb{Z}$-VHS of weight m over Σ°. Then the sheaf cohomology groups $H^k(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ $(k = 0, 1, 2)$ carry a polarized $\mathbb{Z}$-Hodge structure of weight $k+m$ which is functorial with respect to pullbacks and morphisms of VHS. Moreover, these Hodge structures are compatible with Tate twists and the Leray spectral sequence for* projective *morphisms $f : X \to \Sigma$.*

We emphasize that even though Zucker works with polarized $\mathbb{R}$-VHS throughout [Zuc79], his polarized Hodge structures can be refined to $\mathbb{Z}$ as long as the VHS carries a $\mathbb{Z}$-structure, cf. Section 2 in [Zuc79]. His theory also works for Σ° directly. More precisely, if V is a VHS, then the cohomology groups $H^k_{(c)}(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}}$ carry a functorial mixed Hodge structure. They are compatible in the sense that the natural map $H^k_c(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}} \to H^k(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}}$ is a morphism of MHS. In particular, the above Hodge structure on

$$H^1(\Sigma, j_*\mathsf{V}) = \mathrm{im}[H^1_c(\Sigma^\circ, \mathsf{V}) \to H^1(\Sigma^\circ, \mathsf{V})] \tag{16}$$

(see [Loo97]) coincides with the induced one. Our next application of Zucker's theorem is precisely our case of interest.

Lemma 4. *Let $j : \Sigma^\circ = \Sigma - S \to \Sigma$ be as before and V a polarized $\mathbb{Z}$-VHS of weight $m = 2k$ and Tate type over Σ°. Then there exists a commutative diagram*

$$\begin{tikzcd} \hat{S} \arrow[r, hook] \arrow[d] & \hat{\Sigma} \arrow[d, "f"] & \hat{\Sigma}^\circ \arrow[l, hook', "\hat{j}"'] \arrow[d, "f^\circ"] \\ S \arrow[r, hook] & \Sigma & \Sigma^\circ \arrow[l, hook', "j"'] \end{tikzcd} \tag{17}$$

such that f° is a Galois covering and f is branched. Zucker's Hodge structure on $H^1(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ is isogenous to[9] *$H^1(\hat{\Sigma}^\circ, \hat{j}_*\mathsf{V}_0)^W = H^1(\hat{\Sigma}, \mathsf{V}_0)^W$ where W is the covering group of f° and V_0 the typical stalk of $\mathsf{V}_{\mathbb{Z}}$. In particular, $H^1(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ only has types $(k+1, k)$ and $(k, k+1)$.*

Proof. Up to a Tate twist, the $\mathbb{Z}$-VHS V only consists of a local system $\mathsf{V}_{\mathbb{Z}}$ of positive definite lattices so that we only write $\mathsf{V} = \mathsf{V}_{\mathbb{Z}}$. This implies that its monodromy group W has to be finite and we obtain an unbranched Galois covering $f^\circ : \hat{\Sigma}^\circ \to \Sigma^\circ$ with covering group W. Since f° is locally given by $z \mapsto z^k$, it follows that f° can be completed to a branched covering $f : \hat{\Sigma} \to \Sigma$. This yields the diagram (17) as claimed. By construction we have $(f^\circ)^*\mathsf{V} \cong \mathsf{V}_0$, i.e. $\mathsf{V} \cong (f^\circ_*\mathsf{V}_0)^W$. Now the inclusion $i : (f^\circ_*\mathsf{V}_0)^W \hookrightarrow f^\circ_*\mathsf{V}_0$ is obviously a morphism of VHS. Note that this makes sense because $f^\circ_*\mathsf{V}_0$ is again a polarized $\mathbb{Z}$-VHS of Tate type. Moreover, the natural morphism $\phi : H^1(\Sigma^\circ, f^\circ_*\mathsf{V}_0)_{\mathrm{tf}} \to H^1(\hat{\Sigma}^\circ, \mathsf{V}_0)_{\mathrm{tf}}$, induced by the Leray spectral sequence, is a morphism of Hodge structures (cf. Section 15 in [Zuc79]). As f° is finite, ϕ is an isomorphism. By the W-equivariance of f°, these morphisms fit into the commutative diagram

$$\begin{tikzcd} H^1(\Sigma^\circ, f^\circ_*\mathsf{V}_0) \arrow[r, "\phi"] & H^1(\hat{\Sigma}^\circ, \mathsf{V}_0) \\ H^1(\Sigma^\circ, \mathsf{V}) \arrow[u, "i"] \arrow[r, "\phi^W"] & H^1(\hat{\Sigma}^\circ, \mathsf{V}_0)^W \arrow[u, hook] \\ H^1_c(\Sigma^\circ, \mathsf{V}) \arrow[u] \arrow[r, "\psi^W"] & H^1_c(\hat{\Sigma}^\circ, \mathsf{V}_0)^W. \arrow[u] \end{tikzcd} \qquad \mathsf{V} = (f^\circ_*\mathsf{V}_0)^W \tag{18}$$

[9] Note that these cohomology groups are torsion-free.

Here ψ^W is induced from the natural morphism $\psi : H^1_c(\Sigma^\circ, \mathsf{V}) \to H^1_c(\hat{\Sigma}^\circ, \mathsf{V}_0)^W$. Arguing as above, we see that it is compatible with Hodge structures. Thus (18) is a commutative diagram of Hodge structures. Further ψ^W and ϕ^W are isomorphisms over $\mathbb{Q}$ because we can then split off $(f^\circ_* \mathsf{V})^W$. But the lower square in (18) factorizes over

$$H^1(\Sigma, j_* \mathsf{V}) \to H^1(\hat{\Sigma}^\circ, j_* \mathsf{V}_0)^W = H^1(\hat{\Sigma}, \mathsf{V}_0)^W$$

(cf. (16)) which thus has to be an isogeny as well. $\square$

This lemma fits precisely into the previous context. Indeed, the sheaf $p^W_* \mathbf{\Lambda} = (p_* \mathbf{\Lambda})^W$, for $p = p_b$, is a polarizable $\mathbb{Z}$-VHS V away from the branch locus $Br_b \subset \Sigma$ of weight 0 and Tate type. On $\Sigma^\circ = \Sigma - Br_b$ we have $\mathsf{V} = (p^\circ_* \mathbf{\Lambda})^W$. Moreover, the adjunction morphism

$$(p_* \mathbf{\Lambda})^W \longrightarrow j_* j^* (p_* \mathbf{\Lambda})^W = j_* \mathsf{V}$$

is an isomorphism.

Proposition 4. *The $\mathbb{Z}$-Hodge structure of weight 1 corresponding to P_b (whose underlying real torus is $H^1(\Sigma, p^W_* \mathbf{\Lambda}) \otimes S^1$) coincides with Zucker's $\mathbb{Z}$-Hodge structure on $H^1(\Sigma, p^W_* \mathbf{\Lambda})$. Both are isogenous to $H^1(\tilde{\Sigma}, \mathbf{\Lambda})^W$.*

Proof. Recall that $H^1(\Sigma, p^W_* \mathbf{\Lambda})$ is torsion-free. By construction, we further know that $\hat{\Sigma}$ of Lemma 4 coincides with the cameral curve $\tilde{\Sigma}$. Hence the claim follows from that lemma together with the previous remarks. $\square$

3.4. **VHS: Simply-connected case.** We now discuss the simply-connected case, i.e. $G = G_{sc}$ of type Δ. Since it works analogously to the adjoint case, we only summarize the main results. As before we define by $\mathbf{\Lambda}_{sc} := \mathbf{\Lambda}(G_{sc})$ the cocharacter lattice of G_{sc}. As the Hitchin base only sees the Lie algebra, it follows that $\mathbf{B}(\Sigma, G_{sc}) = \mathbf{B}(\Sigma, G_{ad})$ naturally. Then the analogue of Proposition 3 is

Proposition 5. *Let G_{sc} be a simple simply-connected complex Lie group of type Δ and $\mathbf{B}$ the corresponding Hitchin base.*

i) ([DP12]*) If $b \in \mathbf{B}^\circ$, then $\mathcal{T}(b) = \mathcal{T}^\circ(b)$ and*

$$\mathrm{cochar}(P_b) \cong H^1(\Sigma, p^W_* \mathbf{\Lambda}_{sc})_{\mathrm{tf}}.$$

ii) The VHS of weight 1 determined by $\boldsymbol{h}^\circ_{1,sc} : \mathbf{Higgs}^\circ_1(\Sigma, G_{sc}) \to \mathbf{B}^\circ$ is

$$\mathsf{V}^H_{sc} = ((R^1 \boldsymbol{p}_{2,*} \boldsymbol{p}^W_{1,*} \mathbf{\Lambda}_{sc})_{\mathrm{tf}}, \mathcal{F}^\bullet(R^1 \boldsymbol{p}^W_* \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}}))_{|\mathbf{B}^\circ} \cong \mathsf{VHS}^\vee(\boldsymbol{h}^\circ_{1,sc})(-1).$$

The analogue of Proposition 4 is still valid by using Lemma 4.

Remark 9.

a) Even though the adjoint and the simply-connected case are very similar in nature, the cohomology groups $H^1(\Sigma, \mathcal{T})$ behave differently. In fact it can be shown (cf. the proof of Lemma 4.2 in [DP12]) that

$$H^1(\Sigma, p^W_* \mathbf{\Lambda}(G))_{\mathrm{tor}} \cong \begin{cases} 0, & G = Sp(2r, \mathbb{C}), \\ Z(G), & \text{else}. \end{cases} \tag{19}$$

Hence this is always zero for $G = G_{ad}$ but is non-vanishing e.g. for $G = SL(r, \mathbb{C})$, $r \geq 2$.

b) Let ${}^L G_{ad}$ be the Langlands dual group of the simple adjoint complex Lie group G_{ad}, so that

$$\mathbf{\Lambda}({}^L G_{ad}) = \mathbf{\Lambda}_{ad}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{\Lambda}_{ad}, \mathbb{Z})$$

by definition. Moreover, ${}^L G_{ad}$ is a simple simply-connected group. Let

$$\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G_{ad}) \to \mathbf{B}(\Sigma, G_{ad}),$$
$${}^L\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, {}^L G_{ad}) \to \mathbf{B}(\Sigma, {}^L G_{ad})$$

be the corresponding neutral component of the G_{ad}- and ${}^L G_{ad}$-Hitchin system respectively. Applying Proposition 5, we see that $\mathsf{V}(\boldsymbol{h}_1^\circ)$ and $\mathsf{V}({}^L\boldsymbol{h}_1^\circ)$ are (up to a Tate twist) dual VHS. This is a very simple instance of Langlands duality for Hitchin systems ([DP12]) saying that $\boldsymbol{h}_1^\circ$ and ${}^L\boldsymbol{h}_1^\circ$ are dual torus fibrations over $\mathbf{B}^\circ$.

4. Families of quasi-projective Calabi-Yau threefolds over the Hitchin base

As before we denote by $\mathbf{B} = \mathbf{B}(\Sigma, G)$ the Hitchin base for a compact Riemann surface Σ of genus $g(\Sigma) \geq 2$ and a simple complex Lie group G which is either of adjoint type ($G = G_{ad}$) or simply-connected ($G = G_{sc}$)[10]. In this section we construct families of quasi-projective Calabi-Yau threefolds over $\mathbf{B}$. This construction is similar to the ones in [DDP07], [Sze04] but we include an action of graph automorphisms $\mathbf{C}$ (cf. the introduction) and make use of Slodowy slices. We confine ourselves to give the construction and state the main properties of the resulting quasi-projective Calabi-Yau threefolds. For the proofs and detailed comparison between our construction and the one in [DDP07], we refer to [Bec17, Bec].

4.1. **Folding.** Let Δ be an irreducible Dynkin diagram. We follow Slodowy ([Slo80b]) and define the associated symmetry group of Δ via[11]

$$AS(\Delta) := \begin{cases} 1, & \Delta \text{ is of type ADE}, \\ \mathbb{Z}/2\mathbb{Z}, & \Delta \text{ is of type } \mathrm{B}_k, \mathrm{C}_k, \mathrm{F}_4, \\ S_3, & \Delta \text{ is of type } \mathrm{G}_2, \end{cases} \tag{20}$$

for $k \geq 2$. There is a unique irreducible ADE-Dynkin diagram Δ_h such that $AS = AS(\Delta) \subset \mathrm{Aut}(\Delta_h)$ and $\Delta = \Delta_h^{AS}$. Here Δ_h^{AS} stands for the Dynkin diagram which is obtained by taking $AS(\Delta)$-invariants in the root space associated with Δ_h. Restricted to Dynkin diagrams of type $\mathrm{B}_k, \mathrm{C}_k, \mathrm{F}_4, \mathrm{G}_2$ (BCFG-*Dynkin diagrams for short*), we

[10] Note that $\mathbf{B}(\Sigma, G_{ad}) \cong \mathbf{B}(\Sigma, G_{sc})$ canonically, cf. [DP12].

[11] The case, where Δ is an ADE-Dynkin diagram, does not appear in [Slo80b]. But this is a useful convention, cf. Remark 10 and Remark 12.

obtain a bijection

$$\begin{aligned}\{\Delta \ \text{ of type BCFG}\} &\to \{(\Delta_h, \mathbf{C}) \mid \Delta_h \text{ ADE},\ 1 \neq \mathbf{C} \subset \operatorname{Aut}(\Delta_h)\}\\ \Delta &\mapsto (\Delta_h, AS(\Delta))\\ \Delta = \Delta_h^{\mathbf{C}} &\leftarrow\!\shortmid (\Delta_h, \mathbf{C}).\end{aligned} \tag{21}$$

We say that $\Delta = \Delta_h^{\mathbf{C}}$ is obtained from $(\Delta_h, \mathbf{C})$ (or simply Δ_h) by *folding*. For convenience we summarize the corresponding types in the following table (also see Section 6.2 in [Slo80b])

(22)

Δ	Δ_h	$AS(\Delta)$
$\mathrm{B}_{\mathrm{k+1}}$	A_{2k+1}	$\mathbb{Z}/2\mathbb{Z}$
C_k	D_{k+1}	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	S_3

The next definition generalizes (surface) ADE-singularities to include any Dynkin type.

Definition 4. Let Δ be an irreducible Dynkin diagram. A Δ-*singularity* is a pair (Y, H) consisting of

a) a surface singularity $Y = (Y, 0)$ of type Δ_h,
b) a subgroup $H \subset \operatorname{Aut}(Y)$ which is isomorphic to $AS(\Delta)$.

These are subject to the conditions

i) the action of H is free on $Y - \{0\}$,
ii) the induced $AS(\Delta)$-action on the dual resolution graph of its minimal resolution $\hat{Y} \to Y$ coincides with the $AS(\Delta)$-action on Δ_h.

We call a Δ-singularity an ADE-singularity if Δ is of type ADE and a BCFG-*singularity* if Δ is a BCFG-Dynkin diagram.

Remark 10. Of course, this definition goes back to Slodowy (see Section 6.2 in [Slo80b]). His original definition is different and is referred to as simple singularity of type Δ in [Slo80b] if Δ is of type BCFG. By Proposition 6.2 in [Slo80b] the two definitions coincide if Δ is of type BCFG. It is useful to directly incorporate ADE-singularities as well so that we set $AS(\Delta) = 1$ in (20).

Every Δ-singularity (Y, H) is quasi-homogenous, i.e. that Y is a quasi-homogenous singularity. In particular, Y carries a $\mathbb{C}^*$-action that commutes with the H-action. A $\mathbb{C}^*$-deformation of a Δ-singularity (Y, H) is a $\mathbb{C}^* \times H$-deformation $\mathcal{Y} \to B$ such that H acts trivially on the base. It follows from the deformation theory of ADE-singularities that every Δ-singularity (Y, H) has a semi-universal $\mathbb{C}^*$-deformation (see Chapter I.2 in [Slo80b] for more details).

4.2. **Slodowy slices.** Based on ideas of Brieskorn ([Bri71]) and Grothendieck, Slodowy gave a completely Lie-theoretic description of the deformation theory of Δ-singularities and their simultaneous resolutions (over certain fields that are not necessarily of characteristic 0). We briefly recall some of his constructions (over $\mathbb{C}$ only) here to fix notation.

As before let Δ be an irreducible Dynkin diagram and $\mathfrak{g} = \mathfrak{g}(\Delta)$ the corresponding simple complex Lie algebra. For a given subregular nilpotent element $x \in \mathfrak{g}$, there exists an $\mathfrak{sl}_2$-triple (x, y, h) where $h \in \mathfrak{g}$ is semisimple. Then the Slodowy slice through x associated to the triple (x, y, h) is defined as

$$S := x + \ker \operatorname{ad}(y) \subset \mathfrak{g}$$

where $\operatorname{ad}(y)(\bullet) = [\bullet, y]$. Since x is subregular, it follows that $\dim S = r + 2$ for $r = \operatorname{rk}(\mathfrak{g})$. For a fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, we denote by $\chi : \mathfrak{g} \to \mathfrak{t}/W$ its adjoint quotient and

$$\sigma := \chi_{|S} : S \to \mathfrak{t}/W$$

the restriction to the Slodowy slice S. It carries a $\mathbb{C}^*$-action such that σ becomes $\mathbb{C}^*$-equivariant when $\mathfrak{t}/W$ is endowed with *twice* of its usual $\mathbb{C}^*$-weight coming from the quotient $q : \mathfrak{t} \to \mathfrak{t}/W$. Moreover, it can be endowed with an action of $AS(\Delta)$ as follows: Let

$$C(x, h) = \{g \in G_{ad}(\Delta) \mid g \cdot x = x, \quad g \cdot h = h\} \subset G_{ad}(\Delta).$$

In general, this is an infinite group. However, there is a subgroup $\mathbf{C} \subset C(x, h)$ such that

$$\mathbf{C} \cong C(x, h)/C(x, h)^\circ \cong AS(\Delta)$$

where the superscript $^\circ$ denotes the identity component (see Section 7.5 in [Slo80b]). It also acts on S with the following property.

Theorem 11 ([Slo80b])**.** *Let $S \subset \mathfrak{g} = \mathfrak{g}(\Delta)$ be a Slodowy slice through a subregular nilpotent $x \in \mathfrak{g}$ defined by an $\mathfrak{sl}_2$-triple (x, y, h) and $AS(\Delta) \cong \mathbf{C} \subset C(x, h)$ as before. Further let $\sigma : S \to \mathfrak{t}/W$ be the restriction of an adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$.*

i) The surface $Y := \sigma^{-1}(\bar{0}) \subset S$ together with its $\mathbf{C}$-action has a Δ-singularity in $x \in S$. It is the only singularity of Y.

ii) The $\mathbb{C}^* \times \mathbf{C}$-deformation $\sigma : S \to \mathfrak{t}/W$ of Y is a semi-universal $\mathbb{C}^*$-deformation of $(Y, \mathbf{C})$.

Remark 12. Folding is usually a process to go from ADE-Dynkin diagrams to BCFG-Dynkin diagrams. Therefore it might seem counter-intuitive that we set $AS(\Delta) = 1$ if Δ is an ADE-Dynkin diagram but non-trivial if Δ is a BCFG-Dynkin diagram. This corresponds to the *intrinsic approach* to BCFG-singularities by Slodowy, i.e. working with a Slodowy slice $S \subset \mathfrak{g}(\Delta)$. Slodowy also gave an *extrinsic approach* where one works with a Slodowy slice $S_h \subset \mathfrak{g}(\Delta_h)$ for Δ_h as in table 22 ([Slo80b]). All of our constructions below can be performed via the extrinsic approach as well and yield the same results, see [Bec] for more details.

The stratification of $\mathfrak{t}^1/W = \mathfrak{t}^\circ \cup D^s \cup D^l$ introduced in Section 3.1 interacts with the restriction

$$S^1 := \sigma^{-1}(\mathfrak{t}^1/W) \to \mathfrak{t}^1/W \tag{23}$$

of $\sigma : S \to \mathfrak{t}/W$. It coincides with the singularity stratification of $\mathfrak{t}^1/W$ induced by σ meaning that σ is smooth over $\mathfrak{t}^\circ/W$ and

$$\sigma^{-1}(\bar{t}) \text{ has } \begin{cases} \text{an } \mathrm{A}_1\text{-singularity if } \bar{t} \in D^s, \\ \text{an } \mathrm{A}_1 \times \mathrm{A}_1\text{-singularity if } \bar{t} \in D^l \text{ and } \Delta \neq \mathrm{G}_2, \\ \text{an } \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1\text{-singularity if } \bar{t} \in D^l \text{ and } \Delta = \mathrm{G}_2. \end{cases} \tag{24}$$

To give the simultaneous resolution of $\sigma : S \to \mathfrak{t}/W$, recall Grothendieck's simultaneous resolution[12]

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\psi"] \arrow[d, "\tilde{\chi}"] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W \end{tikzcd}$$

of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$. Slodowy showed in [Slo80b] that this diagram restricts to give a simultaneous resolution

$$\begin{tikzcd} \tilde{S} \arrow[r, "\psi"] \arrow[d, "\tilde{\sigma}"] & S \arrow[d, "\sigma"] \\ \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W \end{tikzcd}$$

of $\sigma : S \to \mathfrak{t}/W$. There is a natural $\mathbf{C}$-action on $\tilde{S}$ such that all morphisms in this diagram become $\mathbf{C}$-equivariant. To get $\mathbb{C}^*$-equivariance one has to choose $\mathfrak{t}$ appropriately ([Bec]).

4.3. **Construction of threefolds.** Fix an irreducible Dynkin diagram Δ together with a Slodowy slice $S \subset \mathfrak{g}(\Delta)$ with its $\mathbb{C}^* \times \mathbf{C}$-action. Then we can construct families of surfaces over $\boldsymbol{U} = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W$ as follows: Let $L \in \mathrm{Pic}(\Sigma)$ be a spin bundle, i.e. $L^2 = K_\Sigma$. By the $\mathbb{C}^*$-equivariance[13] of $\sigma : S \to \mathfrak{t}/W$, we obtain a family

$$\boldsymbol{\sigma} : \mathcal{S}_L := L \times_{\mathbb{C}^*} S \to \boldsymbol{U}$$

of surfaces which are $\mathbf{C}$-deformations of the Δ_h-singularity. Similarly to [Sze04], [DDP07], we construct a family $\boldsymbol{\pi}_L : \mathcal{X}_L \to \mathbf{B}$ of threefolds via the diagram

$$\begin{tikzcd} \mathcal{X}_L \arrow[r] \arrow[d, "\boldsymbol{\pi}_{1,L}"] \arrow[dd, bend right=60, "\boldsymbol{\pi}_L"'] & \mathcal{S}_L \arrow[d, "\boldsymbol{\sigma}_L"] \\ \Sigma \times \mathbf{B} \arrow[r, "ev"] \arrow[d, "\boldsymbol{\pi}_2"] & \boldsymbol{U} \\ \mathbf{B}, & \end{tikzcd} \tag{25}$$

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[12] Technically, this is a simultaneous alteration (in the sense of de Jong) because we have to pass to a (branched) covering. However, we keep the common term simultaneous resolution in the following.

[13] We emphasize again that here $\mathbb{C}^*$ acts with *twice* the standard weights on $\mathfrak{t}/W$. With these weights on $\mathfrak{t}/W$, we have $\boldsymbol{U} \cong L \times_{\mathbb{C}^*} \mathfrak{t}/W$.

where the square is cartesian and $\boldsymbol{\pi}_2 : \Sigma \times \mathbf{B} \to \mathbf{B}$ is the projection. Letting $\mathbf{C}$ act trivially on $\boldsymbol{U}$ and $\Sigma \times \mathbf{B}$, all morphisms in the cartesian square of (25) are $\mathbf{C}$-equivariant.
Analogously, the total space $\tilde{S}$ of the simultaneous resolution of $S \to \mathfrak{t}/W$ can be used to construct a *smooth* family $\tilde{\boldsymbol{\pi}}_L : \tilde{\mathcal{X}}_L \to \mathbf{B}$ of threefolds that depends on the chosen simultaneous resolution and the spin bundle L as well. It factorizes over the universal cameral curve by construction,

$$\begin{tikzcd} \tilde{\mathcal{X}}_L \arrow[r, "\tilde{\boldsymbol{\pi}}_1"] \arrow[rr, bend left, "\tilde{\boldsymbol{\pi}}_L"] & \tilde{\mathbf{\Sigma}} \arrow[r, "\boldsymbol{p}"] & \mathbf{B}, \end{tikzcd}$$

and each $\tilde{\pi}_b : \tilde{X}_b \to \tilde{\Sigma}_b$ is a simultaneous resolution of $\pi_b : X_b \to \Sigma$ for $b \in \mathbf{B}^\circ$.

Theorem 13 ([Bec17]). *Let Δ be an irreducible Dynkin diagram, $S \subset \mathfrak{g}(\Delta)$ a Slodowy slice, $\tilde{S} \to S$ a simultaneous resolution and $L \in \mathrm{Pic}(\Sigma)$ a spin bundle of Σ. Then $\boldsymbol{\pi}_L : \mathcal{X}_L \to \mathbf{B}$ as well as $\tilde{\boldsymbol{\pi}}_L : \tilde{\mathcal{X}}_L \to \mathbf{B}$ is an algebraic family of quasi-projective Gorenstein threefolds with $\mathbf{C}$-trivialisable canonical class. The former is smooth over $\mathbf{B}^\circ \subset \mathbf{B}$ whereas the latter is smooth over all of $\mathbf{B}$.*

Sketch of proof. For later purposes we briefly indicate the construction of nowhere-vanishing and $\mathbf{C}$-invariant sections $s_b \in H^0(X_b, K_{X_b})$ (equivalently a $\mathbf{C}$-trivialization of K_{X_b}) and $\tilde{s}_b \in H^0(\tilde{X}_b, K_{\tilde{X}_b})$ for $b \in \mathbf{B}$. In [Bec17] we construct nowhere-vanishing and $\mathbf{C}$-invariant sections

$$\boldsymbol{s} \in H^0(\mathcal{X}, K_{\boldsymbol{\pi}_1} \otimes (\mathrm{pr}_1 \circ \boldsymbol{\pi}_1)^* K_\Sigma),$$
$$\tilde{\boldsymbol{s}} \in H^0(\tilde{\mathcal{X}}, K_{\tilde{\boldsymbol{\pi}}_1} \otimes (\mathrm{pr}_1 \circ \boldsymbol{p}_1)^* K_\Sigma),$$

where $\mathrm{pr}_1 : \Sigma \times \mathbf{B} \to \Sigma$ is the projection. Base change and the adjunction formula imply that

$$(K_{\boldsymbol{\pi}_1} \otimes (\mathrm{pr}_1 \circ \boldsymbol{\pi}_1)^* K_\Sigma)_{|X_b} \cong K_{\pi_b} \otimes \pi_b^* K_\Sigma \cong K_{X_b}$$

and analogously for $\boldsymbol{\pi}_1$ replaced by $\tilde{\boldsymbol{\pi}}_1$. For each $b \in \mathbf{B}$ the restrictions

$$s_b := \boldsymbol{s}_{|X_b} \in H^0(X_b, K_{X_b}), \quad \tilde{s}_b := \tilde{\boldsymbol{s}}_{|X_b} \in H^0(X_b, K_{\tilde{X}_b})$$

therefore $\mathbf{C}$-equivariantly trivialize K_{X_b} and $K_{\tilde{X}_b}$ respectively. □

Remark 14.

a) From now on we fix a spin bundle $L \in \mathrm{Pic}(\Sigma)$ and often drop it from the notation, e.g. $\mathcal{X} = \mathcal{X}_L$, $\boldsymbol{\pi} = \boldsymbol{\pi}_L$. It would be very interesting to understand the relation between different finite choices.

b) A closer analysis shows that in the ADE-case, i.e. $\mathbf{C} = 1$, the families $\mathcal{X}_L \to \mathbf{B}$ coincide with some of the families constructed in [DDP07]. The main novelty of this theorem is hence the BCFG-case, i.e. $\mathbf{C} \neq 1$, in particular the existence of a $\mathbf{C}$-invariant nowhere-vanshing section of the canonical sheaf K_{X_b}, $b \in \mathbf{B}$. Even in these cases, we related these families to then ones appearing in [DDP07]. It is based on the comparison between Slodowy's intrinsic and extrinsic approach (Remark 12). We refer to [Bec] for more details.

Note that the restriction $\boldsymbol{\pi}^\circ : \mathcal{X}^\circ \to \mathbf{B}^\circ$, which is smooth, factors as

$$
\begin{tikzcd}
\mathcal{X}^\circ \arrow[r] \arrow[d, "\boldsymbol{\pi}_1^1"] \arrow[dd, bend right=60, "\boldsymbol{\pi}^\circ"'] & \mathcal{S}^1 \arrow[d, "\boldsymbol{\sigma}^1"] \\
\Sigma \times \mathbf{B}^\circ \arrow[r, "ev"] \arrow[d, "\boldsymbol{\pi}_2^\circ"] & \boldsymbol{U}^1 \\
\mathbf{B}. &
\end{tikzcd}
\tag{26}
$$

and the square is again a fiber product. Here $\mathcal{S}^1 = K_\Sigma \times_{\mathbb{C}^*} S^1$ for $S^1 = \sigma^{-1}(\mathfrak{t}^1/W)$ as in (23). As explained there, not all fibers of $\boldsymbol{\pi}_1^1$ are smooth. This is one of the main differences to [Sze04] which works with the resolved local models $\tilde{S} \to \mathfrak{t}$.

5. Non-compact Calabi-Yau integrable systems

Since the general theory of [DM96a] on compact Calabi-Yau integrable systems does not immediately apply, we need some extra work to show that each family $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ of quasi-projective Calabi-Yau threefolds of Theorem 13 gives rise to an integrable system (at least over $\mathbf{B}^\circ \subset \mathbf{B}$) via their intermediate Jacobians. Some arguments of this section reoccur in Section 7 in the context of mixed Hodge modules. However, it is desirable to have more elementary arguments for constructing these non-compact Calabi-Yau integrable systems. We begin by studying the variation of mixed Hodge structures (VMHS) defined by the middle cohomology groups of $\mathcal{X}^\circ \to \mathbf{B}^\circ$ and then see that the associated intermediate Jacobian fibration over $\mathbf{B}^\circ$ is an integrable system via the methods from Section 2.

5.1. **VMHS of the family of non-compact CY3s.** Let us fix a Slodowy slice $S = x + \ker \operatorname{ad}(y) \subset \mathfrak{g}$, a spin bundle L over Σ and a family $\mathcal{X} = \mathcal{X}_L \to \mathbf{B}$ constructed from these data as in the previous section.

Proposition 6. *The cohomology sheaf* $\mathsf{V}_{\mathbb{Z}}^{CY} := R^3\boldsymbol{\pi}_*^\circ\mathbb{Z}$ *underlies a graded-polarizable* $\mathbb{Z}$*-VMHS*

$$\mathsf{V}^{CY} := (\mathsf{V}_{\mathbb{Z}}^{CY}, \mathbb{W}_\bullet^{CY}, \mathcal{F}_{CY}^\bullet).$$

Proof. This follows from Corollary 1.18. in [BEZ14] or our discussion on Saito's mixed Hodge modules in the next section. Here it is crucial that $\boldsymbol{\pi}$ is quasi-projective. $\square$

The next lemma already appeared in [DDP07] for the ADE-case.

Lemma 5. *Let* $b \in \mathbf{B}^\circ$*. The graded-polarizable* $\mathbb{Z}$*-mixed Hodge structure on* $H^3(X_b, \mathbb{Z})$ *is effective and pure of weight* 1*.*

Recall that a pure Hodge structure H is called *effective* if $H^{pq} = 0$ for $p < 0$ or $q < 0$.

Proof. Consider the Leray spectral sequence for $\pi : X = X_b \to \Sigma$,

$$H^p(\Sigma, R^q\pi_*\mathbb{Z}) \Rightarrow H^{p+q}(X, \mathbb{Z}). \tag{27}$$

Let $\Sigma^\circ \subset \Sigma$ be the locus of smooth fibers of π. Now $\pi^\circ : X^\circ \to \Sigma^\circ$ is C^∞-trivial with fiber (diffeomorphic to) the minimal resolution $\tilde{Y}$ of the Δ-singularity Y. But the latter is homotopic to a bouquet of spheres ([Mil68]) so that $(R^q\pi_*\mathbb{Z})_t = 0$ for

$q \neq 0, 2$. If $t \in F := \Sigma - \Sigma^\circ$, choose a small disc D around t such that $D \cap F = \{t\}$. If D is small enough, we can contract $\pi^{-1}(D)$ to the central fiber $Q_t := \pi^{-1}(t)$ so that $(R^q\pi_*\mathbb{Z})_t = H^q(Q_t, \mathbb{Z})$[14]. But Q_t is homeomorphic to $\tilde{Y}$ with up to three exceptional curves contracted (cf. (24)) so that again $(R^q\pi_*\mathbb{Z})_t = 0$ if $q \neq 0, 2$. Since Σ has cohomological dimension two, it follows that the Leray spectral sequence yields an isomorphism

$$H^3(X, \mathbb{Z}) \cong H^1(\Sigma, R^2\pi_*\mathbb{Z}).$$

Next we observe that $R^2\pi^\circ_*\mathbb{Z}$ carries a polarizable $\mathbb{Z}$-VHS of weight 2 and Tate type. If $j : \Sigma^\circ \to \Sigma$ denotes the open inclusion, then $R^2\pi_*\mathbb{Z} \cong j_*R^2\pi^\circ_*\mathbb{Z}$ so that Zucker's theorem (Theorem 8) implies that $H^1(\Sigma, R^2\pi_*\mathbb{Z})$ carries a functorial polarizable pure $\mathbb{Z}$-Hodge structure of weight $1 + 2 = 3$. It turns out that the Leray spectral sequence for π is compatible with mixed Hodge structures ([Ara05], [PS08], Chapter 6, as well as Section 7 for the approach via Saito's mixed Hodge modules)[15]. Hence the mixed Hodge structure on $H^3(X, \mathbb{Z})$ is in fact pure. It can be seen as in the proof of Lemma 4 that it is effective, i.e. its only (possibly) non-zero H^{pq} are H^{12} and H^{21}. □

Corollary 4. *The graded-polarizable $\mathbb{Z}$-VMHS V^{CY} is pure of weight 3, i.e. $\mathbb{W}^{CY}_\bullet = 0$, and has a second-step Hodge filtration. In particular, it is an* admissible *VMHS.*

The property of *admissibility* is rather technical but important ([SZ85], [Kas86]), not only for VMHS, but also in the theory of mixed Hodge modules. It means that the VMHS degenerates in a controlled way (at infinity). For VHS, this is automatically satisfied ([Sch73]), which explains the second statement of Corollary 4. The upshot of the previous discussion is that we can define the intermediate Jacobians

$$J^2(X_b) = H^3(X_b, \mathbb{C})/(F^2H^3(X_b, \mathbb{C}) + H^3(X_b, \mathbb{Z})), \quad b \in \mathbf{B}^\circ,$$

and these are even abelian varieties. This is in contrast to the compact case where the intermediate Jacobian $J^2(X)$ is projective iff X is a rigid compact Calabi-Yau threefold. Moreover, the intermediate Jacobian fibration

$$\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) := \mathcal{J}(\mathsf{V}^{CY}) \longrightarrow \mathbf{B}^\circ$$

over $\mathbf{B}^\circ$ is a family of abelian varieties (here and in the following we often suppress the necessary Tate twist to make V^{CY} into a VHS of weight 1).
In order to make the relation to BCFG-Hitchin systems, i.e. where the Dynkin diagram of the structure is of type BCFG, we need to consider the $\mathbf{C}$-invariants $(\mathsf{V}^{CY})^{\mathbf{C}} \subset \mathsf{V}^{CY}$ as well (which is an equality in the ADE-case where $\mathbf{C} = 1$). This is a polarizable sub-$\mathbb{Z}$-VHS of weight 3 which again only has a two-step Hodge filtration as V^{CY}. Hence

$$\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) := \mathcal{J}((\mathsf{V}^{CY})^{\mathbf{C}}) \longrightarrow \mathbf{B}^\circ$$

[14] Note that $\pi : X \to \Sigma$ is not proper so that we cannot directly apply base change to conclude the same statement.

[15] We did not see if the methods from [Zuc79] could be applied directly because in that paper only the proper case is discussed.

is a family of abelian varieties. Note that its fibers $J^2_{\mathbf{C}}(X_b)$ do not coincide with the $\mathbf{C}$-invariants $J^2(X_b)^{\mathbf{C}}$ in general because the latter might have several connected components whereas the former is always connected.

5.2. **Period map and abstract Seiberg-Witten differential.** In [DM96a] it is proven that any complete family of compact Calabi-Yau threefolds gives rise to a complex integrable system of index 1 (Example 2). As mentioned earlier, this result does not immediately apply to the above family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ even though the intermediate Jacobian fibration $\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$ is a family of polarizable abelian varieties. The section $\boldsymbol{s} \in H^0(\mathcal{X}, K_{\boldsymbol{\pi}_1} \otimes (\mathrm{pr}_1 \circ \boldsymbol{\pi}_1)^* K_\Sigma)$ of (4.3) yields the period map[16]

$$\rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to \mathsf{V}^{CY}_{\mathcal{O}}, \quad b \mapsto [\boldsymbol{s}_{|X_b}]$$

on $\mathbf{B}^\circ$. By its $\mathbf{C}$-invariance, it actually maps to the $\mathbf{C}$-invariant part of $\mathsf{V}^{CY}_{\mathcal{O}}$.

Proposition 7. *The period map $\rho_{\boldsymbol{s}} \in H^0(\mathbf{B}^\circ, (\mathsf{V}^{CY}_{\mathcal{O}})^{\mathbf{C}})$ is an abstract Seiberg-Witten differential. In particular, $\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ) \to \mathbf{B}^\circ$ carries the structure of an integrable system called* non-compact Calabi-Yau integrable system*.*

This system already appeared in [DDP07] in the ADE-case ($\mathbf{C} = 1$).

Proof. We will see a posteriori that $\rho_{\boldsymbol{s}}$ is an abstract Seiberg-Witten differential, i.e.

$$T\mathbf{B}^\circ \to (\mathcal{F}^2_{CY})^{\mathbf{C}}, \quad v \mapsto \nabla_v \rho_{\boldsymbol{s}},$$

is an isomorphism (see the proof of Theorem 5). Hence $\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$ carries the structure of an integrable system by Proposition 2. □

Remark 15. In [KS14] Kontsevich and Soibelman gave a class of quasi-projective non-singular Calabi-Yau threefolds which yield integrable systems. Their approach is deformation-theoretic and therefore closely related to [DM96a]. However, it is in general difficult to describe the appearing Calabi-Yau threefolds more concretely in the deformation-theoretic approach. We leave it for future work to understand how the above families $\mathcal{X} \to \mathbf{B}$ fit into their framework.

A similar discussion applies to the family $\tilde{\boldsymbol{\pi}} : \tilde{\mathcal{X}} \to \mathbf{B}$. More precisely, the above methods also show that $R^3\tilde{\boldsymbol{\pi}}_*\mathbb{Z}$ carries the structure of a graded-polarizable $\mathbb{Z}$-VMHS which is in fact pure and effective of weight 1 up to a Tate twist (cf. proof of Theorem 5). We denote it by

$$\tilde{\mathsf{V}}^{CY} = (\tilde{\mathsf{V}}^{CY}_{\mathbb{Z}}, \tilde{\mathcal{F}}^\bullet_{CY}). \tag{28}$$

Note that we do not have to restrict to $\mathbf{B}^\circ$ because $\tilde{\boldsymbol{\pi}}$ is smooth.

[16]To save notation we make no notational distinction between a locally free sheaf and the corresponding geometric vector bundle.

6. Isomorphism with the Hitchin system

In the previous section we have seen how each family $\mathcal{X} \to \mathbf{B}$ of quasi-projective Calabi-Yau threefolds over the Hitchin base $\mathbf{B} = \mathbf{B}(\Sigma, G_{ad})$ gives rise to a non-compact Calabi-Yau integrable system over $\mathbf{B}^\circ$. We now show that this integrable system is isomorphic to the Hitchin system $\mathbf{Higgs}_1^\circ(\Sigma, G_{ad}) \to \mathbf{B}^\circ$. In order to do so, we follow the approach in Section 2 and deduce the isomorphism from an isomorphism of VHS.

Theorem 16. *Let Δ be an irreducible Dynkin diagram, $G = G_{ad}$ the simple adjoint complex Lie group with Dynkin diagram $\Delta = (\Delta_h)^{\mathbf{C}}$ as in Section 4.1. Further let $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ any of the families of quasi-projective Gorenstein Calabi-Yau threefolds as constructed in Theorem 13. Then the polarizable $\mathbb{Z}$-VHS V^H_{ad} (see Corollary 2) determined by the neutral component $\boldsymbol{h}_1^\circ : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}^\circ$ of the Hitchin system and $(\mathsf{V}^{CY})^{\mathbf{C}}$ determined by $\boldsymbol{\pi}^\circ : \mathcal{X}^\circ \to \mathbf{B}^\circ$ are isomorphic up to a Tate twist*

$$\mathsf{V}^H(-1) \cong (\mathsf{V}^{CY})^{\mathbf{C}} \tag{29}$$

over the locus $\mathbf{B}^\circ \subset \mathbf{B}$ of smooth cameral curves in the Hitchin base $\mathbf{B} = \mathbf{B}(\Sigma, G)$.

Before we prove Theorem 16 in the next section, let us see how it implies Theorem 1 from the introduction.

Corollary 5. *Keep the notation of Theorem 16. Then there is an isomorphism*

$$\begin{tikzcd} \mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Higgs}_1^\circ(\Sigma, G) \arrow[dl, "\boldsymbol{h}_1^\circ"] \\ & \mathbf{B}^\circ & \end{tikzcd}$$

of integrable systems over $\mathbf{B}^\circ$ that intertwines the cubics.

Remark 17. Unfortunately, in the BCFG-case this result does not yield a family of non-compact Calabi-Yau threefolds over $\mathbf{B}^\circ$, whose intermediate Jacobian fibration is isomorphic to the BCFG-Hitchin system. This will be achieved in [Bec17].

Proof. Theorem 16 implies that the two families of abelian varieties are isomorphic over $\mathbf{B}^\circ$. Recall here that both systems have sections. It remains to prove that the cubics are exchanged.
On both sides the cubic is determined by the (abstract) Seiberg-Witten differential (cf. Proposition 4 and Proposition 3). Hence it suffices to prove that the period map $\rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to (\mathsf{V}^{CY}_{\mathcal{O}})^{\mathbf{C}}$ corresponds to the Seiberg-Witten differential $\boldsymbol{\lambda} : \mathbf{B}^\circ \to \mathsf{V}^H_{\mathcal{O}}$.
We first observe that $\tilde{\mathsf{V}}^{CY}_{\mathbb{C}}$ is given by

$$R^3\tilde{\boldsymbol{\pi}}_*\mathbb{C} \cong R^1\boldsymbol{p}_*R^2\tilde{\boldsymbol{\pi}}_{1,*}\mathbb{C} \cong R^1\boldsymbol{p}_*\mathfrak{t}_h. \tag{30}$$

The first isomorphism follows from the Leray spectral sequence. The second is a consequence of the fact that $\tilde{S} \to \mathfrak{t}$ is C^∞-trivial ([Slo80a]) and $H^2(\tilde{S}_t, \mathbb{C}) \cong \mathfrak{t}_h$. The section $\tilde{\boldsymbol{s}} \in H^0(\tilde{\mathcal{X}}, K_{\tilde{\boldsymbol{\pi}}_1} \otimes (\mathrm{pr}_1 \circ \boldsymbol{p}_1)^* K_\Sigma)$ from (4.3) induces a period map

$$\rho_{\tilde{\boldsymbol{s}}} : \mathbf{B}^\circ \to (\tilde{\mathsf{V}}^{CY}_{\mathcal{O}})^{\mathbf{C}}.$$

It is related to $\rho_{\boldsymbol{s}}$ as follows: The natural map $\tilde{X} \to \mathcal{X}$ induces a $\mathbf{C}$-equivariant morphism

$$\Psi^* : \mathsf{V}_{\mathcal{O}}^{CY} \to \tilde{\mathsf{V}}_{\mathcal{O}}^{CY}.$$

which is in fact a monomorphism. From the construction of $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ (see [Bec],[Bec17]) it follows that $\Psi^* \circ \rho_{\boldsymbol{s}} = \rho_{\tilde{\boldsymbol{s}}}$ (after tensoring with $\mathcal{O}_{\mathbf{B}^\circ}$). By the $\mathbf{C}$-invariance of $\tilde{\boldsymbol{s}}$ and $\mathfrak{t}_h^{\mathbf{C}} = \mathfrak{t}$ the period map can therefore be seen as a map $\rho_{\tilde{\boldsymbol{s}}} : \mathbf{B}^\circ \to R^1\boldsymbol{p}_*\mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ}$. Then we have the equality

$$\rho_{\tilde{\boldsymbol{s}}} = \boldsymbol{\lambda} \in H^0(\mathbf{B}^\circ, \mathsf{V}_{\mathcal{O}}^H)$$

which makes sense because $\mathsf{V}_{\mathcal{O}}^H \subset R^1\boldsymbol{p}_*\mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ}$. This follows from the construction of the Leray spectral sequence for the composition $\tilde{\boldsymbol{\pi}} = \boldsymbol{p} \circ \tilde{\boldsymbol{\pi}}_1$ of *submersions* (cf. [GH94]) and the fact that both $\rho_{\tilde{\boldsymbol{s}}}$ and $\boldsymbol{\lambda}$ are obtained by a base change from the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$. □

Remark 18. As a by-product, we see that the cohomology class $[\boldsymbol{s}_b] \in H^3(X_b, \mathbb{C})$ of the volume form sits in $H^{2,1}(X_b) \cong H^{1,0}(\tilde{\Sigma}_b, \mathbb{C})^W$. This is again in strong contrast to the compact case. Moreover, the previous proof justifies the earlier claim that $\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$ has the structure of an integrable system.

7. Proof of Theorem 16

In this section we prove Theorem 16. The basic idea is to use a global version of the Leray spectral sequence (27) and lift it to $\mathbb{Z}$-VHS (see below for a more precise statement). The latter is technical but by showing that this lifted spectral sequence degenerates, we can deduce Theorem 16. As a first step we show that the global version of (27) degenerates in the category of abelian sheaves.

Lemma 6. *Let $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1 : \mathcal{X}^\circ \to \mathbf{B}^\circ$ be as in (26). Then the Leray spectral sequence degenerates and gives isomorphisms of abelian sheaves*

$$R^3\boldsymbol{\pi}_*^\circ\mathbb{Z} \cong R^1\boldsymbol{\pi}_{2*}^\circ R^2\boldsymbol{\pi}_{1*}^1\mathbb{Z}, \tag{31}$$

Moreover, it yields a morphism

$$R^1\boldsymbol{\pi}_{2*}^\circ((R^2\boldsymbol{\pi}_{1*}^1\mathbb{Z})^{\mathbf{C}}) \to (R^3\boldsymbol{\pi}_*^\circ\mathbb{Z})^{\mathbf{C}}. \tag{32}$$

We will see in Section 7.3 that the morphism (32) is an isomorphism.

Proof. We first consider the case without taking $\mathbf{C}$-invariants. The Leray spectral sequence for $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1$ reads as

$$E_2^{p,q} = R^p\boldsymbol{\pi}_{2*}^\circ(R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z}) \Rightarrow R^{p+q}\boldsymbol{\pi}_*^\circ\mathbb{Z}.$$

We first claim that $R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z} = 0$ for $q \notin \{0, 2\}$. To do so, we consider for each $b \in \mathbf{B}^\circ$ the commutative diagram

$$\begin{tikzcd}
Q_{t,b} \arrow[r] \arrow[d] & X_b \arrow[r] \arrow[d, "\pi_b"] & \mathcal{X}^\circ \arrow[d, "\boldsymbol{\pi}_1^1"] \\
\{t\} \times \{b\} \arrow[r, "i"] \arrow[rr, bend right, "i_{b,t}"] & \Sigma \arrow[r, "i_b"] & \Sigma \times \mathbf{B}^\circ
\end{tikzcd}$$

where each of the squares is a fiber product. Using base change (for locally trivial maps), we obtain

$$i_{b,t}^*(R^q\boldsymbol{\pi}^1_{1*}\mathbb{Z}) \cong i^* i_b^*(R^q\boldsymbol{\pi}^1_{1*}\mathbb{Z}) \cong i^*(R^q\pi_{b*}\mathbb{Z}) = (R^q\pi_{b*}\mathbb{Z})_t.$$

From the local theory (cf. proof of Lemma 5) we know that $(R^q\pi_{b*}\mathbb{Z})_t = 0$ if $q \notin \{0,2\}$ for all $t \in \Sigma$. Hence the claim follows.
As a first consequence we see that $d_2 = 0$ on the E_2-page. To see that the higher differentials d_r, $r \geq 3$, also vanish, observe that the projection $\boldsymbol{\pi}_2^\circ = pr : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ is proper. Hence for any sheaf $\mathcal{F}$ on $\Sigma \times \mathbf{B}^\circ$ we can compute the stalks of $R^p\boldsymbol{\pi}^\circ_{2!}\mathcal{F} = R^p\boldsymbol{\pi}^\circ_{2*}\mathcal{F}$ as

$$R^p\boldsymbol{\pi}^\circ_{2*}\mathcal{F}_b \cong H^p(\Sigma, i_b^*\mathcal{F}).$$

But the cohomological dimension of Σ is 2, so that $R^p\boldsymbol{\pi}^\circ_{2*}\mathcal{F} = 0$ for $p > 2$. This not only implies that $d_r = 0$ for $r \geq 3$ but also $R^3\boldsymbol{\pi}^\circ_{2*}(R^0\boldsymbol{\pi}^1_{1*}\mathbb{Z}) = 0$. Hence the Leray spectral sequence degenerates and

$$R^3\boldsymbol{\pi}^\circ_*\mathbb{Z} \cong R^1\boldsymbol{\pi}^\circ_{2*}R^2\boldsymbol{\pi}^1_{1*}\mathbb{Z}.$$

The morphism (32) is then induced by the inclusion $(R^2\boldsymbol{\pi}^\circ_{2*}\mathbb{Z})^{\mathbf{C}} \hookrightarrow R^2\boldsymbol{\pi}^\circ_{2*}\mathbb{Z}$. $\square$

Using mixed Hodge modules, we will see below that this isomorphism can be lifted to an isomorphism of VHS. To make the connection to the VHS V^H determined by the Hitchin system, we need the next:

Proposition 8. *Let $\tilde{\boldsymbol{U}}^1 \subset \tilde{\boldsymbol{U}}$ and $\boldsymbol{U}^1 \subset \boldsymbol{U}$ be as in (9), (10) and $\mathcal{S}^1 := \boldsymbol{\sigma}^{-1}(\boldsymbol{U}^1) \subset \mathcal{S}$. Further denote by $\boldsymbol{q}^1 : \tilde{\boldsymbol{U}}^1 \to \boldsymbol{U}^1$ and $\boldsymbol{\sigma}^1 : \mathcal{S}^1 \to \boldsymbol{U}^1$ the restrictions of $\boldsymbol{q}$ and $\boldsymbol{\sigma}$ respectively. Then there are isomorphisms of constructible sheaves*

$$R^2\boldsymbol{\sigma}^1_*\mathbb{Z} \cong (\boldsymbol{q}^1_*\boldsymbol{\Lambda}_h)^W, \quad (R^2\boldsymbol{\sigma}^1_*\mathbb{Z})^{\mathbf{C}} \cong (\boldsymbol{q}^1_*\boldsymbol{\Lambda})^W. \tag{33}$$

Proof. First of all we consider the fiberwise case, i.e. $\sigma^1 : S^1 \to \mathfrak{t}^1/W$ and $q^1 : \mathfrak{t}^1 \to \mathfrak{t}^1/W$. From the theory of Δ-singularities ([Slo80a]), we know that the monodromy group of the Δ-singularity $(S_{\bar{0}}, x)$ coincides with the Weyl group $W(\Delta)$. But the former also coincides with the monodromy group of the local system $R^2\sigma^\circ_*\mathbb{Z}$. The stalks of this local system is $\boldsymbol{\Lambda}_h = \boldsymbol{\Lambda}(G_h)$ so that

$$R^2\sigma^\circ_*\mathbb{Z} \cong (q^\circ_*\boldsymbol{\Lambda}_h)^W.$$

Since the W-action commutes with the $\mathbf{C}$-action and $\boldsymbol{\Lambda}_h^{\mathbf{C}} = \boldsymbol{\Lambda} = \boldsymbol{\Lambda}(G)$, we obtain the second isomorphism in (33) over $\mathfrak{t}^\circ/W$ as well.
To get the isomorphism over $\mathfrak{t}^1/W$, let $j : \mathfrak{t}^\circ/W \hookrightarrow \mathfrak{t}^1/W$ be the open inclusion. Using the fact that the complement of $\mathfrak{t}^\circ/W$ in $\mathfrak{t}^1/W$ is smooth, it is not difficult to check that the adjunction morphism

$$a : \mathcal{F} \to j_*j^*\mathcal{F}$$

is an isomorphism for $\mathcal{F} = R^2\sigma^1_*\mathbb{Z}$ or $(q_*\boldsymbol{\Lambda}_h)^W$ (see [Bec], Chapter 1.6). This concludes the fiberwise statement.

For the global case, consider an open $D \subset \Sigma$ such that

$$\begin{array}{ccc} \mathcal{S}_{|D} & \xrightarrow{\cong} & D \times S \\ {\scriptstyle\boldsymbol{\sigma}}\downarrow & & \downarrow{\scriptstyle id\times\sigma} \\ \boldsymbol{U}_{|D} & \xrightarrow{\cong} & D \times \mathfrak{t}/W \\ {\scriptstyle\boldsymbol{q}}\uparrow & & \uparrow{\scriptstyle id\times q} \\ \tilde{\boldsymbol{U}}_{|D} & \xrightarrow{\cong} & D \times \mathfrak{t} \end{array}$$

which exists by construction. The fiberwise considerations imply that

$$R^2(id \times \sigma^\circ)_*\mathbb{Z} \cong ((id \times q^\circ)_*\mathbf{\Lambda}_G)^W. \tag{34}$$

These are local models for $R^2\boldsymbol{\sigma}^\circ_*\mathbb{Z}$ and $(\boldsymbol{q}^\circ_*\mathbf{\Lambda})^W$ respectively (over $\boldsymbol{U}_{|D}$). Since $\boldsymbol{\sigma} : \mathcal{S} \to \boldsymbol{U}$ and $\boldsymbol{q} : \tilde{\boldsymbol{U}} \to \boldsymbol{U}$ are glued via the same cocyle (which is uniquely determined by the spin bundle L of Σ chosen for the construction), the isomorphism (34) can be glued to an isomorphism

$$R^2\boldsymbol{\sigma}^\circ_*\mathbb{Z} \cong (\boldsymbol{q}^\circ_*\mathbf{\Lambda}_h)^W.$$

By pushing forward via $\boldsymbol{j} : \boldsymbol{U}^0 \hookrightarrow \boldsymbol{U}^1$ and arguing as in the local case gives (33). □

Corollary 6. *Each cohomology group $H^3(X_b, \mathbb{Z})$, $b \in \mathbb{Z}$, is torsion-free. In particular, the local system $R^3\pi^\circ_*\mathbb{Z}$ over $\mathbf{B}^\circ$ is a local system of torsion-free abelian groups.*

Proof. Since $\boldsymbol{\pi}^1_1 : \mathcal{X}^\circ \to \Sigma \times \mathbf{B}^\circ$ and $\boldsymbol{p}^1_1 : \tilde{\Sigma}^\circ \to \Sigma \times \mathbf{B}^\circ$ is obtained as pullback of $\boldsymbol{\sigma}^1 : \mathcal{S}^1 \to \boldsymbol{U}^1$ and $\boldsymbol{q}^1 : \tilde{\boldsymbol{U}}^1 \to \boldsymbol{U}^1$ respectively, we see similarly as in the proof of Proposition 8 that

$$R^2\boldsymbol{\pi}^1_*\mathbb{Z} \cong (\boldsymbol{p}^1_*\mathbf{\Lambda}_h)^W, \quad (R^2\pi^1_*\mathbb{Z})^{\mathbf{C}} \cong (\boldsymbol{p}^1_*\mathbf{\Lambda})^W$$

as constructible sheaves. Hence Lemma 6 implies stalkwise that $H^3(X_b, \mathbb{Z}) \cong H^1(\Sigma, p^W_{b,*}\mathbf{\Lambda}_h)$. But the latter group is torsion-free which can be similarly shown as (19) from [DP12] using Proposition 9 below. Note that we cannot directly apply (19) because W acts on $\mathbf{\Lambda}_h$ and not $\mathbf{\Lambda}$. □

Together with Lemma 6, this implies Theorem 16 but only on the level of abelian sheaves. There are two main difficulties in lifting this isomorphism to an isomorphism of V(M)HS:

a) The Hodge filtrations $\mathcal{F}^\bullet$, i.e. holomorphic subbundles, are a datum which cannot be captured by the underlying abelian sheaves as soon as they have more than one step. However, we have to deal with two-step Hodge filtrations.
b) Let $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}^\circ_2 \circ \boldsymbol{\pi}^1_1$ (cf. (26)). Then the fibers of $\boldsymbol{\pi}^1_1 : \mathcal{X}^\circ \to \Sigma \times \mathbf{B}^\circ$ are only generically non-singular, i.e. $R^2\boldsymbol{\pi}^1_{1*}\mathbb{Z}$ is *not* a local system (at least it is constructible). In particular, $R^2\boldsymbol{\pi}^1_{1*}\mathbb{Z}$ cannot underlie a VHS. Hence the Leray spectral sequence for $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}^\circ_2 \circ \boldsymbol{\pi}^1_1$ cannot 'live' in the category of V(M)HS. Moreover, the morphism $\boldsymbol{\pi}^1_1$ is not projective.

In order to obtain an isomorphism of VHS, we employ M. Saito's powerful theory of mixed Hodge modules (MHM) ([Sai88], [Sai90]) which can deal with the above difficulties. The point is that it allows to lift the perverse Leray spectral sequence for the composition $\boldsymbol{\pi}^{\circ} = \boldsymbol{\pi}_2^{\circ} \circ \boldsymbol{\pi}_1^{1}$ to mixed Hodge modules and (admissible) variations of mixed Hodge structures. We emphasize here that we cannot apply this theory to Lemma 6 directly because (31) is (a priori) obtained by the ordinary Leray spectral sequence and not the perverse one.

7.1. **Mixed Hodge modules.** It is beyond the scope of this text to give an introduction to mixed Hodge modules (for a detailed introduction see [Sch14] and [PS08] for an axiomatic account). Intuitively, they can be thought of as perverse sheaves with mixed Hodge structures. In particular, if the underlying perverse sheaf is a local system, then one ends up with a VHS or, more generally, an admissible VMHS (see Theorem 20 and Theorem 23). The huge advantage of mixed Hodge modules over admissible VMHS or VHS is that they admit a full six-functor formalism, at least in the algebraic context. This is analogous to the relation between perverse sheaves and local systems. Saito in fact lifted the full six-functor formalism of perverse sheaves to mixed Hodge modules.

To fix notation, let X be a complex variety. Then we have the following two abelian categories

$\mathrm{HM}(X, w)$: algebraic polarizable pure Hodge modules of weight w on X,
$\mathrm{MHM}(X)$: algebraic polarizable mixed Hodge modules on X.

As the names suggest, Hodge modules are in particular mixed Hodge modules. Both abelian categories admit an *exact* and *faithful* functor

$$\mathrm{rat} : \mathrm{HM}(X, w) \to \mathrm{P}_{\mathbb{Q}}(X), \quad \mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{P}_{\mathbb{Q}}(X).$$

Here $\mathrm{P}_{\mathbb{Q}}(X)$ is the abelian category of perverse sheaves of $\mathbb{Q}$-vector spaces on X, often only denoted by $\mathrm{P}(X)$. It is a full subcategory of the constructible (bounded) derived category $D^b_c(X) = D^b_c(X, \mathbb{Q})$ of X.

Remark 19. Recall that constructiblity here means that a sheaf (resp. the cohomology sheaves of a complex) is constructible with respect to an algebraic stratification. Of course, the condition to be a local system along the strata is with respect to the analytic topology on X.
The situation is analogously for algebraic polarizable (mixed) Hodge modules because they are in fact objects on the analytification of X. Usually, algebraic (mixed) Hodge modules are polarizable by definition (cf. [Sai90], [Sch14]) but we added it here for emphasis. In the following, all (mixed) Hodge modules are assumed to be algebraic if not stated otherwise. However, we sometimes explicitly mention that they are polarizable, e.g. in relation to polarizable V(M)HS (Theorem 20 and Theorem 23).

Now we can make precise what it means that the six-functor formalism lifts to mixed Hodge modules: For example, let $f : X \to Y$ be a morphism of complex varieties. Then there exists a functor $f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ that lifts $Rf_* : D^b_c(X) \to D^b_c(Y)$,

$$\mathrm{rat} \circ f_+ \simeq Rf_* \circ \mathrm{rat}.$$

It is an important theorem that the direct image of projective morphisms (between complex varieties or manifolds) preserves pure Hodge modules, see [Sch14] for an exposition. Before we proceed, let us give three basic examples.

Example 3 ([Sch14])**.**

a) If $X = pt$ is a point, then $\mathrm{MHM}(X)$ is the category of graded-polarizable $\mathbb{Q}$-MHS. The analogous statements holds for $\mathrm{HM}(X, w)$. If $H = (H_{\mathbb{Q}}, W_{\bullet}, F^{\bullet}H_{\mathbb{C}})$ is a graded-polarizable $\mathbb{Q}$-MHS, then
$$\mathrm{rat}(H) = H_{\mathbb{Q}} \in \mathrm{P}(pt) = \mathbb{Q} - \mathrm{mod},$$
so rat associates to a $\mathbb{Q}$-MHS its underlying $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$.

b) Let X be a non-singular complex variety of dimension d_X. Then the constant $\mathbb{Q}$-Hodge module is
$$\mathbb{Q}^{Hdg} = (\mathbb{Q}_X[d_X], K_X, \mathcal{F}_{\bullet}K_X)$$
for the canonical sheaf K_X, seen as a filtered right[17] $\mathcal{D}$-module. The filtration is given by
$$\mathcal{F}_{-d_X-1}K_X = 0, \quad \mathcal{F}_{-d_X}K_X = K_X.$$
It satisfies $\mathrm{rat}(\mathbb{Q}^{Hdg}) = \mathbb{Q}_X[d_X]$ which is an instance of the Riemann-Hilbert correspondence. In fact, this holds in general: (Mixed) Hodge modules are special filtered $\mathcal{D}$-modules M on X and $\mathrm{rat}(M) \in \mathrm{P}(X)$ is the perverse sheaf associated to M via the Riemann-Hilbert correspondence. For our purposes, it is mostly enough to work on the level of perverse sheaves, so that we do not go into more details of the theory of (filtered) $\mathcal{D}$-modules underlying (mixed) Hodge modules.

c) The previous two examples can be combined: If $a : X \to pt$ is the constant map, then $\mathcal{H}^k a_+ \mathbb{Q}_X^{Hdg} \in \mathbb{Q}-\mathrm{MHS}$ coincides with Deligne's mixed Hodge structure on the underlying cohomology groups $H^k(X, \mathbb{Q})$.

To relate our previous discussion to (mixed) Hodge modules, we need the notion of a *smooth* (mixed) Hodge module. This is a (mixed) Hodge module M on X such that $\mathrm{rat}(M) \in \mathrm{P}(X)$ is a local system concentrated in degree $-\dim_{\mathbb{C}} X$.

Theorem 20 (Saito)**.** *Let X be a non-singular complex variety of dimension $d_X = \dim_{\mathbb{C}} X$. Further let $\mathsf{V} = (\mathsf{V}_{\mathbb{Q}}, \mathcal{F}^{\bullet})$ be a* polarizable *$\mathbb{Q}$-VHS of weight w. Then the triple*
$$M(\mathsf{V}) := (K_X \otimes_{\mathcal{O}_X} \mathsf{V}, \mathcal{F}_{\bullet}, \mathsf{V}_{\mathbb{Q}}[d_X]) \in \mathrm{HM}(X, d_X + w),$$
where $\mathcal{F}_k M(\mathsf{V}) = K_X \otimes_{\mathcal{O}_X} \mathcal{F}^{-k-d_X}\mathsf{V}$ defines a polarizable Hodge module of weight $d_X + w$. This yields an equivalence between the category $\mathrm{VHS}^p_{\mathbb{Q}}(X)$ of polarizable $\mathbb{Q}$-VHS and the full subcategory $\mathrm{HM}_{sm}(X) \subset \mathrm{HM}(X)$ of smooth *polarizable Hodge modules.*

Note that this is a generalization of Example 3 b). It follows from the constructions that the faithful functor $\mathrm{rat} : \mathrm{HM}(X) \to \mathrm{P}(X)$ satisfies
$$\mathrm{rat}(M(\mathsf{V})) = \mathsf{V}_{\mathbb{Q}}[d_X].$$

[17] Here we use the convention of [Sch14], which requires to tensor with K_X. See loc. cit. for further discussion and the relation between left and right $\mathcal{D}$-modules.

We say that $\mathsf{V}_{\mathbb{Q}}[d_X]$ *underlies the (smooth) Hodge module* (resp. VHS) $M(\mathsf{V})$ (resp. V).
In general, Hodge modules are generically smooth and there is a way to uniquely extend a polarizable $\mathbb{Q}$-VHS from an open subset $X^\circ \subset X$ to a polarizable Hodge module on X. We only need this result in the special case where $X^\circ = X - D$ is the complement of a *smooth* divisor $D \subset X$ in the non-singular complex variety X. It is due to Saito in the general case and we only add an observation, how this result specializes in the aforementioned situation.

Theorem 21 (Saito)**.** *Let $D \subset X$ be a* smooth *divisor in a non-singular complex variety X and denote by $j : X^\circ \to X$ the inclusion of its complement. Assume that $M = M(\mathsf{V}) \in \mathrm{HM}(X^\circ, d_X)$ corresponds to the polarizable $\mathbb{Q}$-VHS V on X°. Then there exists a unique polarizable Hodge module $\bar{M} \in \mathrm{HM}(X^\circ, d_X)$ such that*

$$\mathrm{rat}(\bar{M}) = j_* \mathsf{V}_{\mathbb{Q}}[d_X] \in \mathrm{P}(X), \tag{35}$$

which is concentrated in degree $-d_X$.

Proof. This works in fact more generally and we only indicate the construction. The general idea is that we can extend the filtered $\mathcal{D}$-module underlying $M(\mathsf{V})$ to all of X using Deligne's extension ([Del70]). This is possible because V has only regular singularities along D. It yields a polarizable Hodge module $\bar{M} \in \mathrm{HM}(X, d_X)$ satisfying

$$\mathrm{rat}(\bar{M}) = j_{!*} \mathsf{V}_{\mathbb{Q}}[d_X]$$

for the intermediate extension $j_{!*}$.
Hence it remains to prove that $j_{!*}\mathsf{V}_{\mathbb{Q}}[d_X] \cong j_*\mathsf{V}_{\mathbb{Q}}[d_X] \in \mathrm{P}(X)$ in case $j : X^\circ \to X$ is the inclusion of the complement of a smooth divisor. This is a local question so that we are reduced to

$$j : U^* \times U \times \cdots \times U \hookrightarrow U \times \cdots \times U$$

where $U^* \subset U \subset \mathbb{C}$ is a (punctured) disk centered around $0 \in \mathbb{C}$. Since we work with local systems, it further suffices to consider the one-dimensional case $j : U^* \hookrightarrow U$, $d_X = 1$. Using Deligne's construction of the intermediate extension (e.g. [EZ08]), the claim follows:

$$j_{!*}\mathsf{V}_{\mathbb{Q}}[1] = \tau_{\leq -1} R j_* \mathsf{V}_{\mathbb{Q}} \simeq (R^0 j_* \mathsf{V}_{\mathbb{Q}})[1] = j_* \mathsf{V}_{\mathbb{Q}}[1].$$

$\square$

Remark 22. This is closely related to Zucker's Theorem 8: Let $j : \Sigma^\circ \hookrightarrow \Sigma$ be the complement of a finite number of points in a Riemann surface Σ. Using the previous theorem, we obtain a Hodge module $\bar{M}(\mathsf{V}) \in \mathrm{HM}(\Sigma, w)$ from any polarizable $\mathbb{Q}$-VHS V of weight $w-1$ that we consider as a Hodge module $M(\mathsf{V}) \in \mathrm{HM}(\Sigma^\circ, w)$. The former has $\mathrm{rat}(\bar{M}(\mathsf{V}) = j_*\mathsf{V}_{\mathbb{Q}}[1]$. Since the constant map $a_\Sigma : \Sigma \to pt$ is projective, it follows that $\mathcal{H}^1 a_+ \bar{M}(\mathsf{V}) \in \mathrm{HM}(pt, w+1)$ which is a pure Hodge structure of weight $w+2$. Its underlying $\mathbb{Q}$-vector space is $H^1(\Sigma, j_*\mathsf{V}_{\mathbb{Q}})$. This Hodge structure coincides with Zucker's Hodge structure on this cohomology group (see [Sch14], Section 17, for more details, especially the direct image theorem for projective morphisms).

In the end, it turns out that all the objects we work with are pure Hodge modules only. However, we need mixed Hodge modules in order to have full functoriality so that we briefly discuss them here as well. The starting point is an analogous result as for smooth polarizable Hodge modules. It extends the latter case.

Theorem 23 ([Sai89])**.** *Let X be a non-singular complex variety of dimension d_X. Moreover, let $\mathrm{VMHS}^p_{ad}(X)$ be the category of admissible graded-polarizable VMHS on X and $\mathrm{MHM}_{sm}(X) \subset \mathrm{MHM}(X)$ be the full subcategory of smooth polarizable mixed Hodge modules on X. Then there is an equivalence*

$$\mathrm{VMHS}^p_{ad}(X) \xrightarrow{\simeq} \mathrm{MHM}_{sm}(X), \quad \mathsf{V} \mapsto M(\mathsf{V}).$$

Mixed Hodge modules admit a full six-functor formalism (in the algebraic context) but we mainly need one functor. Let $f : X \to Y$ be a morphism between (non-singular) complex algebraic varieties. As mentioned above, there exists a (derived) direct image functor[18] $f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ which lifts $Rf_* : D^b_c(X) \to D^b_c(Y)$. In particular, one can construct the Leray spectral sequence for mixed Hodge modules. More precisely, let $f : X \to Y$, $g : Y \to Z$ be morphisms between (non-singular) complex algebraic varieties and $h = g \circ f$ the composition. Then $h_+ = g_+ \circ f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Z))$ and taking cohomology with respect to the standard t-structure on $D^b(\mathrm{MHM})$ yields the Leray spectral sequence

$$\mathcal{H}^k g_+ \mathcal{H}^m f_+ M \Rightarrow \mathcal{H}^{k+m} h_+ M, \quad M \in \mathrm{MHM}(X)$$

in the abelian category $\mathrm{MHM}(X)$. Applying the exact functor $\mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{P}(X)$ gives the perverse Leray spectral sequence

$$ {}^p\mathcal{H}^k Rg_* \, {}^p\mathcal{H}^m Rf_* \mathcal{F} \ \Rightarrow \ {}^p\mathcal{H}^{k+m} Rh_* \mathcal{F} \tag{36}$$

for $\mathcal{F} = \mathrm{rat}(M) \in \mathrm{P}(X)$.

Remark 24. Here we have used the fact that the standard t-structure on $D^b(\mathrm{MHM}(X))$ corresponds to the perverse t-structure on $D^b_c(X)$ under rat. This is a consequence of the Riemann-Hilbert correspondence. There is also an anomalous t-structure (cf. [Sai90] (Section 4), [PS08]) on $D^b(\mathrm{MHM}(X))$ that corresponds to the standard t-structure on $D^b_c(X)$.

7.2. **The** ADE**-case (**$\mathbf{C} = 1$**).** We can now begin with the actual proof of Theorem 16. It is more transparent to begin with the case $\mathbf{C} = 1$, i.e. $\Delta = \Delta_h$ is an irreducible Dynkin diagram of type ADE. The case where $\mathbf{C} \neq 1$, i.e. where Δ is an irreducible Dynkin diagram of type BCFG, is discussed in the next subsection.
To simplify notations for the rest of this subsection, we set

$$X := \mathcal{X}^\circ, \quad D := Br \xhookrightarrow{j} Y := \Sigma \times \mathbf{B}^\circ, \quad Z := \mathbf{B}^\circ. \tag{37}$$

Recall here that D is in fact a smooth divisor in Y, cf. Lemma 1. Further let $f := \boldsymbol{\pi}_1^1 : X \to Y$ and $g := \boldsymbol{\pi}_2 : Y \to Z$ as in (26). Finally, we denote $d_X := \dim_{\mathbb{C}} X$ and analogously for Y and Z.
The next lemma is a 'decomposition theorem' for $f : X \to Y$.

[18]This is notation is non-standard because f_+ usually stands for the direct image of the underlying $\mathcal{D}$-module. However, we consider it useful to have a clear notational distinction between the direct image of (perverse) sheaves and (mixed) Hodge modules.

Lemma 7. *Let $f : X \to Y$ be as before where $X = \mathcal{X}^\circ$ and $Y = \Sigma \times \mathbf{B}^\circ$. Then there is an isomorphism*

$$Rf_*A_X \simeq R^0 f_* A_X[0] \oplus R^2 f_* A_X[-2]$$

in $D^b_c(Y, A)$ where $A = \mathbb{Z}$ or $\mathbb{Q}$.

Proof. We begin with a general claim:

Claim. Let $\mathcal{A}$ be an abelian category and $K^\bullet \in C^b(\mathcal{A})$ a bounded complex such that $H^k(K^\bullet) = 0$ except for $k = 0, 2$. Then $K^\bullet \cong H^0(K^\bullet)[0] \oplus H^2(K^\bullet)[-2]$ in the bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$.

The argument for the claim is straightforward: Denote $K_2^\bullet := H^0(K^\bullet) \oplus H^2(K^\bullet)[-2]$. Then we have a quasi-isomorphism

$$\begin{tikzcd}
K_1^\bullet : \arrow[d, "\varphi"'] & 0 \arrow[r] & K^{-1} \arrow[r, "d^{-1}"] \arrow[d] & \ker d^0 \arrow[r, "0"] \arrow[d] & K^1 \arrow[r, "d_1"] \arrow[d] & \ker d^2 \arrow[r] \arrow[d] & 0 \\
K_2^\bullet : & 0 \arrow[r] & 0 \arrow[r] & H^0(K^\bullet) \arrow[r] & 0 \arrow[r] & H^2(K^\bullet) \arrow[r] & 0
\end{tikzcd}$$

with the obvious maps. On the other hand, there is a natural quasi-isomorphism $\psi : K_1^\bullet \to K^\bullet$ so that we obtain a roof

$$\begin{tikzcd}
& K_1^\bullet \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\
K^\bullet \arrow[rr] & & K_2^\bullet.
\end{tikzcd}$$

Since ψ and φ are quasi-isomorphisms, this defines an isomorphism $K^\bullet \cong K_2^\bullet$ in $D^b(\mathcal{A})$ as claimed. Observe that this argument generalizes as long as $H^{2k+1}(K^\bullet) = 0$, $k \in \mathbb{Z}$.

Now we can apply the claim as follows: Let $A_X \to \mathcal{I}^\bullet$ be an injective resolution so that $Rf_*A_X \simeq f_*\mathcal{I}^\bullet$ in $D^b(Y, A)$. Since[19] $R^k f_* A_X = \mathcal{H}^k(f_*\mathcal{I}^\bullet) = 0$ except for $k = 0, 2$ we can apply the previous claim to $K^\bullet = f_*\mathcal{I}^\bullet$. Thus we obtain an isomorphism

$$Rf_*A_X \cong \mathcal{H}^0(f_*\mathcal{I}^\bullet)[0] \oplus \mathcal{H}^2(f_*\mathcal{I}^\bullet)[-2] = R^0 f_* A[0] \oplus R^2 f_* A[-2]$$

in $D^b(Y, A)$. However, Rf_*A_X lies in $D^b_c(Y, A)$ which is a full subcategory so that the previous isomorphism is in fact an isomorphism in $D^b_c(Y, A)$. $\square$

Proof of Theorem 16. We first work with the constant mixed Hodge module $\mathbb{Q}_X^{Hdg}$ and show that the V(M)HS are isomorphic over $\mathbb{Q}$ (i.e. are isogenous). Further below we argue that all the arguments also work over $\mathbb{Z}$.
The Leray spectral sequence in MHM(Z) for $\mathbb{Q}_X^{Hdg}$ reads as[20]

$$\mathcal{H}^k f_+ \mathcal{H}^m g_+ \mathbb{Q}_X^{Hdg} \Rightarrow \mathcal{H}^{k+m} h_+ \mathbb{Q}_X^{Hdg} \tag{38}$$

[19]Here we follow the usual convention and write $\mathcal{H}^k$ for the ordinary k-th cohomology $\mathcal{H}^k : D^b(Y, A) \to Sh(Y, A)$.

[20]Here we emphasize that our constructions in Section 4 are algebraic (see [Bec] for details). This is crucial because the full six-functor formalism for mixed Hodge modules only works in the algebraic setting.

and we want to show that it degenerates on the E_2-page. Since $\mathrm{rat} : D^b(\mathrm{MHM}(X)) \to \mathrm{P}(X)$ is faithful, it suffices to prove this for the perverse Leray spectral sequence. As it turns out, this works precisely as for the ordinary Leray spectral sequence (Lemma 6). The E_2-terms of the perverse Leray spectral sequence (36) can be computed as follows: First observe by Lemma 7 that

$$Rf_*(\mathbb{Q}[d_X]) \simeq R^0 f_*(\mathbb{Q}[d_X]) \oplus R^2 f_*(\mathbb{Q}[d_X - 2])$$

in $D^b_c(X)$. Next we have seen that $R^k f^\circ_* \mathbb{Q}$ is a local system[21] on $Y - D$. From the second half of Theorem 21, it follows that

$$j_{!*} R^k f^\circ_* \mathbb{Q}[d_Y] \simeq j_* R^k f^\circ_* \mathbb{Q}[d_Y] \simeq R^k f_* \mathbb{Q}[d_Y], \quad k = 0, 2.$$

In particular, $R^k f_* \mathbb{Q}[d_Y]$ is a perverse sheaf for every $k \in \mathbb{Z}$, hence ${}^p\mathcal{H}^m(R^k f_* \mathbb{Q}[d_Y]) = R^k f_* \mathbb{Q}[d_Y]$ for $m = 0$ and vanishes for $m \neq 0$. Noting that $d_X - d_Y = 2$ this yields

$$\begin{aligned} {}^p\mathcal{H}^m(Rf_*(\mathbb{Q}[d_X])) &= {}^p\mathcal{H}^{m+d_X}(Rf_*\mathbb{Q}) \\ &\simeq {}^p\mathcal{H}^{m+d_X}(R^0 f_*\mathbb{Q}[0]) \oplus {}^p\mathcal{H}^{m+d_X}(R^2 f_*\mathbb{Q}[-2]) \\ &= \begin{cases} (R^0 f_*\mathbb{Q})[d_Y], & m = d_Y - d_X = -2 \\ (R^2 f_*\mathbb{Q})[d_Y], & m = d_Y - d_X + 2 = 0 \\ 0, & \text{else.} \end{cases} \end{aligned}$$

In other words, the perverse cohomologies are in this special case concentrated in one degree,

$$(39) \qquad {}^p\mathcal{H}^m(Rf_*(\mathbb{Q}[d_X])) \simeq R^{m+2} f_* \mathbb{Q}[d_Y], \quad \forall m \in \mathbb{Z}.$$

For the next step, consider a local system $\mathcal{L}$ on $Y^\circ = Y - D$. Since $D \subset Y$ is a smooth divisor over Z, it follows that $R^l g_*(j_*\mathcal{L})$ is a local system over Z with typical stalk $H^l(\Sigma, j_{b*}\mathcal{L})$. Here $j_b : D_b = D \cap \Sigma \times \{b\} \hookrightarrow \Sigma$ is the natural inclusion. More precisely, if $b \in Z$ is given, then there exists a small neighborhood $U \subset Z$ and a topological isomorphism

$$(g^{-1}(U), D \cap g^{-1}(U)) \cong (\Sigma \times U, D_b \times U)$$

as pairs. This implies the previous claim and we see from the definition of the perverse t-structure (since we can take as stratification just Z itself) that

$$(40) \qquad {}^p\mathcal{H}^l Rg_*(j_*\mathcal{L}) \simeq \mathcal{H}^{l-d_Z} Rg_*(j_*\mathcal{L})[d_Z] = R^{l-d_Z} g_*(j_*\mathcal{L})[d_Z].$$

Since $j_* R^m f^\circ_* \mathbb{Q} \cong R^m f_* \mathbb{Q}$ the terms ${}^pE_2^{l,m}$ of the perverse Leray spectral sequence can be computed as

$$\begin{aligned} {}^pE_2^{l,m} &= {}^p\mathcal{H}^l Rg_*({}^p\mathcal{H}^m Rf_*\mathbb{Q}[d_X]) \\ &= {}^p\mathcal{H}^l Rg_*(R^{m+2} f_*\mathbb{Q}[d_Y]) && \text{by (39)} \\ &= \mathcal{H}^{l+d_Y-d_Z} Rg_*(R^{m+2} f_*\mathbb{Q})[d_Z] && \text{by (40)} \\ &= (R^{l+1} g_* R^{m+2} f_*\mathbb{Q})[d_Z]. \end{aligned}$$

Hence up to an index shift with respect to the relative dimensions and a degree shift these are the terms of the ordinary Leray spectral sequence. As before, we see that

[21] In fact, it can be shown that $R^0 f_*\mathbb{Q}$ is a local system on all of Y.

this spectral sequence degenerates. Hence the spectral sequence (38) degenerates as well and gives an isomorphism

$$\mathcal{H}^0 h_+ \mathbb{Q}_X^{Hdg} \cong \mathcal{H}^0 g_+ \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg} \tag{41}$$

of *smooth* polarizable mixed Hodge modules. In fact, the right-hand side is a smooth polarizable Hodge module of weight $d_X = d_Z + 3$. Indeed, $M := \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg}$ is a polarizable Hodge module of weight $d_Y + 2$ because it corresponds to a VHS away from the smooth divisor $D \subset Y$ (also cf. Section 7.3). But $g : Y \to X$ is a projective morphism so that $\mathcal{H}^0 g_+ M$ is a polarizable Hodge module of weight $d_Y + 2 = d_Z + 3$ on Z. From the last paragraph we know that it is even smooth. Under the equivalence of Theorem 20, we therefore obtain an isomorphism of VHS of weight 3 that lifts the isomorphism[22]

$$R^3 h_* \mathbb{Q} \cong R^1 g_* R^2 f_* \mathbb{Q}.$$

Finally, we observe that the isomorphisms of Proposition 8 lift to isomorphisms of pure Hodge modules with the help of Theorem 21 and 20. Indeed, the isomorphism (33) pulls back to give an isomorphism $(\boldsymbol{p}_{1*}\boldsymbol{\Lambda})^W_{|Y-D}(-1) \cong R^2\boldsymbol{\pi}_{1*}\mathbb{Z}_{|Y-D}$ of polarizable $\mathbb{Z}$-VHS of weight 2 and Tate type over $Y - D$. This follows because not only the weight filtrations but also the Hodge filtrations are trivial. By Theorem 20 together with Theorem 21 they both can be extended over D to isomorphic pure Hodge modules M_1 and M_2 on Y. The underlying perverse sheaves are $\mathrm{rat}(M_1) = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_{\mathbb{Q}})^W[d_Y]$ and $\mathrm{rat}(M_2) = R^2\boldsymbol{\pi}^1_{1*}\mathbb{Q}[d_Y]$ respectively as follows from (35) together with Proposition 8. But $M_2 = \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg}$ in the notation of (41) so that

$$\mathcal{H}^0 h_+ \mathbb{Q}_X^{Hdg} \cong \mathcal{H}^0 g_+ M_1$$

lifts the isomorphism $R^3 h_* \mathbb{Q} \cong R^1 g_* (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W$ of local systems to (mixed) Hodge modules. Hence we obtain an isomorphism $\mathsf{V}^{CY} \cong \mathsf{V}^H(-1)$ of V(M)HS as claimed in light of Theorem 23. $\square$

In the previous proof, we claimed that everything works over $\mathbb{Z}$ as well. To do so, we have to introduce integral structures on mixed Hodge modules. These are subtle because there are two natural perverse t-structures p and p_+ over $\mathbb{Z}$ that coincide with middle perversity after tensoring with $\mathbb{Q}$ ([BBD82], [Sch15], [Jut09]). Since we have to deal with both of them, we briefly recall their definitions for a topological Hausdorff space X:

[22] Observe that morphisms between perverse sheaves, that are concentrated in one degree, are just sheaf morphisms. This follows because it is true in $D^b_c(X)$ and $\mathrm{P}(X) \subset D^b_c(X)$ is a full subcategory. In particular, this isomorphism is an isomorphism of constructible sheaves.

$$A \in {}^pD^{\leq 0}(X,\mathbb{Z}) \iff \mathcal{H}^n i_S^* A = 0, \text{ for all } n > -\dim S \text{ and each stratum } S.$$
$$A \in {}^pD^{\geq 0}(X,\mathbb{Z}) \iff \mathcal{H}^n i_S^! A = 0, \text{ for all } n < -\dim S \text{ and each stratum } S.$$

$$A \in {}^{p_+}D^{\leq 0}(X,\mathbb{Z}) \iff \forall \text{ stratum } S : \begin{cases} \mathcal{H}^n i_S^* A = 0 \ \forall n > 1 + \dim S, \\ \mathcal{H}^{1-\dim S} i_S^* A \text{ is torsion.} \end{cases}$$

$$A \in {}^{p_+}D^{\geq 0}(X,\mathbb{Z}) \iff \forall \text{ stratum } S : \begin{cases} \mathcal{H}^n i_S^! A = 0 \ \forall n < 0, \\ \mathcal{H}^{-\dim S} i_S^! A \text{ is torsion-free.} \end{cases}$$

Here $i_S : S \hookrightarrow X$ stands for the inclusion of a stratum $S \in X$ of a stratification with respect to which A is constructible. Since both of these perversities are interchanged under Verdier duality, there is no good duality theory for perverse sheaves over $\mathbb{Z}$. At least there exists an intermediate extension $j_{!*}$ and ${}^+j_{!*}$ for p and p_+ respectively.

Example 4. Let X be a non-singular complex variety of $\dim_{\mathbb{C}} X = d_X$. Assume that $\mathcal{L}[0] \in D^b_c(X,\mathbb{Z})$ is a local system (in degree 0) with typical stalk L which is a finitely generated abelian group. As we can take the whole space as a stratum, we see that $\mathcal{L}[d_X]$ is in the heart ${}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ of the perversity p. It is also in the heart ${}^{p_+}D^{\leq 0} \cap {}^{p_+}D^{\geq 0}$ iff L is torsion-free.
As in the proof of Theorem 21, we see that ${}^{(+)}j_{!*}\mathcal{L}[d_X] \simeq j_*\mathcal{L}[d_X]$, if $j : U \to X$ is the inclusion of the complement of a smooth divisor in X.

Following Schnell ([Sch15]), we introduce integral structures for mixed Hodge modules as follows:

Definition 5. Let $M \in D^b(\mathrm{MHM}(X))$ be a complex of mixed Hodge modules. An *integral structure* on M is a constructible sheaf $M_{\mathbb{Z}} \in D^b_c(X,\mathbb{Z})$ such that

$$\mathrm{rat}(M) \simeq M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}_X.$$

It can be shown that mixed Hodge modules with integral structure are compatible with the standard functors like cohomology, see [Sch15].

Example 5. Over a point and a single mixed Hodge structure $(H_{\mathbb{Q}}, W_\bullet, F^\bullet)$, every abelian group $H_{\mathbb{Z}}$ with $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_{\mathbb{Q}}$ is an integral structure. Note that $H_{\mathbb{Z}}$ is allowed to have torsion.
The analogous statement applies to variations of (mixed) Hodge structures considered as (mixed) Hodge modules.

The next lemma gives another, still simple example involving integral structures.

Lemma 8. *Let $h : X \to Y$ be a* locally trivial *fibration such that $Rh_*\mathbb{Z}_X \in D^b_c(Y,\mathbb{Z})$. Then*

$$(Rh_*\mathbb{Z}_X[0]) \otimes \mathbb{Q}_Y[0] \simeq Rh_*(\mathbb{Q}_X[0])$$

in $D^b_c(Y,\mathbb{Z})$. In particular, $h_+\mathbb{Q}^{Hdg} \in D^b(\mathrm{MHM}(Y))$ has a natural integral structure.

Proof. The inclusion $\mathbb{Z}_X[0] \hookrightarrow \mathbb{Q}_X[0]$ gives a natural morphism $Rh_*\mathbb{Z}_X[0] \to Rh_*\mathbb{Q}_X[0]$. Using the $\mathbb{Z}$-flatness of $\mathbb{Q}$ we obtain

$$\Psi : (Rh_*\mathbb{Z}_X[0]) \otimes \mathbb{Q}_Y[0] \longrightarrow Rh_*\mathbb{Q}_X[0] \otimes \mathbb{Q}[0] \simeq Rh_*\mathbb{Q}_X[0].$$

After applying k-th cohomology and taking stalks at some $y \in Y$, we end up with the natural morphism

$$H^k(h^{-1}(y), \mathbb{Z}) \otimes \mathbb{Q} \longrightarrow H^k(h^{-1}(y), \mathbb{Q}).$$

It is an isomorphism because the cohomology groups are finitely generated by assumption. Hence Ψ is a quasi-isomorphism. $\square$

Remark 25. If we work with $Rh_!$ instead of Rh_*, then this lemma clearly holds more general by the projection formula. It is not clear to us, how general the above version holds though. As long as one can compute the stalks of $R^kh_*\mathbb{Z}_X$, it seems to work. Also note that the finiteness condition (which is included in the definition of $D^b_c(X)$) is necessary, as the constant map $f : \mathbb{Z} \to pt$ shows ($\mathbb{Z}$ with the discrete topology).

Taking cohomology is further compatible with integral structures (again cf. [Sch15]). More precisely, let $M \in D^b(\mathrm{MHM}(X))$ which has an integral structure $M_\mathbb{Z} \in D^b_c(X, \mathbb{Z})$. Then we have

$$\mathrm{rat}(\mathcal{H}^k(M)) \simeq {}^p\mathcal{H}^k(\mathrm{rat}(M)) \simeq {}^p\mathcal{H}^k(\mathbb{Q} \otimes_\mathbb{Z} M_\mathbb{Z}) \simeq \mathbb{Q} \otimes_\mathbb{Z} {}^p\mathcal{H}^k(M_\mathbb{Z}).$$

In the last step, we can also use ${}^{p_+}\mathcal{H}^k$ instead because both give to same results after tensoring with $\mathbb{Q}$.

Integral structure for Theorem 16. We take up the notation from (37) so that

$$\begin{tikzcd} X = \mathcal{X}^\circ \arrow[r, "f"] \arrow[rr, bend left, "h"] & Y = \Sigma \times \mathbf{B}^\circ \arrow[r, "g"] & Z = \mathbf{B}^\circ. \end{tikzcd}$$

Since h is topologically locally trivial, we have

$$Rg_*Rf_*\mathbb{Q}_X \simeq Rh_*\mathbb{Q}_X \simeq (Rh_*\mathbb{Z}_X) \otimes_\mathbb{Z} \mathbb{Q}_X \simeq (Rg_*Rf_*\mathbb{Z}_X) \otimes_\mathbb{Z} \mathbb{Q}_X.$$

This in particular shows that the p_+-perverse[23] Leray spectral sequence for the composition $h = g \circ f$ for $\mathbb{Z}$-coefficients becomes the perverse spectral sequence for $\mathbb{Q}$-coefficients after tensoring with $\mathbb{Q}_X$. We can argue as before that the p_+-perverse Leray spectral sequence

$${}^{p_+}\mathcal{H}^k g_* \, {}^{p_+}\mathcal{H}^l f_*\mathbb{Z}_X[d_X] \Rightarrow {}^{p_+}\mathcal{H}^{k+l} h_*\mathbb{Z}_X[d_X]$$

degenerates on the E_2-page. Indeed, using that Lemma 7 holds over $\mathbb{Z}$ and the intermediate extension for p_+ (cf. Example 4), we see as above that the E_2-page of the p_+-perverse Leray sequence for $h = g \circ f$ coincides with the ordinary Leray spectral sequence (up to shifts). But the latter even degenerates over $\mathbb{Z}$ (Lemma 6). Hence we see that the isomorphism $R^3h_*\mathbb{Q}_X \cong R^1g_*R^2f_*\mathbb{Q}_X$ is defined over $\mathbb{Z}$ and is compatible with the corresponding isomorphism of smooth mixed Hodge modules and VMHS. $\square$

[23] We take p_+ because the stalk of $R^3h_*\mathbb{Z}$ are free.

7.3. **The** BCFG**-case ($\mathbf{C} \neq 1$).** The underlying real torus of the (cohomology) intermediate Jacobian $J^2(X)$ of a member $X = X_b$ for $b \in \mathbf{B}^\circ$ is given by

$$J^2(X) = H^3(X, \mathbb{Z}) \otimes S^1.$$

The complex structure on $J^2(X)$ is specified by the pure Hodge structure on $H^3(X, \mathbb{Z})$. By the $\mathbf{C}$-equivariance of $\boldsymbol{\pi}$, each member X of the family $\mathcal{X}$ inherits a $\mathbf{C}$-action. In particular, $\mathbf{C}$ acts on $H^3(X, \mathbb{Z})$ by Hodge morphisms. Since the category of pure (or mixed) Hodge structures is abelian, the $\mathbf{C}$-invariants $H^3(X, \mathbb{Z})^{\mathbf{C}}$ carry a natural pure Hodge structure for $b \in \mathbf{B}^\circ$. It follows that

$$J^2_{\mathbf{C}}(X) := H^3(X, \mathbb{Z})^{\mathbf{C}} \otimes S^1$$

is not only a real subtorus of $J^2(X)$ but is in fact an abelian subvariety. Note that this is a priori not $J^2(X)^{\mathbf{C}}$, since the fixed point set might have several connected components. In the following we want to relate $J^2_{\mathbf{C}}(X) = J^2_{\mathbf{C}}(X_b)$ to the generalized Prym variety $P_b = H^1(\Sigma, \mathcal{T}(b))^\circ$ for $b \in \mathbf{B}^\circ$ and eventually prove a global result as in the ADE-case.

Theorem 26. *Let Δ be a connected Dynkin diagram of type* BCFG*, G the associated simple adjoint complex Lie group and $\mathbf{B} = \mathbf{B}(\Sigma, G)$ the corresponding Hitchin base. If $b \in \mathbf{B}^\circ$, then*

$$P_b \cong J^2_{\mathbf{C}}(X_b)$$

as abelian varieties.

Isomorphic as real tori. We prove Theorem 26 in several steps starting on the real level. To this end, we follow a direct approach and consider the exact sequence

$$0 \longrightarrow \mathcal{F}^{\mathbf{C}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{42}$$

where $\mathcal{F} := R^2\pi_*\mathbb{Z}$ and $\mathcal{G}$ is the quotient. It induces the long exact sequence

(43)
$$\begin{tikzcd} 0 \arrow[r] & H^0(\Sigma, \mathcal{F}^{\mathbf{C}}) \arrow[r] & H^0(\Sigma, \mathcal{F}) \arrow[r] & H^0(\Sigma, \mathcal{G}) \arrow[dll] \\ & H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \arrow[r] & H^1(\Sigma, \mathcal{F}) \arrow[r] & H^1(\Sigma, \mathcal{G}) \arrow[dll] \\ & H^2(\Sigma, \mathcal{F}^{\mathbf{C}}) \arrow[r] & H^2(\Sigma, \mathcal{F}) \arrow[r] & H^2(\Sigma, \mathcal{G}). \end{tikzcd}$$

Since $\mathcal{F}^{\mathbf{C}} \cong (p_*\mathbf{\Lambda})^W$, we can use earlier results to conclude that $H^0(\Sigma, \mathcal{F}^{\mathbf{C}}) = 0$ and $H^2(\Sigma, \mathcal{F}^{\mathbf{C}})$ is torsion.

Lemma 9. *Let $\mathcal{F} = R^2\pi_*\mathbb{Z}$ be as before. Then it has no global sections, $H^0(\Sigma, \mathcal{F}) = 0$.*

Proof. Recall that $\mathcal{F} \cong j_*\mathcal{F}^\circ$ where $\mathcal{F}^\circ = R^2\pi^\circ\mathbb{Z}$ so that $H^0(\Sigma, \mathcal{F}) = \mathbf{\Lambda}_h^{mon}$ for the monodromy group mon of $\mathcal{F}^\circ$. Since $R^2\pi^\circ\mathbb{Z} \cong (p^\circ_*\mathbf{\Lambda}_h)^W$, the monodromy group is

$$mon = W = \langle \prod_{\beta \in \mathbf{C}\cdot\alpha} \rho_\beta \mid \alpha \in R_h \rangle \subset W_h, \quad \rho_\beta = s^\vee_\beta.$$

Note that the elements of a $\mathbf{C}$-orbit are orthogonal to each other so that the ordering in the above product is irrelevant. This also implies that for $\rho_\alpha = \prod_{\beta\in\mathbf{C}\cdot\alpha}\rho_\beta \in W^\vee$, $\alpha \in R_h$, we have

$$\mathbf{\Lambda}_h^{\rho_\alpha} = \bigcap_{\beta\in\mathbf{C}\cdot\alpha} \mathbf{\Lambda}_h^{\rho_\beta}$$

for the hyperplane $\mathbf{\Lambda}_h^{\rho_\alpha} \subset \mathbf{\Lambda}_h$ fixed by ρ_α. Therefore we obtain

$$H^0(\Sigma, R^2\pi_*\mathbb{Z}) \cong \mathbf{\Lambda}_h^W = \mathbf{\Lambda}_h^{W_h} = 0.$$

□

Consequently, the long exact sequence (43) simplifies to the left. At this point, we could go on to further reduce it, e.g. by showing that $H^0(\Sigma, \mathcal{G}) = 0$. Instead we choose the most direct way possible by explicitly showing that the induced map

$$H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^1(\Sigma, \mathcal{F})^{\mathbf{C}} \subset H^1(\Sigma, \mathcal{F})$$

is an isomorphism. To do so, we recall and slightly extend some results from [DP12], Section 6, to describe these cohomology groups.

Let $\mathcal{L}$ be a local system with typical stalk L over $\Sigma^\circ \xhookrightarrow{j} \Sigma$. Then we have the following well-known lemma.

Lemma 10. *Let $\mathcal{L}$ and $j : \Sigma^\circ \hookrightarrow \Sigma$ as before. Then the following holds*

$$\begin{aligned} H^1(\Sigma, j_*\mathcal{L}) &\cong \ker[H^1(\Sigma^\circ, \mathcal{L}) \to H^0(\Sigma, R^1j_*\mathcal{L})] \\ &\cong \mathrm{im}[H^1_c(\Sigma^\circ, \mathcal{L}) \to H^1(\Sigma^\circ, \mathcal{L})]. \end{aligned}$$

Proof. The first description is a consequence of the five-term exact sequence coming from the Leray spectral for the open inclusion $j : \Sigma^\circ \hookrightarrow \Sigma$. Its first three (non-trivial) terms are given by

$$0 \longrightarrow H^1(\Sigma, j_*\mathcal{L}) \longrightarrow H^1(\Sigma^\circ, \mathcal{L}) \xrightarrow{\beta} H^0(\Sigma, R^1j_*\mathcal{L}) \tag{44}$$

yielding the first description. For the second description see [Loo97]. □

Hence as a first step we have to describe $H^1(\Sigma^\circ, \mathcal{L})$. In fact this will be sufficient for our purposes. Let $Br = \{y_1, \ldots, y_n\} \xhookrightarrow{i} \Sigma$ be the branch points, i.e. the complement of Σ°. As in [DP12] it is convenient to add an extra point y_0 to Br, since it simplifies some of the arguments.

Now we can describe the fundamental group of $\Sigma^\circ - \{y_0\}$ as follows: Choose an arc system $\delta_1, \ldots, \delta_{2g}, \gamma_0, \gamma_1, \ldots, \gamma_m$ where the γ_j generate the (local) fundamental group around the puncture y_j. Then we have the well-known description(s)

$$\begin{aligned} \pi_1(\Sigma^\circ - \{y_0\}, \mathfrak{o}) &= \left\langle \delta_1, \ldots, \delta_{2g}, \gamma_0, \ldots, \gamma_m \;\middle|\; \gamma_0 = \prod_{i=1}^{g}[\delta_i, \delta_{i+g}] \prod_{j=0}^{m} \gamma_j \right\rangle \\ &= \langle \delta_1, \ldots, \delta_{2g}, \gamma_1, \ldots, \gamma_m \rangle, \end{aligned}$$

where $\mathfrak{o} \in \Sigma^\circ - \{y_0\}$ is a fixed base point. The second description is reminiscent of the fact that $\Sigma^\circ - \{y_0\}$ is homotopy equivalent to the bouquet of $2g+m$ circles, all attached to the point $\mathfrak{o}$.
We now fix an isomorphism $\mathcal{L}_{\mathfrak{o}} \cong L$ once and for all and denote by $\rho_i = mon(\gamma_i), w_j = mon(\delta_j) \in \mathrm{Aut}(L)$ the monodromy transformation corresponding to γ_i and δ_j respectively. Clearly, since $\mathcal{L}$ is a local system on Σ°, we must have $\rho_0 = mon(\gamma_0) = id_L$.

Remark 27. Clearly, $R^1 j_* \mathcal{L}$ is a skyscraper sheaf supported on $Br = \Sigma - \Sigma^\circ$. By a local computation, it can be shown that $H^1(D_j, \mathcal{L}) = L_{\rho_j}$ are the coinvariants in L where $D_j \subset \Sigma$ is a small disc around $b_j \in Br$. Taking the limit over all such discs yields

$$R^1 j_* \mathcal{L} = \bigoplus_{k=1}^{m} (R^1 j_* \mathcal{L})_{y_k} \cong \bigoplus_{k=1}^{m} L_{\rho_k}.$$

The morphism $\beta : H^1(\Sigma^\circ, \mathcal{L}) \to \bigoplus_k L_{\rho_k}$ in (44) associates to a class its values at the stalks. In particular, β is $\mathbf{C}$-equivariant so that the $\mathbf{C}$-action on $H^1(\Sigma, j_*\mathcal{L}) = \ker \beta$ is induced by that on $H^1(\Sigma^\circ, \mathcal{L})$.

The next proposition is essentially contained in [DP12] where the case $\mathcal{L} = (p^\circ_* \mathbf{\Lambda})^W$ is discussed. It turns out that the method of proof works more generally. We need a more general statement because we work with $\mathcal{L} = R^2 \pi^\circ_* \mathbb{Z} \cong (p^\circ_* \mathbf{\Lambda}_h)^W$ as well.

Proposition 9. *Let $\mathcal{L}$ be a local system over Σ° and $L \cong \mathcal{L}_{\mathfrak{o}}$ its typical stalk. Further let $p : \tilde{\Sigma} \to \Sigma$ be a smooth cameral curve.*

i) There is a natural isomorphism

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong \frac{L^{2g+m}}{(1-w_1, \ldots, 1-w_{2g}, 1-\rho_1, \ldots, 1-\rho_m)L}.$$

ii) Assume additionally that $(p^\circ)^*\mathcal{L} \cong L_{\tilde{\Sigma}^\circ}$ as abelian sheaves. Then there is a non-canonical isomorphism

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong H^1(\Sigma, L) \oplus \frac{L^m}{(1-\rho_1, \ldots, 1-\rho_m)L}.$$

Proof. The first part can be proven as Proposition 6.5. in [DP12] by using that

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong H^1(\pi_1, L), \quad \pi_1 = \pi_1(\Sigma^\circ - \{y_0\}, \mathfrak{o}),$$

still holds. Note again that $\rho_0 = id_L$ gives no contribution.
For the second part, we first observe that

$$\mathcal{L} \cong (p^\circ_* L)^W \tag{45}$$

by the assumption that $\mathcal{L}$ trivializes on $\tilde{\Sigma}^\circ$. In the proof of Proposition 6.5. in [DP12], it was shown that topologically one can assume the following situation: There exists a disc $D \subset \Sigma$ such that $Br \subset D$ and all the γ_j's are contained in D. Moreover, one can assume

$$p^{-1}(\Sigma - D) = \coprod_{w \in W} [\Sigma - D]_w$$

where $[\Sigma - D]_w$ are the connected components which are all isomorphic to $\Sigma - D$ (via p). Then (45) implies that $w_i = id_L$ giving the second (non-canonical) isomorphism. $\square$

Proof of Proposition 26. Lemma 10 and Remark 27 imply that it is sufficient to show that the map

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}^{\mathbf{C}}) \to H^1(\Sigma^\circ - \{y_0\}, \mathcal{L})^{\mathbf{C}} \subset H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}),$$

induced from the inclusion $\mathcal{L}^{\mathbf{C}} = (R^2\pi_*^\circ\mathbb{Z})^{\mathbf{C}} \hookrightarrow \mathcal{L} = R^2\pi_*^\circ\mathbb{Z}$, is an isomorphism. By Proposition 9 this amounts to showing that the natural map

(46)

$$\iota : H^1(\Sigma, \mathbf{\Lambda}) \oplus \frac{\mathbf{\Lambda}^m}{(1-\rho_1, \ldots, 1-\rho_m)\mathbf{\Lambda}} \longrightarrow \left(H^1(\Sigma, \mathbf{\Lambda}_h) \oplus \frac{\mathbf{\Lambda}_h^m}{(1-\rho_1, \ldots, 1-\rho_m)\mathbf{\Lambda}_h} \right)^{\mathbf{C}}$$

is an isomorphism. Here we have fixed isomorphisms $\mathcal{L}_{\mathfrak{o}}^{\mathbf{C}} \cong \mathbf{\Lambda}$ and $\mathcal{L}_{\mathfrak{o}} \cong \mathbf{\Lambda}_h$ as before. Also note that $\rho_j = \rho_{\alpha_j}$ for roots α_j which correspond to the monodromy around y_j. Of course, ι preserves the respective first factors in (46) giving an isomorphism

$$H^1(\Sigma, \mathbf{\Lambda}) \cong H^1(\Sigma, \mathbf{\Lambda}_h)^{\mathbf{C}}.$$

So it remains to check the second factors. For injectivity, assume that $\iota([\lambda_1, \ldots, \lambda_m]) = 0$. This happens iff there exists $\mu \in \mathbf{\Lambda}_h$ such that

$$\lambda_i = (1-\rho_i)\mu = \langle \alpha_i, \mu \rangle \alpha_i^\vee \in \mathbf{\Lambda} \subset \mathbf{\Lambda}_h, \quad \forall i = 1, \ldots, m.$$

So we have to exclude the case that $\langle \alpha_i, \mu \rangle \notin \langle \alpha_i, \mathbf{\Lambda} \rangle \subset \mathbb{Z}$. However, this is impossible because $1 \in \langle \alpha_i, \mathbf{\Lambda} \rangle$ (or $\epsilon_{\alpha_i} = 1$ in the notation of the proof of Proposition 3), since G is adjoint.

For the surjectivity of ι, assume $[\lambda_1, \ldots, \lambda_m]_h \in \mathbf{\Lambda}_h^m/(1-\rho_1, \ldots, 1-\rho_m)\mathbf{\Lambda}_h$ such that

$$\begin{aligned} &\sigma \cdot [\lambda_1, \ldots, \lambda_m]_h = [\lambda_1, \ldots, \lambda_m]_h \\ \Leftrightarrow\ &\sigma \cdot \lambda_i - \lambda_i = \langle \alpha_i, \mu \rangle \alpha_i^\vee, \quad \forall i = 1, \ldots, m \end{aligned}$$

for some $\mu \in \mathbf{\Lambda}_h$ and $\sigma \in \mathbf{C}$ is a generator for the cyclic group $\mathbf{C}$. For the moment assume $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$. Then using $\alpha_i^\vee \in \mathbf{\Lambda} = \mathbf{\Lambda}_h^{\mathbf{C}}$ we have

$$\sigma \cdot (\sigma \cdot \lambda_i - \lambda_i) = \lambda_i - \sigma \cdot \lambda_i = \sigma \cdot \lambda_i - \lambda_i \ \Leftrightarrow\ 2(\sigma \cdot \lambda_i - \lambda_i) = 0.$$

Hence $\lambda_i = \sigma \cdot \lambda_i$ for all $i = 1, \ldots, m$ so that $\lambda_i \in \mathbf{\Lambda}_h^{\mathbf{C}}$. In other words, $[\lambda_1, \ldots, \lambda_m]_h$ is in the image of ι. In case $\mathbf{C} = S_3$ one can argue similarly by taking generators of order 2 and 3. Therefore ι is an isomorphism in all cases. $\square$

Hence we obtain an isomorphism $H^1(\Sigma, R^2\pi_*\mathbb{Z})^{\mathbf{C}}) \cong H^1(\Sigma, (p_*\mathbf{\Lambda})^W)$ and therefore $J^2_{\mathbf{C}}(X_b) \cong P_b$ as *real* tori.

Isomorphic as abelian varieties. We next need to show that the natural isomorphism $H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \to H^1(\Sigma, \mathcal{F})^{\mathbf{C}}$, $\mathcal{F} = R^2\pi_*\mathbb{Z}$, actually is an isomorphism of polarized $\mathbb{Z}$-Hodge structures. This follows from the functoriality of Zucker's Hodge structure: We have the inclusion $(\mathcal{F}^{\mathbf{C}})^\circ \hookrightarrow \mathcal{F}^\circ$ of polarized $\mathbb{Z}$-VHS of weight 2 over Σ°. The induced morphism from above,

$$H^1(\Sigma, j_*(\mathcal{F}^{\mathbf{C}})^\circ) \longrightarrow H^1(\Sigma, j_*\mathcal{F}^\circ)^{\mathbf{C}} \longleftrightarrow H^1(\Sigma, j_*\mathcal{F}^\circ)$$

is therefore a morphism of polarized $\mathbb{Z}$-Hodge structures of weight $2+1=3$. Note that $H^1(\Sigma, j_*\mathcal{F}^\circ)^{\mathbf{C}}$ is a Hodge substructure because $\mathbf{C}$ acts on $\mathcal{F}^\circ$ by Hodge morphisms. In total, we see that

$$H^1(\Sigma, (p_*\mathbf{\Lambda})^W) \cong H^1(\Sigma, (R^2\pi_*\mathbb{Z})^{\mathbf{C}})(1) \cong H^1(\Sigma, R^2\pi_*\mathbb{Z})^{\mathbf{C}}(1)$$

as polarizable $\mathbb{Z}$-Hodge structures of weight 1. Therefore the previous isomorphism $P_b \cong J^2_{\mathbf{C}}(X_b)$ of real tori is in fact an isomorphism of abelian varieties. This concludes the proof of Theorem 26.

Global isomorphism. Using the methods from Section 7.2 we can globalize the previous isomorphisms. As in the ADE-case we introduce the shorthand notation

$$X := \mathcal{X}^\circ, \quad D := Br \xhookrightarrow{j} Y := \Sigma \times \mathbf{B}^\circ, \quad Z := \mathbf{B}^\circ \subset \mathbf{B}.$$

Let $f = \boldsymbol{\pi}_1^1 : X \to Y$ and $g = \boldsymbol{\pi}_2^\circ : Y \to Z$ be the restrictions of $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2 = pr_2$ respectively as well as $h = \boldsymbol{\pi}^\circ = g \circ f$. Further recall the universal (BCFG-)cameral curve $\boldsymbol{p} : \tilde{\mathbf{\Sigma}} \to \mathbf{B}$. It factorizes through the projection $\boldsymbol{p}_1 : \tilde{\mathbf{\Sigma}} \to \Sigma \times \mathbf{B}$ and we denote by $\boldsymbol{p}_1^1 : \tilde{\mathbf{\Sigma}}^\circ \to \Sigma \times \mathbf{B}^\circ$ the corresponding restriction. Then we have an isomorphism

$$(\boldsymbol{p}^\circ_{1*}\mathbf{\Lambda}_h)^W(-1) \cong R^2 f^\circ_*\mathbb{Z} \tag{47}$$

of polarizable $\mathbb{Z}$-VHS of weight 2 and Tate type on $Y-D$. Denote by $M^1, M^2 \in \mathrm{HM}(Y)$ the functorial intermediate extensions of these VHS to pure polarizable Hodge modules over Y of weight d_Y+2 (cf. Theorem 20). Recall that

$$\begin{aligned}\mathrm{rat}(M^1) &= j_*(\boldsymbol{p}^\circ_{1*}\mathbf{\Lambda}_{h,\mathbb{Q}})^W(-1)[d_Y] \cong (\boldsymbol{p}^1_{1*}\mathbf{\Lambda}_{h,\mathbb{Q}})^W(-1)[d_Y],\\ \mathrm{rat}(M^2) &= j_*R^2 f^\circ_*\mathbb{Q}[d_Y] \cong R^2 f_*\mathbb{Q}[d_Y],\end{aligned}$$

since $D \subset Y$ is a smooth divisor. By construction, $M^1_{\mathbb{Z}} = (\boldsymbol{p}^1_{1*}\mathbf{\Lambda}_h)^W(-1)[d_Y]$ and $M^2_{\mathbb{Z}} = R^2 f_*\mathbb{Z}[d_Y]$ are integral structures for M^1 and M^2 respectively. Moreover, the isomorphism (47) extends to give an isomorphism $M_1 \cong M_2$ of pure Hodge modules that is compatible with the integral structures.

Proposition 10. *Let $h : \mathcal{X}^\circ \to \mathbf{B}^\circ$ be the family of quasi-projective non-singular Calabi-Yau threefolds and $g : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ the projection. Then there are natural isomorphisms*

$$\mathcal{H}^0 h_+\mathbb{Q}^{Hdg} \cong \mathcal{H}^0 g_+ M_2 \cong \mathcal{H}^0 g_+ M_1$$

in $\mathrm{HM}(Z, d_Z+3)$ *which is compatible with the integral structures. In particular, the corresponding $\mathbb{Z}$-VHS of weight* 3 *are isomorphic.*

Observe that the last statement makes sense because all the involved Hodge modules are smooth.

Proof. The second isomorphism is immediate, so we are left with the first one. It can be seen as in the ADE-case (Theorem 16) via the Leray spectral sequence. From the perspective of perverse and constructible sheaves, the only difference is that in the BCFG-case the fibers of $\boldsymbol{\pi}_1 : X \to Y$ can have singularities of type A_1, $\mathrm{A}_1 \times \mathrm{A}_1$ or $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, cf. (24). But the (perverse) Leray spectral sequence still degenerates because these fibers again only have cohomology in degree 0 and 2. $\square$

Abbreviate $\mathcal{G} = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_h)^W$ so that $\mathcal{G}^{\mathbf{C}} = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W$. We next give an analogue of Theorem 16 for the BCFG-case which globalizes Theorem 26. To this end, it remains to relate $R^1g_*((\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_h)^W)^{\mathbf{C}}$ with $R^1g_*((\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W)$ which is compatible with the structures as $\mathbb{Z}$-VHS (equivalently smooth Hodge modules). As a warmup, we consider it on the sheaf level:

Lemma 11. *The morphism*

$$\iota : R^1g_*(\mathcal{G}^{\mathbf{C}}) \to (R^1g_*\mathcal{G})^{\mathbf{C}}$$

induced from the inclusion $\mathcal{G}^{\mathbf{C}} \hookrightarrow \mathcal{G}$ *is an isomorphism.*

Proof. Since $g : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ is proper, the above morphism gives

$$H^1(\Sigma, p^W_{b*}\boldsymbol{\Lambda}) \to H^1(\Sigma, p^W_{b*}\boldsymbol{\Lambda}_h)^{\mathbf{C}}$$

on stalks at $b \in \mathbf{B}^\circ$. But this coincides with the morphism from Theorem 26 which is an isomorphism. □

To lift this isomorphism to smooth Hodge modules (equivalently VHS) we observe that $M^1 \in \mathrm{HM}(Y)$ carries a $\mathbf{C}$-action that commutes with rat. As $\mathrm{HM}(Y)$ is an abelian category, $(M^1)^{\mathbf{C}} \subset M^1$ is a Hodge submodule and

$$\mathrm{rat}((M^1)^{\mathbf{C}}) = \mathrm{rat}(M^1)^{\mathbf{C}} = \mathcal{G}^{\mathbf{C}}[d_Z]$$

by construction. The inclusion $(M^1)^{\mathbf{C}} \hookrightarrow M^1$ induces the morphism

$$\iota^{Hdg} : \mathcal{H}^1g_+((M^1)^{\mathbf{C}}) \to \mathcal{H}^1g_+(M^1)^{\mathbf{C}},$$

such that $\mathrm{rat} \circ \iota^{Hdg} = \iota \circ \mathrm{rat}$ (up to a shift).

Proposition 11. ι^{Hdg} *is an isomorphism.*

Proof. Since $\mathrm{HM}(Z)$ is an abelian category, we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{H}^1g_+((M^1)^{\mathbf{C}}) \xrightarrow{\iota^{Hdg}} \mathcal{H}^1g_+(M^1)^{\mathbf{C}} \longrightarrow coK, \longrightarrow 0$$

where $K = \ker(\iota^{Hdg})$ and $coK = \mathrm{coker}(\iota^{Hdg})$. Note that they are itself smooth Hodge modules. Applying the exact functor $\mathrm{rat} : \mathrm{HM}(Z) \to \mathrm{P}(Z)$ yields

$$0 \longrightarrow \mathrm{rat}(K) \longrightarrow R^1g_*(\mathcal{G}^{\mathbf{C}})[d_Z] \xrightarrow{\iota} (R^1g_*\mathcal{G})^{\mathbf{C}}[d_Z] \longrightarrow \mathrm{rat}(coK) \longrightarrow 0,$$

an exact sequence in $\mathrm{P}(Z)$. In particular, this implies $\mathrm{rat}(K) \cong \ker(\iota[d_Z]) = 0$ and $\mathrm{rat}(coK) \cong \mathrm{coker}(\iota[d_Z]) = 0$. But a smooth Hodge module $M \in \mathrm{HM}_{sm}(Z)$ is already zero iff $\mathrm{rat}(M) = 0$. This follows from the equivalence $\mathrm{HM}_{sm}(Z) \simeq \mathrm{VHS}^p_{\mathbb{Q}}(Z)$ because a VHS V is zero iff the underlying local system is zero. Therefore ι^{Hdg} is an isomorphism. □

Note that this does hold with $\mathbb{Z}$-coefficients because all the above isomorphisms are compatible with the $\mathbb{Z}$-structures. This can be seen as in the proof of Theorem 16.

8. The Langlands dual case

In this final section we briefly address the Langlands dual of Corollary 5. Concretely, given a simply-connected simple complex Lie group G_{sc} we show how

$$\boldsymbol{h}_1^\circ : \mathbf{Higgs}_1^\circ(\Sigma, G_{sc}) \to \mathbf{B}^\circ(\Sigma, G_{sc})$$

can be realized via a family $\mathcal{X} \to \mathbf{B}^\circ(\Sigma, G_{sc})$ of non-compact Calabi-Yau threefolds. Since this works analogously as the proof of Corollary 5, we treat this case only briefly.

As before let Δ be an irreducible Dynkin diagram and $(\Delta_h, \mathbf{C})$ the unique pair such that $\Delta = \Delta_h^{\mathbf{C}}$. Let $G = G(\Delta)_{ad}$, $G_h = G(\Delta_h)_{ad}$ be of adjoint type. Further we denote by

$$ {}^LG = G({}^L\Delta)_{sc}, \quad {}^LG_h = G({}^L\Delta_h)_{sc} \tag{48}$$

the simple simply-connected complex Lie group of Langlands dual type ${}^L\Delta$ and ${}^L\Delta_h$ respectively. If we set $\mathbf{\Lambda} := \mathbf{\Lambda}(G)$ and $\mathbf{\Lambda}_h := \mathbf{\Lambda}(G_h)$, then

$$\mathbf{\Lambda}({}^LG) = \mathbf{\Lambda}^\vee, \quad \mathbf{\Lambda}({}^LG_h) = \mathbf{\Lambda}_h^\vee.$$

Note that $\mathbf{\Lambda}_h^\vee$ and $\mathbf{\Lambda}_h$ have the same ADE-Dynkin diagram Δ_h. Finally, since Langlands dual Weyl groups are isomorphic, we only write $W = W(\Delta) = W({}^L\Delta)$ and $W_h = W(\Delta_h) = W({}^L\Delta_h)$.
The lattices $\mathbf{\Lambda}^\vee$ and $\mathbf{\Lambda}_h^\vee$ already occured in Section 3.4. They naturally appear in the context of Δ-singularities. Indeed, if $S \subset \mathfrak{g}(\Delta)$ is a Slodowy slice, then

$$R^2\sigma_!^1\mathbb{Z} \cong (q_*^1\mathbf{\Lambda}_h^\vee)^W, \quad (R^2\sigma_!^1\mathbb{Z})^{\mathbf{C}} = (q_*^1\mathbf{\Lambda})^W$$

for $\sigma^1 : S^1 \to \mathfrak{t}^1/W$ and the quotient $q^1 : \mathfrak{t}^1 \to \mathfrak{t}^1/W$. This follows from the fact that $H_c^2(S_{\bar{t}}, \mathbb{Z}) \cong \mathbf{\Lambda}_h^\vee$. The isomorphisms can be glued as in Proposition 8 to give isomorphisms of constructible sheaves

$$R^2\boldsymbol{\sigma}_!^1\mathbb{Z} \cong (\boldsymbol{q}_*^1\mathbf{\Lambda}_h^\vee)^W \quad (R^2\boldsymbol{\sigma}_!^1\mathbb{Z})^{\mathbf{C}} \cong (\boldsymbol{q}_*^1\mathbf{\Lambda}^\vee)^W \tag{49}$$

over $\boldsymbol{U}^1$.

This already indicates the basic idea to relate (the neutral component of) the Hitchin system $\mathbf{Higgs}_1(\Sigma, {}^LG) \to \mathbf{B} = \mathbf{B}(\Sigma, {}^LG)$ to families of Calabi-Yau threefolds. Note that this idea already appeared in [DDD+06] in the A_1-case. First of all, we recall from [DP12] that $\mathbf{B} = \mathbf{B}(\Sigma, {}^LG) = \mathbf{B}(\Sigma, G)$ canonically. Let $\mathcal{X} \to \mathbf{B}$ be any family of quasi-projective Gorenstein Calabi-Yau threefolds with $\mathbf{C}$-action as in Section 4. Then the middle compactly supported cohomology (or homology) yields a polarizable $\mathbb{Z}$-VHS

$$\mathsf{V}_{CY} = (R^3\pi_!^\circ\mathbb{Z}, \mathcal{F}_{CY}^\bullet)$$

over $\mathbf{B}^\circ$. Indeed, the Leray spectral sequence for $\pi_b : X_b \to \Sigma$, $b \in \mathbf{B}^\circ$, yields an isomorphism

$$H_c^3(X_b, \mathbb{Z}) \cong H^1(\Sigma, R^2\pi_{b,!}\mathbb{Z}) \tag{50}$$

of mixed Hodge structures. The right-hand side is pure implying that V_{CY}, which is a priori a VMHS, is in fact a VHS. It follows that the homology intermediate Jacobians

$$J_2(X_b) := H_c^3(X_b, \mathbb{C})/(F^2H_c^3(X_b, \mathbb{C}) + H_c^3(X_b, \mathbb{Z})), \quad b \in \mathbf{B}^\circ$$

fit into a holomorphic family

$$\mathcal{J}_2(\mathcal{X}^\circ/\mathbf{B}^\circ) := \mathcal{J}(\mathsf{V}_{CY}) \to \mathbf{B}^\circ$$

of abelian varieties. It has a subfamily

$$\mathcal{J}_2^{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) := \mathcal{J}(\mathsf{V}_{CY}^{\mathbf{C}}) \subset \mathcal{J}_2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$$

defined by $\mathbf{C}$-invariants in compactly supported cohomology. Then the analogue of Corollary 5 becomes

Theorem 28. *Let Δ be and irreducible Dynkin diagram and ${}^L G = G({}^L\Delta)_{sc}$ the simple* simply-connected *complex Lie group with Langlands dual Dynkin diagram ${}^L\Delta$. Then there is an isomorphism*

$$\begin{tikzcd} \mathcal{J}_2^{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Higgs}_1^\circ(\Sigma, {}^L G) \arrow[dl, "\boldsymbol{h}_1^\circ"] \\ & \mathbf{B}^\circ & \end{tikzcd}$$

of integrable systems over $\mathbf{B}^\circ \subset \mathbf{B}(\Sigma, H)$ that respects the cubics.

Since Langlands duality is an involution on simple complex Lie groups, we clearly obtain all simple simply-connected complex Lie groups as ${}^L G$ for a unique simple adjoint complex Lie group G.

Proof. We keep the previous notation together with (37). The strategy of the proof is the same as for Corollary 5. We first consider the case $\mathbf{C} = 1$. As remarked before, the local system $R^3 h_! \mathbb{Q}$ over $\mathbf{B}^\circ$ underlies a polarizable $\mathbb{Q}$-VHS. The corresponding mixed Hodge module has $R^3 h_! \mathbb{Z}$ as an integral structure. Note that the stalks have torsion. Indeed, (50) together with (49) implies

$$H_c^3(X_b, \mathbb{Z}) \cong H^1(\Sigma, p_{b*}^W \mathbf{\Lambda}_h^\vee)$$

and the latter group has torsion, cf. Remark 9. Hence we work with the perversity p instead. The same argument as in the proof of Theorem 16) shows that the p-perverse Leray spectral sequence yields an isomorphism

$$R^3 h_! \mathbb{Z}_X \cong R^1 g_! R^2 f_! \mathbb{Z}_X.$$

Since proper direct images can be lifted to mixed Hodge modules, it follows from (49) that

$$\mathsf{V}_{CY} \cong \mathsf{V}_{sc}^H = (R^1 \boldsymbol{p}_{2*}^\circ (\boldsymbol{p}_{1*}^1 \mathbf{\Lambda}^\vee)_{\mathrm{tf}}^W, \mathcal{F}^\bullet)$$

as polarizable $\mathbb{Z}$-VHS of weight 1 over $\mathbf{B}^\circ$ (up to Tate twists). Then we conclude as in the proof of Corollary 5.
Since Proposition 9 applies to the situation at hand, the arguments of Section 7.3 apply to yield the case $\mathbf{C} \neq 1$ as well. □

Remark 29. It is remarkable that the same family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ of Calabi-Yau threefolds encode the Langlands dual (neutral components of) Hitchin systems $\mathbf{Higgs}_1^\circ(\Sigma, G)$ and $\mathbf{Higgs}_1^\circ(\Sigma, {}^L G)$ for any simple adjoint complex Lie group G. In this case, Langlands duality for Hitchin systems can be seen as Verdier duality for the family $\mathcal{X}^\circ \to \mathbf{B}^\circ$.

This is reflected in the fact that we had to work with the two perversities p_+ and p which are exchanged under Verdier duality.

References

[Ara05] Donu Arapura. The Leray spectral sequence is motivic. *Invent. Math.*, 160(3):567–589, 2005.

[Aud08] Michèle Audin. *Hamiltonian systems and their integrability*, volume 15 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2008. Translated from the 2001 French original by Anna Pierrehumbert, Translation edited by Donald Babbitt.

[AvMV04] Mark Adler, Pierre van Moerbeke, and Pol Vanhaecke. *Algebraic integrability, Painlevé geometry and Lie algebras*, volume 47 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004.

[Bal06] David Balduzzi. Donagi-Markman cubic for Hitchin systems. *Math. Res. Lett.*, 13(5-6):923–933, 2006.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[Bec] Florian Beck. Hitchin and Calabi-Yau integrable systems. PhD thesis, Albrecht-Ludwigs Universität Freiburg, 2016, `http://dx.doi.org/10.6094/UNIFR/11668`.

[Bec17] Florian Beck. Calabi-Yau threefolds over Hitchin bases. *in preparation*, 2017.

[BEZ14] Patrick Brosnan and Fouad El Zein. Variations of mixed Hodge structure. In *Hodge theory*, volume 49 of *Math. Notes*, pages 333–409. Princeton Univ. Press, Princeton, NJ, 2014.

[BL99] Christina Birkenhake and Herbert Lange. *Complex tori*, volume 177 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Bri71] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.

[Car80] James A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 107–127. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[CK99] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[DDD$^+$06] D.-E. Diaconescu, R. Dijkgraaf, R. Donagi, C. Hofman, and T. Pantev. Geometric transitions and integrable systems. *Nuclear Phys. B*, 752(3):329–390, 2006.

[DDP07] D. E. Diaconescu, R. Donagi, and T. Pantev. Intermediate Jacobians and *ADE* Hitchin systems. *Math. Res. Lett.*, 14(5):745–756, 2007.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[DG02] R. Y. Donagi and D. Gaitsgory. The gerbe of Higgs bundles. *Transform. Groups*, 7(2):109–153, 2002.

[DM96a] Ron Donagi and Eyal Markman. Cubics, integrable systems, and Calabi-Yau threefolds. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, volume 9 of *Israel Math. Conf. Proc.*, pages 199–221. Bar-Ilan Univ., Ramat Gan, 1996.

[DM96b] Ron Donagi and Eyal Markman. Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 1–119. Springer, Berlin, 1996.

[Don95] Ron Donagi. Spectral covers. In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, volume 28 of *Math. Sci. Res. Inst. Publ.*, pages 65–86. Cambridge Univ. Press, Cambridge, 1995.

[Don97] Ron Y. Donagi. Seiberg-Witten integrable systems. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 3–43. Amer. Math. Soc., Providence, RI, 1997.

[DP12] R. Donagi and T. Pantev. Langlands duality for Hitchin systems. *Invent. Math.*, 189(3):653–735, 2012.

[EZ08] Fouad El Zein. Topology of algebraic morphisms. In *Singularities I*, volume 474 of *Contemp. Math.*, pages 25–84. Amer. Math. Soc., Providence, RI, 2008.

[Fal93] Gerd Faltings. Stable G-bundles and projective connections. *J. Algebraic Geom.*, 2(3):507–568, 1993.

[Fre99] Daniel S. Freed. Special Kähler manifolds. *Comm. Math. Phys.*, 203(1):31–52, 1999.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[GS90] Victor Guillemin and Shlomo Sternberg. *Symplectic techniques in physics*. Cambridge University Press, Cambridge, second edition, 1990.

[HHP10] Claus Hertling, Luuk Hoevenaars, and Hessel Posthuma. Frobenius manifolds, projective special geometry and Hitchin systems. *J. Reine Angew. Math.*, 649:117–165, 2010.

[Hit87a] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[Hit87b] Nigel Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[HT03] Tamás Hausel and Michael Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.

[Jut09] Daniel Juteau. Decomposition numbers for perverse sheaves. *Ann. Inst. Fourier (Grenoble)*, 59(3):1177–1229, 2009.

[Kas86] Masaki Kashiwara. A study of variation of mixed Hodge structure. *Publ. Res. Inst. Math. Sci.*, 22(5):991–1024, 1986.

[KS14] Maxim Kontsevich and Yan Soibelman. Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and mirror symmetry. In *Homological mirror symmetry and tropical geometry*, volume 15 of *Lect. Notes Unione Mat. Ital.*, pages 197–308. Springer, Cham, 2014.

[Loo97] Eduard Looijenga. Cohomology and intersection homology of algebraic varieties. In *Complex algebraic geometry (Park City, UT, 1993)*, volume 3 of *IAS/Park City Math. Ser.*, pages 221–263. Amer. Math. Soc., Providence, RI, 1997.

[Mil68] John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

[Sai88] Morihiko Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995 (1989), 1988.

[Sai89] Morihiko Saito. Mixed Hodge modules and admissible variations. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(6):351–356, 1989.

[Sai90] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.

[Sch73] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

[Sch14] Christian Schnell. An overview of morihiko saito's theory of mixed hodge modules, 2014.

[Sch15] C. Schnell. Torsion points on cohomology support loci: from $\mathcal{D}$-modules to Simpson's theorem. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 405–421. Cambridge Univ. Press, Cambridge, 2015.

[Sco98] Renata Scognamillo. An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups. *Compositio Math.*, 110(1):17–37, 1998.

[Slo80a] Peter Slodowy. *Four lectures on simple groups and singularities*, volume 11 of *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht*. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1980.

[Slo80b] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

[SZ85] Joseph Steenbrink and Steven Zucker. Variation of mixed Hodge structure. I. *Invent. Math.*, 80(3):489–542, 1985.

[Sze04] Balázs Szendrői. Artin group actions on derived categories of threefolds. *J. Reine Angew. Math.*, 572:139–166, 2004.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Zuc79] Steven Zucker. Hodge theory with degenerating coefficients. L_2 cohomology in the Poincaré metric. *Ann. of Math. (2)*, 109(3):415–476, 1979.

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