Topological cycle matroids of infinite graphs

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Abstract

We prove that the topological cycles of an arbitrary infinite graph induce a matroid. This matroid in general is neither finitary nor cofinitary.

1 Introduction

One central aim of infinite matroid theory is to study the connections to infinite graph theory [1, 3, 5, 7, 10, 8, 9, 16]. This approach has not only led us to exciting questions about infinite matroids but also has allowed for new perspectives on infinite graph theory. This paper is part of that approach: We resolve the question for which graphs the topological cycles induce a matroid.

So far there were many competing notions of topological cycle [14]. For each of these notions we determine when the topological cycles induce a matroid. This investigation leads us to a single notion of topological cycle. This notion is strongest in the sense that the theorem that its topological cycles induce a matroid implies the theorems about when the other notions induce matroids. The matroids for this notion are in general neither finitary nor cofinitary and are uncountable in a nontrivial way.

Let us be more precise: Given a graph together with an end boundary, a *topological cycle* is a homeomorphic image of the unit circle in the topological space consisting of the graph together with the boundary. Depending on which end boundary we consider, we get a different notion of topological cycle. The topological cycles *induce* a matroid if their edge sets form the set of circuits of a matroid.

For locally finite graphs, all these end boundaries are the same so that in this case there is only one notion of topological cycle. Bruhn and Diestel showed in this case that the topological cycles induce a matroid by showing that it is the dual of the finite bond matroid [8].

For arbitrary 2-connected graphs, the dual of the finite bond matroid also allows for a description by topological cycles (in the topological space ETOP). However, this matroid is isomorphic to the matroid of a countable graph after deleting loops and parallel edges.

Hence in order to construct matroids that are nontrivially uncountable, we have to consider topological cycles of different topological spaces. One such space is VTOP, which is obtained from the graph by adding the vertex ends. In Figure 1, we depicted a graph whose topological cycles in VTOP do not induce a matroid.

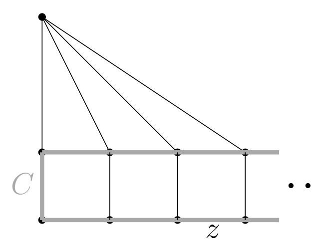

Figure 1: The dominated ladder is obtained from the one ended ladder by adding a vertex that is adjacent to every vertex on the upper side of the ladder. The topological cycles of VTOP of the dominated ladder do not induce a matroid as they violate the elimination axiom (C3): We cannot eliminate all the triangles from the grey cycle C.

The reason why this example works is that the topological cycle C goes through a dominated end. Let DTOP be the topological space obtained from VTOP by deleting the dominated ends. The main result of this paper is the following.

Theorem 1.1. *For any graph, the topological cycles in DTOP induce a matroid.*

The matroids of Theorem 1.1 are in general neither finitary nor cofinitary and are uncountable in a nontrivial way. The proof of Theorem 1.1 involves a new result on the structure of the end space [11] and the theory of trees of matroids [4].

In 1969, Higgs proved that the set of finite cycles and double rays of G is the set of circuits of a matroid if and only if G does not have a subdivision of the Bean-graph [18]. Using Theorem 1.1, we get a result for topological cycles of VTOP in the same spirit.

Corollary 1.2. *The topological cycles of VTOP induce a matroid if and only if G does not have a subdivision of the dominated ladder, which is depicted in Figure 1.*

Theorem 1.1 implies similar results concerning the identification space ITOP and we also extend the main result of [3], see Section 5 for details.

Theorem 1.1 extends to 'Psi-Matroids': Given a set Ψ of ends, by C_Ψ we denote the set of those topological cycles in the topological space obtained from VTOP by deleting the ends not in Ψ. By D_Ψ we denote the set of those bonds that have no end of Ψ in their closure. It is not difficult to show that the set of undominated ends is Borel, see Section 4. Thus the following is a strengthening of Theorem 1.1.

Theorem 1.3. *Let Ψ be a Borel set of ends that only contains undominated ends. Then C_Ψ and D_Ψ are the sets of circuits and cocircuits of a matroid.*

This paper is organised as follows. After giving the necessary background in Section 2, we prove some intermediate results in Section 3. Then we prove Theorem 1.3 in Section 4. Finally, in Section 5 we deduce from it the other theorems mentioned in the Introduction.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [15] unless defined differently. And G always denotes a graph. We denote the complement of a set X by $X^\complement$. Throughout this paper, *even* always means finite and a multiple of 2. An edge set F in a graph is a *cut* if there is a partition of the set of vertices such that F is the set of edges with precisely one endvertex in each partition class. A vertex set *covers* a cut if every edge of the cut is incident with a vertex of that set. A cut is *finitely coverable* if there is a finite vertex set covering it. A *bond* is a minimal nonempty cut.

For us, a *separation* is just an edge set. The *boundary* $\partial(X)$ of a separation X is the set of those vertices adjacent with an edge from X and one from $X^\complement$. The *order* of X is the size of $\partial(X)$. Given a connected subgraph C of G, we denote the set of those edges with at least one endvertex in C by s_C. Given a separation X of finite order and an end ω, then there is a unique component C of $G - \partial(X)$ in which ω lives. We say that ω *lives* in X if $s_C \subseteq X$.

A *tree-decomposition* of G consists of a tree T together with a family of subgraphs $(P_t|t \in V(T))$ of G such that every vertex and edge of G is in at least one of these subgraphs, and such that if v is a vertex of both P_t and P_w, then it is a vertex of each P_u, where u lies on the v-w-path in T. Moreover, each edge of G is contained in precisely one P_t. We call the subgraphs P_t, the *parts* of the tree-decomposition. Sometimes, the 'Moreover'-part is not part of the definition of tree-decomposition. However, both these two definitions give the same concept of tree-decomposition since any tree-decomposition without this additionally property can easily be changed to one with this property by deleting edges from the parts appropriately. Given a directed edge tu of T, the *separation corresponding to tu* is the set of those edges which are in parts P_w, where u lies on the unique t-w-path in T. The *adhesion* of a tree-decomposition is finite if any two adjacent parts intersect finitely. A key tool in our proof is the main result of [11], as follows.

Theorem 2.1. *Every graph G has a tree-decomposition $(T, P_t|t \in V(T))$ of finite adhesion such that the ends of T are the undominated ends of G.*

Remark 2.2. *([11, Remark 6.6]) Let $(T, \leq)$ be the tree order on T as in the proof of Theorem 2.1 where the root r is the smallest element. We remark that we constructed $(T, \leq)$ such that $(T, P_t|t \in V(T))$ has the following additional property: For each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.*

Moreover, we construct $(T, P_t|t \in V(T))$ such that if st and tu are edges of T with $s \leq t \leq u$, then $V(P_s) \cap V(P_t)$ and $V(P_t) \cap V(P_u)$ are disjoint.

Given a part P_t of a tree-decomposition, the *torso* H_t is the multigraph obtained from P_t by adding for each neighbour u of t in the tree a complete graph with vertex set $V(P_t) \cap V(P_u)$.

We denote the set of (vertex-) ends of a graph G by $\Omega(G)$. A vertex v is in the *closure of* an edge set F if there is an infinite fan from v to $V(F)$. An end ω is in the *closure of* an edge set F if every finite order separation X in which ω lives meets F. It is straightforward to show that an end ω is in the closure of an edge set F if and only if every ray (equivalently: some ray) belonging to ω cannot be separated from F by removing finitely many vertices. An end ω *lives in* a component C if it is in the closure of the edge set s_C. A *comb* is a subdivision of the graph obtain from the ray by attaching a leaf at each of its vertices. The set of these newly added vertices is the set of *teeth*. The Star-Comb-Lemma is the following.

Lemma 2.3. *(Diestel [13, Lemma 1.2]) Let U be an infinite set of vertices in a connected graph G. Then either there is a comb with all its teeth in U or a subdivision of the infinite star S with all leaves in U.*

Corollary 2.4. *Every infinite edge set has an end or a vertex in its closure.*

□

2.1 Infinite matroids

An introduction to infinite matroids can be found in [9], whilst the axiomatisation of infinite matroids we work with here is the one introduced in [3]. Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of a groundset E, which can be thought of as the sets of circuits and cocircuits of some matroid, respectively.

(C1) The empty set is not in $\mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(O1) $|C \cap D| \neq 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(O2) For all partitions $E = P \dot\cup Q \dot\cup \{e\}$ either $P + e$ includes an element of $\mathcal{C}$ through e or $Q + e$ includes an element of $\mathcal{D}$ through e.

We follow the convention that if we put a * at an axiom A then this refers to the axiom obtained from A by replacing $\mathcal{C}$ by $\mathcal{D}$, for example (C1*) refers to the axiom that the empty set is not in $\mathcal{D}$. A set $I \subseteq E$ is *independent* if it does not include any nonempty element of $\mathcal{C}$. Given $X \subseteq E$, a *base of X* is a maximal independent subset Y of X.

(IM) Given an independent set I and a superset X, there exists a base of X including I.

The proof of [3, Theorem 4.2] also proves the following:

Theorem 2.5. *Let E be a some set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$. Then there is a matroid M whose set of circuits is $\mathcal{C}$ and whose set of cocircuits is $\mathcal{D}$ if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy (C1), (C1*), (C2), (C2*), (O1), (O2), and (IM).*

Theorem 2.5 shows that the above axioms give an alternative axiomatisation of infinite matroids, which we use in this paper as a definition of infinite matroids. We call elements of $\mathcal{C}$ *circuits* and elements of $\mathcal{D}$ *cocircuits*. The *dual* of $(\mathcal{C}, \mathcal{D})$ is the matroid whose set of circuits is $\mathcal{D}$ and whose set of cocircuits is $\mathcal{C}$.

A matroid $(\mathcal{C}, \mathcal{D})$ is *finitary* if every element of $\mathcal{C}$ is finite, and it is *tame* if every element of $\mathcal{C}$ intersects any element of $\mathcal{D}$ only finitely. An example of a finitary matroid is the *finite-cycle matroids* of a graph G whose circuits are the edge sets of finite cycles of G and whose cocircuits are the bonds of G. We shall need the following lemma:

Lemma 2.6. *[[7, Lemma 2.7]] Suppose that M is a matroid, and $\mathcal{C}$, $\mathcal{C}^*$ are collections of subsets of $E(M)$ such that $\mathcal{C}$ contains every circuit of M, $\mathcal{C}^*$ contains every cocircuit of M, and for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$. Then the set of minimal nonempty elements of $\mathcal{C}$ is the set of circuits of M and the set of minimal nonempty elements of $\mathcal{C}^*$ is the set of cocircuits of M.*

2.2 Trees of presentations

In this subsection, we give a toy version of the definitions of [4], which are just enough to state the results of [4] we need in this paper. A tame matroid is *binary* if every circuit and cocircuit always intersect in an even number of edges.[1]

Roughly, a binary presentation of a tame matroid M is something like a pair of representations over $\mathbb{F}_2$, one of M and of the dual of M, formally:

Definition 2.7. Let E be any set. A *binary presentation* Π on E consists of a pair (V, W) of sets of subsets of E satisfying (02) and are orthogonal, that is, every $o \in V$ intersects any $d \in W$ evenly. We will sometimes denote the first element of Π by V_Π and the second by W_Π. We say that Π *presents* the matroid M if the circuits of M are the minimal nonempty elements of V_Π and the cocircuits of M are the minimal nonempty elements of W_Π.

Given a finitary binary matroid M, let $\overline{V_M}$ be the set of those finite edge sets meeting each cocircuit evenly, and let $\overline{W_M}$ be the set of those (finite or infinite) edge sets meeting each circuit evenly. Then $(\overline{V_M}, \overline{W_M})$ is called the *canonical presentation* of a M.

Definition 2.8. A *tree of binary presentations* $\mathcal{T}$ consists of a tree T, together with functions $\overline{V}$ and $\overline{W}$ assigning to each node t of T a binary presentation $\Pi(t) = (\overline{V}(t), \overline{W}(t))$ on the ground set $E(t)$, such that for any two nodes t and t' of T, if $E(t) \cap E(t')$ is nonempty then tt' is an edge of T.

For any edge tt' of T we set $E(tt') = E(t) \cap E(t')$. We also define the *ground set* of $\mathcal{T}$ to be $E = E(\mathcal{T}) = \left(\bigcup_{t \in V(T)} E(t)\right) \setminus \left(\bigcup_{tt' \in E(T)} E(tt')\right)$.

[1] In [2], it is shown that most of the equivalent characterisations of finite binary matroids extend to tame binary matroids.

We shall refer to the edges which appear in some $E(t)$ but not in E as *dummy edges* of $M(t)$: thus the set of such dummy edges is $\bigcup_{tt' \in E(T)} E(tt')$.

A tree of binary presentations is a *tree of binary finitary presentations* if each presentation $\Pi(t)$ is a canonical presentation of some binary finitary matroid.

Definition 2.9. Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of binary presentations. A *pre-vector* of $\mathcal{T}$ is a pair $(S, \overline{v})$, where S is a subtree of T and $\overline{v}$ is a function sending each node t of S to some $\overline{v}(t) \in \overline{V}(t)$, such that for each $t \in S$ we have $\overline{v}(t)\restriction_{E(tu)} = \overline{v}(u)\restriction_{E(tu)} \neq 0$ if $u \in S$, and $\overline{v}(t)\restriction_{E(tu)} = 0$ otherwise.

The *underlying vector* $\underline{(S, \overline{v})}$ of $(S, \overline{v})$ is the set of those edges in some $\overline{v}(t)$ for some $t \in V(T)$. Now let Ψ be a set of ends of T. A pre-vector $(S, \overline{v})$ is a Ψ-*pre-vector* if all ends of S are in Ψ. The space $V_\Psi(\mathcal{T})$ of Ψ-*vectors* consists of those sets that are a symmetric differences of finitely many underlying vectors of Ψ-pre-vectors.

pre-covectors are defined like pre-vectors with '$\overline{W}(t)$' in place of '$\overline{V}(t)$'. *underlying covectors* are defined similar to underlying vectors. A pre-covector $(S, \overline{w})$ is a Ψ-*pre-covector* if all ends of S are in Ψ. The space $W_\Psi(\mathcal{T})$ of $\Psi^\complement$-*covectors* consists of those sets that are a symmetric differences of finitely many underlying covectors of $\Psi^\complement$-pre-covectors.

Finally, $\Pi_\Psi(\mathcal{T})$ is the pair $(V_\Psi(\mathcal{T}), W_\Psi(\mathcal{T}))$.

The following is a consequence of the main result of [4], Theorem 8.3, and Lemma 6.8.

Theorem 2.10 ([4])**.** *Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of binary finitary presentations and Ψ a Borel set of ends of T, then $\Pi_\Psi(\mathcal{T})$ presents a binary matroid. Moreover, the set of Ψ-vectors and $\Psi^\complement$-covectors satisfy (O1), (O2) and tameness.*

We shall also need the following related lemma, which is a combination of Lemma 6.6 and Lemma 6.8 from [4].

Lemma 2.11 ([4])**.** *Let $\mathcal{T} = (T, M)$ be a tree of binary finitary presentations and Ψ be any set of ends of T. Any Ψ-vectors of $\mathcal{T}$ and any $\Psi^\complement$-covectors of $\mathcal{T}$ are orthogonal.*

3 Ends of graphs

The *simplicial topology of G* is obtained from the disjoint union of copies of the unit interval, one for each edge of G, by identifying two endpoints of these intervals if they correspond to the same vertex.

First we recall the definition of $|G|$ from [14], and then we give an equivalent one using inverse limits. Given a finite set of vertices S and an end ω, by $C(S,\omega)$ we denote the component of $G-S$ in which ω lives. Let $\vec{\epsilon}$ be a function from the set of those edges with exactly one endvertex in $C(S,\omega)$ to $(0,1)$. The set $C_{\vec{\epsilon}}(S,\omega)$ consists of all vertices of $C(S,\omega)$, all ends living in $C(S,\omega)$, the set $e\times(0,1)$ for each edge e with both endvertices in $C(S,\omega)$, together with for each edge f with exactly one endvertex $t(f)$ in $C(S,\omega)$, the set of those points on $f\times(0,1)$ with distance less than $\vec{\epsilon}(f)$ from $t(f)$.

The point space of $|G|$ is the union of $\Omega(G)$, the vertex set $V(G)$ and a set $e\times(0,1)$ for each edge e of G. A basis of this topology consists of the sets $C_{\vec{\epsilon}}(S,\omega)$ together with those sets O that are open considered as sets in the simplicial topology of G. Note that $|G|$ is Hausdorff.

Given a finite vertex set W of G, by $G^+[W]$ we denote the (multi-) graph obtained from G by contracting all edges not incident with a vertex of W. Thus the vertex set of $G^+[W]$ is W together with the set of components of $G-W$. We consider $G^+[W]$ as a topological space endowed with the simplicial topology. If $U\subseteq W$, then there is a continuous surjective map $f[W,U]$ from $G^+[W]$ to $G^+[U]$.

Theorem 3.1. *$|G|$ is the inverse limit of the topological spaces $G^+[W]$ with respect to the maps $f[W,U]$.*

Proof. For each vertex v of G, there is a point in the inverse limit which in the component for $G^+[W]$ takes the vertex whose branch set contains v. This is the point corresponding to the vertex v. Similarly, there are points in the inverse limit corresponding to interior points of edges. All other points in the inverse limit correspond to havens of order $<\infty$ of G. As explained in the appendix of [12], these are precisely the ends of G. Thus $|G|$ and the inverse limit have the same point set. It is straightforward to check that they carry the same topology. □

In particular, $|G|$ has the following universal property: Suppose there is a topological space X and for each finite set W of vertices of G, a continuous function $f_W : X\to G^+[W]$ such that $f[W,U]\circ f_W = f_U$ for every $U\subseteq W$. Then there is a unique continuous function $f : X\to |G|$ such that $\pi_W\circ f = f_W$, where $\pi_W : |G|\to G^+[W]$ is the canonical projection.

A function f from S^1 to $|G|$ is *sparse* if $f^{-1}(v)$ never contains more than one point for each interior point v of an edge, and if there are two distinct points $x,y\in S^1$ with $f(x)=f(y)$, then there are two points z_1 and z_2 in different components of S^1-x-y both of whose f-values are different from $f(x)$ and not equal to interior points of edges.

Let f from S^1 to $|G|$ be a sparse continuous function. Then f meets an edge e in an interior point if and only if it traverses this edge precisely once. The set of those edges e is called the *edge set of* f, denoted by $E(f)$. If f is a topological cycle, we call $E(f)$ a *topological circuit*. An edge set F is *geometrically connected* if F meets every finitely coverable cut b with the property that two components of $G - b$ contain edges of F. Note that if the closure of an edge set F in $|G|$ is connected in $|G|$, then F is geometrically connected.

Lemma 3.2. *A nonempty edge set X is the set of edges of a sparse continuous function f from S^1 to $|G|$ if and only if it meets every finitely coverable cut evenly and is geometrically connected.*

Proof. For the 'only if'-implication, first note that the edge set of f is geometrically connected since connectedness is preserved under continuous images. Second, let F be a finitely coverable cut and let W be a finite vertex set covering it. If there is a sparse continuous function $f : S^1 \to |G|$, then $\pi_W \circ f : G^+[W] \to |G|$ is also continuous and its edge set Y meets F in $X \cap F$. Note that Lemma 3.2 is true with '$G^+[W]$' in place of '$|G|$'. So $X \cap F = Y \cap F$ is even, as F is a cut of $G^+[W]$.

The 'if'-implication is a consequence of Theorem 3.1: Suppose we have a geometrically connected set X meeting every finitely coverable cut evenly. Then for every finite vertex set W, the edge set $X \cap E(G^+[W])$ meets every cut of $G^+[W]$ evenly and is geometrically connected. Hence $X \cap E(G^+[W])$ is the edge set of a sparse continuous function f_W in $G^+[W]$. Each f_W is essentially given by a cyclic order on $E(f_W)$. As each vertex of W is incident with only finitely many vertices of X, the set $E(f_W)$ is finite. Thus we can use a standard compactness argument to ensure that $f_U = f[W, U] \circ f_W$ for every $U \subseteq W$. Then the limit of the f_W is continuous by the universal property of the limit and it is sparse by construction. $\square$

The simplest example of a finitely coverable cut is the set of edges incident with a fixed vertex. Thus the edge set of a sparse continuous function has even degree at each vertex by Lemma 3.2. Thus we get the following.

Corollary 3.3. *Given a sparse continuous function f, then for every finite vertex set W only finitely many components of $G - W$ contain vertices incident with edges of $E(f)$.*

Proof. Let X be the set of those edges of $E(f)$ incident with vertices of W. Note that X is finite by Lemma 3.2. If two components of $G - W$ contain vertices incident with edges of $E(f)$, then s_D intersects X for every component D containing vertices incident with edges of $E(f)$ as $E(f)$ is geometrically connected by Lemma 3.2. Thus there are only finitely many such components D. □

Having Lemma 3.2 and Corollary 3.3 in mind, the set F below can be sought of as the edge set of a topological cycle. Thus the following is an extension of the 'Jumping arc'-Lemma [15]:

Lemma 3.4. *Let F be an edge set meeting every finitely coverable cut evenly such that for every finite vertex set W only finitely many components of $G - W$ contain vertices of $V(F)$. Let b be a cut which does not intersect F evenly. Then there is an end in the closure of both F and b.*

Given a finite vertex set W and a component D of $G - W$, we denote by $v(D)$ the vertex of $G^+[W]$ with branch set D.

Proof. First we show that for every finite vertex set W there is a component D of $G - W$ such that s_D contains infinitely many edges of both F and b. Suppose for a contradiction there is a vertex set W violating this. For a component D of $G - W$, let $X(D)$ be the set of those vertices in D incident with edges of b. Similarly, let $Y(D)$ be the set of those vertices in D incident with edges of F. Let U be the union of W with those $X(D)$ such that $Y(D)$ is infinite and those $Y(D)$ such that $Y(D)$ is finite.

By assumption $Y(D)$ is empty for all but finitely many D. Thus U is finite. Let G' be the graph obtained from $G^+[U]$ by deleting all vertices $v(K)$ for all components K of $G - U$ such that $Y(K)$ is empty.

Since $F \cap E(G')$ has even degree at each vertex of $G^+[U]$, the same is true for G'. On the other hand $b \cap E(G')$ is a cut by construction. Thus it intersects $F \cap E(G')$ evenly. As the intersection of b and F is included in $E(G')$ by construction, we get the desired contradiction.

Hence for every finite vertex set W there is a component D_W of $G - W$ such that s_{D_W} contains infinitely many edges of both F and b. By a standard compactness argument, we can pick the components D_W with the additional property that if $U \subseteq W$, then $f[U, W](v(D_W)) = v(D_U)$. Thus the components D_W define a haven of order $< \infty$ of G, which defines an end ω as explained in the appendix of [12]. By construction the end ω is in the closure of both F and b, completing the proof. □

Lemma 3.5. *Let f be a sparse continuous function from S^1 to $|G|$ and let $x, y \in S^1$ such that $f(x)$ and $f(y)$ are distinct and not interior points of edges. Then for each connected component C of $S^1 - x - y$ there is an edge e_C of G such that $e_C \times (0, 1)$ is included in $f(C)$.*

Proof. We pick a finite vertex set W containing x and y. Clearly, the above lemma is true with '$G^+[W]$' in place of '$|G|$'. Thus for each connected component C of $S^1 - x - y$ there is an edge e_C of G such that $e_C \times (0,1)$ is included in $\pi_W(f(C))$. Hence $e_C \times (0,1)$ is included in $f(C)$. □

4 Proof of Theorem 1.3

Given a connected graph G, we fix a tree-decomposition $(T, P_t | t \in V(T))$ as in Theorem 2.1 that has the additional properties of Remark 2.2. For an undominated end ω of G, we denote the unique end of T in which it lives by $\iota_T(\omega)$. It is straightforward to check that ι_T is a homeomorphism from $\Omega(G)$ restricted to the undominated ends to $\Omega(T)$.

For each $t \in V(T)$, let $M(t)$ be the finite-cycle matroid of the torso H_t. Let $\overline{V}(t) = V_{M(t)}$ and $\overline{W}(t) = W_{M(t)}$. Thus $\overline{V}(t)$ consists of those finite edge sets of H_t that have even degree at every vertex, and $\overline{W}(t)$ consists of the cuts of H_t.

Remark 4.1. *$\mathcal{T} = (T, \overline{V}, \overline{W})$ is a tree of binary finitary presentations.* □

The aim of this section is to prove Theorem 1.3 from the Introduction. For that we have to show for each Borel set Ψ of undominated ends of G that certain sets C_Ψ and D_Ψ are the sets of circuits and cocircuits of a matroid. By Theorem 2.10, we know that $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ presents some matroid. In this section we prove that the circuits and cocircuits of that matroid are given by C_Ψ and D_Ψ.

To build this bridge from $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ to the sets C_Ψ and D_Ψ, we start as follows. We have the two topological spaces $\Omega(G)$ and $\Omega(T)$, which each have their own Borel sets. The next lemma shows that these two systems of Borel sets are compatible:

Lemma 4.2. *The set of dominated ends of G is Borel. In particular, for any set Ψ of undominated ends, Ψ is Borel in $\Omega(G)$ if and only if $\iota_T(\Psi)$ is Borel in $\Omega(T)$.*

To prove this lemma, we need some intermediate lemmas. By $B_k(r)$ we denote the ball of radius k around a fixed vertex r.

Lemma 4.3. *The graph $G[B_k(r)]$ has a spanning tree Y_k of diameter at most $2k+1$.*

Proof. Proving this by induction over k, we may assume that $G[B_{k-1}(r)]$ has a spanning tree Y_{k-1} of diameter at most $2k-1$. Then Y_{k-1} together with all edges joining vertices in $B_k(r) \setminus B_{k-1}(r)$ to vertices in Y_{k-1} is a connected subgraph of $G[B_k(r)]$ with vertex set $B_k(r)$. Let Y_k be any of its spanning trees extending Y_{k-1}. Moreover, Y_k has diameter at most $2k+1$ by construction. □

Lemma 4.4. *Let G be a graph with a fixed vertex r. The set Ω_k of those ends dominated by some vertex in $B_k(r)$ is closed.*

Proof. In order to show that Ω_k is closed, we prove that its complement is open. For that it suffices to find for each ray R not dominated by some vertex in $B_k(r)$ some finite separator S_R disjoint from $B_k(r)$ that separates $B_k(r)$ from a tail of R.

Suppose for a contradiction that there is not such a finite separator S_R. Then we can recursively pick infinitely many $B_k(r)$-R-paths that are vertex-disjoint except possibly their starting vertices. Let U be the set of their starting vertices. The set U is infinite because otherwise some $u \in U$ would dominate R, which is impossible. By Lemma 4.3, $G[B_k(r)]$ has a rayless spanning tree Y_k. Applying the Star-Comb-Lemma [15, Lemma 8.2.2] to Y_k and U, we find a vertex v in $G[B_k(r)]$ together with an infinite fan whose endvertices are in U. Enlarging this fan by infinitely many of the previously chosen $B_k(r)$-R-paths, yields an infinite fan which witnesses that v dominates R, which is the desired contradiction. Thus there is such a finite set R_S for every ray R not dominated by some vertex in $B_k(r)$ and so Ω_k is closed. □

Proof that Lemma 4.4 implies Lemma 4.2. By Lemma 4.4, the set of dominated ends is a countable union of closed sets and thus Borel. □

The next step in our proof of Theorem 1.3 is to give a more combinatorial description of the set C_Ψ defined in the Introduction. For a set A, we denote the set of minimal nonempty elements of A by A^{min}. Given a set Ψ of ends of G, an edge set o is in $\mathcal{C}_\Psi$ if o meets every finitely coverable cut evenly and is geometrically connected. The next lemma implies that $C_\Psi = \mathcal{C}_\Psi^{min}$.

Lemma 4.5. *Given a Borel set Ψ of ends of G, the following are equivalent for some nonempty edge set o.*

1. *$o \in \mathcal{C}_\Psi$;*
2. *o is the edge set of a sparse continuous function from S^1 to $|G|$ that only has ends from Ψ in the closure;*
3. *o is the edge set of a sparse continuous function from S^1 to $|G| \setminus \Psi^{\complement}$.*

In particular, if o is minimal nonempty with one of these properties, then it is minimal nonempty with each of them. Furthermore o is minimal nonempty with one of these properties if and only if o is the edge set of a topological cycle in $|G| \setminus \Psi^{\complement}$.

Proof of Lemma 4.5. Clearly 2 and 3 are equivalent. And 1 and 2 are equivalent by Lemma 3.2. Thus 1,2 and 3 are equivalent.

To see the 'Furthermore'-part, first note that the edge set of a topological cycle in $|G| \setminus \Psi^{\complement}$ is a minimal nonempty edge set satisfying 3. To see the converse, let o be a minimal edge set which is the edge set of a sparse continuous function f from S^1 to $|G| \setminus \Psi^{\complement}$. Suppose for a contradiction that f is not injective. Then there are two distinct points $x, y \in S^1$ with $f(x) = f(y)$. By sparseness of f, there are two points z_1 and z_2 in different components of $S^1 - x - y$ whose f-values are different from $f(x)$. By Lemma 3.5 applied first to x and z_1 and second to x and z_2, for each of the two components C_1 and C_2 of $S^1 - x - y$ there is for each $i = 1, 2$ an edge e_i of G such that $e_i \times (0, 1)$ is included in $f(C_i)$.

We obtain the topological space K from $C_1 \cup \{x, y\} \subseteq S^1$ by identifying x and y. Note that K is homeomorphic to S^1. Moreover, the restriction $\bar{f}$ of f to $C_1 \cup \{x\}$ considered as a map from K to $|G|$ is continuous. However, the edge set of $\bar{f}$ is included in the edge set of f without e_2, violating the minimality of the edge set of f. Thus f is injective, and so o is the edge set of a topological cycle in $|G| \setminus \Psi^{\complement}$, completing the proof.

□

Let $\mathcal{D}_\Psi$ be the set of cuts that do not have an end of Ψ in their closure. Put another way, $d \in \mathcal{D}_\Psi$ if and only if d does not have an end of Ψ in its closure and it meets every finite cycle evenly. Note that $D_\Psi = \mathcal{D}_\Psi^{min}$. The next step in our proof of Theorem 1.3 is to relate $\mathcal{C}_\Psi$ and $\mathcal{D}_\Psi$ to the sets of $\iota_T(\Psi)$-vectors of $\mathcal{T}$ and $\iota_T(\Psi)^{\complement}$-covectors of $\mathcal{T}$.

Lemma 4.6.

1. *The edge set of a finite cycle is an underlying vector of an $\emptyset$-pre-vector of $\mathcal{T}$;*

2. *Any finitely coverable bond is an underlying covector of an $\emptyset$-precovector of $\mathcal{T}$.*

Proof. In this proof we use the tree order $\leq$ on T as in Remark 2.2.

To see the second part, let d be a finitely coverable bond and let $V(G) = A \dot{\cup} B$ be a partition inducing d and let A' be a finite cover of d. Since G is connected, the partition is unique and both A and B are connected.

For $t \in V(T)$, let $x(t)$ be the set of crossing edges of the partition $V(P_t) = (A \cap V(P_t)) \dot\cup (B \cap V(P_t))$ in the torso H_t. Let S be the set of those nodes such that A and B both meet $V(P_t)$.

Our aim is to show that (S, x) is an $\emptyset$-pre-covector of $\mathcal{T}$, which then by construction has underlying set d. By construction, $x(t) \in \overline{W}(t)$. It remains to verify the followings sublemmas.

Sublemma 4.7. *S is connected. Moreover, for each $st \in E(S)$, $x(s)$ contains an edge of the torso H_t.*

Sublemma 4.8. *S is rayless.*

Proof of Sublemma 4.7. It suffices to show for each $st \in E(T)$ separating two vertices of S that $X = V(P_s) \cap V(P_t)$ contains vertices of both A and B. This follows from the fact that A and B are both connected and each has vertices in at least two components of $G - X$. □

Proof of Sublemma 4.8. Suppose for a contradiction that S includes a ray $v_1 v_2 \ldots$. By taking a subray if necessary we may assume that $v_i < v_{i+1}$. As A' is finite, by the 'Moreover'-part of Remark 2.2 there is some m such that for all $w \geq v_m$ the part P_w does not contain vertices of A'. By Remark 2.2, $X_i = \left(\bigcup_{w \geq v_{i+1}} V(P_w)\right) \setminus V(P_i)$ is connected. As $v_{m+2} \in S$, both A and B contain vertices of $P_{v_{m+2}} \subseteq X_m$. Thus X_m contains an edge of d, which is incident with a vertex of A'. This is a contradiction to the choice of m.

□

To see the first part, let o be the edge set of a finite cycle. We shall define for each node $t \in V(T)$ an edge set $x(t)$, which plays a similar role as in the last part. For that we need some preparation. Let $y(t) = o \cap E(P_t)$. Let $st \in E(T)$ with $s < t$. Let $Z(st)$ be the set of those vertices of $V(P_s) \cap V(P_t)$ incident with an odd number of edges of $y(t)$.

Sublemma 4.9. *$|Z(st)|$ is even.*

Proof. The set b of edges joining $V(P_s) \cap V(P_t)$ with $\left(\bigcup_{w \geq t} V(P_w)\right) \setminus V(P_s)$ is a cut. Thus o intersection b evenly. Since $b(st) \subseteq E(P_t)$ by Remark 2.2, the number $|Z(st)|$ has the same parity as $|o \cap b|$ and so is even. □

Thus there is a matching $M(st)$ of $Z(st)$ using only edges from $E(H_s) \cap E(H_t)$. We obtain $x(t)$ from $y(t)$ by adding all the sets $M(st)$ where s is a neighbour of t. Let S be the set of those nodes t where $x(t)$ is nonempty.

Our aim is to show that (S, x) is an $\emptyset$-pre-vector of $\mathcal{T}$, which then by construction has underlying set o. First note that S is finite as $y(t)$ is nonempty for only finitely many t. Thus it remains to verify the following sublemmas.

Sublemma 4.10. *$x(t)$ has even degree at each vertex of H_t.*

Sublemma 4.11. *S is connected. Moreover, for each $st \in E(S)$, $x(s)$ contains an edge of the torso H_t.*

Proof of Sublemma 4.10. By construction $x(t)$ has even degree at all vertices v in $V(H_t) \cap V(H_s)$, where $st \in E(T)$ with $s < t$. Hence if t is maximal in S, then $x(t)$ has even degree at all vertices of H_t. Otherwise the statement follows inductively from the statement for all the upper neighbours. Indeed, let $v \in V(H_t) \setminus V(H_s)$, where $st \in E(T)$ with $s < t$. Then the degree of v in $x(t)$ is congruent modulo 2 to the degree of v in o plus the sum of the degrees of v in $x(u)$, where the sum ranges over all upper neighbours u of t. $\square$

Proof of Sublemma 4.11. It suffices to show for each $st \in E(T)$ separating two vertices of S that $M(st)$ is nonempty. Suppose for a contradiction that $M(st)$ is empty. Let T_s be the component of $T - t$ containing s. The symmetric difference D_s of all $x(u)$ with $u \in T_s$ contains only edges of o and has even degree at each vertex by Sublemma 4.10.

Moreover, T_s contains a vertex v of S. Either P_v contains an edge of o or it has a neighbour w such that $M(vw)$ is nonempty and P_w contains an edge of o. In the later case w is also in T_s. So in either case, D_s is nonempty.

Similarly, we define T_t and D_t, and deduce that D_t is nonempty. Since D_s and D_t are both nonempty, we deduce that o includes two edge disjoint cycles, which is the desired contradiction. $\square$

$\square$

Corollary 4.12. *Every $\Psi^{\complement}$-covector d of $\mathcal{T}$ is in $\mathcal{D}_\Psi$.*

Proof. First note that d has only ends of $\Psi^{\complement}$ in its closure. Moreover d is a cut as it meets every finite cycle evenly by Lemma 4.6 and Lemma 2.11 as $\mathcal{T}$ is tree of binary finitary presentations. $\square$

Let $\mathcal{F}_\Psi$ be the set of those edge sets o meeting every finitely coverable cut evenly such that for every finite vertex set W only finitely many components of $G - W$ contain vertices of $V(o)$. Note that $\mathcal{C}_\Psi \subseteq \mathcal{F}_\Psi$ by Lemma 3.2 and Corollary 3.3.

Lemma 4.13. *Any nonempty $o \in \mathcal{F}_\Psi$ includes a nonempty element of $\mathcal{C}_\Psi$. Hence, $\mathcal{F}_\Psi^{min} = \mathcal{C}_\Psi^{min}$.*

Proof. We say that edges e and f of o are in the *same geometric component* if o meets every finitely coverable cut d such that e and f are in different components of $G - d$. It is straightforward to check that being in the same geometric component is an equivalence relation. Pick some $e \in o$ and let u be its equivalence class. It suffices to show that u is in $\mathcal{C}_\Psi$, which is implies by the following two sublemmas.

Sublemma 4.14. *u is meets every finitely coverable cut evenly.*

Sublemma 4.15. *u is geometrically connected.*

Before proving these two sublemmas, we give a construction that is used in the proof of both these sublemmas. Let $x \in o$ and let b be a finitely coverable cut. For all $z \in b \cap (o \setminus u)$, there is a finitely coverable cut b_z such that x and z are in different components of $G - b_z$. Let $V(G) = A \dot\cup B$ be a partition inducing b, and let $V(G) = A_z \dot\cup B_z$ be a partition inducing b_z such that x has both its endvertices in A_z. Let d be the cut consisting of those edges with precisely one endvertex in the intersection of A and the finitely many A_z. Note that d is finitely coverable. By construction $d \cap u = d \cap o$. Moreover, $b \cap u = d \cap u$ since any $y \in u$ has both its endvertices in A_z.

Proof of Sublemma 4.14. Let b be a finitely coverable cut. Then $b \cap u = d \cap o$, and thus $b \cap u$ has even size. □

Proof of Sublemma 4.15. Let b be a finitely coverable cut such that there are edges x and y of u in different components of $G - b$. Thus there is a partition $V(G) = A \dot\cup B$ inducing b such that x has both endvertices in A and y has both endvertices in B. Then x and y are in different components of $G - d$. As x and y are in the same geometric component, d meets o. Thus b meets u, completing the proof. □

□

Lemma 4.16. *Every Ψ-vector o of $\mathcal{T}$ is in $\mathcal{F}_\Psi$.*

Proof. The set o meets every finitely coverable bond evenly by Lemma 4.6 and Lemma 2.11 as $\mathcal{T}$ is tree of binary finitary presentations. Since every finitely coverable cut is an edge-disjoint union of finitely many finitely coverable bonds, o meets each finitely coverable cut evenly.

The set o is a finite symmetric difference of sets o_i, which are underlying sets of Ψ-pre-vectors $(S_i, \overline{o}_i)$. Note that S_i is locally finite as each $\overline{o}_i$ is finite and for each $xy \in E(S_i)$, the set $\overline{o}_i(x)$ contains an edge of the torso of P_y. It suffices to show that there is no finite vertex set W together with an infinite set $\mathcal{A}$ of components of $G - W$ each containing a vertex of $V(o_i)$.

Suppose for a contradiction there is such a set W. By the 'Moreover'-part of Remark 2.2, there is a rayless subtree Q of T containing all nodes q such that its part P_q contains a vertex of W and the root r of T. For each $A \in \mathcal{A}$, there is an edge z_A in $o_i \cap s_A$. Let t_A be the unique node of T such that $z_A \in P_{t_A}$.

Next we define an edge e_A for each $A \in \mathcal{A}$. If $t_A \in Q$, we pick $e_A = z_A$. Otherwise, let q_A be the last node on the unique t_A-Q-path and u_A be the node before that. By Remark 2.2, P_{u_A} together with the parts above is connected. Thus all these parts are included in A. Thus the nodes u_A are distinct for different A. Moreover, q_A is on the path from t_A to some t_B for some other $b \in \mathcal{A}$. As S_i is connected and $t_A, t_B \in S_i$, it must be that $q_A \in S_i$. So u_A is in S_i, as well. Thus $\bar{o}_i(q_A)$ contains an edge of the torso of P_{u_A}. Pick such an edge for e_A. Summing up, we have picked for each $A \in \mathcal{A}$ an edge e_A in some $\overline{o}_i(q)$ with $q \in Q \cap S_i$ such that all these e_A are distinct.

Note that $S_i \cap Q$ is finite as S_i is locally finite and Q is rayless. Since each e_A is in some of the finite sets $\overline{o}_i(x)$ with $x \in S_i \cap Q$, we get the desired contradiction. □

Theorem 4.17. *Let Ψ be a Borel set of ends of an infinite connected graph G that are all undominated. Then there is a matroid M whose set of circuits is $\mathcal{C}_\Psi^{min}$ and whose set of cocircuits is $\mathcal{D}_\Psi^{min}$.*

Proof. By Lemma 4.2, $\iota_T(\Psi)$ is Borel. Thus we apply Theorem 2.10 to the tree of presentations $\mathcal{T}$, yielding that $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ presents a matroid M. Note that $\mathcal{F}_\Psi$ and $\mathcal{D}_\Psi$ satisfy (01) by Lemma 3.4. Hence by Corollary 4.12 and Lemma 4.16, we can apply Lemma 2.6 to $\mathcal{F}_\Psi$ and $\mathcal{D}_\Psi$ and M. As $\mathcal{F}_\Psi^{min} = \mathcal{C}_\Psi^{min}$ by Lemma 4.13, we get the desired result. □

Proof of Theorem 1.3. By considering distinct connected components separately, we may assume that G is connected. By Lemma 4.5, $\mathcal{C}_\Psi^{min}$ is the set of topological cycles in $|G| \setminus \Psi^\complement$. Thus Theorem 1.3 follows from Theorem 4.17. □

5 Consequences of Theorem 1.3

First, we prove for any graph G that the set of topological circuits is the set of circuits of a matroid if and only if G does not have a subdivision of the dominated ladder H. This theorem was already mentioned in the Introduction, see Corollary 1.2. We start with a couple of preliminary lemmas.

Lemma 5.1. *Let ω be a dominated end of a graph G such that there are two vertex-disjoint rays R and S belonging to ω. Then G has a subdivision of H.*

Proof. Let v be a vertex dominating ω. By taking subrays if necessary, we may assume that v lies on neither R nor S. As R and S belong to the same end, there are infinitely many vertex-disjoint paths $P_1, P_2, \ldots$ from R to S. We may assume that no P_i contains v. Let r_i be the endvertex of P_i on R and s_i be the endvertex of P_i on S. By taking a subsequence of the P_i if necessary, we can ensure that the order in which the r_i appear on R is $r_1, r_2, \ldots$. Similarly, we may assume that the order in which the s_i appear on S is $s_1, s_2, \ldots$.

Let $Q_1, Q_2, \ldots$ be an infinite fan from v to $R \cup S$. So for one of R or S, say R, there is an infinite fan $Q'_1, Q'_2, \ldots$ from v to it that avoids the other ray. As each P_i and each Q'_j is finite, we can inductively construct infinite sets $I, J \subseteq \mathbb{N}$ such that for $i \in I$ and $j \in J$ the paths P_i and Q'_j are vertex-disjoint.

Indeed, just consider the bipartite graph with left hand side $(P_i | i \in \mathbb{N})$ and right hand side $(Q'_j | j \in \mathbb{N})$ and put an edge between two paths P_i and Q'_j if they share a vertex. Now we use that each vertex of this bipartite graph has only finitely many neighbours on the other side to construct an independent set of vertices that intersects both sides infinitely. Indeed, for each finite independent set, there are two vertices, one on the left and one on the right, such that the independent set together with these two vertices is still independent. So there is such an infinite independent set and I is its set of vertices on the left and J is its set of vertices on the right.

Finally, v together with R, S and $(P_i | i \in I)$ and $(Q'_j | j \in J)$ give rise to a subdivision of H, which completes the proof. □

Lemma 5.2. *Let o be a topological circuit that has the end ω in its closure. Then there is a double ray both of whose tails belong to ω.*

This lemma already was proved in [6, Lemma 5.6] in a slightly more general context.

Proof of Corollary 1.2. If G has a subdivision of H, then as explained in the Introduction the topological set of topological circuits violates (C3).

Thus it remains to consider the case that G has no a subdivision of H. Now we apply Theorem 1.3 with Ψ the set of undominated ends, which is Borel by Lemma 4.2.

It suffices to show that every topological circuit o of G is a Ψ-circuit. So let ω be an end in the closure of o. Then by Lemma 5.2 there is a double ray both of whose tails belong to ω. If ω was not in Ψ, then G would have a subdivision of H by Lemma 5.1. Thus ω is in Ψ. As ω was arbitrary, this shows that every end in the closure of o is in Ψ. □

Theorem 1.3 can also be used to extend a central result of [3] from countable graphs to graphs with a normal spanning tree as follows. Given a graph G with a normal spanning tree T, in [3] we constructed the Undomination graph $U = U(G, T)$. This graph has the pleasant property that it has few enough edges to have no dominated end but enough edges to have G as a minor. Moreover there is an inclusion $\tilde{u}$ from the set of ends of G to the set of ends of U. By Theorem 1.3, for every Borel set Ψ, the Ψ-circuits of $U(G, T)$ are the circuits of a matroid. Now we use the following theorem.

Theorem 5.3 ([3, Theorem 9.9]). *Assume that $(U, \tilde{u}(\Psi))$ induces a matroid M. Then (G, Ψ) induces the matroid M/C.*

We refer the reader to [3, Section 3] for a precise definition of when the pair (G, Ψ) consisting of a graph G and an end set Ψ induces the matroid M. Very very roughly, this says that the set of certain 'topological circuits' which only use ends from Ψ is the set of the circuits of M. However the topological space taken there is different from the one we take in this paper, so that the definition of topological circuit there does not match with the definition of topological circuit in this paper. For example, in this different notion a ray starting at a vertex v may also be a circuit if the end it converges to is in Ψ and dominated by v. However these two notions of topological circuit are the same if no vertex is dominated by an end. Thus combining Theorem 5.3 and Theorem 1.3, we get the following.

Corollary 5.4. *Let G be a graph with a normal spanning tree and $\Psi \subseteq \Omega(G)$ such that $\tilde{u}(\Psi)$ is Borel, then (G, Ψ) induces a matroid.*

For example, if we choose Ψ equal to the set of dominated ends, then we get an interesting instance of this corollary: Like Theorem 1.3, this gives a recipe to associate a matroid (which we call $M_I(G)$) to every graph G that has a normal spanning tree which in general is neither finitary nor cofinitary. These two matroids need not be the same. For example, these two matroids differ for the graph obtained from the two side infinite ladder by adding a vertex so that it dominates precisely one of the two ends.

In fact the circuits of the matroid $M_I(G)$ can be described topologically, namely they are the edge sets of topological cycles in the topological space ITOP, see [14] for a definition of ITOP. About ITOP, we shall only need the following fact, which is not difficult to prove: Given a graph G, we denote by G_I, the multigraph obtained from G by identifying any two vertices dominating the same end. It is not difficult to show that G and G_I have the same topological cycles. Thus in order to study when the topological cycles of G induce a matroid, it is enough to study this question for the graphs G_I. In what follows, we show that the underlying simple graphs G'_I of G_I always has a normal spanning tree. This will imply the following:

Corollary 5.5. *The topological cycles of ITOP induce a matroid for every graph.*

Let H' be the graph obtained from the dominated ladder H by adding a clone of the infinite degree vertex of H. Note that G'_I has no subdivision of H'. Thus G'_I has a normal spanning tree due to the following criterion:

Theorem 5.6 (Halin [17])**.** *If G is connected and does not have a subdivision of the completes graph on countably many vertices, then G has a normal spanning tree.*

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