# A TALE OF STARS AND CLIQUES 

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#### Abstract

We show that for an infinitely many natural numbers $k$ there are $k$-uniform hypergraphs which admit a 'rescaling phenomenon' as described in [9]. More precisely, let $\mathcal{A}(k, I, n)$ denote the class of $k$-graphs on $n$ vertices in which the sizes of all pairwise intersections of edges belong to a set $I$. We show that if $k=r t^{2}$ for some $r \geqslant 1$ and $t \geqslant 2$, and $I$ is chosen in some special way, the densest graphs in $\mathcal{A}\left(r t^{2}, I, n\right)$ are either dominated by stars of large degree, or basically, they are ' $t$-thick' $r t^{2}$-graphs in which vertices are partitioned into groups of $t$ vertices each and every edge is a union of $t r$ such groups. It is easy to see that, unlike in stars, the maximum degree of $t$-thick graphs is of a lower order than the number of its edges. Thus, if we study the graphs from $\mathcal{A}\left(r t^{2}, I, n\right)$ with a prescribed number of edges $m$ which minimize the maximum degree, around the value of $m$ which is the number of edges of the largest $t$-thick graph, a rapid, discontinuous phase transition can be observed. Interestingly, these two types of $k$-graphs determine the structure of all hypergraphs in $\mathcal{A}\left(r t^{2}, I, n\right)$. Namely, we show that each such hypergraph can be decomposed into a $t$-thick graph $H_{T}$, a special collection $H_{S}$ of stars, and a sparse 'left-over' graph $H_{R}$.


## §1. Introduction

By a set system we mean a pair $S=(V, \mathcal{E})$ such that $\mathcal{E}$ is a collection of subsets of $V$. The members of $V$ are usually referred to as the vertices of the set system, whilst the members of $\mathcal{E}$ are called edges. If all members of $\mathcal{E}$ are of the same cardinality $k \geqslant 0$ we call $S$ a $k$-uniform hypergraph or, more briefly, a $k$-graph.

Occasionally we identify a hypergraph $H$ with its set of edges, denoting, for example, by $|H|$ the number of edges in $H$. For a given set $I$ of nonnegative integers, we say that a $k$-graph $H$ is $I$-intersecting if $|e \cap f| \in I$ holds for all distinct $e, f \in H$. Starting with the seminal work [4] of Erdős, Ko, and Rado, the study of $I$-intersecting hypergraphs and set systems has a long tradition in extremal combinatorics (see, e.g., [2, 3, 6, 7, 10, 11] for some milestones). Let us remark that sometimes in the literature (e.g., $[2,3,6]$ ) an $I$-intersecting $k$-graph on $n$ vertices is called an $(n, k, I)$-system.

Motivated by the stability of extremal graphs for 3 -uniform loose path of length 3 the first two authors studied $\{0,2,3\}$-intersecting 4 -graphs in [9]. The aim of the present

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article is to extend their results to the more general family $\mathcal{J}(r, t)$ which consists of all $I$-intersecting $r t^{2}$-graphs, where $r \geqslant 1$ and $t \geqslant 2$ are arbitrary integers and

$$
I=\{s: t \mid s \text { or } s \geqslant r t(t-1)\} .
$$

This choice of the set of permissible intersections may look bizarre at first and our main incentive to study it came from the aesthetical merits of the results we hoped to obtain: to explain those, we start from the observation that there are two quite different examples of dense $r t^{2}$-graphs $H \in \mathcal{J}(r, t)$ on $n$ vertices with $\Theta\left(n^{r t}\right)$ edges.

The most obvious one is the full $(r t(t-1))$-star, i.e., a hypergraph $H$ with a distinguished $r t(t-1)$-set $S$ of vertices, called the centre of the star, such that the edges of $H$ are precisely the $r t^{2}$-supersets of $S$. Clearly such a star has exactly $\binom{n-r t^{2}+r t}{r t}$ edges and it can be shown that, for large $n$, it is the unique graph which maximises the number of edges among all graphs from $\mathcal{J}(r, t)$ on $n$ vertices (see Proposition 2.3 below).

However, there exists another natural construction of dense $r t^{2}$-graphs $H \in \mathcal{J}(r, t)$ with $n$ vertices and $\Theta\left(n^{r t}\right)$ edges. It proceeds by splitting the vertex set into $\lfloor n / t\rfloor$ subsets of size $t$ called teams (and a small number of left-over vertices) and to declare an $r t^{2}$-set to be an edge if and only if it is a union of $r t$ teams. We call the resulting hypergraph a thick clique and to its subgraphs we refer as thick hypergraphs. Note that each thick hypergraph has the property that for any two edges $e$ and $f$ the number $|e \cap f|$ is a multiple of $t$ and, hence, it indeed belongs to $\mathcal{J}(r, t)$.

The point that interests us here is that even though both the star and the thick clique have $\Theta\left(n^{r t}\right)$ edges, their maximum vertex degrees are of different orders of magnitude. In fact, while the vertices belonging to the centre of a star have degree $\Omega\left(n^{r t}\right)$, the maximum degree of a thick clique is easily seen to be only $O\left(n^{r t-1}\right)$. Perhaps surprisingly, it turns out that this phenomenon arises in a very "discontinuous" manner: As soon as a graph from $\mathcal{J}(r, t)$ has one edge more than the thick clique, it needs to contain a vertex of degree $\Omega\left(n^{r t}\right)$.

This is the main result of the present work which, crudely, can be stated as follows (for further structural results see Theorems 2.2 and 3.5 below).

Theorem 1.1. For $r \geqslant 1$ and $t \geqslant 2$ there exists an $n_{0}$ such that for every $r t^{2}$-graph $H \in \mathcal{J}(r, t)$ with $n \geqslant n_{0}$ vertices and at least $\binom{[n / t\rfloor}{ r t}+1$ edges we have $\Delta(H) \geqslant m /(3 t)$.

On the other hand, for every $n \geqslant r t^{2}$ a thick clique $H_{0} \in \mathcal{J}(r, t)$ on $n$ vertices has $\binom{\lfloor n / t\rfloor}{ r t}$ edges and $\Delta\left(H_{0}\right)=\binom{[n / t\rfloor-1}{r t-1}$.

The main step in the proof of Theorem 1.1 is a somewhat surprising structural result (see Theorem 2.2 below). It turns out that stars and thick hypergraphs which naturally emerge when we study graphs from $\mathcal{J}(r, t)$ whose density is close to the maximum density $\Theta\left(n^{r t}\right)$, are natural building blocks for all 'not too sparse' graphs from $\mathcal{J}(r, t)$. More specifically,
we show that up to an 'error of lower order,' i.e., up to at most $O\left(n^{r t-1}\right)$ edges, any such hypergraph arises by attaching "non-overlapping" stars to a thick hypergraph.

This article is organised as follows. In the next section we state a precise version of the structure theorem mentioned above. Then, in Section 3, we show that it does indeed imply Theorem 1.1 and give more structural characterisations of dense graphs from $\mathcal{J}(r, t)$ with small maximum degree. Section 4 collects some tools needed for the proof of this structure theorem including a 'decomposition lemma' (see Lemma 4.4 below) that might have some other applications as well. Finally, in Section 5, we prove the structure theorem.

## §2. The structure theorem

We begin this section with some definitions allowing us to formulate a precise version of the structure theorem for $r t^{2}$-graphs $H \in \mathcal{J}(r, t)$.

Let $k \geqslant s \geqslant 0$ be integers. A $k$-uniform hypergraph $H=(V, E)$ with a set $S \subseteq V$ of distinguished vertices of size $|S|=s$ is an $s$-star if $S \subseteq e$ holds for all edges $e \in E$. We call $S$ the centre of the star and $\bigcup_{h \in E}(h \backslash S)$ is referred to as the body of the star. A collection of stars is said to be semi-disjoint if their centres are distinct and their bodies are mutually disjoint. Of course, an $s$-star $H$ on $|V|=n$ vertices can have at most $\binom{n-s}{k-s}$ edges. If this happens we say that $H$ is a full $s$-star and denote it by $S_{n, s}^{k}$.

Next, for a given hypergraph $H=(V, E)$, we say that a subset $W \subseteq V$ of its vertex set is inseparable in $H$, if for all edges $h \in E$ we have $W \cap h \in\{\varnothing, W\}$. Now consider three natural numbers $k$, $t$, and $n$ satisfying $t \mid k$, and suppose that a set $V$ of $n$ vertices is partitioned into $\lfloor n / t\rfloor$ many $t$-subsets called teams and fewer than $t$ further vertices. By $\widetilde{K}_{n, t}^{k}$ we denote the thick $(k, n, t)$-clique, i.e., the $k$-graph on $n$ vertices whose $\binom{\lfloor n / t\rfloor}{ k / t}$ edges are all possible unions of some $k / t$ of these teams. We refer to its subgraphs as $t$-thick or just thick hypergraphs. Evidently the teams are inseparable in $\widetilde{K}_{n, t}^{k}$ and a $k$-graph $H$ on $n$ vertices possessing $\lfloor n / t\rfloor$ mutually disjoint inseparable $t$-sets of vertices is a subhypergraph of the thick clique $\widetilde{K}_{n, t}^{k}$.

Finally, for positive integers $t, \ell$, and $a$, we define a class $\mathcal{F}(t, \ell, a)$ of $\ell t$-graphs as follows.
Definition 2.1. For given natural numbers $t, \ell$, and $a$, we say that a $\ell t$-graph $H=(V, E)$ belongs to the class $\mathcal{F}(t, \ell, a)$ if and only if there are partitions

$$
V=V_{T} \cup V_{S} \cup V_{R} \quad \text { and } \quad H=H_{T} \cup H_{S} \cup H_{R},
$$

such that
(i) $V_{T}$ is a union of inseparable $t$-subsets of $V$, and $H_{T}=H\left[V_{T}\right]$;
(ii) $H_{S}=\left\{h \in H:\left|h \cap V_{S}\right|=\ell\right\}$ consists of semi-disjoint $(\ell(t-1))$-stars with their centres in $V_{T} \cup V_{R}$ and their bodies in $V_{S}$;
(iii) any edge of $H$ that intersects the body of a star $S^{*} \subseteq H_{S}$ contains the centre of $S^{*}$;
(iv) and $\left|H_{R}\right| \leqslant\left|V_{T}\right|\left|V_{S}\right| n^{\ell-3}+\left|V_{R}\right| a n^{\ell-2}$.

Now the structure theorem for $r t^{2}$-graphs $H \in \mathcal{J}(r, t)$ promised in the introduction can be stated as follows.

Theorem 2.2 (Structure Theorem). For all integers $r \geqslant 1$ and $t \geqslant 2$ we have

$$
\mathcal{J}(r, t) \subseteq \mathcal{F}\left(t, r t,\left(r t^{2}\right)^{r^{3} t^{6}}\right) .
$$

The proof of this result is deferred to Section 5 . We conclude this section by pointing out that the structure theorem quickly allows us to determine the extremal $r t^{2}$-graphs from $\mathcal{J}(r, t)$. The following statement shows that the extremal graph, the full $(\ell(t-1))$-star, is unique and stable for this problem.

Proposition 2.3. Given natural numbers $\ell \geqslant t \geqslant 2$, a, and $c$, there exists an integer $n_{*}$ such that every $\ell t$-graph $H=(V, E) \in \mathcal{F}(t, \ell, a)$ with $|V|=n \geqslant n_{*}$ and $m \geqslant\binom{ n-c}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!}$ edges is obtained from an $(\ell(t-1))$-star by adding at most $\binom{c}{\ell t}$ further edges.

Moreover, if $m \geqslant\binom{ n-\ell t}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!}$, then $H$ is an $(\ell(t-1))$-star.
In particular, each $H \in F(t, \ell, a)$ has at most $\binom{n-\ell(t-1)}{\ell}$ edges, and this maximum is achieved only if $H$ is isomorphic to the full $(\ell(t-1))$-star $S_{n, \ell(t-1)}^{\ell t}$.

Proof. Let us choose $n_{*}$ and $D$ such that $n_{*} \gg D \gg \max (t, \ell, a)+c$, so all inequalities below hold for $n \geqslant n_{*}$. Moreover, let $H=(V, E)$ with

$$
|V|=n \geqslant n_{*} \quad \text { and } \quad|E|=m \geqslant\binom{ n-c}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!}
$$

be a $\ell t$-graph from $\mathcal{F}(t, \ell, a)$, and take partitions

$$
V=V_{T} \cup V_{S} \cup V_{R} \quad \text { as well as } \quad H=H_{T} \cup H_{S} \cup H_{R}
$$

exemplifying this.
Clearly

$$
\begin{equation*}
m \leqslant\binom{\left|V_{T}\right| / t}{\ell}+\binom{\left|V_{S}\right|}{\ell}+\left|H_{R}\right| \tag{2.1}
\end{equation*}
$$

where

$$
\left|H_{R}\right| \leqslant\left|V_{T}\right|\left|V_{S}\right| n^{\ell-3}+\left|V_{R}\right| a n^{\ell-2} \leqslant\left(\left|V_{T}\right|+\left|V_{R}\right|\right) a n^{\ell-2} \leqslant a n^{\ell-1}
$$

and so

$$
\begin{equation*}
m \leqslant\binom{\left|V_{S}\right|+\left|V_{T}\right| / t}{\ell}+a n^{\ell-1} \tag{2.2}
\end{equation*}
$$

Assume first that $\left|V_{S}\right|<n-2 D$. Then

$$
\left|V_{S}\right|+\frac{1}{t}\left|V_{T}\right| \leqslant\left|V_{S}\right|+\frac{1}{2}\left(n-\left|V_{S}\right|\right)=\frac{1}{2}\left(n+\left|V_{S}\right|\right) \leqslant n-D
$$

and so, by (2.2),

$$
m \leqslant\binom{ n-D}{\ell}+a n^{\ell-1}<\binom{n-c}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!}
$$

As this contradicts our assumption, we may conclude that $\left|V_{S}\right| \geqslant n-2 D$.
In particular, we have $\left|V_{T}\right|+\left|V_{R}\right| \leqslant 2 D$, and $\left|H_{T} \cup H_{R}\right| \leqslant(2 D a+1) n^{\ell-2}$. Consider the largest star $S^{*}=\left(V^{*}, E^{*}\right)$ in $H_{S}$ and let $s$ be its centre. Then $\left|V^{*}\right| \geqslant n / 2$, since otherwise, by (2.1),

$$
m=\left|H_{S}\right|+\left|H_{T}\right|+\left|H_{R}\right| \leqslant\binom{ n / 2}{\ell}+\binom{n / 2}{\ell}+(2 D a+1) n^{\ell-2}<\binom{n-c}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!} .
$$

Now assuming $\left|V^{*} \cap V_{S}\right| \leqslant n-c-1$ we could use (2.1) again and infer

$$
m \leqslant\binom{ n-c-1}{\ell}+\binom{c+1}{\ell}+(2 D a+1) n^{\ell-2}<\binom{n-c}{\ell}-\frac{n^{\ell-1}}{2(\ell-1)!}
$$

which, again, contradicts our assumption on $m$.
This proves that $\left|V^{*} \cap V_{S}\right| \geqslant n-c$. By Definition 2.1(iii), all edges of $H$ intersecting $V^{*} \cap V_{S}$ contain $s$ and therefore they form an $(\ell(t-1))$-star. Since $\left|V \backslash\left(V^{*} \cap V_{S}\right)\right| \leqslant c$, there can be at most $\binom{c}{\ell t}$ edges not belonging to this star, which establishes our first assertion. The moreover-part follows from the observation that in case $c=\ell t$ the only potential further edge, $V \backslash\left(V^{*} \cap V_{S}\right)$, would still contain $s$ and could thus be adjoined to the star.

## §3. Minimum maximum degree

Let us first start with the proof of Theorem 1.1 which, let us recall, states that in each graph from $\mathcal{J}(r, t)$ with $m>\binom{\lfloor n / t\rfloor}{ r t}$ edges there exists a big star which contains a positive fraction of all edges; moreover thick cliques show that this result is sharp. We prove this result in a slightly stronger form, which gives a better estimate for the size of the biggest star for dense graphs. Besides, it states that each graph from $\mathcal{J}(r, t)$ which has almost $\binom{\lfloor n / t\rfloor}{ r t}$ edges and small maximum degree is thick. Here, for $H \in \mathcal{J}(r, t)$ by $H_{S}$ we denote the subgraph consisting of $(r t(t-1))$-stars whose existence is assured by the Structure Theorem 2.2.

Theorem 3.1. For $r \geqslant 1$ and $t \geqslant 2$ there exists an $n_{0}$ such that, for every $r t^{2}$-graph $H \in \mathcal{J}(r, t)$ with $n \geqslant n_{0}$ vertices and at least $\binom{[n / t\rfloor}{ r t}+1$ edges, $H_{S}$ contains an $(r t(t-1))$-star with at least $\hat{n}$ vertices in the body and at least $m \hat{n} / n-n^{r t-1}>m /(3 t)$ edges, where

$$
\begin{equation*}
\hat{n}=\hat{n}(n, m)=\min \left\{N:\binom{N-1}{r t-1} \geqslant \frac{r t m}{n}-r t n^{r t-2}\right\} \geqslant \frac{n}{t^{r t /(r t-1)}}-(3 r t)^{3 r t}>\frac{2 n}{5 t} . \tag{3.1}
\end{equation*}
$$

On the other hand, if $\binom{[n / t\rfloor}{ r t}-\frac{n^{r t-1}}{2 t^{r t-1}(r t-1)!} \leqslant m \leqslant\binom{[n / t\rfloor}{ r t}$, then each hypergraph $H \in \mathcal{J}(r, t)$ with $m$ edges and $\Delta(H) \leqslant m /(3 t)$ is a subgraph of a thick clique $\widetilde{K}_{n, t}^{r t^{2}}$; in particular, $\Delta(H) \leqslant\binom{[n / t\rfloor-1}{r t-1}$.

Proof. For given integers $t \geqslant 2$, and $r \geqslant 1$, choose $n_{0}$ so large that all inequalities below hold for $n \geqslant n_{0}$. Moreover, let $H \in \mathcal{J}(r, t)$, where the number of edges $m$ satisfies

$$
\begin{equation*}
m \geqslant\binom{\lfloor n / t\rfloor}{ r t}-\frac{n^{r t-1}}{2 t^{r t-1}(r t-1)!}>\frac{n^{r t}}{t^{r t}(r t)!}-\frac{2 r t^{2} n^{r t-1}}{t^{r t}(r t-1)!}>\left(\frac{n}{r t^{2}}\right)^{r t} \tag{3.2}
\end{equation*}
$$

By Theorem 2.2, $H \in \mathcal{F}\left(t, r t,\left(r t^{2}\right)^{r^{3} t^{6}}\right)$, so let us take partitions $V=V_{T} \cup V_{S} \cup V_{R}$ and $H=H_{T} \cup H_{S} \cup H_{R}$ exemplifying this. Then

$$
\begin{equation*}
|H|=\left|H_{T}\right|+\left|H_{S}\right|+\left|H_{R}\right| \leqslant\binom{\left|V_{T}\right| / t}{r t}+\binom{\left|V_{S}\right|}{r t}+\left|V_{T}\right|\left|V_{S}\right| n^{r t-3}+\left|V_{R}\right| C n^{r t-2}, \tag{3.3}
\end{equation*}
$$

where $C=\left(r t^{2}\right)^{r^{3} t^{6}}$. As a straightforward consequence of the above inequality we get the following claim.

Claim 3.2. If $V_{S}=\varnothing$ then $H_{R}=\varnothing$ and $m \leqslant\binom{\lfloor n / t\rfloor}{ r t}$.
Proof. Since $V_{S}=\varnothing$, the vertex set of $H$ is partitioned into sets $V_{T}$ and $V_{R}$, where $\left|V_{T}\right|$ is divisible by $t$ and $\left|V_{R}\right|=n-\left|V_{T}\right|$. Recall that $V_{T}$ consists of $t$-tuples that are inseparable in $H$. Therefore, if $H_{R} \neq \varnothing$ then $\left|V_{R}\right| \geqslant t$ and consequently, by (3.3),

$$
\begin{aligned}
m & =\left|H_{T}\right|+\left|H_{R}\right| \leqslant\binom{\left(n-\left|V_{R}\right|\right) / t}{r t}+0+\left|V_{R}\right| C n^{r t-2} \leqslant\binom{\lfloor n / t\rfloor-1}{r t}+2 t C n^{r t-2} \\
& <\binom{\lfloor n / t\rfloor}{ r t}-\frac{n^{r t-1}}{2 t^{r t-1}(r t-1)!},
\end{aligned}
$$

contrary to (3.2).
Thus we must have $H_{R}=\varnothing$ and, hence,

$$
m=\left|H_{T}\right| \leqslant\binom{\lfloor n / t\rfloor}{ r t} .
$$

It turns out that if $V_{S} \neq \varnothing$, then the largest degree must be large.
Claim 3.3. If $V_{S} \neq \varnothing$, then $H_{S}$ contains an $(r t(t-1))$-star with at least $\hat{n}$ vertices and at least $m \hat{n} / n-n^{r t-1}$ edges, where $\hat{n}$ is defined as in (3.1).

Proof. We start with bounding from below the average degree $\operatorname{ad}(G)$ of the $r t$-graph $G_{S}=\left(V_{S}, E_{S}\right)$ with the set of vertices $V_{S}$ and the set of edges $E_{S}=\left\{h \cap V_{S}: h \in H_{S}\right\}$. Using the upper bound on $\left|H_{R}\right|$ we get

$$
\begin{aligned}
\operatorname{ad}\left(G_{S}\right) & =\frac{r t\left|H_{S}\right|}{\left|V_{S}\right|}=\frac{r t\left(m-\left|H_{R}\right|-\left|H_{T}\right|\right)}{n-\left|V_{R}\right|-\left|V_{T}\right|} \\
& \geqslant \frac{r t m-r t\left|V_{R}\right| C n^{r t-2}-r t\binom{\left(V_{T} \mid / t\right.}{r t}}{n-\left|V_{R}\right|-\left|V_{T}\right|}-\frac{r t\left|V_{T}\right|\left|V_{S}\right| n^{r t-3}}{\left|V_{S}\right|} \\
& =\frac{r t m}{n}+\frac{\frac{r t m}{n}\left(\left|V_{R}\right|+\left|V_{T}\right|\right)-r t\left|V_{R}\right| C n^{r t-2}-r t\binom{\left|V_{T}\right| / t}{r t}}{n-\left|V_{R}\right|-\left|V_{T}\right|}-r t\left|V_{T}\right| n^{r t-3} .
\end{aligned}
$$

Here the numerator of the second fraction is, due to (3.2), at least

$$
r t\left|V_{R}\right| n^{r t-2}\left(\frac{n}{\left(r t^{2}\right)^{r t}}-C\right)+r t\left|V_{T}\right|\left(\frac{1}{n}\binom{\lfloor n / t\rfloor}{ r t}-\frac{n^{r t-2}}{2 t^{r t-1}(r t-1)!}-\frac{1}{\left|V_{T}\right|}\binom{\left|V_{T}\right| / t}{r t}\right)
$$

and because of $\left|V_{T}\right| \leqslant n-t$ and the largeness of $n$ this term is positive. For these reasons we have

$$
\operatorname{ad}\left(G_{S}\right) \geqslant \frac{r t m}{n}-r t n^{r t-2}>\binom{\hat{n}-2}{r t-1}
$$

Now, each vertex $v$ of degree at least $\operatorname{ad}\left(G_{S}\right)$ must be contained in a component with at least $\hat{n}$ vertices and, since $G_{S}$ must contain a component whose average degree is at least $\operatorname{ad}\left(G_{S}\right)$, each such component must have at least $\operatorname{ad}\left(G_{S}\right) \hat{n} /(r t)$ edges.

Finally, we can complete the proof of Theorem 3.1. If $m>\binom{[n / t\rfloor}{ r t}$, then, by Claim 3.2, we have $V_{S} \neq \varnothing$, and the first part of Theorem 3.1 follows directly from Claim 3.3. On the other hand, if $H \in \mathcal{J}(r, t)$ has $m$ edges, where

$$
\binom{\lfloor n / t\rfloor}{ r t}-\frac{n^{r t-1}}{2 t^{r t-1}(r t-1)!} \leqslant m \leqslant\binom{\lfloor n / t\rfloor}{ r t},
$$

and $\Delta(H) \leqslant m /(3 t)$, then Claim 3.3 implies that $V_{S}$ is empty and, by Claim 3.2, $H_{R}$ is empty as well. Thus, $H$ must be a subgraph of a thick $\left(r t^{2}, n, t\right)$-clique $\tilde{K}_{n, t}^{r t^{2}}$.

Once we know that dense graphs from $\mathcal{J}(r, t)$ with $m>\binom{[n / t\rfloor}{ r t}$ contain vertices of large degree one may ask about the structure of graphs which, for a given $m=m(n)$, minimise the maximum degree. A natural conjecture is that they can be expressed as a union of large disjoint $(r t(t-1))$-stars with, perhaps, some limited number of extra edges like those which intersect the centres of these stars in sets whose sizes are multiples of $t$.

In [9] such a result is proved for the family $\mathcal{J}(1,2)$. Namely, it is shown that from each $\{0,2,3\}$-intersecting 4 -graph with $m>\binom{[n / 2\rfloor}{ 2}$ edges that minimises the maximum degree one can remove at most 470 edges to get a 4 -graph which consists of at most 4 stars and, perhaps, some number of isolated vertices. The number 4 here is optimal, while 470 is clearly not and most likely it can be replaced either by 11 , if we want each of the stars to be a 2 -star, or just by 8 , if we are satisfied that each of the four stars is just a 1 -star (for details and discussions of this result see the final remarks in [9]).

The remaining part of this section is devoted to the proof of an analogous result for $\mathcal{J}(r, t)$ in the general case. As we will see shortly, a similar result hold whenever $r=1$, while for $r \geqslant 2$ a weaker yet quite satisfactory characterisation of the extremal graphs can be shown. Nevertheless, in order to state our theorem more precisely, let us introduce some notation, analogous to those used in [9].

Thus, let us define the minimum maximum-degree function of $\mathcal{J}(r, t)$ by setting

$$
f(\mathcal{J}(r, t) ; n, m)=\min \{\Delta(H): H=(V, E) \in \mathcal{J}(r, t),|V|=n, \text { and }|E|=m\}
$$

for all nonnegative integers $n$ and $m$. The corresponding collection of extremal hypergraphs is denoted by $\mathfrak{E}(\mathcal{J}(r, t) ; n, m)$.

Note that the function $f(\mathcal{J}(r, t) ; n, m)$ is always bounded from below by the average degree $r t^{2} m / n$; on the other hand, one can always find a thick graph from $\mathcal{J}(r, t)$ such that the degrees of all vertices, except at most $t-1$, are within distance one from each other. Hence, from Theorem 3.1 it follows that whenever $m \leqslant\binom{[n / t\rfloor}{ r t}$ and $n$ is large enough we have

$$
\left\lceil r t^{2} m / n\right\rceil \leqslant f(\mathcal{J}(r, t) ; n, m) \leqslant\left\lceil r t^{2} m / n\right\rceil+1
$$

i.e., in this range of $m$ the function $f(\mathcal{J}(r, t) ; n, m)$ is determined up to a constant term. Thus, it remains to study the value of $f(\mathcal{J}(r, t) ; n, m)$ and the structure of the extremal graphs from $\mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ for $m>\binom{[n / t\rfloor}{ r t}$. For this we require one more concept.

Let us say that an $(r t(t-1))$-star $S$ with some number $N$ of vertices in its body is heavy if its minimum vertex degree is at least $2 r^{2} t^{3} N^{r t-2}$. The standard process of repeatedly removing vertices of low degree allows us to find in any $(r t(t-1))$-star $S$ a maximal heavy substar $S^{\prime} \subseteq S$ (which is possibly empty). This star $S^{\prime}$ is called the core of $S$, its set of edges is denoted by $\mathrm{cr}_{e}(S)$, and by $\mathrm{cr}_{v}(S)$ we mean the set of vertices forming its body.

The first two parts of the following fact list standard properties of the process by means of which the core is constructed, while its third part explains that cores have a property reminiscent of condition (iii) in Definition 2.1.

Fact 3.4. Let $r \geqslant 1$ and $t \geqslant 2$.
(a) There are integers $n_{0}$ and $c_{0}$ such that if a hypergraph $H \in \mathcal{J}(r, t)$ has $n \geqslant n_{0}$ vertices and $m>\binom{\lfloor n / t\rfloor}{ r t}$ edges, then there is a heavy star $S \subseteq H$ with

$$
|S| \geqslant \frac{m \hat{n}(n, m)}{n}-3 r^{2} t^{3} n^{r t-1} \quad \text { and } \quad\left|\operatorname{cr}_{v}(S)\right| \geqslant \hat{n}(n, m)-c_{0}>\frac{2 n}{5 t}
$$

where $\hat{n}(n, m)$ is the number introduced in (3.1).
(b) For every positive integer a there exists an integer $b$ such that every $(r t(t-1))$-star $S$ with $N$ vertices in its body and $|S| \geqslant\binom{ N-a}{r t}$ satisfies $\left|\operatorname{cr}_{v}(S)\right| \geqslant N-b$.
(c) If $H \in \mathcal{J}(r, t)$ and $S \subseteq H$ is a heavy $(r t(t-1))$-star, then every edge of $H$ intersecting the body of $S$ needs to contain the centre of $S$.

Proof. For the proof of part $(a)$ we take $n_{0}$ to be at least as large as the number provided by Theorem 3.1. We then know that for any $H \in \mathcal{J}(r, t)$ as above there exists a star $\hat{S} \subseteq H$ with $|\hat{S}| \geqslant \frac{m \hat{n}(n, m)}{n}-n^{r t-1}$. Throughout the process yielding $S=\operatorname{cr}_{e}(\hat{S})$ we remove at most $2 r^{2} t^{3} \sum_{i=1}^{n} i^{r t-2}<2 r^{2} t^{3} n^{r t-1}$ edges and thus $S$ has at least the size we claimed. To obtain the desired lower bound on $\left|\operatorname{cr}_{v}(S)\right|$ we use that the definition of $\hat{n}=\hat{n}(n, m)$ implies $\frac{(\hat{n}-1) m}{n}>\binom{\hat{n}-1}{r t}$, whence

$$
\binom{\left|\operatorname{cr}_{v}(S)\right|}{r t} \geqslant|S|>\binom{\hat{n}-1}{r t}-3 r^{2} t^{3} n^{r t-1}>\binom{\hat{n}-c_{0}}{r t}
$$

holds for sufficiently large $c_{0}$ and $n_{0}$.
For the verification of part ( $b$ ) we may take two large constants $b, b^{\prime}$ with $b \gg b^{\prime} \gg a$. There is nothing to prove in case $N \leqslant b$, so let us assume $N>b$ from now on. As above we have $\left|S \backslash \operatorname{cr}_{e}(S)\right| \leqslant 2 r^{2} t^{3} N^{r t-1}$ and, hence,

$$
\left|\operatorname{cr}_{e}(S)\right| \geqslant\binom{ N-a}{r t}-2 r^{2} t^{3} N^{r t-1} \geqslant\binom{ N-b^{\prime}}{r t}
$$

which is only possible if $\left|\operatorname{cr}_{v}(S)\right| \geqslant N-b^{\prime}$.
Finally, let $H$ and $S$ be as in (c), write $B$ for the set of vertices forming the body of $S$, set $N=|B|$, and consider any $e \in H$ intersecting $B$ in some vertex $v$. The minimum degree condition satisfied by $S$ yields $|S| \geqslant 2 r t^{2} N^{r t-1}$. As at most $r t^{2} N^{r t-1}$ edges of $S$ can intersect $e \cap B$, there is an edge $f \in S$ disjoint to $e \cap B$ and, consequently, we have $|e \cap s| \in I$, where $s$ denotes the centre of $S$. Similarly, for every $w \in e \cap B$ distinct from $v$ there are at most $N^{r t-2}$ edges of $S$ containing both $v$ and $w$, and thus there is an edge $f^{\prime} \in S$ with $(e \cap B) \cap f^{\prime}=\{v\}$, which proves $|e \cap s|+1 \in I$. But the only possibility for the consecutive integers $|e \cap s|$ and $|e \cap s|+1$ to belong to $I$ is that $s \subseteq e$, as desired.

The following result describes the structure of $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ quite precisely.
Theorem 3.5. For all integers $t \geqslant 2$ and $r \geqslant 1$, there exist an integer $n_{*}=n_{*}(r, t)$ and constants $c_{i}=c_{i}(r, t), i=1,2,3$, such that from every rt ${ }^{2}$-graph $H=(V, E)$ in $\mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ with $n \geqslant n_{*}$ vertices and $m>\binom{[n / t\rfloor}{ r t}$ edges one can remove at most $c_{1}$ edges to get a graph which consists of $\ell \leqslant 7 t^{2}$ many $(r t(t-1))$-stars $S^{1}, S^{2}, \ldots, S^{\ell}$ and, perhaps, some number of isolated vertices.

Moreover, we can also assume that
(i) $\left|\operatorname{cr}_{v}\left(S^{i}\right)\right| \geqslant n /\left(7 t^{2}\right)$, for $i=1,2, \ldots, \ell-1$;
(ii) $\left|V \backslash \bigcup_{i=1}^{\ell} \mathrm{cr}_{v}\left(S^{i}\right)\right| \leqslant c_{2}$;
(iii) the centres of the stars $S^{1}, \ldots, S^{\ell}$ are pairwise disjoint.

In particular, we can delete from $H$ at most $c_{3} n^{(r-1) t}$ edges to get a union of at most $\ell$ vertex disjoint $(r t(t-1))$-stars.

Proof. Let us assume that $n$ is sufficiently large, $m>\binom{[n / t\rfloor}{ r t}$, and that $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$. The idea for constructing the first $\ell-1$ of the desired stars is to apply Fact 3.4 (a) iteratively, pulling these stars out of $H$ one by one. This process comes to an end when we cannot guarantee anymore to find a star with a sufficiently large core in the remaining part of $H$.

More precisely, we start by setting $H^{1}=H$. Due to Fact $3.4(a)$ there exists a maximal star $S^{1} \subseteq H^{1}$ with $\left|\operatorname{cr}_{v}\left(S^{1}\right)\right| \geqslant 2 n /(5 t)$ and $\left|\operatorname{cr}_{e}\left(S^{1}\right)\right| \geqslant m \hat{n}(n, m) / n-3 r^{2} t^{3} n^{r t-1}$. By $H^{2}$ we denote the hypergraph arising from $H^{1}$ by the deletion of all vertices in $\mathrm{cr}_{v}\left(S^{1}\right)$ and all edges in $S^{1}$. If for some integer $i \geqslant 2$ we have just chosen a star $S^{i-1} \subseteq H^{i-1}$ and
constructed a hypergraph $H^{i} \subseteq H^{i-1}$ with $n_{i}$ vertices and $m_{i}$ edges, we check whether the conditions
(a) $n_{i} \geqslant \hat{n}(n, m)$,
(b) and $m_{i}>\binom{\left\lfloor n_{i} / t\right\rfloor}{ r t}$
are satisfied. If at least one of them fails we set $\ell=i$ and terminate the procedure, the last constructed objects being $S^{\ell-1}$ and $H^{\ell}$. On the other hand, if both conditions hold, the assumptions of Fact $3.4(a)$ are satisfied by $H^{i}$. Thus we find a maximal star $S^{i} \subseteq H^{i}$ with $\left|\operatorname{cr}_{v}\left(S^{i}\right)\right| \geqslant 2 n_{i} /(5 t)$ and $\left|\operatorname{cr}_{e}\left(S^{i}\right)\right| \geqslant m_{i} \hat{n}\left(n_{i}, m_{i}\right) / n_{i}-3 r^{2} t^{3} n_{i}^{r t-1}$. Moreover, we let $H^{i+1}$ denote the hypergraph with vertex set $V\left(H^{i}\right) \backslash \mathrm{cr}_{v}\left(S^{i}\right)$ and edge set $H^{i} \backslash S^{i}$.

Notice that for each $i \in[\ell-1]$ property $(a)$ of the above process entails

$$
\left|\operatorname{cr}_{v}\left(S^{i}\right)\right| \geqslant \frac{2 n_{i}}{5 t} \geqslant \frac{2 \hat{n}(n, m)}{5 t} \geqslant \frac{4 n}{25 t^{2}}>\frac{n}{7 t^{2}},
$$

meaning that condition $(i)$ of the theorem holds. Besides, since $\mathrm{cr}_{v}\left(S^{1}\right), \ldots, \mathrm{cr}_{v}\left(S^{\ell-1}\right)$ are mutually disjoint, it also follows that $\ell \leqslant 7 t^{2}$. Denote the centre of $S^{i}$ by $s_{i}$ for $i \in[\ell-1]$.

Now we investigate in detail the structure of the last hypergraph $H^{\ell}$. The following result is crucial for our argument. It shows, in particular, that our statement about $c_{1}$ holds.

Claim 3.6. There are constants $c_{1}=c_{1}(r, t)$ and $c_{4}=c_{4}(r, t)$ such that after removing at most $c_{1}$ edges from $H^{\ell}$ this hypergraph becomes the union of an $(r t(t-1))$-star with at least $\binom{n_{\ell}-c_{4}}{r t}$ edges and, perhaps, some number of isolated vertices.

Proof. Suppose that the assertion does not hold. Our aim is to get a contradiction with the assumption that $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ by constructing a graph $H^{\prime} \in \mathcal{J}(r, t)$ having the same number of vertices and edges as $H$ but a smaller maximum degree. Choose some absolute constants $c, c^{\prime}, c^{\prime \prime}, c_{1}$, and $c_{4}$ depending only on $r$ and $t$, sufficiently large so that all arguments below will work, and obeying the hierarchy

$$
c_{1} \gg c_{4} \gg c^{\prime \prime} \gg c^{\prime} \gg c .
$$

First case: $n_{\ell}<\hat{n}(n, m)$
Notice that we may assume $n_{\ell} \geqslant c_{4}$, since otherwise an appropriate choice of $c_{1}$ would show that the claim holds with an empty star. Moreover, if $m_{\ell}>\binom{n_{\ell}-c^{\prime \prime}}{r t}$ the desired conclusion can be drawn from Proposition 2.3 (and Theorem 2.2). So we may suppose $m_{\ell} \leqslant\binom{ n_{\ell}-c^{\prime \prime}}{r t}$ from now on.

The hypergraph $H^{\prime}$ will have three kinds of edges. First, there will be stars $\hat{S}^{1}, \ldots, \hat{S}^{\ell-1}$, where each $\hat{S}^{i}$ is obtained from $\mathrm{cr}_{e}\left(S^{i}\right)$ by the omission of a single edge.

Second, there will be edges serving as "substitutes" for the edges in $F^{i}=S^{i} \backslash \mathrm{cr}_{e}\left(S^{i}\right)$ for $i \in[\ell-1]$. The reason for this substitution is that it "cleans up some space" so that in the end $H^{\prime} \in \mathcal{J}(r, t)$ will be true. Let us recall that $\left|F^{i}\right| \leqslant O\left(n^{r t-1}\right)$ follows from the
construction of cores. Hence, there are disjoint subsets $U^{1}, \ldots, U^{\ell-1}$ of the vertex set of $H^{\ell}$ with $\left|U^{i}\right|=\left\lceil\left|F^{i}\right| /\left(\begin{array}{c}\left.\left\lvert\, \begin{array}{c}c c_{v}\left(S^{i}\right) \mid \\ r t-1\end{array}\right.\right) \\ \hline\end{array} \leqslant c\right.\right.$ for $i \in[\ell-1]$. Now instead of $F^{i}$ we put the same number of edges of the type $\{v\} \cup s_{i} \cup f$ into $H^{\prime}$, where $v \in U^{i}$, and $f$ is a subset of the body $\operatorname{cr}_{v}\left(S^{i}\right)$ with $r t-1$ elements.

Third, we include a star with $m_{\ell}+(\ell-1)$ edges into $H^{\prime}$ that uses only vertices of $H^{\ell}$ that are not occupied by the sets $s_{i}$ and $U^{i}$ for $i \in[\ell-1]$. There is enough space for such a star, as at most $\ell\left(r t^{2}+c\right) \leqslant c^{\prime}$ vertices are occupied, $m_{\ell} \leqslant\binom{ n_{\ell}-c^{\prime \prime}}{r t}$, and $n_{\ell}$ is sufficiently large.

It remains to check that we have indeed $\Delta\left(H^{\prime}\right)<\Delta(H)$. The only vertices of $H^{\prime}$ that might be problematic are in the centre of the new star that has just been created.

However, working carefully with the estimates provided by Theorem 3.1 and exploiting that we are in the first case, one checks easily that

$$
\begin{aligned}
\Delta(H) & \geqslant\left|S^{1}\right| \geqslant \frac{m \hat{n}(n, m)}{n}-3 r^{2} t^{3} n^{r t-1} \geqslant\binom{\hat{n}(n, m)-c^{\prime}}{r t} \\
& >\binom{n_{\ell}-c^{\prime \prime}}{r t}+(\ell-1) \geqslant m_{\ell}+(\ell-1) .
\end{aligned}
$$

Thus $H^{\prime}$ contradicts indeed our assumption that $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$.

## Second case: $n_{\ell} \geqslant \hat{n}(n, m)$

This means that the iterative procedure that led us to the stars $S^{1}, \ldots, S^{\ell-1}$ stopped owing to the failure of condition (b), i.e., that

$$
\begin{equation*}
m_{\ell} \leqslant\binom{\left\lfloor n_{\ell} / t\right\rfloor}{ r t} \tag{3.4}
\end{equation*}
$$

One can deal with this case in a very similar way as with the previous one but instead of replacing the edges of $H^{\ell}$ by one large star we replace them by $t$ smaller and mutually disjoint stars with roughly $n_{\ell} / t$ vertices and $m_{\ell} / t$ edges.

The inequality $\frac{m_{\ell}}{t} \leqslant\binom{ n_{\ell} / t-c^{\prime}}{r t}$, which is a direct consequence of (3.4), shows that there is indeed enough space for such stars.

Finally, it remains to check that the hypergraph $H^{\prime}$ generated as above really has a smaller maximum degree than $H$.

If $n_{\ell} \leqslant \frac{t n}{t+1}$ this follows from

$$
\begin{aligned}
\left\lceil\frac{m_{\ell}+(\ell-1)}{t}\right\rceil & \leqslant \frac{1}{t}\binom{n /(t+1)}{r t}+O(1) \leqslant \frac{1}{t}\left(\frac{t}{t+1}\right)^{r t}\binom{n / t}{r t}+O\left(n^{r t-1}\right) \\
& \leqslant \frac{4 m}{9 t}+O\left(n^{r t-1}\right)<\frac{m \hat{n}(n, m)}{n}-3 r^{2} t^{3} n^{r t-1} \leqslant\left|S^{1}\right|
\end{aligned}
$$

The case $\frac{t n}{t+1}<n_{\ell}$, however, is impossible, because due to $n_{\ell} \leqslant n-\operatorname{cr}_{v}\left(S^{1}\right)<\left(1-\frac{2}{5 t}\right) n$ it would entail

$$
\begin{aligned}
m & =\sum_{i=1}^{\ell-1}\left(\left|\mathrm{cr}_{e}\left(S^{i}\right)\right|+\left|F^{i}\right|\right)+m_{\ell} \leqslant \sum_{i=1}^{\ell-1}\binom{\left|\mathrm{cr}_{v}\left(S^{i}\right)\right|}{r t}+\binom{\left\lfloor n_{\ell} / t\right\rfloor}{ r t}+O\left(n^{r t-1}\right) \\
& \leqslant\binom{ n-n_{\ell}}{r t}+\binom{n_{\ell} / t}{r t}+O\left(n^{r t-1}\right)<\binom{\lfloor n / t\rfloor}{ r t}
\end{aligned}
$$

where for the last estimate we used that $(1-x)^{r t}+(x / t)^{r t}<(1 / t)^{r t}$ holds for all real $x \in\left[\frac{t}{t+1}, 1-\frac{2}{5 t}\right]$. But the above estimate contradicts our initial hypothesis about $m$.

Notice that taken together Fact 3.4 (b) and Claim 3.6 show that part (ii) of the theorem holds for a sufficiently large choice of $c_{2}$ and an approriate star $S^{\ell} \subseteq H^{\ell}$. Let $s_{\ell}$ be the centre of $S^{\ell}$.

In order to verify part (iii) let us observe first that each star $S^{i}$ consists of at most

$$
\binom{\left|\mathrm{cr}_{v}\left(S_{i}\right)\right|}{r t}+O\left(n^{r t-1}\right)
$$

edges. It may be helpful to recall each of the $\operatorname{sets}^{\mathrm{cr}_{v}}\left(S^{1}\right), \ldots, \mathrm{cr}_{v}\left(S^{\ell-1}\right)$ has size $\Omega(n)$. We do not know the same about the last star, but at least we may suppose that $\mathrm{cr}_{v}\left(S^{\ell}\right)$ is sufficiently large for otherwise we may ignore this star and proceed. Now let us assume that there are two stars, $S^{i}$ and $S^{j}, 1 \leqslant i<j \leqslant \ell$, which do not have disjoint centres. Then we construct a new hypergraph $H^{\prime} \in \mathcal{J}(r, t)$ out of $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ in the following way. We delete all the edges of the stars $S^{i}$ and $S^{j}$, say of $m_{i}^{\prime}$ and $m_{j}^{\prime}$ edges respectively, and on the vertex set $\mathrm{cr}_{v}\left(S^{i}\right) \cup \mathrm{cr}_{v}\left(S^{j}\right)$ we create an $(r t(t-1))$-star $S^{i j}$ which has $m_{i}^{\prime}+m_{j}^{\prime}-1<\Delta(H)$ edges and which uses as few vertices as possible. Due to $\left|\operatorname{cr}_{v}\left(S^{i}\right)\right| \geqslant \Omega(n)$ and $\left|\operatorname{cr}_{v}\left(S^{j}\right)\right| \geqslant O(1)$ we have

$$
\binom{\left|\operatorname{cr}_{v}\left(S^{i}\right)\right|}{r t}+\binom{\left|\operatorname{cr}_{v}\left(S^{j}\right)\right|}{r t}+O\left(n^{r t-1}\right)<\binom{\left|\operatorname{cr}_{v}\left(S^{i}\right)\right|+\left|\operatorname{cr}_{v}\left(S^{j}\right)\right|-8 r t^{4}}{r t}
$$

and hence $S^{i j}$ uses fewer than $\left|\mathrm{cr}_{v}\left(S^{i}\right)\right|+\left|\mathrm{cr}_{v}\left(S^{j}\right)\right|-7 r t^{4}$ vertices. Now we remove one edge from each of the existing stars $S^{t}$, where $t=1,2, \ldots, \ell$, and $t \neq i, j$, and add $\ell-1 \leqslant 7 t^{2}$ disjoint edges to the hypergraph. Such a hypergraph $H^{\prime} \in \mathcal{J}(r, t)$ has a smaller maximum degree than $H$, which contradicts the fact that $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$. Thus, the centres of the stars $S^{i}, i=1,2, \ldots, \ell$, are pairwise disjoint, as claimed in clause (iii) of the theorem. For later use we record that using Fact $3.4(c)$ one can show that $s_{i} \cap \operatorname{cr}_{v}\left(S^{j}\right)=\varnothing$ holds whenever $i, j \in[\ell]$.

Finally, we need to argue that we can delete from $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ at most $c_{3} n^{(r-1) t}$ additional edges to make $S^{1}, \ldots, S^{\ell}$ vertex disjoint. We contend that it suffices for this purpose to delete all edges intersecting the set $W=\bigcup_{i=1}^{\ell} \mathrm{cr}_{v}\left(S^{i}\right)$ in at most $(r-1) t$ vertices. Notice that due to (ii) the resulting hypergraph $\widetilde{H}$ differs in at most $c_{2}^{r t^{2}} n^{(r-1) t}$ edges from $H$. Owing to of Fact $3.4(c)$ and (iii) there is for every edge $e \in \widetilde{H}$ a unique $i \in[\ell]$
with $e \in S^{i}$; for this $i$ we have $\left|e \cap \operatorname{cr}_{v}\left(S^{i}\right)\right| \geqslant(r-1) t+1$ and, hence, $e$ has at most $t-1$ vertices outside $s_{i} \cup \mathrm{cr}_{v}\left(S^{i}\right)$. These vertices cannot belong to the centre $s_{j}$ of another star, for then $e$ would have a forbidden intersection with every edge in $\mathrm{cr}_{e}\left(S^{j}\right)$. Now assume that one of the vertices in $e \backslash\left(s_{i} \cup \mathrm{cr}_{v}\left(S^{i}\right)\right)$, say $v$, would belong in $\widetilde{H}$ to another star as well. This means that there are an index $j \neq i$ and an edge $f \in S^{j} \cap \widetilde{H}$ with $v \in e \cap f$. Due to $f \in \tilde{H}$ at most $t-1$ vertices of $f$ are outside $s_{j} \cup \operatorname{cr}_{v}\left(S^{j}\right)$ and thus we have $1 \leqslant|e \cap f| \leqslant t-1$, which is absurd. Thus $\widetilde{H}$ is indeed a union of $\ell$ vertex disjoint stars.

Let us comment briefly on the structure of $H \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ described in Theorem 3.5. Once we know Theorem 3.5 the estimate for the number of stars $\ell$ can be easily improved to the optimal $\ell \leqslant\left\lceil t^{r t /(r t-1)}\right\rceil$ (see [9], where a similar argument is used for $r=1, t=2$ ). However, we cannot significantly decrease the number of edges needed to make the stars vertex disjoint. To see this, let us consider the $r t^{2}$-graph $\tilde{H} \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ with vertex set $V=U_{1} \cup U_{2} \cup C_{1} \cup C_{2} \cup T$, where $\left|U_{1}\right|=\left|U_{2}\right|=u,\left|C_{1}\right|=\left|C_{2}\right|=r t(t-1),|T|=t$, whose set of edges consists of:

- $\binom{u}{r t}$ subsets which are unions of $C_{1}$ and some $r t$-element subset of $U_{1}$,
- $\binom{u}{r t}$ subsets which are unions of $C_{2}$ and some $r t$-element subset of $U_{2}$,
- $\binom{u}{(r-1) t}$ subsets which are unions $T, C_{1}$ and some $(r-1) t$-element subset of $U_{1}$,
- $\binom{u}{(r-1) t}$ subsets which are unions $T, C_{2}$ and some $(r-1) t$-element subset of $U_{2}$,
- and a thick clique on $T \cup C_{1} \cup C_{2}$ whose teams are $T$ and partitions of $C_{1}, C_{2}$.

It is easy to see that $\tilde{H} \in \mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ with $n=2 u+2 r t(t-1)+t$ and the appropriate $m$. On the other hand $\tilde{H}$ consists of two stars with centres $C_{1}$ and $C_{2}$ and to make them vertex disjoint one must delete at least $\Omega\left(n^{(r-1) t}\right)$ edges.

Finally, let us notice that from Theorem 3.5 it follows that almost all the edges of dense extremal graphs from $\mathfrak{E}(\mathcal{J}(r, t) ; n, m)$ are contained in at most $\ell$ stars among which $\ell-1$ are roughly equal and only one can be a bit smaller than the others. Having this is mind one can easily compute the scaled extremal function

$$
f_{r t}(x)=\lim _{n \rightarrow \infty} \frac{f\left(\mathcal{J}(r, t) ; n, x\binom{n-r t(t-1)}{r t}\right.}{\binom{n-r t(t-1)}{r t}} .
$$

From Theorems 3.1 and 3.5 we know that the function is well defined for $x \in[0,1] \backslash\left\{t^{-t r}\right\}$. Furthermore, besides the point $x=t^{-t r}$ where it jumps from 0 to some value which is at least $t^{-t^{2} r^{2} /(r t-1)}$, it is continuous everywhere. It is also smooth everywhere except the points $x=j^{1-t r}$ for $j=2,3, \ldots,\left\lceil t^{t r /(r t-1)}\right\rceil-1$ (see [9] where details are worked out for the case $r=1, t=2)$.

## $\S 4$. Tools

The purpose of this section is to gather three statements that will turn out to be useful in the proof of the Structure Theorem. While the first two of them are fairly well known, the third one (see Lemma 4.4 below) could very well be new.
4.1. Divisible set systems. Given a natural number $t \geqslant 2$ we shall say that a set $\operatorname{system}(V, \mathcal{E})$ is $t$-divisible if for any two distinct edges $e, e^{\prime} \in \mathcal{E}$ the size $\left|e \cap e^{\prime}\right|$ of their intersection is a multiple of $t$. The problem to study upper bounds on the size of such set systems with additional assumptions on the behaviour of the sizes of the edges modulo $t$ was first studied, in the particular case $t=2$, by Berlekamp [1], who realised that ideas from linear algebra can be applied in such contexts. At a later occasion we will need a variant of one of his result that was first observed, in more general form, by Babai and Frankl (see [2, Theorem 1]).

Lemma 4.1. Let $(V, \mathcal{E})$ be a $t$-divisible set system for some natural number $t \geqslant 2$. If $|e| \equiv 1(\bmod t)$ holds for all $e \in \mathcal{E}$, then $|\mathcal{E}| \leqslant|V|$.

Proof. Let $p$ denote a prime factor of $t$. We identify the members of $\mathcal{E}$ with vectors from the $|V|$-dimensional vector space $\mathbb{F}_{p}^{V}$ via characteristic functions and contend that the stronger conclusion that $\mathcal{E}$ is linearly independent holds. To see this, one looks at a hypothetical linear dependency $\alpha_{1} e_{1}+\cdots+\alpha_{n} e_{n}=0$ with distinct $e_{i} \in \mathcal{E}$ and certain numbers $\alpha_{i} \in \mathbb{F}_{p} \backslash\{0\}$, where $n \geqslant 1$. Taking the standard scalar product with $e_{1}$ we obtain

$$
0=\left\langle e_{1}, 0\right\rangle=\left\langle e_{1}, \alpha_{1} e_{1}+\cdots+\alpha_{n} e_{n}\right\rangle=\sum_{i=1}^{n} \alpha_{i}\left\langle e_{1}, e_{i}\right\rangle=\alpha_{1}
$$

which is absurd.
4.2. Delta systems. A set system $\mathcal{F}$ is called a sunflower (or a $\Delta$-system) if there exists a (possibly empty) set $S$ of vertices such that the intersection of any two distinct edges of $\mathcal{F}$ is equal to $S$. This constant intersection $S$ is called the kernel of the sunflower.

In 1960 Erdős and Rado [5] proved their "sunflower lemma" saying that any sufficiently large collection of finite sets of bounded size contains big sunflowers.

Theorem 4.2. For all positive integers a and b, any collection of more than b!a ${ }^{b+1}$ sets of cardinality at most $b$ contains a $\Delta$-system with more than a elements.

It should perhaps be pointed out that $b!a^{b+1}$ is not the least number $f(a, b)$ for which this statement is true. In fact, Erdős and Rado themselves stated a marginally better but less clean upper bound on this number in [5, Theorem III], but despite the considerable attention that the problem to improve our understanding of the growth behaviour of this function has received (see e.g., [8]) the progress on this problem has been rather slow. For
the purposes of the present article, however, even knowing the exact value of $f(a, b)$ would be quite immaterial.
4.3. Divisible pairs of set systems. The next result makes use of the following concept.

Definition 4.3. Let $\mathcal{F}$ and $\mathcal{G}$ denote two set systems with the same vertex set $V$ and let $q$ be a positive integer. We say that the pair $(\mathcal{F}, \mathcal{G})$ is $q$-divisible if for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the size $|f \cap g|$ of their intersection is divisible by $q$.

The result that follows will often help us to analyse the structure of divisible pairs.
Lemma 4.4 (Decomposition lemma). Suppose that $k$, $q$ are positive integers and that $\mathcal{F}, \mathcal{G}$ are two set systems with the same vertex set $V$ such that

- all members of $\mathcal{F} \cup \mathcal{G}$ have size at most $k$,
- and the pair $(\mathcal{F}, \mathcal{F} \cup \mathcal{G})$ is $q$-divisible.

Then there is a set system $\mathcal{H}$ on $V$ with the following properties:
(i) $\Delta(\mathcal{H}) \leqslant k^{2 k}$;
(ii) $\mathcal{H}$ is an antichain (that is, $x \ddagger y$ holds for all distinct $x, y \in \mathcal{H}$ );
(iii) Every edge of $\mathcal{F}$ is a disjoint union of edges from $\mathcal{H}$;
(iv) The pair $(\mathcal{H}, \mathcal{H} \cup \mathcal{G})$ is $q$-divisible.

Proof. Without loss of generality we may assume that given $k, q$, and $\mathcal{G}$, the set system $\mathcal{F}$ is maximal with respect to inclusion, i.e., that for every set system $\mathcal{F}^{*} \supsetneqq \mathcal{F}$ with $|f| \leqslant k$ for all $f \in \mathcal{F}^{*}$ the pair $\left(\mathcal{F}^{*}, \mathcal{F}^{*} \cup \mathcal{G}\right)$ fails to be $q$-divisible.

Now we define $\mathcal{H}$ to be the collection of those members of $\mathcal{F} \backslash\{\varnothing\}$ that are minimal with respect to inclusion, i.e., we set

$$
\mathcal{H}=\{h \in \mathcal{F} \backslash\{\varnothing\}: \text { if } f \in \mathcal{F} \text { and } f \neq \varnothing, h, \text { then } f \ddagger h\} .
$$

This choice of $\mathcal{H}$ makes part (ii) obvious and (iv) follows directly from $\mathcal{H} \subseteq \mathcal{F}$.
Assuming that (iii) would be false let $f \in \mathcal{F}$ be chosen with $|f|$ minimum such that $f$ is not expressible as a disjoint union of appropriate edges from $\mathcal{H}$. Since the empty set is equal to the empty union, we have $f \neq \varnothing$. Moreover, $f$ cannot belong to $\mathcal{H}$ and, consequently, there exists some $h \in \mathcal{H}$ with $h \subseteq f$ and $0<|h|<|f|$. Notice that for every $g \in \mathcal{F} \cup \mathcal{G}$ the number $|(f \backslash h) \cap g|=|f \cap g|-|h \cap g|$ is divisible by $q$. Owing to the maximality of $\mathcal{F}$ it follows that $(f \backslash h) \in \mathcal{F}$. But in view of our minimal choice of $f$ this means that $f \backslash h$ is a disjoint union of edges from $\mathcal{H}$ and, hence, so is $f$. Hence, $\mathcal{H}$ must satisfy (iii).

Now it remains to show that $\mathcal{H}$ has bounded maximum degree. Assume for the sake of contradiction, that there exists a vertex $v \in V$ contained in more than $k^{2 k}$ edges of $\mathcal{H}$ and look at the set system

$$
\mathcal{F}_{v}=\{h \in \mathcal{H}: v \in h\} .
$$

Then $\left|\mathcal{F}_{v}\right|>k^{2 k} \geqslant k!k^{k+1}$ and Theorem 4.2 reveals that $\mathcal{F}_{v}$ contains a $\Delta$-system $\mathcal{F}^{*}$ with more than $k$ elements. Denote the kernel of $\mathcal{F}^{*}$ by $e$ and observe that, since $\left|\mathcal{F}^{*}\right|>k$, for all edges $g \in \mathcal{F} \cup \mathcal{G}$, the size of the intersection $|e \cap g|$ is divisible by $q$. Together with the maximality of $\mathcal{F}$ this implies $e \in \mathcal{F}$. Due to $\left|\mathcal{F}^{*}\right| \geqslant 2$ there must exist some $h \in \mathcal{F}^{*} \subseteq \mathcal{H}$ properly containing $e$ and by our definition of $\mathcal{H}$ this is only possible if $e=\varnothing$. But, on the other hand, we certainly have $v \in e$. This contradiction concludes the proof $(i)$ and, hence, the proof of the decomposition lemma.

## §5. Proof of the Structure Theorem

This entire section is dedicated to the proof of Theorem 2.2. Let integers $r \geqslant 1$ and $t \geqslant 2$ as well as an $r t^{2}$-uniform hypergraph $H=(V, E)$ with $|V|=n$ be given such that the size of the intersection of any two edges of $H$ belongs to the set

$$
I=\{s: t \mid s \text { or } s \geqslant r t(t-1)\},
$$

which means $H \in \mathcal{J}(r, t)$. We will show that $H \in \mathcal{F}\left(t, r t,\left(r t^{2}\right)^{r^{3} t^{6}}\right)$.
Let us start by colouring all those subsets $f \subseteq V$ with $|f| \leqslant r t(t-1)+1$ red that are kernels of sunflowers consisting of at least $r t^{2}$ edges of $H$. Recall that the latter condition means that there are to exist at least $r t^{2}$ disjoint sets $f_{1}, \ldots, f_{r t^{2}} \subseteq V$ of size $r t^{2}-|f|$ such that $f \cup f_{i} \in H$ holds for every $i \in\left[r t^{2}\right]$. We denote the set system on $V$ whose edges are the red sets by $H_{\text {red }}$. By $H_{\text {red }}^{*}$ we mean the $(r t(t-1)+1)$-uniform hypergraph on $V$ whose edges are the red $(r t(t-1)+1)$-sets and finally we put $\hat{H}_{\text {red }}=H_{\text {red }} \backslash H_{\text {red }}^{*}$.

Observe that

$$
\begin{equation*}
\text { for any } f, f^{\prime} \in H \cup H_{\text {red }} \text { we have }\left|f \cap f^{\prime}\right| \in I \text {. } \tag{5.1}
\end{equation*}
$$

This is because we can first extend $f$ to an edge $e$ of $H$ with $e \cap f^{\prime}=f \cap f^{\prime}$ and proceeding similarly with $f^{\prime}$ we get an edge $e^{\prime} \in H$ with $e \cap e^{\prime}=f \cap f^{\prime}$, so that $\left|f \cap f^{\prime}\right|=\left|e \cap e^{\prime}\right| \in I$ follows from the assumption that $H$ be $I$-intersecting.

As a consequence of this observation we learn that for any distinct $f, f^{\prime} \in H_{\text {red }}^{*}$ the number $\left|f \cap f^{\prime}\right|$ is divisible by $t$ and in view of Lemma 4.1 it follows that

$$
\begin{equation*}
\left|H_{\mathrm{red}}^{*}\right| \leqslant n . \tag{5.2}
\end{equation*}
$$

Moreover, (5.1) reveals that the pair $\left(\hat{H}_{\text {red }}, \hat{H}_{\text {red }} \cup H_{\text {red }}^{*} \cup H\right)$ is $t$-divisible, which allows us to apply the decomposition lemma (Lemma 4.4) to $r t^{2}, t, \hat{H}_{\text {red }}$, and $H_{\text {red }}^{*} \cup H$ here in place of $k, q, \mathcal{F}$, and $\mathcal{G}$ there. We thus infer the existence of a set system $G$ on $V$ with the following properties:
(a) $\Delta(G) \leqslant\left(r t^{2}\right)^{2 r t^{2}}$;
(b) $G$ is an antichain;
(c) Every edge of $\widehat{H}_{\text {red }}$ is a disjoint union of edges from $G$;
(d) The pair $\left(G, G \cup H_{\text {red }}^{*} \cup H\right)$ is $t$-divisible.

We imagine that the edges of $G$ have been coloured green. The green sets of cardinality $t$ will be referred to as teams. Notice that due to condition $(d)$ the teams are inseparable in $H$ and, moreover, by $(b)$ and $(d)$ each team is disjoint to any other green set.

Now we are ready to decompose $V$ and $H$ in the envisioned way. We start by defining $V_{T}$ to be the union of all teams and setting $H_{T}=H\left[V_{T}\right]$, which guarantees part ( $i$ ) of Definition 2.1.

Preparing the definition of $H_{S}$ we colour a set consisting of $r t(t-1)$ vertices purple if it is the kernel of a Delta system in $H_{\text {red }}^{*}$ of size at least $r t^{2}$. Imitating the proof of (5.1) one checks easily that

$$
\begin{equation*}
\text { if } Y, Y^{\prime} \text { are purple and } f \in H_{\mathrm{red}}^{*} \cup H \text {, then }\left|Y \cap Y^{\prime}\right|,|Y \cap f| \in I . \tag{5.3}
\end{equation*}
$$

Now we define $H_{S}$ to be the collection of all edges $h \in H$ which contain a purple set $Y_{h} \subseteq h$ such that $Y_{h} \cup\{v\} \in H_{\text {red }}^{*}$ holds for each $v \in h \backslash Y_{h}$. Moreover, we set

$$
V_{S}=\bigcup_{h \in H_{S}}\left(h \backslash Y_{h}\right),
$$

which is clearly a subset of $V \backslash V_{T}$. Provided we can show

$$
\begin{equation*}
Y_{h} \cap V_{S}=\varnothing \quad \text { for each } h \in H_{S} \tag{5.4}
\end{equation*}
$$

it will be clear that $H_{S}$ is a union of stars with centres $Y_{h} \subseteq\left(V \backslash V_{S}\right)$ and their bodies in $V_{S}$, as required by Definition 2.1 (ii).

For the proof of (5.4) we assume indirectly that for some $h \in H_{S}$ there is a vertex $v \in Y_{h} \cap V_{S}$. This means that there exists an edge $h^{\prime} \in H_{S}$ with $v \in h^{\prime} \backslash Y_{h^{\prime}}$. But now $\left|Y_{h} \cap Y_{h^{\prime}}\right|$ and $\left|Y_{h} \cap\left(Y_{h^{\prime}} \cup\{v\}\right)\right|$ are two consecutive integers belonging to $I$ by (5.3) and both are at most $\left|Y_{h}\right|=r t(t-1)$, contrary to $t \geqslant 2$. Thereby (5.4) is proved.

Next we observe that if for some $f \in H$ and $h \in H_{S}$ there is a vertex $v \in f \cap h \cap V_{S}$, then the consecutive integers $\left|f \cap Y_{h}\right|$ and $\left|f \cap\left(Y_{h} \cup\{v\}\right)\right|$, again by (5.3), are both in $I$.

Consequently,

$$
\begin{equation*}
\text { if } f \in H \text { and } h \in H_{S} \text { satisfy } f \cap h \cap V_{S} \neq \varnothing \text {, then } Y_{h} \subseteq f \text {. } \tag{5.5}
\end{equation*}
$$

Hence, all stars in $H_{S}$ must be semi-disjoint and we may associate with each vertex $v \in V_{S}$ the set $Y_{v} \in H_{\text {red }}^{*}$ containing $v$ and the centre of the star to which $v$ belongs. With this notation, (5.5) rewrites as

$$
\begin{equation*}
\text { if } f \in H \text { and } v \in f \cap V_{S} \text {, then } Y_{v} \subseteq f . \tag{5.6}
\end{equation*}
$$

Condition (iii) of Definition 2.1 is an immediate consequence of this statement and it also follows that $H_{S} \supseteq\left\{h \in H:\left|h \cap V_{S}\right|=r t\right\}$. The reverse inclusion is implied by (5.4) and thereby condition (ii) is proved as well.

It remains to establish (iv), i.e., that for

$$
V_{R}=V \backslash\left(V_{T} \cup V_{S}\right) \quad \text { and } \quad H_{R}=H \backslash\left(H_{T} \cup H_{S}\right)
$$

we have

$$
\begin{equation*}
\left|H_{R}\right| \leqslant\left|V_{T}\right|\left|V_{S}\right| n^{r t-3}+\left(r t^{2}\right)^{r^{3} t^{6}}\left|V_{R}\right| n^{r t-2} . \tag{5.7}
\end{equation*}
$$

The first step in the proof of this result is to split $H_{R}$ into the two subhypergraphs $H_{S T}$ and $\hat{H}_{R}$ with the intention of proving $\left|H_{S T}\right| \leqslant\left|V_{T}\right|\left|V_{S}\right| n^{r t-3}$ and $\left|\hat{H}_{R}\right| \leqslant\left(r t^{2}\right)^{r^{3} t^{6}}\left|V_{R}\right| n^{r t-2}$.

The family $H_{S T}$ is defined as

$$
H_{S T}=\left\{h \in H_{R}: \text { there are } v \in h \cap V_{S} \text { and a team } g \subseteq h \text { with } Y_{v} \cap g=\varnothing\right\} .
$$

Observe that if $h \in H_{S T}$ and $v, g$ are as in the above definition, then $Y_{v} \subseteq h$ follows from (5.6) and we have $\left|h \backslash\left(Y_{v} \cup g\right)\right|=r t^{2}-1-t-r t(t-1) \leqslant r t-3$. As there are at most $\left|V_{S}\right|$ possibilities for $v,\left|V_{T}\right|$ possibilities for $g$, and $n^{r t-3}$ possibilities for the set $h \backslash\left(Y_{v} \cup g\right)$, it follows that we have indeed

$$
\left|H_{S T}\right| \leqslant\left|V_{T}\right|\left|V_{S}\right| n^{r t-3}
$$

Thus to conclude the argumnt we need to show that the hypergraph $\hat{H}_{R}=H_{R} \backslash H_{S T}$ satisfies

$$
\begin{equation*}
\left|\hat{H}_{R}\right| \leqslant\left(r t^{2}\right)^{r^{3} t^{6}}\left|V_{R}\right| n^{r t-2} \tag{5.8}
\end{equation*}
$$

In the special case $V_{R}=\varnothing$ this can only be true if $\hat{H}_{R}=\varnothing$ holds as well. For that reason it will certainly help us to establish

$$
\begin{equation*}
H_{R} \backslash V_{R} \subseteq H_{S T} \tag{5.9}
\end{equation*}
$$

To verify this, consider any edge $f \in H_{R}$ not meeting $V_{R}$. Owing to $f \notin H_{T}$ there must exist a vertex $v \in f \cap V_{S}$ and (5.6) tells us that $Y_{v} \subseteq f$. Now $f \backslash Y_{v}$ cannot be a subset of $V_{S}$ because (5.6) would then yield $f \in H_{S}$. Together with $f \subseteq\left(V_{S} \cup V_{T}\right)$ this shows that there must be a vertex $x \in V_{T} \cap\left(f \backslash Y_{v}\right)$. This vertex must in turn belong to some team $g \in G$, which is in fact a subset of $f \backslash Y_{v}$. Thereby (5.9) is proved.

Due to the discussion preceeding (5.9) we may henceforth suppose that $V_{R} \neq \varnothing$. Now the idea for proving (5.8) is that we can mark in every edge $h \in \hat{H}_{R}$ at least one vertex from $h \cap V_{R}$ in such a way that every vertex in $V_{R}$ gets marked at most $\left(r t^{2}\right)^{r^{3} t^{6}} n^{r t-2}$ many times. The marking procedure we use depends on the red and green sets contained in $h$ and thus it involves several case distinctions.

In view of property (c) of the green sets, we may write

$$
\begin{equation*}
\widehat{H}_{R}=\hat{H}_{R}^{1} \cup \hat{H}_{R}^{2} \cup \widehat{H}_{R}^{3} \tag{5.10}
\end{equation*}
$$

with

$$
\begin{aligned}
\hat{H}_{R}^{1} & =\left\{h \in \widehat{H}_{R}: h \text { cannot be written as a union of red and green sets }\right\}, \\
\hat{H}_{R}^{2} & =\left\{h \in \widehat{H}_{R}: h \text { is the union of its green subsets }\right\} \\
\text { and } \quad \hat{H}_{R}^{3} & =\left\{h \in \widehat{H}_{R}: h \notin \widehat{H}_{R}^{1} \text { and there is some } f \in H_{\text {red }}^{*} \text { with } f \subseteq h\right\} .
\end{aligned}
$$

Regarding the first of these three hypergraphs, we note that if $h \in \widehat{H}_{R}^{1}$ and $v \in h$ is not contained in any red or green subset of $h$, then $v \in V_{T}$ is impossible due to the inseparability of the teams, $v \in V_{S}$ is impossible by (5.6), and hence we must have $v \in V_{R}$. In other words, if we set

$$
H_{v}=\left\{h \in \hat{H}_{R}: v \in h\right\}
$$

and

$$
\hat{H}^{1}(v)=\left\{h \in H_{v}: \text { there is no } f \in H_{\text {red }} \text { with } v \in f \subseteq h\right\}
$$

for every $v \in V_{R}$, then

$$
\begin{equation*}
\hat{H}_{R}^{1} \subseteq \bigcup_{v \in V_{R}} \hat{H}^{1}(v) . \tag{5.11}
\end{equation*}
$$

According to our plan the hypergraphs $\hat{H}^{1}(v)$ should be of size at most $O\left(n^{r t-2}\right)$ and this is indeed what we prove next.

Fact 5.1. For every $v \in V_{R}$ we have

$$
\left|\hat{H}^{1}(v)\right|<\left(r t^{2}\right)^{2 r t^{2}} n^{r t-2} .
$$

Proof. Assume for the sake of contradiction that $v \in V_{R}$ violates this claim. Then $x=\{v\}$ is an example of a subset of $V$ with $v \in x$ and

$$
\begin{equation*}
\left|\left\{h \in \hat{H}^{1}(v): x \subseteq h\right\}\right| \geqslant\left(r t^{2}\right)^{2\left(r t^{2}-|x|\right)} n^{r t-2} . \tag{5.12}
\end{equation*}
$$

Now let $x \subseteq V$ be a maximal set of vertices with $v \in x$ that satisfies (5.12). As $x \subseteq h$ for some $h \in H$, we must have $|x| \leqslant r t^{2}$. Thus

$$
n^{r t-2} \leqslant\left(r t^{2}\right)^{2\left(r t^{2}-|x|\right)} n^{r t-2} \leqslant\left|\left\{h \in \hat{H}^{1}(v): x \subseteq h\right\}\right| \leqslant\binom{ n-|x|}{r t^{2}-|x|}<n^{r t^{2}-|x|}
$$

and it follows that $|x| \leqslant r t(t-1)+1$. But owing to the definition of $\hat{H}^{1}(v)$ it is not possible for $x$ to be red. This means, in particular, that there is a maximal $\Delta$-system $\mathcal{G} \subseteq \widehat{H}^{1}(v)$ with kernel $x$ and $|\mathcal{G}|<r t^{2}$. The size of the set $B=\bigcup_{h \in \mathcal{G}}(h \backslash x)$ can be bounded by $|B| \leqslant \sum_{h \in \mathcal{G}}|h|<\left(r t^{2}\right)^{2}$ and the maximality of $\mathcal{G}$ implies that every edge $h \in \hat{H}^{1}(v)$ with $x \subseteq h$ intersects $B$. So by averaging and (5.12) there exists a vertex $w \in B$ with

$$
\left|\left\{h \in \widehat{H}^{1}(v):(x \cup\{w\}) \subseteq h\right\}\right| \geqslant \frac{\left(r t^{2}\right)^{2\left(r t^{2}-|x|\right)}}{\left(r t^{2}\right)^{2}} n^{r t-2}=\left(r t^{2}\right)^{2\left(r t^{2}-|x \cup\{w\}|\right)} n^{r t-2}
$$

This inequality tells us that $x \cup\{w\}$ contradicts the maximality of $x$. Thereby Fact 5.1 is proved.

This completes our analysis of $\hat{H}_{R}^{1}$ and we proceed with $\hat{H}_{R}^{2}$. To this end, we shall use the trivial decomposition

$$
\begin{equation*}
\hat{H}_{R}^{2}=\bigcup_{v \in V_{R}} \hat{H}^{2}(v) \tag{5.13}
\end{equation*}
$$

where

$$
\hat{H}^{2}(v)=\left\{h \in H_{v}: h \text { is the union of its green subsets }\right\} .
$$

Fact 5.2. If $v \in V_{R}$, then

$$
\left|\widehat{H}^{2}(v)\right|<\left(r t^{2}\right)^{r^{3} t^{6}-7 r t^{2}} n^{r t-2}
$$

Proof. Consider the auxiliary set system

$$
\mathcal{G}=\left\{x \subseteq V:|x| \leqslant r t^{2} \text { and } G \mid x \text { is connected }\right\}
$$

Utilising property $(a)$ of the green sets and the fact that for every $x \in \mathcal{G}$ there is a spanning sub-setsystem of $G \mid x$ consisting of at most $|x|$ sets we obtain

$$
\Delta(\mathcal{G})<\left(\Delta(G) r t^{2}\right)^{r t^{2}} \leqslant\left(r t^{2}\right)^{2\left(r t^{2}\right)^{2}+r t^{2}}
$$

As every edge of $G$ has at least $t$ vertices, the same is true about $\mathcal{G}$. Moreover, the only possibility for $x \in \mathcal{G}$ to have size exactly $t$ is that it is a team.

Now any given $h \in \hat{H}^{2}(v)$ can be expressed as a disjoint union of edges of $\mathcal{G}$ by looking at the connected components of $G \mid h$. The number of edges from $\mathcal{G}$ appearing in such a decomposition can be at most $r t-1$ because of the remarks from the previous paragraph and as $v$ cannot belong to a team.

Representing each edge $h \in \hat{H}^{2}(v)$ by a selection of one vertex from each of its at most $r t-2$ green components not containing $v$, we learn that indeed

$$
\left|\hat{H}^{2}(v)\right| \leqslant \Delta(\mathcal{G})^{r t-1} n^{r t-2}<\left(r t^{2}\right)^{r^{3} t^{6}-7 r t^{2}} n^{r t-2} .
$$

It remains to deal with the hypergraph $\hat{H}_{R}^{3}$, which may be further decomposed as

$$
\begin{equation*}
\hat{H}_{R}^{3}=\hat{H}_{R}^{3, \mathrm{x}} \cup \hat{H}_{R}^{3, \mathrm{y}} \tag{5.14}
\end{equation*}
$$

where

$$
\widehat{H}_{R}^{3, \mathrm{x}}=\left\{h \in \widehat{H}_{R}^{3}: h \text { is the union of its subsets belonging to } H_{\mathrm{red}}^{*}\right\}
$$

and $\hat{H}_{R}^{3, \mathrm{y}}=\hat{H}_{R}^{3} \backslash \hat{H}_{R}^{3, \mathrm{x}}$. We will estimate the sizes of these two hypergraphs in the two facts that follow. In both proofs we will frequently use the inequality $\left|H_{\text {red }}^{*}\right| \leqslant n$ obtained in (5.2) above without referencing it.

Fact 5.3. We have $\left|\hat{H}_{R}^{3, \mathrm{x}}\right| \leqslant\left(r t^{2}\right)^{r t^{2}}\left|V_{R}\right| n^{r t-2}$.
Proof. In the light of (5.1) there are only two possibilities for an edge $h \in \hat{H}_{R}^{3, \mathrm{x}}$. Either
(i) there are $f, f^{\prime} \subseteq h$ in $H_{\text {red }}^{*}$ such that $\left|f \cap f^{\prime}\right| \leqslant r t(t-1)-t$,
(ii) or there is some $Y_{h} \subseteq h$ of size $r t(t-1)$ such that $Y_{h} \cup\{v\} \in H_{\text {red }}^{*}$ holds for every vertex $v \in h \backslash Y_{h}$.

If $h$ is of type $(i)$ we have $\left|f \cup f^{\prime}\right|=|f|+\left|f^{\prime}\right|-\left|f \cap f^{\prime}\right| \geqslant r t^{2}-r t+t+2$ and hence $\left|h-\left(f \cup f^{\prime}\right)\right| \leqslant r t-t-2 \leqslant r t-4$. As there are at most $n^{2}$ possibilities to choose a pair $f, f^{\prime}$ of two edges from $H_{\text {red }}^{*}$ and at most $n^{r t-4}$ possibilities to choose at most $r t-4$ further vertices in $V$, there can be at most $n^{r t-2}$ edges in $\hat{H}_{R}^{3, \mathrm{x}}$ to which the description ( $i$ ) applies.

Next we note that if $h$ and $Y_{h}$ are as in (ii), then $Y_{h}$ cannot be purple for otherwise $h$ would satisfy the requirements for belonging to $H_{S}$. We will prove below that there are at most $9\left|V_{R}\right|$ such edges in the special case $r=1$ and $t=2$, and at most $\binom{r t^{2}}{r t} \cdot r t^{2} n$ such edges if $r t \geqslant 3$. Due to $V_{R} \neq \varnothing$ this suffices to establish Fact 5.3 in both cases.

Let us consider the case that $r=1$ and $t=2$ first. If $h$ denotes an edge of type ( $i i$ ), then $h \backslash Y_{h} \subseteq V_{R}$ by (5.6), and we may mark any vertex $v \in h \backslash Y_{h}$. Since the triples $Y_{h} \cup\{v\}$ and $h \backslash\{v\}$ are both in $H_{\text {red }}^{*}$, (5.1) implies that $v$ is contained in at most 3 red sets $f \in H_{\text {red }}^{*}$. For none of them $f \backslash\{v\}$ is purple (because $v \notin V_{S}$ ), which in turn means that each of them can be involved at most 3 times in the marking of $v$. Altogether each $v \in V_{R}$ gets marked at most 9 times due to edges of type (ii), wherefore there are indeed at most $9\left|V_{R}\right|$ such edges.

Now suppose that $r t \geqslant 3$ and let $h$ again denote an edge of type (ii). As $Y_{h}$ arises from a member of $H_{\text {red }}^{*}$ by the deletion of a vertex, there are at most $r t^{2} n$ candidates for this set and each of them can be used in at most $\binom{r t^{2}}{r t}$ edges of type (ii), for otherwise it would be purple. This proves the upper bound of $\binom{r t^{2}}{r t} \cdot r t^{2} n$ on the number of such edgses $h$ and the proof of Fact 5.3 is complete.

Fact 5.4. We have $\left|\hat{H}_{R}^{3, \mathrm{y}}\right| \leqslant\left(\left(r t^{2}\right)^{2 r t^{2}}+1\right)\left|V_{R}\right| n^{r t-2}$.
Proof. Consider any edge $h \in \hat{H}_{R}^{3, y}$. Since $h \in \hat{H}_{R}^{3}$, there is a set $f \in H_{\text {red }}^{*}$ with $f \subseteq h$. If possible let $f$ be chosen in such a way that it contains a vertex from $V_{S}$.

By $h \notin \widehat{H}_{R}^{3, \mathrm{x}}$ there exists a vertex $v \in h \backslash f$ that is not contained in any member of $H_{\text {red }}^{*}$ which at the same time happens to be a subset of $h$. Therefore $h \notin \widehat{H}_{R}^{1}$ tells us that there exists a set $\widehat{g} \in \hat{H}_{\text {red }} \cup G$ with $v \in \widehat{g} \subseteq h$. Due to property $(c)$ of $G$ this leads us to a green set $g$ with $v \in g \subseteq h$. From (d) and $v \in g \backslash f$ it follows that $|g \backslash f| \geqslant t$ and hence we have

$$
\begin{equation*}
|h \backslash(f \cup g)| \leqslant r t-t-1 \leqslant r t-3 . \tag{5.15}
\end{equation*}
$$

By (5.6) and the choice of $v$ we have $v \notin V_{S}$ and, hence, $v$ is either in $V_{R}$ or in $V_{T}$. Let us analyse these two possibilities separately.

If $v \in V_{R}$, then we mark it. Property $(a)$ of $G$ tells us that $v$ is contained in at most $\left(r t^{2}\right)^{2 r t^{2}}$ green sets and, using (5.15), one can conclude that in this way each vertex of $V_{R}$ is marked at most $\left(r t^{2}\right)^{2 r t^{2}} n^{r t-2}$ many times.

On the other hand, if $v \in V_{T}$, then $g$ is a team. Appealing to $h \notin H_{S T}$ and the choice of $f$ we learn that $h \cap V_{S}=\varnothing$. But $|h \backslash f| \equiv-1(\bmod t)$ and therefore it is not possible that $h \backslash f$ is entirely covered by teams. Consequently there is a vertex $w \in(h \backslash(f \cup g)) \cap V_{R}$
that can be marked. Now there are at most $n$ possibilities for $f$, for $g$, and for each of the remaining vertices in $h \backslash(g \cup f \cup\{w\})$. Using (5.15) again, we get that in this way each vertex is marked at most $n^{r t-2}$ further times.

Summarising the above estimations one obtains

$$
\left|\hat{H}_{R}^{3, \mathrm{y}}\right| \leqslant\left|V_{R}\right|\left(r t^{2}\right)^{2 r t^{2}} n^{r t-2}+\left|V_{R}\right| n^{r t-2}
$$

Collecting all the above results we get

$$
\left|\hat{H}_{R}^{1}\right| \leqslant\left(r t^{2}\right)^{2 r t^{2}}\left|V_{R}\right| n^{r t-2}
$$

from (5.11) and Fact 5.1,

$$
\left|\hat{H}_{R}^{2}\right| \leqslant\left(r t^{2}\right)^{r^{3} t^{6}-7 r t^{2}}\left|V_{R}\right| n^{r t-2}
$$

from (5.13) and Fact 5.2,

$$
\left|\hat{H}_{R}^{3}\right| \leqslant 2\left(r t^{2}\right)^{2 r t^{2}}\left|V_{R}\right| n^{r t-2}
$$

from (5.14), Fact 5.3, Fact 5.4, and finally

$$
\left|\hat{H}_{R}\right| \leqslant\left(3\left(r t^{2}\right)^{2 r t^{2}}+\left(r t^{2}\right)^{r^{3} t^{6}-7 r t^{2}}\right)\left|V_{R}\right| n^{r t-2} \leqslant\left(r t^{2}\right)^{r^{3} t^{6}}\left|V_{R}\right| n^{r t-2}
$$

from (5.10) and the three previous estimates. This concludes the proof of (5.8) and, hence, the proof of the Structure Theorem 2.2.

## References

[1] E. R. Berlekamp, On subsets with intersections of even cardinality, Canad. Math. Bull. 12 (1969), 471-474, DOI 10.4153/CMB-1969-059-3. MR0249303 $\uparrow 4.1$
[2] L. Babai and P. Frankl, On set intersections, J. Combin. Theory Ser. A 28 (1980), no. 1, 103-105, DOI 10.1016/0097-3165(80)90063-1. MR558879 ¡1, 4.1
[3] M. Deza, P. Erdős, and P. Frankl, Intersection properties of systems of finite sets, Proc. London Math. Soc. (3) $\mathbf{3 6}$ (1978), no. 2, 369-384, DOI 10.1112/plms/s3-36.2.369. MR0476536 $\uparrow 1$
[4] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961), 313-320, DOI 10.1093/qmath/12.1.313. MR0140419 $\uparrow 1$
[5] P. Erdős and R. Rado, Intersection theorems for systems of sets, J. London Math. Soc. 35 (1960), 85-90, DOI 10.1112/jlms/s1-35.1.85. MR0111692 $\uparrow 4.2,4.2$
[6] P. Frankl and Z. Füredi, Forbidding just one intersection, J. Combin. Theory Ser. A 39 (1985), no. 2, 160-176, DOI 10.1016/0097-3165(85)90035-4. MR793269 $\uparrow 1$
[7] P. Frankl and V. Rödl, Forbidden intersections, Trans. Amer. Math. Soc. 300 (1987), no. 1, 259-286, DOI 10.2307/2000598. MR871675 $\uparrow 1$
[8] A. V. Kostochka, A bound of the cardinality of families not containing $\Delta$-systems, The mathematics of Paul Erdős, II, Algorithms Combin., vol. 14, Springer, Berlin, 1997, pp. 229-235, DOI 10.1007/978-3-642-60406-5_19. MR1425216 $\uparrow 4.2$
[9] T. Łuczak and J. Polcyn, Paths in hypergraphs: a rescaling phenomenon, available at arXiv:1706.08465. Submitted. $\uparrow$ (document), 1, 3, 3
[10] D. Mubayi and V. Rödl, Specified intersections, Trans. Amer. Math. Soc. 366 (2014), no. 1, 491-504, DOI 10.1090/S0002-9947-2013-05877-1. MR3118403 个1
[11] R. M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, Combinatorica 4 (1984), no. 2-3, 247-257, DOI 10.1007/BF02579226. MR771733 $\uparrow 1$

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