arXiv:1706.05516v1 [math.DG] 17 Jun 2017

Twist, elementary deformation, and KK correspondence in generalized complex geometry

V. Cortés[1] and L. David[2]

[1]Department of Mathematics
and Center for Mathematical Physics
University of Hamburg
Bundesstraße 55, D-20146 Hamburg, Germany

[2]"Simion Stoilow" Institute of Mathematics of the Romanian Academy
Calea Grivitei 21,
Sector 1, Bucharest, Romania

vicente.cortes@uni-hamburg.de, liana.david@imar.ro

June 17, 2017

Abstract

We define the operations of conformal change and elementary deformation in the setting of generalized complex geometry. Then we apply Swann's twist construction to generalized (almost) complex and Hermitian structures obtained by these operations and establish necessary and sufficient conditions for the Courant integrability of the resulting twisted structures. In particular, we associate to any appropriate generalized Kähler manifold $(M, G, \mathcal{J})$ with a Hamiltonian Killing vector field a new generalized Kähler manifold, depending on the choice of a pair of non-vanishing functions and compatible twist data. We study this construction when $(M, G, \mathcal{J})$ is (diagonal) toric, with emphasis on the four-dimensional case. In particular, we apply it to deformations of the standard flat Kähler metric on $\mathbb{C}^n$, the Fubini-Study Kähler metric on $\mathbb{C}P^2$ and the so called admissible Kähler metrics on Hirzebruch surfaces. As a further application, we recover the KK (Kähler-Kähler) correspondence, which is obtained by specializing to the case of an ordinary Kähler manifold.

Keywords: Generalized complex structures, (toric) generalized Kähler

structures, Hamiltonian vector fields, twist, elementary deformations, Kähler-Kähler correspondence

MSC classification: 53D18 (primary); 53D20, 53C55 (secondary).

Contents

1 Introduction

Swann's twist construction is a powerful method to construct new manifolds with geometrical structures from given ones [19]. It was applied successfully to construct explicit examples of compact SKT, hyper-complex and HKT manifolds. Combined with the so-called elementary deformation of hyper-Kähler structures, it provides an elegant approach to the HK/QK (hyper-Kähler/quaternionic-Kähler) correspondence [18]. The HK/QK correspondence basically associates to a hyper-Kähler manifold with a certain type of Killing vector field a quaternionic-Kähler manifold of the same dimension. It was introduced by Haydys in [13] (see also [15] for recent related work) and was extended to allow for indefinite hyper-Kähler metrics in [1, 2], without losing control over the signature of the resulting quaternionic-Kähler metrics, for which a simple formula was given in [2]. This result was applied in [2] to prove that the one-loop quantum correction of the supergravity c-map metric is quaternionic-Kähler, leading to the first completeness results for the quaternionic-Kähler metric in the HK/QK correspondence (see [6] for the state of the art). The approach from [1, 2] yields also the so-called KK (Kähler-Kähler) correspondence. This is a method to associate to a Kähler manifold with a Hamiltonian Killing vector field a new Kähler manifold, of the same dimension.

Despite a rich literature on the twist construction in complex and Hermitian geometry, applications in generalized complex geometry seem to be missing in the literature. In this paper we fill this gap, by studying how various notions from generalized complex geometry behave under the twist construction. Our main goal is to extend, in the spirit of [18], the KK correspondence to the setting of generalized Kähler geometry.

The plan of the paper is the following. Section 2 is mainly intended to fix notation. Here we recall Swann's twist construction (see Section 2.1) and the material we need from generalized complex geometry (see Section 2.2).

In Section 3 we define two basic algebraic tools for this paper: the conformal change and elementary deformation in the setting of generalized complex geometry (see Definitions 2 and 6). Our definition of elementary deformation is inspired by [18], where elementary deformations of hyper-Kähler structures were introduced. The integrability of a generalized Kähler structure is encoded in the Courant integrability of L_1 and of the intersections $L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$, where L_i $(i = 1, 2)$ are the $(1, 0)$-bundles of its two generalized complex structures. We study how these intersections vary under elementary deformations of generalized almost Hermitian structures (see Lemma 9).

In Section 4 we study how the Courant bracket behaves under the twist construction and we determine conditions on the twist data which ensure that a generalized almost complex structure $\mathcal{J}$ becomes integrable under this construction (see Theorem 16). In Section 4.1 we present various particular cases: when $\mathcal{J}$ is a symplectic structure (this was already considered in [19]); when $\mathcal{J}$ is the deformation of a complex structure by a Poisson bivector Π (when $\Pi = 0$ this was also considered in [19]); when $\mathcal{J}$ is the interpolation between a complex and a symplectic structure; and finally, when $\mathcal{J}$ is a conformal change of a generalized almost complex structure (we shall encounter this situation later in the paper).

In Section 5 we develop what we call the generalized KK correspondence. Its statement in highest generality is Theorem 23. Let $(M, G, \mathcal{J})$ be a generalized Kähler manifold with a Hamiltonian Killing vector field X_0. Let $f, h \in C^\infty(M)$ be two non-vanishing functions, $(G', \mathcal{J})$ the elementary deformation of $(G, \mathcal{J})$ by X_0 and f and $\tau_h(G', \mathcal{J})$ the conformal change of $(G', \mathcal{J})$ by h. Theorem 23 expresses the conditions which ensure that the twist $[\tau_h(G', \mathcal{J})]_W$ of $\tau_h(G', \mathcal{J})$ is generalized Kähler. The idea of the proof is the following. As a first stage, we determine in Section 5.1 conditions which ensure that the twist of a generalized almost Hermitian manifold is generalized Kähler (see Theorem 22). Then we compute various useful Courant brackets in Section 5.2.1, using in an essential way that X_0 is a Hamiltonian Killing vector field. In Sections 5.2.2 and 5.2.3 we apply Theorem 22 to the generalized almost Hermitian manifold $\tau_h(G', \mathcal{J})$ and we conclude the proof of Theorem 23.

In Section 6 we develop examples of the generalized KK correspondence. As a first particular case of Theorem 23 we recover in our setting the KK correspondence of [1] (see Proposition 40). After a short discussion on various conditions from Theorem 23 and their impact on the generalized Kähler manifold $(M, G, \mathcal{J})$, we apply the generalized KK correspondence under the additional assumption that $\mathcal{J}_3 X_0 \in \Omega^1(M)$ (see Proposition 43). In Section 6.1, after a brief review of the theory of toric generalized Kähler manifolds developed in [4], we show that examples of (non-Kähler) generalized Kähler manifolds with such Hamiltonian Killing vector fields exist in the toric setting (see Examples 46). In Section 6.2 we treat in detail the generalized KK correspondence when $(M, G, \mathcal{J})$ is toric and four-dimensional (see Proposition 47, Corollaries 50 and 51). In particular, we apply it to a suitable 1-parameter family of deformations of the Fubini-Study Kähler structure on $\mathbb{C}P^2$ and of the class of admissible Kähler structures on Hirzebruch surfaces $\mathbb{F}_k$ (see Examples 53 and 54).

It would be interesting to study the properties of the generalized Kähler structures produced by the generalized KK correspondence. Owing to the length of this paper, this will be postponed for a future project. Some preliminary observations in this direction are presented in Remark 55.

Acknowledgements: We warmly thank Paul Gauduchon for sharing with us his unpublished manuscripts [9, 10, 11] and for drawing our attention to reference [4]. This work was partly supported by the German Science Foundation (DFG) under the Research Training Group 1670 "Mathematics inspired by String Theory". Part of this work was done during the Humboldt Research Fellowship of L.D. at the University of Hamburg. She thanks Alexander von Humboldt Foundation for financial support and the University of Hamburg for hospitality and great working conditions.

2 Preliminaries

We begin by fixing various conventions we shall use in the paper. We work in the smooth setting. All our manifolds, vector bundles, functions, tensor fields, etc. are smooth. We denote by $\Gamma(E)$ the space of (smooth) sections of a vector bundle $E \to M$. For every manifold M, we denote by pr_T and pr_{T^*} the natural projections from the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$

onto its components TM and T^*M. The Lie derivative in the direction of a vector field X is denoted by $\mathcal{L}_X$. For a real form $\alpha \in \Omega^k(M)$, we use the same notation α for its complex linear extension to any (subbundle) of $(TM)^{\mathbb{C}}$. For a non-degenerate 2-form $\omega \in \Omega^2(M)$, $\omega : TM \to T^*M$ is the map $X \to i_X\omega$. Similarly, $g : TM \to T^*M$, $X \to g(X, \cdot)$ denotes the Riemannian duality defined by a Riemannian metric g. In our conventions, $J^*\alpha := \alpha \circ J$, for any 1-form α and almost complex structure J.

2.1 Twist construction

Swann's twist construction associates to a manifold M with a circle action a new manifold W. Following [19], we now recall this construction. The starting point is a twist data: a tuple (X_0, F, a) with the following properties:

- $X_0 \in \mathfrak{X}(M)$ is a vector field which generates a circle action on M. Along the paper by an invariant tensor field, we mean a tensor field invariant under this action. We denote by $\Gamma(E)^{\mathrm{inv}}$ the space of invariant sections of a tensor bundle $E \to M$;

- $F \in \Omega^2(M)$ is an invariant closed 2-form. It is the curvature of a connection $\mathcal{H}$ on a principal S^1-bundle $\pi : P \to M$. We denote by $X^P \in \mathfrak{X}(P)$ the principal vector field of π and by $\theta \in \Omega^1(M)$ the connection form of $\mathcal{H}$. We denote by $\tilde{X} \in \mathfrak{X}(P)$ the $\mathcal{H}$-horizontal lift of any $X \in \mathfrak{X}(M)$;

- $a \in C^\infty(M)$ is non-vanishing and satisfies

$$da = -i_{X_0}F. \tag{1}$$

In this setting, we assume that the vector field $X_0' := \tilde{X}_0 + \pi^*(a)X^P$ generates a free and proper group action of a 1-dimensional Lie group. Let $W := P/\langle X_0' \rangle$ be the quotient manifold and $\pi_W : P \to W$ the natural projection. Since a is non-vanishing, X_0' is transversal to $\mathcal{H}$. We denote by $\widehat{X} \in \mathcal{H}$ the horizontal lift of any $X \in \mathfrak{X}(W)$. Identifying (by means of the projections $\pi_* : \mathcal{H}_p \to T_{\pi(p)}M$ and $(\pi_W)_* : \mathcal{H}_p \to T_{\pi_W(p)}W$), $T_{\pi(p)}M$ and $T_{\pi_W(p)}W$ with $\mathcal{H}_p$, we can transfer any invariant tensor field A on M to a tensor field (of the same type) A_W on W (the invariance of A ensures that A_W is well-defined). We say that A and A_W are $\mathcal{H}$-related and we write $A_W \sim_{\mathcal{H}} A$. The tensor field A_W is called the twist of A. In particular, $\tilde{X} = \widehat{X}_W$ and, for any invariant form $\alpha \in \Omega^k(M)$, $\pi^*(\alpha)|_{\mathcal{H}} = \pi_W^*(\alpha_W)|_{\mathcal{H}}$. In fact, as proved in [19],

$$\pi_W^*(\alpha_W) = \pi^*(\alpha) - \frac{1}{a}\theta \wedge \pi^*(i_{X_0}\alpha). \tag{2}$$

The exterior derivative behaves under twist as :

$$d\alpha_W \sim_{\mathcal{H}} d_W\alpha := d\alpha - \frac{1}{a}F \wedge i_{X_0}\alpha, \tag{3}$$

for any invariant form $\alpha \in \Omega^k(M)$. In particular, if α is closed, then its twist α_W is also closed if and only if $F \wedge i_{X_0}\alpha = 0$.

Similarly, the Lie bracket behaves under twist as:

$$[X_W, Y_W] \sim_{\mathcal{H}} [X, Y] + \frac{1}{a}F(X, Y)X_0, \tag{4}$$

for any invariant vector fields $X, Y \in \mathfrak{X}(M)$. Using (4), one can show that the twist J_W of an invariant complex structure J on M is integrable if and only if F is of type $(1,1)$ with respect to J (see [19]).

2.2 Generalized complex geometry

Generalized almost complex structures. A generalized almost complex structure on a manifold M is a field of endomorphisms $\mathcal{J} \in \Gamma\mathrm{End}(\mathbb{T}M)$ of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ which satisfies $\mathcal{J}^2 = -\mathrm{Id}$ and is skew-symmetric with respect to the canonical metric $\langle \cdot, \cdot \rangle$ of $\mathbb{T}M$ of neutral signature, defined by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(\eta(X) + \xi(Y)\right), \ X + \xi, Y + \eta \in \mathbb{T}M.$$

The $(1,0)$-bundle $L \subset (\mathbb{T}M)^{\mathbb{C}}$ (the i-eigenbundle) of a generalized almost complex structure $\mathcal{J}$ is maximal isotropic with respect to $\langle \cdot, \cdot \rangle$, satisfies $L \oplus \bar{L} = (\mathbb{T}M)^{\mathbb{C}}$, and (when $\mathcal{J}$ is of constant type), has the form

$$L = L(E, \epsilon) = \{X + \xi \in E \oplus (T^*M)^{\mathbb{C}}, \ \xi|_E = i_X\epsilon\}$$

where $E \subset (TM)^{\mathbb{C}}$ is a complex subbundle and $\epsilon \in \Gamma(\Lambda^2 E^*)$ is a (complex) 2-form, such that $\mathrm{Im}(\epsilon|_\Delta)$ is non-degenerate, where $\Delta \subset TM$ is defined by $\Delta^{\mathbb{C}} := E \cap \bar{E}$. The corank of $E \subset (TM)^{\mathbb{C}}$ is called the type of $\mathcal{J}$. For later use, we note that

$$\epsilon(X, Y) = 2\langle X + \xi, Y \rangle = \xi(Y), \tag{5}$$

for $X + \xi \in L$ and $Y \in E$.

Usual almost complex and almost symplectic structures define generalized almost complex structures by

$$\mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

with $(1,0)$-bundles $L(T^{1,0}M, 0)$ and $L((TM)^{\mathbb{C}}, -i\omega)$ respectively. For any generalized almost complex structure $\mathcal{J}$ and 2-form $B \in \Omega^2(M)$, $\exp(B) \cdot \mathcal{J} := \exp(B) \circ \mathcal{J} \circ \exp(-B)$ is also a generalized almost complex structure, where $\exp(B) \in \mathrm{Aut}(\mathbb{T}M)$ is given by $\exp(B)(X + \xi) := X + \xi + i_X B$, for any $X + \xi \in \mathbb{T}M$. The generalized almost complex structure $\exp(B) \cdot \mathcal{J}$ is called the B-field transformation of $\mathcal{J}$. If $L = L(E, \epsilon)$ is the $(1,0)$-bundle of $\mathcal{J}$, then $L(E, B + \epsilon)$ is the $(1,0)$-bundle of $\exp(B) \cdot \mathcal{J}$.

Generalized complex structures. A generalized almost complex structure $\mathcal{J}$ is integrable (i.e. is a generalized complex structure) if its $(1,0)$-bundle $L \subset (\mathbb{T}M)^{\mathbb{C}}$ is closed under the Courant bracket, given by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d\left(\eta(X) - \xi(Y)\right).$$

This reduces to the usual notions of integrability of almost complex and almost symplectic structures. If $\mathcal{J}$ is integrable and $dB = 0$, then $\exp(B) \cdot \mathcal{J}$ is also integrable. In terms of the $(1,0)$-bundle L, a generalized almost complex structure is integrable if and only if $E \subset (\mathbb{T}M)^{\mathbb{C}}$ is involutive and $d\epsilon = 0$. As in the case of ordinary complex structures, a generalized almost complex structure $\mathcal{J}$ gives rise to a tensor field $N_{\mathcal{J}} \in \Gamma(\Lambda^2 \mathbb{T}^* M \otimes \mathbb{T}M)$ (called the Nijenhuis tensor) defined by

$$N_{\mathcal{J}}(u, v) := [\mathcal{J}u, \mathcal{J}v] - [u, v] - \mathcal{J}([\mathcal{J}u, v] + [u, \mathcal{J}v]) \tag{6}$$

and $\mathcal{J}$ is integrable if and only if $N_{\mathcal{J}} = 0$.

The Courant bracket does not satisfy the Jacobi identity but has the following properties which we shall use in our computations (see e.g. [14]):

$$[u, fv] = f[u, v] + \mathrm{pr}_T(u)(f)v - \langle u, v\rangle df \tag{7}$$

and

$$\mathrm{pr}_T(u)\langle v, w\rangle = \langle [u, v], w\rangle + \langle v, [u, w]\rangle + \langle d\langle u, v\rangle, w\rangle + \langle v, d\langle u, w\rangle\rangle, \tag{8}$$

for all $u, v, w \in \Gamma(\mathbb{T}M)$. Also, the Courant bracket $L_X := [X, \cdot]$ is related to the Lie bracket $\mathcal{L}_X$ by

$$L_X(Y + \eta) = \mathcal{L}_X(Y + \eta) - \frac{1}{2}d(\eta(X)) = \mathcal{L}_X(Y + \eta) - d\langle X, Y + \eta\rangle \tag{9}$$

for any $X \in \mathfrak{X}(M)$ and $Y + \beta \in \Gamma(\mathbb{T}M)$.

Generalized almost Hermitian structures. A generalized almost Hermitian structure [12] on a manifold M is a Hermitian structure $(G, \mathcal{J})$ on the bundle $\mathbb{T}M$, such that $\mathcal{J}$ is skew symmetric with respect to $\langle \cdot, \cdot\rangle$ (i.e. is a generalized almost complex structure on M) and $G^{\mathrm{end}} \in \Gamma\mathrm{End}(\mathbb{T}M)$, defined by

$$G(u, v) = \langle G^{\mathrm{end}} u, v\rangle, \tag{10}$$

satisfies $(G^{\mathrm{end}})^2 = \mathrm{Id}$. The endomorphism G^{end} commutes with $\mathcal{J}$ and is of the form [12]

$$G^{\mathrm{end}} = \begin{pmatrix} A & g^{-1} \\ \sigma & A^* \end{pmatrix}, \tag{11}$$

where g and σ are Riemannian metrics, $A \in \Gamma\mathrm{End}(TM)$ is skew-symmetric with respect to both g and $\sigma = g - bg^{-1}b$, where $b := -gA \in \Omega^2(M)$. It has eigenvalues ± 1 and its associated eigenbundles $C_\pm$ are the graphs of $b \pm g : TM \to T^*M$. By means of the isomorphism $\mathrm{pr}_T|_{C_+} : C_+ \to TM$, the metric g and the 2-form b correspond to $\langle \cdot, \cdot\rangle|_{C_+ \times C_+}$ and, respectively, to $(\cdot, \cdot)|_{C_+ \times C_+}$, where

$$(X + \xi, Y + \eta) := \frac{1}{2}(\xi(Y) - \eta(X)).$$

Let $J_\pm \in \Gamma\mathrm{End}(TM)$ be the g-orthogonal almost complex structures, which correspond to $\mathcal{J}|_{C_\pm}$ via the isomorphisms $\mathrm{pr}_T|_{C_\pm} : C_\pm \to TM$. The generalized almost Hermitian structure $(M, G, \mathcal{J})$ is uniquely determined by the data (J_+, J_-, g, b).

On a generalized almost Hermitian manifold $(M, G, \mathcal{J})$ there is a second generalized almost complex structure $\mathcal{J}_2 := G^{\mathrm{end}}\mathcal{J}$, which commutes with $\mathcal{J}_1 := \mathcal{J}$. The generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are on equal footing: one can alternatively define a generalized almost Hermitian structure as a pair of commuting generalized almost complex structures $(\mathcal{J}_1, \mathcal{J}_2)$, such that the metric G defined by (10) with $G^{\mathrm{end}} := -\mathcal{J}_1\mathcal{J}_2$ is positive definite. In analogy with the quaternionic case, we shall use the notation $\mathcal{J}_3 := \mathcal{J}_1\mathcal{J}_2 = -G^{\mathrm{end}}$. Note that the structures $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ commute and that

$$\mathcal{J}_1^2 = \mathcal{J}_2^2 = -\mathcal{J}_3^2 = -\mathrm{Id}. \tag{12}$$

Any almost Hermitian structure (g, J) determines a generalized almost Hermitian structure with generalized almost complex structures $\mathcal{J}_1 := \mathcal{J}_J$, $\mathcal{J}_2 := \mathcal{J}_\omega$ (where $\omega := g \circ J$ is the Kähler form), metric G and endomorphism G^{end} given by

$$G(X + \xi, Y + \eta) = \frac{1}{2}\left(g(X, Y) + g^{-1}(\xi, \eta)\right), \quad G^{\mathrm{end}} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{13}$$

Generalized Kähler structures. A generalized Kähler structure [12] is a generalized almost Hermitian structure $(G, \mathcal{J})$ on a manifold M for which $\mathcal{J}_1$ and $\mathcal{J}_2$ (defined as above) are generalized complex structures. As proved by Gualtieri [12], a generalized almost Hermitian structure $(G, \mathcal{J})$ is generalized Kähler if and only if $\mathcal{J}_1$ (or L_1) is Courant integrable and the bundles $L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$ are also Courant integrable, where L_i are the $(1, 0)$-bundles of $\mathcal{J}_i$ $(i = 1, 2)$. In terms of the data (J_+, J_-, g, b) associated to $(G, \mathcal{J})$, the generalized Kähler condition is equivalent to the integrability of the almost complex structures $J_\pm$, together with the relation

$$db(X, Y, Z) = d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $\omega_\pm := g(J_\pm \cdot, \cdot)$ are the Kähler forms.

The generalized almost Hermitian structure $(\mathcal{J}_J, \mathcal{J}_\omega)$ defined by an almost Hermitian structure (g, J), with Kähler form ω, is generalized Kähler, if and only if (g, J) is Kähler. The B-field transformation $(\exp(B)\cdot\mathcal{J}_1, \exp(B)\cdot\mathcal{J}_2)$ (with $dB = 0$) of a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is generalized Kähler.

Generalized Kähler structures of symplectic type. Let (M, ω) be a symplectic manifold. The map $\mathcal{J} \to J_+$ is a one to one correspondence from the space of generalized Kähler structures $(G, \mathcal{J})$ with second generalized complex structure $\mathcal{J}_2 = \mathcal{J}_\omega$, and the space of (integrable) complex structures J_+ which tame ω, which means that $\omega(X, J_+X) > 0$, for all $X \neq 0$, and the ω-adjoint of J_+, given by $J_+^{*\omega} = \omega^{-1} \circ J_+^* \circ \omega$, is integrable [8]. The complex structure J_-, Hermitian metric g and 2-form b associated to $(G, \mathcal{J})$ are given by:

$$J_- = -J_+^{*\omega},\ g = -\frac{1}{2}\omega \circ (J_+ + J_-),\ b = -\frac{1}{2}\omega \circ (J_+ - J_-). \tag{14}$$

A generalized Kähler structure with $\mathcal{J}_2 = \mathcal{J}_\omega$ is called of symplectic type.

Hamiltonian vector fields on generalized Kähler manifolds. This is a particular case of the more general notion of Hamiltonian (real) actions of groups on generalized Kähler manifolds [16].

Definition 1. *Let $(M, G, \mathcal{J})$ be a generalized Kähler manifold, with generalized complex structures $\mathcal{J}_1 := \mathcal{J}$ and $\mathcal{J}_2 := G^{\mathrm{end}}\mathcal{J}$. A vector field X_0 is called* Hamiltonian Killing *if $\mathcal{L}_{X_0}(\mathcal{J}) = 0$, $\mathcal{L}_{X_0}(G) = 0$, and there is a function $f^H : M \to \mathbb{R}$ (called the* Hamiltonian function*) such that $\mathcal{J}_2 X_0 = df^H$.*

Let $(M, \mathcal{J}, G)$ be a generalized Kähler manifold with $\mathcal{J}_2 = \mathcal{J}_\omega$ determined by a symplectic form ω. Any Hamiltonian Killing vector field X_0 on (M, ω), with $\mathcal{L}_{X_0}(\mathcal{J}) = 0$, is Hamiltonian Killing on the generalized Kähler manifold $(M, \mathcal{J}, G)$.

3 Algebraic operations in generalized complex geometry

Let M be a manifold and $\tau \in \mathrm{Isom}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$. If $\mathcal{J}$ is a generalized almost complex structure on M, then so is $\tau(\mathcal{J}) := \tau \circ \mathcal{J} \circ \tau^{-1}$. Similarly, if $(G, \mathcal{J})$ is a generalized almost Hermitian structure on M, then $\tau(G, \mathcal{J}) := (G' := (\tau^{-1})^*(G), \mathcal{J}' = \tau \circ \mathcal{J} \circ \tau^{-1})$ is also a generalized almost Hermitian structure, with endomorphism $(G')^{\mathrm{end}} = \tau \circ G^{\mathrm{end}} \circ \tau^{-1}$ and second generalized almost complex structure $\mathcal{J}_2' = \tau \circ \mathcal{J}_2 \circ \tau^{-1}$. We apply these remarks to define the conformal change and elementary deformation in generalized complex geometry.

3.1 Conformal change

Any non-vanishing function $h \in C^\infty(M)$ defines $\tau_h \in \mathrm{Isom}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$, by $\tau_h(X) = hX$, $\tau_h(\xi) = \frac{1}{h}\xi$, for any $X + \xi \in \mathbb{T}M$.

Definition 2. *The generalized almost complex structure $\tau_h(\mathcal{J})$ is called the* conformal change *of $\mathcal{J}$ by $h \in C^\infty(M)$. The generalized almost Hermitian structure $\tau_h(G, \mathcal{J})$ is called the* conformal change *of $(G, \mathcal{J})$ by h.*

Remark 3. i) If $\mathcal{J}$ is a generalized almost complex structure with $(1,0)$-bundle $L = L(E, \epsilon)$, then the $(1,0)$-bundle of $\tau_h(\mathcal{J})$ is $L^h = L(E, \frac{1}{h^2}\epsilon)$. In particular, the conformal change preserves the type of a generalized almost complex structure. Notice that $\tau_h(\mathcal{J}_J) = \mathcal{J}_J$ for every almost complex structure J and $\tau_h(\mathcal{J}_\omega) = \mathcal{J}_{\frac{1}{h^2}\omega}$ for every almost symplectic form ω.

ii) If $(G, \mathcal{J})$ is the generalized almost Hermitian structure defined by a usual almost Hermitian structure (g, J), then $\tau_h(G, \mathcal{J})$ is defined by $(\frac{1}{h^2}g, J)$.

3.2 Elementary deformation

Let $(G, \mathcal{J})$ be a generalized almost Hermitian structure on a manifold M, $X_0 \in \mathfrak{X}(M)$ a non-vanishing vector field and $f \in C^\infty(M)$ a non-vanishing function. In this section we associate to this data a new generalized almost Hermitian structure on M and we study some of its properties. Define

$$\mathcal{S} := \mathrm{span}\{X_0, \mathcal{J}_1 X_0, \mathcal{J}_2 X_0, \mathcal{J}_3 X_0\},$$

where, as usual, $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = G^{\mathrm{end}}\mathcal{J}$ and $\mathcal{J}_3 = -G^{\mathrm{end}} = \mathcal{J}_1\mathcal{J}_2$. Since G is positive definite, X_0, $\mathcal{J}_1 X_0$, $\mathcal{J}_2 X_0$, $\mathcal{J}_3 X_0$ are linearly independent. Moreover,

$$\mathcal{S} = \mathrm{span}_{\mathbb{R}}\{X_0, \mathcal{J}X_0\} + \mathrm{span}_{\mathbb{R}}\{\mathcal{J}_2 X_0, \mathcal{J}_3 X_0\}$$

is a direct sum decomposition into two isotropic planes, which are interchanged by $\mathcal{J}_2$ and $\mathcal{J}_3$. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{S}$ is non-degenerate and the orthogonal complements of $\mathcal{S}$ with respect to G and $\langle \cdot, \cdot \rangle$ coincide and will be denoted by $\mathcal{S}^{\perp}$. For $u \in \mathbb{T}M$ we shall denote by $u^{\mathcal{S}}$ and $u^{\perp}$ its components on $\mathcal{S}$ and $\mathcal{S}^{\perp}$, with respect to the decomposition $\mathbb{T}M = \mathcal{S} \oplus \mathcal{S}^{\perp}$.

Let $\tau_f^{\mathcal{S}} \in \mathrm{Aut}(\mathcal{S})$ be given by

$$X_0 \mapsto fX_0,\ \mathcal{J}X_0 \mapsto f\mathcal{J}X_0,\ \mathcal{J}_2 X_0 \mapsto \frac{1}{f}\mathcal{J}_2 X_0,\ \mathcal{J}_3 X_0 \mapsto \frac{1}{f}\mathcal{J}_3 X_0.$$

Lemma 4. *The automorphism $\tau_f^{\mathcal{S}} \in \mathrm{Aut}(\mathcal{S})$ is an isometry with respect to $\langle \cdot, \cdot \rangle$.*

Decompose $\mathbb{T}M := \mathcal{S}^{\perp} \oplus \mathcal{S}$ and let $\tau := \mathrm{Id}_{\mathcal{S}^{\perp}} + \tau_f^{\mathcal{S}} \in \mathrm{Aut}(\mathbb{T}M)$.

Proposition 5. *Define a metric $G' := (\tau^{-1})^*(G)$ on $\mathbb{T}M$. Then $(G', \mathcal{J})$ is a generalized almost Hermitian structure.*

Proof. The claim follows from the fact that τ is an isometry for $\langle \cdot, \cdot \rangle$ (using Lemma 4) and commutes with $\mathcal{J}$. $\square$

Definition 6. *The generalized almost Hermitian structure $(G', \mathcal{J})$ is called the* elementary deformation *of $(G, \mathcal{J})$ by X_0 and f.*

Remark 7. i) If a generalized almost Hermitian structure is defined by a usual almost Hermitian structure with metric g and almost complex structure J, then its elementary deformation by X_0 and f is also defined by a usual almost Hermitian structure, with the same almost complex structure J and metric g', such that $\mathrm{span}_{\mathbb{R}}\{X_0, JX_0\}$ and $\mathrm{span}_{\mathbb{R}}\{X_0, JX_0\}^{\mathrm{perp}}$ are g'-orthogonal and

$$g' = \frac{1}{f^2} g|_{\mathrm{span}_{\mathbb{R}}\{X_0, JX_0\}} + g|_{\mathrm{span}_{\mathbb{R}}\{X_0, JX_0\}^{\mathrm{perp}}}.$$

Here "perp" refers to the g-orthogonal complement.

ii) More generally, elementary deformations preserve the class of generalized almost Hermitian structures which are B-field transformations of usual almost Hermitian structures. This follows from the fact that elementary deformations leave the first generalized almost complex structure unchanged, together with the fact that any generalized almost Hermitian structure $(G, \mathcal{J})$ for which $\mathcal{J}$ is the B-field transformation of a usual almost complex structure is the B-field transformation of a usual almost Hermitian structure (this is a consequence of relation (6.3), page 76, of [12]).

The next lemma can be checked directly, from the definition of elementary deformation and Lemma 4.

Lemma 8. *The second generalized almost complex structure $\mathcal{J}_2'$ and the endomorphism $(G^{\mathrm{end}})'$ of $(G', \mathcal{J})$ coincide, respectively, with $\mathcal{J}_2$ and G^{end} on $\mathcal{S}^{\perp}$. On $\mathcal{S}$, $\mathcal{J}_2'$ is given by*

$$\begin{aligned}
&X_0 \mapsto \frac{1}{f^2}\mathcal{J}_2 X_0,\ \mathcal{J}X_0 \mapsto \frac{1}{f^2}\mathcal{J}_3 X_0,\\
&\mathcal{J}_2 X_0 \mapsto -f^2 X_0,\ \mathcal{J}_3 X_0 \mapsto -f^2\mathcal{J}X_0
\end{aligned} \tag{15}$$

and $(G^{\mathrm{end}})'$ by

$$\begin{aligned}
&X_0 \mapsto -\frac{1}{f^2}\mathcal{J}_3 X_0,\ \mathcal{J}X_0 \mapsto \frac{1}{f^2}\mathcal{J}_2 X_0,\\
&\mathcal{J}_2 X_0 \mapsto f^2\mathcal{J}X_0,\ \mathcal{J}_3 X_0 \mapsto -f^2 X_0.
\end{aligned}$$

The vector fields $\{X_0, \mathcal{J}X_0, \mathcal{J}_2 X_0, \mathcal{J}_3 X_0\}$ are G'-orthogonal and

$$\begin{aligned}
G'(X_0, X_0) &= G'(\mathcal{J}X_0, \mathcal{J}X_0) = \frac{1}{f^2}G(X_0, X_0)\\
G'(\mathcal{J}_2 X_0, \mathcal{J}_2 X_0) &= G'(\mathcal{J}_3 X_0, \mathcal{J}_3 X_0) = f^2 G(X_0, X_0).
\end{aligned}$$

An important role for the integrability of a generalized almost Hermitian structure $(G, \mathcal{J})$ is played by the intersections $L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$, where L_i are the $(1,0)$-bundles of $\mathcal{J}_i$ (recall Section 2.2). The next lemma describes these intersections for elementary deformations. Let

$$\begin{aligned}
v_f &:= X_0 - i\mathcal{J}X_0 - \frac{1}{f^2}(\mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0)\\
v_{if} &:= X_0 - i\mathcal{J}X_0 + \frac{1}{f^2}(\mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0).
\end{aligned} \tag{16}$$

Lemma 9. *Let $(M, G, \mathcal{J})$ be a generalized almost Hermitian manifold, $X_0 \in \mathfrak{X}(M)$ a non-vanishing vector field and $f \in C^\infty(M)$ a non-vanishing function. Denote by L_i the $(1,0)$-bundles of $\mathcal{J}_i$ $(i = 1, 2)$. Let $(G', \mathcal{J})$ be the elementary deformation of $(G, \mathcal{J})$ by X_0 and f. Let $L_2' = L(E_2', \epsilon_2')$ be the $(1,0)$-bundle of the second generalized almost complex structure $\mathcal{J}_2' = (G^{\mathrm{end}})'\mathcal{J}$ of $(G', \mathcal{J})$. Then*

$$
\begin{aligned}
L_1 \cap L_2' &= \mathrm{span}_{\mathbb{C}}\{v_f\} + L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp},\\
L_1 \cap \bar{L}_2' &= \mathrm{span}_{\mathbb{C}}\{v_{if}\} + L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
\mathrm{pr}_T(L_1 \cap L_2') &= \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(v_f)\} + \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})\\
\mathrm{pr}_T(L_1 \cap \bar{L}_2') &= \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(v_{if})\} + \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})
\end{aligned}
\tag{18}
$$

are all direct sum decompositions, where $\mathcal{S}_{\mathbb{C}}$ is the complexification of $\mathcal{S}$.

Proof. We only prove the statements about $L_1 \cap L_2'$, i.e. the first relation (17) and the first relation (18) (the statements about $L_1 \cap \bar{L}_2'$ can be proved in a similar way). Since $\mathcal{J}_2'$ preserves $\mathcal{S}$ and $\mathcal{S}^\perp$, $L_2' = L_2' \cap \mathcal{S}_{\mathbb{C}} + L_2' \cap \mathcal{S}_{\mathbb{C}}^\perp$. Since $\mathcal{J}$ commutes with $\mathcal{J}_2'$, it preserves L_2'. It also preserves $\mathcal{S}$ and its orthogonal complement $\mathcal{S}^\perp$. Therefore,

$$
L_1 \cap L_2' = L_1 \cap L_2' \cap \mathcal{S}_{\mathbb{C}} + L_1 \cap L_2' \cap \mathcal{S}_{\mathbb{C}}^\perp = L_1 \cap L_2' \cap \mathcal{S}_{\mathbb{C}} + L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp \tag{19}
$$

(direct sums) since $L_1 \cap L_2' \cap \mathcal{S}_{\mathbb{C}}^\perp = L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp$ (because $\mathcal{J}_2' = \mathcal{J}_2$ on $\mathcal{S}^\perp$). On $\mathcal{S}$, $\mathcal{J}_2'$ is given by (15), from where we deduce that

$$
\begin{aligned}
L_2' \cap \mathcal{S}_{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\{X_0 - \frac{i}{f^2}\mathcal{J}_2 X_0, \mathcal{J}X_0 - \frac{i}{f^2}\mathcal{J}_3 X_0\},\\
L_1 \cap L_2' \cap \mathcal{S}_{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\{v_f\}.
\end{aligned}
\tag{20}
$$

Combining the second relation (20) with (19) we obtain the first relation (17).

We now check the first relation (18). By projecting the first relation (17) onto the tangent bundle, we obtain that $\mathrm{pr}_T(v_f)$ and $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp)$

generate $\mathrm{pr}_T(L_1 \cap L_2)$. We will show that they are also transverse. Suppose, by contradiction, that this is not true. Then there is $\xi \in (T^*M)^{\mathbb{C}}$ such that

$$v_f + \xi \in L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}. \tag{21}$$

The condition $v_f + \xi \in L_1$ together with $v_f \in L_1$ implies that $\xi \in L_1$. On the other hand, from the definition of v_f,

$$\mathcal{J}_2 v_f = \mathcal{J}_2 X_0 - i\mathcal{J}_3 X_0 + \frac{1}{f^2}(iX_0 + \mathcal{J} X_0). \tag{22}$$

From (22) and $\mathcal{J}_2(v_f + \xi) = i(v_f + \xi)$ (recall that $v_f + \xi \in L_2$), we obtain

$$\mathcal{J}_2 \xi = i\xi + (\frac{1}{f^2} - 1)(\mathcal{J}_2 X_0 - i\mathcal{J}_3 X_0) + (1 - \frac{1}{f^2})(iX_0 + \mathcal{J} X_0). \tag{23}$$

But $\mathcal{J}\xi = i\xi$ implies $\mathcal{J}_2 \xi = iG^{\mathrm{end}}(\xi)$ and relation (23) becomes

$$G^{\mathrm{end}}\left(\xi + (1 - \frac{1}{f^2})(X_0 - i\mathcal{J} X_0)\right) = \xi + (1 - \frac{1}{f^2})(X_0 - i\mathcal{J} X_0),$$

i.e. $\xi + (1 - \frac{1}{f^2})(X_0 - i\mathcal{J} X_0) \in C_+$ (the $+1$-eigenbundle C_+ of G^{end}). But C_+ is the graph of $(b + g)$, where b and g are the 2-form, respectively the metric of the bi-Hermitian structure associated to $(G, \mathcal{J})$ (see Section 2.2). We obtain that ξ is given by

$$\xi = i(1 - \frac{1}{f^2})\mathrm{pr}_{T^*}(\mathcal{J} X_0) + (1 - \frac{1}{f^2})(b + g)(X_0 - i\mathrm{pr}_T \mathcal{J} X_0). \tag{24}$$

To summarize: we proved that $v_f + \xi \in L_1 \cap L_2$ implies that ξ is given by (24). We now show that the additional condition $v_f + \xi \in \mathcal{S}_{\mathbb{C}}^{\perp}$ from (21), with ξ given by (24), leads to a contradiction. Indeed, suppose, by contradiction, that $v_f + \xi \in \mathcal{S}_{\mathbb{C}}^{\perp}$. In particular, $\langle v_f + \xi, X_0 \rangle = 0$. Taking the real part of this equality and using (24) we obtain

$$\frac{1}{f^2} G(X_0, X_0) + \frac{1}{2}(1 - \frac{1}{f^2}) g(X_0, X_0) = 0. \tag{25}$$

But, using (11),

$$G(X_0, X_0) = \frac{1}{2}\left(g(X_0, X_0) + g^{-1}(b(X_0), b(X_0))\right)$$

and the left hand side of (25) is equal to

$$\frac{1}{2f^2} g^{-1}(b(X_0), b(X_0)) + \frac{1}{2} g(X_0, X_0),$$

which is non-zero (because g is positive definite and X_0 is non-vanishing). We obtain a contradiction. $\square$

Remark 10. i) From (17), the bundle E_2' decomposes as

$$\begin{aligned} E_2' &= \mathrm{pr}_T(\bar{L}_1 \cap L_2') + \mathrm{pr}_T(L_1 \cap L_2') \\ &= \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})\} + \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) + \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \end{aligned} \tag{26}$$

The 2-forms ϵ_2' and ϵ_2 coincide, when both arguments belong to $\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ (the sum of the last two components in the second decomposition (26)). From (5), for any $X \in E_2'$,

$$\epsilon_2'(\mathrm{pr}_T(v_f), X) = 2\langle v_f, X\rangle,\ \epsilon_2'(\mathrm{pr}_T(\bar{v}_{if}), X) = 2\langle \bar{v}_{if}, X\rangle$$

and

$$\epsilon_2'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = 2\langle v_f, \mathrm{pr}_T(\bar{v}_{if})\rangle = -2\langle \bar{v}_{if}, \mathrm{pr}_T(v_f)\rangle.$$

ii) The intersections $L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$ have similar decompositions as in Lemma 9, with v_f replaced by

$$v_1 := X_0 - i\mathcal{J}X_0 - (\mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0) \tag{27}$$

and v_{if} replaced by

$$v_i := X_0 - i\mathcal{J}X_0 + \mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0. \tag{28}$$

That is,

$$\begin{aligned} L_1 \cap L_2 &= \mathrm{span}_{\mathbb{C}}\{v_1\} + L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp} \\ L_1 \cap \bar{L}_2 &= \mathrm{span}_{\mathbb{C}}\{v_i\} + L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}. \end{aligned} \tag{29}$$

From Lemma 9 and Remark 10, we obtain:

Corollary 11. *The elementary deformations preserve the rank of the bundles $L_1 \cap L_2$, $L_1 \cap \bar{L}_2$ and of their projections to $(TM)^{\mathbb{C}}$.*

4 Twist of generalized almost complex structures

From now on, we fix twist data (X_0, F, a) on a manifold M.

Lemma 12. *For any invariant vector field $X \in \mathfrak{X}(M)$ and invariant form $\alpha \in \Omega^k(M)$,*

$$\mathcal{L}_{X_W}\alpha_W \sim_{\mathcal{H}} \mathcal{L}_X\alpha - \frac{1}{a}(i_X F) \wedge i_{X_0}\alpha. \tag{30}$$

Proof. Using relation (2), $\tilde{X} = \widehat{X}_W$ and $\theta(\tilde{X}) = 0$, we obtain

$$\begin{aligned}
\pi_W^*(i_{X_W}(\alpha_W)) &= \pi_W^*(\alpha_W)(\widehat{X}_W, \cdot) = \pi_W^*(\alpha_W)(\tilde{X}, \cdot) \\
&= \left(\pi^*(\alpha) - \frac{1}{a}\theta \wedge \pi^*(i_{X_0}\alpha)\right)(\tilde{X}, \cdot) \\
&= \pi^*(i_X\alpha) + \frac{1}{a}\theta \wedge \pi^*(i_X i_{X_0}\alpha).
\end{aligned} \tag{31}$$

Thus,

$$\pi_W^*(i_{X_W}\alpha_W) = \pi^*(i_X\alpha) + \frac{1}{a}\theta \wedge \pi^*(i_X i_{X_0}\alpha). \tag{32}$$

Pulling back by π_W the Cartan formula

$$\mathcal{L}_{X_W}\alpha_W = i_{X_W}(d\alpha_W) + d(i_{X_W}\alpha_W)$$

and using (32), we can write

$$\begin{aligned}
\pi_W^*(\mathcal{L}_{X_W}(\alpha_W)) &= \pi_W^*(i_{X_W}(d\alpha_W)) + d\pi_W^*(i_{X_W}\alpha_W) \\
&= i_{\widehat{X}_W}\pi_W^*(d\alpha_W) + d\left(\pi^*(i_X\alpha) + \frac{1}{a}\theta \wedge \pi^*(i_X i_{X_0}\alpha)\right).
\end{aligned} \tag{33}$$

Recall from relation (3) that $d\alpha_W \sim_{\mathcal{H}} d_W\alpha$. From this fact, together with (2), we obtain

$$\pi_W^*(d\alpha_W) = \pi^*(d_W\alpha) - \frac{1}{a}\theta \wedge \pi^*(i_{X_0}d_W\alpha).$$

Replacing this relation into (33) and using $da = -i_{X_0}F$ and $d\theta = \pi^*F$, we obtain

$$\begin{aligned}
&\pi_W^*(\mathcal{L}_{X_W}\alpha_W) = \pi^*(\mathcal{L}_X\alpha) - \frac{1}{a}\pi^*((i_X F) \wedge i_{X_0}\alpha) \\
&+ \frac{1}{a}\theta \wedge \pi^*(i_X i_{X_0}d\alpha - d(i_X i_{X_0}\alpha)) - \frac{F(X_0, X)}{a^2}\theta \wedge \pi^*(i_{X_0}\alpha).
\end{aligned}$$

This relation, restricted to $\mathcal{H}$, gives

$$\pi_W^*(\mathcal{L}_{X_W}\alpha_W)|_{\mathcal{H}} = \pi^*(\mathcal{L}_X\alpha)|_{\mathcal{H}} - \frac{1}{a}\pi^*((i_X F)\wedge i_{X_0}\alpha)|_{\mathcal{H}} \tag{34}$$

which is (30). $\square$

The next lemma describes the behaviour of the Courant bracket under twists.

Lemma 13. *For any invariant sections $X+\xi, Y+\eta \in \Gamma(\mathbb{T}M)$, the Courant bracket $[(X+\xi)_W, (Y+\eta)_W]$ is $\mathcal{H}$-related to*

$$[X+\xi, Y+\eta] + \frac{F(X,Y)}{a}X_0 - \frac{\eta(X_0)}{a}i_X F + \frac{\xi(X_0)}{a}i_Y F.$$

Proof. Recall the expression of the Courant bracket:

$$\begin{aligned}[(X+\xi)_W, (Y+\eta)_W] = [X_W, Y_W] + \mathcal{L}_{X_W}\eta_W - \mathcal{L}_{Y_W}\xi_W \\ - \frac{1}{2}d\left(\eta_W(X_W) - \xi_W(Y_W)\right).\end{aligned} \tag{35}$$

From (4), $[X_W, Y_W] \sim_{\mathcal{H}} [X,Y] + \frac{F(X,Y)}{a}X_0$. From (30), $\mathcal{L}_{X_W}\eta_W \sim_{\mathcal{H}} \mathcal{L}_X\eta - \frac{\eta(X_0)}{a}i_X F$ and $\mathcal{L}_{Y_W}\xi_W \sim_{\mathcal{H}} \mathcal{L}_Y\xi - \frac{\xi(X_0)}{a}i_Y F$. From (31), $\pi_W^*(\eta_W(X_W)) = \pi^*(\eta(X))$. Taking the exterior derivative, we obtain $d(\eta_W(X_W)) \sim_{\mathcal{H}} d(\eta(X))$. A similar argument shows that $d(\xi_W(Y_W)) \sim_{\mathcal{H}} d(\xi(Y))$. Combining these facts with (35) we obtain the claim. $\square$

To simplify terminology, we introduce the following definition.

Definition 14. *A subbundle $E \subset (TM)^{\mathbb{C}}$ is called (F,a)-*involutive *if, for any sections $X, Y \in \Gamma(E)$, the complex vector field*

$$[X,Y]^{(F,a)} := [X,Y] + \frac{F(X,Y)}{a}X_0$$

is also a section of E.

If E is (F,a)-involutive and $\alpha \in \Gamma(\Lambda^2 E^*)$ then its twisted exterior differential $d^{(F,a)}\alpha \in \Gamma(\Lambda^3 E^*)$ is defined by

$$\begin{aligned}(d^{(F,a)}\alpha)(X,Y,Z) &:= X(\alpha(Y,Z)) + Z(\alpha(X,Y)) + Y(\alpha(Z,X)) \\ &+ \alpha(Z,[X,Y]^{(F,a)}) + \alpha(X,[Y,Z]^{(F,a)}) + \alpha(Y,[Z,X]^{(F,a)}),\end{aligned} \tag{36}$$

for any $X, Y, Z \in \Gamma(E)$. Remark that, if $E = (TM)^{\mathbb{C}}$, then $d^{(F,a)}\alpha = d_W\alpha$ (the latter defined in (3)).

Definition 15. *A form $\alpha \in \Gamma(\Lambda^2 E^*)$ defined on an (F,a)-involutive bundle E is called (F,a)-*closed *if $d^{(F,a)}\alpha = 0$.*

Using this preliminary material, we now find conditions which ensure that the twist $\mathcal{J}_W$ of an invariant generalized almost complex structure $\mathcal{J}$ on M is integrable. Recall that $\mathcal{J}_W$ is defined by

$$\mathcal{J}_W(u_W) = (\mathcal{J}u)_W,\ u \in \Gamma(\mathbb{T}M)^{\mathrm{inv}}. \tag{37}$$

Let $L = L(E,\epsilon)$ be the $(1,0)$-bundle of $\mathcal{J}$.

Theorem 16. *In this setting, $\mathcal{J}_W$ is integrable if and only if one of the following equivalent conditions holds:*

i) for any invariant sections $X+\xi, Y+\eta$ of L, the section

$$F(X,Y)X_0 - \eta(X_0)i_X F + \xi(X_0)i_Y F + a[X+\xi, Y+\eta] \tag{38}$$

of $(\mathbb{T}M)^{\mathbb{C}}$ is a section of L;

ii) the bundle E is (F,a)-involutive and $\epsilon \in \Gamma(\Lambda^2 E^*)$ is (F,a)-closed.

Proof. A straightforward computation which uses Lemma 13 and relation (37) shows that $N_{\mathcal{J}_W}(u_W, v_W)$, with $u,v \in \Gamma(\mathbb{T}M)^{\mathrm{inv}}$, is $\mathcal{H}$-related to

$$\begin{aligned}
\mathcal{E}(u,v) &:= N_{\mathcal{J}}(u,v) + \frac{1}{a}F(\mathrm{pr}_T(\mathcal{J}u), \mathrm{pr}_T(\mathcal{J}v))X_0 - \frac{\mathrm{pr}_{T^*}(\mathcal{J}v)(X_0)}{a} i_{\mathrm{pr}_T(\mathcal{J}u)}F\\
&+ \frac{\mathrm{pr}_{T^*}(\mathcal{J}u)(X_0)}{a} i_{\mathrm{pr}_T(\mathcal{J}v)}F - \frac{1}{a}F(\mathrm{pr}_T(u), \mathrm{pr}_T(v))X_0 + \frac{(\mathrm{pr}_{T^*}v)(X_0)}{a} i_{\mathrm{pr}_T(u)}F\\
&- \frac{\mathrm{pr}_{T^*}(u)(X_0)}{a} i_{\mathrm{pr}_T(v)}F - \frac{1}{a}F(\mathrm{pr}_T(\mathcal{J}u), \mathrm{pr}_T(v))\mathcal{J}X_0 + \frac{\mathrm{pr}_{T^*}(v)(X_0)}{a}\mathcal{J} i_{\mathrm{pr}_T(\mathcal{J}u)}F\\
&- \frac{\mathrm{pr}_{T^*}(\mathcal{J}u)(X_0)}{a}\mathcal{J} i_{\mathrm{pr}_T(v)}F - \frac{1}{a}F(\mathrm{pr}_T(u), \mathrm{pr}_T(\mathcal{J}v))\mathcal{J}X_0\\
&+ \frac{\mathrm{pr}_{T^*}(\mathcal{J}v)(X_0)}{a}\mathcal{J} i_{\mathrm{pr}_T(u)}F - \frac{\mathrm{pr}_{T^*}(u)(X_0)}{a}\mathcal{J} i_{\mathrm{pr}_T(\mathcal{J}v)}F.
\end{aligned}$$

We obtain that $\mathcal{J}_W$ is integrable if and only if

$$\mathcal{E}(u,v) = 0,\ \forall u,v \in \Gamma(\mathbb{T}M)^{\mathrm{inv}}. \tag{39}$$

We now consider in (39) various cases: a) $u,v \in \Gamma(L)^{\mathrm{inv}}$; b) $u \in \Gamma(L)^{\mathrm{inv}}$, $v \in \Gamma(\bar{L})^{\mathrm{inv}}$; c) $u,v \in \Gamma(\bar{L})^{\mathrm{inv}}$. In case b) relation (39) is automatically satisfied. In case a) relation (39) with $u = X + \xi$ and $v = Y + \eta$ becomes

$$\begin{aligned} & - F(X,Y)(X_0 + i\mathcal{J}X_0) + \eta(X_0)(i_X F + i\mathcal{J} i_X F) - \xi(X_0)(i_Y F + i\mathcal{J} i_Y F) \\ & = a([X+\xi, Y+\eta] + i\mathcal{J}[X+\xi, Y+\eta]), \end{aligned} \tag{40}$$

for any $X+\xi, Y+\eta \in \Gamma(L)^{\mathrm{inv}}$, and in case c) it is equivalent (by conjugation) to (40). But relation (40) is equivalent to claim i).

We now prove claim ii). The $(1,0)$-bundle of $\mathcal{J}_W$ is $L_W := L(E_W, \epsilon_W)$, where $E_W \subset (TW)^{\mathbb{C}}$ is generated (locally) by sections X_W, where $X \in \Gamma(E)^{\mathrm{inv}}$, and $\epsilon_W \in \Gamma(\Lambda^2 E_W^*)$ is given by

$$\epsilon_W(X_W, Y_W) = \epsilon(X,Y)_W,\ \forall X, Y \in \Gamma(E)^{\mathrm{inv}}.$$

The generalized almost complex structure $\mathcal{J}_W$ is integrable if and only if E_W is involutive, i.e. E is (F,a)-involutive (from (4)) and ϵ_W is closed. Using (4) and $X_W(f_W) = (Xf)_W$, for any invariant vector field $X \in \mathfrak{X}(M)$ and invariant function $f \in C^\infty(M)$ (see Lemma 12), we obtain: for any $X, Y, Z \in \Gamma(E)^{\mathrm{inv}}$,

$$\begin{aligned} & X_W(\alpha_W(Y_W, Z_W)) = X_W(\alpha(Y,Z)_W) = (X\alpha(Y,Z))_W \\ & \alpha_W(X_W, [Y_W, Z_W]) = \alpha_W(X_W, ([Y,Z]^{(F,a)})_W) = \alpha(X, [Y,Z]^{(F,a)})_W \end{aligned}$$

and similarly for cyclic permutations of X, Y and Z. We deduce that

$$d\epsilon_W(X_W, Y_W, Z_W) = (d^{(F,a)}\epsilon(X,Y,Z))_W,\ \forall X, Y, Z \in \Gamma(E)^{\mathrm{inv}}.$$

In particular, ϵ_W is closed if and only if ϵ is $d^{(F,a)}$-closed. $\square$

Remark 17. The invariance of $X + \xi$ and $Y + \eta$ is not essential in Theorem 16 i), which can be formulated also without this condition. This is true as $\mathbb{T}M$ admits local frames formed by invariant sections, from relation (7), and from the fact that L is isotropic with respect to $\langle \cdot, \cdot \rangle$.

4.1 Examples

In this subsection we apply Theorem 16 to various particular cases.

4.1.1 Twist of symplectic and (deformations) of complex structures

The conditions which ensure that the integrability of a complex or symplectic structure is preserved under twist were determined in [19] (see Section 2.1). In this section we rediscover these conditions using Theorem 16. We begin with the symplectic case.

Corollary 18. *Let ω be an invariant symplectic form on M and $\mathcal{J}_\omega$ the associated generalized complex structure. Then $(\mathcal{J}_\omega)_W$ is integrable (i.e. ω_W is symplectic) if and only if $F \wedge i_{X_0}\omega = 0$.*

Proof. For any $X \in TM$ and $\xi \in (TM)^*$, $\mathcal{J}_\omega X = i_X\omega$, $\mathcal{J}_\omega \xi = -\omega^{-1}(\xi)$ and the $(1,0)$-bundle of $\mathcal{J}_\omega$ is $L = L((TM)^{\mathbb{C}}, -i\omega)$. The sections of L are of the form $X - i(i_X\omega)$, for any $X \in (TM)^{\mathbb{C}}$. From Theorem 16, $(\mathcal{J}_\omega)_W$ is integrable if and only if, for any complex vectors X, Y,

$$\begin{aligned} &F(Y,X)(X_0 + i(i_{X_0}\omega)) + i\omega(X_0,Y)(i_X F - i\omega^{-1}(i_X F)) \\ &+ i\omega(X,X_0)(i_Y F - i\omega^{-1}(i_Y F)) = 0. \end{aligned} \tag{41}$$

Identifying the TM and T^*M components in (41) we obtain $F \wedge i_{X_0}\omega = 0$, as required. □

We now turn to the complex case. We consider a more general setting, namely a complex structure J on M and $\Pi \in \Gamma(\Lambda^2 TM)$, viewed as a homomorphism from T^*M to TM. Assume that

$$J \circ \Pi = \Pi \circ J^*. \tag{42}$$

From (42), (the complexification) of Π maps $\Lambda^{1,0}M$ to $T^{1,0}M$ and $\Lambda^{0,1}M$ to $T^{0,1}M$ (the type $(1,0)$ and $(0,1)$ of forms/vectors are with respect to J). Let $\sigma := \frac{i}{2}\Pi^{0,2}$, i.e.

$$\sigma : \Lambda^{1,0}M \to T^{1,0}M,\ \sigma(\xi) := \frac{i}{2}\Pi(\xi),\ \forall \xi \in \Lambda^{1,0}M.$$

From (42),

$$\mathcal{J}_{J,\sigma} := \begin{pmatrix} J & \Pi \\ 0 & -J^* \end{pmatrix} \tag{43}$$

is a generalized almost complex structure, with $(0,1)$-bundle given by

$$\bar{L} = \{X + \sigma(\xi) + \xi,\ X \in T^{0,1}M,\ \xi \in \Lambda^{1,0}M\}.$$

It is well-known that $\mathcal{J}_{J,\sigma}$ is integrable if and only if σ is a holomorphic Poisson structure on (M, J) (see [10, 14]). We assume that this holds. Moreover, we assume that J and Π are invariant.

Corollary 19. *i) In the above setting, $(\mathcal{J}_{J,\sigma})_W$ is integrable if and only if*

$$i_X F \in \sigma^{-1}(\mathrm{span}_{\mathbb{C}}\{X_0 - iJX_0\}),\ \forall X \in T^{0,1}M. \tag{44}$$

In particular, if $(\mathcal{J}_{J,\sigma})_W$ is integrable then F is of type $(1,1)$.

ii) If $\mathcal{J}_J$ is the generalized complex structure associated to an invariant complex structure J, then $(\mathcal{J}_J)_W$ is integrable if and only if F is of type $(1,1)$.

Proof. i) From Theorem 16 i), $\mathcal{J}_{J,\sigma}$ is integrable if and only if, for any $X + \sigma(\xi) + \xi, Y + \sigma(\eta) + \eta \in \bar{L}$ (where $X, Y \in T^{0,1}M$ and $\xi, \eta \in \Lambda^{1,0}M$),

$$\begin{aligned} & F(X + \sigma(\xi), Y + \sigma(\eta))(X_0 - i\mathcal{J}_{J,\sigma}X_0) \\ & - \eta(X_0)(i_{X+\sigma(\xi)}F - i\mathcal{J}_{J,\sigma} i_{X+\sigma(\xi)}F) \\ & + \xi(X_0)(i_{Y+\sigma(\eta)}F - i\mathcal{J}_{J,\sigma} i_{Y+\sigma(\eta)}F) = 0. \end{aligned} \tag{45}$$

Using the definition (43) of $\mathcal{J}_{J,\sigma}$ and identifying the vector and covector parts in the above relation, we obtain that (45) is equivalent to

$$\eta(X_0) i_{X+\sigma(\xi)}(F) - \xi(X_0) i_{Y+\sigma(\eta)}F \in \Lambda^{1,0}M \tag{46}$$

together with

$$F(X + \sigma(\xi), Y + \sigma(\eta))(X_0 - iJX_0) = i\Pi\left(\xi(X_0) i_{Y+\sigma(\eta)}F - \eta(X_0) i_{X+\sigma(\xi)}F\right). \tag{47}$$

Taking in (46) $\eta = 0$ we obtain $i_Y F \in \Lambda^{1,0}M$, i.e. F is of type $(1,1)$. This, together with $\sigma(\xi), \sigma(\eta) \in T^{1,0}M$, imply that $i_{\sigma(\xi)}F$ and $i_{\sigma(\eta)}F$ are of type $(0,1)$. Relation (46) is equivalent to

$$F|_{\Lambda^2 T^{1,0}M} = 0,\ \eta(X_0) i_{\sigma(\xi)}F = \xi(X_0) i_{\sigma(\eta)}F. \tag{48}$$

From (48), relation (47) becomes

$$(F(\sigma(\xi), Y) - F(\sigma(\eta), X))\,(X_0 - iJX_0) = i\Pi(\xi(X_0) i_Y F - \eta(X_0) i_X F). \quad (49)$$

We proved that $(\mathcal{J}_{J,\sigma})_W$ is integrable if and only if (48) and (49) hold. We now study these relations. Relation (49) with $\xi = 0$ and $\eta(X_0) = 0$ gives

$$F = 0 \text{ on } T^{0,1}M \wedge \sigma(\mathrm{Ann}(X_0) \cap \Lambda^{1,0}M). \quad (50)$$

If F is of type $(1,1)$, then relation (50) implies the second relation (48), because $\eta(X_0)\xi - \xi(X_0)\eta \in \mathrm{Ann}(X_0) \cap \Lambda^{1,0}M$. Thus, $(\mathcal{J}_J)_W$ is integrable if and only if F is of type $(1,1)$ and relation (49) holds. We assume that F is of type $(1,1)$ and we consider in more detail relation (49):

a) for $\xi = 0$ and $\eta(X_0) = 0$, we saw that it gives (50);

b) for $\xi = 0$ and $\eta(X_0) \neq 0$ it gives

$$\Pi(i_X F) = iF(X, \sigma(\frac{\eta}{\eta(X_0)}))(X_0 - iJX_0),\ \forall X \in T^{0,1}M. \quad (51)$$

Remark that relation (51) implies that

$$i_X F \in \Pi^{-1}(\mathrm{span}_{\mathbb{C}}\{X_0 - iJX_0\}),\ \forall X \in T^{0,1}M \quad (52)$$

and an easy argument shows that the converse is also true, i.e. (51) and (52) are equivalent.

The remaining cases in (49), namely: c) $\xi, \eta \in \Lambda^{1,0}M$ both non-zero with $\xi(X_0) = 0$ and $\eta(X_0) \neq 0$; d) $\xi, \eta \in \mathrm{Ann}(X_0) \cap \Lambda^{1,0}M$ both non-zero; e) $\xi, \eta \in \Lambda^{1,0}M$ with $\xi(X_0) \neq 0$ and $\eta(X_0) \neq 0$, all follow from (50) and (51).

To summarize: we proved that $(\mathcal{J}_{J,\sigma})_W$ is integrable if and only if F is of type $(1,1)$ and relations (50), (51) hold. It is easy to show that these conditions are equivalent to (44). This concludes claim i). Claim ii) follows from claim i), by taking $\Pi = 0$. $\square$

4.1.2 Twist of interpolation between complex and symplectic structures

We now apply Theorem 16 to a family of generalized complex structures, which interpolate between complex and symplectic structures. Let (g, I, J, K)

be an invariant hyper-Kähler structure on M, with Kähler forms ω_I, ω_J and ω_K. For any $t \in [0, \frac{\pi}{2})$, let $\mathcal{J}_t := \sin(t)\mathcal{J}_I + \cos(t)\mathcal{J}_{\omega_J}$. Then $\mathcal{J}_t$ is an (integrable) generalized complex structure (see [12], page 55).

Corollary 20. *The generalized almost complex structure $(\mathcal{J}_t)_W$ is integrable if and only if $F = f(i_{X_0}\omega_K) \wedge (i_{X_0}\omega_J)$, where $f \in C^\infty(M)$ is invariant and $df \wedge (i_{X_0}\omega_K) \wedge (i_{X_0}\omega_J) = 0$.*

Proof. Let $B_t := \tan(t)\omega_K$. As proved in [12], $e^{B_t}\mathcal{J}_t e^{-B_t} = \mathcal{J}_{\sec(t)\omega_J}$. We deduce that the $(1,0)$-bundle of $\mathcal{J}_t$ is $L((TM)^{\mathbb{C}}, -B_t - i\sec(t)\omega_J)$. From Theorem 16, we deduce that $(\mathcal{J}_t)_W$ is integrable if and only if $d^{(F,a)}(B + i\sec(t)\omega_J) = 0$. Using that B and ω_J are closed, and the formula

$$d^{(F,a)}\alpha = d\alpha - \frac{1}{a}F \wedge i_{X_0}\alpha, \quad \forall \alpha \in \Omega^k(M), \tag{53}$$

we obtain that $\mathcal{J}_t$ is integrable if and only if $i_{X_0}(B_t + i\sec(t)\omega_J) \wedge F = 0$, i.e.

$$(i_{X_0}\omega_K) \wedge F = 0, \ (i_{X_0}\omega_J) \wedge F = 0. \tag{54}$$

Relations (54) together with $dF = 0$ imply our claim. □

4.1.3 Twist and conformal change

For the KK correspondence developed later in the paper, we need to understand when the twist of the conformal change of a generalized almost complex structure is integrable. This is done in the next proposition.

Proposition 21. *Let $\mathcal{J}$ be an invariant generalized almost complex structure on M, with $(1,0)$-bundle $L = L(E, \epsilon)$, and $h \in C^\infty(M)$ an invariant non-vanishing function. Then the twist $[\tau_h(\mathcal{J})]_W$ of the conformal change $\tau_h(\mathcal{J})$ of $\mathcal{J}$ by h is integrable if and only if one of the following conditions hold:*

i) for any invariant sections $X + \xi, Y + \eta$ of L, the expression

$$\begin{aligned} &- F(X,Y)X_0 + \eta(X_0)i_X F - \xi(X_0)i_Y F + \frac{2a}{h}\left(X(h)\eta - Y(h)\xi\right) \\ &- \frac{a}{h}(\eta(X) - \xi(Y))dh - a[X + \xi, Y + \eta] \end{aligned} \tag{55}$$

is a section of L;

ii) the bundle E is (F,a)-involutive and

$$d^{(F,a)}\epsilon = \frac{2}{h}\epsilon \wedge dh|_E. \tag{56}$$

Proof. Let L^h be the $(1,0)$-bundle of $\tau_h(\mathcal{J})$ and $\tilde{h} := \frac{1}{h^2}$. Then $X + \xi \in L$ if and only if $X + \tilde{h}\xi \in L^h$. From Theorem 16 i), $[\tau_h(\mathcal{J})]_W$ is integrable if and only if, for any $X + \xi, Y + \eta \in \Gamma(L)^{\mathrm{inv}}$,

$$-F(X,Y)X_0 + (\tilde{h}\eta)(X_0)i_X F - (\tilde{h}\xi)(X_0)i_Y F - a[X + \tilde{h}\xi, Y + \tilde{h}\eta] \in \Gamma(L^h),$$

or

$$\begin{aligned} &- F(X,Y)X_0 + \eta(X_0)i_X F - \xi(X_0)i_Y F - \frac{a}{\tilde{h}}\mathrm{pr}_{T^*}[X + \tilde{h}\xi, Y + \tilde{h}\eta] \\ &- a\mathrm{pr}_T[X + \tilde{h}\xi, Y + \tilde{h}\eta] \in \Gamma(L). \end{aligned} \tag{57}$$

On the other hand,

$$\begin{aligned} &[X + \tilde{h}\xi, Y + \tilde{h}\eta] = [X,Y] + X(\tilde{h})\eta - Y(\tilde{h})\xi + \tilde{h}(\mathcal{L}_X\eta - \mathcal{L}_Y\xi) \\ &- \frac{1}{2}\tilde{h}d(\eta(X) - \xi(Y)) - \frac{1}{2}(\eta(X) - \xi(Y))d\tilde{h}. \end{aligned} \tag{58}$$

Replacing this relation in (57) we obtain (55), as needed. Relation (56) follows from Theorem 16 ii) and $L^h = L(E, \tilde{h}\epsilon)$, using that $d^{(F,a)}(\tilde{h}\epsilon) = (d\tilde{h})|_E \wedge \epsilon + \tilde{h}d^{(F,a)}\epsilon$. $\square$

5 KK correspondence in generalized complex geometry

5.1 Twist of generalized almost Hermitian structures

Let $(G, \mathcal{J})$ be an invariant generalized almost Hermitian structure on M. As usual, we denote by $L_i = L(E_i, \epsilon_i)$ the $(1,0)$-bundles of its generalized almost complex structures $\mathcal{J}_1 = \mathcal{J}$ and $\mathcal{J}_2 = G^{\mathrm{end}}\mathcal{J}$. We define a 2-form

$$\epsilon \in \Gamma(\mathrm{pr}_T(L_1 \cap L_2)^* \wedge (E_1 + E_2)^*)$$

by

$$\epsilon(X, \cdot)|_{E_1} := \epsilon_1(X, \cdot), \quad \epsilon(X, \cdot)|_{E_2} := \epsilon_2(X, \cdot), \tag{59}$$

for any $X \in \mathrm{pr}_T(L_1 \cap L_2)$. The form ϵ is well defined: from (5), $\epsilon_1(X, \cdot) := \xi|_{E_1}$, where $\xi \in (T^*M)^{\mathbb{C}}$ is arbitrary, such that $X + \xi \in L_1$; similarly, $\epsilon_2(X, \cdot) = \eta|_{E_2}$, where $\eta \in (T^*M)^{\mathbb{C}}$ is arbitrary, such that $X + \eta \in L_2$; since $X \in \mathrm{pr}_T(L_1 \cap L_2)$, we can take $\xi = \eta$ and we obtain that $\epsilon_1(X, \cdot)|_{E_1 \cap E_2} = \epsilon_2(X.\cdot)|_{E_1 \cap E_2}$ as claimed. Similarly, the 2-form

$$\tilde{\epsilon} \in \Gamma(\mathrm{pr}_T(L_1 \cap \bar{L}_2)^* \wedge (E_1 + \bar{E}_2)^*)$$

given by

$$\tilde{\epsilon}(X, \cdot)|_{E_1} := \epsilon_1(X, \cdot), \quad \tilde{\epsilon}(X, \cdot)|_{\bar{E}_2} := \bar{\epsilon}_2(X, \cdot), \tag{60}$$

for any $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2)$, is well defined. Above $\bar{\epsilon}_2 \in \Gamma(\Lambda^2 \bar{E}_2^*)$ is defined by

$$\bar{\epsilon}_2(\bar{X}, \bar{Y}) := \overline{\epsilon_2(X, Y)}, \ X, Y \in E_2.$$

Let $h \in C^\infty(M)$ be an invariant, non-vanishing function.

Theorem 22. *In the above setting, the twist $[\tau_h(G, \mathcal{J})]_W$ of the conformal change $\tau_h(G, \mathcal{J})$ of $(G, \mathcal{J})$ by h is generalized Kähler if and only if the following conditions hold:*

i) the bundles E_1, $\mathrm{pr}_T(L_1 \cap L_2)$ and $\mathrm{pr}_T(L_1 \cap \bar{L}_2)$ are (F, a)-involutive. Moreover, for any $X \in \Gamma\mathrm{pr}_T(L_1 \cap L_2)$ and $Y \in \Gamma\mathrm{pr}_T(\bar{L}_1 \cap L_2)$,

$$[X, Y]^{(F,a)} \in \Gamma(E_1 + E_2) \tag{61}$$

and for any $X \in \Gamma\mathrm{pr}_T(L_1 \cap \bar{L}_2)$ and $Y \in \Gamma\mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2)$,

$$[X, Y]^{(F,a)} \in \Gamma(E_1 + \bar{E}_2). \tag{62}$$

ii) the forms ϵ_1, ϵ and $\tilde{\epsilon}$ satisfy the relations

$$\begin{aligned} d^{(F,a)}\epsilon_1 &= \frac{2}{h}\epsilon_1 \wedge dh|_{E_1} \\ d^{(F,a)}\epsilon &= \frac{2}{h}\epsilon \wedge dh \text{ on } \Lambda^2\mathrm{pr}_T(L_1 \cap L_2) \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2) \\ d^{(F,a)}\tilde{\epsilon} &= \frac{2}{h}\tilde{\epsilon} \wedge dh \text{ on } \Lambda^2\mathrm{pr}_T(L_1 \cap \bar{L}_2) \wedge \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2) \end{aligned} \tag{63}$$

(which, owing to i), are well-defined).

Proof. Let $\mathcal{J}_i^h = \tau_h \circ \mathcal{J}_i \circ \tau_h^{-1}$ $(i = 1, 2)$ be the generalized almost complex structures of the conformal change $\tau_h(G, \mathcal{J})$. The $(1,0)$-bundle of $\mathcal{J}_i^h$ is $L_i^h = L(E_i, \tilde{h}\epsilon_i)$, where $\tilde{h} := \frac{1}{h^2}$. Let $(L_i^h)_W$ be the $(1,0)$- bundle of $(\mathcal{J}_i^h)_W$. It is generated by sections of the form u_W, where u is an invariant section of L_i^h. From Gualtieri's characterization of generalized Kähler structures (see Section 2.2), $[\tau_h(G, \mathcal{J})]_W$ is generalized Kähler if and only if the following conditions hold:

a) $(\mathcal{J}_1^h)_W$ is a generalized complex structure;

b) $(L_1^h)_W \cap (L_2^h)_W$ is Courant integrable;

c) $(L_1^h)_W \cap (\bar{L}_2^h)_W$ is Courant integrable.

From Proposition 21, condition a) is equivalent to the (F, a)-involutivity of E_1 and to the first relation (63). From now on we assume that a) holds. Since $X + \xi \in \Gamma(L_i)$ if and only if $X + \tilde{h}\xi \in \Gamma(L_i^h)$, we obtain that condition b) is equivalent to

$$[(X + \tilde{h}\xi)_W, (Y + \tilde{h}\eta)_W] \in \Gamma((L_1^h)_W \cap (L_2^h)_W), \ \forall X + \xi, Y + \eta \in \Gamma(L_1 \cap L_2)^{\mathrm{inv}}. \tag{64}$$

Using Lemma 13, relation (58) and $(L_1^h)_W \cap (L_2^h)_W = (L_1^h \cap L_2^h)_W$, we see that (64) becomes

$$\begin{aligned}&[X, Y] + \frac{F(X,Y)}{a} X_0 + \frac{1}{\tilde{h}}(X(\tilde{h})\eta - Y(\tilde{h})\xi) + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y))\\ &- \frac{1}{2\tilde{h}}(\eta(X) - \xi(Y)) d\tilde{h} - \frac{\eta(X_0)}{a} i_X F + \frac{\xi(X_0)}{a} i_Y F \in \Gamma(L_1 \cap L_2),\end{aligned} \tag{65}$$

for any $X + \xi, Y + \eta \in \Gamma(L_1 \cap L_2)^{\mathrm{inv}}$, which is equivalent to the following two conditions:

• $[X, Y]^{(F,a)}$ is a section of $\mathrm{pr}_T(L_1 \cap L_2)$, for any $X, Y \in \Gamma \mathrm{pr}_T(L_1 \cap L_2)$, i.e. $\mathrm{pr}_T(L_1 \cap L_2)$ is (F, a)-involutive;

• the left hand side of (65) is a section of L_2, i.e. for any $V \in \Gamma(E_2)$,

$$\begin{aligned}&\frac{1}{\tilde{h}}(X(\tilde{h})\eta(V) - Y(\tilde{h})\xi(V)) + (\mathcal{L}_X \eta)(V) - (\mathcal{L}_Y \xi)(V) - \frac{1}{2} V(\eta(X) - \xi(Y))\\ &- \frac{1}{2\tilde{h}}(\eta(X) - \xi(Y)) V(\tilde{h}) - \frac{\eta(X_0)}{a} F(X, V) + \frac{\xi(X_0)}{a} F(Y, V)\\ &= \epsilon_2([X, Y]^{(F,a)}, V).\end{aligned} \tag{66}$$

(From a), the left-hand side of (65) belongs to $\Gamma(L_1)$ as well). Assume now that $\mathrm{pr}_T(L_1 \cap L_2)$ is (F,a)-involutive. Under this assumption we prove that (66) is equivalent to (61) together with the second relation (63). For this, we remark, from $Y+\eta \in \Gamma(L_1 \cap L_2)$ (in particular, $Y+\eta$ is a section of L_2) and $V \in \Gamma(E_2)$,

$$(\mathcal{L}_X \eta)(V) = X(\eta(V)) - \eta([X,V]) = X(\epsilon_2(Y,V)) - \eta([X,V]).$$

Similarly,

$$(\mathcal{L}_Y \xi)(V) = Y(\epsilon_2(X,V)) - \xi([Y,V]).$$

With these relations, (66) becomes

$$\begin{aligned} &-\frac{2}{h}\left(X(h)\epsilon_2(Y,V) + Y(h)\epsilon_2(V,X) + V(h)\epsilon_2(X,Y)\right) \\ &+ X(\epsilon_2(Y,V)) + Y(\epsilon_2(V,X)) + V(\epsilon_2(X,Y)) \\ &+ \eta([V,X]^{(F,a)}) + \xi([Y,V]^{(F,a)}) + \epsilon_2(V,[X,Y]^{(F,a)}) = 0. \end{aligned} \tag{67}$$

Since $X+\xi, Y+\eta \in \Gamma(L_1 \cap L_2)$, the 1-forms ξ and η are determined by X and, respectively, by Y, on $E_1 + E_2$, but outside this bundle they take arbitrary values. Therefore, relation (67) implies that

$$[V,X]^{(F,a)} \in \Gamma(E_1+E_2), \quad \forall X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2),\ V \in \Gamma(E_2), \tag{68}$$

which is equivalent to (61), when $\mathrm{pr}_T(L_1 \cap L_2)$ is (F,a)-involutive (by decomposing $E_2 = \mathrm{pr}_T(L_1 \cap L_2) + \mathrm{pr}_T(\bar{L}_1 \cap L_2)$). Moreover, if $\mathrm{pr}_T(L_1 \cap L_2)$ is (F,a)-involutive and (61) holds, then $d^{(F,a)}\epsilon$ is defined on $\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2) \wedge E_2$ and (67) is equivalent to

$$d^{(F,a)}\epsilon = \frac{2}{h}\epsilon \wedge dh \text{ on } \Lambda^2 \mathrm{pr}_T(L_1 \cap L_2) \wedge E_2. \tag{69}$$

Decomposing $E_2 = \mathrm{pr}_T(L_1 \cap L_2) + \mathrm{pr}_T(\bar{L}_1 \cap L_2)$ again and using the first relation (63) (which holds because condition a) holds) we obtain that (69) is equivalent to the second relation (63).

We proved that if condition a) holds, then condition b) is equivalent to the (F,a)-involutivity of $\mathrm{pr}_T(L_1 \cap L_2)$, together with relation (61) and the second relation (63). A similar argument shows that if condition a) holds, then condition c) is equivalent to the (F,a)-involutivity of $\mathrm{pr}_T(L_1 \cap \bar{L}_2)$, together with relation (62) and the third relation (63). $\square$

5.2 Statement of the KK correspondence

Let $(G, \mathcal{J})$ be an invariant generalized Kähler structure on M, $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = G^{\mathrm{end}}\mathcal{J}$ its generalized complex structures with $(1,0)$-bundles $L_i = L(E_i, \epsilon_i)$ $(i = 1, 2)$. Assume that the vector field X_0 is Hamiltonian Killing on $(M, G, \mathcal{J})$, with Hamiltonian function f^H, and let $f, h \in C^\infty(M)$ be invariant, non-vanishing functions. Let $(G', \mathcal{J})$ be the elementary deformation of $(G, \mathcal{J})$ by X_0 and f. We assume that $f^2 - 1$ is non-vanishing and denote by α the 1-form

$$\alpha := -d\left(\ln \frac{|f^2 - 1|}{f^2 G(X_0, X_0)}\right). \tag{70}$$

Theorem 23. *The twist $[\tau_h(G', \mathcal{J})]_W$ of the conformal change $\tau_h(G', \mathcal{J})$ of $(G', \mathcal{J})$ by h is generalized Kähler if and only if the following conditions i) - v) hold:*

i) The curvature F vanishes on

$$\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \oplus \Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$$

and

$$F(X, Y)X_0 \in \Gamma(E_1), \ \forall X \wedge Y \in \Lambda^2 E_1. \tag{71}$$

ii) For any $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} &[\mathrm{pr}_T(\mathcal{J}_3 X_0), X] + \alpha(X)\mathrm{pr}_T(\mathcal{J}_3 X_0) + \frac{f^2 F(\mathrm{pr}_T(v_f), X)}{a(f^2 - 1)} X_0 \\ &\in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \end{aligned} \tag{72}$$

and for any $X \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} &[\mathrm{pr}_T(\mathcal{J}_3 X_0), X] + \alpha(X)\mathrm{pr}_T(\mathcal{J}_3 X_0) - \frac{f^2 F(\mathrm{pr}_T(v_{if}), X)}{a(f^2 - 1)} X_0 \\ &\in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \end{aligned} \tag{73}$$

iii) For any $X \in \mathrm{pr}_T(\mathcal{S}^{\perp})$,

$$X(af^2h^2) = 0. \tag{74}$$

iv) The following algebraic conditions on ϵ_1 and ϵ_2 hold:

$$(i_{X_0}\epsilon_1) \wedge F = -\frac{2a}{h}\epsilon_1 \wedge dh \text{ on } \Lambda^3 E_1 \tag{75}$$
$$\epsilon_2 \wedge dh = 0 \text{ on } \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}).$$

v) On $\Lambda^2 \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} &d\left(\frac{1-f^2}{f^2 G(X_0,X_0)}\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0)\right) + \frac{f^2-1}{f^2 G(X_0,X_0)}\mathcal{D}_{\mathrm{pr}_T(\mathcal{J}_3 X_0)}\epsilon_2 + \frac{2}{af^2}F \\ &= \frac{2}{hG(X_0,X_0)}\left(\mathrm{pr}_T(v_f)(h)\epsilon_2 + \mathrm{pr}_{T^*}(v_f) \wedge dh\right), \end{aligned} \tag{76}$$

where, for any $X \wedge Y \in \Lambda^2 \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} (\mathcal{D}_{\mathcal{J}_3 X_0}\epsilon_2)(X,Y) :=\ & (\mathrm{pr}_T \mathcal{J}_3 X_0)(\epsilon_2(X,Y)) \\ & - \epsilon_2([\mathrm{pr}_T \mathcal{J}_3 X_0, X] + \alpha(X)\mathrm{pr}_T \mathcal{J}_3 X_0, Y) \\ & - \epsilon_2(X, [\mathrm{pr}_T \mathcal{J}_3 X_0, Y] + \alpha(Y)\mathrm{pr}_T \mathcal{J}_3 X_0). \end{aligned} \tag{77}$$

Now we give more detailed explanations for some of the relations from the above theorem.

Remark 24. i) Relation (71) does not imply, a priori, that $X_0 \in \Gamma(E_1)$ (the form F could vanish on $\Lambda^2 E_1$), but it does imply that the 3-form $(i_{X_0}\epsilon_1) \wedge F$, which appears in the first relation (75), is well defined (in the usual way).

ii) The form $\mathcal{D}_{\mathrm{pr}_T(\mathcal{J}_3 X_0)}\epsilon_2$, as given in (77), is well-defined, owing to condition ii) from Theorem 23: this condition implies that

$$[\mathrm{pr}_T(\mathcal{J}_3 X_0), X] + \alpha(X)\mathrm{pr}_T(\mathcal{J}_3 X_0) \in \Gamma(E_2),\ \forall X \in \Gamma \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$$

(decompose $\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ into the sum of $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $\mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and use that $X_0 \in \Gamma(E_2)$, which holds since $X_0 - i\mathcal{J}_2 X_0 \in \Gamma(L_2)$ and $\mathcal{J}_2 X_0 \in \Omega^1(M)$).

The next sections are devoted to the proof of Theorem 23. The plan is the following. Section 5.2.1 is a preliminary part of the proof. Here we compute various Courant brackets, which will be useful in our argument. In order to prove Theorem 23, we apply Theorem 22 to the generalized almost Hermitian structure $(G', \mathcal{J})$ and the conformal function h. Namely, in Section 5.2.2 we prove that the generalized almost Hermitian structure $(G', \mathcal{J})$ satisfies condition i) from Theorem 22 if and only if conditions i) and ii) from Theorem 23 are satisfied. In Section 5.2.3 we assume that conditions i) and ii) from Theorem 23 are satisfied, and we prove that condition ii) from Theorem 22 (applied to the generalized almost Hermitian structure $(G', \mathcal{J})$ and function h) is equivalent to the remaining conditions iii) - v) from Theorem 23.

5.2.1 Various Courant brackets

We begin by computing the Courant brackets of the canonical basis of $\mathcal{S}$.

Lemma 25. *i) The Courant bracket of X_0 with $\mathcal{J}_i X_0$ ($i = 1, 3$) is given by*

$$L_{X_0}(\mathcal{J} X_0) = L_{X_0}(\mathcal{J}_2 X_0) = 0, \; L_{X_0}(\mathcal{J}_3 X_0) = dG(X_0, X_0). \tag{78}$$

ii) The Courant bracket of $\mathcal{J} X_0$ with $\mathcal{J}_2 X_0$ and $\mathcal{J}_3 X_0$ is given by

$$[\mathcal{J} X_0, \mathcal{J}_2 X_0] = dG(X_0, X_0), \; [\mathcal{J} X_0, \mathcal{J}_3 X_0] = 2\mathcal{J} dG(X_0, X_0). \tag{79}$$

iii) The Courant bracket of $\mathcal{J}_2 X_0$ with $\mathcal{J}_3 X_0$ is trivial:

$$[\mathcal{J}_2 X_0, \mathcal{J}_3 X_0] = 0. \tag{80}$$

Proof. We use relation (9) and $\mathcal{L}_{X_0}(\mathcal{J}) = 0$, $\mathcal{L}_{X_0}(G^{\mathrm{end}}) = 0$ (see Definition 1). The first relation (78) follows from

$$L_{X_0}(\mathcal{J} X_0) = \mathcal{L}_{X_0}(\mathcal{J} X_0) - d\langle X_0, \mathcal{J} X_0 \rangle = 0.$$

In a similar way we obtain the other relations (78). Let us prove (79): from the definition of the Courant bracket and $\mathcal{J}_2 X_0 = df^H$,

$$\begin{aligned} [\mathcal{J} X_0, \mathcal{J}_2 X_0] &= [\mathcal{J} X_0, df^H] = \mathcal{L}_{\mathrm{pr}_T \mathcal{J} X_0}(df^H) - \frac{1}{2} d\left((df^H)(\mathrm{pr}_T \mathcal{J} X_0)\right) \\ &= \frac{1}{2} d\left(df^H(\mathrm{pr}_T \mathcal{J} X_0)\right) = d\langle df^H, \mathcal{J} X_0 \rangle = dG(X_0, X_0), \end{aligned} \tag{81}$$

which is the first relation (79). For the second relation (79) we use that $N_{\mathcal{J}}(X_0, \mathcal{J}_2 X_0) = 0$, i.e.

$$[\mathcal{J} X_0, \mathcal{J}_3 X_0] - [X_0, \mathcal{J}_2 X_0] = \mathcal{J}([\mathcal{J} X_0, \mathcal{J}_2 X_0] + [X_0, \mathcal{J}_3 X_0]). \tag{82}$$

The Courant brackets $[X_0, \mathcal{J}_2 X_0]$ and $[X_0, \mathcal{J}_3 X_0]$ were computed in (78) and the Courant bracket $[\mathcal{J} X_0, \mathcal{J}_2 X_0]$ in (81). Using (82) we obtain the second relation (79). Relation (80) can be proved equally easy. $\square$

Corollary 26. *i) The Courant bracket $[v_f, \bar{v}_{if}]$ is given by*

$$[v_f, \bar{v}_{if}] = -2i\mathrm{pr}_T(v_f)(\frac{1}{f^2})\mathcal{J}_2 X_0 + 4G(X_0, X_0)d(\frac{1}{f^2}). \tag{83}$$

ii) The following relation holds:

$$[\mathrm{pr}_T(v_f), \bar{v}_{if}] - [\mathrm{pr}_T(\bar{v}_{if}), v_f] = -2i\mathrm{pr}_T(v_f)(\frac{1}{f^2})\mathcal{J}_2 X_0 + 4G(X_0, X_0)d(\frac{1}{f^2}). \tag{84}$$

Proof. From the definition of v_{if} and the property (7) of the Courant bracket, we obtain

$$\begin{aligned}[v_f, \bar{v}_{if}] &= -[X_0, v_f] - i[\mathcal{J} X_0, v_f] - \frac{1}{f^2}[\mathcal{J}_3 X_0, v_f] + \pi(v_f)(\frac{1}{f^2})\mathcal{J}_3 X_0 \\ &+ 2G(X_0, X_0)d(\frac{1}{f^2}) + \frac{i}{f^2}[\mathcal{J}_2 X_0, v_f] - i\pi(v_f)(\frac{1}{f^2})\mathcal{J}_2 X_0. \end{aligned} \tag{85}$$

From Lemma 25 we obtain

$$\begin{aligned}[X_0, v_f] &= -d\left(\frac{G(X_0, X_0)}{f^2}\right); \\ [\mathcal{J} X_0, v_f] &= -\frac{1}{f^2}(2\mathcal{J} dG(X_0, X_0) + idG(X_0, X_0)) - \pi(\mathcal{J} X_0)(\frac{1}{f^2})(\mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0) \\ &+ iG(X_0, X_0)d(\frac{1}{f^2}); \\ [\mathcal{J}_2 X_0, v_f] &= idG(X_0, X_0); \\ [\mathcal{J}_3 X_0, v_f] &= -dG(X_0, X_0) + 2i\mathcal{J} dG(X_0, X_0) - \mathrm{pr}_T(\mathcal{J}_3 X_0)(\frac{1}{f^2})(\mathcal{J}_3 X_0 + i\mathcal{J}_2 X_0). \end{aligned}$$

Replacing these relations in (85) we obtain (83). In order to prove (84) we remark, from (83), that $[\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})] = 0$. Since the Courant bracket of any 2-forms is trivial, we obtain that the left hand side of (84) is equal to $[v_f, \bar{v}_{if}]$. From (83) again, we obtain (84). $\square$

Lemma 27. *The Courant bracket L_{X_0} preserves $\Gamma(\mathcal{S}^\perp)$, $\Gamma(L_i \cap \mathcal{S}^\perp_{\mathbb{C}})$ and $\Gamma(\bar{L}_i \cap \mathcal{S}^\perp_{\mathbb{C}})$ $(i = 1, 2)$.*

Proof. We prove the statements which involve $\mathcal{S}^{\perp}$ and L_i (the statements which involve $\bar{L}_i$ can be obtained similarly). Let $w \in \Gamma(\mathcal{S}^{\perp})$. Then $\langle X_0, w\rangle = 0$ and, from (9), $L_{X_0}(w) = \mathcal{L}_{X_0}(w)$. From relation (8) applied to $u := X_0$, $v := w$ and $w := x \in \Gamma(\mathcal{S})$, together with $L_{X_0}(x) = \mathcal{L}_{X_0}(x) - d\langle X_0, x\rangle$, we obtain

$$\langle L_{X_0}(w), x\rangle + \langle w, \mathcal{L}_{X_0}(x)\rangle = 0. \tag{86}$$

For $x \in \{X_0, \mathcal{J}X_0, \mathcal{J}_2X_0, \mathcal{J}_3X_0\}$, $\mathcal{L}_{X_0}(x) = 0$ and from (86) we deduce that $\langle L_{X_0}(w), x\rangle = 0$. We proved that L_{X_0} preserves $\Gamma(\mathcal{S}^{\perp})$. We now prove that L_{X_0} preserves $\Gamma(L_i \cap \mathcal{S}^{\perp}_{\mathbb{C}})$. Without loss of generality, we take $i = 1$ (the argument for $i = 2$ is similar). Let $w \in \Gamma(L_1 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$. As $L_{X_0}(w) \in \Gamma(\mathcal{S}^{\perp}_{\mathbb{C}})$ (from claim i)), we need to show that $L_{X_0}(w) \in \Gamma(L_1)$. Since $\mathcal{L}_{X_0}(\mathcal{J}_1) = 0$, $\mathcal{L}_{X_0}(\mathcal{J}_1 w) = \mathcal{J}_1\mathcal{L}_{X_0}(w)$. But $w, \mathcal{J}_1 w \in \Gamma(\mathcal{S}^{\perp}_{\mathbb{C}})$ which implies that $L_{X_0}(w) = \mathcal{L}_{X_0}(w)$, $L_{X_0}(\mathcal{J}_1 w) = \mathcal{L}_{X_0}(\mathcal{J}_1 w)$. We obtain that $L_{X_0}(\mathcal{J}_1 w) = \mathcal{J}_1 L_{X_0}(w)$. Since $w \in \Gamma(L_1)$, $\mathcal{J}_1 w = iw$ and $iL_{X_0}(w) = \mathcal{J}_1 L_X(w)$, i.e. $L_X(w) \in \Gamma(L_1)$, as required. □

Lemma 28. *i) For any $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$, and*

$$[\mathrm{pr}_T(v_1), X] + \frac{XG(X_0, X_0)}{G(X_0, X_0)}\mathrm{pr}_T(v_1) \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}}). \tag{87}$$

ii) For any $X \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$,

$$[\mathrm{pr}_T(v_i), X] + \frac{XG(X_0, X_0)}{G(X_0, X_0)}\mathrm{pr}_T(v_i) \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}}). \tag{88}$$

Proof. Let $\mathcal{E}(X)$ be the left hand side of (87). Since L_1 and L_2 are Courant integrable and $v_1 \in \Gamma(L_1 \cap L_2)$, we deduce that $\mathcal{E}(X)$ is a section of $\mathrm{pr}_T(L_1 \cap L_2)$. Since $\langle \mathcal{E}(X), df^H\rangle = 0$ (easy check), we obtain (87). Relation (88) can be proved similarly. □

5.2.2 Involutivity of the bundles

Consider the setting from Theorem 23 and let $L_2' = L(E_2', \epsilon_2')$ be the $(1,0)$-bundle of the second generalized almost complex structure $\mathcal{J}_2'$ of $(G', \mathcal{J})$.

Proposition 29. *Condition i) from Theorem 22, applied to the generalized almost Hermitian structure $(G', \mathcal{J})$, holds, if and only if conditions i) and ii) from Theorem 23 hold.*

Part of the statement of Proposition 29 is obvious: since $\mathcal{J}$ is integrable, the bundle E_1 is involutive. We deduce that E_1 is (F, a)-involutive if and only if $F(X, Y)X_0 \in \Gamma(E_1)$, for any $X, Y \in \Gamma(E_1)$, i.e. relation (71) holds. The remaining part of the proof of Proposition 29 is divided into several lemmas, as follows.

Lemma 30. *i) The bundle* $\mathrm{pr}_T(L_1 \cap L_2')$ *is* (F, a)*-involutive if and only if* $F = 0$ *on* $\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and relation (72) holds.*

ii) The bundle $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$ *is* (F, a)*-involutive if and only if* $F = 0$ *on* $\Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and relation (73) holds.*

Proof. Owing to the decomposition of $\mathrm{pr}_T(L_1 \cap L_2')$ given by the first relation (18), the (F, a)-involutivity of $\mathrm{pr}_T(L_1 \cap L_2')$ involves two cases: when both arguments X and Y of $[X, Y]^{(F,a)}$ are sections of $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and, respectively, when one is a section of this bundle and the other is $\mathrm{pr}_T(v_f)$. We begin with the first case. From Lemma 2.1 of [16], $\{df^H\}^{\perp}$ is Courant integrable. Since L_i are Courant integrable, we obtain that also $L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp} = \{df^H\}^{\perp} \cap L_1 \cap L_2$ is Courant integrable, and, in particular, $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ is involutive. If $\mathrm{pr}_T(L_1 \cap L_2')$ is (F, a)-involutive, then

$$F(X, Y)X_0 \in \mathrm{pr}_T(L_1 \cap L_2'),\ \forall X, Y \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \qquad (89)$$

We will show that this relation implies that $F = 0$ on $\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. Suppose that this is not true. Then $X_0 \in \mathrm{pr}_T(L_1 \cap L_2')$ (at least at one point of M; for simplicity, this point will be omitted in our notation) and there is $\lambda \in \mathbb{C}$, $w \in L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}$ and $\xi \in (T^*M)^{\mathbb{C}}$, such that $X_0 = \lambda v_f + w + \xi$, or (from the definition of v_f),

$$(1 - \lambda)X_0 + \lambda i \mathcal{J} X_0 + \frac{\lambda}{f^2}(\mathcal{J}_3 X_0 + i \mathcal{J}_2 X_0) - \xi \in L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}. \qquad (90)$$

In particular, the inner product $\langle \cdot, \cdot \rangle$ of the left hand side of (90) with $\mathcal{J}_2 X_0 = df^H$ vanishes and we obtain $\lambda G(X_0, X_0) = 0$, i.e. $\lambda = 0$ (because G is positive definite). We deduce that $X_0 - \xi \in L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}$. We will now show that this leads to a contradiction. Indeed, since $X_0 - \xi \in L_1 \cap L_2$, we obtain that $X_0 - \xi \in C_+$ (the 1-eigenbundle of G^{end}). Recall now that C_+ is the graph of $b+g$ (where b and g are the 2-forms, respectively the metric of the bi-Hermitian structure associated to $(G,\mathcal{J})$). This implies $\xi = -(b+g)(X_0)$. But then $\langle X_0 - \xi, X_0\rangle = -\frac{1}{2} g(X_0, X_0) \neq 0$ (because g is positive definite) and $X_0 - \xi \notin \mathcal{S}^{\perp}_{\mathbb{C}}$. We obtain a contradiction and we conclude that the first case of the (F,a)-involutivity of $\mathrm{pr}_T(L_1\cap L_2')$ is equivalent to $F=0$ on $\Lambda^2 \mathrm{pr}_T(L_1\cap L_2\cap \mathcal{S}^{\perp}_{\mathbb{C}})$.

We now prove that the second case mentioned above, of the (F,a)-involutivity of $\mathrm{pr}_T(L_1\cap L_2')$, is equivalent to relation (72). For any $X\in \Gamma \mathrm{pr}_T(L_1\cap L_2\cap \mathcal{S}^{\perp}_{\mathbb{C}})$, let

$$\mathcal{F}_f(X) := [\mathrm{pr}_T(v_f), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f) \tag{91}$$

By a standard computation, which uses relation (8), we obtain that $\mathcal{F}_f(X)$ is related to the left hand side $\mathcal{E}(X)$ of (87) by

$$\begin{aligned}\mathcal{F}_f(X) = \mathcal{E}(X) + (1-\frac{1}{f^2})\{[\mathrm{pr}_T(\mathcal{J}_3X_0), X] + \alpha(X)\mathrm{pr}_T(\mathcal{J}_3X_0)\}\\ + \frac{F(\mathrm{pr}_T(v_f), X)}{a}X_0.\end{aligned} \tag{92}$$

We obtain that $[\mathrm{pr}_T(v_f), X]^{(F,a)}\in \Gamma(L_1\cap L_2')$ if and only if $\mathcal{F}_f(X)\in\Gamma(L_1\cap L_2')$, if and only if (from Lemma 28)

$$[\mathrm{pr}_T(\mathcal{J}_3X_0), X] + \alpha(X)\mathrm{pr}_T(\mathcal{J}_3X_0) + \frac{f^2F(\mathrm{pr}_T(v_f), X)}{a(f^2-1)}X_0 \in \Gamma \mathrm{pr}_T(L_1\cap L_2'). \tag{93}$$

In order to conclude the proof of claim i), it remains to show that (93) is equivalent to (72). In order to prove this, let $u\in\Gamma(L_1\cap L_2\cap\mathcal{S}^{\perp}_{\mathbb{C}})$ which projects to X. From (93), there is a 1-form ξ and $\lambda\in\mathbb{C}$ such that

$$[\mathcal{J}_3X_0, u] + \alpha(X)\mathcal{J}_3X_0 \quad + \frac{f^2F(\mathrm{pr}_T(v_f), X)}{a(f^2-1)}X_0 + \lambda v_f + \xi \in \Gamma(L_1\cap L_2\cap\mathcal{S}^{\perp}_{\mathbb{C}}).$$

In particular, the $\langle\cdot,\cdot\rangle$-inner product of the above expression with df^H vanishes. From $\langle [\mathcal{J}_3X_0, u], df^H\rangle = 0$ (which follows from relation (8), with u replaced by $\mathcal{J}_3X_0$, v replaced by u and w replaced by df^H, by using $[df^H, \mathcal{J}_3X_0] = 0$, from Lemma 25), we obtain that $\lambda = 0$, which implies (72). Claim i) follows. Claim ii) can be proved in a similar way. More precisely, one shows that the condition $[X,Y]^{(F,a)} \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}'_2)$, for any $X, Y \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$ is equivalent to $F = 0$ on $\Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$. Then, for any $X \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$, one defines

$$\mathcal{F}'_f(X) := [\mathrm{pr}_T(v_{if}), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)} \mathrm{pr}_T(v_{if}) \tag{94}$$

and shows that it is related to the left hand side $\mathcal{E}'(X)$ of (88) by

$$\begin{aligned}\mathcal{F}'_f(X) = \mathcal{E}'(X) + (\frac{1}{f^2} - 1)\{[\mathrm{pr}_T(\mathcal{J}_3 X_0), X] + \alpha(X) \mathrm{pr}_T(\mathcal{J}_3 X_0)\} \\ + \frac{F(\mathrm{pr}_T(v_{if}), X)}{a} X_0. \end{aligned} \tag{95}$$

From Lemma 28, $\mathcal{E}'(X) \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$. Thus, $[\mathrm{pr}_T(v_{if}), X]^{(F,a)} \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}'_2)$ if and only if

$$[\mathrm{pr}_T(\mathcal{J}_3 X_0), X] + \alpha(X) \mathrm{pr}_T(\mathcal{J}_3 X_0) - \frac{f^2 F(\mathrm{pr}_T(v_{if}), X)}{a(f^2-1)} X_0 \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}'_2) \tag{96}$$

and as before one can show that this is equivalent to (73). $\square$

Remark 31. The proof of the above lemma shows that if $\mathrm{pr}_T(L_1 \cap L'_2)$ and $\mathrm{pr}_T(L_1 \cap \bar{L}'_2)$ are (F,a)-involutive, then, for every $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$,

$$[\mathrm{pr}_T(v_f), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)} \mathrm{pr}_T(v_f) \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}}) \tag{97}$$

and for every $X \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}})$,

$$[\mathrm{pr}_T(v_{if}), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)} \mathrm{pr}_T(v_{if}) \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}^{\perp}_{\mathbb{C}}). \tag{98}$$

Next, we assume that the bundles $\mathrm{pr}_T(L_1 \cap L'_2)$ and $\mathrm{pr}_T(L_1 \cap \bar{L}'_2)$ are (F,a)-involutive. Under these assumptions, we will prove that the conditions (61) and (62) from Theorem 22, applied to L_1 and L'_2, are satisfied. That is, we aim to show that for any $X \in \Gamma \mathrm{pr}_T(L_1 \cap L'_2)$ and $Y \in \Gamma \mathrm{pr}_T(\bar{L}_1 \cap L'_2)$,

$$[X,Y]^{(F,a)} \in \Gamma(E_1 + E'_2) \tag{99}$$

and for any $X \in \Gamma \mathrm{pr}_T(L_1 \cap \bar{L}'_2)$ and $Y \in \Gamma \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}'_2)$,

$$[X,Y]^{(F,a)} \in \Gamma(E_1 + \bar{E}'_2). \tag{100}$$

This will be done in Lemma 32 and in Corollary 33 below, by analysing how the map $(X,Y) \to [X,Y]^{(F,a)}$ behaves with respect to the decompositions of $\mathrm{pr}_T(L_1 \cap L_2')$, $\mathrm{pr}_T(\bar{L}_1 \cap L_2')$ and $E_1 + E_2'$ (for condition (99)) and how it behaves with respect to the decompositions of $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$, $\mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2')$ and $E_1 + \bar{E}_2'$ (for condition (100)). Recall the decompositions (18) of the bundles $\mathrm{pr}_T(L_1 \cap L_2')$ and $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$. By conjugation, $\mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2')$ and $\mathrm{pr}_T(\bar{L}_1 \cap L_2')$ decompose similarly:

$$\begin{aligned} \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2') &= \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(\bar{v}_f)\} + \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \\ \mathrm{pr}_T(\bar{L}_1 \cap L_2') &= \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(\bar{v}_{if})\} + \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \end{aligned} \tag{101}$$

(direct sum decompositions). From Remark 10 i) and $v_f, v_{if} \in L_1$, we obtain

$$\begin{aligned} E_1 + E_2' &= E_1 + \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) + \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(\bar{v}_{if})\} \\ E_1 + \bar{E}_2' &= E_1 + \mathrm{pr}_T(\bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) + \mathrm{span}_{\mathbb{C}}\{\mathrm{pr}_T(\bar{v}_f)\}. \end{aligned} \tag{102}$$

Also,

$$X_0 = \frac{1}{2}\mathrm{pr}_T(v_f + \bar{v}_{if}) \in E_2' \cap \bar{E}_2' \tag{103}$$

and, from Corollary 26 i),

$$[\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})]^{(F,a)} = \frac{F(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))}{a} X_0. \tag{104}$$

Lemma 32. *Suppose that* $\mathrm{pr}_T(L_1 \cap L_2')$ *and* $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$ *are* (F,a)*-involutive. The following statements hold:*

i) For any $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} &[\mathrm{pr}_T(\bar{v}_{if}), X]^{(F,a)} - \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f) - \frac{2F(X_0,X)}{a}X_0 \\ &\in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \end{aligned} \tag{105}$$

ii) For any $X \in \Gamma \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$\begin{aligned} &[\mathrm{pr}_T(v_f), X]^{(F,a)} - \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(\bar{v}_{if}) - \frac{2F(X_0,X)}{a}X_0 \\ &\in \Gamma \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \end{aligned} \tag{106}$$

Proof. Let $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $\mathcal{R}_f(X)$ be the expression from the left hand side of (105). Recall the expression $\mathcal{F}_f(X)$ defined in (91). A straightforward computation shows that

$$\mathcal{R}_f(X) = -\mathcal{F}_f(X) + 2[X_0, X].$$

From Lemmas 27 and (the first relation of) Remark 31, $[X_0, X]$ and $\mathcal{F}_f(X)$ are sections of $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. Relation (105) follows. Relation (106) is obtained similarly, by comparering the the left hand side of (106) with the conjugate of $\mathcal{F}'_f(X)$, defined in (94), and using (the second relation of) Remark 31. □

Corollary 33. *Suppose that* $\mathrm{pr}_T(L_1 \cap L_2')$ *and* $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$ *are* (F, a)*-involutive. Then (99) and (100) are satisfied.*

Proof. From relations (97), (98), (104), (105), (106), their conjugates, and the decompositions (18), (101), (102), it remains to prove two more statements:

a) $[X, Y]^{(F,a)}$ is a section of $E_1 + E_2'$, for any $X \in \Gamma(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Y \in \Gamma(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$;

b) $[X, Y]^{(F,a)}$ is a section of $E_1 + \bar{E}_2'$, for any $X \in \Gamma(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Y \in \Gamma(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$.

To prove these claims, we will show that

$$[X, Y]^{(F,a)} = \mathrm{pr}_T([w_1, w_2]^{\perp}) + \left(\frac{G([w_1, w_2], X_0)}{G(X_0, X_0)} + \frac{F(X, Y)}{a} \right) X_0, \tag{107}$$

for any $w_1 \in \Gamma(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $w_2 \in \Gamma(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. Let $X := \mathrm{pr}_T(w_1)$ and $Y := \mathrm{pr}_T(w_2)$. In order to prove relation (107), we remark that $[w_1, w_2] \in \Gamma(L_2 \cap \{df^H\}^{\perp})$ (as L_2 and $\{df^H\}^{\perp}$ are Courant integrable). This implies, using $df^H = \mathcal{J}_2 X_0$,

$$\begin{aligned} &G([w_1, w_2], \mathcal{J} X_0) = G([w_1, w_2], \mathcal{J}_3 X_0) = 0; \\ &G([w_1, w_2], \mathcal{J}_2 X_0) = -G(\mathcal{J}_2 [w_1, w_2], X_0) = -iG([w_1, w_2], X_0). \end{aligned} \tag{108}$$

From (108) we deduce that

$$[w_1, w_2] = [w_1, w_2]^{\perp} + \frac{G([w_1, w_2], X_0)}{G(X_0, X_0)} (X_0 - i\mathcal{J}_2 X_0), \tag{109}$$

which implies (107). Note that $\mathrm{pr}_T([w_1, w_2]^{\perp}) \in \Gamma \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. From (103) and (107) we obtain

$$[X, Y]^{(F,a)} \in \Gamma(\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) + E_2' \cap \bar{E}_2'). \tag{110}$$

As $\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \subset E_2'$, the statements a) and b) follow from (110) (and its conjugate). $\square$

The proof of Proposition 29 is now completed.

5.2.3 The differential equations on forms

We consider the setting from Theorem 23 and we assume that conditions i) and ii) from this theorem are satisfied. From the previous section, this means that condition i) from Proposition 22, applied to the generalized almost complex structure $(G', \mathcal{J})$, is satisfied. Let $\epsilon' \in \Gamma(\mathrm{pr}_T(L_1 \cap L_2') \wedge (E_1 + E_2'))$ and $\tilde{\epsilon}' \in \Gamma(\mathrm{pr}_T(L_1 \cap \bar{L}_2') \wedge (E_1 + \bar{E}_2'))$ be the two 2-forms associated to $(G', \mathcal{J})$, as at the beginning of Section 5.1. Relations (63) from Theorem 22, applied to the generalized almost Hermitian structure $(G', \mathcal{J})$ and function h, are

$$\begin{aligned}
d^{(F,a)}\epsilon_1 &= \frac{2}{h}\epsilon_1 \wedge dh|_{E_1}\\
d^{(F,a)}\epsilon' &= \frac{2}{h}\epsilon' \wedge dh \text{ on } \Lambda^2 \mathrm{pr}_T(L_1 \cap L_2') \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2')\\
d^{(F,a)}\tilde{\epsilon}' &= \frac{2}{h}\epsilon' \wedge dh \text{ on } \Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2') \wedge \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2').
\end{aligned} \tag{111}$$

In order to conclude the proof of Theorem 23 it remains to show (using the material from the previous section) that relations (111) are equivalent to the remaining conditions iii), iv) and v) from Theorem 23. This will be done in this section. Since $\mathcal{J}$ is integrable, E_1 is involutive and $d\epsilon_1 = 0$. From relation (53) applied to $\alpha := \epsilon_1$, we obtain that the first relation (111) is equivalent to the first relation (75). From now on we assume that these two equivalent relations hold.

Proposition 34. *In this setting, the second and third relation (111) are equivalent to relation (74), the second relation (75) and relation (76).*

We divide the proof of the above proposition into several steps. We begin by computing ϵ' and $\tilde{\epsilon}'$. Recall the decompositions (18) and (102).

Lemma 35. *i) The form* $\epsilon' \in \Gamma((L_1 \cap L_2)^* \wedge (E_1 + E_2')^*)$ *is given by:*

$$\begin{aligned}
&\epsilon'(\mathrm{pr}_T(v_f), X) = \mathrm{pr}_{T^*}(v_f)(X),\ \forall X \in E_1 + \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\\
&\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = \frac{4}{f^2} G(X_0, X_0),\\
&\epsilon'(X,Y) = \epsilon_2(X,Y),\ \forall X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\ Y \in \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\\
&\epsilon'(X,Y) = \epsilon_1(X,Y),\ \forall X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\ Y \in E_1,\\
&\epsilon'(X, \mathrm{pr}_T(\bar{v}_{if})) = -\mathrm{pr}_{T^*}(\bar{v}_{if})(X),\ \forall X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}).
\end{aligned}$$

ii) The form $\tilde{\epsilon}' \in \Gamma((L_1 \cap \bar{L}_2')^* \wedge (E_1 + \bar{E}_2')^*)$ *is given by*

$$\begin{aligned}
&\tilde{\epsilon}'(\mathrm{pr}_T(v_{if}), X) = \mathrm{pr}_{T^*}(v_{if})(X),\ \forall X \in E_1 + \mathrm{pr}_T(\bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\\
&\tilde{\epsilon}'(\mathrm{pr}_T(v_{if}), \mathrm{pr}_T(\bar{v}_f)) = -\frac{4}{f^2} G(X_0, X_0),\\
&\tilde{\epsilon}'(X,Y) = \bar{\epsilon}_2(X,Y),\ \forall X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\ Y \in \mathrm{pr}_T(\bar{L}_2 \cap W_{\mathbb{C}}^{\perp}),\\
&\tilde{\epsilon}'(X,Y) = \epsilon_1(X,Y),\ \forall X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}),\ Y \in E_1,\\
&\tilde{\epsilon}'(X, \mathrm{pr}_T(\bar{v}_f)) = -\mathrm{pr}_{T^*}(\bar{v}_f)(X),\ \forall X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}).
\end{aligned}$$

Proof. The proof is straightforward from definitions. Let us compute for example $\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))$. Using $v_f \in L_1 \cap L_2'$, $\bar{v}_{if} \in \bar{L}_1 \cap L_2'$ and the definition of ϵ', we obtain

$$\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = \epsilon_2'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}) = 2\langle v_f, \mathrm{pr}_T(\bar{v}_{if})\rangle. \tag{112}$$

On the other hand, for any $u, v \in \mathbb{T}M$ orthogonal with respect to $\langle\cdot,\cdot\rangle$, $\langle \mathrm{pr}_T(u), v\rangle + \langle u, \mathrm{pr}_T(v)\rangle = 0$. Replacing in (112) the definitions of v_f and v_{if} and using this remark for $u = v = \mathcal{J}X_0$, $u = v = \mathcal{J}_3 X_0$ and for $u = \mathcal{J}X_0, v = \mathcal{J}_3 X_0$, we obtain that $\langle v_f, \mathrm{pr}_T(\bar{v}_{if})\rangle = \frac{2}{f^2} G(X_0, X_0)$. The required expression for $\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))$ follows from (112). $\square$

For simplicity of notation, let $\mathcal{F}$ be the 2-form on $\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ defined by the left hand side of (76).

Lemma 36. *The* 3*-form* $d^{(F,a)}\epsilon'$ *on* $\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2') \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2')$ *is given by:*

i) for any $X, Y \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Z \in \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\epsilon')(X, Y, Z) = 0; \tag{113}$$

ii) for any $X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\epsilon')(X, \mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = -\frac{4X(af^2)}{af^4} G(X_0, X_0); \tag{114}$$

iii) for any $X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Y \in \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\epsilon')(X, Y, \mathrm{pr}_T(v_f)) = G(X_0, X_0)\mathcal{F}(X, Y); \tag{115}$$

iv) for any $X, Y \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\epsilon')(X, Y, \mathrm{pr}_T(\bar{v}_{if})) = -G(X_0, X_0)\mathcal{F}(X, Y). \tag{116}$$

Proof. For claim i), let $X, Y \in \Gamma\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Z \in \Gamma\mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. Choose $w_1, w_2 \in \Gamma(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ which project to X and Y respectively, and $w_3 \in \Gamma(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ which projects to Z. Since $\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ is involutive (see the proof of Lemma 30) and $F(X, Y) = 0$ (see relation (71)) we obtain that $[X, Y]^{(F,a)} = [X, Y] \in \Gamma\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. From Lemma 35,

$$\epsilon'([X, Y]^{(F,a)}, Z) = \epsilon_2([X, Y], Z). \tag{117}$$

We now compute $\epsilon'([X, Z]^{(F,a)}, Y)$. From relation (109),

$$[X, Z]^{(F,a)} = \mathrm{pr}_T([w_1, w_3]^{\perp}) + \left(\frac{G(X_0, [w_1, w_3])}{G(X_0, X_0)} + \frac{F(X, Z)}{a} \right) X_0. \tag{118}$$

Since $\mathrm{pr}_T([w_1, w_3]^{\perp}) \in \Gamma(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Y \in \Gamma\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$, we obtain

$$\begin{aligned}\epsilon'([X, Z]^{(F,a)}, Y) &= \epsilon_2(\mathrm{pr}_T([w_1, w_3]^{\perp}), Y) \\ &+ \left(\frac{G([w_1, w_3], X_0)}{G(X_0, X_0)} + \frac{F(X, Z)}{a} \right) \epsilon'(X_0, Y).\end{aligned} \tag{119}$$

From Lemma 35 and $Y \in \Gamma\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$2\epsilon'(X_0, Y) = \epsilon'(\mathrm{pr}_T(v_f), Y) + \epsilon'(\mathrm{pr}_T(\bar{v}_{if}), Y) = 2\langle \mathrm{pr}_{T^*}(v_f + \bar{v}_{if}), Y \rangle = 0, \tag{120}$$

where in the last equality we used $\mathrm{pr}_{T^*}(v_f + \bar{v}_{if}) = -\frac{2i}{f^2}\mathcal{J}_2 X_0$ and $Y \in \Gamma\mathrm{pr}_T(\mathcal{S}_{\mathbb{C}}^{\perp})$. We proved that

$$\epsilon'([X,Z]^{(F,a)}, Y) = \epsilon_2([X,Z], Y). \tag{121}$$

Similarly,

$$\epsilon'([Y,Z]^{(F,a)}, X) = \epsilon_2([Y,Z], X). \tag{122}$$

From (117), (121), (122) we obtain

$$(d^{(F,a)}\epsilon')(X,Y,Z) = d\epsilon_2(X,Y,Z) = 0.$$

Claim i) is proved.

We now prove claim ii). Let $X \in \Gamma\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. From Corollary 26 i), $[\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})] = 0$. From relation (120), $\epsilon'(X_0, X) = 0$. We obtain

$$\epsilon'([\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})]^{(F,a)}, X) = 0. \tag{123}$$

Next, we compute $\epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)}, \mathrm{pr}_T(\bar{v}_{if}))$. In order to do this, we add and subtract to $[\mathrm{pr}_T(v_f), X]^{(F,a)}$ the term $\frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f)$:

$$\begin{aligned}\epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)}, \mathrm{pr}_T(\bar{v}_{if})) &= -\frac{XG(X_0,X_0)}{G(X_0,X_0)}\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))\\ &+ \epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})).\end{aligned}$$

From relation (112), $\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = 2\langle v_f, \mathrm{pr}_T(\bar{v}_{if})\rangle$. From relation (97) and Lemma 35, we obtain

$$\begin{aligned}&\epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))\\ &= -2\langle[\mathrm{pr}_T(v_f), X]^{(F,a)} + \frac{XG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(v_f), \mathrm{pr}_{T^*}(\bar{v}_{if})\rangle.\end{aligned}$$

Combining the above relations and using

$$\langle v_f, \mathrm{pr}_T(\bar{v}_{if})\rangle + \langle \mathrm{pr}_T(v_f), \mathrm{pr}_{T^*}(\bar{v}_{if})\rangle = \langle v_f, \bar{v}_{if}\rangle = 0, \tag{124}$$

($v_f, \bar{v}_{if} \in L_2'$ which is isotropic), we obtain

$$\begin{aligned}&\epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)}, \mathrm{pr}_T(\bar{v}_{if}))\\ &= -2\langle[\mathrm{pr}_T(v_f), X], \mathrm{pr}_{T^*}(\bar{v}_{if})\rangle + \frac{2F(\mathrm{pr}_T(v_f), X)}{af^2}G(X_0,X_0).\end{aligned} \tag{125}$$

A similar computation which uses relation (105) shows that

$$\begin{aligned}&\epsilon'([\mathrm{pr}_T(\bar{v}_{if}), X]^{(F,a)}, \mathrm{pr}_T(v_f))\\&= -2\langle [\mathrm{pr}_T(\bar{v}_{if}), X], \mathrm{pr}_{T^*}(v_f)\rangle - \frac{2F(\mathrm{pr}_T(\bar{v}_{if}), X)}{af^2} G(X_0, X_0).\end{aligned} \tag{126}$$

Using Lemma 35 for $\epsilon'(\mathrm{pr}_T(v_f), X)$, $\epsilon'(\mathrm{pr}_T(\bar{v}_{if}), X)$ and $\epsilon'(\mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if}))$, together with relations (123), (125) and (126), we obtain

$$\begin{aligned}&(d^{(F,a)}\epsilon')(X, \mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = -2\langle [\mathrm{pr}_T(\bar{v}_{if}), v_f] - [\mathrm{pr}_T(v_f), \bar{v}_{if}], X\rangle\\&+ \frac{4}{af^2} F(X_0, X) G(X_0, X_0).\end{aligned}$$

From Corollary 26 ii), $i_{X_0}F = -da$ and $X \in \Gamma \mathrm{pr}_T(\mathcal{S}_{\mathbb{C}}^{\perp})$ we obtain claim ii).

Claims iii) and iv) can be proved in a similar way. We only sketch the proof of claim iii). Let $X \in \Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Y \in \Gamma \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. For computing $\epsilon'([\mathrm{pr}_T(v_f), Y]^{(F,a)}, X)$ we use relation (106) (by adding and substracting the term $\frac{YG(X_0,X_0)}{G(X_0,X_0)}\mathrm{pr}_T(\bar{v}_{if}) + \frac{2F(X_0,Y)}{a}X_0$), for computing $\epsilon'([\mathrm{pr}_T(v_f), X]^{(F,a)}, Y)$ we use, as above, relation (97) and for computing $\epsilon'([X, Y]^{(F,a)}, \mathrm{pr}_T(v_f))$ we use (118). Using Lemma 35 for $\epsilon'(X, Y)$, $\epsilon'(X, \mathrm{pr}_T(v_f))$ and $\epsilon'(Y, \mathrm{pr}_T(v_f))$ we finally obtain

$$\begin{aligned}&(d^{(F,a)}\epsilon')(X, Y, \mathrm{pr}_T(v_f)) = -d(\mathrm{pr}_{T^*}(v_f))(X, Y) + (\mathcal{D}_{\mathrm{pr}_T(v_f)}\epsilon_2)(X, Y)\\&- (\frac{dG(X_0, X_0)}{G(X_0, X_0)} \wedge \mathrm{pr}_{T^*}(\bar{v}_{if}))(X, Y) + \frac{2F(X, Y)}{af^2} G(X_0, X_0),\end{aligned} \tag{127}$$

where $\mathcal{D}_{\mathrm{pr}_T(v_f)}(\epsilon_2)(X, Y)$ is defined by

$$\begin{aligned}&(\mathcal{D}_{\mathrm{pr}_T(v_f)}\epsilon_2)(X, Y) := \mathrm{pr}_T(v_f)(\epsilon_2(X, Y))\\&- \epsilon_2([\mathrm{pr}_T(v_f), X] + \frac{XG(X_0, X_0)}{G(X_0, X_0)}\mathrm{pr}_T(v_f), Y)\\&- \epsilon_2(X, [\mathrm{pr}_T(v_f), Y] - \frac{YG(X_0, X_0)}{G(X_0, X_0)}\mathrm{pr}_T(\bar{v}_{if})).\end{aligned} \tag{128}$$

(Remark that $\mathcal{D}_{\mathrm{pr}_T(v_f)}\epsilon_2$ is well-defined, owing to (97), (106) and $X_0 \in \Gamma(E_2)$, which ensure that the maps

$$\Gamma \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \ni X \to [\mathrm{pr}_T(v_f), X] + \frac{XG(X_0, X_0)}{G(X_0, X_0)}\mathrm{pr}_T(v_f)$$

and

$$\Gamma \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \ni Y \to [\mathrm{pr}_T(v_f), Y] - \frac{YG(X_0, X_0)}{G(X_0, X_0)} \mathrm{pr}_T(\bar{v}_{if})$$

take values in $\Gamma(E_2)$). In particular, taking in (127) $f := 1$ and $F := 0$, we obtain

$$\begin{aligned}(d\epsilon_2)(X, Y, \mathrm{pr}_T(v_1)) &= -d(\mathrm{pr}_{T^*}(v_1))(X, Y) + (\mathcal{D}_{\mathrm{pr}_T(v_1)}\epsilon_2)(X, Y) \\ &\quad - (\frac{dG(X_0, X_0)}{G(X_0, X_0)} \wedge \mathrm{pr}_{T^*}(\bar{v}_i))(X, Y),\end{aligned} \tag{129}$$

where $(\mathcal{D}_{\mathrm{pr}_T(v_1)}\epsilon_2)(X, Y)$ is defined by (128), by replacing v_f with v_1 and $\bar{v}_{if}$ with $\bar{v}_i$. But since $d\epsilon_2 = 0$, the right hand side of (129) vanishes. Substracting the right hand side of (129) from (127) we obtain, after a straightforward computation, claim iii). □

By similar computations we obtain $d^{(F,a)}\tilde{\epsilon}'$. Below $\bar{\mathcal{F}}$ is a 2-form on $\mathrm{pr}_T(\bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$, defined by $\bar{\mathcal{F}}(X, Y) := \overline{\mathcal{F}(\bar{X}, \bar{Y})}$, for any $X, Y \in \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$.

Lemma 37. *The 3-form $d^{(F,a)}\tilde{\epsilon}'$ on $\Lambda^2\mathrm{pr}_T(L_1 \cap \bar{L}_2') \wedge \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2')$ is given by:*

i) for any $X, Y \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Z \in \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\tilde{\epsilon}')(X, Y, Z) = 0; \tag{130}$$

ii) for any $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\tilde{\epsilon}')(X, \mathrm{pr}_T(v_{if}), \mathrm{pr}_T(\bar{v}_f)) = \frac{4X(af^2)}{af^4} G(X_0, X_0); \tag{131}$$

iii) for any $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ and $Y \in \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\tilde{\epsilon}')(X, Y, \mathrm{pr}_T(v_{if})) = -G(X_0, X_0)\bar{\mathcal{F}}(X, Y); \tag{132}$$

iv) for any $X, Y \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(d^{(F,a)}\tilde{\epsilon}')(X, Y, \mathrm{pr}_T(\bar{v}_f)) = G(X_0, X_0)\bar{\mathcal{F}}(X, Y); \tag{133}$$

The next two lemmas collect the exterior products $dh \wedge \epsilon'$ and $dh \wedge \tilde{\epsilon}'$. The proofs are straighforward computations.

Lemma 38. *The* 3*-form* $dh \wedge \epsilon'$ *on* $\Lambda^2 \mathrm{pr}_T(L_1 \cap L_2') \wedge \mathrm{pr}_T(\bar{L}_1 \wedge L_2')$ *is given by:*

i) for any $X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \epsilon')(X, \mathrm{pr}_T(v_f), \mathrm{pr}_T(\bar{v}_{if})) = \frac{4G(X_0, X_0)}{f^2} X(h); \tag{134}$$

ii) for any $X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Y \in \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \epsilon')(X, Y, \mathrm{pr}_T(v_f)) = \mathrm{pr}_T(v_f)(h)\epsilon_2(X, Y) + (\mathrm{pr}_{T^*}(v_f) \wedge dh)(X, Y); \tag{135}$$

iii) for any $X, Y \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \epsilon')(X, Y, \mathrm{pr}_T(\bar{v}_{if})) = \mathrm{pr}_T(\bar{v}_{if})(h)\epsilon_2(X, Y) + (\mathrm{pr}_{T^*}(\bar{v}_{if}) \wedge dh)(X, Y); \tag{136}$$

iv) for any $X, Y \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Z \in \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \epsilon')(X, Y, Z) = (dh \wedge \epsilon_2)(X, Y, Z).$$

Similarly:

Lemma 39. *The* 3*-form* $dh \wedge \tilde{\epsilon}'$ *on* $\Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2') \wedge \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2')$ *is given by:*

i) for any $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$.

$$(dh \wedge \tilde{\epsilon}')(X, \mathrm{pr}_T(v_{if}), \mathrm{pr}_T(\bar{v}_f)) = -\frac{4G(X_0, X_0)}{f^2} X(h); \tag{137}$$

ii) for any $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Y \in \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \tilde{\epsilon}')(X, Y, \mathrm{pr}_T(v_{if})) = \mathrm{pr}_T(v_{if})(h)\bar{\epsilon}_2(X, Y) + (\mathrm{pr}_{T^*}(v_{if}) \wedge dh)(X, Y); \tag{138}$$

iii) for any $X, Y \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \tilde{\epsilon}')(X, Y, \mathrm{pr}_T(\bar{v}_f)) = \mathrm{pr}_T(\bar{v}_f)(h)\bar{\epsilon}_2(X, Y) + (\mathrm{pr}_{T^*}(\bar{v}_f) \wedge dh)(X, Y); \tag{139}$$

iv) for any $X, Y \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$ *and* $Z \in \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$,

$$(dh \wedge \tilde{\epsilon}')(X, Y, Z) = (dh \wedge \bar{\epsilon}_2)(X, Y, Z).$$

Using the above lemmas, we now prove Proposition 34 as follows. Suppose that the second and third relation (111) hold. We apply these relations to various types of arguments, according to the decomposition of $\mathrm{pr}_T(L_1 \cap L_2')$, $\mathrm{pr}_T(\bar{L}_1 \cap L_2')$, $\mathrm{pr}_T(L_1 \cap \bar{L}_2')$ and $\mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2')$, and we use Lemmas 36, 37, 38 and 39, to obtain relation (74), the second relation (75) and relation (76).

Let us explain how we obtain relation (74). From the second relation (111), Lemma 36 ii) and Lemma 38 i), we obtain

$$X(af^2h^2) = 0, \tag{140}$$

for any $X \in \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. From the third relation (75), Lemma 37 ii) and Lemma 39 i), we obtain the same relation (140), with $X \in \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp})$. Therefore, (140) holds on

$$\mathrm{pr}_T(L_1 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) = \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) + \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}).$$

By conjugation, using that $L_1 + \bar{L}_1 = (TM)^{\mathbb{C}}$, we obtain (140) for any $X \in \mathrm{pr}(\mathcal{S}^{\perp})$. Relation (74) follows.

We now prove the second relation (75). From the second and third relations (111), Lemma 36 i) together with Lemma 38 iv), and, respectively, Lemma 37 i) together with Lemma 39 iv), we obtain

$$dh \wedge \epsilon_2 = 0 \text{ on } \Lambda^2 \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \tag{141}$$

and, respectively,

$$dh \wedge \bar{\epsilon}_2 = 0 \text{ on } \Lambda^2 \mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(\bar{L}_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}). \tag{142}$$

Relation (141) and the conjugate of (142) give $dh \wedge \epsilon_2 = 0$ on

$$\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) \wedge \mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}).$$

The second relation (75) follows. Finally, one can check that the second and third relation (111) and the remaining statements from Lemmas 36, 37, 38 and 39 give (76). The proof of Proposition 34 and Theorem 23 is completed.

6 Applications, comments and examples

As a first application of Theorem 23 , we recover the KK correspondence from the Kähler setting.

Proposition 40. *Let (M, g, J) be a Kähler manifold with a Hamiltonian Killing vector field X_0. Assume, for simplicity, that the Hamiltonian function f^H of X_0 is positive. Let $X_0^\flat = g(X_0, \cdot)$ be the 1-form dual to X_0. Then the data formed by $F := \omega - \frac{1}{2} dX_0^\flat$ together with*

$$a := -\left(f^H + \frac{g(X_0, X_0)}{2}\right), \ f^2 := \frac{2f^H}{2f^H + g(X_0, X_0)}, \ h^2 := f^H \tag{143}$$

satisfies all the conditions from Theorem 23 and gives rise to a Kähler manifold $[\tau_h(g', J)]_W$. (The assignment $(M, g, J, X_0) \to [\tau_h(g', J)]_W$ is known as the KK correspondence *[1, 2].)*

Proof. We consider (M, g, J) as a generalized Kähler manifold and we apply Theorem 23. As $\mathcal{J}_3 X_0 = -X_0^\flat$ projects trivially on TM and $F = 0$ on $\Lambda^2 T^{1,0} M$ (because X_0 is Hamiltonian Killing), the first two conditions of Theorem 23 are satisfied. Since $af^2 h^2 = -(f^H)^2$, its differential annihilates $\mathrm{pr}_T(\mathcal{S}^\perp) = \mathrm{Ker}\{df^H, df^H \circ J\}$ and condition iii) from Theorem 23 is satisfied. The first relation (75) is trivially satisfied ($\epsilon_1 = 0$), and the second relation (75) is satisfied as well, because h depends only on f^H and therefore dh annihilates $\mathrm{pr}_T(\mathcal{S}^\perp)$. Relation (76) follows from the definition of F. $\square$

In order to apply Theorem 23 in the generalized (non-Kähler) setting, it is natural to look, in view of condition i) from this theorem, for examples with $F = 0$ on $\Lambda^2 E_1$. But when $F = 0$ on $\Lambda^2 E_1$, the first relation (75) from Theorem 23 becomes $\epsilon_1 \wedge dh = 0$ on $\Lambda^3 E_1$. When the function h is not constant, this imposes strong restrictions on the generalized Kähler manifold, as the next lemma shows.

Lemma 41. *Let $(M, G, \mathcal{J})$ be a generalized Kähler manifold and $L(E_1, \epsilon_1)$ the $(1, 0)$-bundle of $\mathcal{J}$. If there is a form $\beta \in \Omega^1(M)$ (non-trivial at any point), such that $\epsilon_1 \wedge \beta = 0$ on $\Lambda^3 E_1$, then either $(G, \mathcal{J})$ is the B-field transformation of a Kähler structure or* $\mathrm{rank}(\Delta_1) = 2$*, where Δ_1 is the set of real points of $(\Delta_1)^{\mathbb{C}} := E_1 \cap \bar{E}_1$.*

Proof. The proof is simple and we skip the details. Assume that $\mathrm{rank}(\Delta_1) \neq 2$. The condition $\epsilon_1 \wedge \beta = 0$ implies that $\Delta_1 = 0$, i.e. $\mathcal{J}$ is the B-field transformation of a complex structure. From Remark 7 ii), $(G, \mathcal{J})$ is the B-field transformation of a Kähler structure. $\square$

In Proposition 43 below we apply Theorem 23, with $h := 1$, to produce new generalized Kähler manifolds from given ones (not necessarily Kähler). We need the following lemma.

Lemma 42. *Let $(M, G, \mathcal{J})$ be a generalized Kähler manifold, with a Hamiltonian Killing vector field X_0, such that $\mathcal{J}_3 X_0 \in \Omega^1(M)$. Then $d(\mathcal{J}_3 X_0) = 0$ on $\Lambda^2 E_1$, where $L(E_1, \epsilon_1)$ is the $(1,0)$-bundle of $\mathcal{J}$.*

Proof. From the Cartan formula for $d(\mathcal{J}_3 X_0)$, we obtain (using $\mathcal{J}_3 X_0 \in \Omega^1(M)$), for any $u, v \in \Gamma(\mathbb{T}M)$,

$$d(\mathcal{J}_3 X_0)(\mathrm{pr}_T(u), \mathrm{pr}_T(v)) = -2\left(\mathrm{pr}_T(u)\langle df^H, \mathcal{J}v\rangle - \mathrm{pr}_T(v)\langle df^H, \mathcal{J}u\rangle\right) + 2G(X_0, [u, v]). \tag{144}$$

From (8), with v replaced by df^H and w replaced by $\mathcal{J}v$, we obtain

$$\mathrm{pr}_T(u)\langle df^H, \mathcal{J}v\rangle = \langle [u, df^H], \mathcal{J}v\rangle + \langle df^H, [u, \mathcal{J}v]\rangle + \langle d\langle u, df^H\rangle, \mathcal{J}v\rangle.$$

On the other hand, from the definition of the Courant bracket, $[u, df^H] = \frac{1}{2} d(\mathrm{pr}_T(u)(f^H))$ and we deduce that

$$\mathrm{pr}_T(u)\langle df^H, \mathcal{J}v\rangle = \frac{1}{2}\mathrm{pr}_T(u)\mathrm{pr}_T(\mathcal{J}v)(f^H).$$

Combining this relation with (144) we obtain

$$d(\mathcal{J}_3 X_0)(\mathrm{pr}_T(u), \mathrm{pr}_T(v)) = -\mathrm{pr}_T(u)\mathrm{pr}_T(\mathcal{J}v)(f^H) + \mathrm{pr}_T(v)\mathrm{pr}_T(\mathcal{J}u)(f^H) + 2\langle df^H, \mathcal{J}[u, v]\rangle. \tag{145}$$

Replacing in (145) u, v by $\mathcal{J}u$, $\mathcal{J}v$ and using $N_{\mathcal{J}}(u, v) = 0$ we obtain

$$d(\mathcal{J}_3 X_0)(\mathrm{pr}_T(\mathcal{J}u), \mathrm{pr}_T(\mathcal{J}v)) = d(\mathcal{J}_3 X_0)(\mathrm{pr}_T(u), \mathrm{pr}_T(v)),$$

which implies our claim. $\square$

Proposition 43. *Let $(M, G, \mathcal{J})$ be a generalized Kähler manifold with a Hamiltonian Killing vector field X_0, such that $\mathcal{J}_3 X_0 \in \Omega^1(M)$. Consider the elementary deformation $(G', \mathcal{J})$ of $(G, \mathcal{J})$ by the vector field X_0 and function*

$$f := \frac{1}{(1 + K(f^H)G(X_0, X_0))^{1/2}},$$

where $K = K(f^H)$ is any smooth, non-negative function, depending only on the Hamiltonian function f^H of X_0. Consider the twist data with curvature F and function a given by

$$F := -\frac{1}{2} d(K(f^H)\mathcal{J}_3 X_0), \ a := 1 + K(f^H)G(X_0, X_0). \tag{146}$$

Then the twist $[(\mathcal{J}, G')]_W$ is generalized Kähler.

Proof. Since $\mathcal{J}_3 X_0 \in \Omega^1(M)$, for any $u, v \in \mathbb{T}M$,

$$(df^H \wedge \mathcal{J}_3 X_0)(\mathrm{pr}_T(u), \mathrm{pr}_T(v)) = 4\left(\langle df^H, u\rangle\langle \mathcal{J}_3 X_0, v\rangle - \langle df^H, v\rangle\langle \mathcal{J}_3 X_0, u\rangle\right)$$

and

$$(df^H \wedge \mathcal{J}_3 X_0)(\mathrm{pr}_T(\mathcal{J}u), \mathrm{pr}_T(\mathcal{J}v)) = (df^H \wedge \mathcal{J}_3)(\mathrm{pr}_T(u), \mathrm{pr}_T(v)),$$

i.e. $df^H \wedge \mathcal{J}_3 X_0 = 0$ on $\Lambda^2 E_1$. From Lemma 42, also $d(\mathcal{J}_3 X_0) = 0$ on $\Lambda^2 E_1$. We deduce that $F = 0$ on $\Lambda^2 E_1$. The 1-form α defined by (70) is given by $\alpha = -\frac{K'(f^H)}{K(f^H)} df^H$, it annihilates $\mathrm{pr}_T(\mathcal{S}^\perp)$, and $af^2 = 1$. The conditions from Theorem 23 can be checked easily. (We remark that relation (76) holds on the entire $\Lambda^2(TM)^{\mathbb{C}}$, not only on $\Lambda^2 \mathrm{pr}_T(L_2 \cap \mathcal{S}^\perp_{\mathbb{C}})$). ◻

Examples of generalized Kähler manifolds $(M, G, \mathcal{J})$ with a Hamiltonian Killing vector field X_0 satisfying $\mathcal{J}_3 X_0 \in \Omega^1(M)$ can be found in the toric setting, which will be treated in the next section (see Example 46).

6.1 Examples: toric generalized Kähler manifolds

Following [4], we briefly recall the local description of diagonal toric generalized Kähler structures, in terms of a strictly convex function τ (the symplectic potential) and a skew-symmetric matrix C. This is a generalization of the local description of toric Kähler structures, which can be recovered as a particular case, when $C = 0$.

Let $(M, \omega, \mathbb{T}^n)$ be a $2n$-dimensional toric symplectic manifold, that is, a symplectic manifold M of dimension $2n$ with a Hamiltonian action of the torus $\mathbb{T}^n$. Recall from Section 2.2 that generalized complex structures $\mathcal{J}$, with the property that $(\mathcal{J}, \mathcal{J}_\omega)$ is a generalized Kähler structure, are in one to one correspondence with complex structures J_+ which tame ω, and whose ω-adjoint $J_+^{*\omega}$ is integrable. We assume that $\mathcal{J}$ (equivalently, J_+) is $\mathbb{T}^n$-invariant and that $J_+\mathcal{K} = J_+^{*\omega}\mathcal{K}$, where $\mathcal{K}$ is the distribution tangent to the orbits of the $\mathbb{T}^n$-action. Such toric generalized Kähler structures are called diagonal. We restrict to the open subset (also denoted by M) of M where $\mathcal{K}$ has constant rank n. Choose a basis of the Lie algebra of $\mathbb{T}^n$ and let $K_i \in \mathfrak{X}(M)$ be the associated fundamental vector fields generated by the $\mathbb{T}^n$-action. As $\{K_i, J_+K_j\}$ commute, we may choose local coordinates (t^i, u^i), such that

$$K_i = -\frac{\partial}{\partial t^i},\ J_+K_i = \frac{\partial}{\partial u^i},\quad 1 \le i \le n.$$

For any $1 \le i \le n$, let μ^i be the moment map of K_i: $i_{K_i}\omega = d\mu^i$. As $\mathrm{span}_{\mathbb{R}}\{d\mu^i\} = \mathrm{span}_{\mathbb{R}}\{du^i\}$, $du^i = \sum_{j=1}^n \Psi_{ij} d\mu^j$, for some functions Ψ_{ij} which depend only on the moment coordinates $\{\mu^i\}$. In the coordinate system (t^i, μ^i), ω, J_+ and $J_- = -J_+^{*\omega}$ are given by:

$$\omega = \sum_{i=1}^n d\mu^i \wedge dt^i, \tag{147}$$

and

$$\begin{aligned} J_+ &= \sum_{i,j=1}^n \Psi_{ij}\frac{\partial}{\partial t^i} \otimes d\mu^j - \sum_{i,j=1}^n \Psi^{ij}\frac{\partial}{\partial \mu^i} \otimes dt^j \\ J_- &= \sum_{i,j=1}^n \Psi_{ji}\frac{\partial}{\partial t^i} \otimes d\mu^j - \sum_{i,j=1}^n \Psi^{ji}\frac{\partial}{\partial \mu^i} \otimes dt^j, \end{aligned} \tag{148}$$

where $(\Psi^{ij}) := (\Psi^{-1})_{ij}$. Since $J_\pm$ are integrable, $\Psi = (\Psi_{ij})$ is of the form

$$\Psi_{ij} = \frac{\partial^2 \tau}{\partial \mu^i \partial \mu^j} + C_{ij}, \tag{149}$$

where $\tau = \tau(\mu^i)$ is strictly convex (i.e. has positive definite Hessian) and $C := (C_{ij})$ is a (constant) skew-symmetric matrix (see Theorem 6 of [4]). Conversely, any strictly convex function $\tau = \tau(\mu^i)$ together with a skew-symmetric matrix C, define, via (147), (148) and (149), a diagonal generalized Kähler structure. The function τ is referred to as the symplectic potential.

Notation 44. i) All toric generalized Kähler manifolds we are concerned with are diagonal. To simplify terminology, from now on we omit the word 'diagonal' when referring to them.

ii) The superscripts "s" and "a" used below mean the symmetric, respectively, the skew-symmetric parts of a matrix.

Lemma 45. *The Hamiltonian Killing vector field $X_0 := K_1 = -\frac{\partial}{\partial t^1}$ on the the toric generalized Kähler manifold associated to (M, J_+, ω) satisfies*

$$\begin{aligned}
\mathrm{pr}_T(\mathcal{J}_3 X_0) &= \sum_{r=1}^{n} \left((\Psi^{-1})^a[(\Psi^{-1})^s]^{-1}\right)_{1r} \frac{\partial}{\partial t^r} \\
\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0) &= \sum_{r=1}^{n} \left((\Psi^{-1})^s - (\Psi^{-1})^a[(\Psi^{-1})^s]^{-1}(\Psi^{-1})^a\right)_{r1} dt^r \\
G(X_0, X_0) &= \frac{1}{2}\left(\Psi^{-1} - (\Psi^{-1})^a[(\Psi^{-1})^s]^{-1}(\Psi^{-1})^a\right)_{11}.
\end{aligned} \tag{150}$$

Proof. Recall that $\mathcal{J}_3 = -G^{\mathrm{end}}$ and that G^{end} is given by (11), in terms of the Riemannian metric g and 2-form b of the generalized Kähler structure. The first two relations (150) follow from the expressions of g and b given by (14), combined with (147) and (148). The expression of $G(X_0, X_0)$ is computed from $\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0)$, using $G(X_0, X_0) = \frac{1}{2}\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0)(\frac{\partial}{\partial t^1})$. □

Using Lemma 45, we construct examples of toric generalized Kähler (non-Kähler) manifolds, with a Hamiltonian Killing vector field X_0, for which $\mathcal{J}_3 X_0$ is a 1-form, as required by Proposition 43. We remark that such examples do not exist in four dimensions: with $X_0 = -\frac{\partial}{\partial t^1}$, we obtain, from the first relation (150),

$$\mathrm{pr}_T(\mathcal{J}_3 X_0) = -\frac{C_{12}}{\det(\Psi) - C_{12}^2}\left((\Psi_{12})^s \frac{\partial}{\partial t^1} + \Psi_{22}\frac{\partial}{\partial t^2}\right), \tag{151}$$

where $(\Psi_{jk})^s := (\Psi^s)_{jk}$. We deduce that $\mathrm{pr}_T(\mathcal{J}_3 X_0) = 0$ if and only if $C_{12} = 0$, i.e. the generalized Kähler manifold is Kähler. The four dimensional case will be treated separately in the next section.

Example 46. i) Let $\tau : (\mathbb{R}^{>0})^n \to \mathbb{R}$, defined by $\tau = \sum_{j=1}^n \mu^j \log(\mu^j)$, be the symplectic potential of the standard Kähler metric on $\mathbb{C}^n$ $(n \geq 3)$. Let $C := (C_{ij}) \in M_n(\mathbb{R})$ be any skew-symmetric matrix, with $C_{1j} = C_{j1} = 0$, for any $1 \leq j \leq n$. With these choices, the matrix function Ψ defined by (149) satisfies $(\Psi^{-1})_{1i} = (\Psi^{-1})_{i1} = 0$, for any $2 \leq i \leq n$, and the Hamiltonian Killing vector field $X_0 = -\frac{\partial}{\partial t^1}$ on the toric generalized Kähler manifold defined by τ and C satisfies $\mathrm{pr}_T(\mathcal{J}_3 X_0) = 0$.

ii) For a six-dimensional toric generalized Kähler manifold, with symplectic potential τ and skew-symmetric matrix C, the vector field $X_0 = -\frac{\partial}{\partial t^1}$ satisfies $\mathrm{pr}_T(\mathcal{J}_3 X_0) = 0$ if and only if

$$C_{12}\frac{\partial \tau}{\partial \mu^3} - C_{13}\frac{\partial \tau}{\partial \mu^2} + C_{23}\frac{\partial \tau}{\partial \mu^1} \tag{152}$$

depends only on μ^1. This is the case if and only if

a) $C = 0$ or

b) τ admits a separation of variables $\tau = \tau_1(\mu^1) + \tau_2(\mu^2, \mu^3)$ and $C_{12} = C_{13} = 0$.

For example, the symplectic potential

$$\tau := \sum_{i=1}^{3} (\mu^i + c)\log(\mu^i + c) + (c - \mu^2 - \mu^3)\log(c - \mu^2 - \mu^3)$$

defined on

$$\Delta := \{(\mu^1, \mu^2, \mu^3) \in \mathbb{R}^3 : \mu^1 + c > 0,\ \mu^2 + c > 0,\ \mu^3 + c > 0,\ \mu^2 + \mu^3 < c\}$$

(where $c > 0$), together with any skew-symmetric matrix C for which $C_{12} = C_{13} = 0$, satisfy b) and hence (152).

6.2 The four-dimensional case

In this final section we consider the setting formed by a four-dimensional toric generalized Kähler (non-Kähler) manifold $(M, G, \mathcal{J})$, defined by a symplectic potential $\tau = \tau(\mu^1, \mu^2)$, skew-symmetric matrix $C = (C_{ij})$ and matrix valued function $\Psi = \mathrm{Hess}(\tau) + C$, as in the previous section. As $(M, G, \mathcal{J})$ is non-Kähler, $C_{12} \neq 0$. Let f and h be two non-vanishing functions on M, independent of t^1. Our aim is to describe all such data, together with the twist data (X_0, F, a) with $X_0 := -\frac{\partial}{\partial t^1}$, for which the conditions from Theorem 23 are satisfied. Remark that $(G, \mathcal{J})$, f and h are X_0-invariant. In addition, we assume that h depends only on μ^1 (the moment map of X_0). This hypothesis is natural, in view of the second relation (75).

Proposition 47. *In the above setting, the conditions from Theorem 23 are satisfied if and only if:*

i) the function f depends only on μ^1;

ii) the function a and the curvature form F of the twist data are given by $a = k_0 h^{-2}$ (where $k_0 \in \mathbb{R} \setminus \{0\}$) and

$$F = \frac{2ah'}{h} d\mu^1 \wedge dt^1 + \left(\frac{(\Psi_{12})^s}{\Psi_{22}} (\lambda - \frac{2ah'}{h}) d\mu^1 + \lambda d\mu^2 \right) \wedge dt^2, \tag{153}$$

where $\lambda \in C^\infty(M)$ is independent of t^1;

iii) the following relations hold:

$$\begin{aligned} \frac{\partial}{\partial \mu^2} \left(\frac{(\Psi_{12})^s}{\Psi_{22}} \right) &= -\frac{1}{(f^2 - 1)a} \left(\lambda + \frac{2ah' f^2}{h} \right) \\ \frac{(\Psi_{12})^s}{\Psi_{22}} \frac{\partial \lambda}{\partial \mu^2} - \frac{\partial \lambda}{\partial \mu^1} &= \frac{1}{(f^2 - 1)a} \left(\lambda + \frac{2ah' f^2}{h} \right) \left(\lambda - \frac{2ah'}{h} \right). \end{aligned} \tag{154}$$

Proof. We divide the proof of Proposition 47 in several steps. Let $L = L(E_1, \epsilon_1)$ be the $(1,0)$-bundle of $\mathcal{J}$ and $(J_\pm, \omega, g, b)$ the complex structures, Riemannian metric and 2-form on M associated to $(G, \mathcal{J})$, as in Section 2.2. As $E_1 = (T^{1,0}M)^{J_+} + (T^{1,0}M)^{J_-}$, cf. Section 2.2,

$$\begin{aligned} (T^{1,0}M)^{J_+} &= \mathrm{span}_{\mathbb{C}} \left\{ v_j^+ := \frac{\partial}{\partial \mu^j} - i \Psi_{kj} \frac{\partial}{\partial t^k},\ j = 1, 2 \right\} \\ (T^{1,0}M)^{J_-} &= \mathrm{span}_{\mathbb{C}} \left\{ v_j^- := \frac{\partial}{\partial \mu^j} - i \Psi_{jk} \frac{\partial}{\partial t^k},\ j = 1, 2 \right\} \end{aligned} \tag{155}$$

and $C_{12} \neq 0$, we obtain that $E_1 = (TM)^{\mathbb{C}}$. (Note that here and in the following we are using Einstein's summation convention.) More precisely,

$$\frac{\partial}{\partial t^1} = \frac{i}{2C_{12}}(v_2^+ - v_2^-),\ \frac{\partial}{\partial t^2} = -\frac{i}{2C_{12}}(v_1^+ - v_1^-);$$
$$\frac{\partial}{\partial \mu^1} = (1 + \frac{\Psi_{21}}{2C_{12}})v_1^+ - \frac{\Psi_{21}}{2C_{12}}v_1^- - \frac{\Psi_{11}}{2C_{12}}v_2^+ + \frac{\Psi_{11}}{2C_{12}}v_2^-;$$
$$\frac{\partial}{\partial \mu^2} = \frac{\Psi_{22}}{2C_{12}}v_1^+ - \frac{\Psi_{22}}{2C_{12}}v_1^- + (1 - \frac{\Psi_{12}}{2C_{12}})v_2^+ + \frac{\Psi_{12}}{2C_{12}}v_2^-. \tag{156}$$

Lemma 48. *i) The form ϵ_1 is given by*

$$\begin{aligned}\epsilon_1 = & \frac{1}{C_{12}}\left(dt^1 \wedge dt^2 - \det(\Psi) d\mu^1 \wedge d\mu^2\right) \\ & + i\left((1 + \frac{\Psi_{21}}{C_{12}})dt^1 \wedge d\mu^1 + \frac{\Psi_{22}}{C_{12}}dt^1 \wedge d\mu^2 - \frac{\Psi_{11}}{C_{12}}dt^2 \wedge d\mu^1\right) \\ & + i(1 - \frac{\Psi_{12}}{C_{12}})dt^2 \wedge d\mu^2. \end{aligned} \tag{157}$$

ii) A real 2-form F satisfies the first relation (75), with ϵ_1 given by (157) and $h, a \in C^\infty(M)$, if and only if F is of the form (153), for a function $\lambda \in C^\infty(M)$.

iii) The relation $i_{X_0}F = -da$ holds if and only if $a = k_0 h^{-2}$, where $k_0 \in \mathbb{R} \setminus \{0\}$ is arbitrary.

Proof. From the definition of $J_\pm$ in Section 2.2 it follows that

$$L_1 \cap C_\pm = \{X + (b \pm g)(X),\ X \in (T^{1,0}M)^{J_\pm}\}, \tag{158}$$

which together with $v_j^\pm \in (T^{1,0}M)^{J_\pm}$ implies that $i_{v_j^\pm}\epsilon_1 = (b \pm g)(v_j^\pm)$. From the definition of $v_j^\pm$ and relations (14), (147) and (148),

$$b(v_j^+) = (\Psi_{kj})^a d\mu^k - i\Psi_{kj}(\Psi^{rk})^a dt^r,\ g(v_j^+) = (\Psi_{kj})^s d\mu^k - i\Psi_{kj}(\Psi^{rk})^s dt^r$$
$$b(v_j^-) = (\Psi_{kj})^a d\mu^k - i\Psi_{jk}(\Psi^{rk})^a dt^r,\ g(v_j^-) = (\Psi_{kj})^s d\mu^k - i\Psi_{jk}(\Psi^{rk})^s dt^r,$$

from where we deduce that

$$i_{v_j^+}\epsilon_1 = \Psi_{kj}d\mu^k - idt^j,\ i_{v_j^-}\epsilon_1 = -\Psi_{jk}d\mu^k + idt^j. \tag{159}$$

Combining (156) with (159) we obtain (157). Claim ii) follows from (157), by computing

$$i_{X_0}\epsilon_1 = -\frac{1}{C_{12}}(i(\Psi_{k2})^s d\mu^k + dt^2)$$

and identifying the real and imaginary parts in the first relation (75). Claim iii) can be checked using the expression (153) of F and that $h = h(\mu^1)$. Moreover, $k_0 \neq 0$ as the function a is non-vanishing. □

With the above preliminary lemma, we now proceed to the proof of Proposition 47. The first relation (75) from the statement of Theorem 23 is satisfied if and only if F is of the form (153). From now on, we assume that F is of this form. Then the function a is given as in claim iii) of Lemma 48. Since

$$\mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) = (T^{1,0}M)^{J_+} \cap \mathrm{Ker}\{d\mu^1\}$$
$$\mathrm{pr}_T(L_1 \cap \bar{L}_2 \cap \mathcal{S}_{\mathbb{C}}^{\perp}) = (T^{1,0}M)^{J_-} \cap \mathrm{Ker}\{d\mu^1\}$$

are of rank one, generated by v_2^+ and v_2^- respectively, and $E_1 = (TM)^{\mathbb{C}}$, condition i) from Theorem 23 is satisfied. We now consider condition ii) from this theorem.

Lemma 49. *Condition ii) from Theorem 23 holds, with F given by (153), if and only if f depends only on μ^1 and the first relation (154) is satisfied.*

Proof. Condition ii) from Theorem 23 holds if and only if

$$\mathcal{E}_1 := [\mathrm{pr}_T(\mathcal{J}_3 X_0), v_2^+] + \alpha(v_2^+)\mathrm{pr}_T(\mathcal{J}_3 X_0) + \frac{f^2 F(\mathrm{pr}_T(v_f), v_2^+)}{a(f^2-1)} X_0 \qquad (160)$$

is a multiple of v_2^+, and

$$\mathcal{E}_2 := [\mathrm{pr}_T(\mathcal{J}_3 X_0), v_2^-] + \alpha(v_2^-)\mathrm{pr}_T(\mathcal{J}_3 X_0) - \frac{f^2 F(\mathrm{pr}_T(v_{if}), v_2^-)}{a(f^2-1)} X_0 \qquad (161)$$

is a multiple of v_2^-. From (151) and the definitions of $v_2^{\pm}$, these conditions are equivalent to $\mathcal{E}_1 = \mathcal{E}_2 = 0$ (as $\mathcal{E}_i$ are linear combinations of $\frac{\partial}{\partial t^1}$ and $\frac{\partial}{\partial t^2}$, while $v_2^{\pm}$ involve also $\frac{\partial}{\partial \mu^2}$) or to

$$\frac{\partial E_1}{\mu^2} - \alpha(v_2^+)E_1 + \frac{f^2 F(\mathrm{pr}_T(v_f), v_2^+)}{a(f^2-1)} = 0$$
$$\frac{\partial E_1}{\mu^2} - \alpha(v_2^-)E_1 - \frac{f^2 F(\mathrm{pr}_T(v_{if}), v_2^-)}{a(f^2-1)} = 0$$
$$\frac{\partial E_2}{\partial \mu^2} - \alpha(v_2^+)E_2 = \frac{\partial E_2}{\partial \mu^2} - \alpha(v_2^-)E_2 = 0, \tag{162}$$

where

$$E_1 := -\frac{C_{12}(\Psi_{12})^s}{\det(\Psi) - C_{12}^2};\ E_2 := -\frac{C_{12}\Psi_{22}}{\det(\Psi) - C_{12}^2} \tag{163}$$

are the coordinates of $\mathrm{pr}_T(\mathcal{J}_3 X_0)$. The third relations (162) are equivalent to

$$\alpha(\frac{\partial}{\partial t^1}) = \alpha(\frac{\partial}{\partial t^2}) = 0,\ \alpha(\frac{\partial}{\partial \mu^2}) = \frac{1}{E_2}\frac{\partial E_2}{\partial \mu^2}. \tag{164}$$

Let $H := \frac{f^2-1}{f^2 G(X_0,X_0)}$, so that $\alpha := -\frac{dH}{H}$. The first two relations (164) are equivalent to f - independent of t^i (recall that $G(X_0, X_0)$ is independent of t^i) and the third relation (164) is equivalent to HE_2 - independent of μ^2. From Lemma 45,

$$\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0) = \frac{1}{\det(\Psi) - C_{12}^2}(\Psi_{22} dt^1 - (\Psi_{12})^s dt^2)$$
$$G(X_0, X_0) = \frac{\Psi_{22}}{2(\det(\Psi) - C_{12}^2)}. \tag{165}$$

and from the second relation (165) we obtain that $HE_2 = -\frac{2C_{12}(f^2-1)}{f^2}$. We proved that relations (164) hold if and only if f depends only on μ^1, as required.

We now study the first two relations (162). From (164), $\alpha(v_2^+) = \alpha(v_2^-) = \frac{1}{E_2}\frac{\partial E_2}{\partial \mu^2}$ and the first two relations (162) are equivalent to

$$\mathrm{Im} F(\mathrm{pr}_T(v_f), v_2^+) = 0,\ \mathrm{Im} F(\mathrm{pr}_T(v_{if}), v_2^-) = 0;$$
$$F(\mathrm{pr}_T(v_f), v_2^+) = -F(\mathrm{pr}_T(v_{if}), v_2^-);$$
$$\frac{\partial}{\partial \mu^2}\left(\frac{E_1}{E_2}\right) = -\frac{f^2 \mathrm{Re} F(\mathrm{pr}_T(v_f), v_2^+)}{a(f^2-1)E_2}. \tag{166}$$

To study these relations, we need to find $\mathrm{pr}_T(v_f)$ and $\mathrm{pr}_T(v_{if})$. Recall that $\mathrm{pr}_T(\mathcal{J}_3 X_0)$ was computed in (151) and $\mathrm{pr}_T(\mathcal{J}_2 X_0) = 0$. We now compute $\mathrm{pr}_T(\mathcal{J} X_0)$. Since $X_0 + G^{\mathrm{end}}(X_0) \in C_+$, from the definition of J_+ we obtain

$$J_+(X_0 + \mathrm{pr}_T G^{\mathrm{end}}(X_0)) = \mathrm{pr}_T(\mathcal{J} X_0 + \mathcal{J}_2 X_0) = \mathrm{pr}_T(\mathcal{J} X_0),$$

i.e. $\mathrm{pr}_T(\mathcal{J} X_0) = J_+(X_0 - \mathrm{pr}_T(\mathcal{J}_3 X_0))$. From this relation and (151), we obtain

$$\mathrm{pr}_T(\mathcal{J} X_0) = \frac{1}{\det(\Psi) - C_{12}^2}\left(\Psi_{22}\frac{\partial}{\partial \mu^1} - (\Psi_{12})^s \frac{\partial}{\partial \mu^2}\right). \tag{167}$$

The first two relations (166) are equivalent to

$$F(\mathrm{pr}_T(\mathcal{J} X_0), \frac{\partial}{\partial \mu^2}) + \Psi_{k2} F(X_0 - \frac{1}{f^2}\mathrm{pr}_T(\mathcal{J}_3 X_0), \frac{\partial}{\partial t^k}) = 0$$
$$F(\mathrm{pr}_T(\mathcal{J} X_0), \frac{\partial}{\partial \mu^2}) + \Psi_{2k} F(X_0 + \frac{1}{f^2}\mathrm{pr}_T(\mathcal{J}_3 X_0), \frac{\partial}{\partial t^k}) = 0$$

and are satisfied (from the expressions of F, $\mathrm{pr}_T(\mathcal{J} X_0)$ and $\mathrm{pr}_T(\mathcal{J}_3 X_0)$), and the third relation (166) is equivalent to

$$2\Psi_{22} F(\mathrm{pr}_T(\mathcal{J} X_0), \frac{\partial}{\partial t^2}) + (\Psi_{12} + \Psi_{21}) F(\mathrm{pr}_T(\mathcal{J} X_0), \frac{\partial}{\partial t^1}) = 0$$

and is satisfied as well (again, from the expressions of F and $\mathrm{pr}_T(\mathcal{J} X_0)$). The fourth relation (166) becomes the first relation (154). $\square$

According to Lemma 49, we assume that f depends only on μ^1 and that the first relation (154) is satisfied. Since f, h depend only on μ^1 and $a = k_0 h^{-2}$, the function af^2h^2 depends only on μ^1 as well and $X(af^2h^2) = 0$, for any $X \in \mathrm{pr}_T(\mathcal{S}^\perp)$. Therefore, condition iii) from Theorem 23 is satisfied. Condition iv) from this theorem is also satisfied: the first relation (75) follows from Lemma 48 ii) and the second relation (75) is a consequence of the fact that h depends only on μ^1. It remains to study condition iv) of Theorem 23. Since

$$\mathrm{pr}_T(L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp) = \mathrm{pr}_T(L_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp) + \mathrm{pr}_T(\bar{L}_1 \cap L_2 \cap \mathcal{S}_{\mathbb{C}}^\perp)$$

is generated by v_2^+ and $\overline{v_2^-}$, relation (76) holds if and only if it holds when applied to the pair $(v_2^+, \overline{v_2^-})$. From the first relation (165),

$$d\left(\mathrm{pr}_{T^*}(\mathcal{J}_3 X_0)\right)(v_2^+, \overline{v_2^-}) = -2i\frac{\partial}{\partial \mu^2}\left(\frac{(\Psi_{12})^s}{\Psi_{22}}\right)\frac{(\Psi_{22})^2}{\det(\Psi) - C_{12}^2}.$$

Computations similar to those from the proof of Lemma 49 show that relation (76) is equivalent to the first relation (154). Moreover, a straightforward computation shows that F, defined by (153), is closed, if and only if the second relation (154) holds and λ is independent of t^1. This concludes the proof of Proposition 47. $\square$

We now consider the setting of Proposition 47 with $\lambda := \lambda_0$ constant. Under this assumption, relations (154) become

$$\begin{aligned}
&(1-f^2)a\frac{\partial}{\partial\mu^2}\left(\frac{(\Psi_{12})^s}{\Psi_{22}}\right) = \lambda_0 + \frac{2ah'f^2}{h}\\
&(\lambda_0 + \frac{2ah'f^2}{h})(\lambda_0 - \frac{2ah'}{h}) = 0.
\end{aligned} \tag{168}$$

The functions h and $a = k_0 h^{-2}$ depend only on μ^1. By Proposition 47, f has to depend only on μ^1. From our hypothesis of the generalized KK correspondence, $f^2 - 1$ is non-vanishing. From the first relation (168), we obtain that $\frac{\partial}{\partial\mu^2}\left(\frac{(\Psi_{12})^s}{\Psi_{22}}\right)$ has to depend only on μ^1. On the other hand, from the second relation (168) we distinguish two cases, according to the vanishing of the two factors from its left hand side. Corollary 50 and 51 below correspond to $\lambda_0 + \frac{2ah'f^2}{h} = 0$, respectively to $\lambda_0 = \frac{2ah'}{h}$. Their proofs are straightforward, from Proposition 47, and will be omitted.

Corollary 50. *Consider a 4-dimensional toric generalized Kähler structure, determined by a matrix valued function Ψ, such that $\frac{\partial}{\partial\mu^2}\left(\frac{(\Psi_{12})^s}{\Psi_{22}}\right) = 0$. Let $h = h(\mu^1)$ be any smooth non-vanishing function with h' non-vanishing and $\lambda_0, k_0 \in \mathbb{R}\setminus\{0\}$, such that $\frac{\lambda_0 h^3}{k_0 h'}$ is (at any point) negative and different from -1. Choose*

$$X_0 := -\frac{\partial}{\partial t^1},\ a := k_0 h^{-2},\ f^2 := -\frac{\lambda_0 h^3}{2k_0 h'} \tag{169}$$

and F of the form (153), with $\lambda := \lambda_0$. Then all conditions from Theorem 23 are satisfied.

Corollary 51. *Consider a 4-dimensional toric generalized Kähler structure, determined by a matrix valued function Ψ, such that $\frac{\partial}{\partial\mu^2}\left(\frac{(\Psi_{12})^s}{\Psi_{22}}\right)$ depends* *only on μ^1. Choose $X_0 := -\frac{\partial}{\partial t^1}$,*

$$a := k_0 k_1 - \lambda_0 \mu^1,\ h^2 := \frac{-k_0}{\lambda_0 \mu^1 - k_0 k_1},\ f^2 := \frac{(\mu^1 - \frac{k_0 k_1}{\lambda_0}) \frac{\partial}{\partial \mu^2} \left(\frac{(\Psi_{12})^s}{\Psi_{22}} \right) + 1}{(\mu^1 - \frac{k_0 k_1}{\lambda_0}) \frac{\partial}{\partial \mu^2} \left(\frac{(\Psi_{12})^s}{\Psi_{22}} \right) - 1}, \tag{170}$$

and F given by (153) with $\lambda := \lambda_0$. Then all conditions from Theorem 23 are satisfied. Above $k_0, \lambda_0 \in \mathbb{R} \setminus \{0\}$, $k_1 \in \mathbb{R}$, and we assume that the defining expression for a in (170) is non-vanishing, while the defining expressions for h^2 and f^2 are positive.

The conditions from Proposition 47 (and Corollaries 50 and 51), on the toric generalized Kähler structure, involve only the symmetric part of Ψ, i.e. the symplectic potential τ. There are various interesting symplectic potentials for which $\frac{\partial}{\partial \mu^2}(\frac{\tau_{12}}{\tau_{22}})$ depends only on μ^1, as required by Corollaries 50 and 51 (here $\tau_{ij} := \frac{\partial^2 \tau}{\partial \tau^i \tau^j}$):

Example 52. Let $c \in \mathbb{R}$ and $k \in \mathbb{R}^{>0}$. The symplectic potential $\tau(\mu^1, \mu^2) := e^{\mu^1}(e^{\mu^2 + c} + k)$ defined on $\mathbb{R}^2$ satisfies $\frac{\partial}{\partial \mu^2} \left(\frac{(\Psi_{12})^s}{\Psi_{22}} \right) = 0$.

Example 53. (Symplectic potential of the Fubini-Study metric). Consider the symplectic potential of the Fubini Study metric on $\mathbb{C}P^2$, of constant holomorphic sectional curvature equal to 2:

$$\tau(\mu^1, \mu^2) := \sum_{i=1}^{2} (\mu^i + \frac{1}{3}) \log(\mu^i + \frac{1}{3}) + (\frac{1}{3} - \sum_{i=1}^{2} \mu^i) \log(\frac{1}{3} - \sum_{i=1}^{2} \mu^i) \tag{171}$$

defined on $\Delta := \{(\mu^1, \mu^2) \in \mathbb{R}^2 :\ \mu^1 + 1/3 > 0,\ \mu^2 + 1/3 > 0,\ \mu^1 + \mu^2 < 1/3\}$. Then $\frac{\partial}{\partial \mu^2}(\frac{\tau_{12}}{\tau_{22}}) = \frac{1}{2/3 - \mu^1}$ depends only on μ^1. In the notation of Corollary 51, choose $k_0, k_1 \in \mathbb{R}$ such that $k_0 < 0$ and $k_0 k_1 < -\frac{4}{3}$, and let $\lambda_0 := 1$. Then

$$h^2 = -\frac{k_0}{\mu^1 - k_0 k_1},\ f^2 = \frac{2/3 - k_0 k_1}{2\mu^1 - (2/3 + k_0 k_1)}. \tag{172}$$

The choice of k_0, k_1 ensure that the defining expressions for h^2 and f^2 in (172) are positive on Δ. Also $a = -\mu^1 + k_0 k_1$ is negative on Δ (in particular, non-vanishing).

Example 54. (Symplectic potential of admissible metrics on Hirzebruch surfaces). The k-Hirzebruch surface $\mathbb{F}_k := \mathbb{P}(1 \oplus \mathcal{O}(-k))$ (where k is a positive integer) admits a class of toric Kähler metrics (called admissible), which generalize Calabi's extremal Kähler metrics [5]. A detailed description of these metrics can be found in [3, 9, 11]. They are defined in terms of a smooth function (the momentum profile) $\Theta : [a, b] \to \mathbb{R}$, positive on (a, b), which satisfies the boundary conditions

$$\Theta(a) = 0,\ \Theta(b) = 0,\ \Theta'(a) = 2,\ \Theta'(b) = -2$$

(where $b > a > 0$). The symplectic potential

$$\tau : \{(\mu^1, \mu^2) \in \mathbb{R}^2 :\ \mu^2 > 0,\ k\mu^1 - \mu^2 > 0, -\mu^1 + b > 0,\ \mu^1 - a > 0\} \to \mathbb{R}$$

of the admissible Kähler metric with momentum profile Θ satisfies

$$\tau_{11} = \frac{1}{\Theta(\mu^1)} + \frac{k\mu^2}{2\mu^1(k\mu^1 - \mu^2)}, \tau_{12} = \frac{-k}{2(k\mu^1 - \mu^2)},\ \tau_{22} = \frac{k\mu^1}{2\mu^2(k\mu^1 - \mu^2)}.$$

We obtain that $\frac{\partial}{\partial \mu^2}(\frac{\tau_{12}}{\tau_{22}}) = -\frac{1}{\mu^1}$ depends only on μ^1. In the notation of Corollary 51, let $\lambda_0 := 1$. Then f and h are given by

$$f^2 = \frac{-k_0 k_1}{2\mu^1 - k_0 k_1},\ h^2 = \frac{-k_0}{\mu^1 - k_0 k_1}. \tag{173}$$

Choosing $k_0, k_1 \in \mathbb{R}^{>0}$ with $k_0 k_1 > 2b$, we obtain that the defining expressions for f^2, h^2 in (173) and $a = k_0 k_1 - \mu^1$ are positive.

Remark 55. It would be interesting to study the properties of the generalized Kähler structures $[\tau_h(G', \mathcal{J})]_W$ produced by Proposition 47. Here we only remark that they are not of symplectic type. The manifold W inherits the vector fields $(\frac{\partial}{\partial t^1})_W$ and $(\frac{\partial}{\partial t^2})_W$, which commute (from (4), as $\frac{\partial}{\partial t^1}$ and $\frac{\partial}{\partial t^2}$ commute and $F(\frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^2}) = 0$). However, the abelian Lie algebra generated by $(\frac{\partial}{\partial t^1})_W$ and $(\frac{\partial}{\partial t^2})_W$ does not necessarily preserve the generalized Kähler structure $[\tau_h(G', \mathcal{J})]_W$. Further investigations in this direction are needed.

References

[1] D. V. Alekseevsky, V. Cortes, and T. Mohaupt: *Conification of Kähler and Hyper-Kähler manifolds*, Comm. Math. Physics, **324** (2), p. 637-655 (2013).

[2] D. V. Alekseevsky, V. Cortes, and M. Dyckmanns, T. Mohaupt: *Quaternionic-Kähler metrics associated with special Kähler manifolds*, J. Geom. Physics **92**, p. 271-287 (2015).

[3] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, C. W. Tonnesen-Friedman: *Hamiltonian 2-forms in Kähler geometry, III: Extremal metrics and stability*, Invent. Math. 173 (2008), p. 547-601.

[4] L. Boulanger: *Toric generalized Kähler structures*, arxiv: 1509.06785v2.

[5] E. Calabi: *Extremal Kähler metrics*, in *Seminar of Differential Geometry*, ed. S. T. Yau, Annals of Mathematics Studies 102, Princeton University Press (1982), p. 259-290.

[6] V. Cortes, M. Dyckmanns, and S. Suhr: *Completeness of projective special Kähler and quaternionic Kähler manifolds*, to appear (accepted 26/12/2016) in "New perspectives in differential geometry: special metrics and quaternionic geometry" in honour of Simon Salamon (Rome, 16-20 November 2015), arXiv:1607.07232 [math.DG].

[7] M. Crainic: *Generalized complex structures and Lie brackets*, Bull. Braz. Math. Soc. **42** (4), p. 559-578 (2011).

[8] N. Enrietti, A. Fino, G. Grantcharov: *Tamed symplectic forms and generalized geometry*, J. Geom. Physics **71** (2013), p. 103-116.

[9] P. Gauduchon: *Hirzebruch surfaces and weighted projective planes*, unpublished notes.

[10] P. Gauduchon: *Symplectic bihermitian structures*, unpublished notes.

[11] P. Gauduchon: *Calabi extremal Kähler metrics*, Lecture Notes in progress.

[12] M. Gualtieri: *Generalized complex geometry*, Ph.D thesis, University of Oxford, 2003; arxiv:math/0401221.

[13] A. Haydys: *Hyper-Kähler and quaternionic-Kähler manifolds with S^1-symmetry*, J. Geom. Physics **58** (3), p. 293-306 (2008).

[14] N. Hitchin: *Lectures on generalized geometry*, arXiv:1008.0973 [math.DG].

[15] N. Hitchin: *On the hyper-Kähler/quaternion-Kähler correspondence*, Comm. Math. Physics **324** (1), p. 77-106 (2013).

[16] Y. Lin and S. Tolman: *Symmetries in Generalized Kähler Geometry*, Comm. Math. Physics **268** (1), p. 119-222 (2006).

[17] Z. J. Liu, A. Weinstein, and P. Xu: *Manin Triple for Lie bialgebroids*, J. Diff. Geom. **45**, p. 547-574 (1997).

[18] O. Macia and A. Swann: *Elementary deformations and the hyperKähler-quaternionic Kähler correspondence*, Proceedings of ICM Satellite Conference on Real and Complex Submanifolds, 10-12 August 2014, Daejon, ed. Y. J. Suh, J. Berndt, Y. Ohnita, B. H. Kim, H, Lee, Springer Proceedings in Mathematics and Statistics 106 (2014), p. 229-347.

[19] A. Swann: *Twisting Hermitian and Hypercomplex Geometries*, Duke Math J., **155** (2), p. 403-431 (2010).