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Hochschild Cohomology and the Modular Group

Simon Lentner, Svea Nora Mierach,
Christoph Schweigert, Yorck Sommerhäuser

Abstract

It has been shown in previous work that the modular group acts projectively on the center of a factorizable ribbon Hopf algebra. The center is the zeroth Hochschild cohomology group. In this article, we extend this projective action of the modular group to an arbitrary Hochschild cohomology group of a factorizable ribbon Hopf algebra, in fact up to homotopy even to a projective action on the entire Hochschild cochain complex.

Introduction

An important idea coming from conformal field theory is that modular categories lead to projective representations of mapping class groups of surfaces (see [BK], [G], [T] and the references cited therein). At least for certain aspects of this construction, it is not necessary that the category under consideration is semisimple. For a particularly simple surface, the torus, the mapping class group is the homogeneous modular group of two-times-two matrices with integer entries and determinant one. By applying these ideas in the case of the representation category of a factorizable ribbon Hopf algebra, which is not required to be semisimple, we obtain a projective representation of the homogeneous modular group on the center of this Hopf algebra (see for example [CW1], [CW2], [Ke], [KL], [LM] and [T]). As the center is the zeroth Hochschild cohomology group of the Hopf algebra, it is natural to ask whether there is a corresponding action on the higher cohomology groups. In this article, we answer this question affirmatively by showing that the modular group acts, projectively and up to homotopy, even on the entire Hochschild cochain complex.

The article is organized as follows: In the first section, we briefly review the Hochschild cohomology of an algebra A with coefficients in an A-bimodule M, as found for example in [W]. We then construct in Proposition 1.3 a particular homotopy between two cochain maps that will be important later for the verification of the defining relations of the modular group. In the second section, we turn to the case where the algebra A is a Hopf algebra and introduce a way to modify the bimodule structure of M while leaving the Hochschild cohomology groups essentially unchanged. In the third section, we turn to the case where A is a factorizable ribbon Hopf algebra and recall the action of the modular group on its center. In particular, we introduce the Radford and the Drinfel'd map. Our treatment here follows largely the exposition in [SZ], to which the reader is referred for references to the original work. In the fourth section, we take advantage of our modification of the bimodule structure introduced in the second section to generalize the Radford and the Drinfel'd map to the Hochschild cochain complex. In the fifth and final section, we use these maps to generalize the action of the modular group on the center to an action on all Hochschild cohomology groups of our factorizable ribbon Hopf algebra.

We will always work over a base field that is denoted by K, and all unadorned tensor products are taken over K. The dual of a vector space V is denoted by $V^* := \mathrm{Hom}_K(V, K)$.

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1 Hochschild Cohomology

We begin by briefly recalling the approach to Hochschild cohomology via the standard resolution. Further details can be found for example in [CE, Chap. IX] or [W, Chap. 9]. We consider an associative algebra A over our base field K and an A-bimodule M. As in [CE, Chap. IX, § 3, p. 167], we assume that the left and the right action of A on M become equal when restricted to K, so that an A-bimodule is the same as a module over $A \otimes A^{\mathrm{op}}$. Here A^{op} denotes the opposite algebra, in which the product is modified by interchanging the factors.

Definition 1.1 For an integer $n > 0$, we call $C^n(A, M) := \mathrm{Hom}_K(A^{\otimes n}, M)$ the space of cochains, and extend this definition to all integers by setting $C^0(A, M) := M$ and $C^n(A, M) := 0$ for $n < 0$. For $n > 0$ and $i = 0, \ldots, n$, we define the coface maps $\partial_i^{n-1} \colon C^{n-1}(A, M) \to C^n(A, M)$ as

$$\partial_i^{n-1}(f)(a_1 \otimes \ldots \otimes a_n) := \begin{cases} a_1.f(a_2 \otimes \ldots \otimes a_n) & \text{if } i = 0, \\ f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) & \text{if } 0 < i < n, \\ f(a_1 \otimes \ldots \otimes a_{n-1}).a_n & \text{if } i = n. \end{cases}$$

Using these maps, we define the coboundary operator $d^{n-1} \colon C^{n-1}(A, M) \to C^n(A, M)$, which is also called the differential, as $d^{n-1} := \sum_{i=0}^{n} (-1)^i \partial_i^{n-1}$, and extend this definition to negative numbers by setting $d^n = 0$ for $n < 0$. We then get a cochain complex

$$\cdots \xrightarrow{d^{-2}} 0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} \mathrm{Hom}_K(A, M) \xrightarrow{d^1} \mathrm{Hom}_K(A \otimes A, M) \xrightarrow{d^2} \cdots$$

that we briefly denote by $C(A, M)$. The n-th Hochschild cohomology group of the algebra A with coefficients in the bimodule M is defined as the n-th cohomology group of this cochain complex, i.e.,

$$HH^n(A, M) := H^n(C(A, M), d).$$

We note that for finite-dimensional separable algebras, and therefore in particular for finite-dimensional semisimple algebras over fields of characteristic zero, the higher Hochschild cohomology groups $HH^n(A, M)$ for $n \geq 1$ vanish, as shown for example in [CE, Chap. IX, Thm. 7.10, p. 179].

The following special cases will be particularly important in the sequel:

Example 1.2 For the zeroth Hochschild cohomology group, we find

$$\begin{aligned} HH^0(A, M) &= \ker(d^0 \colon C^0(A, M) \to C^1(A, M)) \\ &= \ker(\partial_0^0 - \partial_1^0 \colon C^0(A, M) \to C^1(A, M), m \mapsto (a \mapsto a.m - m.a)) \\ &= \{m \in M \mid a.m = m.a \text{ for all } a \in A\}, \end{aligned}$$

a set that is often called the space of invariants of M, for example in [CE, Chap. IX, § 4, p. 170] or [Ka, Sec. 1.1, p. 2]. For $M = A$, where the bimodule structure is given by multiplication, we get in particular that

$$HH^0(A, A) = Z(A),$$

the center of the algebra A.

For any bimodule M, the dual space $M^* = \mathrm{Hom}_K(M, K)$ is again a bimodule with respect to the action $(a.\varphi.b)(m) = \varphi(b.m.a)$. According to the preceding computation, we then have

$$HH^0(A, M^*) = \{\varphi \in M^* \mid \varphi(m.a) = \varphi(a.m) \text{ for all } a \in A \text{ and all } m \in M\}.$$

By composition on the left, any bimodule homomorphism $g \colon M \to N$ induces a homomorphism

$$g_* : C^n(A, M) \to C^n(A, N),\ f \mapsto g \circ f$$

between the cochain groups, where in general we use a lower star for the map induced by composition on the left and an upper star for the map induced by composition on the right. Because these homomorphisms g_* commute with the coboundary operators, they can be combined to a cochain map. An element $c \in Z(A)$ in the center of A gives rise to two natural choices for g on every bimodule M, namely the left and right actions

$$l_c^M : M \to M,\ m \mapsto c.m \qquad \text{and} \qquad r_c^M : M \to M,\ m \mapsto m.c.$$

The induced maps on the Hochschild cochain complex are related as follows:

Proposition 1.3 The cochain maps $(l_c^M)_*$ and $(r_c^M)_*$ are homotopic.

Proof: For $n \geq 0$, we define $h^{n+1} \colon C^{n+1}(A, M) \to C^n(A, M)$ as

$$\begin{aligned} h^{n+1}(f)(a_1 \otimes \ldots \otimes a_n) &:= \sum_{j=0}^{n} (-1)^j f(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n) \\ &= f(c \otimes a_1 \otimes \ldots \otimes a_n) + \sum_{j=1}^{n-1} (-1)^j f(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n) \\ &\quad + (-1)^n f(a_1 \otimes \ldots \otimes a_n \otimes c). \end{aligned}$$

In particular, we have $h^1(f) := f(c)$. For $n \leq 0$, we define $h^n := 0$, and claim that $h = (h^n)_{n\in\mathbb{Z}}$ is a homotopy between $(l_c^M)_*$ and $(r_c^M)_*$. To prove this, we have to show that

$$(d^{n-1} \circ h^n + h^{n+1} \circ d^n)(f)(a_1 \otimes \ldots \otimes a_n) = c.f(a_1 \otimes \ldots \otimes a_n) - f(a_1 \otimes \ldots \otimes a_n).c$$

for all $f \in C^n(A, M) = \mathrm{Hom}_K(A^{\otimes n}, M)$ and $a_1, \ldots, a_n \in A$. We first show this for the cases involving h^1. For $n = 0$, we have as in Example 1.2 above that

$$(h^1 \circ d^0)(m) = d^0(m)(c) = c.m - m.c$$

for all $m \in M$. For $n = 1$, we need to consider $f \in \mathrm{Hom}_K(A, M)$ and have

$$\begin{aligned}
&(d^0 \circ h^1 + h^2 \circ d^1)(f)(a) = a.h^1(f) - h^1(f).a + d^1(f)(c \otimes a) - d^1(f)(a \otimes c) \\
&= a.f(c) - f(c).a + c.f(a) - f(ca) + f(c).a - a.f(c) + f(ac) - f(a).c = c.f(a) - f(a).c
\end{aligned}$$

for all $a \in A$, because c is central.

We now turn to the general case, where $n \geq 2$. For $f \in C^n(A, M)$ and $a_1, \ldots, a_n \in A$, we have that $d^{n-1}(h^n(f))(a_1 \otimes \ldots \otimes a_n)$ is given by the sum

$$\begin{aligned}
d^{n-1}(h^n(f))(a_1 \otimes \ldots \otimes a_n) &= a_1.h^n(f)(a_2 \otimes \ldots \otimes a_n) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i h^n(f)(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) + (-1)^n h^n(f)(a_1 \otimes \ldots \otimes a_{n-1}).a_n \\
&= \sum_{j=1}^{n} (-1)^{j-1} a_1.f(a_2 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n) \\
&\quad + t_1 + t_2 + (-1)^n \sum_{j=0}^{n-1} (-1)^j f(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_{n-1}).a_n,
\end{aligned}$$

where for the second equality we have broken the middle sum into two terms, namely the term

$$\begin{aligned}
t_1 &:= \sum_{0 \leq j < i \leq n-1} (-1)^{i+j} f(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\
&= \sum_{0 \leq j < i \leq n-1} (-1)^{i+j} \partial_{i+1}^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n)
\end{aligned}$$

and the term

$$\begin{aligned}
t_2 &:= \sum_{1 \leq i \leq j \leq n-1} (-1)^{i+j} f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{j+1} \otimes c \otimes a_{j+2} \otimes \ldots \otimes a_n) \\
&= \sum_{1 \leq i \leq j \leq n-1} (-1)^{i+j} \partial_i^n(f)(a_1 \otimes \ldots \otimes a_{j+1} \otimes c \otimes a_{j+2} \otimes \ldots \otimes a_n) \\
&= \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \partial_i^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n).
\end{aligned}$$

On the other hand, $h^{n+1}(d^n(f))(a_1 \otimes \ldots \otimes a_n)$ is given by the sum

$$\begin{aligned}h^{n+1}(d^n(f))(a_1 \otimes \ldots \otimes a_n) &= \sum_{j=0}^{n}(-1)^j d^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n)\\ &= \sum_{j=0}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\partial_i^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n).\end{aligned}$$

In the preceding sum, the term for $i = 0$ can be written in the form

$$\sum_{j=0}^{n}(-1)^j \partial_0^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n)$$
$$= c.f(a_1 \otimes \ldots \otimes a_n) - a_1.h^n(f)(a_2 \otimes \ldots \otimes a_n).$$

Looking at the term for $i = n+1$, we get similarly that

$$\sum_{j=0}^{n}(-1)^j(-1)^{n+1}\partial_{n+1}^n(f)(a_1 \otimes \ldots \otimes a_j \otimes c \otimes a_{j+1} \otimes \ldots \otimes a_n)$$
$$= (-1)^{n+1}h^n(f)(a_1 \otimes \ldots \otimes a_{n-1}).a_n - f(a_1 \otimes \ldots \otimes a_n).c.$$

In the remaining terms, we have $1 \leq i \leq n$. The sum of the terms with $1 \leq i < j \leq n$ is equal to $-t_2$. The sum of the terms with $i = j$ is

$$\sum_{i=1}^{n} f(a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i c \otimes a_{i+1} \otimes \ldots \otimes a_n),$$

while the sum of the terms with $i = j+1$ is

$$-\sum_{j=0}^{n-1} f(a_1 \otimes \ldots \otimes a_j \otimes c a_{j+1} \otimes a_{j+2} \otimes \ldots \otimes a_n).$$

Because c is central, these two sums cancel each other. Finally, there is the sum of the terms with $0 < j+1 < i \leq n$, which is equal to $-t_1$. Combining all these terms, we find that

$$\begin{aligned}h^{n+1}(d^n(f))(a_1 \otimes \ldots \otimes a_n) &= c.f(a_1 \otimes \ldots \otimes a_n) - a_1.h^n(f)(a_2 \otimes \ldots \otimes a_n)\\ &\quad + (-1)^{n+1}h^n(f)(a_1 \otimes \ldots \otimes a_{n-1}).a_n - f(a_1 \otimes \ldots \otimes a_n).c - t_2 - t_1\\ &= c.f(a_1 \otimes \ldots \otimes a_n) - d^{n-1}(h^n(f))(a_1 \otimes \ldots \otimes a_n) - f(a_1 \otimes \ldots \otimes a_n).c,\end{aligned}$$

which implies our assertion. □

We note that a similar homotopy for Hochschild homology is described in [L, Par. 1.1.5, p. 10; Exerc. 1.1.2, p. 15]. We also note that the preceding proposition can be understood from a more abstract and less computational point of view: In our definition above, we have realized the Hochschild cohomology groups $HH^n(A, M)$ as the groups $\mathrm{Ext}^n_{A\otimes A^{\mathrm{op}}}(A, M)$ by using a special resolution of A as an A-bimodule, or equivalently as an $A \otimes A^{\mathrm{op}}$-module, a resolution that is called the standard resolution in [CE, Chap. IX, § 6, p. 174f] and the bar resolution in [L, Par. 1.1.12, p. 12]. But in fact we can work with a general projective resolution

$$A \xleftarrow{\xi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

of A as an A-bimodule, which we briefly denote by P. As already pointed out above, the fact that c is central implies that the maps $l_c^{P_n}$ and $r_c^{P_n}$ are bimodule homomorphisms. Because ξ and the boundary operators d_n are bimodule homomorphisms, the maps $l_c^{P_n}$ commute with them, and therefore lift the left multiplication of c on A to the entire resolution:

$$\begin{tikzcd}
A \arrow[d, "l_c^A"] & P_0 \arrow[l, "\xi"'] \arrow[d, "l_c^{P_0}"] & P_1 \arrow[l, "d_1"'] \arrow[d, "l_c^{P_1}"] & \cdots \arrow[l, "d_2"'] \\
A & P_0 \arrow[l, "\xi"'] & P_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"']
\end{tikzcd}$$

Analogously, we can lift the right multiplication of c on A to the entire resolution:

$$\begin{tikzcd}
A \arrow[d, "r_c^A"] & P_0 \arrow[l, "\xi"'] \arrow[d, "r_c^{P_0}"] & P_1 \arrow[l, "d_1"'] \arrow[d, "r_c^{P_1}"] & \cdots \arrow[l, "d_2"'] \\
A & P_0 \arrow[l, "\xi"'] & P_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"']
\end{tikzcd}$$

Because c is central, we have $r_c^A = l_c^A$. Therefore, the comparison theorem found in [ML, Chap. III, Thm. 6.1, p. 87] or [W, Thm. 2.2.6, p. 35] yields that the chain maps $l_c^P = (l_c^{P_n})$ and $r_c^P = (r_c^{P_n})$ are chain homotopic.

The contravariant functor $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(-, M)$ coming from our bimodule M turns this homotopy of chain maps into a homotopy of cochain maps, so that we get that the cochain maps $(l_c^P)^*$ and $(r_c^P)^*$ are cochain homotopic. But we have $(l_c^P)^* = (l_c^M)_*$: For $f \in \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P_n, M)$ and $p \in P_n$, we have

$$((l_c^M)_*(f))(p) = l_c^M(f(p)) = c.f(p) = f(c.p) = f(l_c^{P_n}(p)) = ((l_c^{P_n})^*(f))(p)$$

A similar computation shows that $(r_c^P)^* = (r_c^M)_*$, which completes our second, resolution-independent proof of the proposition.

We note that generalizations of this proposition can be found in the literature, for example in [SS, Cor. 1.3, p. 709]. However, we will only need the above form of the proposition in the sequel.

2 Hochschild Cohomology of Hopf Algebras

We now turn to the case where the algebra A is a Hopf algebra. We will denote the coproduct of A by Δ, its counit by ε, and its antipode by S. For the coproduct of $a \in A$, we will use Heyneman-Sweedler notation in the form $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Because A is a Hopf algebra, every A-bimodule M can be considered as a right A-module via the right adjoint action

$$\mathrm{ad}\colon M \otimes A \to M,\ m \otimes a \mapsto \mathrm{ad}(m \otimes a),$$

which is defined as $\mathrm{ad}(m \otimes a) := S(a_{(1)}).m.a_{(2)}$. We denote M by M_{ad} if it is considered as a right A-module in this way.

In general, a right A-module N becomes an A-bimodule with respect to the trivial left action, i.e., the action defined as $a.n := \varepsilon(a)n$. We denote N by ${}_{\varepsilon}N$ if it is considered as a bimodule in this way. By combining the two operations, we can associate with an A-bimodule M the A-bimodule ${}_{\varepsilon}M_{\mathrm{ad}} := {}_{\varepsilon}(M_{\mathrm{ad}})$. As it turns out, the Hochschild cochain complexes determined by these two bimodules are isomorphic:

Proposition 2.1 The maps $\Omega^n\colon C^n(A, {}_{\varepsilon}M_{\mathrm{ad}}) \to C^n(A, M)$ defined via the formula

$$\Omega^n(f)(a_1 \otimes \ldots \otimes a_n) = a_{1(1)} \ldots a_{n(1)}.f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})$$

give rise to an isomorphism $\Omega = (\Omega^n)$ between the Hochschild cochain complex of ${}_{\varepsilon}M_{\mathrm{ad}}$ and the Hochschild cochain complex of M.

Proof: We first note that Ω^n is bijective with inverse

$$(\Omega^n)^{-1}(f)(a_1 \otimes \ldots \otimes a_n) := S(a_{n(1)}) \ldots S(a_{1(1)}).f(a_{1(2)} \otimes \ldots \otimes a_{n(2)}),$$

because for $f \in C^n(A, {}_{\varepsilon}M_{\mathrm{ad}}) = \mathrm{Hom}_K(A^{\otimes n}, {}_{\varepsilon}M_{\mathrm{ad}})$, we have

$$\begin{aligned}(\Omega^n)^{-1}(\Omega^n(f))(a_1 \otimes \ldots \otimes a_n) &= S(a_{n(1)}) \ldots S(a_{1(1)}).\Omega^n(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)})\\ &= S(a_{n(1)}) \ldots S(a_{1(1)})a_{1(2)} \ldots a_{n(2)}.f(a_{1(3)} \otimes \ldots \otimes a_{n(3)}) = f(a_1 \otimes \ldots \otimes a_n),\end{aligned}$$

and the relation $\Omega^n \circ (\Omega^n)^{-1} = \mathrm{id}_{C^n(A,M)}$ follows analogously.

For $f \in C^{n-1}(A, {}_{\varepsilon}M_{\mathrm{ad}})$ and $a_1, \ldots, a_n \in A$, we have on the one hand

$$\begin{aligned}d^{n-1}(\Omega^{n-1}(f))(a_1 \otimes \ldots \otimes a_n) &= a_1.\Omega^{n-1}(f)(a_2 \otimes \ldots \otimes a_n)\\ &\quad + \sum_{i=1}^{n-1}(-1)^i \Omega^{n-1}(f)(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n)\\ &\quad + (-1)^n(\Omega^{n-1}(f)(a_1 \otimes \ldots \otimes a_{n-1})).a_n\\ &= a_1 a_{2(1)} \ldots a_{n(1)}.f(a_{2(2)} \otimes \ldots \otimes a_{n(2)})\\ &\quad + \sum_{i=1}^{n-1}(-1)^i a_{1(1)} \ldots (a_i a_{i+1})_{(1)} \ldots a_{n(1)}.f(a_{1(2)} \otimes \ldots \otimes (a_i a_{i+1})_{(2)} \otimes \ldots \otimes a_{n(2)})\\ &\quad + (-1)^n a_{1(1)} \ldots a_{n-1(1)}.f(a_{1(2)} \otimes \ldots \otimes a_{n-1(2)}).a_n\end{aligned}$$

and on the other hand

$$
\begin{aligned}
\Omega^n(d^{n-1}(f))(a_1 \otimes \ldots \otimes a_n) &= a_{1(1)} \ldots a_{n(1)}.d^{n-1}(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)}) \\
= a_{1(1)} \ldots a_{n(1)}.\varepsilon(a_{1(2)}) f(a_{2(2)} \otimes \ldots \otimes a_{n(2)}) \\
+ \sum_{i=1}^{n-1} (-1)^i a_{1(1)} \ldots a_{n(1)}.f(a_{1(2)} \otimes \ldots \otimes a_{i(2)} a_{i+1(2)} \otimes \ldots \otimes a_{n(2)}) \\
+ (-1)^n a_{1(1)} \ldots a_{n(1)}. \left(S(a_{n(2)}).f(a_{1(2)} \otimes \ldots \otimes a_{n-1(2)}).a_{n(3)}\right),
\end{aligned}
$$

where we have used for the last summand that

$$
\operatorname{ad}(f(a_1 \otimes \ldots \otimes a_{n-1}) \otimes a_n) = S(a_{n(1)}).f(a_1 \otimes \ldots \otimes a_{n-1}).a_{n(2)}
$$

according to the definition of the right adjoint action. Because both expressions agree, Ω is a cochain map, which establishes our assertion. $\square$

If the antipode of A is bijective, the coopposite Hopf algebra A^{cop}, in which the product remains unaltered, but the coproduct is modified by interchanging the tensor factors, is a Hopf algebra, and its antipode is the inverse of the antipode of A. For an A-bimodule M, we denote the right adjoint action that arises from this Hopf algebra structure by cad; in terms of the original structure elements, this action is given by the formula

$$
\operatorname{cad}(m \otimes a) := S^{-1}(a_{(2)}).m.a_{(1)}.
$$

If we apply the preceding proposition to this situation, we obtain the following corollary:

Corollary 2.2 If the antipode of A is bijective, the maps $\Omega'^n \colon C^n(A, {}_{\varepsilon}M_{\mathrm{cad}}) \to C^n(A, M)$ defined via the formula

$$
\Omega'^n(f)(a_1 \otimes \ldots \otimes a_n) = a_{1(2)} \ldots a_{n(2)}.f(a_{1(1)} \otimes \ldots \otimes a_{n(1)})
$$

give rise to an isomorphism $\Omega' = (\Omega'^n)$ between the Hochschild cochain complex of ${}_{\varepsilon}M_{\mathrm{cad}}$ and the Hochschild cochain complex of M.

We record that Ω'^n is bijective with inverse

$$
(\Omega'^n)^{-1}(f)(a_1 \otimes \ldots \otimes a_n) := S^{-1}(a_{n(2)}) \ldots S^{-1}(a_{1(2)}).f(a_{1(1)} \otimes \ldots \otimes a_{n(1)}),
$$

as we had seen in the proof of our proposition.

Proposition 2.1 generalizes a result found in [FS, Sec. 1, p. 2862f]. We note that further results related to this proposition can be found in the literature: In the case where the Hopf algebra is a group ring, the argument is contained in [EM, § 5, p. 60f], one of the foundational articles for group cohomology. A homology version of the proposition can be found in [FT, Prop. (2.4), p. 488], at least in the case where the bimodule is the underlying algebra. Similar statements for cohomology appear in [GK, Par. 5.5, p. 197] and [PW, Lem. 12, p. 591]. These last two references, however, rather state a combination of Proposition 2.1 with the following lemma:

Lemma 2.3 For a right A-module N, we have $HH^n(A, {}_\varepsilon N) \cong \mathrm{Ext}^n_A(K, N)$, where the base field K is given the trivial right A-module structure via the counit ε.

Proof: If $P = (P_n)$ is a projective resolution of A as a left $A \otimes A^{\mathrm{op}}$-module, we know from [CE, Chap. X, Thm. 2.1, p. 185] that $K \otimes_A P := (K \otimes_A P_n)$ is a projective resolution of K as a right A-module. Therefore $\mathrm{Ext}^n_A(K, N)$ is the n-th cohomology group of the cochain complex formed by the cochain groups $\mathrm{Hom}_A(K \otimes_A P_n, N)$. But the cochain map

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P_n, {}_\varepsilon N) \to \mathrm{Hom}_A(K \otimes_A P_n, N),\ f \mapsto (\lambda \otimes p \mapsto \lambda f(p))$$

with inverse

$$\mathrm{Hom}_A(K \otimes_A P_n, N) \to \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P_n, {}_\varepsilon N),\ g \mapsto (p \mapsto g(1_K \otimes p))$$

shows that this complex is isomorphic to the cochain complex of the cochain groups $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P_n, {}_\varepsilon N)$, whose cohomology groups are $\mathrm{Ext}^n_{A\otimes A^{\mathrm{op}}}(A, {}_\varepsilon N) = HH^n(A, {}_\varepsilon N)$. $\square$

3 The Action on the Center

We now turn to the case of a factorizable ribbon Hopf algebra A with R-matrix R and ribbon element v. Even though the R-matrix is in general not a pure tensor, we use the notation $R = R_1 \otimes R_2$. If τ denotes the flip map, we therefore have $\tau(R) = R_2 \otimes R_1$. This element in turn can be used to introduce the monodromy matrix $Q := \tau(R)R$, and as for the R-matrix, we write $Q = Q_1 \otimes Q_2$. An important role will be played by the Drinfel'd and Radford map, which are defined as follows:

Definition 3.1 We call the map

$$\bar{\Phi} \colon A^* \to A,\ \varphi \mapsto \varphi(Q_1)Q_2$$

the Drinfel'd map, and define the subalgebra

$$\bar{C}(A) := \{\varphi \in A^* \mid \varphi(bS^{-2}(a)) = \varphi(ab) \text{ for all } a, b \in A\},$$

whose elements we call generalized class functions. With the help of a nonzero right integral $\rho \in A^*$, we introduce the Radford map

$$\iota \colon A \to A^*,\ a \mapsto \rho_{(1)}(a)\rho_{(2)}.$$

By definition, A is factorizable if and only if $\bar{\Phi}$ is bijective, which implies in particular that A is finite-dimensional. The basic properties of the Drinfel'd map can be found in [SZ, Par. 3.2, p. 26], and the basic properties of the Radford map can also be found there, namely in [SZ, Par. 4.1, p. 35]. In particular, the Drinfel'd map restricts to an algebra isomorphism from $\bar{C}(A)$ to $Z(A)$, the center of A, while the Radford map restricts to a K-linear isomorphism from $Z(A)$ to $\bar{C}(A)$. A consequence of this last fact is that $\rho = \iota(1_A) \in \bar{C}(A)$, which is a special case of a general result found in [R, Thm. 10.5.4, p. 307] that arises when combined with [R, Prop. 12.4.2, p. 405].

Following [SZ, Par. 4.1, p. 35], we introduce the endomorphism $\mathfrak{S} := S \circ \bar{\Phi} \circ \iota$ of A, where as before S denotes the antipode of A. For $a \in A$, we have explicitly

$$\mathfrak{S}(a) = S(\bar{\Phi}(\rho_{(1)}(a)\rho_{(2)})) = S(\rho_{(1)}(a)\rho_{(2)}(Q_1)Q_2) = \rho(aQ_1)\, S(Q_2),$$

or $\mathfrak{S}(a) = \rho(aR_2R'_1)\, S(R_1R'_2)$ if we insert the definition of the monodromy matrix by using a second copy R' of the R-matrix.

As in [SZ, Par. 4.3, p. 37], we introduce a second such map, namely the multiplication

$$\mathfrak{T}\colon A \to A,\ a \mapsto va$$

with the ribbon element $v \in A$. The endomorphisms $\mathfrak{S}$ and $\mathfrak{T}$ will be used to encode the action of the two generators of the modular group described below.

We will need a third endomorphism of A, namely the antipode of the transmutation of A. The transmutation of a quasitriangular Hopf algebra was described by S. Majid in several articles, among them [M1], and is discussed in his monograph [M2]. It has the same underlying vector space as A, in fact even the same algebra structure. In the version that we are using, the antipode $\underline{S}$ of the transmutation is given by

$$\underline{S}(a) = S(S(R_{1(1)})aR_{1(2)})R_2.$$

This variant arises from the one given in [M2, Ex. 9.4.9, p. 504] by replacing A with $A^{\mathrm{op\,cop}}$. If $u := S(R_2)R_1$ is the Drinfel'd element of A, then the element $S(u) = R_1S(R_2)$ is the Drinfel'd element of $A^{\mathrm{op\,cop}}$. Therefore, the alternative form of $\underline{S}$ given in [M2, Eq. (9.42), p. 507] becomes in our case

$$\underline{S}(a) = R_1S(a)S(R_2)S(u^{-1}).$$

These three endomorphisms are related as follows:

Proposition 3.2 The maps $\mathfrak{S}$ and $\mathfrak{T}$ satisfy the relations

$$\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\, \mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1} \quad \text{and} \quad \mathfrak{S}^2 = (\rho \otimes \rho)(Q)\, \underline{S}^{-1}.$$

Proof: A proof of the first relation can be found in [SZ, Prop. 4.3, p. 37]. To prove the second relation, we use four copies R, R', R'' and R''' of the R-matrix. Because $\rho \in \bar{C}(A)$, the map $\mathfrak{S}$ is alternatively given by

$$\mathfrak{S}(a) = \rho(aR_2R'_1)\, S(R'_2)S(R_1) = \rho(aR_2S^{-1}(R'_1))\, R'_2S(R_1) = \rho(S(R'_1)aR_2)\, R'_2S(R_1),$$

where we have used the fact $(S \otimes S)(R) = R$ proved in [M, Prop. 10.1.8, p. 180]. From [SZ, Prop. 4.1, p. 35], we know that $\mathfrak{S}$ is A-linear with respect to the right adjoint action. Therefore, we have

$$\underline{S}(\mathfrak{S}(a)) = S(\mathfrak{S}(S(R''_{1(1)})aR''_{1(2)}))R''_2 = \rho(S(R'_1)S(R''_{1(1)})aR''_{1(2)}R_2)\, S(R'_2S(R_1))R''_2.$$

If we use one of the axioms for the R-matrix, namely [M, Eq. 10.1.6, p. 180], and the fact that the antipode is antimultiplicative, this equation can be rewritten in the form

$$\begin{aligned}\underline{S}(\mathfrak{S}(a)) &= \rho(S(R'_1)S(R'''_1)aR''_1R_2)\,S^2(R_1)S(R'_2)R'''_2R''_2 \\ &= \rho(R'_1S(R'''_1)aR''_1R_2)\,S^2(R_1)R'_2R'''_2R''_2.\end{aligned}$$

Another fact proved in [M, Prop. 10.1.8, p. 180] is that $(S \otimes \mathrm{id})(R) = R^{-1}$, so that this equation reduces to

$$\underline{S}(\mathfrak{S}(a)) = \rho(aR''_1R_2)\,S^2(R_1)R''_2 = (\rho \otimes \rho)(Q)\,\mathfrak{S}^{-1}(a),$$

where the last step follows from [SZ, Prop. 4.2, p. 36]. Our claim is a minor rearrangement of this equation. $\square$

We would like to emphasize that this proposition is not new; rather, it is a variant of [LM, Thm. 4.4, p. 523]. We also note that it follows directly from another elementary property of R-matrices also proved in [M, Prop. 10.1.8, p. 180] that $\underline{S}$ agrees with the ordinary antipode S on the center of A, and because the square of the antipode is given by conjugation with the Drinfel'd element u, as shown in [M, Prop. 10.1.4, p. 179], we have $S = S^{-1}$ on the center. Therefore, the second relation in the previous proposition generalizes [SZ, Cor. 4.2, p. 37].

The fact that the square of $\underline{S}$ restricts to the identity on the center can also be seen from the fact that, in general, it is given by the right adjoint action of our ribbon element:

Lemma 3.3 For all $a \in A$, we have $\underline{S}^2(a) = \mathrm{ad}(a \otimes v)$.

Proof: With the help of the alternative form of $\underline{S}$, we get

$$\begin{aligned}\underline{S}^2(a) &= \underline{S}(R_1S(a)S(R_2)S(u^{-1})) = R'_1S(R_1S(a)S(R_2)S(u^{-1}))S(R'_2)S(u^{-1}) \\ &= R'_1S^2(u^{-1})S^2(R_2)S^2(a)S(R_1)S(R'_2)S(u^{-1}) = R'_1u^{-1}S^2(R_2a)S(u^{-1}R'_2R_1).\end{aligned}$$

Using the definition of the monodromy matrix Q, the basic properties of ribbon elements found in [SZ, Par. 4.3, p. 37] and the above-mentioned fact that the square of the antipode is given by conjugation with the Drinfel'd element u, this becomes

$$\begin{aligned}\underline{S}^2(a) &= Q_2au^{-1}S(u^{-1}Q_1) = S^2(Q_2)au^{-1}S(u^{-1}S^2(Q_1)) = S^2(Q_2)au^{-1}S(Q_1u^{-1}) \\ &= S^2(Q_2)au^{-1}S(u^{-1})S(Q_1) = S^2(Q_2)av^2S(Q_1) = S(Q_1)av^2Q_2 = S(v_{(1)})av_{(2)}\end{aligned}$$

as asserted. $\square$

We note that this equation is stated in [Ke, Eq. (2.60), p. 370], at least in the case of Drinfel'd doubles. A version in the framework of coends can be found in [Ly, Cor. 3.10, p. 306].

The proposition above implies that the (homogeneous) modular group

$$\mathrm{SL}(2,\mathbb{Z}) = \{M \in \mathrm{GL}(2,\mathbb{Z}) \mid \det(M) = 1\}$$

acts projectively on the center of A. The modular group is generated by the two elements $\mathfrak{s} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathfrak{t} := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which satisfy the relations

$$\mathfrak{s}^4 = 1 \quad \text{and} \quad \mathfrak{s}\mathfrak{t}\mathfrak{s} = \mathfrak{t}^{-1}\mathfrak{s}\mathfrak{t}^{-1},$$

and these relations are defining, as shown for example in [FR, Thm. 3.2.3.2, p. 97], [KT, Thm. A.2, p. 312] or [Ma, Sec. II.1, Thm. 8, p. 53].

If we denote the projective space associated to $Z(A)$ by $P(Z(A))$ and the automorphisms of this projective space arising from $\mathfrak{S}$ and $\mathfrak{T}$ by $P(\mathfrak{S})$ and $P(\mathfrak{T})$, the above proposition implies immediately the following fact:

Corollary 3.4 There is a unique homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{PGL}(Z(A))$ that maps $\mathfrak{s}$ to $P(\mathfrak{S})$ and $\mathfrak{t}$ to $P(\mathfrak{T})$.

This result holds for any ribbon element $v \in A$ and any nonzero right integral $\rho \in A^*$. As shown in [R, Cor. 12.4.4, p. 407], we have $\rho(v) \neq 0$; this is obviously also a consequence of the proposition above. Because right integrals are only unique up to scalar multiples, we can choose a right integral that satisfies $\rho(v) = 1$; following [SZ, Def. 4.4, p. 39], we call such a right integral ribbon-normalized with respect to v. If we use a ribbon-normalized right integral, the proposition above shows that the action of the modular group on the center is linear, and not only projective, if and only if $(\rho \otimes \rho)(Q) = \pm 1$. By [SZ, Lem. 4.4, p. 39], this condition is equivalent to the condition $\rho(v^{-1}) = \pm 1$.

4 The Radford and the Drinfel'd Map for Complexes

We remain in the situation described in Section 3 and consider a factorizable ribbon Hopf algebra A with R-matrix R and ribbon element v. Our first goal is to generalize the Radford map, the Drinfel'd map and the antipode to cochain maps of Hochschild cochain complexes. We begin with the Radford map, for which this is particularly easy.

By $A_{S^{-2}}$, we denote A considered as an A-bimodule with the left action given by multiplication, but the right action modified via the square of the inverse antipode, so that the right action is given by $b.a := bS^{-2}(a)$ for $a, b \in A$. As explained in Example 1.2, we then have

$$HH^0(A, (A_{S^{-2}})^*) = \{\varphi \in A^* \mid \varphi(bS^{-2}(a)) = \varphi(ab) \text{ for all } a, b \in A\} = \bar{C}(A),$$

the algebra of generalized class functions introduced in Definition 3.1. The bimodule $A_{S^{-2}}$ is related to the Radford map in the following way:

Proposition 4.1 The Radford map ι is a bimodule isomorphism from A to $(A_{S^{-2}})^*$.

Proof: By [M, Thm. 2.1.3, p. 18], A is a Frobenius algebra with Frobenius homomorphism ρ, so that ι is bijective. It is a bimodule homomorphism because

$$\iota(a_1 a a_2)(b) = \rho(a_1 a a_2 b) = \rho(a a_2 b S^{-2}(a_1)) = \iota(a)(a_2 b S^{-2}(a_1)) = (a_1.\iota(a).a_2)(b)$$

for all $a, a_1, a_2, b \in A$, where the second equality holds because $\rho \in \bar{C}(A)$, a fact already pointed out in Section 3. $\square$

Because bimodule isomorphisms induce isomorphisms between the corresponding Hochschild cochain complexes, this proposition enables us to generalize the Radford map to a cochain map as follows:

Definition 4.2 We define the Radford map for Hochschild cochain complexes as the cochain map from $C(A, A)$ to $C(A, (A_{S^{-2}})^*)$ with components

$$\iota^n \colon C^n(A, A) \to C^n(A, (A_{S^{-2}})^*), \ f \mapsto \iota \circ f.$$

In other words, we set $\iota^n := \iota_*$, the composition with ι on the left.

In order to compare this definition with the treatment of the Drinfel'd map and the antipode below, it will be important to relate this cochain map to another one defined between different cochain complexes. From Proposition 2.1, we get a cochain map $\Omega = (\Omega^n)$ from the cochain complex $C^n(A, {}_\varepsilon A_{\mathrm{ad}})$ to the cochain complex $C^n(A, A)$, but also a cochain map from the cochain complex $C^n(A, {}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}})$ to the cochain complex $C^n(A, (A_{S^{-2}})^*)$, which we denote by $\Omega'' = (\Omega''^n)$. The bimodule ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}}$ admits a slightly simpler description: For $\varphi \in (A_{S^{-2}})^*$, $a \in A$ and $b \in A_{S^{-2}}$, we have

$$\begin{aligned} \mathrm{ad}(\varphi \otimes a)(b) &= (S(a_{(1)}).\varphi.a_{(2)})(b) = \varphi(a_{(2)}.b.S(a_{(1)})) \\ &= \varphi(a_{(2)} b S^{-1}(a_{(1)})) = \varphi_{(1)}(a_{(2)})\varphi_{(2)}(b)\varphi_{(3)}(S^{-1}(a_{(1)})), \end{aligned}$$

which shows that the right adjoint action in ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}}$ coincides with the right coadjoint action of the coopposite Hopf algebra A^{cop}, which we denote by

$$\mathrm{coad} \colon A^* \otimes A \to A^*, \ \varphi \otimes a \mapsto \varphi_{(1)}(a_{(2)})\varphi_{(3)}(S^{-1}(a_{(1)}))\, \varphi_{(2)}.$$

In other words, we have ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}} = {}_\varepsilon(A^*)_{\mathrm{coad}}$. The Radford map now relates the two isomorphisms Ω and Ω'' as follows:

Lemma 4.3 The diagram

$$\begin{tikzcd} C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) \arrow[r, "\Omega''^n"] & C^n(A, (A_{S^{-2}})^*) \\ C^n(A, {}_\varepsilon A_{\mathrm{ad}}) \arrow[u, "\iota_*"] \arrow[r, "\Omega^n"] & C^n(A, A) \arrow[u, "\iota_*"] \end{tikzcd}$$

commutes.

Proof: We first note that it follows from Proposition 4.1 above that the Radford map ι is also a bimodule isomorphism from ${}_\varepsilon A_{\mathrm{ad}}$ to ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}} = {}_\varepsilon(A^*)_{\mathrm{coad}}$, so that the map on the left is well-defined. For $f \in C^n(A, {}_\varepsilon A_{\mathrm{ad}}) = \mathrm{Hom}_K(A^{\otimes n}, {}_\varepsilon A_{\mathrm{ad}})$ and $a_1, \ldots, a_n, b \in A$, we now have on the one hand

$$\begin{aligned} \iota(\Omega^n(f)(a_1 \otimes \ldots \otimes a_n))(b) &= \iota(a_{1(1)} \ldots a_{n(1)} f(a_{1(2)} \otimes \ldots \otimes a_{n(2)}))(b) \\ &= \rho(a_{1(1)} \ldots a_{n(1)} f(a_{1(2)} \otimes \ldots \otimes a_{n(2)}) b). \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
((\Omega''^n \circ \iota_*)(f)(a_1 \otimes \ldots \otimes a_n))(b) &= (a_{1(1)} \ldots a_{n(1)}.\iota_*(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)}))(b) \\
&= \iota_*(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)})(b.a_{1(1)} \ldots a_{n(1)}) \\
&= \iota(f(a_{1(2)} \otimes \ldots \otimes a_{n(2)}))(bS^{-2}(a_{1(1)} \ldots a_{n(1)})) \\
&= \rho(f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})bS^{-2}(a_{1(1)} \ldots a_{n(1)})).
\end{aligned}
$$

Since $\rho \in \bar{C}(A)$, these expressions are equal. $\square$

To generalize the Drinfel'd map to a cochain map between Hochschild cochain complexes, we first recall from [SZ, Par. 3.2, p. 26] that the Drinfel'd map $\bar{\Phi}$ is a bimodule isomorphism between ${}_\varepsilon(A^*)_{\mathrm{coad}}$ and ${}_\varepsilon A_{\mathrm{cad}}$, so that we obtain an isomorphism of cochain complexes

$$\bar{\Phi}_* \colon C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) \to C^n(A, {}_\varepsilon A_{\mathrm{cad}}), \ f \mapsto \bar{\Phi} \circ f$$

by composing with $\bar{\Phi}$ on the left. Now the isomorphism Ω' from Corollary 2.2 enables us to obtain a cochain map between the original cochain complexes:

Definition 4.4 We define the Drinfel'd map for Hochschild cochain complexes as the cochain map from $C(A, (A_{S^{-2}})^*)$ to $C(A, A)$ with components $\bar{\Phi}^n := \Omega'^n \circ \bar{\Phi}_* \circ (\Omega''^n)^{-1}$. In other words, it is the unique cochain map whose components make the diagram

$$
\begin{array}{ccc}
C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) & \xrightarrow{\Omega''^n} & C^n(A, (A_{S^{-2}})^*) \\
\Big\downarrow{\scriptstyle \bar{\Phi}_*} & & \Big\downarrow{\scriptstyle \bar{\Phi}^n} \\
C^n(A, {}_\varepsilon A_{\mathrm{cad}}) & \xrightarrow{\Omega'^n} & C^n(A, A)
\end{array}
$$

commutative.

With the help of the monodromy matrix Q, the map $\bar{\Phi}^n$ can be calculated explicitly: For $f \in C^n(A, (A_{S^{-2}})^*) = \mathrm{Hom}_K(A^{\otimes n}, (A_{S^{-2}})^*)$ and $a_1, \ldots, a_n \in A$, we have

$$
\begin{aligned}
\bar{\Phi}^n(f)(a_1 \otimes \ldots \otimes a_n) &= (\Omega'^n \circ \bar{\Phi}_* \circ (\Omega''^n)^{-1})(f)(a_1 \otimes \ldots \otimes a_n) \\
&= a_{1(2)} \ldots a_{n(2)} \bar{\Phi}((\Omega''^n)^{-1}(f)(a_{1(1)} \otimes \ldots \otimes a_{n(1)})) \\
&= a_{1(3)} \ldots a_{n(3)} \bar{\Phi}(S(a_{n(1)}) \ldots S(a_{1(1)}).f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})) \\
&= a_{1(3)} \ldots a_{n(3)} (S(a_{n(1)}) \ldots S(a_{1(1)}).f(a_{1(2)} \otimes \ldots \otimes a_{n(2)}))(Q_1)Q_2 \\
&= a_{1(3)} \ldots a_{n(3)} f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})(Q_1.(S(a_{n(1)}) \ldots S(a_{1(1)})))Q_2 \\
&= f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})(Q_1 S^{-1}(a_{n(1)}) \ldots S^{-1}(a_{1(1)}))\, a_{1(3)} \ldots a_{n(3)} Q_2.
\end{aligned}
$$

In a similar way, we can generalize the antipode S to a cochain map between Hochschild cochain complexes: Since we have

$$S(\mathrm{cad}(b \otimes a)) = S(S^{-1}(a_{(2)})ba_{(1)}) = S(a_{(1)})S(b)a_{(2)} = \mathrm{ad}(S(b) \otimes a),$$

the antipode is a bimodule isomorphism from ${}_{\varepsilon}A_{\text{cad}}$ to ${}_{\varepsilon}A_{\text{ad}}$. Composition with S therefore yields an isomorphism

$$S_*\colon C^n(A, {}_{\varepsilon}A_{\text{cad}}) \to C^n(A, {}_{\varepsilon}A_{\text{ad}}),\ f \mapsto S \circ f$$

of cochain complexes. Now the isomorphisms Ω and Ω' from Section 2 enable us to obtain a cochain map between the original cochain complexes:

Definition 4.5 We define the antipode map for Hochschild cochain complexes as the cochain map from $C(A,A)$ to itself with components $S^n := \Omega^n \circ S_* \circ (\Omega'^n)^{-1}$. In other words, it is the unique cochain map whose components make the diagram

$$\begin{array}{ccc} C^n(A, {}_{\varepsilon}A_{\text{cad}}) & \xrightarrow{\Omega'^n} & C^n(A,A) \\ \Big\downarrow{\scriptstyle S_*} & & \Big\downarrow{\scriptstyle S^n} \\ C^n(A, {}_{\varepsilon}A_{\text{ad}}) & \xrightarrow{\Omega^n} & C^n(A,A) \end{array}$$

commutative.

As in the case of the Drinfel'd map, there is an explicit expression for the antipode map for cochain complexes: For $f \in C^n(A,A) = \operatorname{Hom}_K(A^{\otimes n}, A)$ and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned} S^n(f)(a_1 \otimes \ldots \otimes a_n) &= (\Omega^n \circ S_* \circ (\Omega'^n)^{-1})(f)(a_1 \otimes \ldots \otimes a_n) \\ &= a_{1(1)} \ldots a_{n(1)} S((\Omega'^n)^{-1}(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)})) \\ &= a_{1(1)} \ldots a_{n(1)} S(S^{-1}(a_{n(3)}) \ldots S^{-1}(a_{1(3)}) f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})) \\ &= a_{1(1)} \ldots a_{n(1)} S(f(a_{1(2)} \otimes \ldots \otimes a_{n(2)})) a_{1(3)} \ldots a_{n(3)}. \end{aligned}$$

5 The Action on the Hochschild Cochain Complex

We still remain in the situation described in Section 3 and Section 4. Our goal is to use the Radford map, the Drinfel'd map and the antipode map for Hochschild cochain complexes introduced in Section 4 in order to construct a projective action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on each Hochschild cohomology group $HH^n(A,A)$ in such a way that the action on the zeroth Hochschild cohomology group, which is, as we saw in Example 1.2, equal to the center $Z(A)$, coincides with the action described in Section 3. Up to homotopy, we will in fact construct a projective action of the modular group on the entire Hochschild cochain complex.

To define a projective representation of the modular group, we have to specify the images of the generators $\mathfrak{s}$ and $\mathfrak{t}$ introduced in Section 3 and prove that they satisfy the defining relations stated there. For the first generator $\mathfrak{s}$, we use the same approach as in Section 3 and map it to the composition of the Radford map, the Drinfel'd map and the antipode:

Definition 5.1 We define $\mathfrak{S}^n\colon C^n(A,A) \to C^n(A,A)$ as $\mathfrak{S}^n := S^n \circ \bar{\Phi}^n \circ \iota^n$.

Because the cochain complex versions of the Radford map, the Drinfel'd map and the antipode are cochain isomorphisms by construction, the maps $\mathfrak{S}^n$ are the components of a cochain automorphism of the Hochschild cochain complex. Its basic property is the following:

Lemma 5.2 The diagram

$$\begin{array}{ccc} C^n(A, {}_\varepsilon A_{\text{ad}}) & \xrightarrow{\Omega^n} & C^n(A,A) \\ \downarrow{\scriptstyle \mathfrak{S}_*} & & \downarrow{\scriptstyle \mathfrak{S}^n} \\ C^n(A, {}_\varepsilon A_{\text{ad}}) & \xrightarrow{\Omega^n} & C^n(A,A) \end{array}$$

commutes.

Proof: This is immediate from Lemma 4.3, Definition 4.4 and Definition 4.5: We have

$$\begin{aligned} \mathfrak{S}^n \circ \Omega^n &= S^n \circ \bar{\Phi}^n \circ \iota^n \circ \Omega^n = S^n \circ \bar{\Phi}^n \circ \Omega''^n \circ \iota_* \\ &= S^n \circ \Omega'^n \circ \bar{\Phi}_* \circ \iota_* = \Omega^n \circ S_* \circ \bar{\Phi}_* \circ \iota_* = \Omega^n \circ (S \circ \bar{\Phi} \circ \iota)_* = \Omega^n \circ \mathfrak{S}_* \end{aligned}$$

since successive composition with ι, $\bar{\Phi}$ and S is the same as composition with $\mathfrak{S}$. It may be noted that $\mathfrak{S}$, as the composition of the bimodule isomorphisms ι, $\bar{\Phi}$ and S, is a bimodule automorphism of ${}_\varepsilon A_{\text{ad}}$, so that the map $\mathfrak{S}_*$ on the left is indeed the component of a cochain map. □

It is not difficult to compute $\mathfrak{S}^n$ explicitly in terms of the monodromy matrix Q:

Corollary 5.3 For $f \in C^n(A,A) = \operatorname{Hom}_K(A^{\otimes n}, A)$ and $a_1, \ldots, a_n \in A$, we have

$$\mathfrak{S}^n(f)(a_1 \otimes \ldots \otimes a_n) = \rho\left(S(a_{n(2)}) \ldots S(a_{1(2)}) f(a_{1(3)} \otimes \ldots \otimes a_{n(3)}) Q_1\right) a_{1(1)} \ldots a_{n(1)} S(Q_2).$$

Proof: We have seen in Section 3 that $\mathfrak{S}(a) = \rho(aQ_1)S(Q_2)$, so that we get

$$\begin{aligned} \mathfrak{S}^n(f)(a_1 \otimes \ldots \otimes a_n) &= (\Omega^n \circ \mathfrak{S}_* \circ (\Omega^n)^{-1})(f)(a_1 \otimes \ldots \otimes a_n) \\ &= a_{1(1)} \ldots a_{n(1)} \mathfrak{S}((\Omega^n)^{-1}(f)(a_{1(2)} \otimes \ldots \otimes a_{n(2)})) \\ &= a_{1(1)} \ldots a_{n(1)} \mathfrak{S}(S(a_{n(2)}) \ldots S(a_{1(2)}) f(a_{1(3)} \otimes \ldots \otimes a_{n(3)})) \\ &= \rho\left(S(a_{n(2)}) \ldots S(a_{1(2)}) f(a_{1(3)} \otimes \ldots \otimes a_{n(3)}) Q_1\right) a_{1(1)} \ldots a_{n(1)} S(Q_2) \end{aligned}$$

from the preceding lemma. □

For the second generator $\mathfrak{t}$ of the modular group, we also proceed as in Section 3 and let it act on the cochain groups by multiplication with the ribbon element v:

Definition 5.4 We define $\mathfrak{T}^n \colon C^n(A,A) \to C^n(A,A)$ as

$$\mathfrak{T}^n(f)(a_1 \otimes \ldots \otimes a_n) := v f(a_1 \otimes \ldots \otimes a_n)$$

for $f \in C^n(A,A) = \operatorname{Hom}_K(A^{\otimes n}, A)$. In other words, using the map $\mathfrak{T}$ introduced in Section 3, we set $\mathfrak{T}^n := \mathfrak{T}_*$, the composition with $\mathfrak{T}$ on the left.

Because v is central, the maps $\mathfrak{T}^n$ commute with the differentials d^n and therefore constitute the components of a cochain map. The centrality of v also implies that the diagram

$$\begin{array}{ccc} C^n(A, {}_\varepsilon A_{\text{ad}}) & \xrightarrow{\Omega^n} & C^n(A, A) \\ \Big\downarrow \mathfrak{T}_* & & \Big\downarrow \mathfrak{T}^n \\ C^n(A, {}_\varepsilon A_{\text{ad}}) & \xrightarrow{\Omega^n} & C^n(A, A) \end{array}$$

is commutative. Here, it is understood that $\mathfrak{T}_*$ is also given on $C^n(A, {}_\varepsilon A_{\text{ad}})$ by multiplication with v, and not via the left or right action of v on ${}_\varepsilon A_{\text{ad}}$.

The key result that relates these maps to the modular group is the following theorem:

Theorem 5.5

1. We have $\mathfrak{S}^n \circ \mathfrak{T}^n \circ \mathfrak{S}^n = \rho(v)\, (\mathfrak{T}^n)^{-1} \circ \mathfrak{S}^n \circ (\mathfrak{T}^n)^{-1}$.

2. The cochain maps with components $(\mathfrak{S}^n)^4$ and $((\rho \otimes \rho)(Q))^2\, \mathrm{id}_{C^n(A,A)}$ are homotopic.

Proof: As recalled in Proposition 3.2, we have $\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\, \mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1}$. Combining the commutativity of the preceding diagram with Lemma 5.2, we therefore get

$$\begin{aligned} \mathfrak{S}^n \circ \mathfrak{T}^n \circ \mathfrak{S}^n \circ \Omega^n &= \Omega^n \circ \mathfrak{S}_* \circ \mathfrak{T}_* \circ \mathfrak{S}_* = \Omega^n \circ (\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S})_* \\ &= \rho(v)\, \Omega^n \circ (\mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1})_* = \rho(v)\, \Omega^n \circ \mathfrak{T}_*^{-1} \circ \mathfrak{S}_* \circ \mathfrak{T}_*^{-1} \\ &= \rho(v)\, (\mathfrak{T}^n)^{-1} \circ \mathfrak{S}^n \circ (\mathfrak{T}^n)^{-1} \circ \Omega^n. \end{aligned}$$

Because Ω^n is bijective, this proves our first assertion.

To prove the second assertion, it suffices to show that the cochain maps with components $(\Omega^n)^{-1} \circ (\mathfrak{S}^n)^4 \circ \Omega^n$ and $((\rho \otimes \rho)(Q))^2\, \mathrm{id}_{C^n(A, {}_\varepsilon A_{\text{ad}})}$ are homotopic, because (Ω^n) is an isomorphism of cochain complexes. By Lemma 5.2, we have $(\Omega^n)^{-1} \circ (\mathfrak{S}^n)^4 \circ \Omega^n = (\mathfrak{S}^4)_*$, and we also have $\mathfrak{S}^2 = (\rho \otimes \rho)(Q)\, \underline{S}^{-1}$ by Proposition 3.2. Therefore our second assertion will hold if we can show that the cochain map $(\underline{S}^{-2})_*$ is homotopic to the identity on the cochain complex $C(A, {}_\varepsilon A_{\text{ad}})$, or equivalently that the cochain map $(\underline{S}^2)_*$ is homotopic to the identity.

We know from Lemma 3.3 that $\underline{S}^2(a) = \mathrm{ad}(a \otimes v)$ for all $a \in A$. Because by definition the right action on the bimodule $M := {}_\varepsilon A_{\text{ad}}$ is given by the right adjoint action, this means in the notation of Proposition 1.3 that $(\underline{S}^2)_* = (r_v^M)_*$. Now Proposition 1.3 states that $(r_v^M)_*$ is homotopic to $(l_v^M)_*$. But $(l_v^M)_*$ is the identity: For $f \in C^n(A, M) = \mathrm{Hom}_K(A^{\otimes n}, {}_\varepsilon A_{\text{ad}})$ and $a_1, \ldots, a_n \in A$, we have

$$(l_v^M)_*(f)(a_1 \otimes \ldots \otimes a_n) = v.f(a_1 \otimes \ldots \otimes a_n) = \varepsilon(v) f(a_1 \otimes \ldots \otimes a_n) = f(a_1 \otimes \ldots \otimes a_n)$$

because $\varepsilon(v) = 1$. $\square$

As a consequence, we can generalize the projective action of the modular group on the center $Z(A)$ obtained in Corollary 3.4, which is by Example 1.2 equal to $HH^0(A, A)$, to an arbitrary Hochschild cohomology group $HH^n(A, A)$. For this, we denote the automorphisms of $HH^n(A, A)$ induced by the cochain maps $(\mathfrak{S}^n)$ and $(\mathfrak{T}^n)$ by $\overline{\mathfrak{S}^n}$ and $\overline{\mathfrak{T}^n}$, respectively, and by $P(\overline{\mathfrak{S}^n})$ and $P(\overline{\mathfrak{T}^n})$ we denote the corresponding automorphisms of the projective space $P(HH^n(A, A))$. We then have the following generalization of Corollary 3.4:

Corollary 5.6 There is a unique homomorphism from $\mathrm{SL}(2, \mathbb{Z})$ to $\mathrm{PGL}(HH^n(A, A))$ that maps $\mathfrak{s}$ to $P(\overline{\mathfrak{S}^n})$ and $\mathfrak{t}$ to $P(\overline{\mathfrak{T}^n})$.

Exactly as in the analogous discussion at the end of Section 3, this representation of the modular group is linear, and not only projective, if the unique ribbon-normalized right integral ρ satisfies $\rho(v^{-1}) = \pm 1$, or equivalently $(\rho \otimes \rho)(Q) = \pm 1$.

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