# A COUNTEREXAMPLE TO MONTGOMERY'S CONJECTURE ON DYNAMIC COLOURINGS OF REGULAR GRAPHS

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Abstract. A *dynamic colouring* of a graph is a proper colouring in which no neighbourhood of a non-leaf vertex is monochromatic. The *dynamic colouring number* $\chi_2(G)$ of a graph $G$ is the least number of colours needed for a dynamic colouring of $G$.

Montgomery conjectured that $\chi_2(G) \leq \chi(G) + 2$ for all regular graphs $G$, which would significantly improve the best current upper bound $\chi_2(G) \leq 2\chi(G)$. In this note, however, we show that this last upper bound is sharp by constructing, for every integer $n \geq 2$, a regular graph $G$ with $\chi(G) = n$ but $\chi_2(G) = 2n$. In particular, this disproves Montgomery's conjecture.

## 1. Introduction

A *proper $n$-colouring* of a graph $G$ is a map $c\colon V(G) \to [n]$ such that there is no edge $uv$ with $c(u) = c(v)$. An *$r$-dynamic $n$-colouring* is a proper $n$-colouring $c$ of $G$ such that for each vertex $v \in V(G)$ at least $\min\{r, d(v)\}$ colours are used on $N(v)$. The *$r$-dynamic chromatic number* $\chi_r(G)$, introduced by Montgomery [13], is the smallest $n$ such that $G$ has an $r$-dynamic $n$-colouring. Problems on $r$-dynamic colourings of graphs are an active area of research in graph theory, with many recent papers investigating properties of $r$-dynamic colourings, e.g. [2, 6, 10, 11, 12].

Of particular interest is the 2-dynamic chromatic number, which is often simply called the *dynamic chromatic number*. Note that $\chi_2(G) - \chi(G)$ can be arbitrarily large, which can be seen by considering subdivisions of complete graphs. However, in the case of regular graphs, Montgomery [13] conjectured the following:

**Conjecture 1** (Montgomery's Conjecture)**.** *If $G$ is a regular graph, then $\chi_2(G) \leq \chi(G) + 2$.*

Ahadi, Akbari, Deghan, and Ghanbari [1] conjectured further that if $\chi(G) \geq 4$ then $\chi(G) = \chi_2(G)$. However Alishahi [5] disproved the stronger conjecture by constructing graphs $G$ with $\chi(G) = n$ such that $\chi_2(G) \geq \chi(G) + 1$ for each $n \geq 2$.

There are several results showing that Conjecture 1 holds for certain classes of regular graphs. For example, Montgomery [13] showed that Conjecture 1 holds for claw-free graphs, and Akbari, Ghanbari and Jahanbekam [3] showed that Conjecture 1 holds for $d$-regular bipartite graphs with $d \geq 4$. Alishahi [5] verified Conjecture 1 if $G$ has diameter at most 2 and chromatic number at least 4.

There are also results bounding the dynamic chromatic number of regular graphs in terms of other invariants of the graph. For example, Dehghan, and Ahadi [8] showed that if $G$ is regular, then $\chi_2(G) \leq \chi(G) + 2\log\alpha(G) + 3$, where $\alpha$ is the independence number of the graph. Alishahi [4] showed that if $G$ is $d$-regular,

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then $\chi_2(G) \leq \chi(G) + 14.06 \log d + 1$, and Taherkhani [17] improved this bound to $\chi_2(G) \leq \chi(G) + 5.437 \log d + 2$. The best general bound for $\chi_2(G)$ depending only on $\chi(G)$ was given by Alishahi [4] who showed that $\chi_2(G) \leq 2\chi(G)$ for all regular graphs $G$.

in this paper we show that for every value of $\chi(G)$ there exists a graph for which this inequality is sharp, giving an infinite family of counterexamples to Montgomery's Conjecture.

**Theorem 1.1.** *For every natural number $n \geq 2$, there exists a regular graph $G_n$ with $\chi(G_n) = n$ and $\chi_2(G_n) = 2\chi(G_n)$.*

In general, the $r$-dynamic chromatic number of regular graphs cannot be bounded by $r\chi_r(G)$ as was shown by Jahanbekam, Kim, O, and West [9]. They showed that if $d = r$ then for infinitely many values of $r$ there exists a $d$-regular graph $G$ such that $\chi_r(G) \geq r^{1.377}\chi(G)$. However, they also showed that $\chi_r(G) \leq r\chi(G)$ for $d \geq (3 + o(1))\, r \log r$. Here we show that this bound is sharp even for arbitrarily large values of $d$.

**Theorem 1.2.** *For all natural numbers $r, n, \delta \geq 2$, there exists a $d$-regular graph $G = G(r, n, \delta)$ with $d > \delta$, $\chi(G) = n$ and $\chi_r(G) = r\chi(G)$.*

The dynamic chromatic number also has connections to the chromatic number of total dominating sets, which was observed by Alishahi [4]. We say a set $D \subset V(G)$ is a *total dominating set* of $G$ if every vertex in $V(G)$ is adjacent to some vertex of $D$. Let us write

$$\gamma(G) := \min\{\chi(G[D]) \,:\, D \text{ a total dominating set of } G\}.$$

In the 1970s, Berge (unpublished, see [7]) and independently Payan [14] conjectured that every regular graph contains two disjoint maximal independent sets. Payan [15, 16] showed that this is equivalent to $\gamma(G) = 2$ for every regular graph $G$ and disproved the conjecture by constructing regular graphs $G_p$ with $\gamma(G_p) > p$ for every $p \in \mathbb{N}$. His graphs have the property that $\chi(G_p) = 4p + 1$ and $\gamma(G_p) = p + 1$. The graphs we construct here are new examples showing that $\gamma(G)$ is unbounded for regular graphs. Moreover, our graphs have the even stronger property that $\chi(G) = \gamma(G)$.

**Theorem 1.3.** *For every natural number $n \geq 2$, there exists a regular graph $G_n$ with $\chi(G_n) = n = \gamma(G_n)$.*

## 2. Construction of the counterexamples

Our aim is to build, for fixed natural numbers $r, n, \delta \geq 2$, a $d$-regular graph $G := G(r, n, \delta)$ with $d > \delta$ and $\chi(G) = n$ but $\chi_r(G) = rn$ and $\gamma(G) = n$. The graph $G$ will consist of a disjoint union of many complete $n$-partite graphs, together with some supplementary vertices whose neighbourhoods are carefully chosen subsets of that disjoint union.

More formally, letting $m := \max\{\binom{rn-1}{r-1}n^2, \delta\}$ and $N := \binom{m-1}{n-1}$, we take a vertex $v_{i,j,k}$ for each $i \in [n]$, $j \in [N]$ and $k \in [m]$ and we add an edge from $v_{i,j,k}$ to $v_{i',j',k'}$ whenever $i \neq i'$ and $k = k'$. Additionally we take vertices $s_{i,X}$ for every $i \in [n]$ and every $X \subseteq [m]$ of size $n$, and we join $s_{i,X}$ to $v_{i',j,k}$ whenever $i = i'$ and $k \in X$ (see Fig 1). We call the graph with these vertices and edges $G$.

**Lemma 2.1.** *$G$ is $(nN)$-regular.*

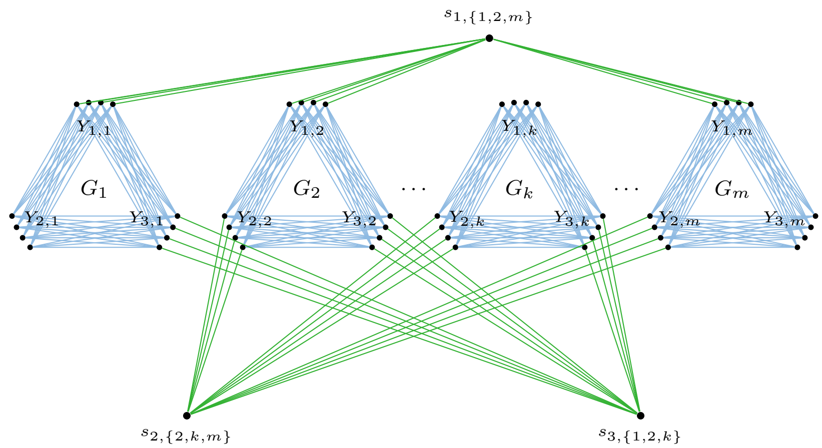


FIGURE 1. $G$ is composed of $m$ disjoint complete $n$-partite graphs $(G_k)_{k\in[m]}$ each with parts $(Y_{i,k})_{i\in[n]}$ of size $N$, along with vertices $s_{i,X}$ for every $(i, X) \in [n] \times \binom{m}{n}$, where $s_{i,X}$ is connected to every vertex in $\bigcup_{k\in X} Y_{i,k}$.

*Proof.* It is clear that vertices of the form $s_{i,X}$ have degree $nN$. Now we consider the neighbours of a vertex of the form $v_{i,j,k}$. This vertex has $(n-1)N$ neighbours of the form $v_{i',j',k'}$, and it has $N$ neighbours of the form $s_{i,X}$ since there are $\binom{m-1}{n-1}$ subsets $X$ of $[m]$ of size $n$ containing $k$. So in total it has $nN$ neighbours. □

**Lemma 2.2.** $\chi(G) = n$.

*Proof.* The chromatic number is at least $n$, since the subgraph on the $v_{i,1,1}$ with $i \in [n]$ is complete. But $G$ can be properly coloured with $n$ colours by assigning each vertex $v_{i,j,k}$ the colour $i$ and each vertex $s_{i,X}$ a colour other than $i$. □

**Lemma 2.3.** $\chi_r(G) = rn$.

*Proof.* We can $r$-dynamically colour $G$ with $rn$ colours by assigning each vertex $v_{i,j,k}$ with $j < r$ the colour $nj + i$, each vertex $v_{i,j,k}$ with $j \geq r$ the colour $i$, and each vertex $s_{i,X}$ a colour which is not congruent to $i$ modulo $n$.

Now consider a proper colouring of $G$ with at most $rn-1$ colours. We shall show that this colouring is not $r$-dynamic. For each $k \in [m]$ and each colour $c$, there is at most one $i \in [n]$ such that for some $j$ the vertex $v_{i,j,k}$ has colour $c$. Since there are only $rn - 1$ colours, there must be some $i_k \in [n]$ and some set $C_k$ of colours of size at most $r - 1$ such that all $v_{i_k,j,k}$ are coloured with a colour from $C_k$. By the pigeonhole principle, since $m > (n-1)n\binom{rn-1}{r-1}$, there is a set $X \subseteq [m]$ of size $n$ such that all $i_k$ with $k \in X$ take some common value $i$ and all $C_k$ with $k \in X$ take some common value $C$. But then all neighbours of $s_{i,X}$ are coloured with colours from $C$, and so the colouring is not $r$-dynamic. □

**Lemma 2.4.** $\gamma(G) = n$.

*Proof.* Consider a set $S \subseteq V(G)$ with $\chi(G[S]) \leq n-1$. For each $k \in [m]$ there exists at least one $i_k \in [n]$ such that $S$ contains none of the $v_{i_k,j,k}$. Since $m > (n-1)n$, by the pigeonhole principle, there exists a set $X \subseteq [m]$ of size $n$ such that all $i_k$

with $k \in X$ take some common value $i$. Now none of the neighbours of $s_{i,X}$ are in $S$, so $S$ is not a total dominating set. □

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