Quarter-pinched Einstein metrics interpolating between real and complex hyperbolic metrics

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Abstract

We show that the one-loop quantum deformation of the universal hypermultiplet provides a family of complete 1/4-pinched negatively curved quaternionic Kähler (i.e. half conformally flat Einstein) metrics g^c , $c \ge 0$, on \mathbb{R}^4 . The metric g^0 is the complex hyperbolic metric whereas the family $(g^c)_{c>0}$ is equivalent to a family of metrics $(h^b)_{b>0}$ depending on b = 1/c and smoothly extending to b = 0 for which h^0 is the real hyperbolic metric. In this sense the one-loop deformation interpolates between the real and the complex hyperbolic metrics. We also determine the (singular) conformal structure at infinity for the above families.

Keywords: quaternionic Kähler manifolds, Einstein deformations, negative sectional curvature, quarter pinching

MSC classification: 53C26.

Introduction

Einstein deformations of rank one symmetric spaces of non-compact type have been considered by various authors, see [P, L, B1, B2] and references therein. In particular, LeBrun has shown that the quaternionic hyperbolic metric on the smooth manifold \mathbb{R}^{4n} admits deformations by complete quaternionic Kähler metrics. These metrics are constructed using deformations of the twistor data and depend on functional parameters. However, the sectional curvature of the deformed metrics does not seem to have been studied. In previous work [ACM, ACDM] a geometric construction of a class of quaternionic Kähler manifolds of negative scalar curvature was described. The manifolds in this class are obtained from projective special Kähler manifolds and come in one-parameter families. In string theory, such families can be interpreted as perturbative quantum corrections to the hypermultiplet moduli space metric [RSV]. The one-parameter families are known as one-loop deformations of the supergravity c-map metrics. The simplest example corresponds to the case when the initial projective special Kähler manifold is a point. In that case one obtains the family of metrics

$$g^{c} = \frac{1}{4\rho^{2}} \left[\frac{\rho + 2c}{\rho + c} d\rho^{2} + \frac{\rho + c}{\rho + 2c} (d\tilde{\phi} + \zeta^{0} d\tilde{\zeta}_{0} - \tilde{\zeta}_{0} d\zeta^{0})^{2} + 2(\rho + 2c) \left((d\tilde{\zeta}_{0})^{2} + (d\zeta^{0})^{2} \right) \right], \qquad (0.1)$$

where $(\rho, \tilde{\phi}, \zeta^0, \tilde{\zeta}_0)$ are standard coordinates on the manifold $M := \mathbb{R}^{>0} \times \mathbb{R}^3 \cong \mathbb{R}^4$ and $c \ge 0$. This is a deformation of the complex hyperbolic metric g^0 (known as the universal hypermultiplet metric in the physics literature [RSV]) by complete quaternionic Kähler¹ metrics, see [ACDM, Remark 8]. Using the c-map and its one-loop deformation it is also possible to deform higher rank quaternionic Kähler symmetric spaces and, more generally, quaternionic Kähler homogeneous spaces by families of complete quaternionic Kähler metrics depending on one or several parameters [CDS, CDJL].

In this paper we prove that the metrics (0.1) are all negatively curved and $\frac{1}{4}$ -pinched, see Theorem 8. By similar calculations, we also show that Pedersen's deformation of the real hyperbolic 4-space², which depends on a parameter $m^2 \geq 0$, has negative curvature if $m^2 < 1$, see Theorem 11. These are presumably the first examples of non-locally symmetric complete Einstein four-manifolds of negative curvature. For the family (0.1), we show in Section 1 that the limit $c \to \infty$ is well-defined after a suitable change of coordinates and parameter, and that it is given by the real hyperbolic metric. Furthermore, we perform another change of coordinates in order to analyze the conformal structure at infinity. We find in Section 2 that the conformal structure induced by g^c (for $0 < c < \infty$) on the boundary sphere S^3 is singular precisely at a single point p_{∞} , which we can consider as the south pole, where it has a double pole. The point p_{∞} is also a special point with respect to the asymptotic behaviour of the metric. In fact, the metric g^c (considered as a metric on the 4-ball B^4 with boundary S^3) is asymptotic to the real hyperbolic metric on the complement in B^4 of any neighborhood of p_{∞} but it is not near p_{∞} . These observations show that the family of metrics g^c cannot be obtained as an Einstein deformation induced

¹Recall that in dimension four quaternionic Kähler manifolds are defined as half conformally flat Einstein manifolds.

²This deformation is induced by a deformation of the standard conformal structure of S^3 at the boundary of the real hyperbolic space by a rescaling along the fibres of the Hopf fibration [P].

by a deformation of the conformal structure at the boundary in the spirit of [B1].

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1 The limit $c \to \infty$

We introduce a second one-parameter family of metrics given by

$$h^{b} = \frac{1}{4\rho'^{2}} \left[\frac{b\rho' + 2}{b\rho' + 1} d\rho'^{2} + \frac{b\rho' + 1}{b\rho' + 2} (d\tilde{\phi}' + b\zeta'^{0} d\tilde{\zeta}_{0}' - b\tilde{\zeta}_{0}' d\zeta'^{0})^{2} + 2(b\rho' + 2) \left((d\tilde{\zeta}_{0}')^{2} + (d\zeta'^{0})^{2} \right) \right],$$
(1.1)

where b > 0. This is in fact equivalent to the one-loop deformation g^c for c > 0 under the identifications c = 1/b and $(\rho, \tilde{\phi}, \zeta^0, \tilde{\zeta}_0) = (\rho', \tilde{\phi}', \sqrt{b} \zeta'^0, \sqrt{b} \tilde{\zeta}_0')$. But now the family can be extended to the b = 0 case. This implies that after the above parameter-dependent coordinate transformation the $c \to \infty$ limit of the one-loop deformation g^c is indeed well-defined and is given by the metric

$$h^{0} = \frac{1}{4\rho'^{2}} \left[2 \,\mathrm{d}\rho'^{2} + \frac{1}{2} \,\mathrm{d}\tilde{\phi}'^{2} + 4(\mathrm{d}\tilde{\zeta}'_{0})^{2} + 4(\mathrm{d}\zeta'^{0})^{2} \right],\tag{1.2}$$

which has constant curvature -2.

2 Asymptotics and conformal structure at infinity

We would like to determine the conformal structure of the family of metrics $(g^c)_{c\geq 0}$ on the sphere at the boundary of M. In our coordinates, this consists of the hyperplane at $\rho = 0$, along with a point at infinity p_{∞} . In order to be able to directly see the singularity at p_{∞} , we consider the following change of coordinates:

$$\rho = \Re\left(\frac{1-z_1}{1+z_1}\right) - \left|\frac{z_2}{z_1+1}\right|^2 = \frac{1-|z_1|^2-|z_2|^2}{|z_1+1|^2},$$

$$\tilde{\phi} = -\Im\left(\frac{1-z_1}{1+z_1}\right), \quad \zeta := \zeta^0 + i\tilde{\zeta}_0 = \frac{\sqrt{2}\,z_2}{1+z_1}.$$
(2.1)

This is indeed a diffeomorphism from $M = \mathbb{R}_{>0} \times \mathbb{R}^3 = \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$ to the unit ball $B^2_{\mathbb{C}}$ in \mathbb{C}^2 , as it admits the following (smooth) inverse:

$$z_{1} = \frac{1 - \left(\rho + |\zeta|^{2}/2 - i\tilde{\phi}\right)}{1 + \left(\rho + |\zeta|^{2}/2 - i\tilde{\phi}\right)}, \quad z_{2} = \frac{\sqrt{2}\,\zeta}{1 + \left(\rho + |\zeta|^{2}/2 - i\tilde{\phi}\right)}.$$
(2.2)

As a result of the above change of coordinates, the boundary is mapped to the unit sphere $S^3 \subset \mathbb{C}^2$, and p_{∞} is mapped to the south pole $(z_1, z_2) = (-1, 0)$. We have the following proposition.

Proposition 1. In the coordinates introduced in (2.1), the conformal structure at the boundary $[g^c|_{\partial M}]$, for c > 0 is singular at p_{∞} $(z_1 = -1)$ and away from the singularity is given by the nondegenerate conformal structure:

$$[g^{c}|_{\partial M}] = \left[\left(2 \Re \left(d \left(\frac{1-z_{1}}{1+z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1+\overline{z}_{1}} \right) d \left(\frac{z_{2}}{1+z_{1}} \right) \right)^{2} + \frac{1}{2} \Im \left(d \left(\frac{1-z_{1}}{1+z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1+\overline{z}_{1}} \right) d \left(\frac{z_{2}}{1+z_{1}} \right) \right)^{2} + 4c \left| d \left(\frac{z_{2}}{1+z_{1}} \right) \right|^{2} \right|_{\partial M} \right].$$

$$(2.3)$$

Meanwhile the conformal structure for c = 0 is supported only on the CR distribution \mathcal{D} on S^3 and is given by

$$\left[g^{0}|_{\mathcal{D}\times\mathcal{D}}\right] = \left[\left.\left(\left|\mathrm{d}\left(\frac{z_{2}}{1+z_{1}}\right)\right|^{2}\right)\right|_{\mathcal{D}\times\mathcal{D}}\right].$$
(2.4)

Proof. For any $c \ge 0$, the metric g^c in the new coordinates is given by

$$g^{c} = \frac{1}{4\rho^{2}} \left[\frac{\rho + 2c}{\rho + c} \Re \left(d \left(\frac{1 - z_{1}}{1 + z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1 + \overline{z}_{1}} \right) d \left(\frac{z_{2}}{1 + z_{1}} \right) \right)^{2} + \frac{\rho + c}{\rho + 2c} \Im \left(d \left(\frac{1 - z_{1}}{1 + z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1 + \overline{z}_{1}} \right) d \left(\frac{z_{2}}{1 + z_{1}} \right) \right)^{2} + 2(\rho + 2c) \left| d \left(\frac{z_{2}}{1 + z_{1}} \right) \right|^{2} \right],$$
(2.5)

where now $\rho = \frac{1-|z_1|^2-|z_2|^2}{|z_1+1|^2}$ is considered as a function of (z_1, z_2) . The above metric is well-defined and nondegenerate when $|z_1|^2 + |z_2|^2 < 1$. Moreover we see that for c > 0, the conformal structure at the boundary $[g^c|_{\partial M}] = [(4\rho^2 g^c)|_{\partial M}]$ is singular at $z_1 = -1$. Away from the singularity, it may be computed to be the following:

$$[g^{c}|_{\partial M}] = \left[\left(2 \Re \left(\mathrm{d} \left(\frac{1-z_{1}}{1+z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1+\overline{z}_{1}} \right) \mathrm{d} \left(\frac{z_{2}}{1+z_{1}} \right) \right)^{2} + \frac{1}{2} \Im \left(\mathrm{d} \left(\frac{1-z_{1}}{1+z_{1}} \right) - \left(\frac{2 \overline{z}_{2}}{1+\overline{z}_{1}} \right) \mathrm{d} \left(\frac{z_{2}}{1+z_{1}} \right) \right)^{2} + 4c \left| \mathrm{d} \left(\frac{z_{2}}{1+z_{1}} \right) \right|^{2} \right) \right|_{\partial M} \right].$$

Meanwhile, in the case c = 0, the (rescaled) metric in (2.5) becomes:

$$2\rho g^{0} = \frac{1}{2\rho} \left| d\left(\frac{1-z_{1}}{1+z_{1}}\right) - \left(\frac{2\,\overline{z}_{2}}{1+\overline{z}_{1}}\right) d\left(\frac{z_{2}}{1+z_{1}}\right) \right|^{2} + \left| d\left(\frac{z_{2}}{1+z_{1}}\right) \right|^{2}.$$
 (2.6)

The second term stays finite at the boundary but the first term blows up, except on its kernel, which may be verified to be spanned by the following two vector fields:

$$\overline{z}_2 \frac{\partial}{\partial z_1} - \left(\frac{1 - |z_2|^2 + \overline{z}_1}{1 + z_1}\right) \frac{\partial}{\partial z_2}, \quad z_2 \frac{\partial}{\partial \overline{z}_1} - \left(\frac{1 - |z_2|^2 + z_1}{1 + \overline{z}_1}\right) \frac{\partial}{\partial \overline{z}_2}.$$

At the boundary, the above become vector fields spanning the CR distribution \mathcal{D} on S^3 :

$$\overline{z}_2 \frac{\partial}{\partial z_1} - \overline{z}_1 \frac{\partial}{\partial z_2}, \quad z_2 \frac{\partial}{\partial \overline{z}_1} - z_1 \frac{\partial}{\partial \overline{z}_2}.$$

The conformal structure at the boundary $[g^0|_{\mathcal{D}\times\mathcal{D}}]$ is defined as the nondegenerate conformal structure on \mathcal{D} obtained by keeping only the finite term in the above decomposition (2.6), see [B1]. Thus the conformal structure $[g^0|_{\partial M}]$ is supported only on the CR distribution \mathcal{D} and is given by

$$\left[g^{0}|_{\mathcal{D}\times\mathcal{D}}\right] := \left[(2\rho g^{0})|_{\mathcal{D}\times\mathcal{D}}\right] = \left[\left(\left|d\left(\frac{z_{2}}{1+z_{1}}\right)\right|^{2}\right)\right|_{\mathcal{D}\times\mathcal{D}}\right].$$

We would also like to determine the conformal structure of the family of metrics $(h^b)_{b\geq 0}$ on the sphere at the boundary of M. As in the case of g^c above, in order to directly see the singularity at the point at infinity p_{∞} , we again carry out a change of coordinates that maps $M = \mathbb{R}_{>0} \times \mathbb{R}^3$ to the unit ball $B^4_{\mathbb{R}}$ in \mathbb{R}^4 :

$$\rho' = \frac{1 - w^2 - x^2 - y^2 - z^2}{(1 + w)^2 + x^2 + y^2 + z^2}, \quad \tilde{\phi}' = \frac{4x}{(1 + w)^2 + x^2 + y^2 + z^2},$$

$$\zeta'^0 = \frac{\sqrt{2}y}{(1 + w)^2 + x^2 + y^2 + z^2}, \quad \tilde{\zeta}'_0 = \frac{\sqrt{2}z}{(1 + w)^2 + x^2 + y^2 + z^2}.$$
(2.7)

This is indeed a diffeomorphism, with (smooth) inverse given by

$$w = \frac{1 - {\rho'}^2 - \tilde{\phi}'^2/4 - 2({\zeta'}^0)^2 - 2\tilde{\zeta}_0'^2}{(1 + {\rho'})^2 + \tilde{\phi}'^2/4 + 2({\zeta'}^0)^2 + 2\tilde{\zeta}_0'^2}, \quad x = \frac{\tilde{\phi}'}{(1 + {\rho'})^2 + \tilde{\phi}'^2/4 + 2({\zeta'}^0)^2 + 2\tilde{\zeta}_0'^2}, \quad z = \frac{2\sqrt{2}\tilde{\zeta}_0'}{(1 + {\rho'})^2 + \tilde{\phi}'^2/4 + 2({\zeta'}^0)^2 + 2\tilde{\zeta}_0'^2}, \quad (2.8)$$

As a result of the above change of coordinates, the boundary is mapped to the unit sphere $S^3 \subset \mathbb{R}^4$, and the point at infinity p_{∞} is mapped to the south pole (w, x, y, z) = (-1, 0, 0, 0).

Proposition 2. In the coordinates introduced in (2.7), the conformal structure $[h^b|_{\partial M}]$ on the boundary sphere for b > 0 is singular at w = -1 and away from the singularity is given by

$$\begin{bmatrix} h^{b}|_{\partial M} \end{bmatrix} = \left[\left(\frac{1}{2} \left(d \left(\frac{2x}{1+w} \right) + \frac{b}{2} \left(\frac{y}{1+w} \right) d \left(\frac{z}{1+w} \right) - \frac{b}{2} \left(\frac{z}{1+w} \right) d \left(\frac{y}{1+w} \right) \right)^{2} + 2 \left(\left(d \left(\frac{y}{1+w} \right) \right)^{2} + \left(d \left(\frac{z}{1+w} \right) \right)^{2} \right) \right) \right|_{\partial M} \right].$$
(2.9)

Moreover, for b = 0, the conformal structure $[h^0|_{\partial M}]$ is the standard conformal structure on S^3 .

Proof. At the boundary and away from the south pole, we have $w^2 + x^2 + y^2 + z^2 = 1$ and $w \neq -1$. So, the restrictions of the coordinate functions $\rho', \tilde{\phi}', \zeta'^0, \tilde{\zeta}'_0$ to ∂M are given as functions of w, x, y, z as follows:

$$\rho'|_{\partial M} = 0, \quad \tilde{\phi}'|_{\partial M} = \frac{2x}{1+w}, \quad \zeta'^0|_{\partial M} = \frac{y}{\sqrt{2}(1+w)}, \quad \tilde{\zeta}'_0|_{\partial M} = \frac{z}{\sqrt{2}(1+w)}.$$
 (2.10)

A straightforward substitution therefore yields

$$(4\rho'^{2}h^{b})|_{\partial M} = \left(\frac{1}{2}\left(\mathrm{d}\left(\frac{2x}{1+w}\right) + \frac{b}{2}\left(\frac{y}{1+w}\right)\mathrm{d}\left(\frac{z}{1+w}\right) - \frac{b}{2}\left(\frac{z}{1+w}\right)\mathrm{d}\left(\frac{y}{1+w}\right)\right)^{2} + 2\left(\left(\mathrm{d}\left(\frac{y}{1+w}\right)\right)^{2} + \left(\mathrm{d}\left(\frac{z}{1+w}\right)\right)^{2}\right)\right)\Big|_{\partial M}.$$

$$(2.11)$$

The conformal structure is nondegenerate with a double pole at the south pole (i.e. w = -1) for b > 0. When b = 0, the above becomes:

$$\begin{bmatrix} h^{0}|_{\partial M} \end{bmatrix} = \left[\left(2 \left(d \left(\frac{x}{1+w} \right) \right)^{2} + 2 \left(d \left(\frac{y}{1+w} \right) \right)^{2} + 2 \left(d \left(\frac{z}{1+w} \right) \right)^{2} \right) \Big|_{\partial M} \right] \\ = \left[\left(\frac{2 (dx^{2} + dy^{2} + dz^{2})}{(1+w)^{2}} + \frac{2 (x^{2} + y^{2} + z^{2}) dw^{2}}{(1+w)^{4}} - \frac{4 (x \, dx + y \, dy + z \, dz) \, dw}{(1+w)^{3}} \right) \Big|_{\partial M} \right] \\ = \left[\left(\frac{2 (dx^{2} + dy^{2} + dz^{2})}{(1+w)^{2}} + \frac{2 (1-w^{2}) dw^{2}}{(1+w)^{4}} + \frac{4 (w \, dw) \, dw}{(1+w)^{3}} \right) \Big|_{\partial M} \right] \\ = \left[\left(\frac{2 (dw^{2} + dx^{2} + dy^{2} + dz^{2})}{(1+w)^{2}} \right) \Big|_{\partial M} \right] = \left[\left(dw^{2} + dx^{2} + dy^{2} + dz^{2} \right) \Big|_{\partial M} \right].$$

$$(2.12)$$

This is the standard conformal structure on S^3 i.e. the conformal class to which the restriction of the Euclidean metric on \mathbb{R}^4 to S^3 belongs.

3 Computation of the curvature tensor

In order to compute the curvature associated with the metric in (0.1), we make use of the Cartan formalism. In this formalism, we choose an orthonormal frame $(e_I)_{I=1,...,4}$ and denote the dual co-frame by (θ^I) so that $g^c = \sum_I \theta^I \otimes \theta^I$. The way we have presented the metric in (0.1) suggests an obvious choice, namely

$$\theta^{1} := F(\rho) \,\mathrm{d}\rho, \qquad \theta^{2} := G(\rho)(\mathrm{d}\tilde{\phi} + \zeta^{0}\mathrm{d}\tilde{\zeta}_{0} - \tilde{\zeta}_{0}\mathrm{d}\zeta^{0}), \theta^{3} := H(\rho) \,\mathrm{d}\tilde{\zeta}_{0}, \qquad \theta^{4} := H(\rho) \,\mathrm{d}\zeta^{0}.$$

$$(3.1)$$

where $F(\rho), G(\rho), H(\rho)$ are functions of ρ given by

$$F(\rho) = \frac{1}{2\rho} \sqrt{\frac{\rho + 2c}{\rho + c}}, \quad G(\rho) = \frac{1}{2\rho} \sqrt{\frac{\rho + c}{\rho + 2c}}, \quad H(\rho) = \frac{\sqrt{2(\rho + 2c)}}{2\rho}.$$
 (3.2)

The $\mathfrak{so}(4)$ -valued connection 1-form $\omega = (\omega_J^I)$ and curvature 2-form $\Omega = (\Omega_J^I)$ corresponding to the Levi-Civita connection ∇ and its curvature tensor R are defined by

$$\nabla_v e_I = \sum_J \omega_I^J(v) e_J, \quad \Omega_I^J(v, w) = g^c(R(v, w) e_I, e_J), \tag{3.3}$$

for any vector vector fields v and w. The forms ω_I^J and Ω_I^J can be calculated through the Cartan structural equations:

$$\mathrm{d}\theta^{I} = \sum_{J} \theta^{J} \wedge \omega_{J}^{I}, \quad \mathrm{d}\omega_{J}^{I} = \Omega_{J}^{I} + \sum_{K} \omega_{J}^{K} \wedge \omega_{K}^{I}.$$
(3.4)

In fact, the first equation is equivalent to the vanishing of torsion and determines the forms $\omega_I^J = -\omega_J^I$ uniquely. We now gather together the results of the calculation in the following two lemmata. We omit the proofs, which consist of just checking the structure equations.

Lemma 3. The connection 1-forms ω_J^I in (3.4) are given by

$$\omega_{2}^{1} = -\omega_{1}^{2} = \frac{1}{F(\rho)} \frac{2\rho^{2} + 5c\rho + 4c^{2}}{2\rho(\rho + c)(\rho + 2c)} \theta^{2}, \quad \omega_{3}^{1} = -\omega_{1}^{3} = \frac{1}{F(\rho)} \frac{\rho + 4c}{2\rho(\rho + 2c)} \theta^{3},$$

$$\omega_{4}^{1} = -\omega_{1}^{4} = \frac{1}{F(\rho)} \frac{\rho + 4c}{2\rho(\rho + 2c)} \theta^{4}, \qquad \omega_{3}^{2} = -\omega_{2}^{3} = -\frac{1}{F(\rho)} \frac{1}{2(\rho + 2c)} \theta^{4}, \qquad (3.5)$$

$$\omega_{4}^{2} = -\omega_{2}^{4} = \frac{1}{F(\rho)} \frac{1}{2(\rho + 2c)} \theta^{3}, \qquad \omega_{4}^{3} = -\omega_{3}^{4} = \frac{1}{F(\rho)} \frac{1}{2(\rho + 2c)} \theta^{2}.$$

Lemma 4. The curvature 2-forms Ω_J^I in (3.4) are given by

$$\Omega_{2}^{1} = -\Omega_{1}^{2} = -A_{\mathrm{I}}(\rho) \,\theta^{1} \wedge \theta^{2} + 2A_{\mathrm{III}}(\rho) \,\theta^{3} \wedge \theta^{4},
\Omega_{3}^{1} = -\Omega_{1}^{3} = -A_{\mathrm{II}}(\rho) \,\theta^{1} \wedge \theta^{3} + A_{\mathrm{III}}(\rho) \,\theta^{2} \wedge \theta^{4},
\Omega_{4}^{1} = -\Omega_{1}^{4} = -A_{\mathrm{II}}(\rho) \theta^{1} \wedge \theta^{4} - A_{\mathrm{III}}(\rho) \,\theta^{2} \wedge \theta^{3},
\Omega_{3}^{2} = -\Omega_{2}^{3} = -A_{\mathrm{III}}(\rho) \,\theta^{1} \wedge \theta^{4} - A_{\mathrm{II}}(\rho) \,\theta^{2} \wedge \theta^{3},
\Omega_{4}^{2} = -\Omega_{2}^{4} = A_{\mathrm{III}}(\rho) \,\theta^{1} \wedge \theta^{3} - A_{\mathrm{II}}(\rho) \,\theta^{2} \wedge \theta^{4},
\Omega_{4}^{3} = -\Omega_{3}^{4} = 2A_{\mathrm{III}}(\rho) \,\theta^{1} \wedge \theta^{2} - A_{\mathrm{I}}(\rho) \,\theta^{3} \wedge \theta^{4},$$
(3.6)

where A_{I} , A_{II} , and A_{III} are given by

$$A_{\rm I}(\rho) := \frac{4\rho^3 + 12c\rho^2 + 24c^2\rho + 16c^3}{(\rho + 2c)^3},$$

$$A_{\rm II}(\rho) := \frac{\rho^3 + 12c\rho^2 + 24c^2\rho + 16c^3}{(\rho + 2c)^3},$$

$$A_{\rm III}(\rho) := -\frac{\rho^3}{(\rho + 2c)^3}.$$

(3.7)

4 Eigenspaces of the curvature operator

In this section we consider the curvature operator $\mathscr{R}: \Lambda^2 TM \to \Lambda^2 TM$ which is defined by

$$g^{c}(\mathscr{R}X \wedge Y, Z \wedge W) = g^{c}(R(X, Y)W, Z),$$

where on the left-hand side g^c denotes the scalar product on bi-vectors which is induced by the Riemannian metric g^c :

$$g^{c}(X \wedge Y, Z \wedge W) = g^{c}(X, Z)g^{c}(Y, W) - g^{c}(X, W)g^{c}(Y, Z).$$

Identifying vector with co-vectors by means of the metric, we will consider the curvature operator as a map

$$\mathscr{R}: \Lambda^2 T^* M \to \Lambda^2 T^* M. \tag{4.1}$$

As such it maps $\theta^I \wedge \theta^J$ to Ω^I_J . The endomorphism \mathscr{R} is self-adjoint with respect to (the metric on $\Lambda^2 T^* M$ induced by) g^c . It follows, that all eigenvalues are real and that there exists an orthonormal eigenbasis.

Proposition 5. The following (anti-)self-dual 2-forms

$$\alpha_{JKL}^{\pm} = \theta^1 \wedge \theta^J \pm \theta^K \wedge \theta^L, \qquad (4.2)$$

where (J, K, L) is a cyclic permutation of (2, 3, 4), form an eigenbasis of the curvature operator (4.1) of the one-loop deformation (0.1). The corresponding eigenvalues λ_{JKL}^{\pm} are

$$\lambda_{234}^{+} = -\frac{6\rho^{3} + 12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}},$$

$$\lambda_{234}^{-} = \lambda_{423}^{-} = -\frac{2\rho^{3} + 12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}},$$

$$\lambda_{342}^{+} = \lambda_{423}^{+} = -\frac{12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}}.$$
(4.3)

In particular, when $c \neq 0$, the above depends only on the ratio $\tilde{\rho} := \rho/c$:

$$\lambda_{234}^{+} = -\frac{6\tilde{\rho}^{3} + 12\tilde{\rho}^{2} + 24\tilde{\rho} + 16}{(\tilde{\rho} + 2)^{3}},$$

$$\lambda_{234}^{-} = \lambda_{342}^{-} = \lambda_{423}^{-} = -\frac{2\tilde{\rho}^{3} + 12\tilde{\rho}^{2} + 24\tilde{\rho} + 16}{(\tilde{\rho} + 2)^{3}},$$

$$\lambda_{342}^{+} = \lambda_{423}^{+} = -\frac{12\tilde{\rho}^{2} + 24\tilde{\rho} + 16}{(\tilde{\rho} + 2)^{3}}.$$
(4.4)

Proof: From Lemma 4 we see that \mathscr{R} is block diagonal, whereby the bundle $\Lambda^2 T^* M$ of 2-forms decomposes into three invariant subbundles $\Lambda^2_{234}T^*M$, $\Lambda^2_{342}T^*M$, and $\Lambda^2_{423}T^*M$, where $\Lambda^2_{JKL}T^*M$ denotes the span of $\theta^1 \wedge \theta^J$ and $\theta^K \wedge \theta^L$. By inspection, we may read off the two eigen-2-forms α^{\pm}_{JKL} in $\Lambda^2_{JKL}T^*M$. The corresponding eigenvalues are

$$\lambda_{234}^{+} = -A_{\rm I} + 2A_{\rm III} = -\frac{6\rho^{3} + 12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}},$$

$$\lambda_{234}^{-} = -A_{\rm I} - 2A_{\rm III} = -\frac{2\rho^{3} + 12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}},$$

$$\lambda_{342}^{-} = \lambda_{423}^{-} = -A_{\rm II} + A_{\rm III} = -\frac{2\rho^{3} + 12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}},$$

$$\lambda_{342}^{+} = \lambda_{423}^{+} = -A_{\rm II} - A_{\rm III} = -\frac{12c\rho^{2} + 24c^{2}\rho + 16c^{3}}{(\rho + 2c)^{3}}.$$

(4.5)

5 Sectional curvature and pinching of the one-loop deformation

Since any element of $\Lambda^2 TM$ can be written as a linear combination of eigenvectors of \mathscr{R} , the sectional curvature

$$K(\Pi) = g^c(\mathscr{R}u \wedge v, u \wedge v)$$

of a plane $\Pi \subset TM$ with orthonormal basis (u, v) can be written as a convex linear combination of the eigenvalues of \mathscr{R} . So the spectrum of \mathscr{R} , determined in Lemma 5, shall provide bounds on K.

In order to obtain the pointwise maximum and minimum of the sectional curvature one has to minimise and maximise $g^c(\Re\alpha, \alpha)$ subject to the conditions $\alpha \wedge \alpha = 0$ (decomposability) and $g^c(\alpha, \alpha) = 1$. This leads us to the following lemma

Lemma 6. For any point $p \in M$, we have the following bounds for the sectional curvature

of the one-loop deformation (0.1):

$$\max_{\Pi \subset T_p M} K(\Pi) = \frac{1}{2} (\max\{\lambda_{234}^+(p), \lambda_{342}^+(p), \lambda_{423}^+(p)\} + \max\{\lambda_{234}^-(p), \lambda_{342}^-(p), \lambda_{423}^-(p)\}),$$

$$\min_{\Pi \subset T_p M} K(\Pi) = \frac{1}{2} (\min\{\lambda_{234}^+(p), \lambda_{342}^+(p), \lambda_{423}^+(p)\} + \min\{\lambda_{234}^-(p), \lambda_{342}^-(p), \lambda_{423}^-(p)\}).$$
(5.1)

Proof. We consider a general 2-form α written in terms of the eigen-2-forms as follows

$$\alpha = \sum_{\epsilon,(J,K,L)} a^{\epsilon}_{JKL} \alpha^{\epsilon}_{JKL}, \qquad (5.2)$$

where (J, K, L) runs over the cyclic permutations of (2, 3, 4), and ϵ runs over the values \pm . By decomposing α into its self-dual and anti-self-dual parts, we see that two equations $\alpha \wedge \alpha = 0$ and $g^{c}(\alpha, \alpha) = 1$ are together equivalent to

$$(a_{234}^+)^2 + (a_{342}^+)^2 + (a_{423}^+)^2 = \frac{1}{4},$$

$$(a_{234}^-)^2 + (a_{342}^-)^2 + (a_{423}^-)^2 = \frac{1}{4}.$$
(5.3)

On plugging (5.2) into $g^{c}(\mathscr{R}\alpha, \alpha)$, we find that

$$K(\Pi) = \frac{1}{2} \left[4(a_{234}^+)^2 \lambda_{234}^+ + 4(a_{342}^+)^2 \lambda_{324}^+ + 4(a_{423}^+)^2 \lambda_{423}^+ \right] + \frac{1}{2} \left[4(a_{234}^-)^2 \lambda_{234}^- + 4(a_{342}^-)^2 \lambda_{324}^- + 4(a_{423}^-)^2 \lambda_{423}^- \right].$$
(5.4)

Under the constraint (5.3), the expressions within the square brackets are each convex combinations of three eigenvalues of \mathscr{R} . Therefore in order to maximise or minimise $K(\Pi)$ we need to respectively maximise or minimise these convex combinations separately. \Box

In the limit $\tilde{\rho} \to 0$, all the eigenvalues become -2 as for the real hyperbolic space $\mathbb{R}\mathbf{H}^4$ with constant negative sectional curvature -2. Meanwhile, in the limit $\tilde{\rho} \to \infty$, the pointwise maximum of the sectional curvature is -1 and the pointwise minimum is -4, giving a pinching of 1/4 as for the complex hyperbolic plane $\mathbb{C}\mathbf{H}^2$.

The interpolation of the pinching between these two limits is described in the following proposition.

Proposition 7. The pointwise pinching of the metric g^c for c > 0 at a point $p = (c\tilde{\rho}, \tilde{\phi}, \tilde{\zeta}_0, \zeta^0) \in M$ is given by

$$\delta_p := \frac{\max\{K(\Pi) \mid \Pi \subset T_p M\}}{\min\{K(\Pi) \mid \Pi \subset T_p M\}} = \frac{\tilde{\rho}^3 + 12\tilde{\rho}^2 + 24\tilde{\rho} + 16}{4\tilde{\rho}^3 + 12\tilde{\rho}^2 + 24\tilde{\rho} + 16}.$$
(5.5)

Proof. We note that we have $\lambda_{234}^+ < \lambda_{234}^- = \lambda_{342}^- = \lambda_{423}^- < \lambda_{342}^+ = \lambda_{423}^+$ for all $\tilde{\rho} > 0$. So, we have for all $p \in M$

$$\begin{aligned} \max\{\lambda_{234}^+(p), \lambda_{342}^+(p), \lambda_{423}^+(p)\} &= \lambda_{342}^+(p) = \lambda_{423}^+(p), \\ \min\{\lambda_{234}^+(p), \lambda_{342}^+(p), \lambda_{423}^+(p)\} &= \lambda_{234}^+, \\ \max\{\lambda_{234}^-(p), \lambda_{342}^-(p), \lambda_{423}^-(p)\} &= \lambda_{234}^-(p) = \lambda_{342}^-(p) = \lambda_{423}^-(p), \\ \min\{\lambda_{234}^-(p), \lambda_{342}^-(p), \lambda_{423}^-(p)\} &= \lambda_{234}^-(p) = \lambda_{342}^-(p) = \lambda_{423}^-(p). \end{aligned}$$

It now follows from Lemma 6 that the pointwise pinching at $p = (c\tilde{\rho}, \tilde{\phi}, \tilde{\zeta}_0, \zeta^0)$ is given by

$$\delta_p = \frac{\lambda_{342}^+(p) + \lambda_{234}^-(p)}{\lambda_{234}^+(p) + \lambda_{234}^-(p)} = \frac{\tilde{\rho}^3 + 12\tilde{\rho}^2 + 24\tilde{\rho} + 16}{4\tilde{\rho}^3 + 12\tilde{\rho}^2 + 24\tilde{\rho} + 16},$$

as was to be shown.

Now that we have a concrete expression for the pointwise pinching, we can derive our main result.

Theorem 8. For the one-loop deformation g^c , c > 0, the pinching function $p \mapsto \delta_p$ defined in (5.5) satisfies $\frac{1}{4} < \delta < 1$ and attains the boundary values asymptotically when $\tilde{\rho} = \rho/c$ approaches 0 or ∞ , respectively, which is to say, M is everywhere (at least) "quarter-pinched".

Proof. For any $\tilde{\rho} > 0$, we see that

$$1 > \delta_p = \frac{1}{4} + \frac{9\tilde{\rho}^2 + 18\tilde{\rho} + 12}{4\tilde{\rho}^3 + 12\tilde{\rho}^2 + 24\tilde{\rho} + 16} > \frac{1}{4},\tag{5.6}$$

and that both boundary values are attained asymptotically.

6 Pedersen metric

We now consider the Pedersen metric defined on the unit ball $B^4_{\mathbb{R}}$ as discussed in [P]:

$$\kappa^{m} = \frac{1}{(1-\varrho^{2})^{2}} \left(\frac{1+m^{2}\varrho^{2}}{1+m^{2}\varrho^{4}} \,\mathrm{d}\varrho^{2} + \varrho^{2}(1+m^{2}\varrho^{2}) \left(\sigma_{1}^{2}+\sigma_{2}^{2}\right) + \frac{\varrho^{2}(1+m^{2}\varrho^{4})}{1+m^{2}\varrho^{2}} \,\sigma_{3}^{2} \right), \quad (6.1)$$

where the boundary is the sphere at $\rho = 1$ and $\sigma_1, \sigma_2, \sigma_3$ are the three left-invariant 1-forms on S^3 satisfying $d\sigma_i = \sum_{j,k} \varepsilon_{ijk} \sigma_j \wedge \sigma_k$. As in the case of the 1-loop deformed universal hypermultiplet metric, there is an obvious choice of an orthonormal co-frame (θ^I) , given by

$$\theta^{1} = \frac{\varrho}{(1-\varrho^{2})}\sqrt{1+m^{2}\varrho^{2}}\,\sigma_{1}, \quad \theta^{2} = \frac{\varrho}{(1-\varrho^{2})}\sqrt{1+m^{2}\varrho^{2}}\,\sigma_{2},$$

$$\theta^{3} = \frac{\varrho}{(1-\varrho^{2})}\sqrt{\frac{1+m^{2}\varrho^{4}}{1+m^{2}\varrho^{2}}}\,\sigma_{3}, \quad \theta^{4} = \frac{1}{(1-\varrho^{2})}\sqrt{\frac{1+m^{2}\varrho^{2}}{1+m^{2}\varrho^{4}}}\,\mathrm{d}\varrho.$$
 (6.2)

The steps in the previous sections for the calculation of the eigenvalues and an eigenbasis of the curvature operator $\mathscr{R} : \Lambda^2 T^* M \to \Lambda^2 T^* M$ may be repeated for the Pedersen metric. We summarize the results in the next proposition.

Proposition 9. The following (anti-)self-dual 2-forms

$$\beta_{IJK}^{\pm} := \theta^I \wedge \theta^J \pm \theta^K \wedge \theta^4 \tag{6.3}$$

where (I, J, K) is a cyclic permutation of (1, 2, 3), form an eigenbasis of the curvature operator \mathscr{R} of the Pedersen metric (6.1). The corresponding eigenvalues ν_{IJK}^{\pm} are

$$\nu_{123}^{+} = \nu_{231}^{+} = \nu_{312}^{+} = -4,$$

$$\nu_{123}^{-} = -4 \left(1 - \frac{2m^2 \left(1 - \varrho^2\right)^3}{\left(m^2 \varrho^2 + 1\right)^3} \right),$$

$$\nu_{231}^{-} = \nu_{312}^{-} = -4 \left(1 + \frac{m^2 \left(1 - \varrho^2\right)^3}{\left(m^2 \varrho^2 + 1\right)^3} \right).$$
(6.4)

In order to obtain the pointwise maximum and minimum of the sectional curvature one has to minimise and maximise $\kappa^m(\mathscr{R}\beta,\beta)$ subject to the conditions $\beta \wedge \beta = 0$ (decomposability) and $\kappa^m(\beta,\beta) = 1$. Again, this calculation proceeds exactly as earlier and so we just summarise the result in the following proposition.

Proposition 10. The pointwise maximum and pointwise minimum of the sectional curvature of the Pedersen metric is given by

$$\max_{\Pi \subset T_p M} K(\Pi) = -4 \left(1 - \frac{m^2 \left(1 - \varrho^2\right)^3}{\left(m^2 \varrho^2 + 1\right)^3} \right),$$

$$\min_{\Pi \subset T_p M} K(\Pi) = -4 \left(1 + \frac{m^2 \left(1 - \varrho^2\right)^3}{2 \left(m^2 \varrho^2 + 1\right)^3} \right).$$
(6.5)

In particular, a straightforward rearrangement shows that the pointwise maximum $\max_{\Pi \subset T_pM} K(\Pi)$ becomes nonnegative when the following condition holds:

$$\varrho^2 \le \frac{\sqrt[3]{m^2 - 1}}{m^2 + \sqrt[3]{m^2}}.$$
(6.6)

Note that this condition cannot hold if $m^2 < 1$. As a consequence we have the following result.

Theorem 11. The Pedersen metric (6.1) has negative sectional curvature if and only if $m^2 < 1$. For $m^2 > 1$ (respectively $m^2 = 1$) there are negative as well as positive (respectively zero) sectional curvatures near (respectively at) the origin $\rho = 0$.

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