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Orbifold Construction for Topological Field Theories

Christoph Schweigert and Lukas Woike

Fachbereich Mathematik, Universität Hamburg
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D – 20 146 Hamburg

Abstract. We formulate an orbifold construction directly at the level of non-extended topological field theories, i.e. we associate to an equivariant topological field theory an ordinary topological field theory. The construction is functorial. It is based on a reformulation of equivariant topological field theories, followed by taking invariants or, in a geometric formulation, parallel sections.

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Contents

1 Introduction and summary

In several contexts, the following strategy has proven to be appropriate to construct new mathematical objects from a given one: For a fixed (and for our purposes always finite) group G find an embedding of the given object into a G-equivariant one. In the terminology of of physics, this is is often referred to as the *addition of twisted sectors*. Then taking invariants in the appropriate sense yields a new object, the *orbifold object*. In [DVVV89, p. 495], this has been phrased as follows: "First the idea of an orbifold clearly implies that we keep only the G-invariant states in the original Hilbert space $\mathscr{H}_0$. However, [...] we also have to include twisted sectors."

Examples include the construction of Frobenius algebras from G-equivariant Frobenius algebras, see [Kau02], and the construction of modular tensor categories from G-equivariant modular tensor categories, see [Kir04]. A prominent special case of the latter is the orbifoldization of the trivial modular tensor category, i.e. of the category of finite-dimensional vector spaces. One way of adding twisted sectors yields a category of finite-dimensional G-graded vector spaces; taking invariants in the appropriate sense gives the representation category of the Drinfeld double of G, see [Kir04, Example 5.2], which is an algebraic object of independent interest.

The study of these algebraic structures can be rather involved. However, they are often accessible from a different point of view in the sense that they define a topological field theory in a certain dimension. Since topological field theories are sometimes easier to manipulate than the mere algebraic objects, this perspective can lead to to conceptual insights or simplifications of proofs. We will see some examples in this article.

A topological field theory is a symmetric monoidal functor

$$\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

from (some version of) the n-dimensional bordism category to the category of (complex) vector spaces or, more generally, to any symmetric monoidal category. Results exhibiting the strong relation between topological field theories and algebraic objects include the classification of two-dimensional topological field theories by commutative Frobenius algebras, see [Kock03], and the classification of extended three-dimensional topological field theories by modular tensor categories, see [BDSPV15].

Adding twisted sectors to a topological field theory $Y : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ amounts to a factorization

$$Y : \mathbf{Cob}(n) \xrightarrow{\text{triv}} G\text{-}\mathbf{Cob}(n) \xrightarrow{Z} \mathbf{Vect}_{\mathbb{C}}$$

into symmetric monoidal functors, where G-$\mathbf{Cob}(n)$ is a G-equivariant version of the bordism category, in which the bordisms are equipped with principal G-bundles, and $\text{triv} : \mathbf{Cob}(n) \longrightarrow G\text{-}\mathbf{Cob}(n)$ equips all manifolds with the trivial G-bundle. The symmetric monoidal functor $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ is a so-called *G-equivariant topological field theory* and a special case of a *homotopy quantum field theory* in the sense of [Tur10b], see also Section 2 of this article.

Hence, we set up our construction, taking the equivariant theory Z as an input. More precisely, we have to assign to a G-equivariant topological field theory

$$Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

in a natural way an ordinary topological field theory

$$\frac{Z}{G} : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}},$$

called the *orbifold theory of Z*. The operation $Z \longmapsto Z/G$ should correspond to a sum over twisted sectors and taking invariants.

We make this precise for a class of models: This article is a first step and uses the framework of monoidal categories and non-extended topological field theories. However, the essential ideas should not depend on orientability and should generalize to extended topological field theories. For our main result, no assumptions on the dimension are necessary.

We formulate the orbifold construction at the level of topological field theories as a two-step procedure. This is the content of Section 3:

(1) In the first step we produce from the equivariant theory Z a symmetric monoidal functor $\widehat{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ from the cobordism category to an auxiliary symmetric monoidal category defined using flat vector bundles over essentially finite groupoids (alternatively, representations of essentially finite groupoids) and spans of groupoids (Section 3.2-3.4). The functor $\widehat{Z}$ still remembers the relevant aspects of the G-equivariance of Z.

(2) By taking parallel sections (in other words, invariants) of these flat vector bundles and using integrals over homotopy fibers, we define a symmetric monoidal functor $\text{Par} : \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ (Section 3.5-Section 3.6).

This allows us to define the orbifold theory in Section 3.7 as the concatenation

$$\frac{Z}{G} : \mathbf{Cob}(n) \xrightarrow{\widehat{Z}} \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd} \xrightarrow{\text{Par}} \mathbf{Vect}_{\mathbb{C}}.$$

In this way, we obtain a functor $Z \longmapsto Z/G$ from the groupoid of G-equivariant topological field theories to the groupoid of topological field theories. Contrary to a possible guess, this functor cannot be obtained as an adjoint of the pullback along the forgetful functor $G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Cob}(n)$ as explained in Example 3.48. To a closed oriented n-dimensional manifold the orbifold theory Z/G assigns the invariant

$$\frac{Z}{G}(M) = \int_{\mathbf{PBun}_G(M)} Z(M, P)\, \mathrm{d}P,$$

where $Z(M, ?)$ is seen as an invariant function on the groupoid of G-bundles over M that can be integrated with respect to groupoid cardinality (Proposition 3.47) over the groupoid of G-bundles on M and hence over all twisted sectors. The concept of integrating over all twisted sectors is also reflected in the definition of the orbifold theory on objects (Proposition 3.45 and 3.46) and morphisms (Proposition 3.45) in the bordism category.

The subsequent sections the article are devoted to the computation of orbifold theories in various cases:

- The orbifold theory of the primitive homotopy quantum field theory twisted by a cocycle (in the sense of [Tur10b], I.2.1) is the Dijkgraaf-Witten theory twisted by this cocycle (Example 3.48). The corresponding extended topological field theory provides the twisted Drinfeld double.
- The orbifold construction for one-dimensional equivariant theories amounts to taking invariants of group representations, see Section 4.1. As a byproduct, we obtain orthogonality relations for characters.
- For two-dimensional G-equivariant theories a classification by G-crossed Frobenius algebras due to [Tur10b] is available. We describe in Section 4.2 the orbifold construction in dimension two on the level of Frobenius objects by using the notion of an orbifold Frobenius algebra in the sense of [Kau02].
- In Section 5.2 we prove that we can associate to a morphism between two presheaves in groupoids fulfilling certain additional requirements an equivariant topolological field theory. The corresponding orbifold theory is computed in Section 5.4. As a special case we find the orbifold theory of the J-equivariant Dijkgraaf-Witten theory Z_λ constructed in [MNS12] from a short exact sequence $0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$ of finite groups: The orbifold theory of Z_λ is proven to be isomorphic to the Dijkgraaf-Witten theory Z_H for the group H, i.e.

  $$\frac{Z_\lambda}{J} \cong Z_H. \qquad (*)$$

  Considered at the level of invariants this result is a particular incarnation of Cavalieri's principle for groupoids (Proposition A.14).

Finally in Section 6 we construct a pushforward operation for equivariant topological field theories along a group morphism and identify the orbifoldization of G-equivariant topological field theories with the pushforward along the unique group morphism $G \longrightarrow 1$ to the trivial group.

In view of the continuation of this work we remark that it is obviously desirable to have an extension of the geometric orbifold construction to extended equivariant topological field theories. In particular, in the 3-2-1-dimensional case this should yield a geometric underpinning of the algebraic orbifold construction in [Kir04].

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2 Homotopy quantum field theories and equivariant topological field theories

In this introductory section we define topological field theories with arbitrary target following [Tur10b] and explain how they give rise to representations of mapping groupoids. In Section 2.3 we specialize to aspherical targets to obtain equivariant topological field theories.

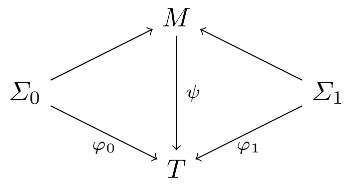
2.1 Topological field theories with arbitrary target space

Topological field theories are symmetric monoidal functors on the bordism category $\mathbf{Cob}(n)$, see [Kock03]. Replacing bordisms by bordisms together with a map into some fixed topological space T, yields a natural generalization of the bordism category. This leads to the notion of an (oriented) topological field theory with target space T, which is essentially taken from [Tur10b]:

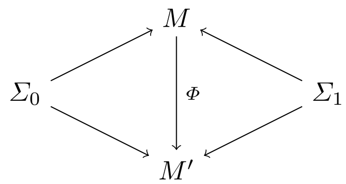
Definition 2.1 – Bordism category for arbitrary target space. Let $n \geq 1$. For a non-empty topological space T the *category T-$\mathbf{Cob}(n)$ of n-dimensional bordisms carrying maps with target space T* is defined in the following way:

(1) Objects are pairs (Σ, φ), where Σ is an $n-1$-dimensional oriented closed manifold (hence an object in $\mathbf{Cob}(n)$) and $\varphi : \Sigma \longrightarrow T$ a continuous map. By *manifold* we always mean smooth finite-dimensional manifold. A continuous map will often just be referred to as map.

(2) A morphism $(M, \psi) : (\Sigma_0, \varphi_0) \longrightarrow (\Sigma_1, \varphi_1)$ is an equivalence class of pairs of oriented compact bordisms $M : \Sigma_0 \longrightarrow \Sigma_1$ and continuous maps $\psi : M \longrightarrow T$ such that the diagram of continuous maps



commutes. The unlabeled arrows are the embeddings of the boundary components into M. Two such pairs (M, ψ) and (M', ψ') are defined to be equivalent if there is an orientation-preserving diffeomorphism $\Phi : M \longrightarrow M'$ making the diagram



commute such that additionally $\psi = \psi' \circ \Phi$.

The identity of (Σ, φ) is represented by the cylinder over Σ carrying the trivial homotopy $\varphi \simeq \varphi$. Composition is by gluing of bordisms and maps, respectively. Just like $\mathbf{Cob}(n)$, the category T-$\mathbf{Cob}(n)$ carries the structure of a symmetric monoidal category with duals.

Definition 2.2 – Topological field theory with target space and homotopy quantum field theory. An *n-dimensional topological field theory over a field K with target space T* is a symmetric monoidal functor

$$Z : T\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K .$$

Again, the target space is always assumed to be non-empty. If Z is additionally *homotopy invariant*, i.e. $Z(M, \psi_0) = Z(M, \psi_1)$ for morphisms $(M, \psi_0), (M, \psi_1)$ in T-$\mathbf{Cob}(n)$ with $\psi_0 \simeq \psi_1$ relative ∂M, we call Z, following [Tur10b], an *n-dimensional homotopy quantum field theory with target space T*.

Remarks 2.3.

(a) If T is the one-point-space (the terminal object in the category **Top** of topological spaces), then T-$\mathbf{Cob}(n) \cong \mathbf{Cob}(n)$ and we obtain an ordinary topological field theory.

(b) Here and in the sequel the manifolds involved in the definition of $\mathbf{Cob}(n)$ or T-$\mathbf{Cob}(n)$ are always oriented. Therefore, one could specify all theories as *oriented*. Since we only consider this case, we will drop this additional adjective.

(c) Every object (Σ, φ) in T-$\mathbf{Cob}(n)$ has a dual. More precisely, the dual of an object (Σ, φ) is $(\overline{\Sigma}, \overline{\varphi})$, where $\overline{\Sigma}$ is Σ with orientation reversed and $\overline{\varphi}$ is φ now seen as a map $\overline{\Sigma} \longrightarrow T$. Since Z is monoidal, $Z(\Sigma, \varphi)$ also has a dual. This implies that $Z(\Sigma, \varphi)$ is finite-dimensional.

(d) If M is a closed oriented n-dimensional manifold and $\varphi : M \longrightarrow T$ a continuous map, then an n-dimensional topological field theory $Z : T$-$\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$ assigns to (M, φ) a number in K since (M, φ) can be seen as morphism $(\varnothing, \bullet) \longrightarrow (\varnothing, \bullet)$, with $\bullet$ the unique map $\varnothing \longrightarrow T$, and $(\varnothing, \bullet)$ is sent to K (up to natural isomorphism). This number is a diffeomorphism invariant of M. If Z is a homotopy quantum field theory, it is also an invariant of the homotopy class of φ.

(e) In the above definition, $\mathbf{Vect}_{\mathbb{C}}$ can be replaced by any other symmetric monoidal category.

(f) The homotopy invariance property can be built into the source category: Denote by T-$\mathbf{Cob}(n)/\simeq$ the symmetric monoidal category, which arises from T-$\mathbf{Cob}(n)$ by identifying morphisms (M, ψ) and (M, ψ') if $\psi \simeq \psi'$ relative ∂M. Then a homotopy quantum field theory with target T can be equivalently described as a symmetric monoidal functor T-$\mathbf{Cob}(n)/\simeq \longrightarrow \mathbf{Vect}_{\mathbb{C}}$.

We can consider monoidal natural transformations between different topological field theories with the same target space. According to the following well-known result these are always isomorphisms:

Proposition 2.4. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal categories, where $\mathcal{C}$ has duals. Then every monoidal natural transformation between F and G is an isomorphism. Con-*

sequently, the category $\mathrm{Sym}(\mathcal{C},\mathcal{D})$ *of symmetric monoidal functors with monoidal natural transformations as morphisms is a groupoid.*

Definition 2.5 – Category of homotopy quantum field theories. For any target space T we define the *category* $\mathrm{HSym}(T\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)$ *of n-dimensional homotopy quantum field theories with target T* to be the full subcategory of $\mathrm{Sym}(T\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)$ consisting of those functors having the homotopy invariance property from Definition 2.2.

By Remark 2.3, (c) T-$\mathbf{Cob}(n)$ has duals, so Proposition 2.4 implies:

Corollary 2.6. *For any target space T the category* $\mathrm{HSym}(T\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)$ *of homotopy quantum field theories with target T is a groupoid.*

Remark 2.7. In [Tur10b] a pointed version T_*-$\mathbf{Cob}(n)$ of the category T-$\mathbf{Cob}(n)$ is used for a fixed basepoint $t_0\in T$: Objects are pairs of a closed oriented $n-1$-dimensional pointed manifold Σ (*pointed* means that all components are equipped with a basepoint) and a pointed map $\varphi:\Sigma\longrightarrow T$. The morphisms, their composition and the monoidal structure are defined just as in T-$\mathbf{Cob}(n)$, i.e. the forgetful functor

$$U: T_*\text{-}\mathbf{Cob}(n)\longrightarrow T\text{-}\mathbf{Cob}(n)$$

is a symmetric monoidal embedding. We will prove now that U is even an equivalence in case T is path-connected: To show essential surjectivity in that case, let (Σ,φ) be an object in T-$\mathbf{Cob}(n)$. It suffices to show that for any choice of basepoints on the components of Σ there is a pointed map $\psi:\Sigma\longrightarrow T$ which is homotopic to φ (not necessarily pointed homotopic). For this we may assume that Σ is connected. Denote by x_0 the basepoint on the single component of Σ and equip the unit interval $I=[0,1]$ with the basepoint $\{0\}$. Then the wedge sum (the coproduct in the category of pointed spaces) of Σ and I can be identified with

$$\Sigma\vee I=(\Sigma\times\{0\})\cup(\{x_0\}\times I)\subset\Sigma\times I.$$

Since the inclusion $\{x_0\}\longrightarrow\Sigma$ is a cofibration, there is a retraction $r:\Sigma\times I\longrightarrow\Sigma\vee I$ by [Bre93, VII, Theorem 1.3]. This allows us to define a map

$$\mu:\Sigma\longrightarrow\Sigma\vee I,\quad x\longmapsto r(x,1).$$

By path-connectedness of T we can choose a path $\gamma:[0,1]\longrightarrow T$ from $t':=\varphi(x_0)$ to t_0 and define the map

$$\psi:\Sigma\xrightarrow{\mu}\Sigma\vee I\xrightarrow{(\varphi,\gamma)}T$$

sending x_0 to t_0. This map is homotopic to φ by the homotopy defined by

$$h_t(x):=(\varphi,\gamma)\circ r(x,1-t)\quad\text{for all}\quad t\in I,\quad x\in\Sigma.$$

This concludes the proof that the forgetful functor $U: T_*\text{-}\mathbf{Cob}(n)\longrightarrow T\text{-}\mathbf{Cob}(n)$ is a symmetric monoidal equivalence. It induces a symmetric monoidal equivalence

$$T_*\text{-}\mathbf{Cob}(n)/\simeq\;\longrightarrow T\text{-}\mathbf{Cob}(n)/\simeq$$

between the corresponding categories which identify morphisms which consist of maps homotopic relative boundary on the same underlying bordism, see Remark 2.3, (f). This shows that the category of pointed homotopy quantum field theories with target space T in the sense of [Tur10b] is equivalent to the category of homotopy quantum field theories with target space T used in this article if T is path-connected.

2.2 Groupoid representations from evaluation on the cylinder

Evaluating an ordinary topological field theory on the cylinder, seen as the identity in the bordism category, yields the identity map. For a homotopy quantum field theory, however, we can place maps into T on the cylinder representing a homotopy between the maps on the incoming and outgoing boundary; and the theory assigns maps to these homotopies, which might be non-trivial. To be more precise, we get representations of groupoids. Here, a representation of a groupoid Γ (on vector spaces over a field K) is a functor $\rho:\Gamma\longrightarrow\mathbf{Vect}_K$. The groupoids we want to represent are of the following form:

Definition 2.8 – Mapping groupoid. For topological spaces X and Y we denote by $\Pi(X,Y)$ the *mapping groupoid of X and Y* which has continuous maps $X\longrightarrow Y$ as objects and equivalence classes of homotopies of such maps as morphisms. Here, we consider homotopies $h,h':X\times[0,1]\longrightarrow Y$ between f and g to be equivalent if they are homotopic relative the boundary $X\times\{0,1\}$ of the cylinder $X\times[0,1]$.

Remarks 2.9.

(a) We could also define $\Pi(X,Y)$ to be the fundamental groupoid $\Pi(\mathscr{C}(X,Y))$ of the mapping space $\mathscr{C}(X,Y)$ equipped with the compact-open topology.

(b) For the one-point-space $\star$ the groupoid $\Pi(\star, X)$ is the fundamental groupoid of X.

(c) For a continuous map $\varphi : X \longrightarrow Y$ the group $\mathrm{Aut}(\varphi)$ of automorphisms of φ in $\Pi(X,Y)$ is the group of equivalence classes of homotopies $h : X \times [0,1] \longrightarrow Y$ from φ to φ. Since $h_0 = h_1$, h gives rise to a map $\bar{h} : X \times \mathbb{S}^1 \longrightarrow Y$. Obviously, two homotopies $h, h' : X \times [0,1] \longrightarrow Y$ from φ to itself are equivalent iff $\bar{h} \simeq \bar{h}'$ relative $X \times \{0\} \subset X \times \mathbb{S}^1$, where $0 \in \mathbb{S}^1$ is some basepoint.

(d) A morphism $h : f \longrightarrow g$ in $\Pi(X,Y)$, i.e. an equivalence class of homotopies $f \overset{h}{\simeq} g$, has an inverse denoted by $h^- : g \longrightarrow f$. It is obtained by reading h backwards.

Many of the constructions discussed in this paper will rely on the following elementary observation.

Proposition 2.10. *A homotopy quantum field theory $Z : T\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$ provides for each object Σ in $\mathbf{Cob}(n)$ a representation*

$$\begin{aligned} \varrho_\Sigma : \Pi(\Sigma, T) &\longrightarrow \mathbf{Vect}_K \\ (\varphi : \Sigma \longrightarrow T) &\longmapsto Z(\Sigma, \varphi), \\ (\varphi \overset{h}{\simeq} \psi) &\longmapsto Z(\Sigma \times [0,1], h). \end{aligned}$$

of the mapping groupoid $\Pi(\Sigma, T)$. In particular, Z provides a representation $\mathrm{Aut}(\varphi) \longrightarrow \mathrm{Aut}(Z(\Sigma,\varphi))$ of the automorphism group $\mathrm{Aut}(\varphi)$ of any continuous map $\varphi : \Sigma \longrightarrow T$. Its character is given by

$$\mathrm{Aut}(\varphi) \longrightarrow K, \quad (\varphi \overset{h}{\simeq} \varphi) \longmapsto Z(\Sigma \times \mathbb{S}^1, \bar{h}).$$

Proof. The only non-trivial point is the well-definedness of $Z(\Sigma \times [0,1], h)$, but this follows from the homotopy invariance of Z. For the character formula observe that

$$\operatorname{tr} Z(\Sigma \times [0,1], h) = Z(\Sigma \times \mathbb{S}^1, \bar{h}) \qquad (*)$$

for $\varphi \overset{h}{\simeq} \varphi$, where $Z(\Sigma \times \mathbb{S}^1, \bar{h}) \in K$ is the invariant that Z assigns to the closed n-dimensional manifold $\Sigma \times \mathbb{S}^1$ together with the map $\bar{h} : \Sigma \times \mathbb{S}^1 \longrightarrow T$ induced by h, see Remark 2.9, (c). □

A homotopy quantum field theory with target T assigns to two morphisms (M, ψ) and (M, ξ) in $T\text{-}\mathbf{Cob}(n)$ the same linear map if $\psi \simeq \xi$ relative ∂M. If ψ and ξ are homotopic, but possibly not relative boundary, we have the following result:

Proposition 2.11. *Let $Z : T\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$ be a homotopy quantum field theory with target T. If $\psi \overset{h}{\simeq} \xi : M \longrightarrow T$ for two morphisms (M, ψ) and (M, ξ) in $T\text{-}\mathbf{Cob}(n)$, then the square*

$$\begin{array}{ccc} Z(\Sigma_0, \psi|_{\Sigma_0}) & \xrightarrow{Z(\Sigma_0 \times [0,1], h|_{\Sigma_0}) = \varrho_{\Sigma_0}(h|_{\Sigma_0})} & Z(\Sigma_0, \xi|_{\Sigma_0}) \\ {\scriptstyle Z(M,\psi)} \downarrow & & \downarrow {\scriptstyle Z(M,\xi)} \\ Z(\Sigma_1, \psi|_{\Sigma_1}) & \xrightarrow{Z(\Sigma_1 \times [0,1], h|_{\Sigma_1}) = \varrho_{\Sigma_1}(h|_{\Sigma_1})} & Z(\Sigma_1, \xi|_{\Sigma_1}) \end{array}$$

commutes.

Proof. We construct a homotopy $H : M \times [0,1] \longrightarrow T$ relative ∂M starting at ψ. In order to define the maps $H_t : M \longrightarrow T$, we choose $(\Sigma_1 \times [0,1]) \cup_{\Sigma_1} M \cup_{\Sigma_1} (\Sigma_0 \times [0,1])$ as a representative for the bordism class M, where $\cup_?$ denotes the gluing of manifolds (we are gluing in two cylinders). Now let H_t be the map $(\Sigma_1 \times [0,1]) \cup_{\Sigma_1} M \cup_{\Sigma_0} (\Sigma_0 \times [0,1]) \longrightarrow T$ obtained by gluing together the maps

$$\begin{aligned} \alpha_t : \Sigma_0 \times [0,1] \longrightarrow T, &\quad (x,s) \longmapsto h(x, st), \\ h_t : M \longrightarrow T, &\quad (x,s) \longmapsto h(x,t), \\ \beta_t : \Sigma_1 \times [0,1] \longrightarrow T, &\quad (x,s) \longmapsto h(x, (1-s)t). \end{aligned}$$

Hence, H is a homotopy relative boundary from ψ to the map

$$M \cong (\Sigma_1 \times [0,1]) \cup_{\Sigma_1} M \cup_{\Sigma_0} (\Sigma_0 \times [0,1]) \longrightarrow T$$

obtained by gluing $h|^{-}_{\Sigma_1}$, ξ and $h|_{\Sigma_0}$. Using homotopy invariance and functoriality, we obtain

$$Z(M,\psi) = Z(\Sigma_1 \times [0,1], h|^{-}_{\Sigma_1}) \circ Z(M,\xi) \circ Z(\Sigma_0 \times [0,1], h|_{\Sigma_0}).$$

From Proposition 2.10 we deduce that $Z(\Sigma_1 \times [0,1], h|^{-}_{\Sigma_1}) = Z(\Sigma_1 \times [0,1], h|_{\Sigma_1})^{-1}$. □

2.3 Aspherical targets

One particular choice for a target space of a homotopy quantum field theory is the classifying space of a finite group. This leads to equivariant topological field theories as discussed in [Tur10b, I.3]. In this subsection we explain how maps to the classifying space and homotopies between them can be understood as principal fiber bundles and gauge transformations.

Definition 2.12 – Equivariant topological field theory. Let G be a group, which, if needed, is always seen as discrete topological group. Then we define a *G-equivariant topological field theory* to be a homotopy quantum field theory with the classifying space BG of G as a target (hence by definition, an equivariant topological field theory is homotopy invariant).

Remark 2.13. If we set $G\text{-}\mathbf{Cob}(n) := BG\text{-}\mathbf{Cob}(n)$, a G-equivariant topological theory is a symmetric monoidal functor

$$Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$$

fulfilling the homotopy invariance property.

Let us recall the most elementary facts about classifying spaces, as a reference see e.g. [tD08]: For a topological group G, there is a space BG, called the *classifying space*, and a numerable G-bundle $EG \longrightarrow BG$ over it, called the *universal G-bundle*, such that for any space X the map

$$[X, BG] \longrightarrow \pi_0(\mathbf{PBun}^{\mathrm{num}}_G(X)), \quad [f] \longmapsto [f^*EG] \tag{$*$}$$

is a natural bijection between homotopy classes of maps $X \longrightarrow BG$ and the isomorphism classes of the groupoid $\mathbf{PBun}^{\mathrm{num}}_G(X)$ of numerable G-bundles over X. Equivalently, BG represents the cofunctor $\pi_0(\mathbf{PBun}^{\mathrm{num}}_G(?))$ from the category of topological spaces and homotopy classes of maps to the category of sets, which also implies that BG is unique up to homotopy equivalence. If G is discrete, an elegant way to obtain a model for BG is to see G as groupoid $\star//G$ with one object $\star$ and automorphism group G. We can now apply the nerve functor $B : \mathbf{Cat} \longrightarrow \mathbf{sSet}$ from the category of small categories to the category of simplicial sets and the geometric realization $|?| : \mathbf{sSet} \longrightarrow \mathbf{Top}$. The result $|B(\star//G)|$ is a model for BG, see [GJ99]. Since BG is an Eilenberg-MacLane space $K(G,1)$, and therefore aspherical, G-equivariant topological field theories are said to have an *aspherical target*.

The bijections $(*)$ suggest that we have to understand the maps into BG occuring in the definition of $G\text{-}\mathbf{Cob}(n)$ as G-bundles (canonically equipped with a connection since G is discrete). This point of view has a clearer geometric motivation than the approach from Definition 2.12 following [Tur10b]. The latter, however, has technical advantages since it makes sense to require classifying maps to be equal, whereas equality of bundles is problematic. In order to reconcile both approaches, $(*)$ is not enough because these bijections do not take gauge transformations into account. Instead, we need natural equivalences of groupoids

$$\Pi(M, BG) \cong \mathbf{PBun}_G(M) \tag{$\ddagger$}$$

for all manifolds M.

In order to define the equivalence $(\ddagger)$ for a discrete group G we use the canonical flat connection on the bundle $EG \longrightarrow BG$:

Theorem 2.14. *Let G be a discrete group. Then for any manifold M there is a canonical equivalence*

$$\begin{aligned} K : \Pi(M, BG) &\longrightarrow \mathbf{PBun}_G(M), \\ \varphi &\longmapsto \varphi^*EG, \\ (\varphi \overset{H}{\simeq} \psi) &\longmapsto (K(H) : \varphi^*EG \longrightarrow \psi^*EG), \end{aligned}$$

where $K(H) : \varphi^*EG \longrightarrow \psi^*EG$ is the isomorphism of G-bundles sending for $x \in M$ a point $(x,p) \in (\varphi^*EG)_x = EG_{\varphi(x)}$ to the point $(x, \wp_{H_x}(p)) \in (\psi^*EG)_x = EG_{\psi(x)}$ obtained by parallel transport $\wp_{H_x}$ in EG along the path $H_x : [0,1] \ni t \longmapsto H_t(x)$.

Proof. Since the parallel transport operators of a flat connection are invariant under homotopy of the path relative to the endpoints, K is well-defined. The functoriality is obvious. By the universal property of $\pi : EG \longrightarrow BG$, the functor K yields a bijection on isomorphism classes. In particular, K is essentially surjective. It remains to show that K is fully faithful. For this consider maps $\varphi, \psi : M \longrightarrow BG$ and the map

$$\mathrm{Hom}_{\Pi(M,BG)}(\varphi,\psi) \longrightarrow \mathrm{Hom}_{\mathbf{PBun}_G(M)}(\varphi^* EG, \psi^* EG) \tag{$*$}$$

induced by K. By

$$\begin{aligned} \Phi &: \varphi^* EG = \{(x,p) \in M \times EG \mid p \in EG_{\varphi(x)}\} \longrightarrow EG, \\ \Psi &: \psi^* EG = \{(x,p) \in M \times EG \mid p \in EG_{\psi(x)}\} \longrightarrow EG \end{aligned} \tag{$\ddagger$}$$

we denote the canonical G-equivariant maps coming from the projection to the second factor.

- First we prove surjectivity of $(*)$. If $f : \varphi^* EG \longrightarrow \psi^* EG$ is a morphism of G-bundles over M, we obtain two G-equivariant maps $\Phi, \Psi \circ f : \varphi^* EG \longrightarrow EG$, which by the properties of EG are homotopic through G-equivariant maps $\Lambda_t : \varphi^* EG \longrightarrow EG$. These equivariant maps cover the maps $H_t : M \longrightarrow BG$ constituting a homotopy H from φ to ψ. We find

$$K(H)(x,p) = (x, \wp_{H_x}(p)) = (x, \Lambda_1(x,p)) = (x, \Psi \circ f(x,p)) = f(x,p) \quad \text{for all} \quad (x,p) \in \varphi^* EG,$$

  where in the last step we used that the first component of $f(x,p)$ is x. This proves $K(H) = f$ and hence surjectivity of $(*)$.

- In order to prove injectivity of $(*)$, it suffices to prove that a homotopy H from φ to φ with $K(H) = \mathrm{id}_{\varphi^* EG}$ is homotopic relative boundary to the identity homotopy of φ. Equivalently, we have to show that $[0,1] \ni t \longmapsto H_t$ represents the trivial element in $\pi_1(\mathscr{C}_P(M,BG),\varphi)$ for $P := \varphi^* EG$, where $\mathscr{C}_P(M,BG)$ is the space of maps $M \longrightarrow BG$ classifying P (as all mapping spaces in this article $\mathscr{C}_P(M,BG)$ is equipped with the compact-open topology). To this end, we use Theorem 3.4 in Chapter 7 of [Hus94], which states that the map

$$\Xi : \mathscr{C}(P,EG)^G \longrightarrow \mathscr{C}_P(X,BG)$$

  from G-equivariant maps $P \longrightarrow EG$ to classifying maps of $P = \varphi^* EG$ assigning to a G-equivariant map $P \longrightarrow EG$ the corresponding classifying map is a universal $\mathrm{Aut}(P)$-bundle. In particular, the total space $\mathscr{C}(P,EG)^G$ is contractible. In the next step observe $\Xi(\Phi) = \varphi$ for the map Φ which already appeared in $(\ddagger)$. Given the contractibility of $\mathscr{C}(P,EG)^G$, we only need to give a closed Ξ-lift of $[0,1] \ni t \longmapsto H_t$ to a path in $\mathscr{C}(P,EG)^G$ starting and ending at Φ. Such a lift is given by $[0,1] \ni t \longmapsto H_t^*$, where we define $H_t^* : P \longrightarrow EG$ by

$$H_t^*(x,p) := \wp_{[0,t]\ni s \longmapsto H_s(t)}(p) \quad \text{for all} \quad p \in E_{\varphi(x)}, \quad x \in M. \qquad \square$$

Remarks 2.15.

(a) Note that although, to the knowledge of the authors, this result is not formulated in the above way in the literature, it is known as indicated in [Hei04, 1]. The statement can also be proven using simplicial methods and the adjunction between nerve and homotopy category.

(b) The above result may be wrong if G is non-discrete. For example, the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ is a model for $B\,\mathrm{U}(1)$. Since $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$, we obtain in the case that M is the one-point space $\star$ that $\Pi(\star, B\,\mathrm{U}(1)) \cong \Pi(B\,\mathrm{U}(1)) \cong \bullet//1$, where $\bullet//1$ is the groupoid with one object and trivial automorphism group. But the groupoid $\mathbf{PBun}_{\mathrm{U}(1)}(\star)$ of $\mathrm{U}(1)$-bundles over the one-point-space is the groupoid $\mathrm{U}(1)$-**Tor** of $\mathrm{U}(1)$-torsors, which is equivalent to $\bullet//\,\mathrm{U}(1)$. The reason behind this failure is that homotopies can only be turned into isomorphisms of $\mathrm{U}(1)$-bundles if a connection is specified.

In the sequel we will, having Theorem 2.14 in mind, jump back and forth between the description of G-bundles using the total space perspective, classifying maps or parallel transport operators.

3 Orbifold construction

To set up the orbifold construction we follow the plan outlined in the introduction: After discussing vector bundles over groupoids and their parallel sections in Section 3.1, we define a symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$ that comprises all representations of essentially finite groupoids in Section 3.2 and a certain type of preaheaves in Section 3.3 to give a reformulation of the notion of an equivariant topological field theory in Section 3.4. Using the pushforward maps introduced in Section 3.5 we define the parallel section functor in Section 3.6. In Section 3.7

all these ingredients are put together to yield the orbifold construction.

3.1 Parallel sections of vector bundles on groupoids

For a representation $\varrho : G \longrightarrow \mathrm{Aut}(V)$ of a group G on a K-vector space V the invariants of V form the subspace

$$V^G := \{v \in V \mid g.v := \varrho(g)v = v \text{ for all } g \in G\},$$

whereas the coinvariants of V form the quotient

$$V_G := \frac{V}{\mathrm{span}\{v - g.v \mid v \in V, g \in G\}}.$$

Obviously, we can see ϱ as a functor from the groupoid $\star//G$ with one object and automorphism group G to $\mathbf{Vect}_K$. Now V^G is a limit for ϱ, and V_G is a colimit of ϱ. For groupoid representations an obvious and well-known generalization is true.

Proposition 3.1. *Let Γ be a small groupoid and $\varrho : \Gamma \longrightarrow \mathbf{Vect}_K$ a representation. Then its limit and colimit are given by*

$$\lim \varrho \cong \prod_{[x]\in\pi_0(\Gamma)} \varrho(x)^{\mathrm{Aut}(x)},$$

$$\operatorname{colim} \varrho \cong \bigoplus_{[x]\in\pi_0(\Gamma)} \varrho(x)_{\mathrm{Aut}(x)},$$

respectively. There is a canonical map $\lim \varrho \longrightarrow \operatorname{colim} \varrho$. If Γ is essentially finite (Definition A.5) and K of characteristic zero, this map is an isomorphism.

The formulae from Proposition 3.1 are very explicit, but not too useful since they use chosen representatives of the isomorphism classes. We will now discuss a very convenient realization of the limit of a representation by seeing a groupoid representation as a vector bundle over a groupoid (see Proposition 3.6 below). All the notions occuring in the following definition are directly transferred from the ordinary theory of vector bundles with connection. They also appear in the context of groupoid representations in [Wil05].

Definition 3.2 – Vector bundle over a groupoid. Let K be a field and Γ a small groupoid. We will view a representation $\varrho : \Gamma \longrightarrow \mathbf{FinVect}_K$ of Γ on finite-dimensional K-vector spaces as a *K-vector bundle ϱ over Γ (with flat connection)*. We call the vector space $\varrho(x)$ the *fibre* over $x \in \Gamma$. We will not define directly what a connection on ϱ is, but we will define its parallel transport: For a morphism $g : x \longrightarrow y$ in Γ we call the operator $\varrho(g) : \varrho(x) \longrightarrow \varrho(y)$ the *parallel transport* of ϱ along g. A morphism $\eta : \varrho \longrightarrow \xi$ of vector bundles over Γ is a natural transformation of the corresponding functors $\Gamma \longrightarrow \mathbf{Vect}_K$ (this means that η is an intertwiner for the parallel transport operators). The category of K-vector bundles over Γ is denoted by $\mathbf{VecBun}_K(\Gamma)$. It coincides, of course, with the functor category $[\Gamma, \mathbf{FinVect}_K]$.

Remarks 3.3.

(a) So whenever it is convenient, we will see groupoid representations as vector bundles over groupoids and morphisms of representations (also called intertwiners) as vector bundle morphisms.

(b) The category $\mathbf{VecBun}_K(\Gamma)$ of K-vector bundles over some groupoid Γ inherits from $\mathbf{FinVect}_K$ the structure of a symmetric monoidal category with duals. The monoidal unit $\mathbb{I}_\Gamma$ assigns to every $x \in \Gamma$ the vector space K and to every morphism in Γ the identity on K.

For a vector bundle over a groupoid there is the notion of a parallel section, sometimes also called *invariant* or *flat sections*, see [Wil05]. The relation to parallel sections in the geometric sense is obvious.

Definition 3.4 – Parallel section of a vector bundle over a groupoid. Let ϱ be a K-vector bundle over a groupoid Γ. A *parallel section* of ϱ is a function s on Γ with $s(x) \in \varrho(x)$ for $x \in \Gamma$ such that for any morphism $g : x \longrightarrow y$ the equation

$$s(y) = \varrho(g)s(x)$$

holds. By

$$\mathrm{Par}\,\varrho := \{s : \Gamma \longrightarrow K \mid s \text{ parallel section}\}$$

we denote the *vector space of parallel sections of ϱ*.

Remark 3.5. The following well-known reformulation of the above definition is useful: The vector space $\operatorname{Par}\varrho$ coincides with the vector space $\operatorname{Hom}_{\mathbf{VecBun}_K(\Gamma)}(\mathbb{I}_\Gamma, \varrho)$ of morphisms (i.e. vector bundle morphisms) from the monoidal unit $\mathbb{I}_\Gamma$ to ϱ. Hence, we obtain

$$\operatorname{Par}\varrho = \operatorname{Hom}_{\mathbf{VecBun}_K(\Gamma)}(\mathbb{I}_\Gamma, \varrho) = \int_\Gamma \operatorname{Hom}_{\mathbf{Vect}_K}(\mathbb{I}_\Gamma(x), \varrho(x))\,\mathrm{d}x,$$

where in the last step we have used that we can express the space of natural transformations between functors via an end, see [ML98, IX.5].

The following result is the analogue of the well-known *holonomy principle* from the theory of vector bundles with connection.

Proposition 3.6 – Holonomy principle. *Let ϱ be a K-vector bundle over a small groupoid Γ. Then the vector space* $\operatorname{Par}\varrho$ *of parallel sections of ϱ is the limit of ϱ. By the functoriality of the limit, taking parallel sections extends to a functor*

$$\operatorname{Par}_\Gamma : \mathbf{VecBun}_K(\Gamma) \longrightarrow \mathbf{Vect}_K.$$

Proof. By using $\mathbb{I}_\Gamma(x) = K$ for all $x \in \Gamma$ in Remark 3.5 we see that the end describing $\operatorname{Par}\varrho$ is mute in the first variable and hence reduces to a limit. □

Remarks 3.7.

(a) Note that $\operatorname{Par}_\Gamma$ is not (yet) the parallel section functor needed for the orbifold construction, which will be introduced in Section 3.6 and which is defined on a different category.

(b) Let $\lambda : \varrho \longrightarrow \xi$ be a morphism of K-vector bundles over a small groupoid Γ. Then the image of λ under $\operatorname{Par}_\Gamma$ will be denoted by

$$\lambda_* : \operatorname{Par}\varrho \longrightarrow \operatorname{Par}\xi, \quad s \longmapsto \lambda_* s,$$

and is explicitly given by $(\lambda_* s)(x) := \lambda_x s(x)$ for $x \in \Gamma$.

Just like vector bundles over topological spaces, vector bundles over groupoids admit pullbacks.

Proposition 3.8 – Pullback of vector bundles over groupoids. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between small groupoids and ϱ a K-vector bundle over Ω. Then $\Phi^*\varrho := \varrho \circ \Phi : \Gamma \longrightarrow \mathbf{Vect}_K$ is a vector bundle over Γ. This provides a pullback functor*

$$\Phi^* : \mathbf{VecBun}_K(\Omega) \longrightarrow \mathbf{VecBun}_K(\Gamma)$$

(a) *The pullback functors obey the composition law $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$, where $\Psi : \Omega \longrightarrow \Lambda$ is another functor between small groupoids.*

(b) *Let $\Phi' : \Gamma \longrightarrow \Omega$ be another functor between small groupoids and $\eta : \Phi \Rightarrow \Phi'$ a natural isomorphism. Then η induces an isomorphism $\varrho(\eta) : \Phi^*\varrho \longrightarrow {\Phi'}^*\varrho$ consisting of the maps $(\varrho(\eta_x))_{x\in\Gamma}$.*

(c) *The functor Φ induces a linear map*

$$\Phi^* : \operatorname{Par}\varrho \longrightarrow \operatorname{Par}\Phi^*\varrho, \quad s \longmapsto \Phi^* s,$$

where $(\Phi^* s)(x) = s(\Phi(x))$ for all $x \in \Gamma$. Such a map is called a pullback map.

(d) *Just like the pullback functors the pullback maps obey the composition law $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$.*

(e) *The pullback maps are natural in the sense that they provide a natural transformation*

$$\operatorname{Par}_\Omega \longrightarrow \operatorname{Par}_\Gamma \circ \Phi^*,$$

i.e. for any morphism $\lambda : \varrho \longrightarrow \xi$ of vector bundles over Ω the square

$$\begin{array}{ccc} \operatorname{Par}\varrho & \xrightarrow{\Phi^*} & \operatorname{Par}\Phi^*\varrho \\ {\scriptstyle\lambda_*}\big\downarrow & & \big\downarrow{\scriptstyle(\Phi^*\lambda)_*} \\ \operatorname{Par}\xi & \xrightarrow{\Phi^*} & \operatorname{Par}\Phi^*\xi \end{array}$$

commutes.

Proof. It is obvious that Φ yields a functor $\Phi^* : \mathbf{VecBun}_K(\Omega) \longrightarrow \mathbf{VecBun}_K(\Gamma)$, and so is the composition law (a). The remaining assertions all follow immediately from the definitions. As an illustration let us prove (e): For the map $\Phi^* : \operatorname{Par}\varrho \longrightarrow \operatorname{Par}\Phi^*\varrho$ to be well-defined we need to check that $\Phi^* s$ is again parallel if s is. Indeed, for a morphism $g : x \longrightarrow y$ in Γ we obtain

$$(\Phi^* s)(y) = s(\Phi(y)) = \varrho(\Phi(g))s(x) = (\Phi^*\varrho)(g)s(x)$$

since s is parallel. □

Remark 3.9. We denoted both a functor and a linear map by Φ^*. But as long as source and target are specified, there should be no confusion.

3.2 The symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$

The symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$ is a crucial ingredient for the generalization of equivariant topological field theories in Section 3.4. It can be seen as an equivariant version of the span category in [Mor11] and [Mor15].

In order to define $\mathbf{VecBun}_K\mathbf{Grpd}$ we need the notion of a homotopy pullback.

Definition 3.10 – Span and cospan of groupoids. A *span of groupoids* is a diagram

$$\Gamma \longleftarrow \Lambda \longrightarrow \Omega$$

of groupoids and functors between them. Dually, a *cospan of groupoids* is a diagram

$$\Gamma \longrightarrow \Lambda \longleftarrow \Omega$$

of groupoids and functors between them.

A prescription of how to compute homotopy pullbacks can be extracted from the theory of homotopy limits, see for instance [Rie14]. For our purposes, the following model for the homotopy pullback of groupoids suffices:

Definition 3.11 – Homotopy pullback. To every cospan $\Gamma \xrightarrow{\Phi} \Omega \xleftarrow{\Psi} \Lambda$ of groupoids we can associate a groupoid $\Gamma \times_\Omega \Lambda$, called the *weak pullback* or *weak pullback* of $\Gamma \xrightarrow{\Phi} \Omega \xleftarrow{\Psi} \Lambda$: Objects are triples $(x, y, \eta_{x,y})$, where $x \in \Gamma$, $y \in \Lambda$ and $\Phi(x) \overset{\eta_{x,y}}{\cong} \Psi(y)$. A morphism $(x, y, \eta_{x,y}) \longrightarrow (x', y', \eta_{x',y'})$ is a pair (g, h) of morphisms $g : x \longrightarrow x'$ and $h : y \longrightarrow y'$ such that the square

$$\begin{array}{ccc}
\Phi(x) & \xrightarrow{\eta_{x,y}} & \Psi(y) \\
{\scriptstyle \Phi(g)}\downarrow & & \downarrow{\scriptstyle \Psi(h)} \\
\Phi(x') & \xrightarrow[\eta_{x',y'}]{} & \Psi(y')
\end{array}$$

commutes. We have the obvious projection functors $\pi_\Gamma : \Gamma \times_\Omega \Lambda \longrightarrow \Gamma$ and $\pi_\Lambda : \Gamma \times_\Omega \Lambda \longrightarrow \Lambda$. The assignment $\Gamma \times_\Omega \Lambda \ni (x, y, \eta_{x,y}) \longmapsto \eta_{x,y}$ defines a natural isomorphism $\Phi \circ \pi_\Gamma \Rightarrow \Psi \circ \pi_\Lambda$, i.e. the square

$$\begin{array}{ccc}
\Gamma \times_\Omega \Lambda & \xrightarrow{\pi_\Gamma} & \Gamma \\
{\scriptstyle \pi_\Lambda}\downarrow & \overset{\eta}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \Phi} \\
\Lambda & \xrightarrow[\Psi]{} & \Omega
\end{array}$$

weakly commutes.

Definition 3.12 – The category $\mathbf{VecBun}_K\mathbf{Grpd}$. For a field K define the category $\mathbf{VecBun}_K\mathbf{Grpd}$ as follows:

(a) Objects are pairs (Γ, ϱ), where $\varrho : \Gamma \longrightarrow \mathbf{FinVect}_K$ is a vector bundle over an essentially finite groupoid Γ.

(b) A morphism $(\Gamma, \varrho) \longrightarrow (\Omega, \xi)$ is a equivalence class of pairs (Λ, λ), where $\Gamma \xleftarrow{r_0} \Lambda \xrightarrow{r_1} \Omega$ is a span of essentially finite groupoids and $\lambda : r_0^*\varrho \longrightarrow r_1^*\xi$ is a morphism in $\mathbf{VecBun}_K(\Lambda)$. Here, an equivalence

$(\Lambda, \lambda) \longrightarrow (\Lambda', \lambda')$ of spans from Γ to Ω is an equivalence $\Phi : \Lambda \longrightarrow \Lambda'$ together with natural isomorphisms

$$\eta_0 : r_0 \Longrightarrow r_0' \circ \Phi,$$
$$\eta_1 : r_1 \Longrightarrow r_1' \circ \Phi,$$

such that the diagram

$$\begin{array}{ccccc} & & \Lambda & & \\ & \swarrow^{r_0} & \overset{\eta_0}{\Swarrow} \ \downarrow \Phi \ \overset{\eta_1}{\Searrow} & {}^{r_1}\searrow & \\ \Gamma & & & & \Omega \\ & \nwarrow_{r_0'} & & {}_{r_1'}\nearrow & \\ & & \Lambda' & & \end{array}$$

weakly commutes up to η_0 and η_1 and such that the square

$$\begin{array}{ccc} r_0^*\varrho & \xrightarrow{\quad\lambda\quad} & r_1^*\xi \\ {\scriptstyle \varrho(\eta_0)}\downarrow & & \downarrow{\scriptstyle \xi(\eta_1)} \\ (r_0' \circ \Phi)^*\varrho = \Phi^* {r_0'}^* \varrho & \xrightarrow{\quad\Phi^*\lambda'\quad} & (r_1' \circ \Phi)^*\xi = \Phi^* {r_1'}^* \xi \end{array}$$

commutes (note that the operations from Proposition 3.8 enter here). So we can write (the representative of) a morphism $(\Gamma \xleftarrow{r_0} \Lambda \xrightarrow{r_1} \Omega, \lambda : r_0^*\varrho \longrightarrow r_1\xi)$ as

$$\begin{array}{ccc} & \Lambda & \\ {}^{r_0}\swarrow & & \searrow^{r_1} \\ \Gamma & \overset{\lambda}{\Longrightarrow} & \Omega \\ {}_{\varrho}\searrow & & \swarrow_{\xi} \\ & \mathbf{Vect}_K & \end{array}$$

.

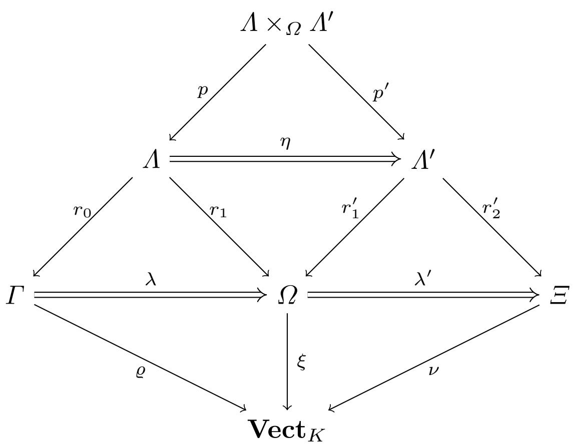
(c) The composition of the morphisms $(\Gamma, \rho) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Omega, \xi)$ and $(\Omega, \xi) \xleftarrow{r_1'} (\Lambda', \lambda') \xrightarrow{r_2'} (\Xi, \nu)$ is defined to be equivalence class of the span $\Gamma \longleftarrow \Lambda \times_\Omega \Lambda' \longrightarrow \Omega$ defined by the diagram

$$\begin{array}{ccccccc} & & & \Lambda \times_\Omega \Lambda' & & & \\ & & {}^{p}\swarrow & & \searrow^{p'} & & \\ & \Lambda & & \overset{\eta}{\Longrightarrow} & & \Lambda' & \\ {}^{r_0}\swarrow & & \searrow^{r_1} & & {}^{r_1'}\swarrow & & \searrow^{r_2'} \\ \Gamma & & & \Omega & & & \Xi \end{array},$$

where $\Lambda \times_\Omega \Lambda'$ is the weak pullback coming with the projections p and p' and the natural isomorphism $\eta : r_1 \circ p \Rightarrow r_1' \circ p'$. The needed morphism $\lambda' \times_\Omega \lambda : (r_0 \circ p)^*\varrho \longrightarrow (r_2 \circ p')^*\nu$ is defined as the composition

$$(r_0 \circ p)^*\varrho = p^* r_0^* \varrho \xrightarrow{p^*\lambda} p^* r_1^* \xi = (r_1 \circ p)^*\xi \xrightarrow{\xi(\eta)} (r_1' \circ p')^*\xi = {p'}^* {r_1'}^* \xi \xrightarrow{{p'}^*\lambda'} {p'}^* {r_2'}^* \nu = (r_2 \circ p')^*\nu.$$

Here $\xi(\eta) : (r_1 \circ p)^*\xi \longrightarrow (r_1' \circ p')^*\xi$ is the morphism of vector bundles induced from $\eta : r_1 \circ p \Rightarrow r_1' \circ p'$ according to Proposition 3.8, (b). Note that $\lambda' \times_\Omega \lambda : (r_0 \circ p)^*\varrho \longrightarrow (r_2 \circ p')^*\nu$ is the natural transformation of functors $\Lambda \times_\Omega \Lambda' \longrightarrow \mathbf{Vect}_K$ obtained by composing all the natural transformations in the diagram



The category $\mathbf{VecBun}_K\mathbf{Grpd}$ carries in a natural way the structure of a monoidal category. The tensor product is analogous to the external product known from K-theory, see [Hat09].

Definition 3.13. We define the structure of a symmetric monoidal category on $\mathbf{VecBun}_K\mathbf{Grpd}$ as follows:

(a) For objects (Γ, ϱ) and (Ω, ξ) the tensor product $(\Gamma, \varrho) \otimes (\Omega, \xi)$, which we will also denote as $\varrho \mathbin{\widehat{\otimes}} \xi$ in case we want to suppress the groupoids, is the vector bundle over $\Gamma \times \Omega$ given by

$$(\varrho \mathbin{\widehat{\otimes}} \xi)(x, y) := \varrho(x) \otimes \xi(y) \quad \text{for all} \quad x \in \Gamma, \quad y \in \Omega$$

and

$$(\varrho \mathbin{\widehat{\otimes}} \xi)(g, h) := \varrho(g) \otimes \xi(h) \quad \text{for all} \quad g : x \longrightarrow x', \quad h : y \longrightarrow y'.$$

The fiberwise tensor products occuring in these definitions are tensor products over the field K. The definition of the tensor product

$$\otimes : \mathbf{VecBun}_K\mathbf{Grpd} \times \mathbf{VecBun}_K\mathbf{Grpd} \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$$

on morphisms is obtained in an analogous way (Cartesian products on the span level combined with tensor products of vector bundles). The monoidal unit $\mathbb{I}$ of $\mathbf{VecBun}_K\mathbf{Grpd}$ is the vector bundle over the groupoid $\star//1$ with one object and trivial automorphism group on K.

(b) The category $\mathbf{VecBun}_K\mathbf{Grpd}$ also inherits a symmetric braiding. Explicitly, the braiding isomorphism $(\Gamma, \varrho) \otimes (\Omega, \xi) \longrightarrow (\Omega, \xi) \otimes (\Gamma, \varrho)$ is the class of the span

$$\Gamma \times \Omega \longleftarrow \Gamma \times \Omega \xrightarrow{T} \Omega \times \Gamma,$$

where the first functor is the identity and the second switches the factors, together with the morphism $\varrho \mathbin{\widehat{\otimes}} \xi \longrightarrow T^*(\xi \mathbin{\widehat{\otimes}} \varrho)$ consisting of the obvious flip maps

$$\varrho(x) \otimes \xi(y) \longrightarrow \xi(y) \otimes \varrho(x).$$

Remarks 3.14.

(a) In order to write down the braiding we could also use the span $\Gamma \times \Omega \xleftarrow{T} \Omega \times \Gamma \longrightarrow \Omega \times \Gamma$ with the associated intertwiner changed accordingly. Both spans (and the intertwiners) represent the same morphism. Hence, the definition of the braiding does not make any choices on where to apply the switch functor.

(b) The category $\mathbf{VecBun}_K\mathbf{Grpd}$ admits a functor $U : \mathbf{VecBun}_K\mathbf{Grpd} \longrightarrow \mathbf{SpanGrpd}$ to the category of spans of essentially finite groupoids as defined in [Mor11] which forgets the vector bundles and morphisms between them. We will refer to this functor as *projection to the span part.*

The duality is inherited from $\mathbf{FinVect}_K$.

Proposition 3.15. *The symmetric monoidal category* $\mathbf{VecBun}_K\mathbf{Grpd}$ *has coinciding left and right duals.*

Proof. For an object $\varrho : \Gamma \longrightarrow \mathbf{Vect}_\mathbb{C}$ in $\mathbf{VecBun}_K\mathbf{Grpd}$ the left and right dual object is given by the object $\varrho^* : \Gamma \longrightarrow \mathbf{Vect}_\mathbb{C}$, where $\varrho^*(x)$ is the dual vector space $\varrho(x)^*$ and $\varrho^*(g)$ for a morphism $g : x \longrightarrow y$ is the dual map $\varrho\left(g^{-1}\right)^* : \varrho(x)^* \longrightarrow \varrho(y)^*$ of $\varrho\left(g^{-1}\right) : \varrho(y) \longrightarrow \varrho(x)$. The right evaluation $d : \varrho \mathbin{\widehat{\otimes}} \varrho^* = (\Gamma, \varrho) \otimes (\Gamma, \varrho^*) \longrightarrow \mathbb{I}$ has span part $\Gamma \times \Gamma \xleftarrow{\Delta} \Gamma \longrightarrow \star//1$, where Δ is the diagonal. To fully specify d we need to exhibit a morphism $\Delta^*(\varrho \mathbin{\widehat{\otimes}} \varrho^*) = \varrho \otimes \varrho^* \longrightarrow \mathbb{I}_\Gamma$ in $\mathbf{VecBun}_K(\Gamma)$. Note that $\varrho \otimes \varrho^*$ is now the tensor product in $\mathbf{VecBun}_K(\Gamma)$ and

$\mathbb{I}_\Gamma$ the respective monoidal unit. We choose this needed intertwiner to be the one from $\mathbf{VecBun}_K(\Gamma)$, i.e. the obvious evaluation map. The other dualities are defined similarly. The snake identities are easily verified. □

3.3 Homotopy invariant presheaves

We will introduce now a certain type of presheaves in groupoids that will be used in Section 3.4 to implement equivariance in topological field theories. Usually, a presheaf in groupoids is defined as a (weak) cofunctor from the category of smooth manifolds and smooth maps to the bicategory of (small) groupoids, functors and natural transformations, see for instance [KS06] and [NS11]. We will use a similar notion, but we will require that the presheaf is also defined on homotopies.

Definition 3.16 – Homotopy invariant presheaf. Let **Man** be the 2-category of smooth manifolds (possibly with boundary), smooth maps and equivalence classes of homotopies between such maps. A *homotopy invariant presheaf* is a weak functor $\Gamma : \mathbf{Man}^{\mathrm{opp}} \longrightarrow \mathbf{Grpd}$ into the 2-category of (essentially) small groupoids, functors and natural isomorphisms. Every homotopy invariant presheaf is, in particular, a presheaf. A morphism of homotopy invariant presheaves is a morphism of the underlying presheaves, i.e. a weak natural transformation of functors.

Remarks 3.17.

(a) We write also $f^* = \Gamma(f)$ for any smooth map f. If $\iota : \Sigma \longrightarrow M$ is an inclusion, we sometimes write $x|_\Sigma$ instead of $\iota^* x$ for $x \in \Gamma(M)$.

(b) We call a homotopy invariant presheaf a *homotopy invariant stack* if its underlying presheaf is a stack (with respect to open covers). The stack property means that for a manifold M the groupoid $\Gamma(M)$ can be computed from local data. We will, however, not need the notion of a stack too much, but instead a much simpler gluing condition, see below in Definition 3.18. For the proper definition of a stack using the descent condition we refer to [KS06] and [NS11].

Definition 3.18. We define the following properties of a homotopy invariant presheaf $\Gamma : \mathbf{Man}^{\mathrm{opp}} \longrightarrow \mathbf{Grpd}$:

(a) We call Γ *additive* if for all manifolds Σ and Σ' the inclusions $\iota : \Sigma \longrightarrow \Sigma \coprod \Sigma'$ and $\iota' : \Sigma' \longrightarrow \Sigma \coprod \Sigma'$ induce an equivalence

$$\Gamma\left(\Sigma \coprod \Sigma'\right) \xrightarrow{\iota^* \times \iota'^*} \Gamma(\Sigma) \times \Gamma(\Sigma').$$

and if $\Gamma(\varnothing)$ is naturally equivalent to the trivial groupoid with one object.

(b) We call Γ *essentially finite* if $\Gamma(K)$ is essentially finite for every compact manifold (with boundary).

(c) We say that Γ satisfies the *gluing property (with respect to bordisms)* if for morphisms $M : \Sigma_0 \longrightarrow \Sigma_1$ and $M' : \Sigma_1 \longrightarrow \Sigma_2$ in $\mathbf{Cob}(n)$ for $n \geq 1$ the inclusions $j : M \longrightarrow M' \circ M$ and $j' : M' \longrightarrow M' \circ M$ induce an equivalence of groupoids

$$\Gamma(M' \circ M) \xrightarrow{j^* \times j'^*} \Gamma(M) \times_{\Gamma(\Sigma_1)} \Gamma(M'),$$

where $\Gamma(M) \times_{\Gamma(\Sigma_1)} \Gamma(M')$ is the homotopy pullback of the cospan $\Gamma(M) \longrightarrow \Gamma(\Sigma_1) \longleftarrow \Gamma(M')$.

Remarks 3.19.

(a) The gluing property implies additivity. However, for the construction of n-dimensional topological field theories, the additivity will be important for $n-1$-dimensional manifolds (it will correspond to the monoidal structure), whereas the gluing will be relevant for n-dimensional manifolds. Therefore, we keep these axioms separated.

(b) If a homotopy invariant presheaf is a stack, then it satisfies the gluing condition. In fact, in this case the gluing condition is just the descent condition for $M' \circ M$ and the cover $\{M, M'\}$ with intersection Σ_1. Strictly speaking, M and M' are not open in $M' \circ M$, but there is an open cover with strong deformation retraction onto $\{M, M'\}$. To see this, extend M and M' in direction of some unit normal of Σ. By homotopy invariance, evaluating the presheaf on this open cover yields the same as evaluating it on $\{M, M'\}$.

Example 3.20. The well-known properties of the bundle stack now give us a very important example fitting into the framework set up in Definition 3.18: For a discrete group G the homotopy invariant stacks

$$\mathbf{PBun}_G(?), \Pi(?, BG), [\Pi(?), G\text{-}\mathbf{Tor}] : \mathbf{Man}^{\mathrm{opp}} \longrightarrow \mathbf{Grpd}$$

are canonically equivalent. All three are additive and satisfy the gluing condition. If G is finite, they are essentially finite. For the proof we recall that the descent condition for stacks formalizes the description of bundles in terms of transition functions, which entails that $\mathbf{PBun}_G(?)$ is a stack. The equivalences from Theorem 2.14 yield an equivalence of presheaves

$$\mathbf{PBun}_G(?) \cong \Pi(?, BG),$$

which, in particular, implies that $\Pi(?, BG)$ is a stack as well. Another classical result, namely the classification of flat bundles, yields an equivalence $\mathbf{PBun}_G(?) \cong [\Pi(?), G\text{-}\mathbf{Tor}]$ of presheaves, which proves that $[\Pi(?), G\text{-}\mathbf{Tor}]$ is also a stack. The stacks $\Pi(?, BG)$ and $[\Pi(?), G\text{-}\mathbf{Tor}]$ are homotopy invariant (it is clear how they have to be defined on homotopies). Using Theorem 2.14 the homotopy invariance carries over to $\mathbf{PBun}_G(?)$. The additivity is clear, the gluing condition can be derived using (b). It remains to show that for a compact manifold K and a finite group G the groupoid $[\Pi(K), G\text{-}\mathbf{Tor}]$ is essentially finite, but this follows from Corollary A.10.

There should be generalizations of this example involving twisted and relative bundles. In the present paper we refrain from developing this point.

3.4 Equivariant topological field theories – a reformulation

Having introduced the symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$ in Section 3.2 we can reformulate equivariant topological field theories.

According to Remark 2.3, (e) an n-dimensional topological field theory with values in the symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$ is a symmetric monoidal functor

$$Z : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}.$$

By Proposition 2.4 the category $\mathrm{Sym}(\mathbf{Cob}(n), \mathbf{VecBun}_K\mathbf{Grpd})$ of $\mathbf{VecBun}_K\mathbf{Grpd}$-valued topological field theories is a groupoid.

Note that a $\mathbf{VecBun}_K\mathbf{Grpd}$-valued topological field theory $Z : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ associates to a boundary component a vector bundle over a groupoid and to a bordism a morphism between pullback bundles. We will now describe the case where the equivariance data, i.e. the vector bundles over groupoids and the vector bundle morphisms, comes from a fixed additive, essentially finite homotopy-invariant presheaf Γ satisfying the gluing property (see Section 3.3):

Definition 3.21. Let Γ be an additive, essentially finite homotopy-invariant presheaf satisfying the gluing property and $Z : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ a $\mathbf{VecBun}_K\mathbf{Grpd}$-valued topological field theory. We call Z an *n-dimensional Γ-equivariant topological field theory* if the concatenation

$$\mathbf{Cob}(n) \xrightarrow{Z} \mathbf{VecBun}_K\mathbf{Grpd} \xrightarrow{U} \mathbf{SpanGrpd}$$

with the projection to the span part (Remark 3.14, (b)) is naturally equivalent to the functor

$$\mathbf{Cob}(n) \longrightarrow \mathbf{SpanGrpd}$$

induced by Γ, i.e. if there are natural equivalences $UZ(\Sigma) \cong \Gamma(\Sigma)$ for every object in $\mathbf{Cob}(n)$ and $UZ(M) \cong \Gamma(M)$ for every morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$ such that the diagram of groupoids

$$\begin{array}{ccccc}
\Gamma(\Sigma_0) & \longleftarrow & \Gamma(M) & \longrightarrow & \Gamma(\Sigma_1) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
UZ(\Sigma_0) & \longleftarrow & UZ(M) & \longrightarrow & UZ(\Sigma_1)
\end{array}$$

weakly commutes.

Remark 3.22. Whenever $Z : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ is Γ-equivariant, we can assume without loss of generality that for any morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$ the $\mathbf{Cob}(n)$ the strict equality

$$UZ(M) = (\Gamma(\Sigma_0) \longleftarrow \Gamma(M) \longrightarrow \Gamma(\Sigma_1))$$

holds.

For a finite group G the stacks $\Pi(?, BG)$ and $\mathbf{PBun}_G(?)$ are equivalent and fulfill the requirements on Γ in the above definition (see Example 3.20). So it makes sense to consider a $\Pi(?, BG)$- or $\mathbf{PBun}_G(?)$-equivariant equivariant topological field theory. This is just a symmetric monoidal functor $Z : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$

such that the span part of $Z(M)$ for a morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ is given by

$$\Pi(\Sigma_0, BG) \longleftarrow \Pi(M, BG) \longrightarrow \Pi(\Sigma_1, BG)$$

or, equivalently,

$$\mathbf{PBun}_G(\Sigma_0) \longleftarrow \mathbf{PBun}_G(M) \longrightarrow \mathbf{PBun}_G(\Sigma_1),$$

where the arrows are restriction functors. For the new notion to be justified we should be able to relate such $\Pi(?, BG)$- or $\mathbf{PBun}_G(?)$-equivariant equivariant topological field theories to G-equivariant topological field theories according to Definition 2.12.

Theorem 3.23. *Let G be a finite group. Then the notion of a G-equivariant topological field theory in the sense of Definition 2.12 is equivalent to the notion of a $\Pi(?, BG)$-equivariant (or equivalently $\mathbf{PBun}_G(?)$-equivariant) topological field theory in the sense of Definition 3.21 in the following way:*

(a) *To a G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ in the sense of Definition 2.12 we can associate in a canonical way a $\Pi(?, BG)$-equivariant topological field theory $\widehat{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ which assigns to an object Σ in $\mathbf{Cob}(n)$ the vector bundle ϱ_Σ from Proposition 2.10.*

(b) *To a $\Pi(?, BG)$-equivariant topological field theory $Y : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ in the sense of Definition 3.21 we can associate in a canonical way a G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$.*

Up to isomorphism the procedures are inverse to each other.

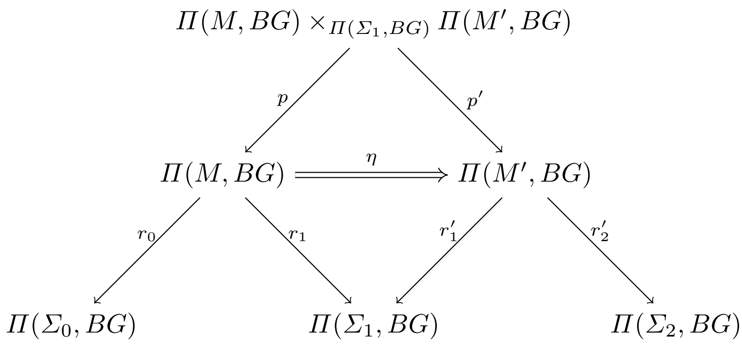
Remarks 3.24.

(a) The statement is not meant to be an equivalence between the category of G-equivariant topological field theories and some other functor category. At this point we only observe that a G-equivariant topological field theory in the sense of Definition 2.12 consists of the same data and properties as a $\Pi(?, BG)$-equivariant topological field theory in the sense of Definition 3.21. So far we have ignored morphisms between different theories (which in both cases are natural monoidal transformations). We address this issue in Proposition 3.26.

(b) In the rest of this article we will freely change between the description of a G-equivariant topological field theory as a symmetric monoidal functor $G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$ or a $\Pi(?, BG)$-equivariant symmetric functor $\mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$.

Proof.

(a) First let $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$ be a G-equivariant topological field theory. We need to prove that Z gives rise to a $\Pi(?, BG)$-equivariant topological field theory $\widehat{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$. For an object Σ in $\mathbf{Cob}(n)$ we define $\widehat{Z}(\Sigma)$ to be the representation $\varrho_\Sigma : \Pi(\Sigma, BG) \longrightarrow \mathbf{Vect}_K$ obtained from Z in Proposition 2.10. If $M : \Sigma_0 \longrightarrow \Sigma_1$ is a morphism in $\mathbf{Cob}(n)$, we define $\widehat{Z}(M)$ to be the span $\Pi(\Sigma_0, BG) \xleftarrow{r_0} \Pi(M, BG) \xrightarrow{r_1} \Pi(\Sigma_1, BG)$ together with the morphism $r_0^*\widehat{Z}(\Sigma_0) = r_0^*\varrho_{\Sigma_0} \longrightarrow r_1^*\widehat{Z}(\Sigma_1) = r_1^*\varrho_{\Sigma_1}$ given by the maps $Z(M, \psi) : Z(\Sigma_0, \psi|_{\Sigma_0}) \longrightarrow Z(\Sigma_1, \psi|_{\Sigma_1})$. This is really an intertwiner for the given representations as follows from Proposition 2.11. We will denote this intertwiner by $Z(M, ?)$. The functoriality of $\widehat{Z}$ follows from the functoriality of Z. More precisely, let $M : \Sigma_0 \longrightarrow \Sigma_1$ and $M' : \Sigma_1 \longrightarrow \Sigma_2$ be two morphisms in $\mathbf{Cob}(n)$. The span part of $\widehat{Z}(M') \circ \widehat{Z}(M)$ is by definition the outer span of



together with

$$\lambda : (r_0 \circ p)^*\varrho_{\Sigma_0} \xrightarrow{p^* Z(M,?)} (r_1 \circ p)^*\varrho_{\Sigma_1} \xrightarrow{\varrho_{\Sigma_1}(\eta)} (r_1' \circ p')^*\varrho_{\Sigma_1} \xrightarrow{p'^* Z(M',?)} (r_2 \circ p')^*\varrho_{\Sigma_2}. \qquad (*)$$

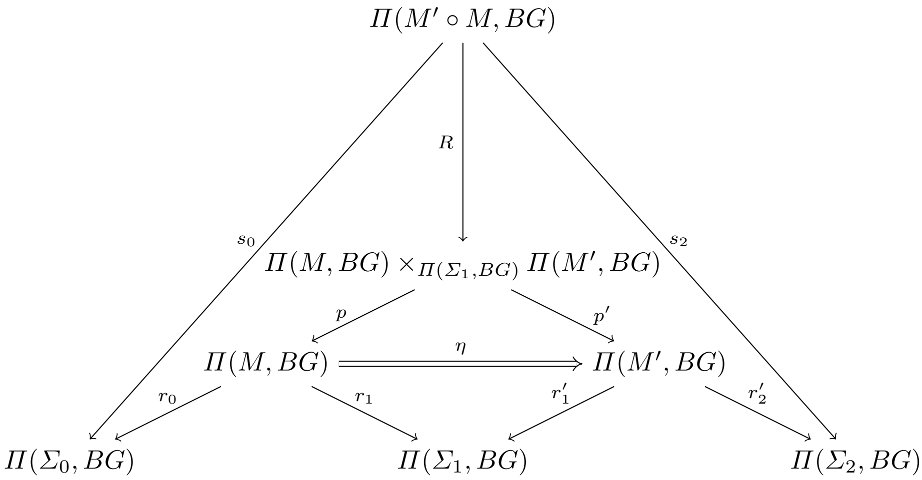
We need to show that this coincides with $\widehat{Z}(M' \circ M)$. For this we can use that $\Pi(?, BG)$ satisfies the gluing property. It entails that the span part of $\widehat{Z}(M') \circ \widehat{Z}(M)$ can also be represented by

$$\Pi(\Sigma_0, BG) \xleftarrow{s_0} \Pi(M' \circ M, BG) \xrightarrow{s_2} \Pi(\Sigma_2, BG)$$

with the arrows coming from restriction. The relation between these spans is given by the restriction equivalence

$$R : \Pi(M' \circ M, BG) \longrightarrow \Pi(M, BG) \times_{\Pi(\Sigma_1, BG)} \Pi(M', BG) \tag{‡}$$

appearing in the diagram



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in which the triangles containing s_0 and s_2 strictly commute. We will now use the rules from Definition 3.12 to compute the intertwiner corresponding to $(*)$, but now with respect to the span (‡). We claim that this intertwiner is $Z(M' \circ M, ?)$, which would yield the gluing law $\widehat{Z}(M' \circ M) = \widehat{Z}(M') \circ \widehat{Z}(M)$. We only have to show that the diagram

$$\begin{array}{ccc} s_0^* \varrho_{\Sigma_0} & \xrightarrow{\quad Z(M' \circ M, ?) \quad} & s_2^* \varrho_{\Sigma_2} \\ \big\downarrow{=} & & \big\downarrow{=} \\ (r_0 \circ p \circ R)^* \varrho_{\Sigma_0} = R^*(r_0 \circ p)^* \varrho_{\Sigma_0} & \xrightarrow{R^*\lambda} & (r_2' \circ p' \circ R)^* \varrho_{\Sigma_2} = R^*(r_2' \circ p')^* \varrho_{\Sigma_2} \end{array}$$

commutes, where

$$R^*\lambda = R^*(p'^* Z(M', ?) \circ \varrho_{\Sigma_1}(\eta) \circ p^* Z(M, ?)) = Z(M', ?) \circ R^*(\varrho_{\Sigma_1}(\eta)) \circ Z(M, ?).$$

By functoriality of Z this is $Z(M' \circ M, ?)$ if we can prove that

$$R^*(\varrho_{\Sigma_1}(\eta)) : R^*(r_1 \circ p)^* \varrho_{\Sigma_1} \longrightarrow R^*(r_1' \circ p')^* \varrho_{\Sigma_1}$$

is the identity intertwiner of $s_1^* \varrho_{\Sigma_1}$ for the restriction functor $s_1 : \Pi(M' \circ M, BG) \longrightarrow \Pi(\Sigma_1, BG)$. This follows from the fact that the pullback of η along R is the identity transformation. In summary, the functoriality from $\widehat{Z}$ follows from the functoriality of Z. Moreover, the monoidal structure of Z yields a monoidal structure for $\widehat{Z}$ in a natural way if we take into account that $\Pi(?, BG)$ is additive.

(b) If we are given a $\Pi(?, BG)$-equivariant topological field $Y : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$, we just have to read backwards the steps given in (a) to obtain a G-equivariant field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_K$. First note that $\Pi(?, BG)$-equivariance of Y means that for an object Σ in $\mathbf{Cob}(n)$ we get a representation $Y(\Sigma)$ of $\Pi(\Sigma, BG)$. Evaluation at a map $\varphi : \Sigma \longrightarrow BG$ yields a vector space, which we define to be $Z(\Sigma, \varphi)$. To a morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ the theory Y assigns the span $\Pi(\Sigma_0, BG) \xleftarrow{r_0} \Pi(M, BG) \xrightarrow{r_1} \Pi(\Sigma_1, BG)$ together with a morphism $r_0^* Y(\Sigma_0) \longrightarrow r_1^* Y(\Sigma_1)$ consisting of linear maps

$$Z(M, \psi) : Z(\Sigma_0, \psi|_{\Sigma_0}) = Y(\Sigma_0)(\psi|_{\Sigma_0}) \longrightarrow Z(\Sigma_1, \psi|_{\Sigma_1}) = Y(\Sigma_1)(\psi|_{\Sigma_1})$$

for all maps $\psi : M \longrightarrow BG$. This defines Z on morphisms. Functoriality and monoidality of Z follow from the properties of Y by the same arguments as in (a). Again, the gluing property of $\Pi(?, BG)$ is crucial. The given constructions are inverse to each other. □

Remark 3.25. Looking at the above proof we see that a functor $\mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ with span part being $\Pi(?, BG)$ is really just a reformulation of the concept of a G-equivariant topological field theory. Let us give advantages of the new formulation:

- Given a G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$, we can only glue maps into BG if they strictly coincide at the boundary. If they coincide only up to homotopy, we can glue in cylinders by hand to put the homotopy on it, but this is not properly formalized. In the category $\mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ this is done by allowing the natural isomorphisms in the weak pullback square to give a contribution.

- We will see later, for instance in the proof of Theorem 3.40, that using $\mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ allows us to work diagrammatically rather than exclusively with formulae.

Proposition 3.26. *For any finite group G the correspondence of Theorem 3.23 extends to a functor*

$$\widehat{?}\ :\ \mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)\longrightarrow \mathrm{Sym}(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$$

from the category of G-equivariant topological field theories to the category of symmetric monoidal functors from $\mathbf{Cob}(n)$ to $\mathbf{VecBun}_K\mathbf{Grpd}$. All objects in the image are $\Pi(?,BG)$-equivariant.

Remark 3.27. Note that on the left hand side we impose separately the requirement that all functors are homotopy invariant. This is not the case of the right hand side because the homotopy invariance property has been built in. It enters in the naturality squares in Proposition 2.11 that form the intertwiners needed for the right hand side, see the proof of Theorem 3.23.

Proof. Let η be a morphism $Z\longrightarrow Z'$ between G-equivariant topological field theories, i.e. a natural monoidal transformation. We have to specify how η gives rise to a natural monoidal transformation of the functors $\widehat{Z},\widehat{Z'}$: $\mathbf{Cob}(n)\longrightarrow \mathbf{VecBun}_K\mathbf{Grpd}$ which correspond to Z and Z' in the sense of (the proof of) Theorem 3.23. To this end, we have to define a morphism $(\Pi(\Sigma,BG),\varrho_\Sigma)\longrightarrow(\Pi(\Sigma,BG),\varrho'_\Sigma)$ in $\mathbf{VecBun}_K\mathbf{Grpd}$ for all Σ in $\mathbf{Cob}(n)$, where ϱ_Σ and ϱ'_Σ are the representations of $\Pi(\Sigma,BG)$ coming from Z and Z' respectively. Such a morphism has a span part, but we choose it to be trivial, i.e. the identity span $\Pi(\Sigma,BG)\longleftarrow\Pi(\Sigma,BG)\longrightarrow\Pi(\Sigma,BG)$. Now a morphism $(\Pi(\Sigma,BG),\varrho_\Sigma)\longrightarrow(\Pi(\Sigma,BG),\varrho'_\Sigma)$ having this span part is just an intertwiner $\varrho_\Sigma\longrightarrow\varrho'_\Sigma$. We define it to be the intertwiner having the components $(\eta_{(\Sigma,\varphi)})_{\varphi\in\Pi(\Sigma,BG)}$. It is easy to see that this yields a morphism $\widehat{Z}\longrightarrow\widehat{Z'}$. The functoriality and monoidality of these assignments is obvious. □

Remark 3.28. The functor $\mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)\longrightarrow\mathrm{Sym}(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$ is not going to be an embedding (it is not fully faithful) because in the image we only find morphisms which are the identity in the span part. Of course, not all morphisms will be of that type. However, the functor is faithful, i.e. injective on morphism spaces. So if we define $\mathrm{Sym}'(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$ to be the category $\mathrm{Sym}(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$, but now only with morphisms being identities in the span part, then

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)\longrightarrow\mathrm{Sym}'(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$$

is an embedding. If moreover we denote by

$$\mathrm{Sym}'_G(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})\subset\mathrm{Sym}'(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$$

the full subcategory consisting of all $\Pi(?,BG)$-equivariant theories, then

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_K)\longrightarrow\mathrm{Sym}'_G(\mathbf{Cob}(n),\mathbf{VecBun}_K\mathbf{Grpd})$$

is an equivalence. This restriction in the codomain, however, is not needed in the context of the orbifold construction.

The following two examples illustrate the advantages of the reformulation of equivariant topological field theories given in this section. These examples will be used to illustrate the orbifold construction in low dimensions (Section 5).

Example 3.29 – *Classification of one-dimensional equivariant topological field theories.* Using our reformulation of equivariant topological field theories in this section we can give a very short proof of the classification result for one-dimensional equivariant topological field theories mentioned in [Tur10b, I.1.4]. This result refers to the pointed version of the theory, but is also valid for the unpointed one by Remark 2.7.

First denote by $\mathbf{core}(\mathcal{C})$ the *core* of category $\mathcal{C}$. It has the same objects as $\mathcal{C}$ and the isomorphisms of $\mathcal{C}$ as morphisms.

Now for a finite group G the groupoid of G-equivariant topological field theories $G\text{-}\mathbf{Cob}(1)\longrightarrow\mathbf{Vect}_K$ is equivalent to the core of the category of finite-dimensional K-representations of G.

Indeed, by Theorem 3.23 G-equivariant topological field theories $G\text{-}\mathbf{Cob}(1)\longrightarrow\mathbf{Vect}_K$ correspond to symmetric monoidal functors $Z:\mathbf{Cob}(1)\longrightarrow\mathbf{VecBun}_K\mathbf{Grpd}$, for which the groupoid part is given by the restriction of $\Pi(?,BG)$. Such symmetric monoidal functors are classified by evaluation on the positively oriented point, where they give a representation of $\Pi(\star,BG)=\Pi(BG)\cong\star//G$, so a representation ϱ_Z of G on a finite-dimensional complex vector space. From Remark 3.28 we know that the assignment $Z \longmapsto \varrho_Z$ yields an equivalence between $\mathrm{HSym}(G\text{-}\mathbf{Cob}(1), \mathbf{Vect}_K)$ to the groupoid of functors $\star//G \longrightarrow \mathbf{FinVect}_K$. A morphism between two such functors $\varrho, \varrho' : \star//G \longrightarrow \mathbf{FinVect}_K$ is the identity span $\star//G \longleftarrow \star//G \longrightarrow \star//G$ together with an equivalence between ϱ and ϱ' seen as representations. This yields the claim.

Example 3.30 – *Crossed Frobenius G-algebras.* We use the reformulation of G-equivariant topological field theories of Theorem 3.23 and Proposition 3.26 to motivate the notion of a crossed Frobenius G-algebra as appearing in [Tur10b, II, 3.2]: For this note that [Kock03, Theorem 3.6.19] applied to the symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$ yields an equivalence

$$\mathrm{Sym}(\mathbf{Cob}(2), \mathbf{VecBun}_K\mathbf{Grpd}) \cong \mathbf{cFrob}(\mathbf{VecBun}_K\mathbf{Grpd})$$

between the groupoid of two-dimensional $\mathbf{VecBun}_K\mathbf{Grpd}$-valued topological field theories and the groupoid of commutative Frobenius algebras in $\mathbf{cFrob}(\mathbf{VecBun}_K\mathbf{Grpd})$.

Concatenating with the functor from Proposition 3.26 we obtain a functor

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(2), \mathbf{Vect}_K) \longrightarrow \mathrm{Sym}(\mathbf{Cob}(2), \mathbf{VecBun}_K\mathbf{Grpd}) \cong \mathbf{cFrob}(\mathbf{VecBun}_K\mathbf{Grpd}) \qquad (*)$$

assigning to a two-dimensional G-equivariant topological field theory a commutative Frobenius algebra in the symmetric monoidal category $\mathbf{VecBun}_K\mathbf{Grpd}$. It remains to describe the commutative Frobenius objects in the image explicitly: For a finite group G let $Z : G\text{-}\mathbf{Cob}(2) \longrightarrow \mathbf{Vect}_K$ be a two-dimensional G-equivariant topological field theory. Then the image of Z under $(*)$ is a commutative Frobenius algebra $\mathfrak{A}$ in $\mathbf{VecBun}_K\mathbf{Grpd}$ admitting of the following form: As a functor $\mathfrak{A}$ is of the form

$$\varrho_{\mathfrak{A}} : \varPi(\mathbb{S}^1, BG) \cong G//G \longrightarrow \mathbf{Vect}_K.$$

By $\mathfrak{A}$ we denote the direct sum $\mathfrak{A} = \bigoplus_{g\in G} \mathfrak{A}_g$, where $\mathfrak{A}_g := \varrho_{\mathfrak{A}}(g)$ for an object $g \in G$. For $g \in G$ and $v \in \mathfrak{A}_h$ we use the abbreviation $g.v := \varrho_{\mathfrak{A}}(g)v \in \mathfrak{A}_{ghg^{-1}}$. We see $\mathfrak{A}$ and hence also $\mathfrak{A} \otimes \mathfrak{A}$ as a G-representation. We can specify the Frobenius structure by giving the multiplication and the pairing (for a detailed account on the equivalent ways to describe Frobenius structures see [FS08]):

(a) The multiplication is a linear map

$$\mu : \mathfrak{A} \otimes \mathfrak{A} \longrightarrow \mathfrak{A}, \quad v \otimes w \longmapsto vw$$

carrying $\mathfrak{A}_g \otimes \mathfrak{A}_h$ to $\mathfrak{A}_{gh}$. It is associative, unital with unit in $\mathfrak{A}_1$ and intertwines with the G-action.

(b) The pairing is a linear map

$$\kappa : \mathfrak{A} \otimes \mathfrak{A} \longrightarrow K,$$

which is also an intertwiner and hence G-invariant (G acts trivially on K). We have $\kappa|_{\mathfrak{A}_g \otimes \mathfrak{A}_h} = 0$ if $h \neq g^{-1}$. The pairing is non-degenerate, i.e. $\kappa(v, w) = 0$ for all $v \in \mathfrak{A}$ implies $w = 0$.

This summarizes the Frobenius structure on $\mathfrak{A}$.

(c) The commutativity constraint is given by

$$vw = (g.w)(g.v) \quad \text{for all} \quad v \in \mathfrak{A}_g, \quad w \in \mathfrak{A}.$$

For the proof of the above statements we also have to evaluate the stack of G-bundles on various morphisms in $\mathbf{Cob}(2)$, see [Mor15, 4.2] for all the spans of groupoids needed. Moreover, let us fix notation for the multiplication functor $M : (G \times G)//G \longrightarrow G//G$, the flip of factors $T : G//G \times G//G \longrightarrow G//G \times G//G$ and the obvious functor $B : (G \times G)//G \longrightarrow G//G \times G//G$. A discussion of all the span of groupoids needed can be found in [Mor15, 4.2].

We now obtain the multiplication in (a) by evaluation Z on the pair of pants $P : \mathbb{S}^1 \coprod \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. This yields the span

$$G//G \times G//G \xleftarrow{B} (G \times G)//G \xrightarrow{M} G//G$$

together with a vector bundle morphism, i.e. an intertwiner $B^*\varrho_{\mathfrak{A}} \longrightarrow M^*\varrho_{\mathfrak{A}}$ of representations consisting of linear maps $\mathfrak{A}_g \otimes \mathfrak{A}_h \longrightarrow \mathfrak{A}_{gh}$ giving us the multiplication. The multiplication is associative and unital (this is always true for Frobenius objects). The intertwiner property implies that the multiplication intertwines with the G-action. The unit and the pairing are obtained in a similar way. It is necessarily non-degenerate because it belongs to a Frobenius structure.

We have given the multiplication and the pairing, hence the Frobenius structure is completely specified. Since we are only interested in commutative Frobenius objects (commutativity is defined here with respect to a certain braiding; $\mathfrak{A}$ need not be commutative as an algebra), we still have to spell out explicitly the commutativity constraint. To this end, we consider the braiding isomorphism

$$C : \mathbb{S}^1 \coprod \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \coprod \mathbb{S}^1.$$

It is sent to the span

$$G//G \times G//G \xleftarrow{\mathrm{id}_{G//G\times G//G}} G//G \times G//G \xrightarrow{T} G//G \times G//G$$

together with the usual flip maps $\tau : \mathfrak{A}_g \otimes \mathfrak{A}_h \longrightarrow \mathfrak{A}_h \otimes \mathfrak{A}_g$. We have to compose this span with the span for the pair of pants already encountered above. Since on the level of bordisms the result is again a pair of pants, a convenient representative in the class of the weak pullback is

$$\begin{array}{ccccccc} & & & (G\times G)//G & & & \\ & & \swarrow^{B} & & \searrow^{F} & & \\ & G//G\times G//G & & \xRightarrow{\eta} & & (G\times G)//G & \\ \swarrow^{\mathrm{id}} & & \searrow^{T} & & \swarrow^{B} & & \searrow^{M} \\ G//G\times G//G & & & G//G\times G//G & & & G//G \end{array} ,$$

where F is the functor sending (g, h) to (ghg^{-1}, g) and being the identity on morphisms seen as group elements. The natural isomorphism η relates $TB(g, h) = (h, g)$ and $BF(g, h) = (ghg^{-1}, g)$ by (g, g). The representative is convenient since the outer span is the span assigned to the pair of pants (observe $MF = M$). By the definition of the composition in $\mathbf{VecBun}_K\mathbf{Grpd}$ we read off that the commutativity constraint is given by the commutativity of the square

$$\begin{array}{ccc} \mathfrak{A}_g \otimes \mathfrak{A}_h & \xrightarrow{\mu} & \mathfrak{A}_{gh} \\ {\scriptstyle\tau}\downarrow & & \uparrow{\scriptstyle\mu} \\ \mathfrak{A}_h \otimes \mathfrak{A}_g & \xrightarrow{g\otimes g} & \mathfrak{A}_{ghg^{-1}} \otimes \mathfrak{A}_g \end{array} ,$$

which is just our claim.

We will call a commutative Frobenius object in $\mathbf{VecBun}_K\mathbf{Grpd}$ of this special form a *pre-crossed Frobenius G-algebra*. We also get the suitable notion of morphism: The image of a natural monoidal transformation $\eta : Z \longrightarrow Z'$ of two-dimensional G-equivariant topological field theories under

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(2), \mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(\mathbf{Cob}(2), \mathbf{VecBun}_K\mathbf{Grpd}) \cong \mathbf{cFrob}(\mathbf{VecBun}_K\mathbf{Grpd})$$

amounts to a morphism of their pre-crossed Frobenius G-algebras $\mathfrak{A}$ and $\mathfrak{A}'$. The explicit description of such a morphism is as follows: It is an isomorphism $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}'$ of unital algebras preserving the grading, the pairing and intertwining with the G-action.

We could ask whether the functor from two-dimensional G-equivariant topological field theories to pre-crossed Frobenius G-algebras is an equivalence. Rephrasing Theorem 3.1 in [Tur10b] in the language of this section, it will be an equivalence once we restrict in range to those pre-crossed Frobenius G-algebras satisfying the following additional properties:

(F1) Self-invariance of twisted sectors: Any element $g \in G$ acts trivially on $\mathfrak{A}_g$, i.e. $g.v = v$ for $v \in \mathfrak{A}_g$. This entails that the commutativity constraint takes the form

$$vw = (g.w)v \quad \text{for all} \quad v \in \mathfrak{A}_g, \quad w \in \mathfrak{A}.$$

(F2) Trace property: For $g, h \in G$ and $v \in \mathfrak{A}_{ghg^{-1}h^{-1}}$ the equality

$$\mathrm{tr}_{\mathfrak{A}_g}\, vh = \mathrm{tr}_{\mathfrak{A}_h}\, g^{-1}v$$

holds, where v is the multiplication map by v from the left and g and h denote the automorphisms coming from the action with these elements.

If a pre-crossed Frobenius G-algebra satisfies these axioms, it is called *crossed Frobenius G-algebra*. In summary, Theorem 3.1 in [Tur10b] implies that

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(2), \mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(\mathbf{Cob}(2), \mathbf{VecBun}_K\mathbf{Grpd}) \cong \mathbf{cFrob}(\mathbf{VecBun}_K\mathbf{Grpd})$$

becomes an equivalence once restricted in range to crossed Frobenius G-algebras.

3.5 Pushforward maps via integration over homotopy fibers

We continue the preparation for the definition of the parallel section functor $\mathrm{Par} : \mathbf{VecBun}_K\mathbf{Grpd} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ by defining pushforward maps. Recall first that a functor $\Phi : \Gamma \longrightarrow \Omega$ between small groupoids and a vector bundle ϱ over Ω yield a pullback map

$$\Phi^* : \mathrm{Par}\, \varrho \longrightarrow \mathrm{Par}\, \Phi^* \varrho$$

by Proposition 3.8, (e). For the orbifold construction we will need another natural map running in the opposite direction. It will be constructed using integrals with respect to groupoid cardinality over homotopy fibers.

The concept of the groupoid cardinality

$$|\Gamma| := \sum_{[x] \in \pi_0(\Gamma)} \frac{1}{|\mathrm{Aut}(x)|}$$

of an essentially finite groupoid Γ is recalled in Appendix A. Seen as a measure on the set $\pi_0(\Gamma)$ of isomorphism classes of Γ, it gives rise to the following notion of an integral:

Definition 3.31 – Integral of invariant functions over groupoids with respect to groupoid cardinality. An *invariant function* f on a groupoid Γ with values in a vector space V over a field of characteristic zero is the assignment of a vector $f(x) \in V$ to each $x \in \Gamma$ such that $f(x) = f(y)$ if $x \cong y$ in Γ. If Γ is essentially finite, we define by

$$\int_\Gamma f = \int_\Gamma f(x)\, \mathrm{d}x := \sum_{[x] \in \pi_0(\Gamma)} \frac{f(x)}{|\mathrm{Aut}(x)|} \ \in \ V$$

the *integral of f over Γ*.

The properties of this integral are collected in Appendix A.

The groupoids over which we need to integrate arise as homotopy fibers.

Definition 3.32 – Homotopy fiber. For a functor $\Phi : \Gamma \longrightarrow \Omega$ and $y \in \Omega$ the homotopy pullback (Definition 3.11)

$$\begin{array}{ccc} \Phi^{-1}[y] & \longrightarrow & \Gamma \\ \big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ \star & \xrightarrow[y]{} & \Omega \end{array}$$

is called the *homotopy fiber* over y. Here $\star$ is the terminal object in the category of simplicial sets. The natural transformation contained in the square is suppressed in the notation.

Remark 3.33. It is worth writing out the definition of the homotopy fiber $\Phi^{-1}[y]$: Objects are pairs (x, g), where $x \in \Gamma$ and $g : \Phi(x) \longrightarrow y$. A morphism $(x, g) \longrightarrow (x', g')$ in $\Phi^{-1}[y]$ is a morphism $h : x \longrightarrow x'$ in Γ such that the triangle

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\Phi(h)} & \Phi(x') \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g'} & \\ & y & \end{array}$$

commute. So $\Phi^{-1}[y]$ is the groupoid consisting of 'preimages of y up to isomorphism' and could therefore also be called *preimage groupoid of y*. There is an obvious forgetful functor $\Phi^{-1}[y] \longrightarrow \Gamma_y$ to the full subgroupoid Γ_y of Γ consisting of objects x with $\Phi(x) \cong y$. This forgetful functor is a $|\mathrm{Aut}(y)|$-fold covering, see Appendix A for the proof and the definition of coverings of groupoids.

We can now finally define the pushforward map.

Definition 3.34 – Pushforward map. Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between essentially finite groupoids and K a field of characteristic zero. Then for any K-vector bundle ϱ over Ω we define the *pushforward map*

$$\Phi_* : \mathrm{Par}\, \Phi^* \varrho \longrightarrow \mathrm{Par}\, \varrho$$

by

$$(\Phi_* s)(y) = \sum_{[x,g]\in\pi_0(\Phi^{-1}[y])} \frac{\varrho(g)s(x)}{|\mathrm{Aut}(x,g)|} = \int_{\Phi^{-1}[y]} \varrho(g)s(x)\,\mathrm{d}(x,g) \in \varrho(y)$$

for any parallel section s of $\Phi^*\varrho$ and $y \in \Omega$.

Remarks 3.35.

(a) Obviously, $\mathrm{Aut}(x,g) \cong \mathrm{Aut}_0(x) := \ker(\mathrm{Aut}(x) \longrightarrow \mathrm{Aut}(\Phi(x)))$ and hence

$$(\Phi_* s)(y) = \sum_{[x,g]\in\pi_0(\Phi^{-1}[y])} \frac{\varrho(g)s(x)}{|\mathrm{Aut}_0(x)|}.$$

(b) If $\Omega = \Gamma$ and $\Phi = \mathrm{id}_\Gamma$, then $\Phi^{-1}[y]$ is connected and $|\mathrm{Aut}_0(x)| = 1$. This implies $\mathrm{id}_{\Gamma *} = \mathrm{id}_{\mathrm{Par}\,\varrho}$.

For pushforward maps we can derive another formula that will be useful in the sequel.

Corollary 3.36. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between essentially finite groupoids and K a field of characteristic zero. Then for any K-vector bundle ϱ over Ω and any parallel section s of $\Phi^*\varrho$ the formula*

$$(\Phi_* s)(y) = \sum_{\substack{[x]\in\pi_0(\Gamma)\\ \Phi(x)\overset{h}{\cong} y}} \sum_{g\in\mathrm{Aut}(y)} \frac{\varrho(g)\varrho(h)s(x)}{|\mathrm{Aut}(x)|} \quad \text{for all} \quad y \in \Omega$$

holds. Here in the first sum $\Phi(x) \overset{h}{\cong} y$ expresses the condition that such an isomorphism exists. Its choice is irrelevant. Note that we do not sum over all h.

Proof. For $x \in \Gamma$ denote by $\Phi_x^{-1}[y]$ the full subgroupoid of all $(x',g') \in \Phi^{-1}[y]$ with $x' \cong x$. Then

$$\Phi^{-1}[y] = \coprod_{[x]\in\pi_0(\Gamma)} \Phi_x^{-1}[y].$$

By Definition 3.34 it suffices to show

$$\int_{\Phi_x^{-1}[y]} \varrho(g')s(x')\,\mathrm{d}(x',g') = \sum_{g\in\mathrm{Aut}(y)} \frac{\varrho(g)\varrho(h)s(x)}{|\mathrm{Aut}(x)|}$$

for some arbitrary $h : \Phi(x) \longrightarrow y$. For this we can assume that all objects in $\Phi_x^{-1}[y]$ are of the form (x,g') for some $g' : \Phi(x) \longrightarrow y$. Hence, they can all be uniquely written as (x,gh) for $g \in \mathrm{Aut}(y)$ and some arbitrary, but fixed $h : \Phi(x) \longrightarrow y$. This entails

$$\int_{\Phi_x^{-1}[y]} \varrho(g')s(x')\,\mathrm{d}(x',g') = \sum_{[x',g']=[x,gh]\in\pi_0(\Phi_x^{-1}[y])} \frac{\varrho(g)\varrho(h)s(x)}{|\mathrm{Aut}_0(x)|}.$$

Obviously, $\Phi_x^{-1}[y]$ is the action groupoid of the action

$$(x,gh).a := (x, gh\Phi(a)) \quad \text{for all} \quad a \in \mathrm{Aut}(x)$$

of $\mathrm{Aut}(x)$ on the object set of $\Phi_x^{-1}[y]$. Hence, we obtain

$$\int_{\Phi_x^{-1}[y]} \varrho(g')s(x')\,\mathrm{d}(x',g') = \sum_{g\in\mathrm{Aut}(y)} \frac{\varrho(g)\varrho(h)s(x)}{|\mathrm{Aut}_0(x)||[x,gh]|},$$

where the factor $|[x,gh]|$ corrects the overcounting resulting from the summation over all objects in contrast to the summation over all isomorphism classes. Since $\mathrm{Aut}_0(x)$ is the stabilizer group of (x,gh), the orbit theorem yields $|\mathrm{Aut}_0(x)||[x,gh]| = |\mathrm{Aut}(x)|$ and hence the claim. □

The formulae for pushforward maps simplify significantly for equivalences.

Corollary 3.37. *Let $\Phi : \Gamma \longrightarrow \Omega$ be an equivalence between essentially finite groupoids and K a field of characteristic zero. Then for any K-vector bundle ϱ over Ω and any parallel section s of $\Phi^*\varrho$ the formula*

$$(\Phi_* s)(y) = \varrho(h)s(x) \quad \text{for all} \quad y \in \Omega$$

holds, where $x \in \Gamma$ is any object and $h : \Phi(x) \longrightarrow y$ any isomorphism.

For pullback maps we derived a composition law and a naturality condition in Proposition 3.8. Both results have analogues for pushforward maps.

Proposition 3.38. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between essentially finite groupoids and K a field of characteristic zero.*

(a) *The pushforward maps obey the composition law $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$, where $\Psi : \Omega \longrightarrow \Lambda$ is another functor between essentially finite groupoids.*

(b) *The pushforward maps are natural in the sense that they provide a natural transformation*

$$\mathrm{Par}_\Gamma \circ \Phi^* \longrightarrow \mathrm{Par}_\Omega,$$

i.e. for any morphism $\lambda : \varrho \longrightarrow \xi$ of vector bundles over Ω the square

$$\begin{array}{ccc} \mathrm{Par}\,\Phi^*\varrho & \xrightarrow{\Phi_*} & \mathrm{Par}\,\varrho \\ {\scriptstyle (\Phi^*\lambda)_*}\downarrow & & \downarrow{\scriptstyle \lambda_*} \\ \mathrm{Par}\,\Phi^*\xi & \xrightarrow{\Phi_*} & \mathrm{Par}\,\xi \end{array}$$

commutes.

Proof.

(a) For a vector bundle ϱ on Λ, $s \in \mathrm{Par}(\Psi \circ \Phi)^*\varrho$ and $z \in \Lambda$ we find

$$\begin{aligned} ((\Psi \circ \Phi)_* s)(z) &= \int_{(\Psi\circ\Phi)^{-1}[z]} \varrho(g)s(x)\,\mathrm{d}(x,g) \\ &= \sum_{[y,h]\in\pi_0(\Psi^{-1}[z])} \int_{(\Psi\circ\Phi)^{-1}_{(y,h)}[z]} \varrho(g)s(x)\,\mathrm{d}(x,g) \\ &= \int_{\Psi^{-1}[z]} |\mathrm{Aut}_0(y)| \int_{(\Psi\circ\Phi)^{-1}_{(y,h)}[z]} \varrho(g)s(x)\,\mathrm{d}(x,g)\,\mathrm{d}(y,h), \end{aligned} \qquad (*)$$

where $(\Psi \circ \Phi)^{-1}_{(y,h)}[z]$ is the full subgroupoid of $(\Psi \circ \Phi)^{-1}[z]$ consisting of all (x, g) such that $(\Phi(x), g) \cong (y, h)$. For a fixed $(y, h) \in \Psi^{-1}[z]$ we define the functor

$$\begin{aligned} Q : \Phi^{-1}[y] &\longrightarrow (\Psi \circ \Phi)^{-1}_{(y,h)}[z], \\ (x, a) &\longmapsto (x, h\Psi(a)), \end{aligned}$$

which is the identity on morphisms (if these are seen as morphisms in Γ). A direct computations shows that Q is a $|\mathrm{Aut}_0(y)|$-fold covering. Now application of Proposition A.16 to the invariant function

$$f : (\Psi \circ \Phi)^{-1}_{(y,h)}[z] \longrightarrow \varrho(z), \quad (x, g) \longmapsto \varrho(g)s(x)$$

yields

$$|\mathrm{Aut}_0(y)| \int_{(\Psi\circ\Phi)^{-1}_{(y,h)}[z]} \varrho(g)s(x)\,\mathrm{d}(x,g) = \int_{\Phi^{-1}[y]} (Q^*f)(x,a)\,\mathrm{d}(x,a) = \int_{\Phi^{-1}[y]} \varrho(h\Psi(a))s(x)\,\mathrm{d}(x,a).$$

In view of $(*)$ this entails

$$\begin{aligned} ((\Psi \circ \Phi)_* s)(z) &= \int_{\Psi^{-1}[z]} \int_{\Phi^{-1}[y]} \varrho(h\Psi(g))s(x)\,\mathrm{d}(x,g)\,\mathrm{d}(y,h) \\ &= \int_{\Psi^{-1}[z]} \varrho(h) \int_{\Phi^{-1}[y]} \varrho(\Psi(g))s(x)\,\mathrm{d}(x,g)\,\mathrm{d}(y,h) \\ &= \int_{\Psi^{-1}[z]} \varrho(h)(\Phi_* s)(y)\,\mathrm{d}(y,h) \\ &= ((\Psi_* \circ \Phi_*)s)(z). \end{aligned}$$

(b) For $s \in \mathrm{Par}\,\Phi^*\varrho$ and $y \in \Omega$ we use Definition 3.34 and the naturality of λ to find

$$(\Phi_*(\Phi^*\lambda)_* s)(y) = \int_{\Phi^{-1}[y]} \xi(g)\lambda_{\Phi(x)}s(\Phi(x))\,\mathrm{d}(x,g) = \int_{\Phi^{-1}[y]} \lambda_y\varrho(g)s(\Phi(x))\,\mathrm{d}(x,g) = (\lambda_*\Phi_* s)(y).$$ □

3.6 The parallel section functor

Finally, as the crucial ingredient of the orbifold construction, we will now define the symmetric monoidal functor

$$\mathrm{Par} : \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd} \longrightarrow \mathbf{Vect}_{\mathbb{C}}.$$

It sends vector bundles to their spaces of parallel sections. On morphisms it uses a summation inspired by Dijkgraaf-Witten theories. The crucial result needed for the proof is an equivariant Beck-Chevalley condition. The name is justified since it reduces in the non-equivariant case to the well-known Beck-Chevalley condition. For a dicussion in the context of Dijkgraaf-Witten theory see [Mor11, Appendix A.2] or Section 5.1 of this article.

Proposition 3.39 – Equivariant Beck-Chevalley condition. *Let K be a field of characteristic zero. For the homotopy pullback*

$$\begin{array}{ccc}
\Gamma \times_{\Omega} \Lambda & \xrightarrow{\pi_\Gamma} & \Gamma \\
{\scriptstyle \pi_\Lambda}\downarrow & \overset{\eta}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \Phi} \\
\Lambda & \xrightarrow[\Psi]{} & \Omega
\end{array}$$

of a cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ of essentially finite groupoids and any K-vector bundle ϱ over Ω the pentagon relating different pull-push combinations

$$\begin{array}{ccccc}
\mathrm{Par}\,\Phi^*\varrho & & \xrightarrow{\quad \Phi_* \quad} & & \mathrm{Par}\,\varrho \\
{\scriptstyle \pi_\Gamma^*}\downarrow & & & & \\
\mathrm{Par}\,\pi_\Gamma^*\Phi^*\varrho & & & & \downarrow{\scriptstyle \Psi^*} \\
& {\scriptstyle \varrho(\eta)_*}\searrow & & & \\
& & \mathrm{Par}\,\pi_\Lambda^*\Psi^*\varrho & \xrightarrow{\pi_{\Lambda *}} & \mathrm{Par}\,\Psi^*\varrho
\end{array}$$

commutes.

Proof. For $s \in \mathrm{Par}\,\Phi^*\varrho$ and $y \in \Lambda$ we compute

$$(\pi_{\Lambda *}\varrho(\eta)_*\pi_\Gamma^* s)(y) = \int_{\pi_\Lambda^{-1}[y]} \varrho(\Psi(g))(\varrho(\alpha)\pi_\Gamma^* s)(x, y', \alpha)\,\mathrm{d}(x, y', \alpha, g) = \int_{\pi_\Lambda^{-1}[y]} \varrho(\Psi(g)\alpha)s(x)\,\mathrm{d}(x, y', \alpha, g). \qquad (*)$$

and

$$(\Psi^*\Phi_* s)(y) = \int_{\Phi^{-1}[\Psi(y)]} \varrho(h)s(x)\,\mathrm{d}(x, h). \qquad (\ddagger)$$

By the fiberwise characterization of homotopy pullbacks in [CPS06, 5.2] we find the weakly commutative diagram

$$\begin{array}{ccccc}
\pi_\Gamma^{-1}[y] & \longrightarrow & \Gamma \times_\Omega \Lambda & \xrightarrow{\pi_\Gamma} & \Gamma \\
{\scriptstyle \Theta}\downarrow & & {\scriptstyle \pi_\Lambda}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\Psi^{-1}[\Phi(y)] & \longrightarrow & \Lambda & \xrightarrow[\Psi]{} & \Omega
\end{array}$$

containing the equivalence Θ. Applying the transformation formula (Proposition A.13) to this equivalence shows that the integrals in $(*)$ and $(\ddagger)$ are equal. □

Finally, we introduce the symmetric monoidal functor needed for the orbifold construction.

Theorem 3.40 – Parallel section functor. *For a field K of characteristic zero the assignment*

$$\begin{aligned} \mathrm{Par} \;:\; \mathbf{VecBun}_K\mathbf{Grpd} &\longrightarrow \mathbf{Vect}_K \\ (\varGamma, \varrho) &\longmapsto \mathrm{Par}\,\varrho \\ \Big((\varGamma_0, \varrho_0) \xleftarrow{r_0} (\varLambda, \lambda) \xrightarrow{r_1} (\varGamma_1, \varrho_1) \Big) &\longmapsto (r_{1*}\lambda_* r_0^* : \mathrm{Par}\,\varrho_0 \longrightarrow \mathrm{Par}\,\varrho_1) \end{aligned}$$

yields a symmetric monoidal functor. We will refer to this functor as parallel section functor.

Proof.

(i) It follows directly from the definitions that Par sends identities to identities (see Remark 3.35, (b)). To conclude the proof of functoriality we verify the composition law

$$\mathrm{Par}((\varLambda', \lambda') \circ (\varLambda, \lambda)) = \mathrm{Par}(\varLambda', \lambda') \circ \mathrm{Par}(\varLambda, \lambda) \qquad (*)$$

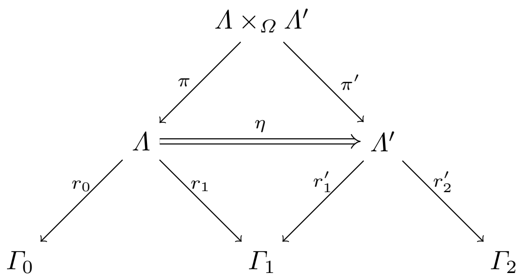
for morphisms

$$(\varGamma_0, \varrho_0) \xleftarrow{r_0} (\varLambda, \lambda) \xrightarrow{r_1} (\varGamma_1, \varrho_1)$$

and

$$(\varGamma_1, \varrho_1) \xleftarrow{r_1'} (\varLambda', \lambda') \xrightarrow{r_2'} (\varGamma_2, \varrho_2)$$

in $\mathbf{VecBun}_K\mathbf{Grpd}$: By definition of composition in $\mathbf{VecBun}_K\mathbf{Grpd}$ the composed morphism $(\varLambda', \lambda') \circ (\varLambda, \lambda)$ is the class of the outer span in



together with the intertwiner

$$(r_0 \circ \pi)^* \varrho_0 = \pi^* r_0^* \varrho_0 \xrightarrow{\pi^*\lambda} \pi^* r_1^* \varrho_1 \xrightarrow{\varrho_1(\eta)} \pi'^* r_1'^* \varrho_1 \xrightarrow{\pi'^*\lambda'} \pi'^* r_2'^* \varrho_2 = (r_2' \circ \pi')^* \varrho_2 .$$

The proof of the composition law is now rather short because by using the properties of the pullback and pushforward maps previously established we find

$$\begin{aligned} \mathrm{Par}((\varLambda', \lambda') \circ (\varLambda, \lambda)) &= (r_2' \circ \pi')_* (\pi'^* \lambda')_* \varrho_1(\eta)_* (\pi^*\lambda)_* (r_0 \circ \pi)^* \\ &= r_{2*}' \pi_*' (\pi'^* \lambda')_* \varrho_1(\eta)_* (\pi^*\lambda)_* \pi^* r_0^* \quad \begin{pmatrix} \text{Proposition 3.8, (d) and} \\ \text{Proposition 3.38, (a)} \end{pmatrix} \\ &= r_{2*}' \pi_*' (\pi'^* \lambda')_* \varrho_1(\eta)_* \pi^* \lambda_* r_0^* \quad \text{(Proposition 3.8, (e))} \\ &= r_{2*}' \lambda_*' \pi_*' \varrho_1(\eta)_* \pi^* \lambda_* r_0^* \quad \text{(Proposition 3.38, (b))} \\ &= r_{2*}' \lambda_*' r_1'^* r_{1*} \lambda_* r_0^* \quad \text{(Proposition 3.39)} \\ &= \mathrm{Par}(\varLambda', \lambda') \circ \mathrm{Par}(\varLambda, \lambda). \end{aligned}$$

This proves $(*)$ and hence the functoriality of Par. It is worth noting that this proof uses a strategy quite similar to the one used in the proof of Theorem 5.6.

(ii) We still have to endow Par with the structure of a symmetric monoidal functor. Recall for this the monoidal structure of $\mathbf{VecBun}_K\mathbf{Grpd}$ from Definition 3.12. The parallel sections of the monoidal unit $\mathbb{I}$ are obviously functions $\star \longrightarrow K$, which proves that $\mathrm{Par}\,\mathbb{I}$ is naturally isomorphic to K. For vector bundles ϱ over $\varGamma$ and ξ over $\varOmega$ we need a natural isomorphism

$$\mathrm{Par}\,\varrho \otimes \mathrm{Par}\,\xi \longrightarrow \mathrm{Par}(\varrho \mathbin{\widehat{\otimes}} \xi) = \mathrm{Par}\left((\varGamma, \varrho) \otimes (\varOmega, \xi)\right) . \qquad (\ddagger)$$

We find this morphism by observing that for parallel sections s and r of ϱ and ξ, respectively, $s \otimes r$ is a parallel section of $\varrho \otimes \xi$. This yields the map $(\ddagger)$, which is obviously a natural monomorphism. Using the holonomy principle (Proposition 3.6) we can reduce the proof that $(\ddagger)$ is an isomorphism to the statement that for finite groups G and H acting on vector spaces V and W over K, respectively, we have $V^G \otimes W^H \cong (V \otimes W)^{G \times H}$ by the obvious inclusion map. Indeed, this can directly be verified, which concludes the proof of monoidality. It is straightforward to verify that the symmetry requirement is fulfilled. □

Remarks 3.41.

(a) Writing out the above definition of the parallel section functor explicitly yields for a parallel section s of ϱ_0 and $x_1 \in \Gamma_1$

$$(\mathrm{Par}(\Lambda, \lambda)s)(x_1) = \int_{r_1^{-1}[x_1]} \varrho_1(g)\lambda_y s(r_0(y))\,\mathrm{d}(y,g).$$

Using Corollary 3.36 we can also write

$$(\mathrm{Par}(\Lambda, \lambda)s)(x_1) = \sum_{\substack{[y]\in\pi_0(\Lambda) \\ r_1(y)\overset{h}{\cong}x_1}} \sum_{g\in\mathrm{Aut}(x_1)} \frac{\varrho(g)\varrho(h)s(r_0(y))}{|\mathrm{Aut}(y)|},$$

where the choice of h is irrelevant and we do not sum over h.

(b) In the following case a strictification of (a) is possible: Let us assume that $r_1 : \Lambda \longrightarrow \Gamma_1$ is an isofibration (Remark A.2), then for $y \in \Lambda$ with $r_1(y) \cong x_1$ we can find $y' \in \Lambda$ with $r_1(y') = x_1$, i.e. we can turn equalities holding up to isomorphism into strict ones. Using the identity as morphism $r_1(y') \longrightarrow x_1$ we find in this case the strict version of the formula from (a)

$$(\mathrm{Par}(\Lambda, \lambda)s)(x_1) = \sum_{\substack{[y]\in\pi_0(\Lambda) \\ r_1(y)=x_1}} \sum_{g\in\mathrm{Aut}(x_1)} \frac{\varrho(g)s(r_0(y))}{|\mathrm{Aut}(y)|},$$

where $r_1(y) = x_1$ means that the representatives are chosen such that this equality holds, but the result does not depend on the representatives.

(c) The isofibration property is always fulfilled in the cases we are interested in because for G-equivariant topological field theories all relevant functors between groupoids will be restriction functors

$$\iota^* : \Pi(M, BG) \longrightarrow \Pi(\Sigma, BG)$$

for the inclusion $\iota : \Sigma \longrightarrow M$ of a collection of boundary components of some oriented compact manifold M with boundary. Since ι is a cofibration, we can extend homotopies of maps $\Sigma \longrightarrow BG$ to homotopies of maps $M \longrightarrow BG$. This implies that ι^* automatically has the needed lifting property.

3.7 The orbifold construction

We have now established all the ingredients of the orbifold construction. Although the construction works for any field of characteristic zero, we restrict from now on to $K = \mathbb{C}$.

Definition 3.42 – Orbifold construction for G-equivariant topological field theories. Let

$$Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

be a G-equivariant topological field theory. Then the *orbifold theory* Z/G of Z is defined as the concatenation of symmetric monoidal functors

$$\frac{Z}{G} : \mathbf{Cob}(n) \xrightarrow{\widehat{Z}} \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd} \xrightarrow{\mathrm{Par}} \mathbf{Vect}_{\mathbb{C}},$$

where $\widehat{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ is the $\Pi(?, BG)$-equivariant topological field theory corresponding to Z in the sense of Proposition 3.26.

Remark 3.43. The process of forming the orbifold theory will be understood as the functor

$$\frac{?}{G} : \mathrm{HSym}(G\text{-}\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}) \xrightarrow{\widehat{?}} \mathrm{Sym}(\mathbf{Cob}(n), \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}) \xrightarrow{\mathrm{Par}_*} \mathrm{Sym}(\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}).$$

The functor $\widehat{?} : \mathrm{HSym}(G\text{-}\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(\mathbf{Cob}(n), \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd})$ comes from Proposition 3.26. So the orbifold construction is itself functorial: It is a functor between groupoids of field theories.

This finishes the construction of an orbifold theory $Z/G : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ for a G-equivariant theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$. We now give a very explicit description of the orbifold theory and derive its most important properties. First of all, the following result holds by construction:

Proposition 3.44. *For a G-equivariant theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ the orbifold theory Z/G associates to an object Σ in $\mathbf{Cob}(n)$ the vector space $\mathrm{Par}\,\varrho_\Sigma$ of parallel sections of the vector bundle ϱ_Σ over $\Pi(\Sigma, BG)$ given* *by Z and Proposition 2.10, i.e.*

$$\frac{Z}{G}(\varSigma) = \operatorname{Par} \varrho_\varSigma .$$

Explicitly,

$$\frac{Z}{G}(\varSigma) \cong \bigoplus_{[\varphi]\in[\varSigma,BG]} Z(\varSigma,\varphi)^{\operatorname{Aut}(\varphi)} \cong \bigoplus_{[\varphi]\in[\varSigma,BG]} Z(\varSigma,\varphi)_{\operatorname{Aut}(\varphi)} . \qquad (*)$$

This realizes the old idea of summing over twisted sectors and taking invariants that was mentioned in the introduction.

The next result is a description of the orbifold construction on morphisms:

Proposition 3.45. *Let G be a finite group and $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ an n-dimensional G-equivariant topological field theory. Then for a morphism $M : \varSigma_0 \longrightarrow \varSigma_1$ the formula*

$$\left(\frac{Z}{G}(M)s\right)(\varphi_1) = \int_{r_1^{-1}[\varphi_1]} Z((\varSigma_1 \times [0,1]) \circ M, h \cup \psi) s(\psi|_{\varSigma_0}) \, \mathrm{d}(\psi,h)$$

$$\text{for all} \quad s \in \frac{Z}{G}(\varSigma_0) = \operatorname{Par}\varrho_{\varSigma_0}, \quad \varphi_1 : \varSigma_1 \longrightarrow BG$$

holds. It expresses $Z/G(M)$ as an integral with respect to groupoid cardinality. Here $r_1 : \varPi(M,BG) \longrightarrow \varPi(\varSigma_1,BG)$ is the restriction functor, $h \cup \psi : (\varSigma_1 \times [0,1]) \circ M \longrightarrow BG$ is the function on M with a cylinder glued to it obtained from $\psi : M \longrightarrow BG$ and a homotopy $\psi|_{\varSigma_1} \overset{h}{\simeq} \varphi_1$. As an alternative we can use the formula

$$\left(\frac{Z}{G}(M)s\right)(\varphi_1) = \sum_{\substack{[\psi]\in[M,BG] \\ \psi|_{\varSigma_1}=\varphi_1}} \sum_{\varphi_1 \overset{h}{\simeq} \varphi_1 \in \operatorname{Aut}(\varphi_1)} \frac{Z((\varSigma_1 \times [0,1]) \circ M, h \cup \psi) s(\psi|_{\varSigma_0})}{|\operatorname{Aut}(\psi)|},$$

where for the first sum the representatives are chosen such that $\psi|_{\varSigma_1} = \varphi_1$ holds strictly, but the result is independent of the representatives.

Proof. The first formula is an immediate consequence of Remark 3.41, (a). For the second formula we use Remark 3.41, (b), which is justified by Remark 3.41, (c). □

Taking into account Remark 3.5 and the fact that $Z(\varnothing)$ is canonically isomorphic to the ground field, we can express the orbifold construction on objects:

Proposition 3.46. *Let G be a finite group and $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ an n-dimensional G-equivariant topological field theory. To a closed $n-1$-dimensional oriented manifold $\varSigma$ the orbifold theory Z/G assigns the vector space given by the end*

$$\frac{Z}{G}(\varSigma) = \int_{\varPi(\varSigma,BG)} \operatorname{Hom}_{\mathbf{Vect}_\mathbb{C}}(Z(\varnothing), Z(\varSigma,\varphi)) \, \mathrm{d}\varphi .$$

Finally, we can compute the n-manifold invariants of Z/G by using Proposition 3.45 or Remark 3.41, (b):

Proposition 3.47. *Let G be a finite group and $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ an n-dimensional G-equivariant topological field theory. For a closed oriented n-dimensional manifold the invariant $Z/G(M)$ is given by the integral of the invariant function*

$$\varPi(M,BG) \ni \psi \longmapsto Z(M,\psi) \in \mathbb{C}$$

over $\varPi(M,BG)$ with respect to groupoid cardinality, i.e.

$$\frac{Z}{G}(M) = \int_{\varPi(M,BG)} Z(M,\psi) \, \mathrm{d}\psi .$$

Note how in the preceding result the sum over all twisted sectors is realized by an integral with respect to groupoid cardinality.

Example 3.48 – *Twisted version of the orbifold construction.* Let G be a finite group and BG_θ be the n-dimensional primitive G-equivariant theory associated to a cocycle $\theta \in Z^n(BG;\mathrm{U}(1))$ as defined in [Tur10b, I.2.1]. We can now define the functor

$$\mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}) \xrightarrow{?\otimes BG_\theta} \mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}) \xrightarrow{?/G} \mathrm{Sym}(\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}})$$

which first takes the tensor product of a given equivariant topological field theory with the primitive theory associated to θ and then orbifoldizes. We call this functor the *θ-twisted orbifold construction*.

Let us make the following observations:

(1) If we apply the θ-twisted orbifold construction to the trivial G-equivariant theory, which assigns identities between complex lines to all morphisms with maps into BG, then we obtain the orbifold theory BG_θ/G of BG_θ. The ordinary topological field theory BG_θ/G is commonly referred to as *θ-twisted Dijkgraaf-Witten theory* (see e.g. [FQ93], [Mor15] or Remark 5.7, (c) of this article). It assigns to a closed oriented n-dimensional manifold M the number

$$\frac{BG_\theta}{G}(M) = \int_{\Pi(M,BG)} \langle \psi^*\theta, \mu_M\rangle \, \mathrm{d}\psi,$$

where $\mu_M \in H_n(M)$ is the fundamental class of M. For an object Σ in $\mathbf{Cob}(n)$ we obtain

$$\frac{BG_\theta}{G}(\Sigma) \cong \bigoplus_{[\varphi]\in[\Sigma,BG]} BG_\theta(\Sigma,\varphi)^{\mathrm{Aut}(\varphi)}.$$

(2) If we denote by $U : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Cob}(n)$ the forgetful functor, then

$$\frac{?}{G}\circ U^* \in \mathrm{End}(\mathrm{Sym}(\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}))$$

assigns to the trivial n-dimensional topological field theory the Dijkgraaf-Witten theory for the group G as follows from (1) in the case $\theta = 0$. For general G this implies that $(?/G)\circ U^*$ is not naturally isomorphic to the identity. So $?/G$ cannot be an adjoint to the pullback along U because a pair of adjoint functors between groupoids always form an equivalence.

4 Example: The orbifold construction in low dimensions

In low dimensions, by which we mean dimension one and two, classification results for topological field theories are well-known. The same is true for equivariant topological field theories, see [Tur10b]. In this section we compute the orbifold theory in low dimensions in terms of classifying objects.

4.1 One-dimensional equivariant topological field theories

In Example 3.29 we recalled that the groupoid of one-dimensional G-equivariant equivariant topological field theories is equivalent to the core of the representation category of G on finite vector spaces. If $Z : G\text{-}\mathbf{Cob}(1) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ is an equivariant topological field theory classified by a representation ϱ of G on a finite-dimensional complex vector space V, then its orbifold theory, which is an ordinary topological field theory, is classified by the vector space, which it assigns to the positively oriented point. Using Proposition 3.44 we obtain:

Theorem 4.1 – Orbifold construction for one-dimensional equivariant topological field theories. *Let G be a finite group and $Z : G\text{-}\mathbf{Cob}(1) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ a one-dimensional G-equivariant topological field theory with classifying representation $\varrho : G \longrightarrow \mathrm{Aut}(V)$ in the sense of Example 3.29. Then the orbifold theory Z/G is determined by the evaluation on the positively oriented point, where it yields the space V^G of invariants.*

Remark 4.2. Maybe more interesting than the one-dimensional orbifold construction itself is the insight it provides on why the orbifold construction cannot be obtained as adjoint of the pullback along the forgetful functor $U : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Cob}(n)$: Using Example 3.29 and Theorem 4.1 we see that the one-dimensional orbifold construction is the functor

$$(?)^G : \mathbf{core}\,(\mathbf{FinRep}_{\mathbb{C}}(G)) \longrightarrow \mathbf{core}\,(\mathbf{FinVect}_{\mathbb{C}}) \qquad (*)$$

taking invariants. If we forgot about taking the core, we would be left with

$$(?)^G : \mathbf{FinRep}_{\mathbb{C}}(G) \longrightarrow \mathbf{FinVect}_{\mathbb{C}},$$

which would be the two-sided adjoint of the functor $\mathbf{FinVect}_{\mathbb{C}} \longrightarrow \mathbf{FinRep}_{\mathbb{C}}(G)$ forming the trivial representation for a given vector space (this would correspond to the pullback functor along the forgetful functor $G\text{-}\mathbf{Cob}(1) \longrightarrow \mathbf{Cob}(1)$). This adjunction, however, does not persist if we take cores because the natural bijections of the adjunction do not take isomorphisms to isomorphisms. Hence, $(*)$ is not an adjoint for the pullback along the forgetful functor $G\text{-}\mathbf{Cob}(1) \longrightarrow \mathbf{Cob}(1)$ – which is what we have seen more generally in Example 3.48.

Example 4.3. As an application of Theorem 4.1 we recover the orthogonality relations for characters: Let $\varrho : G \longrightarrow \operatorname{Aut}(V)$ be a representation of a finite group G on some finite-dimensional complex vector space V. Then the character χ of ϱ provides an invariant function on the action groupoid $G//G$. Observe further that by Theorem 4.1 ϱ classifies a one-dimensional G-equivariant topological field theory Z. By Proposition 3.47 its orbifold theory assigns to $\mathbb{S}^1$ the invariant

$$\frac{Z}{G}(\mathbb{S}^1) = \int_{\Pi(\mathbb{S}^1, BG)} Z(\mathbb{S}^1, \varphi)\, \mathrm{d}\varphi = \int_{G//G} \operatorname{tr} \varrho(g)\, \mathrm{d}g = \int_{G//G} \chi,$$

where we have applied the transformation formula (Proposition A.13) to an equivalence $\Pi(\mathbb{S}^1, BG) \cong G//G$ and used Example 3.29 and the character formula from Proposition 2.10. But a general well-known argument shows that any one-dimensional topological field theory assigns to $\mathbb{S}^1$ the dimension of the vector space assigned to a point. Together with Theorem 4.1 this yields $Z/G(\mathbb{S}^1) = \dim V^G$, which proves

$$\int_{G//G} \chi = \dim V^G.$$

4.2 Orbifold construction for two-dimensional equivariant topological field theories

For two-dimensional equivariant topological field theories we can write down the orbifold construction on the level of classifying objects as well, i.e. by using the classification of G-equivariant topological field theories by crossed Frobenius G-algebras due to [Tur10b] as recalled in Example 3.30. The orbifold theory is an ordinary two-dimensional topological field theory and hence equivalent to a commutative Frobenius algebra. Our goal in this subsection is to determine this Frobenius algebra.

Definition 4.4 – Parallel sections of a crossed Frobenius G-algebra. Let G be a finite group and $\mathfrak{A}$ be a crossed Frobenius G-algebra. A *parallel section* of $\mathfrak{A}$ is a parallel section of the underlying vector bundle over $G//G$, i.e. a family $s = (s(g))_{g \in G}$ of vectors $s(g) \in \mathfrak{A}_g$ with $s(hgh^{-1}) = h.s(g)$ for all $g, h \in G$. We denote the *vector space of parallel sections* of $\mathfrak{A}$ by $\mathfrak{A}/G$.

Theorem 4.5 – Orbifold construction for two-dimensional equivariant topological field theories. *Let G be a finite group, $Z : G\text{-}\mathbf{Cob}(2) \longrightarrow \mathbf{Vect}_\mathbb{C}$ a two-dimensional G-equivariant topological field theory and $\mathfrak{A}$ its crossed Frobenius G-algebra. Then the orbifold theory $Z/G : G\text{-}\mathbf{Cob}(2) \longrightarrow \mathbf{Vect}_\mathbb{C}$ is classified by the commutative Frobenius algebra which, as a vector space, is given by the vector space of parallel section $\mathfrak{A}/G$ of $\mathfrak{A}$. The multiplication is given by*

$$(ss')(g) = \sum_{\substack{a,b \in G\\ ab=g}} s(a)s'(b) \quad \text{for all} \quad s, s' \in \mathfrak{A}/G, \quad g \in G,$$

the unit is the parallel section with $s(1) = 1$ and $s(g) = 0$ for $g \neq 1$, and the pairing is given by

$$\kappa(s, s') = \frac{1}{|G|} \sum_{g \in G} \kappa(s(g), s(g^{-1})) \quad \text{for all} \quad s, s' \in \mathfrak{A}/G.$$

The Frobenius algebra $\mathfrak{A}/G$ is called the orbifold Frobenius algebra of $\mathfrak{A}$.

Remarks 4.6.

(a) To reduce notational complexity we refrain from introducing additional symbols for the multiplication, pairing etc. of the orbifold algebra.

(b) The term orbifold algebra is not only justified by the above assertion, but also used in the literature: By [Kau02, Proposition 2.1.3] the invariants of a crossed Frobenius G-algebra naturally form a Frobenius algebra. This is exactly the idea underlying Definition 4.4 because the holonomy principle allows us to identify invariants with parallel sections. In fact, on the level of vector spaces we have the non-canonical isomorphism

$$\mathfrak{A}/G \cong \bigoplus_{[g] \in G/G} \mathfrak{A}_g^{\operatorname{Aut}(g)}.$$

In summary, the orbifold construction is designed in such a way that it relates in dimension two to known orbifold constructions for Frobenius algebras.

Proof. We know that the orbifold theory is a two-dimensional topological field theory. By the classification result for two-dimensional topological field theories it can be equivalently described by the commutative Frobenius algebra obtained by evaluation on the circle. Since the orbifold theory on objects is given by forming spaces of parallel sections, we deduce that this commutative Frobenius algebra, as a vector space, is $\mathfrak{A}/G$. The multiplication is obtained by evaluation on the pair of pants. In Example 3.30 we have seen that the application of the stack of G-bundles to this bordism yields the span

$$G//G \times G//G \xleftarrow{B} (G\times G)//G \xrightarrow{M} G//G,$$

where B is the obvious functor and M the multiplication. Using the explicit formula for the orbifold construction (Proposition 3.45) we find

$$(ss')(g) = \sum_{\substack{[a,b]\in(G\times G)//G\\ ab=g\\ h\in\mathrm{Aut}(g)}} \frac{h.(s(a)s'(b))}{|\mathrm{Aut}(a,b)|} \quad \text{for all} \quad s,s'\in\mathfrak{A}/G, \quad g\in G,$$

where the representatives are chosen such that $ab=g$ holds strictly. Now denote by Γ_g the full subgroupoid of $(G\times G)//G$ of all (a,b) with $ab=g$ (in fact, this is equivalent to the full subgroupoid of all (a,b) with $ab=cgc^{-1}$ for some $c\in G$), define

$$f(s,s',g)(a,b) := \sum_{h\in\mathrm{Aut}(g)} h.(s(a)s'(b)) \quad \text{for all} \quad (a,b)\in\Gamma_g$$

and observe that $f(s,s',g):\Gamma_g \longrightarrow \mathfrak{A}_g$ is an invariant function on Γ_g. Having introduced this notation we obtain

$$(ss')(g) = \int_{\Gamma_g} f(s,s',g) = \sum_{[a,b]\in\Gamma_g} \frac{1}{|\mathrm{Aut}(a,b)|} \sum_{h\in\mathrm{Aut}(g)} s(hah^{-1})s'(hbh^{-1}). \qquad (\ddagger)$$

The groupoid Γ_g is the action groupoid of the action of $\mathrm{Aut}(g)$ on the underlying object set by conjugation. Recalling the classical orbit theorem stating that the map

$$\mathrm{Aut}(g)\ni h\longmapsto (hah^{-1},hbh^{-1})\in\mathcal{O}(a,b)$$

induces a bijection $\mathrm{Aut}(g)/\mathrm{Aut}(a,b)\cong\mathcal{O}(a,b)$ we see that we can replace the inner sum by

$$\sum_{h\in\mathrm{Aut}(g)} s(hah^{-1})s'(hbh^{-1}) = |\mathrm{Aut}(a,b)| \sum_{(x,y)\in\mathcal{O}(a,b)} s(x)s'(y).$$

Using this together with (‡) proves the formula

$$(ss')(g) = \sum_{[a,b]\in\Gamma_g}\sum_{(x,y)\in\mathcal{O}(a,b)} s(x)s'(y) = \sum_{\substack{a,b\in G\\ ab=g}} s(a)s'(b)$$

for the multiplication. The formula for the pairing can be derived by first computing the counit of the orbifold Frobenius algebra. For this we have to use the span $G//G \longleftarrow \star//G \longrightarrow \star//1$, see Example 3.30. Using again the explicit formula for the orbifold construction (Proposition 3.45) we find $\varepsilon(s)=\varepsilon(s(1))/|G|$. Concatenating multiplication and counit we obtain the formula for the pairing. We still have to compute the unit. We could do this by considering another span and using our formula for the orbifold construction, but we can also argue that, by the general construction, we already know that the multiplication has a unit – and hence a unique one. An easy computation shows that the unit described in the above assertion is indeed a unit. □

5 Applications to equivariant Dijkgraaf-Witten theories

As another class of examples we will now investigate the equivariant Dijkgraaf-Witten theories constructed in [MNS12]. In Section 5.2 we give an alternative, slightly generalized construction of this theory. The relation to the results of [MNS12] will be made precise in Section 5.3. The rest of the chapter is devoted to the computation of the orbifold theory for the equivariant Dijkgraaf-Witten theory. The final result is given in Theorem 5.14. It is applied to the case treated in [MNS12] in Corollary 5.15 and can be seen as a preparatory step towards relating the geometric orbifold construction developed in this article to the algebraic orbifoldization of modular tensor categories in the sense of [Kir04].

5.1 Topological field theories from presheaves – the Dijkgraaf-Witten model

Before turning to the equivariant case let us recall (a slight generalization of) the non-equivariant Dijkgraaf-Witten model: The stack of bundles for a finite group gives rise to a topological field theory, the so-called *Dijkgraaf-Witten theory* based on the ideas in [DW90]. Over the years the theory has seen numerous conceptional clarifications and extensions ranging from e.g. [FQ93] to [Mor15].

The definition of the Dijkgraaf-Witten theory relies on the following state space functor and the linearization of spans of essentially finite groupoids, see for instance [BHW10]. The state space functor takes values in finite-dimensional Hilbert spaces and hence allows us to reverse arrows by forming adjoint maps.

Definition 5.1 – State space functor. The *state space functor*

$$\mathscr{H} : \mathbf{FinGrpd}^{\mathrm{opp}} \longrightarrow \mathbf{FinHilb}$$

is a cofunctor from the category of essentially finite groupoids to the category of finite-dimensional complex Hilbert spaces (morphisms in the latter category are linear maps, not necessarily isometries). It assigns to an essentially finite groupoid $\varGamma$ the complex vector space of complex-valued invariant functions on $\varGamma$ (they can be seen as functions $\pi_0(\varGamma) \longrightarrow \mathbb{C}$). The scalar product of invariant functions f and g on $\varGamma$ is defined by

$$\langle f, g\rangle := \int_\varGamma \overline{f} g = \sum_{[x]\in\pi_0(\varGamma)} \frac{\overline{f(x)} g(x)}{|\mathrm{Aut}(x)|}.$$

A functor $\varPhi : \varGamma \longrightarrow \varOmega$ is sent to the pullback map $\varPhi^* = \mathscr{H}(\varPhi) : \mathscr{H}(\varOmega) \longrightarrow \mathscr{H}(\varGamma)$ given by

$$(\varPhi^* f)(x) := f(\varPhi(x)) \quad \text{for all} \quad f \in \mathscr{H}(\varOmega), \quad x \in \varGamma.$$

We call $\mathscr{H}(\varGamma)$ the *state space* of the essentially finite groupoid $\varGamma$.

Remarks 5.2.

(a) Let x be an object of an essentially finite groupoid $\varGamma$. Then we denote by $\delta_{[x]} \in \mathscr{H}(\varGamma)$ the invariant function on $\varGamma$ with

$$\delta_{[x]}(x') = \begin{cases} 1, & \text{if } x \cong x' \\ 0, & \text{else} \end{cases} \qquad \text{for all} \quad x' \in \varGamma.$$

The family $(\delta_{[x]})_{[x]\in\pi_0(\varGamma)}$ forms a canonical basis of $\mathscr{H}(\varGamma)$.

(b) Naturally isomorphic functors between essentially finite groupoids induce the same maps between the state spaces.

(c) The key point about the scalar products on the state spaces is that they give us the possibility to reverse arrows by forming adjoint maps, which crucially enters in Theorem 5.6. We will denote an adjunction by a †. Additionally, we use the following notation: If $\varPhi : \varGamma \longrightarrow \varOmega$ is a functor between essentially finite groupoids, we write

$$\varPhi_* := \mathscr{H}(\varPhi)^\dagger = (\varPhi^*)^\dagger : \mathscr{H}(\varGamma) \longrightarrow \mathscr{H}(\varOmega).$$

A direct computation shows

$$(\varPhi_* f)(y) = |\mathrm{Aut}(y)| \sum_{\substack{[x]\in\pi_0(\varGamma) \\ \varPhi(x)\cong y}} \frac{f(x)}{|\mathrm{Aut}(x)|}.$$

Using Remark 5.2, (b) we can easily prove the following result:

Lemma 5.3. *The state space functor maps an equivalence $\varPhi : \varGamma \longrightarrow \varOmega$ of essentially finite groupoids to a unitary map $\varPhi^* : \mathscr{H}(\varOmega) \longrightarrow \mathscr{H}(\varGamma)$. The adjoint (and inverse) preserves the canonical basis in the sense that it satisfies $\varPhi_*\delta_{[x]} = \delta_{[\varPhi(x)]}$ for all $x \in \varGamma$.*

Adjunction and homotopy fibers are closely related.

Proposition 5.4. *Let $\varPhi : \varGamma \longrightarrow \varOmega$ a functor between essentially finite groupoids. For any invariant function $f : \varGamma \longrightarrow \mathbb{C}$ the formula*

$$(\varPhi_* f)(y) = \int_{\varPhi^{-1}[y]} Q_y^* f$$

holds, where $Q_y : \Phi^{-1}[y] \longrightarrow \Gamma_y$ is the $|\mathrm{Aut}(y)|$-fold forgetful covering associated to the homotopy fiber $\Phi^{-1}[y]$.

Proof. From the adjunction formula given in Remark 5.2, (c) we already know

$$(\Phi_* f)(y) = |\mathrm{Aut}(y)| \int_{\Gamma_y} f.$$

Now Proposition A.16 yields the assertion. □

In the original paper [DW90], the Dijkgraaf-Witten theory is not explicitly described as a symmetric monoidal functor. The categorical description was worked out in [FQ93] and [Mor15]. The strategy is to see morphisms $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$ as cospans $\Sigma_0 \longrightarrow M \longleftarrow \Sigma_1$ and feed them into the stack $\mathbf{PBun}_G(?)$ for some finite group G. The spans of groupoids we obtain are linearized using the state space functor. It makes sense to replace $\mathbf{PBun}_G(?)$ be any presheaf satisfying the properties in Definition 3.18. This will allow for a very elegant proof of the functoriality of Dijkgraaf-Witten theory. Some of the essential ideas in the proof of Theorem 3.40 are inspired by this.

For the proof of the functoriality of Dijkgraaf-Witten type topological field theories we will use a Beck-Chevalley condition, see [Mor11, Appendix A.2] for a discussion in the context of Dijkgraaf-Witten theories. A generalization was given in Proposition 3.39.

Corollary 5.5 – Beck-Chevalley condition. *For any homotopy pullback*

$$\begin{array}{ccc} \Gamma \times_\Omega \Lambda & \xrightarrow{\pi_\Gamma} & \Gamma \\ {\scriptstyle \pi_\Lambda}\downarrow & \overset{\eta}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \Phi} \\ \Lambda & \xrightarrow[\Psi]{} & \Omega \end{array}$$

of a cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ of essentially finite groupoids, the equality of linear maps

$$\Psi^* \Phi_* = \pi_{\Lambda *} \pi_\Gamma^* : \mathscr{H}(\Gamma) \longrightarrow \mathscr{H}(\Lambda)$$

holds.

Theorem 5.6 – Untwisted Dijkgraaf-Witten theory. *Every additive, essentially finite homotopy-invariant presheaf Γ satisfying the gluing condition with respect to bordisms gives rise to a topological field theory Z_Γ of arbitrary dimension $n \geq 1$: For any object Σ in $\mathbf{Cob}(n)$ set*

$$Z_\Gamma(\Sigma) := \mathscr{H}\Gamma(\Sigma),$$

and for every morphism $\Sigma_0 \xrightarrow{\iota_0} M \xleftarrow{\iota_1} \Sigma_1$ in $\mathbf{Cob}(n)$ define $Z_\Gamma(M)$ to be the pull-push map

$$Z_\Gamma(M) : \mathscr{H}\Gamma(\Sigma_0) \xrightarrow{\Gamma(\iota_0)^* = \mathscr{H}\Gamma(\iota_0)} \mathscr{H}\Gamma(M) \xrightarrow{\Gamma(\iota_1)_* = \mathscr{H}\Gamma(\iota_1)^\dagger} \mathscr{H}\Gamma(\Sigma_1),$$

where the adjunction is taken with respect to the scalar products on the state spaces.

Proof.

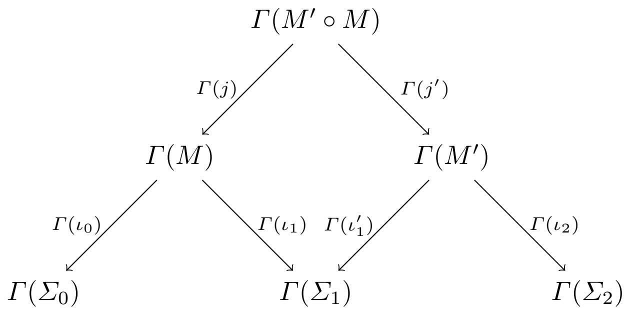
(i) As a first step we prove that Z_Γ is a functor. In order to show that it maps identities to identities, we observe that the inclusion of both the top and the bottom of the cylinder $\Sigma \times [0,1]$ over some object Σ in $\mathbf{Cob}(n)$ is a homotopy equivalence inducing a equivalence $\Gamma(\Sigma \times [0,1]) \cong \Gamma(\Sigma)$. Hence, from Lemma 5.3 we get immediately $Z_\Gamma(\Sigma \times [0,1]) = \mathrm{id}_{\mathscr{H}(\Gamma(\Sigma))}$. Next we prove

$$Z_\Gamma(M' \circ M) = Z_\Gamma(M') \circ Z_\Gamma(M)$$

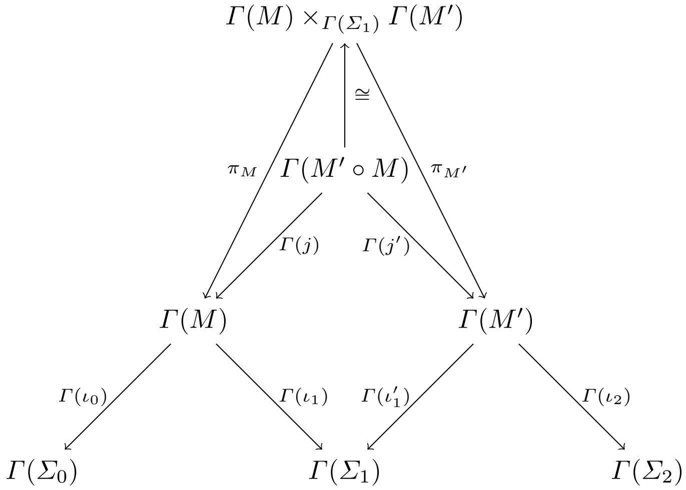
for morphisms $M : \Sigma_0 \longrightarrow \Sigma_1$ and $M' : \Sigma_1 \longrightarrow \Sigma_2$ in $\mathbf{Cob}(n)$. These give us the commutative diagram

$$\begin{array}{ccccccc} & & & M' \circ M & & & \\ & & {\scriptstyle j}\nearrow & & \nwarrow{\scriptstyle j'} & & \\ & M & & & & M' & \\ {\scriptstyle \iota_0}\nearrow & & \nwarrow{\scriptstyle \iota_1} & & {\scriptstyle \iota_1'}\nearrow & & \nwarrow{\scriptstyle \iota_2} \\ \Sigma_0 & & & \Sigma_1 & & & \Sigma_2 \end{array}$$

of embeddings between manifolds, which upon applying Γ yields the diagram of groupoids



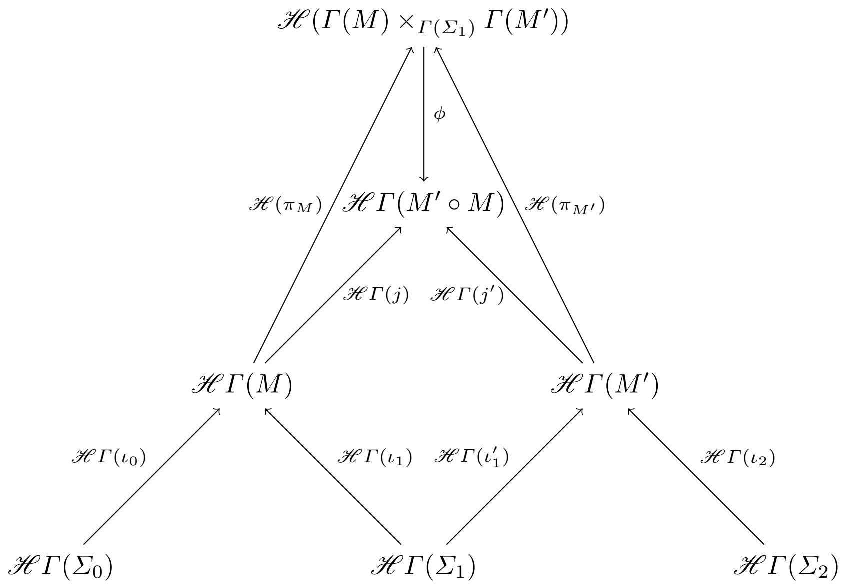
commutative up to natural isomorphism. From the gluing property of Γ we deduce that the diagram



commutes up to natural isomorphism, where $\cong$ denotes the canonical equivalence

$$\Gamma(j) \times \Gamma(j') : \Gamma(M' \circ M) \longrightarrow \Gamma(M) \times_{\Gamma(\Sigma_1)} \Gamma(M')$$

and π_M and $\pi_{M'}$ are the projection functors. Application of the state space functor yields the commutative diagram



(∗)

in which the map ϕ induced by $\Gamma(j) \times \Gamma(j') : \Gamma(M' \circ M) \longrightarrow \Gamma(M) \times_{\Gamma(\Sigma_1)} \Gamma(M')$ is unitary by Lemma 5.3. Using the equality

$$\mathscr{H}(\pi_{M'})^\dagger \mathscr{H}(\pi_M) = \mathscr{H}\Gamma(\iota_1')\mathscr{H}\Gamma(\iota_1)^\dagger \tag{‡}$$

following from Corollary 5.5 we obtain

$$\begin{aligned} Z_\Gamma(M' \circ M) &\overset{(*)}{=} (\mathscr{H}\Gamma(j')\mathscr{H}\Gamma(\iota_2))^\dagger \mathscr{H}\Gamma(j)\mathscr{H}\Gamma(\iota_0) \\ &= \mathscr{H}\Gamma(\iota_2)^\dagger \mathscr{H}\Gamma(j')^\dagger \mathscr{H}\Gamma(j)\mathscr{H}\Gamma(\iota_0) \\ &\overset{(*)}{=} \mathscr{H}\Gamma(\iota_2)^\dagger \mathscr{H}(\pi_{M'})^\dagger \phi^\dagger \phi \mathscr{H}(\pi_M)\mathscr{H}\Gamma(\iota_0) \\ &= \mathscr{H}\Gamma(\iota_2)^\dagger \mathscr{H}(\pi_{M'})^\dagger \mathscr{H}(\pi_M)\mathscr{H}\Gamma(\iota_0) \quad (\phi \text{ is unitary}) \\ &\overset{(\ddagger)}{=} \mathscr{H}\Gamma(\iota_2)^\dagger \mathscr{H}\Gamma(\iota_1')\mathscr{H}\Gamma(\iota_1)^\dagger \mathscr{H}\Gamma(\iota_0) \\ &= Z_\Gamma(M')Z_\Gamma(M). \end{aligned}$$

(ii) Finally, using the fact that Γ is additive, we obtain the data to turn Z_Γ into a monoidal functor. It is easily seen to be symmetric. □

Remarks 5.7.

(a) The topological field theory $Z_\Gamma : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ assigns to a closed oriented n-dimensional manifold M the invariant $|\Gamma(M)|$.

(b) The original Dijkgraaf-Witten theory corresponds to the case $\Gamma = \mathbf{PBun}_G(?)$ for some finite group G.

(c) There is a little more sophisticated version of the n-dimensional G-Dijkgraaf-Witten theory using a twist by a cocycle $\theta \in H^n(BG; \mathrm{U}(1))$ playing the role of an action functional. This version of the theory is also described in [FQ93] and [Mor15]. The above version corresponds to the theory with trivial cocycle and is therefore sometimes called *untwisted*.

(d) For a morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$, $x_0 \in \Gamma(\Sigma_0)$ and $x_1 \in \Gamma(\Sigma_1)$ we find the formula

$$(Z_\Gamma(M)\delta_{[x_0]})(x_1) = |\mathrm{Aut}(x_1)||\Gamma_{x_0,x_1}(M)|,$$

where $\Gamma_{x_0,x_1}(M)$ is the full subgroupoid of $\Gamma(M)$ of all $x \in \Gamma(M)$ with $x|_{\Sigma_0} \cong x_0$ and $x|_{\Sigma_1} \cong x_1$.

Example 5.8. Let Γ be an additive, essentially finite homotopy invariant presheaf Γ satisfying the gluing condition with respect to bordisms and Ω a finitely generated groupoid, then the presheaf $[\Omega, \Gamma]$ also meets these requirements. Hence it gives rise to a topological field theory $Z_{[\Omega,\Gamma]} : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$.

5.2 Equivariant topological field theories from morphisms of homotopy invariant presheaves

In Section 5.1 it was explained how to associate a topological field theory of Dijkgraaf-Witten type to a certain type of homotopy invariant presheaves. We will now construct an equivariant field theory from a *morphism* of such homotopy invariant presheaves.

Definition 5.9 – Vector bundles over groupoids constructed from transport functors. Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between small groupoids. For a morphism $g : y_0 \longrightarrow y_1$ in Ω we define a *transport functor* $\wp^\Phi(g) : \Phi^{-1}[y_0] \longrightarrow \Phi^{-1}[y_1]$ between the corresponding homotopy fibers by sending an object (x_0, h_0) in $\Phi^{-1}[y_0]$ to (x_0, gh_0). A morphism $k : (x_0, h_0) \longrightarrow (x_0', h_0')$ is sent to k seen as morphism $(x_0, gh_0) \longrightarrow (x_0', gh_0')$. This allows us to define a functor

$$\begin{aligned} \wp^\Phi : \Omega &\longrightarrow \mathbf{Grpd} \\ y &\longmapsto \Phi^{-1}[y] \\ (g : y_0 \longrightarrow y_1) &\longmapsto (\wp^\Phi(g) : \Phi^{-1}[y_0] \longrightarrow \Phi^{-1}[y_1]), \end{aligned}$$

which we will also refer to as *transport functor*. Assume now that Γ and Ω are essentially finite, then $\wp^\Phi$ is a functor $\wp^\Phi : \Omega \longrightarrow \mathbf{FinGrpd}$, and we get a vector bundle of Ω by

$$\varrho^\Phi : \Omega \xrightarrow{\wp^\Phi} \mathbf{FinGrpd} \xrightarrow{\mathscr{H}} \mathbf{FinHilb}^{\mathrm{opp}} \xrightarrow{\dagger} \mathbf{FinHilb},$$

where $\mathscr{H}$ is the state space functor (Definition 5.1) and the dagger is the adjunction endocofunctor in the category of finite-dimensional Hilbert spaces.

Let Γ and Ω be additive, essentially finite homotopy invariant presheaves satisfying the gluing condition and $\Phi : \Gamma \longrightarrow \Omega$ a morphism of presheaves. We intend to associate to Φ an Ω-equivariant topological field theory

$$Z_\Phi : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_\mathbb{C}\mathbf{Grpd}.$$

Let us specify the ingredients:

- To an object Σ in $\mathbf{Cob}(n)$ we assign the vector bundle

$$Z_\Phi(\Sigma) := \varrho^{\Phi(\Sigma)} : \Omega(\Sigma) \longrightarrow \mathbf{FinHilb}$$

over $\Omega(\Sigma)$ that the functor $\Phi(\Sigma) : \Gamma(\Sigma) \longrightarrow \Omega(\Sigma)$ gives rise to in the sense of Definition 5.9.

- To a morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$ we assign the span $\Omega(\Sigma_0) \xleftarrow{r_0} \Omega(M) \xrightarrow{r_1} \Omega(\Sigma_1)$, where the restriction functors come from Ω. Additionally, we need a morphism $\lambda(M) : r_0^* Z(\Sigma_0) \longrightarrow r_1^* Z(\Sigma_1)$ of vector bundles over groupoids. It is given by family $(\lambda(M))_{y \in \Omega(M)}$ of pull-push maps

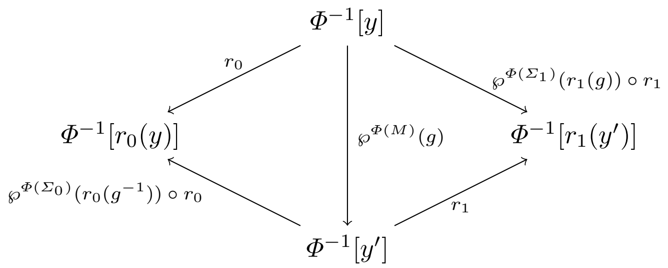
$$\lambda(M)_y : Z_\Phi(\Sigma_0)(r_0(y)) = \mathscr{H}(\Phi^{-1}[r_0(y)]) \xrightarrow{r_0^*} \mathscr{H}(\Phi^{-1}[y]) \xrightarrow{r_{1*}} Z_\Phi(\Sigma_1)(r_1(y)) = \mathscr{H}(\Phi^{-1}[r_1(y)]),$$

where $\mathscr{H}$ is the state space functor from Definition 5.1. Note that by a slight abuse of notation we use the same symbol for the restriction functor $r_0 : \Omega(M) \longrightarrow \Omega(\Sigma)$ and the functor $\Phi^{-1}[y] \longrightarrow \Phi^{-1}[r_0(y)]$ induced by it in an obvious way.

To prove that $\lambda(M) : r_0^* Z(\Sigma_0) \longrightarrow r_1^* Z(\Sigma_1)$ is in fact a morphism of vector bundles, we need to show that for any morphism $g : y \longrightarrow y'$ in $\Omega(M)$ the diagram

$$\begin{array}{ccccc} \mathscr{H}\Phi^{-1}[r_0(y)] & \xrightarrow{r_0^*} & \mathscr{H}\Phi^{-1}[y] & \xrightarrow{r_{1*}} & \mathscr{H}\Phi^{-1}[r_1(y)] \\ {\scriptstyle \wp^{\Phi(\Sigma_0)}(r_0(g))_* = \wp^{\Phi(\Sigma_0)}(r_0(g^{-1}))^*} \downarrow & & & & \downarrow {\scriptstyle \wp^{\Phi(\Sigma_1)}(r_1(g))_*} \\ \mathscr{H}\Phi^{-1}[r_0(y')] & \xrightarrow{r_0^*} & \mathscr{H}\Phi^{-1}[y'] & \xrightarrow{r_{1*}} & \mathscr{H}\Phi^{-1}[r_1(y')] \end{array} \qquad (*)$$

commutes. Indeed, consider the equivalence of spans



and observe that the linearization of the upper span is

$$\wp^{\Phi(\Sigma_1)}(r_1(g))_* r_{1*} r_0^*,$$

whereas the linearization of the lower span is

$$r_{1*} r_0^* \wp^{\Phi(\Sigma_1)}(r_1(g^{-1}))^* = r_{1*} r_0^* \wp^{\Phi(\Sigma_1)}(r_1(g))_*.$$

This proves the commutativity of $(*)$.

Theorem 5.10. *Let Γ and Ω be additive, essentially finite homotopy invariant presheaves satisfying the gluing condition and $\Phi : \Gamma \longrightarrow \Omega$ a morphism of presheaves. Then the associated*

$$Z_\Phi : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$$

is an Ω-equivariant topological field theory.

Proof.

(i) In the first step we prove functoriality of Z_Φ. To this end, we observe that for $y \in \Omega(\Sigma \times [0,1])$ the map $\lambda(\Sigma \times [0,1])$ given by

$$Z(\Sigma)(r_0(y)) = \mathscr{H}(\Phi^{-1}[r_0(y)]) \xrightarrow{r_0^*} \mathscr{H}(\Phi^{-1}[y]) \xrightarrow{r_{1*}} Z(\Sigma)(r_1(y)) = \mathscr{H}(\Phi^{-1}[r_1(y)])$$

is the identity, since the inclusion of top and bottom of the cylinder are homotopy equivalences turning, by homotopy invariance of Ω, the restriction functors r_0 and r_1 into equivalences and hence r_0^* into a unitary operator adjoint to r_{1*}. Consequently, Z_Φ preserves identities. To prove the composition law consider two morphisms $M : \Sigma_0 \longrightarrow \Sigma_1$ and $M' : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$. The morphism $Z_\Phi(M' \circ M)$ is given by an equivalence class of a span of groupoids and a vector bundle morphism. Since Ω satisfies the gluing property, we can represent this span by the outer span of the diagram

$$\begin{array}{ccccccc} & & & \Omega(M)\times_{\Omega(\Sigma_1)}\Omega(M') & & & \\ & & \swarrow^{p} & & \searrow^{p'} & & \\ & \Omega(\Sigma_1) & & \xRightarrow{\eta} & & \Omega(M) & \\ \swarrow^{r_0} & & \searrow^{r_1} & & \swarrow^{r_1'} & & \searrow^{r_2'} \\ \Omega(\Sigma_0) & & & \Omega(\Sigma_2) & & & \Omega(M') \end{array} .$$

The second part of $Z_\Phi(M' \circ M)$, namely the vector bundle morphism

$$\lambda(M' \circ M) : (r_0 \circ p)^* Z_\Phi(\Sigma_0) \longrightarrow (r_2' \circ p')^* Z_\Phi(\Sigma_2),$$

consists of the linear maps

$$\lambda(M' \circ M)_{(y,y',y)} : Z_\Phi(\Sigma_0)(r_0(y)) \longrightarrow Z_\Phi(\Sigma_2)(r_2'(y')) \quad \text{for all} \quad (y, y', g) \in \Omega(M) \times_{\Omega(\Sigma_1)} \Omega(M')$$

obtained, as we will show below in part (ii) of this proof, by linearizing the outer span of

$$\begin{array}{ccccccc} & & & \Phi^{-1}[x]\times_{\Phi^{-1}[r_1'(y')]}\Phi^{-1}[y'] & & & \\ & & \swarrow^{q} & & \searrow^{q'} & & \\ & \Phi^{-1}[y] & & \Longrightarrow & & \Phi^{-1}[y'] & \\ \swarrow^{r_0} & & \searrow^{r_1}\ \Phi^{-1}[r_1(y)] \ \searrow^{\wp^{\Phi(\Sigma_1)}(g)} & & \swarrow^{r_1'} & & \searrow^{r_2'} \\ \Phi^{-1}[r_0(y)] & & & \Phi^{-1}[r_1'(y')] & & & \Phi^{-1}[r_2'(y')] \end{array} . \qquad (\ddagger)$$

The weak pullback square commutes, as usual, up to natural isomorphism. Now the computation

$$\begin{aligned} \lambda(M' \circ M)_{(y,y',g)} &= (r_2' \circ q')_*(r_0 \circ q)^* \\ &= r_{2*}' q_*' q^* r_0^* \\ &= r_{2*} {r_1'}^* \wp^{\Phi(\Sigma_1)}(g)_* r_{1*} r_0^* \quad \text{(Beck-Chevalley condition, Corollary 5.5)} \\ &= \lambda(M)_y \wp^{\Phi(\Sigma_1)}(g)_* \lambda(M')_{y'} \end{aligned}$$

concludes the proof of functoriality.

(ii) In order to prove that we can use ($\ddagger$) for the computation of λ, we observe that, since the square

$$\begin{array}{ccc} \Gamma(M' \circ M) & \xrightarrow{\cong} & \Gamma(M) \times_{\Gamma(\Sigma_1)} \Gamma(M') \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi \times \Phi} \\ \Omega(M' \circ M) & \xrightarrow{\cong} & \Omega(M) \times_{\Omega(\Sigma_1)} \Omega(M') \end{array}$$

containing the gluing equivalences coming from restriction is (weakly) commutative, we can compute λ by linearizing the span

$$\Phi^{-1}[r_0(y)] \xleftarrow{r_0} \Phi^{-1}[y] \longleftarrow (\Phi \times \Phi)^{-1}[y, y', g] \longrightarrow \Phi^{-1}[y'] \xrightarrow{r_2'} \Phi^{-1}[r_2'(y')]$$

with the inner unlabeled arrows being projections. This span is equivalent to the one given in ($\ddagger$). To see this, we will just write down explicitly the definition of $(\Phi \times \Phi)^{-1}[y, y', g]$ and observe that it can be canonically identified with $\Phi^{-1}[y] \times_{\Phi^{-1}[r_1'(y')]} \Phi^{-1}[y']$. Indeed, an object of $(\Phi \times \Phi)^{-1}[y, y', g]$ is an object (x, x', k) in

$\Gamma(M)\times_{\Gamma(\Sigma_1)}\Gamma(M')$, i.e. $x|_{\Sigma_1}\overset{k}{\cong} x'|_{\Sigma_1}$, together with a morphism

$$(h,h'):\left(\Phi(x),\Phi(x'),\Phi(x|_{\Sigma_1})\overset{\Phi(k)}{\cong}\Phi(x'|_{\Sigma_1})\right)\longrightarrow(y,y',g),$$

i.e. $h:\Phi(x)\longrightarrow y$ and $h':\Phi(x')\longrightarrow y'$ such that the square

$$\begin{array}{ccc} \Phi(x|_{\Sigma_1}) & \xrightarrow{\Phi(k)} & \Phi(x'|_{\Sigma_1}) \\ {\scriptstyle h|_{\Sigma_1}}\downarrow & & \downarrow{\scriptstyle h'|_{\Sigma_1}} \\ y|_{\Sigma_1} & \xrightarrow{g} & y'|_{\Sigma_1} \end{array}$$

commutes. Writing out the definition of $\Phi^{-1}[y]\times_{\Phi^{-1}[r_1'(y')]}\Phi^{-1}[y']$ yields just the same. Hence, it is justified to linearize (‡) to obtain λ.

(iii) The monoidal structure of Z_Φ comes from the requirement that Γ and Ω are additive. □

5.3 Specializing to bundle stacks – equivariant Dijkgraaf-Witten theory

In this subsection we apply Theorem 5.10 to an imprtant class of examples: Consider the bundle stacks $\mathbf{PBun}_H(?)$ and $\mathbf{PBun}_J(?)$ for two finite groups H and J. Given a morphism $\lambda: H\longrightarrow J$ we can extend any H-bundle along λ to a J-bundle. This yields an extension functor

$$\lambda_*^M:\mathbf{PBun}_H(M)\longrightarrow\mathbf{PBun}_J(M)$$

for every manifold (we will write λ_* instead of λ_*^M if the manifold is clear). It assigns to an H-bundle Q over M the J-bundle $\lambda_*^M(Q)$ which is defined as the associated bundle $Q\times_H J$, where H acts on J by $h.j:=\lambda(h)j$ for all $h\in H$ and $j\in J$.

These extension functors are natural in the sense that they yield a morphism

$$\lambda_*:\mathbf{PBun}_H(?)\longrightarrow\mathbf{PBun}_J(?) \qquad (*)$$

of stacks. The diagram

$$\begin{array}{ccc} \Pi(M,BH) & \xrightarrow{B\lambda_*} & \Pi(M,BJ) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathbf{PBun}_H(M) & \xrightarrow{\lambda_*} & \mathbf{PBun}_J(M) \end{array}$$

containing the canonical equivalences from Theorem 2.14 and the functor obtained from the continuous map $B\lambda: BH\longrightarrow BJ$ given by λ and the functoriality of the classifying space construction is easily seen to commute weakly. The following result suggests to concentrate on the morphisms of stacks arising in this way from group morphisms.

Proposition 5.11. *For discrete groups H and J denote by $\mathrm{Hom}(H,J)//J$ the action groupoid of the action of J on morphisms $H\longrightarrow J$ by conjugation. Then for the corresponding stacks $\Pi(?,BH)$ and $\Pi(?,BJ)$ on topological spaces there is an equivalence*

$$\mathrm{Hom}(H,J)//J\cong\mathrm{Hom}\left(\Pi(?,BH),\Pi(?,BJ)\right)$$

of groupoids sending a group morphism $\lambda: H\longrightarrow J$ to the morphism $B\lambda_*:\Pi(?,BH)\longrightarrow\Pi(?,BJ)$. Hence, morphisms $\Pi(?,BH)\longrightarrow\Pi(?,BJ)$ are classified up to isomorphism by conjugacy classes of group morphisms $H\longrightarrow J$.

Proof. There is an obvious functor $\Pi(BH,BJ)\longrightarrow\mathrm{Hom}\left(\Pi(?,BH),\Pi(?,BJ)\right)$ sending a continuous map $f: BH\longrightarrow BJ$ to the morphism $f_*:\Pi(?,BH)\longrightarrow\Pi(?,BJ)$. This functor is an equivalence (even an isomorphism) of groupoids by a (generalization of) Yoneda's Lemma. The assertion follows now from

$$\Pi(BH,BJ)\cong\mathbf{PBun}_J(BH)\cong\mathrm{Hom}(\pi_1(BH),J)//J\cong\mathrm{Hom}(H,J)//J,$$

where we succesively used Theorem 2.14, the holonomy classification of flat bundles and $\pi_1(BH)\cong H$. □

Applying Theorem 5.10 to the extension morphism $(*)$ we obtain an equivariant field theory.

Theorem 5.12. *To $n\geq 1$ and any morphism $\lambda: H\longrightarrow J$ of finite groups the construction of Theorem 5.10 associates a J-equivariant n-dimensional topological field theory Z_λ.*

In order to compare Z_λ to the (untwisted, non-extended version of the) J-equivariant Dijkgraaf-Witten theory constructed in [MNS12] from a short exact sequence

$$0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$$

of finite groups, we observe that the groupoid of P-twisted H-bundles introduced in [MNS12, Definition 3.5] is canonically equivalent to the homotopy fiber $\lambda_*^{-1}[P]$. Upon linearization and adjunction these equivalences yield:

Theorem 5.13. *Let $0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$ be a short exact sequence of finite groups. Then the J-equivariant topological field theory Z_λ from Theorem 5.12 is canonically isomorphic (by a monoidal natural isomorphism) to the J-equivariant Dijkgraaf-Witten theory $J\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ associated to this sequence in [MNS12].*

Note that in Theorem 5.12, in contrast to [MNS12], surjectivity of λ is not needed.

5.4 Orbifold construction for Dijkgraaf-Witten theories

The equivariant Dijkgraaf-Witten theory provides another opportunity to relate the geometric orbifold construction of this article to existing concepts of orbifoldization. In [MNS12] the J-equivariant Dijkgraaf-Witten theory associated to a short exact sequence $0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$ of finite groups is constructed as a three-dimensional extended topological field theory. Upon evaluation on the circle, this theory yields an J-equivariant modular category; for this category, a (purely algebraic) orbifold construction is available, see [Kir04]. It can be shown [MNS12] that the orbifold category is the modular category associated to the Dijkgraaf-Witten theory for the group H. We now demonstrate that the geometric orbifold construction of this article yields, at the level of non-extended topological field theories, the same orbifold theory.

For the invariants assigned by the orbifold theory Z_λ/J to a closed oriented manifold M of top dimension, this is a consequence of Cavalieri's principle: By Proposition 3.47 this invariant is given by

$$\frac{Z_\lambda}{J}(M) = \int_{\mathbf{PBun}_J(M)} Z_\lambda(M,P)\,\mathrm{d}P = \int_{\mathbf{PBun}_J(M)} |\lambda_*^{-1}[P]|\,\mathrm{d}P = |\mathbf{PBun}_H(M)|,$$

where Cavalieri's principle (Proposition A.14) enters in the last step. On the right hand side we see the invariant the Dijkgraaf-Witten theory for the group H would assign to M.

To show that this result holds beyond invariants, we compute the orbifold theory of the equivariant theory from Theorem 5.12.

Theorem 5.14. *Let Γ and Ω be additive, essentially finite homotopy invariant presheaves satisfying the gluing condition and $\Phi : \Gamma \longrightarrow \Omega$ a morphism of presheaves. Then the orbifold theory Z_Φ/Ω of the Ω-equivariant topological field theory Z_Φ is canonically isomorphic (by a monoidal natural isomorphism) to the topological field theory Z_Γ from Theorem 5.6, i.e.*

$$\frac{Z_\Phi}{\Omega} \cong Z_\Gamma .$$

Proof.

(i) We begin by defining the isomorphism $\eta : Z_\Gamma \longrightarrow Z_\Phi/\Omega$. It consists of the maps

$$\eta_\Sigma : Z_\Gamma(\Sigma) = \mathscr{H}\Gamma(\Sigma) \longrightarrow \operatorname{Par} \varrho^{\Phi(\Sigma)} = \frac{Z_\Phi}{\Omega}(\Sigma)$$

defined as follows: Let $f : \Gamma(\Sigma) \longrightarrow \mathbb{C}$ be an invariant function and $y \in \Omega(\Sigma)$. Then the forgetful covering $q_y : \Phi^{-1}[y] \longrightarrow \Gamma_y(\Sigma)$ and the inclusion $\iota_y : \Gamma_y(\Sigma) \longrightarrow \Gamma(\Sigma)$ yield a functor $\ell_y : \Phi^{-1}[y] \xrightarrow{q_y} \Gamma_y(\Sigma) \xrightarrow{\iota_y} \Gamma(\Sigma)$ and hence, by pullback, a map

$$\ell_y^* : \mathscr{H}\Gamma(\Sigma) \longrightarrow \mathscr{H}\Phi^{-1}[y].$$

Set now

$$(\eta_\Sigma f)(y) := \ell_y^* f \in \mathscr{H}\Phi^{-1}[y]$$

to define $\eta_\Sigma f$. An easy computation shows that the section $\eta_\Sigma f$ is parallel. At the end of the proof η will be a morphism $\eta : Z_\Gamma \longrightarrow Z_\Phi/\Omega$, i.e. a natural monoidal transformation, and hence automatically an isomorphism. The explicit form of the inverse, however, will simplify the rest of the proof: For a parallel section s of the vector bundle $\wp^{\Phi(\Sigma)}$ and $x \in \Gamma(\Sigma)$ we set

$$(\nu_\Sigma s)(x) := s(\Phi(x))(x, \mathrm{id}_{\Phi(x)}).$$

Note that $s(\Phi(x)) \in \mathscr{H}\Phi^{-1}[\Phi(x)]$, so it makes sense to evaluate $s(\Phi(x))$ at $(x, \mathrm{id}_{\Phi(x)}) \in \Phi^{-1}[\Phi(x)]$. This defines an invariant function on $\Gamma(\Sigma)$, as is easily verified. Another direct computation shows that ν_Σ is the inverse of η_Σ.

(ii) To prove naturality of η consider a morphism $M : \Sigma_1 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$. To show that the square

$$\begin{array}{ccc} Z_\Gamma(\Sigma_0) & \xrightarrow{Z_\Gamma(M)} & Z_\Gamma(\Sigma_1) \\ {\scriptstyle \eta_{\Sigma_0}}\downarrow & & \downarrow{\scriptstyle \eta_{\Sigma_1}} \\ \frac{Z_\Phi}{\Omega}(\Sigma_0) & \xrightarrow{\frac{Z_\Phi}{\Omega}(M)} & \frac{Z_\Phi}{\Omega}(\Sigma_1) \end{array} \tag{$*$}$$

commutes, let $x_0 \in \Gamma(\Sigma_0)$ and $x_1 \in \Gamma(\Sigma_1)$. Then

$$\left(\nu_{\Sigma_1} \circ \frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(x_1) = \left(\frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(\Phi(x_1))(x_1, \mathrm{id}_{\Phi(x_1)}). \tag{$\ddagger$}$$

The map $Z_\Phi/\Omega(M)$ is given by

$$\begin{aligned}\left(\frac{Z_\Phi}{\Omega}(M)s\right)(\Phi(x_1)) &= \sum_{\substack{[y]\in\pi_0(\Omega(M)) \\ y|_{\Sigma_1}\overset{a}{\cong}\Phi(x_1) \\ g\in\mathrm{Aut}(\Phi(x_1))}} \frac{\wp^{\Phi(\Sigma_1)}(g)_*\wp^{\Phi(\Sigma_1)}(a)_* Z_\Phi(M,y)s(y|_{\Sigma_0})}{|\mathrm{Aut}(y)|} \\ &= \sum_{\substack{[y]\in\pi_0(\Omega(M)) \\ y|_{\Sigma_1}\overset{a}{\cong}\Phi(x_1) \\ g\in\mathrm{Aut}(\Phi(x_1))}} \frac{\wp^{\Phi(\Sigma_1)}\left(a^{-1}g^{-1}\right)^* Z_\Phi(M,y)s(y|_{\Sigma_0})}{|\mathrm{Aut}(y)|} \\ &= \sum_{\substack{[y]\in\pi_0(\Omega(M)) \\ y|_{\Sigma_1}\overset{a}{\cong}\Phi(x_1) \\ g\in\mathrm{Aut}(\Phi(x_1))}} \frac{\wp^{\Phi(\Sigma_1)}\left(a^{-1}g\right)^* Z_\Phi(M,y)s(y|_{\Sigma_0})}{|\mathrm{Aut}(y)|}\end{aligned}$$

for every $s \in Z_\Phi/\Omega(\Sigma)$, where the morphism $a : y|_{\Sigma_1} \longrightarrow \Phi(x_1)$ is arbitrary, see Remark 3.41, (a). By the definition of $Z_\Phi(M, y)$ we obtain for $(x_1, k) \in \Phi^{-1}[y|_{\Sigma_1}]$

$$Z_\Phi(M,y)s(y|_{\Sigma_0})(x_1,k) = |\mathrm{Aut}(x_1,k)| \sum_{\substack{[x,h]\in\pi_0(\Phi^{-1}(y)) \\ (x,h)|_{\Sigma_1}\cong(x_1,k)}} \frac{s(y|_{\Sigma_0})((x,h)|_{\Sigma_0})}{|\mathrm{Aut}(x,h)|}.$$

Specializing to $s = \eta_{\Sigma_0}\delta_{[x_0]}$ we find

$$\left((\eta_{\Sigma_0}\delta_{[x_0]}(y|_{\Sigma_0}))\right)((x,h)|_{\Sigma_0}) = \left(\ell^*_{y|_{\Sigma_0}}\delta_{[x_0]}\right)((x,h)|_{\Sigma_0}) = \delta_{[x_0],[x|_{\Sigma_0}]}.$$

Taking all this into account ($\ddagger$) simplifies to

$$\left(\nu_{\Sigma_1} \circ \frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(x_1) = \sum_{\substack{[y]\in\pi_0(\Omega(M)) \\ y|_{\Sigma_1}\cong\Phi(x_1) \\ g\in\mathrm{Aut}(\Phi(x_1))}} \frac{|\mathrm{Aut}(x_1,g)|}{|\mathrm{Aut}(y)|} \sum_{\substack{[x,h]\in\pi_0(\Phi^{-1}(y)) \\ (x,h)|_{\Sigma_1}\cong(x_1,g) \\ x|_{\Sigma_0}\cong x_0}} \frac{1}{|\mathrm{Aut}(x,h)|}.$$

The morphism $a : y|_{\Sigma_1} \longrightarrow \Phi(x_1)$ does not appear anymore because $\wp^{\Phi(\Sigma)}(a)$ is an equivalence and does not change the size of automorphism groups. Note that in the outer sum we can drop the requirement $y|_{\Sigma_1} \cong \Phi(x_1)$ since in the inner sum we require anyway that $y|_{\Sigma_1} \cong \Phi(x)|_{\Sigma_1} \cong \Phi(x|_{\Sigma_1}) \cong \Phi(x_1)$. Moreover, for $g \in \mathrm{Aut}(\Phi(x_1))$ the equivalence $\wp^{\Phi(\Sigma_1)}(g) : \Phi^{-1}[\Phi(x_1)] \longrightarrow \Phi^{-1}[\Phi(x_1)]$ maps the pair $(x_1, \mathrm{id}_{\Phi(x_1)})$ to (x_1, g), which implies $|\mathrm{Aut}((x_1, \mathrm{id}_{\Phi(x_1)}))| = |\mathrm{Aut}(x_1, g)|$ and hence

$$\left(\nu_{\Sigma_1} \circ \frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(x_1) = \sum_{\substack{[y]\in\pi_0(\Omega(M)) \\ g\in\mathrm{Aut}(\Phi(x_1))}} \frac{|\mathrm{Aut}(x_1,\mathrm{id}_{\Phi(x_1)})|}{|\mathrm{Aut}(y)|} \sum_{\substack{[x,h]\in\pi_0(\Phi^{-1}(y)) \\ (x,h)|_{\Sigma_1}\cong(x_1,g) \\ x|_{\Sigma_0}\cong x_0}} \frac{1}{|\mathrm{Aut}(x,h)|}.$$

Every $[x,h] \in \pi_0(\Phi^{-1}[y])$ with $x|_{\Sigma_0} \cong x_0$ occurs in the sum because $(x,h)|_{\Sigma_1} \cong (x_1,g)$ for suitable g is definitely true. The number of objects (x_1,g') isomorphic to this (x_1,g) determines how often this particular $[x,h]$-contribution is counted. Let Π_{x_1} be the full subgroupoid of $\Phi^{-1}[\Phi(x_1)]$ of all objects of the form (x_1,g), where x_1 is fixed and $g \in \mathrm{Aut}(\Phi(x_1))$. The counting number for $[x,h]$ is the number of objects in Π_{x_1} isomorphic to $(x_1, h|_{\Sigma_1})$. As above, we can argue that this number does not depend on h, so it is given by $|[x_1, \mathrm{id}_{\Phi(x_1)}]|$. But Π_{x_1} is the action groupoid for the action

$$k.(x_1,g) := (x_1, g\Phi(k)^{-1}) \quad \text{for all} \quad k \in \mathrm{Aut}(x_1), \quad g \in \mathrm{Aut}(\Phi(x_1))$$

of $\mathrm{Aut}(x_1)$ on (the object set of) Π_{x_1}. Therefore, by the orbit theorem

$$|[x_1, \mathrm{id}_{\Phi(x_1)}]| = \frac{|\mathrm{Aut}(x_1)|}{|\mathrm{Aut}(x_1, \mathrm{id}_{\Phi(x_1)})|}.$$

Using this we find

$$\left(\nu_{\Sigma_1} \circ \frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(x_1) = \sum_{[y]\in\pi_0(\Omega(M))} \frac{|\mathrm{Aut}(x_1)|}{|\mathrm{Aut}(y)|} \underbrace{\sum_{\substack{[x,h]\in\pi_0(\Phi^{-1}(y)) \\ x|_{\Sigma_1}\cong x_1 \\ x|_{\Sigma_0}\cong x_0}} \frac{1}{|\mathrm{Aut}(x,h)|}}_{=:I}.$$

The inner sum I is the cardinality of the groupoid $\Phi^{-1}_{x_0,x_1}[y]$ having the forgetful $|\mathrm{Aut}(y)|$-fold covering onto $\Gamma_{x_0,x_1,y}(M)$ by Lemma A.4, where by the subscripts we indicate the obvious requirements on the restrictions on the boundary, i.e. $\Gamma_{x_0,x_1,y}(M)$, for instance, is the full subgroupoid of $\Gamma(M)$ consisting of all $x \in \Gamma(M)$ with $x|_{\Sigma_0} \cong x_0$ and $x|_{\Sigma_1} \cong x_1$ and $\Phi(x) \cong y$. Using now the covering property of the groupoid cardinality we see $I = |\mathrm{Aut}(y)||\Gamma_{x_0,x_1,y}(M)|$ and hence

$$\left(\nu_{\Sigma_1} \circ \frac{Z_\Phi}{\Omega}(M) \circ \eta_{\Sigma_0}\delta_{[x_0]}\right)(x_1) = \sum_{[y]\in\pi_0(\Omega(M))} |\mathrm{Aut}(x_1)||\Gamma_{x_0,x_1,y}(M)| = |\mathrm{Aut}(x_1)||\Gamma_{x_0,x_1}(M)|.$$

Recalling the definition of the Dijkgraaf-Witten model in Theorem 5.6, see also Remark 5.7, (d), this entails the commutativity of (∗). Hence, $\eta : Z_\Gamma \longrightarrow Z_\Phi/\Omega$ is a natural isomorphism. It is clearly monoidal. □

The above result and Theorem 5.13 immediately imply the following:

Corollary 5.15. *The orbifold theory of the equivariant Dijkgraaf-Witten theory*

$$Z_\lambda : J\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$$

associated to a short exact sequence $0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$ *is canonically isomorphic to the Dijkgraaf-Witten theory* $Z_H : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ *for the group* H*, i.e.*

$$\frac{Z_\lambda}{J} \cong Z_H.$$

Remark 5.16. By Theorem 3.23 the equivariant Dijkgraaf-Witten theory Z_λ gives us a functor $\mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_\mathbb{C}\mathbf{Grpd}$ which assigns to an object Σ in $\mathbf{Cob}(n)$ a representation of $\Pi(\Sigma, BJ) \cong \mathbf{PBun}_J(\Sigma)$ sending a J-bundle P over Σ to $\mathscr{H}\lambda_*^{-1}[P]$. Hence, the representation naturally takes values in Hilbert spaces, and all morphisms in $\mathbf{PBun}_J(\Sigma)$ are easily seen to act as unitary maps. In this case we naturally obtain a Hilbert space structure on the space $Z_\lambda/J(\Sigma)$ of parallel sections as well. We can assign such a representation and a Hilbert space of parallel section not only to objects in $\mathbf{Cob}(n)$, but to all manifolds and therefore in particular to morphisms in $\mathbf{Cob}(n)$.

If we are given now a morphism $M : \Sigma_0 \longrightarrow \Sigma_1$ in $\mathbf{Cob}(n)$, the restriction of bundles to the boundaries yields linear maps

$$\frac{Z_\lambda}{J}(\Sigma_0) \xrightarrow{u_0} \frac{Z_\lambda}{J}(M) \xleftarrow{u_1} \frac{Z_\lambda}{J}(\Sigma_1).$$

The pull-push map $u_1^\dagger u_0$, where the adjunction is taken with respect to the scalar products just specified, can be shown to be equal to the map the orbifold theory Z_λ/J assigns to M. Thus, in the case where the equivariant theory arises from a linearization process of spans of groupoids, the orbifold construction can be seen as a second linearization process. This implies that the orbifold theory is obtained by interweaving two linearization processes for spans of groupoids.

6 A generalized orbifold construction

A morphism $\lambda : G \longrightarrow H$ of finite groups yields a symmetric monoidal functor G-$\mathbf{Cob}(n) \longrightarrow H$-$\mathbf{Cob}(n)$ and hence a pullback functor

$$\lambda^* : \mathrm{HSym}(H\text{-}\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{HSym}(G\text{-}\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}).$$

However, it is less obvious how to get a functor in the opposite direction, i.e. a pushforward along a group morphism. In this section we construct such a pushforward operation and explain how it generalizes the orbifold construction.

6.1 The pushforward of equivariant topological field theories along group morphisms

As for the orbifold construction we use a two-step procedure: The first step is a generalization of the techniques from Section 3.4, the second step uses the parallel section functor we already know.

For the first step, i.e. the generalization of the techniques from Section 3.4, let $\lambda : G \longrightarrow H$ be a morphism of finite groups and

$$\lambda_* : \Pi(?, BG) \longrightarrow \Pi(?, BH)$$

the induced stack morphism. From a G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ we would like to construct a homotopy invariant symmetric monoidal functor $\widehat{Z}^\lambda : H\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$. In the second step we concatenate with the parallel section functor to obtain the H-equivariant topological field theory $\lambda_* Z := \mathrm{Par}\, \widehat{Z}^\lambda$.

Given Z, the functor $\widehat{Z}^\lambda$ is defined as follows:

- To an object (Σ, φ) in H-$\mathbf{Cob}(n)$ we assign the homotopy fiber $\lambda_*^{-1}[\varphi]$ defined by the homotopy pullback square

$$\begin{array}{ccc} \lambda_*^{-1}[\varphi] & \xrightarrow{\;q\;} & \Pi(\Sigma, BG) \\ \big\downarrow & & \big\downarrow \lambda_* \\ \star & \xrightarrow[\;\varphi\;]{} & \Pi(\Sigma, BH) \end{array},$$

  and the pullback vector bundle $q^* \varrho_\Sigma : \lambda_*^{-1}[\varphi] \longrightarrow \mathbf{Vect}_{\mathbb{C}}$, where ϱ_Σ is the vector bundle that Z gives rise to in the sense of Proposition 2.10.

- To a morphism $(M, \psi) : (\Sigma_0, \varphi_0) \longrightarrow (\Sigma_1, \varphi_1)$ in H-$\mathbf{Cob}(n)$ we assign
  - ▷ the span

  $$\lambda_*^{-1}[\varphi_0] \xleftarrow{r_0} \lambda_*^{-1}[\psi] \xrightarrow{r_1} \lambda_*^{-1}[\varphi_1],$$

    in which the needed functors are induced by restriction, i.e. the functors

  $$\begin{aligned} q_0 : \lambda_*^{-1}[\varphi_0] &\longrightarrow \Pi(\Sigma_0, BG), \\ q : \lambda_*^{-1}[\psi] &\longrightarrow \Pi(M, BG), \\ q_1 : \lambda_*^{-1}[\varphi_0] &\longrightarrow \Pi(\Sigma_1, BG) \end{aligned}$$

    and the restriction functors

  $$\Pi(\Sigma_0, BG) \xleftarrow{s_0} \Pi(M, BG) \xrightarrow{s_1} \Pi(\Sigma_1, BG)$$

    fulfill $q_0 r_0 = s_0 q$ and $q_1 r_1 = s_1 q$,
  - ▷ and the intertwiner

  $$q^* Z(M, ?) : r_0^* q_0^* \varrho_{\Sigma_0} = q^* s_0^* \varrho_{\Sigma_0} \longrightarrow q^* s_1^* \varrho_{\Sigma_1} = r_1^* q_1^* \varrho_{\Sigma_1}$$

    obtained as the pullback of the intertwiner $Z(M, ?) : s_0^* \varrho_{\Sigma_0} \longrightarrow s_1^* \varrho_{\Sigma_1}$ that the functor $\widehat{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ assigns to $M : \Sigma_0 \longrightarrow \Sigma_1$, see Theorem 3.23.

We can now generalize Theorem 3.23 and Proposition 3.26:

Theorem 6.1. *For any morphism $\lambda : G \longrightarrow H$ of finite groups the assignment $Z \longmapsto \widehat{Z}^\lambda$ extends to a functor*

$$\widehat{?}^\lambda : \mathrm{HSym}(G\text{-}\mathbf{Cob}(n), \mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{HSym}(H\text{-}\mathbf{Cob}(n), \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}).$$

Proof. We will only prove that for a given G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ the definitions above make $\widehat{Z}^\lambda$ into a homotopy invariant symmetric monoidal functor $H\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$. Once this is established, the functoriality of $\widehat{?}^\lambda$ will be clear.

(i) Clearly, $\widehat{Z}^\lambda$ respects identities, so we turn directly to the gluing law: For morphisms $(M,\psi) : (\Sigma_0,\varphi_0) \longrightarrow (\Sigma_1,\varphi_1)$ and $(M',\psi') : (\Sigma_1,\varphi_1) \longrightarrow (\Sigma_2,\varphi_2)$ in $H\text{-}\mathbf{Cob}(n)$ the span part of the composition of $\widehat{Z}^\lambda(M,\psi)$ and $\widehat{Z}^\lambda(M',\psi')$ in $\mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}$ is the homotopy pullback of

$$\lambda_*^{-1}[\psi] \xrightarrow{r_1} \lambda_*^{-1}[\varphi_1] \xleftarrow{r_1'} \lambda_*^{-1}[\psi'],$$

which by the universal property of the homotopy pullback can be seen to be naturally equivalent to $\lambda_*^{-1}[\psi \cup \psi']$, where $\psi \cup \psi' : M' \circ M \longrightarrow BG$ is the map that ψ and ψ' give rise to since they coincide on Σ_1. Next we observe that the span

$$\lambda_*^{-1}[\psi] \longleftarrow \lambda_*^{-1}[\psi \cup \psi'] \longrightarrow \lambda_*^{-1}[\psi']$$

coming from restriction is the span part of the image of the composition of (M,ψ) and (M,ψ') under $\widehat{Z}^\lambda$. The gluing law can now be verified as in the proof of Theorem 3.23.

(ii) The monoidal structure and the symmetry requirement is defined and checked, respectively, as in the proof of Theorem 3.23 by using the additivity of $\Pi(?,BG)$.

(iii) For the proof of the homotopy invariance consider two morphisms $(M,\psi),(M,\psi') : (\Sigma_0,\varphi_0) \longrightarrow (\Sigma_1,\varphi_1)$. A homotopy $\psi \simeq \psi'$ relative ∂M induces an isomorphism $\lambda_*^{-1}[\psi] \longrightarrow \lambda_*^{-1}[\psi']$, see Definition 5.9, making the diagram

$$\begin{array}{ccccc} & & \lambda_*^{-1}[\psi] & & \\ & \swarrow & \downarrow & \searrow & \\ \lambda_*^{-1}[\varphi_0] & & & & \lambda_*^{-1}[\varphi_1] \\ & \nwarrow & \downarrow & \nearrow & \\ & & \lambda_*^{-1}[\psi'] & & \end{array}$$

commute. This together with the homotopy invariance of Z implies the homotopy invariance of $\widehat{Z}^\lambda$. □

Definition 6.2 – Pushforward of an equivariant topological field theory along a group morphism. For a morphism $\lambda : G \longrightarrow H$ of finite groups we define the *pushforward of G-equivariant topological field theories along λ* as the functor

$$\lambda_* : \mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}) \xrightarrow{\widehat{?}^\lambda} \mathrm{HSym}(H\text{-}\mathbf{Cob}(n),\mathbf{VecBun}_{\mathbb{C}}\mathbf{Grpd}) \xrightarrow{\mathrm{Par}_*} \mathrm{HSym}(H\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}).$$

The orbifold construction corresponds to the case $H = 1$. The description of the pushforward generalizes the description of the orbifold theory in Section 3.7:

Proposition 6.3. *For a morphism $\lambda : G \longrightarrow H$ and a G-equivariant topological field theory $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ the pushforward $\lambda_* Z : H\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ admits the following description:*

(a) *For any object (Σ,φ) in $H\text{-}\mathbf{Cob}(n)$*

$$(\lambda_* Z)(\Sigma,\varphi) := \mathrm{Par}(\Phi^{-1}[\varphi] \longrightarrow \Pi(\Sigma,BG) \xrightarrow{\varrho_\Sigma} \mathbf{Vect}_{\mathbb{C}}).$$

(b) *For any morphism $(M,\psi) : (\Sigma_0,\varphi_0) \longrightarrow (\Sigma_1,\varphi_1)$ in $H\text{-}\mathbf{Cob}(n)$*

$$(\lambda_* Z)(M,\psi)(\alpha_1,a_1) = \int_{r_1^{-1}[\alpha_1,a_1]} Z(\Sigma_1 \times [0,1],g) Z(M,\beta) s(\beta|_{\Sigma_0}, b|_{\Sigma_0})\, \mathrm{d}(\beta,b,g)$$
$$\text{for all} \quad s \in (\lambda_* Z)(\Sigma_0,\varphi_0), \quad (\alpha_1,a_1) \in \lambda_*^{-1}[\varphi_1].$$

In the particular case $\Sigma_0 = \Sigma_1 = \Sigma$ and $M = \Sigma \times [0,1]$, i.e. if ψ is a homotopy $\varphi_0 \overset{h}{\simeq} \varphi_1$, this reduces to

$$(\lambda_* Z)(M,h)(\alpha_1,a_1) = s(\alpha_1, h^- * a_1),$$

where $*$ denotes the composition of homotopies.

(c) *For any n-dimensional closed oriented manifold M and a map $\psi : M \longrightarrow BH$*

$$(\lambda_* Z)(M, \psi) = \int_{\lambda_*^{-1}[\psi]} Z(M, \phi)\, \mathrm{d}(\phi, h).$$

Example 6.4. For a finite group G denote by $\lambda : 1 \longrightarrow G$ the unique group morphism from the trivial group to G. Then for any topological field theory $Z : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ seen as a 1-equivariant theory we get a G-equivariant theory $\lambda_* Z$. For any object (Σ, φ) in G-$\mathbf{Cob}(n)$ we find

$$(\lambda_* Z)(\Sigma, \varphi) = \operatorname{Par}\left(\lambda_*^{-1}[\varphi] \longrightarrow \star \xrightarrow{Z(\Sigma)} \mathbf{Vect}_\mathbb{C} \right).$$

If we see φ as a G-bundle over Σ, then the homotopy fiber $\lambda_*^{-1}[\varphi]$ is discrete with the object set being the trivializations of φ (so the fiber is empty if φ is not trivial), which implies

$$(\lambda_* Z)(\Sigma, \varphi) = \bigoplus_{\#\text{trivializations of } \varphi} Z(\Sigma).$$

For an n-dimensional closed oriented manifold M and a map $\psi : M \longrightarrow BG$ we obtain using Proposition 6.3, (c)

$$(\lambda_* Z)(M, \psi) = Z(M) \cdot \#\text{trivializations of } \psi.$$

By pulling back along the functor $\operatorname{triv} : \mathbf{Cob}(n) \longrightarrow G\text{-}\mathbf{Cob}(n)$ equipping all manifolds with the trivial G-bundle we obtain a topological field theory $Y := \operatorname{triv}^* \lambda_* Z : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ which assigns to an n-dimensional closed oriented manifold M the number

$$Y(M) = Z(M)|G|^{\#\text{number of components of } M}.$$

In the case where Z is the trivial invertible topological field theory, $Y(M)$ just gives us the exponentiated number of components of M.

6.2 The composition law for the pushforward operation

For morphisms $\lambda : G \longrightarrow H$ and $\mu : H \longrightarrow J$ of finite groups it is reasonable to ask whether the pushforward operations on equivariant topological field theories obey

$$(\mu \circ \lambda)_* \cong \mu_* \circ \lambda_*.$$

Since the push operations are not left-adjoint to the pullback functors, such a relation does not hold automatically. However, we will show below in Theorem 6.6 that this composition law holds. Let us first investigate the situation at the level of manifold invariants. For this we need the following assertion:

Lemma 6.5. *For functors $\Phi : \Gamma \longrightarrow \Omega$ and $\Psi : \Omega \longrightarrow \Lambda$ between groupoids, there is a canonical equivalence*

$$(\Psi \circ \Phi)^{-1}[z] \cong \Psi^{-1}[z] \times_\Omega \Gamma.$$

Proof. This is a consequence of the pasting law for (homotopy) pullback squares. □

If now $Z : G\text{-}\mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_\mathbb{C}$ is a G-equivariant field theory and M a closed n-dimensional oriented manifold together with a map $\psi : M \longrightarrow BJ$, we obtain

$$((\mu \circ \lambda)_* Z)(M, \psi) = \int_{(\mu\circ\lambda)_*^{-1}[\psi]} Z(M, \phi)\, \mathrm{d}(\phi, g)$$

by Proposition 6.3, (c). Using Lemma 6.5 and the transformation formula (Proposition A.13) we find

$$((\mu \circ \lambda)_* Z)(M, \psi) = \int_{\mu_*^{-1}[\psi] \times_{\Pi(M,BH)} \Pi(M,BG)} Z(M, \alpha)\, \mathrm{d}(\beta, h, \alpha, g).$$

Next we apply the generalized Cavalieri's principle (Proposition A.15) to the projection functor

$$P : \mu_*^{-1}[\psi] \times_{\Pi(M,BH)} \Pi(M, BG) \longrightarrow \mu_*^{-1}[\psi].$$

Since by the universal property of the homotopy pullback the fiber $P^{-1}[\beta, h]$ of P over some $(\beta, h) \in \mu_*^{-1}[\psi]$ is just the fiber $\lambda_*^{-1}[\beta]$ of λ_* over β we find the desired relation for invariants

$$((\mu\circ\lambda)_*Z)(M,\psi) = \int_{\mu_*^{-1}[\psi]}\int_{\lambda_*^{-1}[\beta]} Z(M,\alpha)\mathrm{d}(\alpha,g)\,\mathrm{d}(\beta,h) = \int_{\mu_*^{-1}[\psi]} (\lambda_*Z)(M,\beta)\,\mathrm{d}(\beta,h) = ((\mu_*\circ\lambda_*)Z)(M,\psi).$$

We will show now by a rather long computation that this result extends beyond invariants:

Theorem 6.6. *For morphisms $\lambda : G \longrightarrow H$ and $\mu : H \longrightarrow J$ of finite groups the pushforward operations on equivariant topological field theories obey*

$$(\mu\circ\lambda)_* \cong \mu_*\circ\lambda_* : \mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}) \longrightarrow \mathrm{HSym}(J\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}}).$$

Proof. For any G-equivariant topological field theory Z we need to specify a monoidal natural transformation

$$\eta_Z : (\mu\circ\lambda)_*Z \longrightarrow \mu_*\lambda_*Z.$$

Its components are linear maps

$$\eta_Z^{(\Sigma,\varphi)} : ((\mu\circ\lambda)_*Z)(\Sigma,\varphi) \longrightarrow (\mu_*\lambda_*Z)(\Sigma,\varphi)$$

for all objects (Σ,φ) in J-$\mathbf{Cob}(n)$ which map a parallel section s of the vector bundle defined by Z over $(\mu\circ\lambda)_*^{-1}[\varphi]$ to the parallel section $\eta_Z^{(\Sigma,\varphi)}s$ defined as follows: By definition the value of $\eta_Z^{(\Sigma,\varphi)}s$ on (α,a) in $\mu_*^{-1}[\varphi]$, i.e. $\alpha : \Sigma \longrightarrow BH$ and $B\mu\circ\alpha \overset{a}{\simeq} \varphi$, needs to be a parallel section of $\lambda_*Z(\Sigma,\alpha)$. We set

$$\left(\eta_Z^{(\Sigma,\varphi)}s(\alpha,a)\right)(\beta,b) := s(\beta, a*(B\mu\circ b)) \quad \text{for all} \quad (\beta,b)\in\lambda_*^{-1}[\alpha],$$

where $\circ$ denotes composition of maps and $*$ composition of homotopies. One easily checks that $\eta_Z^{(\Sigma,\varphi)}$ is well-defined. Once we establish that η_Z is a monoidal natural transformation, this automatically implies its invertibility. Knowing the inverse already, however, will simplify the proof. In fact, the inverse of $\eta_Z^{(\Sigma,\varphi)}$ is the map $\omega_Z^{(\Sigma,\varphi)}$ which sends $s \in (\mu_*\lambda_*Z)(\Sigma,\varphi)$ to

$$\left(\omega_Z^{(\Sigma,\varphi)}s\right)(\gamma,c) := (s(B\lambda\circ\gamma,c))(\gamma,\mathrm{id}_{B\lambda\circ\gamma}) \quad \text{for all} \quad (\gamma,c)\in(\mu\circ\lambda)_*^{-1}[\varphi].$$

Again, one checks that $\omega_Z^{(\Sigma,\varphi)}$ is well-defined and the inverse to $\eta_Z^{(\Sigma,\varphi)}$. In order to verify that the maps $\eta_Z^{(\Sigma,\varphi)}$ form a natural transformation we consider a morphism $(M,\psi) : (\Sigma_0,\varphi_0) \longrightarrow (\Sigma_1,\varphi_1)$ in J-$\mathbf{Cob}(n)$ and show that the square

$$\begin{array}{ccc} ((\mu\circ\lambda)_*Z)(\Sigma_0,\varphi_0) & \xrightarrow{((\mu\circ\lambda)_*Z)(M,\psi)} & ((\mu\circ\lambda)_*Z)(\Sigma_1,\varphi_1) \\ \uparrow{\scriptstyle \omega_Z^{(\Sigma_0,\varphi_0)}} & & \downarrow{\scriptstyle \eta_Z^{(\Sigma_1,\varphi_1)}} \\ (\mu_*\lambda_*Z)(\Sigma_0,\varphi_0) & \xrightarrow{(\mu_*\lambda_*Z)(M,\psi)} & (\mu_*\lambda_*Z)(\Sigma_1,\varphi_1) \end{array} \qquad (*)$$

commutes: To this end, let $s\in(\mu_*\lambda_*Z)(\Sigma_0,\varphi_0)$, $(\alpha,a)\in\mu_*^{-1}[\varphi_1]$ and $(\beta,b)\in\lambda_*^{-1}[\alpha]$. Then by definition

$$\left(\left(\eta_Z^{(\Sigma_1,\varphi_1)}(\mu\circ\lambda)_*Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\alpha,a)\right)(\beta,b) = \left((\mu\circ\lambda)_*Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\beta,a*(B\mu\circ b)).$$

We compute the expression on the right hand side by using Proposition 6.3, (b) and the restriction functor $r_1 : (\mu\circ\lambda)_*^{-1}[\psi] \longrightarrow (\mu\circ\lambda)_*^{-1}[\varphi_1]$ and obtain

$$\begin{aligned} &\left(\left(\eta_Z^{(\Sigma_1,\varphi_1)}(\mu\circ\lambda)_*Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\alpha,a)\right)(\beta,b) \\ &= \int_{r_1^{-1}[\beta,a*(B\mu\circ b)]} Z(\Sigma_1\times[0,1],g)Z(M,\gamma)\left(\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\gamma|_{\Sigma_0},c|_{\Sigma_0})\,\mathrm{d}(\gamma,c,g) \\ &= \int_{r_1^{-1}[\beta,a*(B\mu\circ b)]} Z(\Sigma_1\times[0,1],g)Z(M,\gamma)s(B\lambda\circ\gamma|_{\Sigma_0},c|_{\Sigma_0})(\gamma|_{\Sigma_0},\mathrm{id}_{B\lambda\circ\gamma|_{\Sigma_0}})\,\mathrm{d}(\gamma,c,g). \end{aligned}$$

To continue the computation we use the restriction functor $\ell_1 : \mu_*^{-1}[\psi] \longrightarrow \mu_*^{-1}[\varphi_1]$ and observe that there is a functor

$$F : r_1^{-1}[\beta,a*(B\mu\circ b)] \longrightarrow \ell_1^{-1}[\alpha,a]$$

induced by the functor

$$r_1^{-1}[\beta,a*(B\mu\circ b)] \longrightarrow (\mu\circ\lambda)_*^{-1}[\psi] \cong \mu_*^{-1}[\psi]\times_{\Pi(M,BH)}\Pi(M,BG) \longrightarrow \mu_*^{-1}[\psi]$$

and the universal property of the homotopy fiber. Here the middle equivalence comes from Lemma 6.5. By the generalized Cavalieri's principle (Proposition A.15) we obtain

$$\begin{aligned}&\left(\left(\eta_Z^{(\Sigma_1,\varphi_1)}(\mu\circ\lambda)_* Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\alpha,a)\right)(\beta,b)\\ &=\int_{\ell_1^{-1}[\alpha,a]}\int_{F^{-1}[\delta,d,h]} Z(\Sigma_1\times[0,1],g)Z(M,\gamma)s(B\lambda\circ\gamma|_{\Sigma_0},c|_{\Sigma_0})(\gamma|_{\Sigma_0},\mathrm{id}_{B\lambda\circ\gamma|_{\Sigma_0}})\,\mathrm{d}(\gamma,c,g,u)\,\mathrm{d}(\delta,d,h)\\ &=\int_{\ell_1^{-1}[\alpha,a]}\int_{F^{-1}[\delta,d,h]} Z(\Sigma_1\times[0,1],g)Z(M,\gamma)s(\delta|_{\Sigma_0},d|_{\Sigma_0})(\gamma|_{\Sigma_0},u|_{\Sigma_0})\,\mathrm{d}(\gamma,c,g,u)\,\mathrm{d}(\delta,d,h),\end{aligned}$$

where in the last step we have used that s is parallel. Next observe that $F^{-1}[\delta,d,h]$ is equivalent to the homotopy fiber $k_1^{-1}[\beta,h^-*b]$ of yet another restriction functor $k_1:\lambda_*^{-1}[\delta]\longrightarrow\lambda_*^{-1}[\delta|_{\Sigma_1}]$, which together with the transformation formula (Proposition A.13) yields

$$\begin{aligned}&\left(\left(\eta_Z^{(\Sigma_1,\varphi_1)}(\mu\circ\lambda)_* Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\alpha,a)\right)(\beta,b)\\ &=\int_{\ell_1^{-1}[\alpha,a]}\int_{k_1^{-1}[\beta,h^-*b]} Z(\Sigma_1\times[0,1],v)Z(M,\varepsilon)s(\delta|_{\Sigma_0},d|_{\Sigma_0})(\varepsilon|_{\Sigma_0},e|_{\Sigma_0})\,\mathrm{d}(\varepsilon,e,v)\,\mathrm{d}(\delta,d,h)\end{aligned}$$

By applying Proposition 6.3, (b) again we find

$$\begin{aligned}&\left(\left(\eta_Z^{(\Sigma_1,\varphi_1)}(\mu\circ\lambda)_* Z(M,\psi)\omega_Z^{(\Sigma_0,\varphi_0)}s\right)(\alpha,a)\right)(\beta,b)\\ &=\int_{\ell_1^{-1}[\alpha,a]}((\lambda_*Z)(M,\delta)s(\delta|_{\Sigma_0},d|_{\Sigma_0}))(\beta,h^-*b)\,\mathrm{d}(\delta,d,h)\\ &=\int_{\ell_1^{-1}[\alpha,a]}(\lambda_*Z)(\Sigma_1\times[0,1],h)(\lambda_*Z)(M,\delta)s(\delta|_{\Sigma_0},d|_{\Sigma_0}))(\beta,b)\,\mathrm{d}(\delta,d,h)\\ &=(((\mu_*\lambda_*Z)(M,\psi)s)(\alpha,a))\,(\beta,b).\end{aligned}$$

This proves commutativity of $(*)$ and ensures that the maps $\eta_Z^{(\Sigma,\varphi)}$ form a natural transformation $\eta_Z:(\mu\circ\lambda)_*Z\longrightarrow\mu_*\lambda_*Z$, which is easily seen to be monoidal. Moreover, the assignment $Z\longmapsto\eta_Z$ is clearly functorial in Z. □

Remark 6.7. The isomorphisms $(\mu\circ\lambda)_*\cong\mu_*\circ\lambda_*$ appearing in Theorem 6.6 fulfill coherence conditions, i.e. they are part of the data of a 2-functor

$$\mathbf{FinGrp}\longrightarrow\mathbf{Grpd}$$

from the the category of finite groups (seen as a bicategory with only trivial 2-morphisms) to the bicategory of groupoids, functors and natural isomorphisms. This 2-functor sends a finite group G to the groupoid of G-equivariant topological field theories and a group morphism $\lambda:G\longrightarrow H$ to the push functor λ_*.

Corollary 6.8. *Let G be a finite group. Then the orbifold construction $?/G:\mathrm{HSym}(G\text{-}\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}})\longrightarrow\mathrm{Sym}(\mathbf{Cob}(n),\mathbf{Vect}_{\mathbb{C}})$ is essentially surjective, i.e. any topological field theory can be seen as an orbifold theory of an equivariant topological field theory for any given group.*

Proof. Let $Z:\mathbf{Cob}(n)\longrightarrow\mathbf{Vect}_{\mathbb{C}}$ be a topological field theory. If $\iota:1\longrightarrow G$ is the unique morphism from to trivial group to G, then ι_*Z is a G-equivariant field theory (the one discussed in Example 6.4) satisfying by Theorem 6.6

$$\frac{\iota_*Z}{G}=\pi_*\iota_*Z\cong(\pi\circ\iota)_*Z=Z$$

with the unique group morphism $\pi:G\longrightarrow 1$. □

Example 6.9. Using the composition law from Theorem 6.6 we can give a simple computation of the orbifold theory of the J-equivariant Dijkgraaf-Witten theory $Z_\lambda:J\text{-}\mathbf{Cob}(n)\longrightarrow\mathbf{Vect}_{\mathbb{C}}$ associated to a short exact sequence $0\longrightarrow G\longrightarrow H\xrightarrow{\lambda}J\longrightarrow 0$ of finite groups (Corollary 5.15): If we denote the trivial H-equivariant theory by $BH:H\text{-}\mathbf{Cob}(n)\longrightarrow\mathbf{Vect}_{\mathbb{C}}$, then $\lambda_*BH\cong Z_\lambda$ by the definition of the pushforward. Using the group morphism

$\pi^J : J \longrightarrow 1$ we find

$$\begin{aligned} \frac{Z_\lambda}{J} &\cong \pi^J_* Z_\lambda \\ &\cong \pi^J_* \lambda_* BH \\ &\cong (\pi^J \circ \lambda)_* BH \quad \text{(Theorem 6.6)} \\ &= \pi^H_* BH \quad \text{with} \quad \pi^H : H \longrightarrow \{1\} \\ &\cong \frac{BH}{H} \\ &\cong Z_H \quad \text{(Example 3.48)}, \end{aligned}$$

where Z_H is the Dijkgraaf-Witten theory for the group H.

A Groupoid cardinality and its integration theory

The groupoid cardinality is a rational number that is assigned to an essentially finite groupoid. For more background on the groupoid cardinality we refer to [BHW10].

For the characterization of groupoid cardinality we need the notion of a covering of groupoids. Coverings of simplicial sets are defined in [GZ67, Appendix I.2]. If we specialize to nerves of groupoids and take into account that these are always 2-coskeletal we arrive at the definition of covering of groupoids below. By $\triangle_n$ we will denote the standard simplex. Note that $\triangle_0 = \star$ is the terminal object in the category of simplicial sets. We also denote this object by 0 whenever we want to identify it with the zero vertex of $\triangle_1$ via the inclusion $0 \hookrightarrow \triangle_1$

Definition A.1 – Covering of groupoids. A functor $Q : \Gamma \longrightarrow \Omega$ between (small) groupoids is called *covering* if it is surjective on objects and if in any commuting square in **sSet** of the form

$$\begin{array}{ccc} 0 & \longrightarrow & B\Gamma \\ \big\downarrow & \overset{\exists !}{\nearrow} & \big\downarrow BQ \\ \triangle_1 & \longrightarrow & B\Omega \end{array}$$

the indicated lift exists and is unique (*unique path lifting property*). The latter condition can be reformulated by requiring that for every morphism $g : y_0 \longrightarrow y_1$ in Ω and any given x_0 with $Q(x_0) = y_0$, there is a unique morphism $g^* : x_0 \longrightarrow x_1$ such that $Q(g^*) = g$. We say that a covering $Q : \Gamma \longrightarrow \Omega$ is *n-fold* or *n-sheeted* if $Q^{-1}(y)$ contains n objects for every $y \in \Omega$.

Remark A.2. In the so-called *canonical model structure* on the category of (small) groupoids, see [Bou89, 14.1], the fibrations are the so-called *isofibrations*, i.e. functors $Q : \Gamma \longrightarrow \Omega$ between groupoids such that the lift in the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & B\Gamma \\ \big\downarrow & \nearrow & \big\downarrow BQ \\ \triangle_1 & \longrightarrow & B\Omega \end{array}$$

always exists. Uniqueness is not required. Hence, a covering is a special type of isofibration.

Example A.3. For a manifold M and a Lie group G denote by $\mathbf{PBun}^{\nabla}_G(M)$ the groupoid of G-bundles over M with connection. Then the functor $U : \mathbf{PBun}^{\nabla}_G(M) \longrightarrow \mathbf{PBun}_G(M)$ forgetting the connection is a covering.

Homotopy fibers (Definition 3.32) give us a source of coverings.

Lemma A.4. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between small groupoids. Then for $y \in \Omega$ the forgetful functor $\Phi^{-1}[y] \longrightarrow \Gamma_y$ is a $|\mathrm{Aut}(y)|$-fold covering.*

Definition A.5 – Essentially finite groupoid. A groupoid Γ is called *essentially finite* if $\pi_0(\Gamma)$ and $\mathrm{Aut}(x)$ for every $x \in \Gamma$ are finite. By **FinGrpd** we denote the *category of essentially finite groupoids*.

Both the definition as well as the most important properties of groupoid cardinality can be summarized as follows:

Proposition A.6 – Groupoid cardinality. *There is exactly one assignment $|?|$ of a rational number to each essentially finite groupoid satisfying the following conditions:*

(N) *For the groupoid $\star$ with one object and trivial automorphism group we have $|\star| = 1$.*
(E) *For equivalent essentially finite groupoids Γ and Ω the equality $|\Gamma| = |\Omega|$ holds.*
(U) *For the disjoint union $\Gamma \coprod \Omega$ of essentially finite groupoids we have $|\Gamma \coprod \Omega| = |\Gamma| + |\Omega|$.*
(C) *For any n-fold covering $Q : \Gamma \longrightarrow \Omega$ of essentially finite groupoids we have $|\Gamma| = n|\Omega|$.*

This number is called groupoid cardinality. For an essentially finite groupoid Γ it can be computed by

$$|\Gamma| = \sum_{[x]\in\pi_0(\Gamma)} \frac{1}{|\mathrm{Aut}(x)|}.$$

Moreover, for essentially finite groupoids Γ and Ω the product $\Gamma \times \Omega$ is also essentially finite and its groupoid cardinality is given by $|\Gamma \times \Omega| = |\Gamma||\Omega|$.

From a given essentially finite groupoid one can construct new essentially finite groupoids using functor groupoids. To this end, the following notion is helpful:

Definition A.7 – Finitely generated groupoid. A groupoid Γ is called *finitely generated* if $\pi_0(\Gamma)$ is a finite set and $\mathrm{Aut}(x)$ is a finitely generated group for all $x \in \Omega$.

Example A.8. For a compact manifold M (with boundary) and any basepoint $x \in M$ the fundamental group $\pi_1(M, x)$ is finitely generated by Lemma 1.2 in [Sa96]. Since M has finitely many connected components, we can deduce that the fundamental groupoid $\Pi(M)$ is finitely generated.

The following observation appears in [Mor15]:

Lemma A.9. *For an finitely generated groupoid Γ and an essentially finite groupoid Ω the functor groupoid $[\Gamma, \Omega]$ is essentially finite.*

This result and Example A.8 immediately imply:

Corollary A.10. *For any compact manifold M (with boundary) and any essentially finite groupoid Ω the functor groupoid $[\Pi(M), \Omega]$ is essentially finite.*

The groupoid cardinality gives rise to a rather primitive, but useful integration theory.

Definition A.11 – Integral of invariant functions over groupoids with respect to groupoid cardinality. An *invariant function* f on a groupoid Γ with values in a vector space V over a field of characteristic zero is the assignment of a vector $f(x) \in V$ to each $x \in \Gamma$ such that $f(x) = f(y)$ if $x \cong y$ in Γ. If Γ is essentially finite, we define by

$$\int_\Gamma f = \int_\Gamma f(x)\,\mathrm{d}x := \sum_{[x]\in\pi_0(\Gamma)} \frac{f(x)}{|\mathrm{Aut}(x)|} \in V$$

the *integral of f over Γ*.

Remarks A.12.

(a) The integral is a linear functional on the vector space of invariant functions.
(b) It is clear that an invariant function f on a groupoid Ω can be pulled back along a functor $\Phi : \Gamma \longrightarrow \Omega$ between groupoids. The result $\Phi^* f$ is an invariant function on Γ.

We need the following results that we call, in allusion to results known for instance from Lebesgue integration theory, transformation formula and Cavalieri's principle. The transformation formula explains how the integral behaves under pullback along equivalences. Its proof follows directly from the definitions.

Proposition A.13 – Transformation formula. *Let $\Phi : \Gamma \longrightarrow \Omega$ be an equivalence of essentially finite groupoids and $f : \Omega \longrightarrow V$ an invariant function taking values in a vector space V over a field of characteristic* *zero. Then the transformation formula*

$$\int_{\Gamma} \Phi^* f = \int_{\Omega} f$$

holds.

Cavalieri's principle from ordinary integration theory states that we can compute the volume of an object by summing (or rather integrating) the volume of its slices. The same is possible for groupoid cardinality instead of the volume.

Proposition A.14 – Cavalieri's principle. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor of essentially finite groupoids. Then*

$$|\Gamma| = \int_{\Omega} |\Phi^{-1}[y]| \,\mathrm{d}y.$$

Proof. Since $\Gamma \cong \coprod_{[y]\in\pi_0(\Omega)} \Gamma_y$, we obtain $|\Gamma| = \sum_{[y]\in\pi_0(\Omega)} |\Gamma_y|$ using the property (U) of the groupoid cardinality. According to Lemma A.4, the forgetful functor $\Phi^{-1}[y] \longrightarrow \Gamma_y$ is a $|\mathrm{Aut}(y)|$-fold covering, which together with the property (C) of the groupoid cardinality implies

$$|\Gamma_y| = \frac{|\Phi^{-1}[y]|}{|\mathrm{Aut}(y)|}.$$

Putting these observations together, we obtain

$$|\Gamma| = \sum_{[y]\in\pi_0(\Omega)} |\Gamma_y| = \sum_{[y]\in\pi_0(\Omega)} \frac{|\Phi^{-1}[y]|}{|\mathrm{Aut}(y)|} = \int_{\Omega} |\Phi^{-1}[y]| \,\mathrm{d}y. \qquad \square$$

Proposition A.15 – Generalized Cavalieri's principle. *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor of essentially finite groupoids and $f : \Gamma \longrightarrow V$ an invariant function taking values in a vector space over characteristic zero. Then*

$$\int_{\Gamma} f = \int_{\Omega} \int_{\Phi^{-1}[y]} q_y^* f \,\mathrm{d}y,$$

where $q_y : \Phi^{-1}[y] \longrightarrow \Gamma$ is the canonical functor from the homotopy fiber over y to Γ.

Proof. Without loos of generality we may assume that V is the ground field and $f = \delta_{[x]}$ for some $x \in \Gamma$. Then we can restrict Φ to the full subgroupoid Γ_x of objects in Γ isomorphic to x and obtain

$$\int_{\Gamma} f = |\Gamma_x| = \int_{\Omega} \left| (\Phi|_{\Gamma_x})^{-1} [y] \right| \mathrm{d}y$$

by Cavalieri's principle. But $(\Phi|_{\Gamma_x})^{-1}[y]$ is the full subgroupoid of $\Phi^{-1}[y]$ of all objects projecting under $q_y : \Phi^{-1}[y] \longrightarrow \Gamma$ to an object isomorphic to x. This implies $\left| (\Phi|_{\Gamma_x})^{-1} [y] \right| = \int_{\Phi^{-1}[y]} q_y^* f$ and proves the claim. $\square$

The integral of invariant functions with respect to groupoid cardinality is also compatible with coverings.

Proposition A.16. *Let $Q : \Gamma \longrightarrow \Omega$ be an n-fold covering of essentially finite groupoids. Then for any invariant function $f : \Omega \longrightarrow V$ taking values in a vector space V over a field K of characteristic zero the integral formula*

$$\int_{\Gamma} Q^* f = n \int_{\Omega} f$$

holds.

Proof. Repeating the argument of the preceding proofs we can assume without loss of generality that $V = K$ and $f = \delta_{[y_0]}$ for some $y_0 \in \Omega$. Let Ω_0 be the full subgroupoid of Ω of all objects equal to y_0 (this is $y_0 // \mathrm{Aut}(y_0)$) and $Q_0 : \Gamma_0 \longrightarrow \Omega_0$ the restriction of Q (it is also an n-fold covering). Using the covering property of the groupoid cardinality we obtain

$$\int_{\Gamma} Q^* f = |\Gamma_0| = n|\Omega_0| = \frac{n}{|\mathrm{Aut}(y_0)|} = n \int_{\Omega} f. \qquad \square$$

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