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Projective objects and the modified trace in factorisable finite tensor categories

Azat M. Gainutdinov [a,b] and Ingo Runkel [b] *

[a] *Laboratoire de Mathématiques et Physique Théorique CNRS, Université de Tours, Parc de Grammont, 37200 Tours, France*

[b] *Fachbereich Mathematik, Universität Hamburg Bundesstraße 55, 20146 Hamburg, Germany*

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Abstract

For $\mathcal{C}$ a factorisable and pivotal finite tensor category over an algebraically closed field of characteristic zero we show:

1. $\mathcal{C}$ always contains a simple projective object;
2. if $\mathcal{C}$ is in addition ribbon, the internal characters of projective modules span a submodule for the projective $SL(2,\mathbb{Z})$-action;
3. the action of the Grothendieck ring of $\mathcal{C}$ on the span of internal characters of projective objects can be diagonalised;
4. the linearised Grothendieck ring of $\mathcal{C}$ is semisimple iff $\mathcal{C}$ is semisimple.

Results 1–3 remain true in positive characteristic under an extra assumption. Result 1 implies that the tensor ideal of projective objects in $\mathcal{C}$ carries a unique-up-to-scalars modified trace function. We express the modified trace of open Hopf links coloured by projectives in terms of S-matrix elements. Furthermore, we give a Verlinde-like formula for the decomposition of tensor products of projective objects which uses only the modular S-transformation restricted to internal characters of projective objects.

We compute the modified trace in the example of symplectic fermion categories, and we illustrate how the Verlinde-like formula for projective objects can be applied there.

*Emails: azat.gainutdinov@lmpt.univ-tours.fr, ingo.runkel@uni-hamburg.de

Contents

1 Introduction

Let H be a finite-dimensional quasi-triangular Hopf algebra over a field k and denote by R its universal R-matrix. H is called *factorisable* if the map $H^* \to H : f \mapsto (f \otimes id)(R_{21}R)$ (the *Drinfeld map* [Dr]) is bijective. If H is factorisable and ribbon, the centre $Z(H)$ of H carries a projective representation of $SL(2,\mathbb{Z})$, the mapping class group of the torus [LM]. In this case, the Drinfeld map furthermore allows one to give a faithful representation of the k-linear Grothendieck ring of Rep(H) on $Z(H)$ [Dr].

A factorisable Hopf algebra H can be endowed with a central form $\nu : H \to k$ such that it becomes a symmetric Frobenius algebra [CW, Rem. 3.1]. The central form ν induces an isomorphism between $Z(H)$ and the space $C(H)$ of all central forms on H:

$$Z(H) \longrightarrow C(H) \quad , \quad z \longmapsto \nu(z \cdot -) \ . \tag{1.1}$$

Characters of finite-dimensional H-modules are examples of central forms. We are particularly interested in the preimage in $Z(H)$ of the characters of projective modules. This subspace is called the *Higman ideal* Hig(H) (see e.g. [Br]).

Denote by Rep(H) the category of finite-dimensional representations of H. In [CW], Cohen and Westreich prove, amongst other things, the following remarkable results.

Theorem 1.1 ([CW])**.** *Let H be a factorisable ribbon Hopf algebra over an algebraically closed field of characteristic zero.*

1. *There is at least one simple and projective H-module.*
2. $\mathrm{Hig}(H)$ *is an* $SL(2,\mathbb{Z})$*-submodule of* $Z(H)$.
3. $\mathrm{Hig}(H)$ *is a submodule for the action of the Grothendieck ring of* $\mathrm{Rep}(H)$ *on* $Z(H)$*, and this action can be diagonalised on* $\mathrm{Hig}(H)$ *("the fusion rules can be diagonalised on* $\mathrm{Hig}(H)$*").*

Part 2 generalises [La1, Thm. 5.5], [La2, Cor. 4.1] established for small quantum groups associated to simple Lie algebras of ADE type.

The motivation of the present paper is to give a generalisation of these results to factorisable finite tensor categories. Let us describe our setting in more detail.

Let k be a field. An abelian k-linear category is called *finite* if it is equivalent to the category of finite-dimensional representations of a finite-dimensional k-algebra. A *finite tensor category* is a finite abelian category which is rigid monoidal with bilinear tensor product and simple tensor unit. An important class of examples of finite tensor categories are the categories $\mathrm{Rep}(H)$ for finite-dimensional Hopf algebras H.

For a braided finite tensor category one can write down at least four natural non-degeneracy conditions for the braiding. These have only recently been shown to all be equivalent [Sh3], and we briefly review this result in Section 2. We refer to braided finite tensor categories satisfying these equivalent conditions as *factorisable*. Indeed, for a quasi-triangular finite-dimensional Hopf algebra H, $\mathrm{Rep}(H)$ is factorisable iff H is factorisable [Ly1].

Our first main result generalises part 1 of Theorem 1.1 (see Theorem 3.4).

Theorem 1.2. *A factorisable and pivotal finite tensor category over an algebraically closed field of characteristic zero contains a simple projective object.*

Remark 1.3. One can formulate the above theorem for positive characteristic if one adds an extra assumption, which we call "Condition P" (Section 3). Namely, we say that a finite braided tensor category satisfies Condition P if there exists a projective object P such that $[P]$ is not nilpotent in the linearised Grothendieck ring (that is, in the Grothendieck ring tensored with the field). In characteristic zero, Condition P is always satisfied (Lemma 3.2). Theorems 1.2, 1.4 and part 1 of Theorem 1.5 remain true if instead of requiring the field to be of characteristic zero, one demands that the category satisfies Condition P (but for part 2 of Theorem 1.5 we still need to require characteristic zero). The details are given in the body of the paper.

One noteworthy consequence is that categories $\mathcal{C}$ as in the above theorem allow for a unique-up-to-scalars non-zero modified trace function on the tensor ideal $\mathcal{P}roj(\mathcal{C})$, that is, on the full subcategory of all projective objects in $\mathcal{C}$ [GKPM1, GPMV, GKPM2], see Section 4.

A modified trace on $\mathcal{P}roj(\mathcal{C})$ is a family of k-linear maps $t_P : \mathrm{End}(P) \to k$, $P \in \mathcal{P}roj(\mathcal{C})$, subject to a cyclicity and partial trace condition. Recall that the categorical trace defined in terms of the pivotal structure on $\mathcal{C}$ vanishes identically on $\mathcal{P}roj(\mathcal{C})$ unless $\mathcal{C}$ is semisimple (Remark 4.6). In contrast to this, for the modified trace the pairings $\mathcal{C}(P,Q) \times \mathcal{C}(Q,P) \to k$, $(f,g) \mapsto t_Q(f \circ g)$ are non-degenerate for all $P, Q \in \mathcal{P}roj(\mathcal{C})$ (see [CGPM] and Proposition 4.2 below). In particular, the t_P turn $\mathcal{P}roj(\mathcal{C})$ into a Calabi-Yau category (Definition 5.4).

One can use modified traces to define new link invariants in finite ribbon categories which are not accessible via the usual quantum traces [GPMT]. However, we will not pursue this point in the present paper.

Let $\mathcal{A}$ be a finite abelian category over a field k and assume that $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau. In Section 5 we give a categorical definition of the Higman ideal as an ideal

$$\mathrm{Hig}(\mathcal{A}) \;\subset\; \mathrm{End}(Id_{\mathcal{A}}) \tag{1.2}$$

in the algebra of natural endomorphisms of the identity functor on $\mathcal{A}$. Recall that for algebras, $Z(A) \cong \mathrm{End}(Id_{\text{mod-}A})$, i.e. the centre of A is isomorphic as a k-algebra to the algebra of natural endomorphisms of the identity functor on the category of finite-dimensional (right, say) A-modules. The definition of $\mathrm{Hig}(\mathcal{A})$ is such that via this isomorphism, $\mathrm{Hig}(A)$ gets mapped to $\mathrm{Hig}(\text{mod-}A)$.

As in Theorem 1.2 above, let $\mathcal{C}$ be a factorisable and pivotal finite tensor category over an algebraically closed field k of characteristic zero. Let G be a projective generator of $\mathcal{C}$ and $E = \mathrm{End}(G)$. Then $\mathcal{C}$ is equivalent as a k-linear category to mod-E. As discussed above, we obtain a modified trace $t_G : E \to k$ which turns E into a symmetric Frobenius algebra, and as in (1.1) we get isomorphisms

$$\mathrm{End}(Id_{\mathcal{C}}) \longrightarrow Z(E) \longrightarrow C(E) \quad , \quad \eta \longmapsto \eta_G \longmapsto t_G(\eta_G \circ -) \; . \tag{1.3}$$

Our second main result is Theorem 6.1, which links characters of E-modules to the modified trace over certain elements in $\mathrm{End}(Id_{\mathcal{C}})$ obtained from the internal character of the corresponding object in $\mathcal{C}$. The details of this are too lengthy for this introduction, and we refer to Section 6.

Theorem 6.1 is important for the following as it implies that $\mathrm{Hig}(\mathcal{C})$ is spanned by the images of internal characters of projective objects (Proposition 7.1).

For a factorisable finite ribbon category one obtains a projective $SL(2,\mathbb{Z})$-action on $\mathrm{End}(Id_{\mathcal{C}})$, and in fact actions of all surface mapping class groups on appropriate Hom-spaces [Ly2].

Our third main result generalises part 2 of Theorem 1.1 (see Corollary 8.5)

Theorem 1.4. *Let $\mathcal{C}$ be a factorisable finite ribbon category over an algebraically closed field of characteristic zero. Then* $\mathrm{Hig}(\mathcal{C})$ *is an* $SL(2,\mathbb{Z})$*-submodule of* $\mathrm{End}(Id_{\mathcal{C}})$*.*

We note that under the conditions of this theorem, $\mathrm{Hig}(\mathcal{C}) = \mathrm{End}(Id_{\mathcal{C}})$ iff $\mathcal{C}$ is semisimple (Proposition 5.10).

Still under the conditions of the above theorem, one can define an injective ring homomorphism from the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ to $\mathrm{End}(Id_{\mathcal{C}})$ [Sh2] (see Section 2). This, in particular, turns $\mathrm{End}(Id_{\mathcal{C}})$ into a faithful $\mathrm{Gr}(\mathcal{C})$-module by defining the representation map $\rho^L : \mathrm{Gr}(\mathcal{C}) \to \mathrm{End}_k(\mathrm{End}(Id_{\mathcal{C}}))$ to be left multiplication. We show the following statement, the first part of which generalises part 3 of Theorem 1.1 (see Proposition 8.6).

Theorem 1.5. *Let $\mathcal{C}$ be a factorisable and pivotal finite tensor category over an algebraically closed field of characteristic zero.*

1. *The restriction of ρ^L to the submodule $\mathrm{Hig}(\mathcal{C})$ is diagonalisable.*
2. *The action ρ^L can be diagonalised on $\mathrm{End}(Id_{\mathcal{C}})$ if and only if $\mathcal{C}$ is semisimple.*

See Proposition 8.6 for an explicit choice of basis diagonalising the $\mathrm{Gr}(\mathcal{C})$-action. In particular, the k-linear Grothendieck ring $\mathrm{Gr}_k(\mathcal{C})$ is semisimple iff $\mathcal{C}$ is semisimple (Corollary 8.7).

We also investigate in Section 8 how the restriction of the projective $SL(2,\mathbb{Z})$-action to $\mathrm{Hig}(\mathcal{C})$ can be used to gain information about the decomposition of tensor products of projective objects, see Proposition 8.8 for details. This can be thought of as a non-semisimple variant of the Verlinde formula [Ve]. See e.g. [FHST, Fu, GR2, CG] for discussions of the Verlinde formula in the finitely non-semisimple setting, and [CW] for a related result in the context of factorisable ribbon Hopf algebras. As a corollary of the projective $SL(2,\mathbb{Z})$-action on $\mathrm{Hig}(\mathcal{C})$, we also give a purely categorical counterpart of the conjectural relation between modular properties of pseudo-trace functions and modified trace of Hopf link operators [CG] — we prove a special case of this (namely when all labels are projective) in Proposition 8.10.

Remark 1.6.

1. A semisimple factorisable finite ribbon category is a modular tensor category [Tu, BK]. Since the qualifier "modular" refers to the projective action of the modular group $SL(2,\mathbb{Z})$, it would also be reasonable to refer to modular tensor categories in the sense of [Tu, BK] as *modular fusion categories*, and to call factorisable finite ribbon categories *modular finite tensor categories*. An important application of modular fusion categories is that they are precisely the data needed to construct certain three-dimensional topological field theories [RT, Tu, BDSPV]. For modular finite tensor categories in the above sense, a corresponding result is not known. However, a version of three-dimensional topological field theory build form such categories is studied in [KL].

2. So far, we have motivated our results from the theory of Hopf algebras. Another reason to look at factorisable finite tensor categories is provided by vertex operator algebras. A particularly benign class of vertex operator algebras V are the so-called C_2-cofinite ones, for which we in addition require that they are simple, non-negatively graded and isomorphic to their contragredient module. Let V be such a vertex operator algebra.

   If $\mathrm{Rep}(V)$, the category of (quasi-finite dimensional, generalised) V-modules, is in addition semisimple, it is in fact a modular fusion category [Hu2]. What is more, it is proved in [Zh] that the characters of V-modules form a projective representation of the modular group, which by [Hu1] agrees with the representation obtained from the modular fusion category Rep(V).

If Rep(V) is not semisimple, it is known from [HLZ, Hu3] that Rep(V) satisfies all properties of a finite braided tensor category except for rigidity (which is, however, conjectured in [Hu4, GR2] and proven in a different setting for W_p models in [TW]). It is furthermore conjectured that also in the non-semisimple case, Rep(V) is factorisable [GR2, CG]. For the modular group action one now has to pass to pseudo-trace functions [Mi, AN], and it is tempting to conjecture that this modular group action agrees with the one on $\mathrm{End}(Id_{\mathrm{Rep}(V)})$ discussed above, see [GR2, Conj. 5.10] for a precise formulation. For so-called symplectic fermion conformal field theories [Ka, GK, Ab], the modular group actions obtained from pseudo-trace functions and from Rep(V) have been compared in [GR1, FGR2] and they indeed agree.

If these conjectures were true, Theorems 1.2 and 1.4 would imply that such a V always has at least one simple and projective module, and that the characters (not the pseudo-trace functions) of projective modules transform into each other under the modular group action. Both properties hold in the very few non-semisimple examples where they can be verified [Ka, FHST, FGST, NT, Ab].

The study of properties of factorisable tensor categories and in particular of Rep(V) is also important input for the construction of bulk conformal field theory correlators in the non-semisimple setting as in [FS1, FSS, FS2].

We conclude the paper in Section 9 with the computation of the modified trace in a class of examples of factorisable finite ribbon categories [DR1, Ru] arising in symplectic fermion conformal field theories. We also illustrate in this example how the projective $SL(2,\mathbb{Z})$-action on the Higman ideal can be used to obtain information about the tensor product of projective objects.

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Convention

Through this paper, k denotes an algebraically closed field, possibly of positive characteristic.[1]

[1] Some definitions and results of this paper do not actually require algebraic closedness of k. For example, algebraic closedness is not required in the first half of Section 2, up to and excluding Theorem 2.6, or in the definition of a modified right trace in Section 4, etc. But to avoid confusion we prefer to require algebraic closedness for the entire paper.

2 Factorisable finite tensor categories

In this section we introduce notation and recall some definitions and results that will be used later, such as the different characterisations of a factorisable finite tensor category. The reader is invited to just skim through this section and return to it when necessary.

For $\mathcal{A}$ an essentially small abelian category we write (see e.g. [EGNO] for more details)

- $\mathrm{Irr}(\mathcal{A})$: a choice of representatives for the isomorphism classes of simple objects of $\mathcal{A}$, i.e. every simple object is isomorphic to one and only one element of $\mathrm{Irr}(\mathcal{A})$,
- $\mathcal{P}roj(\mathcal{A})$: the full subcategory of $\mathcal{A}$ consisting of all projective objects,
- $\mathrm{Gr}(\mathcal{A})$: the Grothendieck group of $\mathcal{A}$. We write $[X] \in \mathrm{Gr}(\mathcal{A})$ for the class of $X \in \mathcal{A}$ in the Grothendieck group,
- $\mathrm{Gr}_k(\mathcal{A})$: if $\mathcal{A}$ is k-linear we set $\mathrm{Gr}_k(\mathcal{A}) := k \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathcal{A})$. Note that the canonical ring homomorphism $\mathrm{Gr}(\mathcal{A}) \to \mathrm{Gr}_k(\mathcal{A})$ is injective iff k has characteristic zero.

A k-linear abelian category $\mathcal{A}$ is called *locally finite* if all Hom-spaces are finite dimensional over k and every object of $\mathcal{A}$ has finite length. A locally finite abelian category is called *finite* if $\mathrm{Irr}(\mathcal{A})$ is finite and every simple object has a projective cover. For a finite abelian category $\mathcal{A}$ we set

- P_U: a choice of projective cover $\pi_U : P_U \to U$ for each simple $U \in \mathrm{Irr}(\mathcal{A})$.

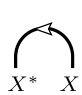
One can show that a k-linear abelian category is finite iff it is equivalent as a k-linear category to the category of finite-dimensional modules over a finite-dimensional k-algebra.

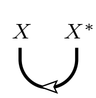
For a finite abelian category $\mathcal{A}$ we have $\mathrm{Gr}(\mathcal{A}) = \bigoplus_{U \in \mathrm{Irr}(\mathcal{A})} \mathbb{Z}\,[U]$. (On the other hand, if infinite direct sums are allowed in $\mathcal{A}$ we have $\mathrm{Gr}(\mathcal{A}) = 0$ because of the short exact sequence $X \to \bigoplus_{\mathbb{N}} X \to \bigoplus_{\mathbb{N}} X$.)

A monoidal category $\mathcal{M}$ is called *rigid* if for each $X \in \mathcal{M}$ there is $X^* \in \mathcal{M}$ (the *left dual*) and ${}^*X$ (the *right dual*), together with morphisms

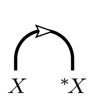
$$\begin{aligned} &\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}\ , && \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*\ , \\ &\widetilde{\mathrm{ev}}_X : X \otimes {}^*X \to \mathbf{1}\ , && \widetilde{\mathrm{coev}}_X : \mathbf{1} \to {}^*X \otimes X\ , \end{aligned} \tag{2.1}$$

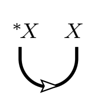
where $\mathbf{1} \in \mathcal{M}$ is the tensor unit. These morphisms have to satisfy the zig-zag identities, see e.g. [EGNO] for more details. We will later use string diagram notation. Our diagrams will be read bottom to top and the above duality morphisms will be written as



$= \mathrm{ev}_X$,


$= \mathrm{coev}_X$,



$= \widetilde{\mathrm{ev}}_X$,


$= \widetilde{\mathrm{coev}}_X$. (2.2)

A monoidal category $\mathcal{M}$ with left duals is *pivotal* if it is equipped with a natural monoidal isomorphism

$$\delta : Id_{\mathcal{M}} \to (-)^{**} . \tag{2.3}$$

It is then automatically rigid via ${}^*X := X^*$ and (we omit the tensor product between objects for better readability)

$$\begin{aligned} \widetilde{\mathrm{ev}}_X &= \Big[XX^* \xrightarrow{\delta_X \otimes id} X^{**}X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbf{1} \Big] , \\ \widetilde{\mathrm{coev}}_X &= \Big[\mathbf{1} \xrightarrow{\mathrm{coev}_{X^*}} X^*X^{**} \xrightarrow{id \otimes \delta_X^{-1}} X^*X \Big] . \end{aligned} \tag{2.4}$$

Let $\mathcal{M}$ be a monoidal category with right duals. Below we will make use of the functor ${}^*(-)\otimes(-) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, $(X,Y) \mapsto {}^*X \otimes Y$, and of dinatural transformations between this functor and the constant functor with value X for some $X \in \mathcal{M}$. We refer to [ML] for dinatural transformations in general and to [FS1, Sec. 4] or [FGR1, Sec. 3] for these specific functors.

The following lemma is a slight generalisation of [GKPM1, Lem. 2.5.1] which in turn follows [De]. It is proven in the same way as in [GKPM1] and we relegate the proof to Appendix A.

Lemma 2.1. *Let $\mathcal{M}$ be an abelian monoidal category with right duals and biexact tensor product functor. Suppose that there is a projective object $P \in \mathcal{M}$ and a surjection $p : P \to \mathbf{1}$. Let $X \in \mathcal{M}$ and let η be a dinatural transformation from ${}^*(-) \otimes (-)$ to X. Consider the following commuting diagram in $\mathcal{M}$ with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array} \tag{2.5}$$

In this case, the following equality of morphisms $\mathbf{1} \to X$ holds for any choice of endomorphisms a, b, and c that make (2.5) commute:

$$\eta_B \circ (id_{{}^*B} \otimes b) \circ \widetilde{\mathrm{coev}}_B \;=\; \eta_A \circ (id_{{}^*A} \otimes a) \circ \widetilde{\mathrm{coev}}_A + \eta_C \circ (id_{{}^*C} \otimes c) \circ \widetilde{\mathrm{coev}}_C \;. \tag{2.6}$$

Remark 2.2.

1. A corresponding statement as that in Lemma 2.1 holds for a dinatural transformation ξ from $(-) \otimes (-)^*$ to X. In this case, for a, b, c as in (2.5) we have

$$\xi_B \circ (b \otimes id_{B^*}) \circ \mathrm{coev}_B \;=\; \xi_A \circ (a \otimes id_{A^*}) \circ \mathrm{coev}_A + \xi_C \circ (c \otimes id_{C^*}) \circ \mathrm{coev}_C \;. \tag{2.7}$$

2. As a consequence of Lemma 2.1, the morphism $\eta_A \circ \widetilde{\mathrm{coev}}_A \in \mathcal{M}(\mathbf{1}, X)$ only depends on the class $[A] \in \mathrm{Gr}(\mathcal{M})$. Indeed, if one chooses a, b, c in (2.5) to be identities, then (2.6) states that given a short exact sequence $0 \to A \to B \to C \to 0$ we have the identity

$$\eta_B \circ \widetilde{\mathrm{coev}}_B \;=\; \eta_A \circ \widetilde{\mathrm{coev}}_A + \eta_C \circ \widetilde{\mathrm{coev}}_C \tag{2.8}$$

of morphisms in $\mathcal{M}(\mathbf{1}, X)$. For example, if $\mathcal{M}$ is in addition pivotal, one can take $X = \mathbf{1}$ and the dinatural transformation $\eta_A = \mathrm{ev}_A$. One obtains the statement that the quantum dimension only depends on the class in $\mathrm{Gr}(\mathcal{M})$.

A *finite tensor category* $\mathcal{C}$ is a category which (see [EO])

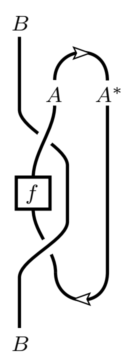
- is a k-linear finite abelian category,
- is a rigid monoidal category with k-bilinear tensor product functor,
- has a simple tensor unit.

The tensor product functor of an abelian rigid monoidal category is automatically biadditive and biexact [EGNO, Prop. 4.2.1]. A finite tensor category is called *unimodular* if $(P_{\mathbf{1}})^* \cong P_{\mathbf{1}}$, see [ENO, EGNO] for more details.

The following technical corollary to Lemma 2.1 generalises [GKPM1, Cor. 2.5.2] and will be needed later. The proof is given in Appendix A.

Corollary 2.3. *Let $\mathcal{C}$ be a finite tensor category over k. Let $X \in \mathcal{C}$ and let η be a dinatural transformation from ${}^*(-) \otimes (-)$ to X and ξ from $(-) \otimes (-)^*$ to X. Let $A \in \mathcal{C}$ and $f \in \mathrm{End}(A)$ be such that for all simple $U \in \mathcal{C}$ and all $u : A \to U$ we have $u \circ f = 0$. Then:*

1. $\eta_A \circ (id \otimes f) \circ \widetilde{\mathrm{coev}}_A = 0$ *and* $\xi_A \circ (f \otimes id) \circ \mathrm{coev}_A = 0$.
2. *If $\mathcal{C}$ is in addition braided and pivotal, then for any $B \in \mathcal{C}$,*



$$= 0 \ . \tag{2.9}$$

For the rest of this section, we fix

- $\mathcal{C}$: a braided and pivotal finite tensor category over k.

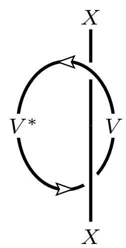
Our notation for the coherence and braiding isomorphisms in $\mathcal{C}$ is, for $U, V, W \in \mathcal{C}$,

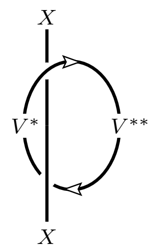
$$\begin{aligned} \alpha_{U,V,W} &: U \otimes (V \otimes W) \longrightarrow (U \otimes V) \otimes W && \text{(associator)} \ , \\ \lambda_U &: \mathbf{1} \otimes U \longrightarrow U \quad , \quad \rho_U : U \otimes \mathbf{1} \longrightarrow U && \text{(unit isomorphisms)} \ , \\ c_{U,V} &: U \otimes V \longrightarrow V \otimes U && \text{(braiding)} \ . \end{aligned} \tag{2.10}$$

We write $\mathrm{End}(Id_{\mathcal{C}})$ for the natural endomorphisms of the identity functor on $\mathcal{C}$.

Given $V \in \mathcal{C}$, consider the natural transformation $\sigma(V) \in \mathrm{End}(Id_{\mathcal{C}})$: for all $X \in \mathcal{C}$,

$$\sigma(V)_X \;=\; \big[X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\widetilde{\mathrm{coev}}_V \otimes id} (V^*V)X \xrightarrow{\sim} V^*(VX) \xrightarrow{id\otimes(c^{-1}_{V,X}\circ c^{-1}_{X,V})} V^*(VX) \xrightarrow{\sim} (V^*V)X \xrightarrow{\mathrm{ev}_V \otimes id} \mathbf{1}X \xrightarrow{\sim} X\big]$$

$=$ 

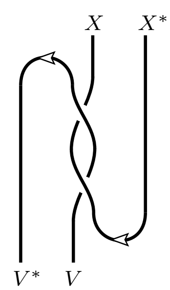
$\overset{(*)}{=}$ 

, (2.11)

where in (*) we used properties of the braiding to deform the string diagram, as well as pivotality to exchange left and right duality morphisms. This natural transformation appears in [Tu, Sec. I.1.5] and also in [CGPM, CG] where it is called "general Hopf link" and "open Hopf link operator", respectively (see [GR2, Rem. 3.10] for the conventions used here).

Lemma 2.4. *$\sigma(V)$ only depends on the class of V in $\mathrm{Gr}(\mathcal{C})$, and the resulting map $\mathrm{Gr}(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}})$ is a ring homomorphism.*

Proof. Let $X \in \mathcal{C}$ be arbitrary and consider the dinatural transformation from $(-)^* \otimes (-)$ to $X \otimes X^*$ given by (see also [FGR1, Sec. 4.4])

$\eta_V \;:=$ 

. (2.12)

Note that $(\sigma(V)_X \otimes id_{X^*}) \circ \mathrm{coev}_X = \eta_V \circ \widetilde{\mathrm{coev}}_V$. By Lemma 2.1 (or rather Remark 2.2 (2)), the right hand side factors through $\mathrm{Gr}(\mathcal{C})$, hence so does σ. That the resulting map is a ring map follows from the identity $\sigma(V \otimes W)_X = \sigma(V)_X \circ \sigma(W)_X$, which one can verify by a straightforward calculation [Tu, Sec. 4.5] (see also [CG, Sec. 3.1]). □

By abuse of notation, we also use the symbol σ for the resulting ring homomorphism

$$\sigma : \mathrm{Gr}(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}}) \quad , \quad \sigma([V])_X \text{ is given by (2.11)} . \tag{2.13}$$

Let $\mathcal{L}$ be the coend (see e.g. [KL] or the review in [FS1, Sec. 4] or in [FGR1, Sec. 3])

$$\mathcal{L} \;=\; \int^{X\in\mathcal{C}} X^* \otimes X \tag{2.14}$$

and denote by $\iota_X : X^* \otimes X \to \mathcal{L}$, $X \in \mathcal{C}$, the corresponding dinatural transformation. The coend $\mathcal{L}$ exists since $\mathcal{C}$ is a finite tensor category (see e.g. [KL, Cor. 5.1.8]). It carries the structure of a Hopf algebra in $\mathcal{C}$ and is equipped with a Hopf pairing $\omega : \mathcal{L} \otimes \mathcal{L} \to \mathbf{1}$ [Ly1]. We denote the product, coproduct, unit, counit and antipode of the Hopf algebra $\mathcal{L}$ by $\mu_{\mathcal{L}}$, $\Delta_{\mathcal{L}}$, $\eta_{\mathcal{L}}$, $\varepsilon_{\mathcal{L}}$, $S_{\mathcal{L}}$, respectively (we use the conventions in [FGR1, Sec. 3]).

If $\mathcal{C}$ is unimodular, the coend $\mathcal{L}$ admits a two-sided integral $\Lambda_{\mathcal{L}} : \mathbf{1} \to \mathcal{L}$ [Sh1, Thm. 6.8].[2] The integral is unique up to a scalar factor. Dually, $\mathcal{L}$ admits a two-sided cointegral $\Lambda^{co}_{\mathcal{L}} : \mathcal{L} \to \mathbf{1}$ which can be normalised such that $\Lambda^{co}_{\mathcal{L}} \circ \Lambda_{\mathcal{L}} = id_{\mathbf{1}}$ (see e.g. [KL, Sec. 4.2.3]).

We will need the linear maps

$$\mathrm{End}(Id_{\mathcal{C}}) \xrightarrow{\psi} \mathcal{C}(\mathcal{L}, \mathbf{1}) \underset{\Omega}{\overset{\rho}{\rightleftarrows}} \mathcal{C}(\mathbf{1}, \mathcal{L}) \quad , \tag{2.15}$$

which are defined as follows.

- For $\alpha \in \mathrm{End}(Id_{\mathcal{C}})$, the value $\psi(\alpha)$ is defined uniquely by the universal property of the coend $(\mathcal{L}, \iota)$ via, for all $X \in \mathcal{C}$,

$$\psi(\alpha) \circ \iota_X = \mathrm{ev}_X \circ (id \otimes \alpha_X) \ . \tag{2.16}$$

- Ω is defined via the Hopf pairing of $\mathcal{L}$ as, for $f : \mathbf{1} \to \mathcal{L}$,

$$\Omega(f) = \big[\, \mathcal{L} \xrightarrow{\sim} \mathbf{1}\mathcal{L} \xrightarrow{f \otimes id} \mathcal{L}\mathcal{L} \xrightarrow{\omega} \mathbf{1} \,\big] \ . \tag{2.17}$$

- The definition of ρ requires $\mathcal{C}$ to be unimodular and the choice of a non-zero integral $\Lambda_{\mathcal{L}} : 1 \to \mathcal{L}$. In this case we set, for $g : \mathcal{L} \to \mathbf{1}$,

$$\rho(g) = \big[\, \mathbf{1} \xrightarrow{\Lambda_{\mathcal{L}}} \mathcal{L} \xrightarrow{\Delta_{\mathcal{L}}} \mathcal{L}\mathcal{L} \xrightarrow{g \otimes id} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L} \,\big] \ . \tag{2.18}$$

The bialgebra structure on $\mathcal{L}$ allows one to endow $\mathcal{C}(\mathbf{1}, \mathcal{L})$ and $\mathcal{C}(\mathcal{L}, \mathbf{1})$ with the structure of a k-algebra. $\mathrm{End}(Id_{\mathcal{C}})$ is equally a k-algebra. We have:

Lemma 2.5. *1. ψ is an isomorphism of k-algebras.*

2. Ω is a k-algebra homomorphism (but not necessarily an isomorphism).

3. For $\mathcal{C}$ unimodular, ρ is an isomorphism (but not necessarily a k-algebra homomorphism).

For the proof of this lemma, see e.g. [Ly1], [KL, Prop. 5.2.5], [KL, Cor. 4.2.13] and [Sh3, Sec. 3.1] (and also [GR2, Sec. 2]). One verifies that the inverses of ψ and ρ are, for $X \in \mathcal{C}$, $f : \mathcal{L} \to \mathbf{1}$, $g : \mathbf{1} \to \mathcal{L}$,

$$\psi^{-1}(f)_X \;=\; \big[X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\mathrm{coev}_X \otimes id} (XX^*)X \xrightarrow{\sim} X(X^*X) \xrightarrow{id \otimes \iota_X} X\mathcal{L} \xrightarrow{id \otimes f} X\mathbf{1} \xrightarrow{\sim} X\big] \ ,$$

[2] For factorisable $\mathcal{C}$ (see Definition 2.9 below), the existence of two-sided integrals has been shown in [Ly1, Thm. 6.11], see also [KL, Cor. 5.2.11].

$$\rho^{-1}(g) \;=\; \big[\,\mathcal{L} \xrightarrow{\sim} \mathbf{1}\mathcal{L} \xrightarrow{g\otimes id} \mathcal{L}\mathcal{L} \xrightarrow{S_{\mathcal{L}}\otimes id} \mathcal{L}\mathcal{L} \xrightarrow{\mu_{\mathcal{L}}} \mathcal{L} \xrightarrow{\Lambda^{\mathrm{co}}_{\mathcal{L}}} \mathbf{1}\,\big]\;. \tag{2.19}$$

The *internal character* of $V \in \mathcal{C}$ is the element $\chi_V \in \mathcal{C}(\mathbf{1},\mathcal{L})$ given by [FS1, Sh2] (we use the convention in [GR2])

$$\chi_V \;=\; \big[\,\mathbf{1} \xrightarrow{\widetilde{\mathrm{coev}}_V} V^* \otimes V \xrightarrow{\iota_V} \mathcal{L}\,\big]\;. \tag{2.20}$$

By Lemma 2.1, this map factors through the Grothendieck ring, i.e. we obtain a map

$$\chi:\ \mathrm{Gr}(\mathcal{C}) \to \mathcal{C}(\mathbf{1},\mathcal{L}), \qquad [V] \mapsto \chi_V\;. \tag{2.21}$$

By abuse of notation, we will denote the induced map $\mathrm{Gr}_k(\mathcal{C}) \to \mathcal{C}(\mathbf{1},\mathcal{L})$, $1 \otimes_{\mathbb{Z}} [V] \mapsto \chi_V$ by χ as well.

Theorem 2.6 ([FS1, Sec. 4.5] and [Sh2, Thm. 3.11, Prop. 3.14, Cor. 4.2, Thm. 5.12])**.**

1. *The linear map $\chi : \mathrm{Gr}_k(\mathcal{C}) \to \mathcal{C}(\mathbf{1},\mathcal{L})$ is an injective k-algebra homomorphism.*
2. *Suppose in addition that $\mathcal{C}$ is unimodular. Then χ is surjective iff $\mathcal{C}$ is semisimple.*

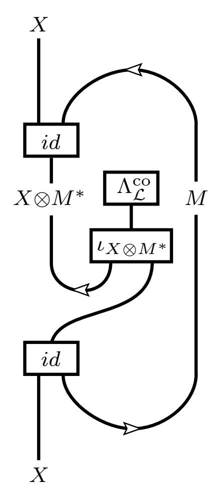
This theorem remains true for $\mathcal{C}$ not braided (but still a pivotal finite tensor category over an algebraically closed field).

Define, for $\mathcal{C}$ unimodular and $M \in \mathcal{C}$,

$$\phi_M \;:=\; \psi^{-1}\big(\rho^{-1}(\chi_M)\big) \;\in\; \mathrm{End}(Id_{\mathcal{C}})\;. \tag{2.22}$$

After substituting the definitions, one can check that, for all $X \in \mathcal{C}$,

$$(\phi_M)_X \;=\; \tag{2.23}$$



As χ_M only depends on $[M] \in \mathrm{Gr}(\mathcal{C})$, so does ϕ_M. Since ψ and ρ are isomorphisms, by Theorem 2.6 the set $\{\,\phi_U \,|\, U \in \mathrm{Irr}(\mathcal{C})\,\}$ is linearly independent in $\mathrm{End}(Id_{\mathcal{C}})$. We note that since ρ is not necessarily an algebra map, neither is the linear map $\mathrm{Gr}_k(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}})$, $[M] \mapsto \phi_M$.

We define the linear map $\mathcal{S}_\mathcal{C} : \mathrm{End}(Id_\mathcal{C}) \to \mathrm{End}(Id_\mathcal{C})$, the *modular S-transformation*, as

$$\mathcal{S}_\mathcal{C} \;=\; \Big[\ \mathrm{End}(Id_\mathcal{C}) \xrightarrow{\psi} \mathcal{C}(\mathcal{L},\mathbf{1}) \xrightarrow{\rho} \mathcal{C}(\mathbf{1},\mathcal{L}) \xrightarrow{\Omega} \mathcal{C}(\mathcal{L},\mathbf{1}) \xrightarrow{\psi^{-1}} \mathrm{End}(Id_\mathcal{C})\ \Big]\ . \tag{2.24}$$

We have seen that ψ and Ω are algebra maps, while ρ is in general not. Thus in general $\mathcal{S}_\mathcal{C}$ is not an algebra map, either. Recall the definition of σ in (2.13). A straightforward calculation shows that $\sigma([M]) = \psi^{-1} \circ \Omega(\chi_M)$. Combining this with (2.22) and (2.24) gives, for all $M \in \mathcal{C}$,

$$\sigma([M]) \;=\; \mathcal{S}_\mathcal{C}(\phi_M)\ , \tag{2.25}$$

cf. [GR2, Rem. 3.10]. In particular, by Lemma 2.4, the combination $[M] \mapsto \mathcal{S}_\mathcal{C}(\phi_M)$ is an algebra homomorphism from $\mathrm{Gr}_k(\mathcal{C})$ to $\mathrm{End}(Id_\mathcal{C})$.

Remark 2.7. Since the coend $\mathcal{L}$ is unique up to unique isomorphism, and the cointegral $\Lambda^{\mathrm{co}}_\mathcal{L}$ is unique up to sign, it is straightforward to verify that the ϕ_M, $M \in \mathcal{C}$ and $\mathcal{S}_\mathcal{C}$ depend on the choice of $(\mathcal{L}, \Lambda^{\mathrm{co}}_\mathcal{L})$ only up to an overall sign, see [FGR1, Prop. 5.3]. In particular, $\mathcal{S}_\mathcal{C}(\phi_M)$ is independent of the choice of $(\mathcal{L}, \Lambda^{\mathrm{co}}_\mathcal{L})$, as we already saw explicitly in (2.25).

Next we introduce four equivalent non-degeneracy requirement for the braiding. (This does not require a pivotal structure, hence we use the letter $\mathcal{D}$ and reserve $\mathcal{C}$ for the pivotal case as declared above.)

Theorem 2.8 ([Sh3, Thm. 1.1])**.** *Let $\mathcal{D}$ be a braided finite tensor category over an algebraically closed field. The following conditions are equivalent:*

1. *Every transparent object in $\mathcal{D}$ is isomorphic to a direct sum of tensor units. ($T \in \mathcal{D}$ is* transparent *if for all $X \in \mathcal{D}$, $c_{X,T} \circ c_{T,X} = id_{T\otimes X}$.)*
2. *The canonical braided monoidal functor $\mathcal{D} \boxtimes \overline{\mathcal{D}} \to \mathcal{Z}(\mathcal{D})$ is an equivalence. (Here, $\boxtimes$ is the Deligne product, $\overline{\mathcal{D}}$ is the same tensor category as $\mathcal{D}$, but has inverse braiding, and $\mathcal{Z}(\mathcal{D})$ is the Drinfeld centre of $\mathcal{D}$.)*
3. *The pairing $\omega : \mathcal{L}\otimes\mathcal{L} \to \mathbf{1}$ is non-degenerate (in the sense that there exists a copairing $\mathbf{1} \to \mathcal{L} \otimes \mathcal{L}$).*
4. *Ω is an isomorphism.*

Definition 2.9. $\mathcal{D}$ as in Theorem 2.8 is called *factorisable* if it satisfies the equivalent conditions 1–4 there.

In [ENO, Prop. 4.5] the following result is proved (using formulation 2 of factorisability in Theorem 2.8):

Theorem 2.10. *Let $\mathcal{C}$ be factorisable. Then $\mathcal{C}$ is unimodular.*

In particular, for factorisable $\mathcal{C}$ the Hopf algebra given by the coend $\mathcal{L}$ admits integrals and cointegrals. Moreover, in this case we can normalise the integral such that

$$\omega \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) = id_{\mathbf{1}} \ , \tag{2.26}$$

see e.g. [KL, Sec. 5.2.3]. Since the space of integrals is one-dimensional, this determines $\Lambda_{\mathcal{L}}$ up to a sign.

Remark 2.11. Suppose $\mathcal{C}$ is factorisable. Then by the above theorem, $\mathcal{C}$ is unimodular and we have the isomorphism ρ from (2.18) at our disposal (Lemma 2.5). Hence the linear endomorphism $\mathcal{S}_{\mathcal{C}}$ of $\mathrm{End}(Id_{\mathcal{C}})$ in (2.24) is defined, and by condition 4 in Theorem 2.8, $\mathcal{S}_{\mathcal{C}}$ is invertible.

Using Lemma 2.4 and (2.22), (2.25), we get the following corollary to Theorem 2.6.

Corollary 2.12. *Let $\mathcal{C}$ be factorisable. Then the algebra map $\sigma\colon \mathrm{Gr}_k(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}})$ is injective. If in addition $\mathrm{char}(k) = 0$, the ring homomorphism $\sigma\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}})$ is injective, too.*

The following important theorem is proved in [Ly1], see also [FGR1, Sec. 5.1] for a summary in the notation used here.

Theorem 2.13. *Let $\mathcal{C}$ be factorisable and ribbon with ribbon twist $\theta \in \mathrm{End}(Id_{\mathcal{C}})$. Then there is a projective representation of $SL(2,\mathbb{Z})$ on $\mathrm{End}(Id_{\mathcal{C}})$ for which the S and T generators act as*

$$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \longmapsto \mathcal{S}_{\mathcal{C}} \quad , \quad \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \longmapsto \theta \circ (-) \quad . \tag{2.27}$$

Remark 2.14. For $\mathcal{C} = \mathrm{Rep}\, H$ for H a finite-dimensional ribbon Hopf algebra over k, we have $\mathbf{1} = k$, and the coend is given by $\mathcal{L} = H^*$ with the coadjoint action [Ly2, Ke2]. Let S be the antipode of H, R the universal R-matrix, v the ribbon element, and $u = \sum_{(R)} S(R_2)R_1$ the Drinfeld element. We will (for the sake of this remark) identify $\mathrm{End}(Id_{\mathcal{C}})$ with the centre $Z(H)$ of H. Then we have:

- The internal characters χ_V are the q-characters: the images $\chi_V(1)$ are linear forms on H invariant under the coadjoint action. They are the trace functions $\chi_V(1) = \mathrm{Tr}_V(u^{-1}v\,\cdot\,)$ introduced and studied in [Dr], see also [Ke1].

- The map $S \circ \psi^{-1} \circ \Omega$ is then *the Drinfeld mapping* from the space of q-characters to $Z(H)$ given by $\chi(\cdot) \mapsto \sum_{(M)} \chi(M_1)M_2$ for the monodromy matrix $M = R_{21}R$, see [Dr, Prop. 1.2]. The central elements $\sigma([V])_H$ (where H is the left regular H-module) are the images of $\mathrm{Tr}_V(u^{-1}v\,\cdot\,)$ under the Drinfeld mapping composed with S^{-1}.

- The map $S \circ \psi^{-1} \circ \rho^{-1}$ is *the Radford mapping* from the space of q-characters to $Z(H)$ given by $\chi(\cdot) \mapsto (\chi \otimes id)(\Delta(\boldsymbol{c}))$, for the (co)integral $\boldsymbol{c} \in H$ (that can be computed from $\Lambda^{\mathrm{co}}_{\mathcal{L}}$ using the duality maps), see [Ra] for properties of this map. The central elements $(\phi_V)_H$ from (2.23) are the images of $\mathrm{Tr}_V(u^{-1}v\,\cdot\,)$ under the Radford mapping composed with S^{-1}, see also [FGR1, Rem. 7.10].

- The modular S-transformation on a quasi-triangular Hopf algebra was introduced in [LM] following the categorical construction of [Ly1]. The images of $\mathrm{Tr}_V(u^{-1}v\,\cdot\,)$ under the Drinfeld and Radford mappings are related by the modular S-transformation [Ke1, Sec. 2], see [FGST, Sec. 5] for the statement in this form. This result is generalised by (2.25).

3 Existence of a simple projective object

Throughout the rest of this paper, the following technical condition will play an important role.

> **Condition P**: A finite tensor category $\mathcal{M}$ over k satisfies *Condition P* if there exists a projective object $P \in \mathcal{M}$ such that $[P]$ is not nilpotent in the linearised Grothendieck ring $\mathrm{Gr}_k(\mathcal{M})$.

Two important classes of categories which satisfy Condition P are described in the next two lemmas.

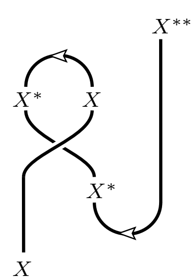
Lemma 3.1. *Every semisimple finite tensor category $\mathcal{M}$ satisfies Condition P.*

Proof. Since $\mathcal{M}$ is semisimple, every simple object is projective. Furthermore, for every simple object $U \in \mathcal{M}$, $[U]$ is non-zero in $\mathrm{Gr}_k(\mathcal{M})$. Hence we may take $P = \mathbf{1}$, the tensor unit. Then for all $m > 0$, $[\mathbf{1}]^m = [\mathbf{1}] \neq 0$ in $\mathrm{Gr}_k(\mathcal{M})$. □

Lemma 3.2. *Let $\mathcal{C}$ be a locally finite braided tensor category over some field. For all $X \in \mathcal{C}$, $X \neq 0$ and $m > 0$ we have $[X]^m \neq 0$ in $\mathrm{Gr}(\mathcal{C})$. If in addition $\mathcal{C}$ is finite and if the field is of characteristic zero, then $\mathcal{C}$ satisfies Condition P.*

Proof. Since by assumption all objects have finite length, we have $\mathrm{Gr}(\mathcal{C}) = \bigoplus_{U\in\mathrm{Irr}(\mathcal{C})} \mathbb{Z}[U]$. Thus for $X \in \mathcal{C}$ it follows that $X = 0$ iff $[X] = 0$.

The map $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ is non-zero, hence $X^* \otimes X \neq 0$, and so $[X^*][X] \neq 0$. Iterating this argument shows $(X^* \otimes X)^* \otimes (X^* \otimes X) \neq 0$, i.e. $([X^*][X])^2 \neq 0$, etc. We used here Drinfeld's canonical isomorphism $u_X : X \to X^{**}$ given by

u_X =  . (3.1)

Using commutativity of $\mathrm{Gr}(\mathcal{C})$, after m steps we find $[X^*]^m[X]^m \neq 0$.

As a finite $\mathcal{C}$ has enough projectives, it contains a non-zero projective object P. Since in characteristic zero, the canonical ring homomorphism $\mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}_k(\mathcal{C})$ is injective, $[P]^m \neq 0$ for all $m > 0$ also holds in $\mathrm{Gr}_k(\mathcal{C})$. □

Remark 3.3. In positive characteristic, Condition P may or may not be satisfied. For example, suppose that k is of characteristic p and let G be a finite group. If p does not divide $|G|$, the category $k[G]$-mod of finite-dimensional $k[G]$-modules is semisimple and hence satisfies Condition P by Lemma 3.1.

On the other hand, if G is a p-group the trivial $k[G]$-module is up to isomorphism the unique simple $k[G]$-module (see e.g. the corollary to Proposition 26 in Section 8.3 of [Se]). The projective cover of the trivial $k[G]$-module is $P := k[G]$ (see [Se, Sec. 15.6]). The image of P in Gr_k is $[P] = |G| \cdot [k] = 0$, as p divides $|G|$. By the previous observations, every projective $k[G]$-module is isomorphic to a direct sum of P's, and so $k[G]$-mod does not satisfy Condition P.

We can now state our first main result:

Theorem 3.4. *Let $\mathcal{C}$ be a factorisable and pivotal finite tensor category over k. If $\mathcal{C}$ satisfies Condition P, it contains a simple projective object.*

The proof relies on a series of lemmas and will be given at the end of this section. The idea is very simple: we show that in the absence of simple projectives, the injective algebra homomorphism σ from Corollary 2.12 would map the class $[P]$ of a projective object to a nilpotent natural endomorphism, which is a contradiction to Condition P.

Remark 3.5.

1. Theorem 3.4 is a generalisation of a result by Cohen and Westreich [CW, Cor. 3.6]. There, the authors show that a factorisable ribbon Hopf algebra H over an algebraically closed field of characteristic zero has an irreducible projective module. The proof strategy in [CW] is different from ours: in [CW] it is observed that Hopf algebras with the above properties are also symmetric algebras (see Section 4 for the definition) and that fact is used to show existence of a simple projective module. Our approach is in some sense opposite: we first show that there exists a simple projective module and then in Section 4 use the theory of modified traces of [GKPM1] to deduce the existence of certain symmetric algebras. In this sense, applying Theorem 3.4 to $\mathcal{C} = \mathrm{Rep}(H)$ gives an alternative proof of the result in [CW] (in addition to having the benefit of not being restricted to characteristic zero).

2. In characteristic p, a category $\mathcal{C}$ as in Theorem 3.4 may or may not satisfy Condition P. To see this, we use the observations in Remark 3.3. Let A be a finite abelian group. The Drinfeld double $D(k[A])$ is a factorisable Hopf algebra, and hence $D(k[A])$-mod is a factorisable and pivotal finite tensor category over k. But as algebras, $D(k[A]) \cong k(A) \otimes_k k[A]$, and $k(A) = k^{|A|}$ is semisimple.

   Suppose that $\mathrm{char}(k) = p$ does not divide $|A|$. Then $D(k[A])$-mod is semisimple and hence satisfies Condition P (Lemma 3.1).

   On the other hand, if A is a p-group, the indecomposable projective modules in $D(k[A])$-mod are isomorphic to a simple $k(A)$-module tensored with $k[A]$. As in Remark 3.3, each of these have zero image in $\mathrm{Gr}_k(\mathcal{C})$.

3. The example in part 2 (combined with Remark 3.3) also shows that for $\mathrm{char}(k) = p$ and A a finite abelian p-group, $D(k[A])$-mod does not contain a simple projective object. Thus we cannot drop Condition P from Theorem 3.4.

We now turn to the proof of Theorem 3.4. We first gather the tools to see that $\sigma([P])$ is nilpotent for projective P unless there is a simple projective object.

Lemma 3.6. *Let $\mathcal{A}$ be an abelian category and let η be a natural endomorphism of the identity functor. If η is zero on all simple objects, then it is nilpotent on all objects of finite length.*

Proof. We use induction on the length of an object. Given an object B of finite length > 1, choose a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that A, C have smaller length. Then

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_C} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array} \tag{3.2}$$

commutes. By induction hypothesis, η_A and η_C are nilpotent. Hence there is $m > 0$ such that

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\eta_B)^m} & & \downarrow{\scriptstyle 0} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array} \tag{3.3}$$

commutes. Then $g \circ (\eta_B)^m = 0$ and $(\eta_B)^m \circ f = 0$, and since the image of f is the kernel of g we have the inclusions $\mathrm{im}(\eta_B)^m \subset \mathrm{im} f \subset \ker(\eta_B)^m$. This shows that $(\eta_B)^{2m} = 0$. □

Corollary 3.7. *Let $\mathcal{A}$ be a finite abelian category over some field and let $\eta \in \mathrm{End}(Id_{\mathcal{A}})$. If η is zero on all simple objects, then η is nilpotent, i.e. there is $m > 0$ such that $\eta^m = 0$.*

Proof. By finiteness, $\mathcal{A}$ contains a projective generator G (of finite length). By Lemma 3.6, $(\eta_G)^m = 0$ for some m. Since η_G determines η, it follows that $\eta^m = 0$. □

Lemma 3.8. *Let $\mathcal{C}$ be a pivotal finite braided tensor category over some field and let $P, U \in \mathcal{C}$ with P projective and U non-projective and simple. Then $\sigma([P])_U = 0$.*

Proof. By (2.11), $\sigma([P])_U$ is a composition of maps $U \to P^* \otimes (U \otimes P) \to U$. Since $P^* \otimes (U \otimes P)$ is projective, the map $P^* \otimes (U \otimes P) \to U$ factors as $P^* \otimes (U \otimes P) \to P_U \to U$, for P_U the projective cover of U. Thus $\sigma([P])_U$ can be written as a composition $U \to P_U \to U$. By assumption $P_U \ncong U$, and so this composition is zero. □

Combining the above results, the proof of Theorem 3.4 is now a straightforward consequence of Corollary 2.12:

Proof of Theorem 3.4. [3] Suppose that $\mathcal{C}$ does not contain a simple projective object. By Condition P, we can find a projective object $P \in \mathcal{C}$ such that $[P]^m \neq 0$ in $\mathrm{Gr}_k(\mathcal{C})$ for all $m > 0$.

By Corollary 2.12, $\sigma : \mathrm{Gr}_k(\mathcal{C}) \to \mathrm{End}(Id_{\mathcal{C}})$ is an injective algebra homomorphism. In particular, since $[P]^m \neq 0$ we find that for each $m > 0$, $\sigma([P])^m = \sigma([P]^m) \neq 0$. But by Lemma 3.8, the natural transformation $\sigma([P])$ is zero on all non-projective simples. By assumption, all simples in $\mathcal{C}$ are non-projective, and so Corollary 3.7 implies that $\sigma([P])$ is nilpotent, which is a contradiction.

Hence $\mathcal{C}$ must contain a simple projective object. □

4 Non-degenerate trace on the projective ideal

In this section we will review some aspects of the theory of modified traces from [GKPM1, GPMV, GKPM2]. In particular, we will see that the existence of a simple projective object established in Theorem 3.4 guarantees the existence of a unique-up-to-scalars non-zero modified trace on the projective ideal, and that this trace induces non-degenerate pairings on Hom-spaces.

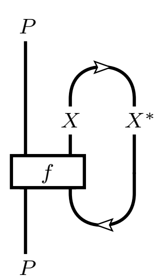
Let $\mathcal{C}$ be a pivotal finite tensor category over k. The full subcategory $\mathcal{P}roj(\mathcal{C})$ of all projective objects in $\mathcal{C}$ is a tensor ideal. Following [GPMV], a *modified right trace* on $\mathcal{P}roj(\mathcal{C})$ is a family of k-linear functions $\{t_P : \mathrm{End}(P) \to k \,|\, P \in \mathcal{P}roj(\mathcal{C})\}$ such that

1. *(partial trace)* For all $P \in Proj(\mathcal{C})$, $X \in \mathcal{C}$ and $f \in \mathrm{End}(P \otimes X)$ we have

$$t_{P\otimes X}(f) \;=\; t_P(\,tr^r_X(f)\,) \tag{4.1}$$

where tr^r_X denotes the partial right trace

$$tr^r_X(f) \;:=\; \big[\,P \xrightarrow{\sim} P\mathbf{1} \xrightarrow{id\otimes \mathrm{coev}_X} P(XX^*) \xrightarrow{\sim} (PX)X^* \xrightarrow{f\otimes id} (PX)X^* \xrightarrow{\sim} P(XX^*) \xrightarrow{id\otimes \widetilde{\mathrm{ev}}_X} P\mathbf{1} \xrightarrow{\sim} P\,\big]$$

$=$  . (4.2)

2. *(cyclicity)* For all $P, P' \in \mathcal{P}roj(\mathcal{C})$, $f : P \to P'$, $g : P' \to P$ we have

$$t_{P'}(f \circ g) = t_P(g \circ f)\ . \tag{4.3}$$

[3] We thank Victor Ostrik for suggesting a simplification of our first version of the proof.

Given a modified right trace, one can define a family of pairings by setting, for all $M \in \mathcal{C}$, $P \in \mathcal{P}roj(\mathcal{C})$,

$$\mathcal{C}(M,P) \times \mathcal{C}(P,M) \to k \quad , \quad (f,g) \mapsto t_P(f \circ g) \ . \tag{4.4}$$

Note that the cyclicity property (4.3) only applies when also M is projective.

Remark 4.1. There are also natural notions of left and left-right (or two-sided) modified traces for a pivotal category, see [GPMV] for details. For $\mathcal{C}$ in addition ribbon, each modified right trace on $\mathcal{P}roj(\mathcal{C})$ is also a two-sided modified trace [GKPM1, Thm. 3.3.1]. In this case, one can construct an isotopy invariant of ribbon graphs in $\mathbb{R}^3$, where at least one strand is coloured by a projective object P of $\mathcal{C}$, by cutting the strand and computing t_P of the corresponding endomorphism of P, see [GPMT, Thm. 3]. (There, the modified trace was used implicitly as the modified dimension for simple projectives.)

In this paper we will only use modified right traces, and we will refer to these just as "modified traces".

Proposition 4.2. *Let $\mathcal{C}$ be a unimodular pivotal finite tensor category over k. Suppose that $\mathcal{C}$ contains a simple projective object.*

1. *There exists a nonzero modified trace on $\mathcal{P}roj(\mathcal{C})$, and this modified trace is unique up to scalar multiples.*
2. *For any choice of non-zero modified trace t on $\mathcal{P}roj(\mathcal{C})$ and for all $M \in \mathcal{C}$, $P \in \mathcal{P}roj(\mathcal{C})$, the pairing (4.4) is non-degenerate.*

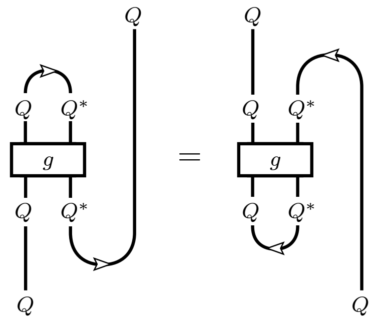
The first part of the proposition is proved in [GKPM2, Cor. 3.2.1][4]. We will review the argument below as we hope it to be helpful to have the more general construction of [GPMV, GKPM2] specialised to the present setting, and since we will need the explicit construction of the modified trace below anyway. The second part of the proposition is proved in [CGPM, Prop. 6.6] for the Hopf algebra $\overline{U}_q^H sl(2)$, but the same proof works in general as we also review below.

Before turning to the proof of Proposition 4.2, we need some preparation. For the rest of this section, we fix:

- $\mathcal{C}$: a unimodular pivotal finite tensor category over k,
- Q: a simple projective object in $\mathcal{C}$ (in particular we assume that such a Q exists).

[4] In [GKPM2, Sect. 3.2] it is assumed that every indecomposable object of $\mathcal{C}$ is absolutely indecomposable (see [GKPM2] for definitions). This condition does typically not hold (e.g. in the example in Section 9). However, the arguments in [GKPM2, Sect. 3.2] actually only require this condition on $\mathcal{P}roj(\mathcal{C})$, where it holds for any finite tensor category over an algebraically closed field.

Lemma 4.3 ([GKPM2, Thm. 3.1.3 & Sec. 3.2]). *For all $g \in \mathrm{End}(Q \otimes Q^*)$ we have*



. (4.5)

Proof. Write $Q \otimes Q^* = \bigoplus_{U \in \mathrm{Irr}(\mathcal{C})} P_U^{\oplus m_U}$ and let $i_U : P_U^{\oplus m_U} \to Q \otimes Q^*$ and $p_U : Q \otimes Q^* \to P_U^{\oplus m_U}$ be the corresponding embedding and projection maps. The resulting idempotent will be denoted by $e_U = i_U \circ p_U \in \mathrm{End}(Q \otimes Q^*)$. Since $\mathcal{C}(Q \otimes Q^*, \mathbf{1}) \cong \mathcal{C}(Q, Q)$ is one-dimensional (Q being simple and k algebraically closed), we have $m_{\mathbf{1}} = 1$.

Since $P_{\mathbf{1}}$ is indecomposable, there is an $\alpha \in k$ and a nilpotent element $n \in \mathrm{End}(P_{\mathbf{1}})$ such that (using again algebraically closedness of k)[5]

$$p_{\mathbf{1}} \circ g \circ i_{\mathbf{1}} \;=\; \alpha \cdot id_{P_{\mathbf{1}}} + n \; . \tag{4.6}$$

For all $u : \mathbf{1} \to P_{\mathbf{1}}$ and $v : P_{\mathbf{1}} \to \mathbf{1}$ we have

$$n \circ u = 0 \qquad \text{and} \qquad v \circ n = 0 \; . \tag{4.7}$$

To see $n \circ u = 0$, first note that by unimodularity (i.e. since $(P_{\mathbf{1}})^* \cong P_{\mathbf{1}}$) $\mathcal{C}(\mathbf{1}, P_{\mathbf{1}})$ is one-dimensional. Suppose $u \neq 0$ and hence u forms a basis of $\mathcal{C}(\mathbf{1}, P_{\mathbf{1}})$. Then there is $\lambda \in k$ such that $n \circ u = \lambda \cdot u$. But n is nilpotent, say $n^m = 0$, and so applying n on both sides of the latter equality m times we get $\lambda^m = 0$ and therefore $\lambda = 0$. For v the argument is analogous.

Next insert the identity $g = \sum_{U,V} e_U \circ g \circ e_V$ into (4.5). Since for $U \ncong \mathbf{1}$, $\mathcal{C}(P_U, \mathbf{1}) = 0$ and $\mathcal{C}(\mathbf{1}, P_U) = 0$ (using unimodularity), only the term with $U = V = \mathbf{1}$ will contribute on either side of (4.5).

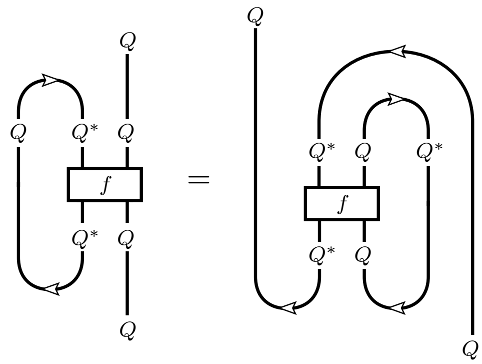
The claim of the lemma now follows from the observation that taking $u = p_{\mathbf{1}} \circ \mathrm{coev}_Q$ and $v = \widetilde{\mathrm{ev}}_Q \circ i_{\mathbf{1}}$ we have by (4.6) and (4.7)

$$\widetilde{\mathrm{ev}}_Q \circ e_{\mathbf{1}} \circ g \circ e_{\mathbf{1}} = \alpha \cdot \widetilde{\mathrm{ev}}_Q \quad , \quad e_{\mathbf{1}} \circ g \circ e_{\mathbf{1}} \circ \mathrm{coev}_Q = \alpha \cdot \mathrm{coev}_Q \; , \tag{4.8}$$

where we used that $\widetilde{\mathrm{ev}}_Q \circ e_{\mathbf{1}} = \widetilde{\mathrm{ev}}_Q$ and $e_{\mathbf{1}} \circ \mathrm{coev}_Q = \mathrm{coev}_Q$ (because $(P_{\mathbf{1}})^* \cong P_{\mathbf{1}}$). Both the left and right hand sides of (4.5) are therefore equal to $\alpha \, id_Q$. □

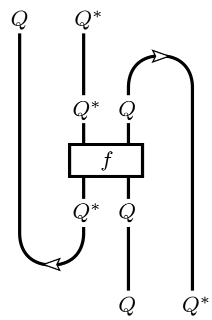
[5] Since $P_{\mathbf{1}}$ is indecomposable, by Fitting's Lemma an element $x \in \mathrm{End}(P_{\mathbf{1}})$ is either nilpotent or invertible. As k is algebraically closed, $x - \alpha \cdot id$ is not invertible for some $\alpha \in k$ (as the k-linear map $x \circ (-)$ on $\mathrm{End}(P_{\mathbf{1}})$ has an eigenvector), hence it is nilpotent, and therefore $x = \alpha \cdot id + n$, as we state.

The above lemma will be used in the following form: for all $f \in \mathrm{End}(Q^* \otimes Q)$,



. (4.9)

This is a consequence of (4.5) upon substituting

$g =$ 

. (4.10)

Let us fix a non-zero linear function for the given simple projective Q,

$$t_Q \ : \ \mathrm{End}(Q) \to k \ . \tag{4.11}$$

Since $\mathrm{End}(Q) = k\, id_Q$, such a function is unique up to a constant, and it is uniquely determined by its value $t_Q(id_Q) \in k^\times$.

Let $P \in \mathcal{P}roj(\mathcal{C})$ and choose an object $X \in \mathcal{C}$ such that there is a surjection $p : Q \otimes X \to P$. Such an X always exists, for example $X = Q^* \otimes P$. As P is projective, there is a morphism $i : P \to Q \otimes X$ such that $p \circ i = id_P$. That is, i, p realise P as a direct summand of $Q \otimes X$. Consider the function

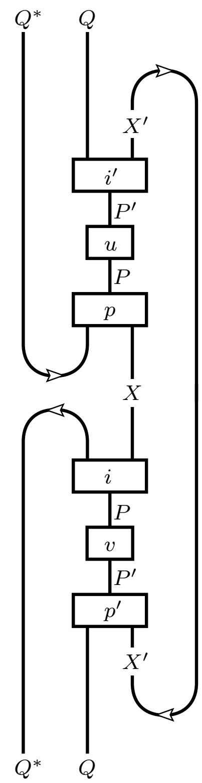
$$t_P : \mathrm{End}(P) \to k \quad , \quad h \mapsto t_Q\big(tr^r_X(i \circ h \circ p)\big) \ . \tag{4.12}$$

The next lemma shows in particular that this notation is indeed consistent with (4.11) when setting $P = Q$ (in the lemma, take $X = \mathbf{1}$ and i, p (inverse) unit isomorphisms).

Lemma 4.4. *The function t_P in (4.12) is independent of the choice of X and of $p : Q \otimes X \to P$, $i : P \to Q \otimes X$.*

Proof. This proof is taken from [GPMV, Sec. 4.5]. The argument is more general than needed for the statement of the lemma, but we can reuse it later to show cyclicity.

Let $P' \in \mathcal{P}roj(\mathcal{C})$ and pick X', $p' : Q \otimes X' \to P'$, $i' : P' \to Q \otimes X'$. For $u : P \to P'$ and $v : P' \to P$ arbitrary, consider the morphism $f \in \mathrm{End}(Q^* \otimes Q)$ given by

$f \; =$  . (4.13)

It is easy to check that inserting f into (4.9) results in the identity

$$tr^r_{X'}(i' \circ u \circ v \circ p') = tr^r_X(i \circ v \circ u \circ p) \;, \tag{4.14}$$

where for RHS we used cyclicity of the categorical trace, the zig-zag axiom and that $p' \circ i' = id_{P'}$. For $P = P'$ and $u = h$, $v = id_P$, this shows the statement of the lemma. □

Proof of Proposition 4.2.

Part 1. (following [GPMV, Sec. 4.5])

Existence: By Lemma 4.4 we have a family of functions $\big(t_P : \mathrm{End}(P) \to k\big)_{P \in \mathcal{P}roj(\mathcal{C})}$. This family is not identically zero since by the definition in (4.11), t_Q is not zero. By (4.14) in the proof of Lemma 4.4, the t_P satisfy the cyclicity requirement (4.3).

For the partial trace property (4.1) choose $Y \in \mathcal{C}$ such that there are $p : Q \otimes Y \to P$, $i : P \to Q \otimes Y$, $p \circ i = id_P$, in order to compute t_P from the definition in (4.12) with X replaced by Y. Then fix

$$p' = \big[\, Q(YX) \xrightarrow{\sim} (QY)X \xrightarrow{p \otimes id_X} PX \,\big] \quad , \quad i' = \big[\, PX \xrightarrow{i \otimes id_X} (QY)X \xrightarrow{\sim} Q(YX) \,\big] \tag{4.15}$$

to compute $t_{P \otimes X}$ from (4.12). A short calculation shows (4.1).

Uniqueness: First note that for the modified trace t_P in (4.12) (or in fact for any modified trace) we have, for $f \in \mathrm{End}(Q \otimes X)$,

$$tr^r_X(f) = \frac{t_{Q\otimes X}(f)}{t_Q(id_Q)} \cdot id_Q \ . \tag{4.16}$$

To see this, apply t_Q to both sides. Let now $\big(t'_P : \mathrm{End}(P) \to k\big)_{P \in \mathcal{P}roj(\mathcal{C})}$ be a modified trace. Let $P \in \mathcal{P}roj(\mathcal{C})$ and $h \in \mathrm{End}(P)$ be given. Pick X, $p : Q \otimes X \to P$, $i : P \to Q \otimes X$ such that $p \circ i = id_P$ and compute

$$\begin{aligned} t'_P(h) \ &= \ t'_P(h \circ p \circ i) \overset{\text{cycl.}}{=} t'_{Q\otimes X}(i \circ h \circ p) \overset{\text{part.tr.}}{=} t'_Q(tr^r_X(i \circ h \circ p)) \\ &\overset{(4.16)}{=} \frac{t_{Q\otimes X}(i \circ h \circ p)}{t_Q(id_Q)} \cdot t'_Q(id_Q) \overset{\text{cycl.}}{=} \frac{t'_Q(id_Q)}{t_Q(id_Q)} \cdot t_P(h) \ . \end{aligned} \tag{4.17}$$

Part 2. (following the proof of [CGPM, Prop. 6.6])
We will show non-degeneracy in the first argument of the pairing (4.4). Non-degeneracy in the second argument can be seen analogously.

Let $M \in \mathcal{C}$, $P \in \mathcal{P}roj(\mathcal{C})$ and $f \in \mathcal{C}(M,P)$, $f \neq 0$, be given. We need to show that there is a $g \in \mathcal{C}(P,M)$ such that $t_P(f \circ g) \neq 0$. Pick X, $p : Q \otimes X \to P$, $i : P \to Q \otimes X$ such that $p \circ i = id_P$. Then also $i \circ f \neq 0$. Define

$$u \ := \ \big[\, MX^* \xrightarrow{(i\circ f)\otimes id} (QX)X^* \xrightarrow{\sim} Q(XX^*) \xrightarrow{id\otimes \widetilde{\mathrm{ev}}_X} Q\mathbf{1} \xrightarrow{\sim} Q \,\big] \ . \tag{4.18}$$

Using the zig-zag identity, we can recover $i \circ f$ from u as

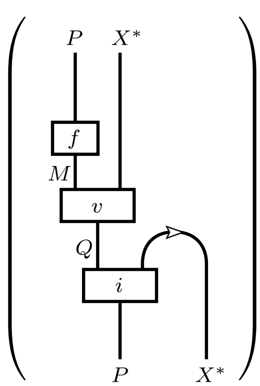
$$i \circ f \ = \ \big[\, M \xrightarrow{\sim} M\mathbf{1} \xrightarrow{id\otimes \widetilde{\mathrm{coev}}_X} M(X^*X) \xrightarrow{\sim} (MX^*)X \xrightarrow{u\otimes id_X} QX \,\big] \ . \tag{4.19}$$

In particular, since $i \circ f \neq 0$, we must have $u \neq 0$. Since Q is simple, u is surjective. Since Q is projective, there is $v : Q \to M \otimes X^*$ such that $u \circ v = id_Q$. Define

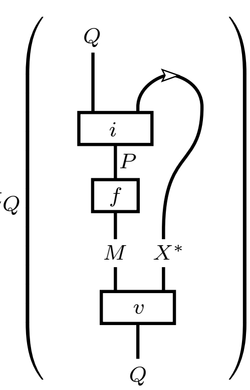
$$g \ := \ \big[\, P \xrightarrow{i} QX \xrightarrow{v\otimes id} (MX^*)X \xrightarrow{\sim} M(X^*X) \xrightarrow{id\otimes \mathrm{ev}_X} M\mathbf{1} \xrightarrow{\sim} M \,\big] \ . \tag{4.20}$$

Then

$t_P(f \circ g) \overset{\text{part.tr.}}{=} t_{P\otimes X^*}$



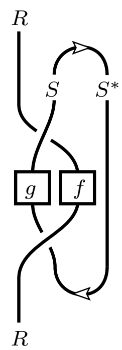
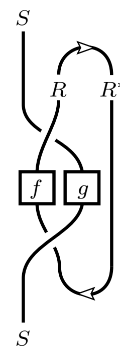
$\overset{\text{cycl}}{=} t_Q$



$$= \ t_Q(u \circ v) \ = \ t_Q(id_Q) \ \neq \ 0 \ , \tag{4.21}$$

where for the first equality we used the partial trace property (4.1) on RHS and then the pivotal structure in (2.4). □

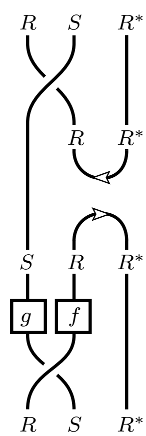
Lemma 4.5. *Suppose $\mathcal{C}$ is in addition braided. Let t be a modified trace on $\mathcal{C}$ and let $R, S \in \mathcal{P}roj(\mathcal{C})$ and $f \in \mathrm{End}(R)$, $g \in \mathrm{End}(S)$. Then*

t_R () $=$ t_S () . (4.22)

Proof. One computes

$$\text{RHS of (4.22)} \overset{(*)}{=} t_{R(SR^*)}(h) \overset{\text{part tr.}}{=} t_R(tr^r_{SR^*}(h)) \overset{(**)}{=} \text{LHS of (4.22)}\ , \tag{4.23}$$

where

h $=$  . (4.24)

Step (*) follows from cyclicity of the modified trace, as well as naturality of the braiding. In step (**) we used that the partial trace over $S \otimes R^*$ amounts to closing the double string, see (4.2). In the last equality, to straighten the line coloured by R we also used (2.4) and the property of the pivotal structure that $\delta_R^* = (\delta_{R^*})^{-1}$. □

Remark 4.6.

1. The result of Proposition 4.2 is remarkable, because the categorical trace defined on all of $\mathcal{C}$ via the pivotal structure vanishes on $\mathcal{P}roj(\mathcal{C})$ unless $\mathcal{C}$ is semisimple. Indeed, $\mathcal{C}$ is semisimple iff $P_{\mathbf{1}} = \mathbf{1}$, which in turn is equivalent to the existence of f, g such that $\mathbf{1} \xrightarrow{f} P_{\mathbf{1}} \xrightarrow{g} \mathbf{1}$ is non-zero. But for any projective object P, the categorical trace

$$\mathbf{1} \xrightarrow{\mathrm{coev}_P} P \otimes P^* \xrightarrow{f \otimes id} P \otimes P^* \xrightarrow{\widetilde{\mathrm{ev}}_P} \mathbf{1}\ , \tag{4.25}$$

factors through $P_{\mathbf{1}}$ in this way.

2. For each $P \in \mathcal{P}roj(\mathcal{C})$, t_P turns the k-algebra $\mathrm{End}(P)$ into a symmetric Frobenius algebra, or a symmetric algebra for short. We will review some aspects of symmetric algebras and their categories of modules in Section 5.

Since a factorisable finite tensor category is automatically unimodular [ENO, Prop. 4.5], we state the following consequence of Theorem 3.4 and Proposition 4.2 for later use:

Corollary 4.7. *A factorisable and pivotal finite tensor category $\mathcal{C}$ over k which satisfies Condition P admits an up-to-scalar unique modified trace on $\mathcal{P}roj(\mathcal{C})$. For each non-zero modified trace, the pairings (4.4) are non-degenerate.*

This result generalises the existence of a modified trace for factorisable Hopf algebras (over an algebraically closed field of characteristic zero) proven in [GKPM2, Thm. 4.3.1].

Remark 4.8. The converse of Corollary 4.7 does not hold, i.e. it is not true that a pivotal finite tensor category $\mathcal{C}$ over k which satisfies Condition P and admits a non-zero modified trace on $\mathcal{P}roj(\mathcal{C})$ can be equipped with a braiding that makes it factorisable. The simplest example is the category of super-vector spaces, which only admits symmetric braidings. A non-semisimple example is provided by the restricted quantum group $\overline{U}_q sl(2)$. This is a unimodular Hopf algebra which has a simple projective module but whose category of modules does not admit a braiding [FGST, KS, GR1].

5 Symmetric algebras and ideals in End($Id_{\mathcal{C}}$)

In this section we will collect some facts about symmetric algebras following [Br, CW] and translate them into categorical statements. In the end we give implications for categories with modified trace as in the previous section.

For a k-algebra A, denote by

$$C(A) \;=\; \left\{ \varphi : A \to k \,\middle|\, \varphi(ab) = \varphi(ba) \text{ for all } a, b \in A \right\} \tag{5.1}$$

the space of *central forms* on A. A *symmetric algebra* A over k (aka a *symmetric Frobenius algebra*) is a finite-dimensional k-algebra together with a central form ε such that the induced pairing $(a, b) \mapsto \varepsilon(ab)$ is non-degenerate. In this case the map

$$\zeta : Z(A) \to C(A) \quad , \quad z \mapsto \varepsilon(z \cdot (-)) \tag{5.2}$$

is an isomorphism, and the invertible elements of $Z(A)$ are precisely mapped to those elements of $C(A)$ that induce a non-degenerate pairing (see e.g. [Br, Lem. 2.5]).

Let (A, ε) be a symmetric algebra and denote by $\gamma \in A \otimes A$, $\gamma = \sum_{(\gamma)} \gamma' \otimes \gamma''$, the copairing, that is, for all $a \in A$ we have $a = \sum_{(\gamma)} \varepsilon(a\gamma')\gamma'' = \sum_{(\gamma)} \gamma' \varepsilon(\gamma'' a)$. We recall that the copairing is unique and that, since ε is central, the copairing is symmetric. Define the map

$$\tau : A \to Z(A) \quad , \quad a \mapsto \sum_{(\gamma)} \gamma' a \gamma'' \; . \tag{5.3}$$

It is easy to check that the image of τ does indeed lie in the centre of A (see e.g. [Br, Sec. 3.A]). The argument uses that the copairing γ of a non-degenerate invariant pairing satisfies $\sum_{(\gamma)}(a\gamma')\otimes\gamma'' = \sum_{(\gamma)}\gamma'\otimes(\gamma''a)$ for all $a\in A$.

We will need the following chains of inclusions in $C(A)$ and $Z(A)$, compatible with the map ζ from (5.2) [CW]:

$$\begin{array}{ccccc} Z(A) & \supset & \mathrm{Rey}(A) & \supset & \mathrm{Hig}(A) \\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\zeta} \\ C(A) & \supset & R(A) & \supset & I(A) \end{array} \tag{5.4}$$

The individual subsets are defined as follows:

- $\mathrm{Rey}(A) = \mathrm{ann}_{Z(A)}(\mathrm{Jac}(A))$ is the *Reynolds ideal.* (Symmetry of A is not required for this definition.) Here, $\mathrm{Jac}(A)$ is the Jacobson radical of A. $\mathrm{Rey}(A)$ is an ideal in $Z(A)$.

- $\mathrm{Hig}(A) = \mathrm{im}(\tau)$ is the *Higman ideal* or *projective centre*[6]. It is an ideal in $Z(A)$ as $z\tau(a) = \tau(za)$ for $z\in Z(A)$, and by [HHKM, Lem. 4.1][7] it is contained in $\mathrm{Rey}(A)$. Note that even though τ depends on the choice of a non-degenerate central form ε, $\mathrm{im}(\tau)$ does not. Indeed, any two ε, ε' would be related by an invertible $z\in Z(A)$ as $\varepsilon'(a) = \varepsilon(za)$, and one checks that $\tau'(a) = \tau(z^{-1}a)$.

- $R(A) = \mathrm{span}_k\{\,\chi_M \,|\, M \text{ right } A\text{-module}\}$, where $\chi_M(a) = \mathrm{tr}_M(a)$, the trace in M over the linear map given by acting with $a\in A$. That $\zeta(\mathrm{Rey}(A)) = R(A)$ is shown in [Lo, Thm. 1.6] (for finite-dimensional not necessarily symmetric algebras).

- $I(A) = \mathrm{span}_k\{\,\chi_P \,|\, P \text{ projective right } A\text{-module}\}$. In [CW, Prop. 2.1] it is shown that $\zeta(\mathrm{Hig}(A)) = I(A)$.

Lemma 5.1. *For an algebra A over some field, $Z(A) = \mathrm{Rey}(A)$ if and only if A is semisimple.*

Proof. If A is semisimple, $\mathrm{Jac}(A) = 0$ and hence $Z(A) = \mathrm{Rey}(A)$. Conversely, if $Z(A) = \mathrm{Rey}(A)$ then $1\in\mathrm{Rey}(A)$ and since $\mathrm{Jac}(A)$ annihilates $\mathrm{Rey}(A)$ we must have $\mathrm{Jac}(A) = 0$. □

The *Cartan matrix* of a finite-dimensional algebra A over some field is the $\mathrm{Irr}(\text{mod-}A)\times\mathrm{Irr}(\text{mod-}A)$ matrix $\mathsf{C}(A)$ with entries

$$\mathsf{C}(A)_{U,V} \;=\; [P_U : V]\;, \tag{5.5}$$

[6] The name "projective centre" can be motivated from the observation that $\mathrm{Hig}(A)$ can be described as all elements in $Z(A)$ such that the corresponding endomorphism of the A-A-bimodule A factors through a projective A-A-bimodule [LZZ, Prop. 2.3 & 2.4].

[7] In [HHKM, Sec. 4] the underlying field is assumed to be algebraically closed, but this is not used in the proof of Lemma 4.1 in [HHKM].

that is, the multiplicity of the simple A-module V in the composition series of the projective cover P_U of U.

Let e_U, $U \in \mathrm{Irr}(\text{mod-}A)$ be a choice of primitive idempotents such that $e_U A \cong P_U$ as right A-modules. If the underlying field is given by k, which we assumed to be algebraically closed in the outset of the paper, the endomorphism spaces of simple A-modules are one-dimensional and we can write

$$\mathsf{C}(A)_{U,V} \;=\; \dim_k \mathrm{Hom}_A(P_V, P_U) \;=\; \dim_k(\, e_U A e_V \,)\;. \tag{5.6}$$

The last equality follows by noting that every right A-module map $f : e_V A \to e_U A$ is given by left multiplication with an element $e_U a e_V$, $a \in A$. Namely, in one direction we set $f \mapsto f(e_V)e_V \in e_U A e_V$, while in the other direction $e_U a e_V \mapsto e_U a e_V \cdot (-)$ which is in $\mathrm{Hom}_A(e_V A, e_U A)$, and one checks that both the maps are inverses of each other.

Proposition 5.2. *Let A be a symmetric algebra over k. Then:*

1. *The Cartan matrix of A is symmetric, $\mathsf{C}(A)_{U,V} = \mathsf{C}(A)_{V,U}$.*

2. *Let $n = |\,\mathrm{Irr}(\text{mod-}A)|$ and let $\widehat{\mathsf{C}} \in \mathrm{Mat}_n(k)$ be the image of $\mathsf{C}(A)$ under the canonical homomorphism $\mathrm{Mat}_n(\mathbb{Z}) \to \mathrm{Mat}_n(k)$. We have*

$$\mathrm{rank}\big(\widehat{\mathsf{C}}\big) \;=\; \dim_k \mathrm{Hig}(A)\;. \tag{5.7}$$

Proof. For part 1 one checks that the non-degenerate pairing on A descends to a non-degenerate pairing between $e_U A e_V$ and $e_V A e_U$. Indeed, for every $a \in A$ such that $e_U a e_V \neq 0$ there exists $b \in A$ such that $(e_U a e_V, b) \neq 0$. But then also $(e_U a e_V, e_V b e_U) \neq 0$ since $\varepsilon(e_U a e_V b) = \varepsilon(e_U e_U a e_V e_V b) = \varepsilon(e_U a e_V e_V b e_U)$. Hence, $\dim_k(e_U A e_V) = \dim_k(e_V A e_U)$. The more surprising part 2 is proved in [LZZ, Cor. 2.7]. □

Note that if k has characteristic 0, then $\mathrm{rank}(\tilde{\mathsf{C}})$ is just the rank of $\mathsf{C}(A)$ over $\mathbb{Q}$.

We now give a categorical formulation of the inclusions $Z(A) \supset \mathrm{Rey}(A) \supset \mathrm{Hig}(A)$ in (5.4), which will be applicable in particular in the presence of a modified trace as in Section 4. For the rest of this section, let us fix

- $\mathcal{A}$: a finite abelian category over k,
- G: a projective generator of $\mathcal{A}$,
- $E = \mathrm{End}(G)$ the k-algebra of endomorphisms of G.

Recall the Hom-tensor adjoint equivalence

$$\mathcal{A} \underset{-\otimes_E G}{\overset{\mathcal{A}(G,-)}{\rightleftarrows}} \text{mod-}E \tag{5.8}$$

of k-linear categories between $\mathcal{A}$ and the category mod-E of finite-dimensional right E-modules (see e.g. [EGNO, Sect. 1.8]).

It is known that being symmetric is a Morita-invariant property of an algebra, as is the chain of ideals $Z(E) \supset \mathrm{Rey}(E) \supset \mathrm{Hig}(E)$ (see e.g. [HHKM, Br, LZZ] for related results). We can therefore use the equivalence in (5.8) and the ideals in (5.4) for the choice $A = E$ and for a given projective generator G to define a chain of ideals in $\mathrm{End}(Id_{\mathcal{A}})$ as:

$$\begin{array}{ccccc} \mathrm{End}(Id_{\mathcal{A}}) & \supset & \mathrm{Rey}(\mathcal{A}) & \supset & \mathrm{Hig}(\mathcal{A}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ Z(E) & & \mathrm{Rey}(E) & & \mathrm{Hig}(E) \end{array} \tag{5.9}$$

Rather than showing directly that the above definition is independent of the choice of G, we will now give a different definition of $\mathrm{Rey}(\mathcal{A})$ and $\mathrm{Hig}(\mathcal{A})$ which does not require a choice of a projective generator. We show in Propositions 5.3 and 5.8 that this definition agrees with the one above.

Rey($\mathcal{A}$)

For each $P \in \mathcal{P}roj(\mathcal{A})$ define the subspace $J_P \subset \mathrm{End}(P)$ as follows:

$$J_P \;=\; \big\{\, f : P \to P \,\big|\, \forall U \in \mathcal{A} \text{ simple, } u \in \mathcal{A}(P,U) : u \circ f = 0 \,\big\}\ . \tag{5.10}$$

By definition, the Jacobson radical $\mathrm{Jac}(R)$ of a ring R consists of all $r \in R$ which act as zero on all simple R-modules. Since $G \in \mathcal{A}$ is a projective generator, $\mathcal{A}(G,U)$, where U runs over all simple $U \in \mathcal{A}$, gives all simple right E-modules up to isomorphism (by the equivalence (5.8)). From (5.10) we therefore get

$$J_G \;=\; \mathrm{Jac}(E)\ . \tag{5.11}$$

In the case $\mathcal{A} = \text{mod-}A$ we can take $G = A_A$, so that $\mathrm{End}(G) = A$ and hence $J_A = \mathrm{Jac}(A)$.

Using the subspaces (5.10), we define

$$\mathrm{Rey}(\mathcal{A}) \;=\; \big\{\, \eta \in \mathrm{End}(Id_{\mathcal{A}}) \,\big|\, \forall\, P \in \mathcal{P}roj(\mathcal{A}), f \in J_P : \eta_P \circ f = 0 \,\big\}\ . \tag{5.12}$$

From this definition it is immediate that $\mathrm{Rey}(\mathcal{A})$ is an ideal.

Recall that for a projective generator $G \in \mathcal{A}$, the map

$$\xi : \mathrm{End}(Id_{\mathcal{A}}) \to Z(E) \quad , \quad \eta \mapsto \eta_G \tag{5.13}$$

is an algebra isomorphism (that η_G is central in E follows from the naturality of η; that (5.13) is a bijection is clear in the case $\mathcal{A} = \text{mod-}E$, for general $\mathcal{A}$ use the equivalence (5.8)). We can use this to relate $\mathrm{Rey}(\mathcal{A})$ to the previous definition in terms of algebras:

Proposition 5.3. *Let A be a finite-dimensional k-algebra. Then ξ in (5.13) restricts to an isomorphism of algebras $\xi : \mathrm{Rey}(\text{mod-}A) \to \mathrm{Rey}(A)$.*

Proof. We set $\mathcal{A} = \text{mod-}A$ and choose the projective generator $G = A_A$ and thus $E = A$. We have to show that $\xi(\text{Rey}(\mathcal{A})) = \text{Rey}(E)$.

"$\subset$": Let $\eta \in \text{Rey}(\mathcal{A})$. Then $\xi(\eta) = \eta_G$. By definition of $\text{Rey}(\mathcal{A})$ we have in particular that for all $f \in J_G = \text{Jac}(E)$ (recall (5.11)), $\eta_G \circ f = 0$. Thus η_G annihilates $\text{Jac}(E)$, i.e. $\eta_G \in \text{Rey}(E)$.

"$\supset$": Conversely, let $r \in \text{Rey}(E)$ and let $\eta = \xi^{-1}(r) \in \text{End}(Id_{\mathcal{C}})$. Let $P \in \mathcal{P}roj(\mathcal{C})$, and $f \in J_P$. We need to show $\eta_P \circ f = 0$. Pick a surjection $p : G^{\oplus n} \to P$ and a right-inverse $i : P \to G^{\oplus n}$. We have $i \circ f \circ p \in J_{G^{\oplus n}}$ since for all $u : G^{\oplus n} \to U$, where U is simple, $u \circ i \circ f \circ p = 0$ since $u \circ i : P \to U$ and $f \in J_P$. Note then that $r^{\oplus n} : G^{\oplus n} \to G^{\oplus n}$ and $r^{\oplus n} \in \text{Rey}(\text{End}(G^{\oplus n}))$, therefore $r^{\oplus n} \circ i \circ f \circ p = 0$. Since $r^{\oplus n} = \eta_{G^{\oplus n}}$ we get

$$0 = \eta_{G^{\oplus n}} \circ i \circ f \circ p = i \circ \eta_P \circ f \circ p \; . \tag{5.14}$$

But i is injective and p surjective, and so $\eta_P \circ f = 0$, as required. □

Hig($\mathcal{A}$)

The condition replacing that the algebra is symmetric will be that $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau (see e.g. [He] for a recent treatment of the relation between symmetric Frobenius algebras and Calabi-Yau categories in the semisimple case):

Definition 5.4. A k-linear additive category $\mathcal{K}$ with finite-dimensional Hom-spaces is called *Calabi-Yau* if it is equipped with a family of k-linear maps (the *trace maps*)

$$\big(\, \mathsf{t}_M : \text{End}(M) \to k \,\big)_{M \in \mathcal{K}} \tag{5.15}$$

such that for all $M, N \in \mathcal{K}$,

1. *(cyclicity)* for all $f : M \to N$, $g : N \to M$ we have $\mathsf{t}_M(g \circ f) = \mathsf{t}_N(f \circ g)$.
2. *(non-degeneracy)* the pairing $(-,-) : \mathcal{K}(M,N) \times \mathcal{K}(N,M) \to k$, $(f,g) = \mathsf{t}_N(f \circ g)$ is non-degenerate.

We denote by $\text{CY}(\mathcal{K})$ the set of all families of trace maps $(\mathsf{t}_M)_{M \in \mathcal{K}}$ which turn $\mathcal{K}$ into a Calabi-Yau category.

A Calabi-Yau category with one object is the same as a symmetric Frobenius algebra. The generalisation of the isomorphism (5.2) to the present setting is:

Lemma 5.5. *Let $\mathcal{K}$ be as in Definition 5.4 and let $\text{End}(Id_{\mathcal{K}})^{\times}$ be the subset of invertible endo-transformations. Then for each $(\mathsf{t}_M)_{M \in \mathcal{K}} \in \text{CY}(\mathcal{K})$ the map*

$$\text{End}(Id_{\mathcal{K}})^{\times} \longrightarrow \text{CY}(\mathcal{K}) \quad , \quad \eta \;\longmapsto\; \big(f \mapsto \mathsf{t}_M(\eta_M \circ f)\big) \tag{5.16}$$

is a bijection.

Proof. Clearly, $(\mathtt{t}_M(\eta_M \circ -))_{M\in\mathcal{K}}$ defines a family of cyclic and non-degenerate traces (for non-degeneracy, one notes that $\eta_M \circ g$ is non-zero for any non-zero g because η_M is invertible). Conversely, given $(\tilde{\mathtt{t}}_M)_{M\in\mathcal{K}} \in \mathrm{CY}(\mathcal{K})$, by non-degeneracy for each $M \in \mathcal{K}$ there exists a unique $\eta_M \in \mathrm{End}(M)^\times$ such that $\tilde{\mathtt{t}}_M(-) = \mathtt{t}_M(\eta_M \circ -)$. Rewriting the cyclicity condition $\tilde{\mathtt{t}}_M(g\circ f) = \tilde{\mathtt{t}}_N(f \circ g)$ for all $f : M \to N$ and $g : N \to M$ in terms of the family $\mathtt{t}_M$ and using its cyclicity property shows $\mathtt{t}_M\big(g\circ(\eta_N \circ f - f\circ \eta_M)\big) = 0$ for any g, and thus by non-degeneracy of $\mathtt{t}_M$ we have $\eta_N \circ f = f \circ \eta_M$ for all M, N, and f. □

Suppose now that the finite abelian category $\mathcal{A}$ is such that $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau with trace maps $(\mathtt{t}_P)_{P\in\mathcal{P}roj(\mathcal{A})}$. For the pairings of morphisms between the projective objects $P, R \in \mathcal{P}roj(\mathcal{A})$ we write

$$(-,-)_{PR} : \mathcal{A}(P,R) \times \mathcal{A}(R,P) \to k \quad , \quad (f,g)_{PR} = \mathtt{t}_R(f\circ g) \ . \tag{5.17}$$

Denote by $\gamma_{PR} \in \mathcal{A}(R,P)\otimes\mathcal{A}(P,R)$ the corresponding (unique) copairing, that is $\gamma_{PR} = \sum_{(\gamma_{PR})} \gamma'_{PR} \otimes \gamma''_{PR}$ and for all $x : P \to R$, $y : R \to P$,

$$\sum_{(\gamma_{PR})} \big(x\,,\gamma'_{PR}\big)_{PR}\,\gamma''_{PR} \;=\; x \qquad , \qquad \sum_{(\gamma_{PR})} \gamma'_{PR}\,\big(\gamma''_{PR}\,,y\big)_{PR} \;=\; y \ . \tag{5.18}$$

We need the following lemma, whose proof is given in Appendix B.

Lemma 5.6. *For each $R \in \mathcal{P}roj(\mathcal{A})$ and $y \in \mathrm{End}(R)$ there exists a unique $\eta \in \mathrm{End}(Id_{\mathcal{A}})$ such that $\eta_P = \sum_{(\gamma_{PR})} \gamma'_{PR}\circ y \circ \gamma''_{PR}$ for all $P \in \mathcal{P}roj(\mathcal{A})$.*

This lemma allows us to define, for each $R \in \mathcal{P}roj(\mathcal{A})$, a k-linear map $\tau_R : \mathrm{End}(R) \to \mathrm{End}(Id_{\mathcal{A}})$ by

$$\tau_R(y) = \eta \quad , \quad \text{where for all } P \in \mathcal{P}roj(\mathcal{A}) \quad \eta_P = \sum_{(\gamma_{PR})} \gamma'_{PR}\circ y \circ \gamma''_{PR} \ . \tag{5.19}$$

These maps are the analogue of the map τ from (5.3). Indeed, $\mathrm{Hig}(\mathcal{A})$ is now defined as the joint image of all the maps τ_R:

$$\mathrm{Hig}(\mathcal{A}) = \big\{\, \eta \in \mathrm{End}(Id_{\mathcal{A}}) \,\big|\, \exists\, R \in \mathcal{P}roj(\mathcal{A})\,,\, y \in \mathrm{End}(R) \,:\, \eta = \tau_R(y) \,\big\} \ . \tag{5.20}$$

The point of the above definition is to avoid the choice of a projective generator. To actually compute $\mathrm{Hig}(\mathcal{A})$, choosing a projective generator G may nonetheless be useful. For example, we show in Corollary B.3 that $\mathrm{Hig}(\mathcal{A}) = \mathrm{im}(\tau_G)$.

As was the case for the Higman ideal of a symmetric algebra, also $\mathrm{Hig}(\mathcal{A})$ is independent of the choice of traces:

Proposition 5.7. *Let $\mathcal{A}$ be a finite abelian category over k such that $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau. Then $\mathrm{Hig}(\mathcal{A})$ is independent of the choice of the family of traces $(\mathtt{t}_P)_{P\in\mathcal{P}roj(\mathcal{A})}$.*

Proof. Let $\mathsf{t}, \tilde{\mathsf{t}} \in \mathrm{CY}(\mathcal{P}roj(\mathcal{A}))$. By Lemma 5.5 there is $\eta \in \mathrm{End}(Id_{\mathcal{P}roj(\mathcal{A})})^\times$ such that $\tilde{\mathsf{t}}_P(f) = \mathsf{t}_P(\eta_P \circ f)$ for all $P \in \mathcal{P}roj(\mathcal{A})$ and $f \in \mathrm{End}(P)$. Let γ_{PR} and $\tilde{\gamma}_{PR}$ be the copairings arising from t and $\tilde{\mathsf{t}}$ as in (5.18). If we write $\gamma_{PR} = \sum_{(\gamma_{PR})} \gamma'_{PR} \otimes \gamma''_{PR}$, it is easy to see that

$$\tilde{\gamma}_{PR} \;=\; \sum_{(\gamma_{PR})} (\gamma'_{PR} \circ \eta_R^{-1}) \otimes \gamma''_{PR} \;=\; \sum_{(\gamma_{PR})} \gamma'_{PR} \otimes (\eta_R^{-1} \circ \gamma''_{PR}) \ . \tag{5.21}$$

Write $\tilde{\tau}_R$ for the map (5.19) computed from $\tilde{\gamma}$. It is immediate from the above equality that for $y \in \mathrm{End}(R)$,

$$\tilde{\tau}_R(y) = \tau_R(\eta_R^{-1} \circ y) \ . \tag{5.22}$$

Thus indeed the joint image of the τ_R is independent of the choice of traces t_P. □

The next proposition makes the connection between the definition (5.20) of $\mathrm{Hig}(\mathcal{A})$ and $\mathrm{Hig}(A)$ for a symmetric algebra A when taking $\mathcal{A} = \text{mod-}A$. The proof has been relegated to Appendix B.

Proposition 5.8. *Let A be a finite-dimensional k-algebra.*

1. *A admits a central form turning it into a symmetric algebra if and only if $\mathcal{P}roj(\text{mod-}A)$ admits a family of traces turning it into a Calabi-Yau category.*

2. *If A is symmetric, the map ξ from (5.13) for $G = A_A$ restricts to an isomorphism of algebras $\xi : \mathrm{Hig}(\text{mod-}A) \to \mathrm{Hig}(A)$.*

Combining this proposition and Proposition 5.3 with the equivalence (5.8) and the inclusions (5.4) we get that we indeed have the inclusions (5.9).

As for algebras, the Cartan matrix of $\mathcal{A}$ is the $\mathrm{Irr}(\mathcal{A})\times \mathrm{Irr}(\mathcal{A})$-matrix $\mathsf{C}(\mathcal{A})$ with entries $\mathsf{C}(\mathcal{A})_{UV} = [P_U : V]$, i.e. in the Grothendieck group $\mathrm{Gr}(\mathcal{A})$ we have

$$[P_U] \;=\; \sum_{V \in \mathrm{Irr}(\mathcal{A})} \mathsf{C}(\mathcal{A})_{UV}\,[V] \ . \tag{5.23}$$

Since k is algebraically closed, we can write

$$\mathsf{C}(\mathcal{A})_{UV} \;=\; \dim_k \mathcal{A}(P_V, P_U) \ . \tag{5.24}$$

In view of Proposition 5.8 and the equivalence (5.8), we have the following corollary to Proposition 5.2.

Corollary 5.9. *Let $\mathcal{A}$ be a finite abelian category over k such that $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau. Then*

1. *The Cartan matrix of $\mathcal{A}$ is symmetric, $\mathsf{C}(\mathcal{A})_{U,V} = \mathsf{C}(\mathcal{A})_{V,U}$.*

2. *Let* $n = |\operatorname{Irr}(\mathcal{A})|$ *and let* $\widehat{\mathsf{C}} \in \operatorname{Mat}_n(k)$ *be the image of* $\mathsf{C}(\mathcal{A})$ *under the canonical homomorphism* $\operatorname{Mat}_n(\mathbb{Z}) \to \operatorname{Mat}_n(k)$*. We have*

$$\operatorname{rank}\big(\widehat{\mathsf{C}}\big) \;=\; \dim_k \operatorname{Hig}(\mathcal{A}) \ . \tag{5.25}$$

In the context of this paper we are particularly interested in the case of factorisable and pivotal finite tensor categories $\mathcal{C}$. In this case we will later give a more direct description of $\operatorname{Rey}(\mathcal{C})$ and $\operatorname{Hig}(\mathcal{C})$ in terms of the ϕ_M defined in (2.22), see Proposition 7.1 below. The next proposition shows in particular that Corollary 5.9 is applicable to such $\mathcal{C}$.

Proposition 5.10. *Let* $\mathcal{C}$ *be a factorisable and pivotal finite tensor category over* k *which satisfies Condition P.*

1. $\mathcal{P}roj(\mathcal{C})$ *admits a family of traces turning it into a Calabi-Yau category.*

2. $\mathcal{C}$ *is semisimple if and only if any one of the inclusions* $\operatorname{End}(Id_{\mathcal{C}}) \supset \operatorname{Rey}(\mathcal{C}) \supset \operatorname{Hig}(\mathcal{C})$ *is an equality.*

Proof. Part 1 is immediate from Corollary 4.7 by setting $\mathsf{t}_P := t_P$ for any $P \in \mathcal{P}roj(\mathcal{C})$, where t is the modified trace on $\mathcal{P}roj(\mathcal{C})$. For Part 2, the equivalence of semisimplicity to $\operatorname{End}(Id_{\mathcal{C}}) = \operatorname{Rey}(\mathcal{C})$ follows from Proposition 5.3 and Lemma 5.1.

To show that $\operatorname{Rey}(\mathcal{C}) = \operatorname{Hig}(\mathcal{C})$ is equivalent to semisimplicity of $\mathcal{C}$, we pick a projective generator G and write $E = \operatorname{End}(G)$. Via the equivalence (5.8) to mod-E, and by Propositions 5.3 and 5.8, the equality $\operatorname{Rey}(\mathcal{C}) = \operatorname{Hig}(\mathcal{C})$ is equivalent to $\operatorname{Rey}(E) = \operatorname{Hig}(E)$, and by the isomorphisms in (5.4) in turn to $R(E) = I(E)$.

Let $\widehat{\mathsf{C}}$ be the image of $\mathsf{C}(\mathcal{C})$ in $\operatorname{Mat}_n(k)$ as in Corollary 5.9. We now show:

Claim: $R(E) = I(E)$ iff $\widehat{\mathsf{C}}$ is non-degenerate.

This claim completes the proof of part 2, since by [EGNO, Thm. 6.6.1], a pivotal finite tensor category over k is semisimple if and only if $\widehat{\mathsf{C}}$ is non-degenerate.

Proof of claim: Recall from (5.4) that the characters χ_M of E-modules M span $R(E)$ (over k) and those of projective E-modules span $I(E)$. Using again the equivalence (5.8), we have $\chi_{\tilde{P}_U} = \sum_{V\in\operatorname{Irr}(\mathcal{C})} \widehat{\mathsf{C}}_{UV}\, \chi_{\tilde{V}}$, where $\tilde{X} := \mathcal{C}(G,X)$ is the image of X under the equivalence (5.8). In particular, the image of the k-linear map described by the matrix $\widehat{\mathsf{C}}$ in the basis $\{\chi_{\tilde{V}}\}_{V\in\operatorname{Irr}(\mathcal{C})}$ of $R(E)$, is precisely $I(E)$. We conclude that $R(E) = I(E)$ is equivalent to non-degeneracy of $\widehat{\mathsf{C}}$, proving the claim. □

Part 2 of the proposition also follows from [Sh2, Cor. 4.2 & Thm. 5.12] together with Proposition 7.1 below. [8]

[8] In [CW, Cor. 2.3] it is stated (among other things) that for a symmetric algebra A over an algebraically closed field of characteristic zero, $R(A) = I(A)$ implies that A is semisimple. This would also imply part 2 of Proposition 5.10. However this statement is not true, as illustrated by the following counterexample (due to Ehud Meir): Take $A = k[X]/\langle X^2\rangle$ and $\varepsilon(1) = 0$, $\varepsilon(X) = 1$. Then A is a symmetric algebra.

6 Characters and the modified trace

In this section we fix

- $\mathcal{C}$: a factorisable and pivotal finite tensor category over k which satisfies Condition P given in the beginning of Section 3,
- t : a choice of non-zero modified trace on $\mathcal{P}roj(C)$ via Corollary 4.7,
- G : a projective generator for $\mathcal{C}$,
- $E := \mathrm{End}(G)$, a symmetric k-algebra via t_G.

Recall from (2.22) the natural endomorphism $\phi_M \in \mathrm{End}(Id_\mathcal{C})$ assigned to each $M \in \mathcal{C}$. Note that ϕ_M is uniquely determined by $(\phi_M)_G \in E$, and that in fact $(\phi_M)_G \in Z(E)$.

For a given $M \in \mathcal{C}$ we can now define two central forms on E. Let $x \in E$. The first central form uses the Hom-tensor equivalence (5.8) and is given by $x \mapsto tr_{\mathcal{C}(G,M)}(x)$, the trace in the right E-module $\mathcal{C}(G, M)$ over the right action of x. The second form is $x \mapsto t_G((\phi_M)_G \circ x)$. The aim of this section is to prove our second main result:

Theorem 6.1. *The modified trace $t_{P_\mathbf{1}}((\phi_\mathbf{1})_{P_\mathbf{1}})$ is non-zero, and for all $M \in \mathcal{C}$ the following equality of central forms on E holds:*

$$tr_{\mathcal{C}(G,M)}(-) \;=\; \frac{t_G((\phi_M)_G \circ -)}{t_{P_\mathbf{1}}((\phi_\mathbf{1})_{P_\mathbf{1}})} \ . \tag{6.1}$$

The proof relies on several intermediate results, which we present now.

Lemma 6.2. *Let $f : A \to B$ and suppose that for all $X \in \mathcal{C}$ we have*

$$(f \otimes id_X) \circ c_{X,A} \circ c_{A,X} = f \otimes id_X \ . \tag{6.2}$$

Then there is $m > 0$ and $a : A \to \mathbf{1}^{\oplus m}$, $b : \mathbf{1}^{\oplus m} \to B$ such that $f = b \circ a$.

Proof. Write $f = \big[A \xrightarrow{a} M \xrightarrow{b} B\big]$, where a is epi and b is mono. Rewrite (6.2) as

$$(b \otimes id_X) \circ c_{X,M} \circ c_{M,X} \circ (a \otimes id_X) = (b \otimes id_X) \circ (a \otimes id_X) \ . \tag{6.3}$$

By exactness of $\otimes$, also $a \otimes id_X$ is epi and $b \otimes id_X$ is mono. We conclude that $c_{X,M} \circ c_{M,X} = id_{M \otimes X}$ for all M. From characterisation 1 of factorisability in Theorem 2.8 it follows that $M \cong \mathbf{1}^{\oplus m}$ for some m. □

Its unique simple module is k and the projective cover of k is $P_k = A$. We have $\chi_A = 2 \cdot \chi_k$. Thus $I(A) = R(A)$, but A is not semisimple.

In [CW], Corollary 2.3 is only used in Theorem 2.9, and there only the remaining parts of Corollary 2.3 are relevant, so that the above counterexample does not affect the other results in [CW] (we thank Miriam Cohen and Sara Westreich for correspondence on this point).

On the other hand, it is shown in [Lo, Thm. 2] that if A is in addition a Hopf algebra such that the square of the antipode is inner, $R(A) = I(A)$ does indeed imply that A is semisimple. This is a special case of [EGNO, Thm. 6.6.1].

Recall the coend $\mathcal{L}$ with its family ι_X of dinatural transformations and its non-zero cointegral $\Lambda^{\text{co}}_{\mathcal{L}}$ from Section 2.

Lemma 6.3. *For each $X \in \mathcal{C}$ there exist $m > 0$, $a : X \to \mathbf{1}^{\oplus m}$, $b : \mathbf{1}^{\oplus m} \to X$ (possibly zero) such that*

$$\begin{aligned} \Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_X &= \text{ev}_X \circ \big(id_{X^*} \otimes (b \circ a)\big) \\ &= \sum_{\alpha=1}^{m} \big[X^*X \xrightarrow{(b_\alpha)^* \otimes a_\alpha} \mathbf{1}^*\mathbf{1} \xrightarrow{\text{ev}_{\mathbf{1}}} \mathbf{1}\big] \end{aligned} \tag{6.4}$$

where $a_\alpha : X \to \mathbf{1}$, $b_\alpha : \mathbf{1} \to X$ are the components of a, b.

Proof. By definition of the cointegral, we have

$$\big[\mathcal{L} \xrightarrow{\Delta_{\mathcal{L}}} \mathcal{L}\mathcal{L} \xrightarrow{\Lambda^{\text{co}}_{\mathcal{L}} \otimes id_{\mathcal{L}}} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\big] = \big[\mathcal{L} \xrightarrow{\Lambda^{\text{co}}_{\mathcal{L}}} \mathbf{1} \xrightarrow{\eta_{\mathcal{L}}} \mathcal{L}\big] . \tag{6.5}$$

Combining this with the pairing $\omega_{\mathcal{L}}$ we obtain the following equality:

$$\big[\mathcal{L}\mathcal{L} \xrightarrow{\Delta_{\mathcal{L}} \otimes id_{\mathcal{L}}} (\mathcal{L}\mathcal{L})\mathcal{L} \xrightarrow{\sim} \mathcal{L}(\mathcal{L}\mathcal{L}) \xrightarrow{\Lambda^{\text{co}}_{\mathcal{L}} \otimes \omega_{\mathcal{L}}} \mathbf{1}\mathbf{1}\big] = \big[\mathcal{L}\mathcal{L} \xrightarrow{\Lambda^{\text{co}}_{\mathcal{L}} \otimes \varepsilon_{\mathcal{L}}} \mathbf{1}\mathbf{1}\big] , \tag{6.6}$$

where we used the property of the Hopf pairing that $\omega_{\mathcal{L}} \circ (\eta_{\mathcal{L}} \otimes id_{\mathcal{L}}) \circ \lambda^{-1}_{\mathcal{L}} = \varepsilon_{\mathcal{L}}$. After pre-composing with $\iota_X \otimes \iota_Y$ and substituting the defining equations, a short calculation shows

$$\begin{aligned} &\big[(X^*X)(Y^*Y) \xrightarrow{\sim} X^*((XY^*)Y) \xrightarrow{id \otimes (c_{Y^*,X} \circ c_{X,Y^*}) \otimes id} X^*((XY^*)Y) \\ &\qquad\qquad \xrightarrow{\sim} (X^*X)(Y^*Y) \xrightarrow{(\Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_X) \otimes \text{ev}_Y} \mathbf{1}\mathbf{1}\big] \\ &= \big[(X^*X)(Y^*Y) \xrightarrow{(\Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_X) \otimes \text{ev}_Y} \mathbf{1}\mathbf{1}\big] \end{aligned} \tag{6.7}$$

Using the isomorphism $\mathcal{C}(X^* \otimes M, N) \xrightarrow{\sim} \mathcal{C}(M, X \otimes N)$,

$$h \longmapsto \big[M \xrightarrow{\sim} \mathbf{1}M \xrightarrow{\text{coev}_X \otimes id} (XX^*)M \xrightarrow{\sim} X(X^*M) \xrightarrow{id \otimes h} XN\big] , \tag{6.8}$$

and similarly for $\mathcal{C}(M \otimes Y, N) \xrightarrow{\sim} \mathcal{C}(M, N \otimes Y^*)$, the latter equality can be brought to the form $(f \otimes id_{Y^*}) \circ c_{Y^*,X} \circ c_{X,Y^*} = f \otimes id_{Y^*}$ with

$$f = \big[X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\text{coev}_X \otimes id} (XX^*)X \xrightarrow{\sim} X(X^*X) \xrightarrow{id \otimes (\Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_X)} X\mathbf{1} \xrightarrow{\sim} X\big] . \tag{6.9}$$

Lemma 6.2 now implies the claim. □

Corollary 6.4. *For $U \in \mathcal{C}$ simple, $U \not\cong \mathbf{1}$, we have $\Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_{P_U} = 0$. Furthermore, there exists a unique $\nu_{\mathbf{1}} : \mathbf{1} \to P_{\mathbf{1}}$ such that*

$$\Lambda^{\text{co}}_{\mathcal{L}} \circ \iota_{P_{\mathbf{1}}} = \text{ev}_{\mathbf{1}} \circ (\nu^*_{\mathbf{1}} \otimes \pi_{\mathbf{1}}) , \tag{6.10}$$

where $\pi_{\mathbf{1}} : P_{\mathbf{1}} \to \mathbf{1}$ is the canonical projection of the projective cover.

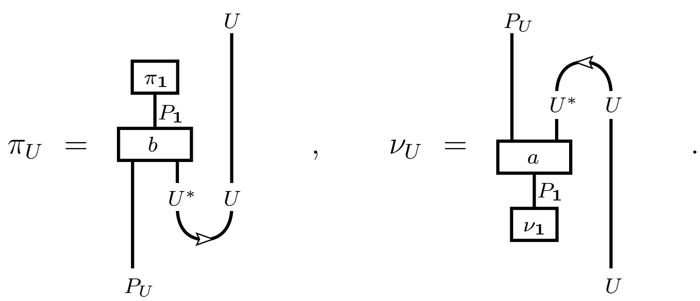
Proof. This follows from Lemma 6.3 and the fact that $\mathcal{C}(P_U, \mathbf{1})$ and $\mathcal{C}(\mathbf{1}, P_U)$ are both zero-dimensional for $U \ncong \mathbf{1}$ and one-dimensional for $U \cong \mathbf{1}$ (by unimodularity of $\mathcal{C}$). $\square$

Recall the natural endomorphism ϕ_M in (2.23). The next proposition computes its values on projective objects and also shows that $\nu_{\mathbf{1}}$ (and all ν_U defined there) are non-zero.

Proposition 6.5. *Let $U, V \in \mathrm{Irr}(\mathcal{C})$. Then $(\phi_U)_{P_U} \neq 0$ and there exists a unique non-zero $\nu_U : U \to P_U$ such that*

$$(\phi_U)_{P_V} = \begin{cases} 0 & , \quad V \neq U \\ \nu_U \circ \pi_U & , \quad V = U \end{cases} \tag{6.11}$$

where $\pi_U : P_U \to U$ is the canonical projection of the projective cover, and where $\nu_{\mathbf{1}}$ is the same as in Corollary 6.4. Moreover, there exists a choice of $a : P_{\mathbf{1}} \to P_U \otimes U^*$, $b : P_U \otimes U^* \to P_{\mathbf{1}}$ which realise $P_{\mathbf{1}}$ as a direct summand of $P_U \otimes U^*$ (i.e. $b \circ a = id_{P_{\mathbf{1}}}$), such that



(6.12)

Proof. In the expression (2.23) for $(\phi_M)_X$ set $M = U$ and $X = P_V$. Write $P_V \otimes U^* \cong \bigoplus_{W \in \mathrm{Irr}(\mathcal{C})} P_W^{\oplus n_W}$ and pick $a_{W,\alpha} \in \mathcal{C}(P_W, P_V \otimes U^*)$ and $b_{W,\alpha} \in \mathcal{C}(P_V \otimes U^*, P_W)$, $\alpha = 1, \dots, n_W$, which realise the direct sum decomposition:

$$b_{W,\alpha} \circ a_{W',\alpha'} = \delta_{W,W'}\,\delta_{\alpha,\alpha'}\, id_{P_W} \quad , \quad \sum_{W \in \mathrm{Irr}(\mathcal{C})} \sum_{\alpha=1}^{n_W} a_{W,\alpha} \circ b_{W,\alpha} = id_{P_V \otimes U^*} \ . \tag{6.13}$$

For the dinatural transformation $\iota_{P_V \otimes U^*}$ in (2.23) consider the equalities

$$\begin{aligned} \iota_{P_V \otimes U^*} &= \sum_{W \in \mathrm{Irr}(\mathcal{C})} \sum_{\alpha=1}^{n_W} \iota_{P_V \otimes U^*} \circ \big(id_{(P_V \otimes U^*)^*} \otimes (a_{W,\alpha} \circ b_{W,\alpha})\big) \\ &= \sum_{W \in \mathrm{Irr}(\mathcal{C})} \sum_{\alpha=1}^{n_W} \iota_{P_W} \circ (a^*_{W,\alpha} \otimes b_{W,\alpha}) \end{aligned} \tag{6.14}$$

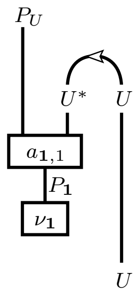
From Corollary 6.4, $\Lambda^{\mathrm{co}}_{\mathcal{L}} \circ \iota_{P_W} = 0$ for $W \ncong \mathbf{1}$, and so in the expression for $\Lambda^{\mathrm{co}}_{\mathcal{L}} \circ \iota_{P_V \otimes U^*}$ the sum over W in (6.14) reduces to $W = \mathbf{1}$. Since $\dim \mathcal{C}(P_W, R) = \delta_{W,R}$ for $W, R \in \mathrm{Irr}(\mathcal{C})$, the multiplicities n_W are given by $n_W = \dim \mathcal{C}(P_V \otimes U^*, W)$. In particular,

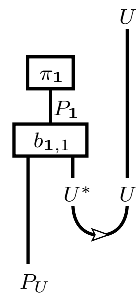
$$n_{\mathbf{1}} = \dim \mathcal{C}(P_V \otimes U^*, \mathbf{1}) = \dim \mathcal{C}(P_V, U) = \delta_{U,V} \ . \tag{6.15}$$

So in (6.14) the sum over α is zero for $U \neq V$ and reduces to $\alpha = 1$ for $U = V$. Together with (6.10) we obtain

$$\Lambda^{\mathrm{co}}_{\mathcal{L}} \circ \iota_{P_V \otimes U^*} \;=\; \delta_{U,V}\ \mathrm{ev}_{\mathbf{1}} \circ \big((a_{\mathbf{1},1} \circ \nu_{\mathbf{1}})^* \otimes (\pi_{\mathbf{1}} \circ b_{\mathbf{1},1}) \big)\ . \tag{6.16}$$

Substituting this into (2.23) yields $(\phi_U)_{P_V} = \delta_{U,V}\, \tilde{\nu} \circ \tilde{\pi}$ where

$\tilde{\nu} \;=$ 

, $\tilde{\pi} \;=$ 

. (6.17)

It remains to show that we can achieve $\tilde{\pi} = \pi_U$, the chosen projection $P_U \to U$, and that for $U = \mathbf{1}$, $\tilde{\nu}$ agrees with $\nu_{\mathbf{1}}$ in (6.10).

By Theorem 2.6, in particular $\chi_U \neq 0$. Since ψ and ρ are isomorphisms, also $\phi_U \neq 0$, and thus $\tilde{\nu}, \tilde{\pi} \neq 0$ (or otherwise ϕ_U would vanish on all indecomposable projectives and hence be zero). Since $\mathcal{C}(P_U, U)$ is one-dimensional, by rescaling $b_{\mathbf{1},1}$ if necessary (and thus rescaling $a_{\mathbf{1},1}$ accordingly by the inverse factor) one can achieve $\tilde{\pi} = \pi_U$. This finally gives (6.12).

For $U = V = \mathbf{1}$, one can alternatively compute $(\phi_{\mathbf{1}})_{P_{\mathbf{1}}}$ directly from (2.23) and (6.10). A short calculation confirms that $\nu_{\mathbf{1}}$ from (6.10) agrees with $\tilde{\nu}$. □

Corollary 6.6. *Let $U, V \in \mathrm{Irr}(\mathcal{C})$, then $(\phi_U)_V = 0$ unless $U = V$ and U is projective.*

Proof. By naturality of ϕ_U we have $\pi_V \circ (\phi_U)_{P_V} = (\phi_U)_V \circ \pi_V$. Applying (6.11) to LHS of the last equality and using $\pi_V \circ \nu_V = 0$ for V non-projective, we get the statement. □

Proof of Theorem 6.1. We will show $t_G((\phi_M)_G \circ -) = z \cdot tr_{\mathcal{C}(G,M)}(-)$ and then determine $z \in k$ to be $t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}})$. Write $G = \bigoplus_{U \in \mathrm{Irr}(\mathcal{C})} P_U^{\oplus n_U}$ and let $e_{U,\alpha}$, $U \in \mathrm{Irr}(\mathcal{C})$, $\alpha = 1, \dots, n_U$ be the corresponding primitive orthogonal idempotents. Pick $j_{U,\alpha} : P_U \to G$, $q_{U,\alpha} : G \to P_U$ such that

$$q_{U,\alpha} \circ j_{M,\beta} = \delta_{U,M}\, \delta_{\alpha,\beta}\, id_{P_U} \qquad \text{and} \qquad j_{U,\alpha} \circ q_{U,\alpha} = e_{U,\alpha}\ . \tag{6.18}$$

As for any finite-dimensional associative algebra, the quotient $E/\mathrm{Jac}(E)$ is a direct sum over $U \in \mathrm{Irr}(\mathcal{C})$ of $n_U \times n_U$-matrix algebras, and the $j_{U,\alpha} \circ q_{U,\beta}$ form a basis in these matrix algebras. Therefore, the endomorphism algebra E decomposes (as a vector space) as

$$E \;=\; \bigoplus_{U,\alpha,\beta} k\, j_{U,\alpha} \circ q_{U,\beta} \;\oplus\; \mathrm{Jac}(E)\ . \tag{6.19}$$

To show $t_G((\phi_M)_G \circ -) = z \cdot \mathrm{tr}_{\mathcal{C}(G,M)}(-)$ it is enough to consider $M \in \mathrm{Irr}(\mathcal{C})$. In this case, $\tilde{M} := \mathcal{C}(G, M)$ is an irreducible E-module of dimension n_M. The trace $\mathrm{tr}_{\tilde{M}}(-)$ vanishes on $\mathrm{Jac}(E)$ while on semisimple part it gives

$$\mathrm{tr}_{\tilde{M}}(j_{U,\alpha} \circ q_{U,\beta}) = \delta_{U,M}\, \delta_{\alpha,\beta}\ . \tag{6.20}$$

It remains to check that evaluating $t_G((\phi_M)_G \circ -)$ on $j_{U,\alpha} \circ q_{U,\beta}$ leads to the same result, up to a factor of $t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}})$. We have, for all $U \in \mathrm{Irr}(\mathcal{C})$,

$$(\phi_U)_G \overset{(6.18)}{=} \sum_{V \in \mathrm{Irr}(\mathcal{C})} \sum_{\alpha=1}^{n_V} j_{V,\alpha} \circ q_{V,\alpha} \circ (\phi_U)_G \overset{\phi_U \text{ nat.}}{=} \sum_{V \in \mathrm{Irr}(\mathcal{C})} \sum_{\alpha=1}^{n_V} j_{V,\alpha} \circ (\phi_U)_{P_V} \circ q_{V,\alpha}$$

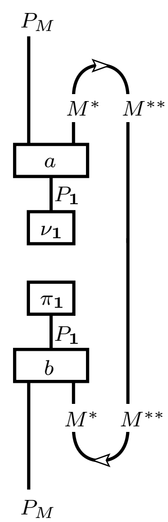
$$\overset{\text{Prop. 6.5}}{=} \sum_{\alpha=1}^{n_U} j_{U,\alpha} \circ \nu_U \circ \pi_U \circ q_{U,\alpha} \ . \tag{6.21}$$

By (5.11) the map $\pi_U \circ q_{U,\alpha} \colon G \to U$ annihilates $\mathrm{Jac}(E)$ and therefore $(\phi_U)_G \circ \mathrm{Jac}(E) = 0$. Thus also $t_G((\phi_M)_G \circ -)$ vanishes on $\mathrm{Jac}(E)$. Furthermore,

$$t_G((\phi_M)_G \circ j_{U,\alpha} \circ q_{U,\beta}) = \delta_{U,M}\, \delta_{\alpha,\beta}\, z \quad , \quad \text{where} \quad z = t_{P_M}((\phi_M)_{P_M}) \ , \tag{6.22}$$

and where we used first cyclicity of t, then naturality of ϕ_M and then (6.18) and (6.11). To arrive at $z = t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}})$, we first observe that

$(\phi_M)_{P_M} =$



. (6.23)

To see this, first use (6.11) and substitute the explicit expressions for ν_M, π_M from (6.12). Next use the pivotal structure to insert $id_M = \big[M \xrightarrow{\delta_M} M^{**} \xrightarrow{\delta_M^{-1}} M\big]$ and combine δ, δ^{-1} with the evaluation and coevaluation maps as in (2.4), thereby replacing M by M^{**} in the right-most strand of the above diagram. We can now compute

$$t_{P_M}\big((\phi_M)_{P_M}\big) \overset{(*)}{=} t_{P_M \otimes M^*}\big(a \circ \nu_{\mathbf{1}} \circ \pi_{\mathbf{1}} \circ b\big)$$

$$\overset{\text{cycl.}}{=} t_{P_{\mathbf{1}}}\big(\nu_{\mathbf{1}} \circ \pi_{\mathbf{1}} \circ b \circ a\big) \overset{(6.11)}{=} t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}}) \ , \tag{6.24}$$

where in (*) we have rewritten (6.23) as a partial trace and used compatibility of the modified trace with partial traces in (4.1).

So far we proved $t_G((\phi_M)_G \circ -) = z \cdot tr_{\tilde{M}}(-)$ with $z = t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}})$. It remains to show that $z \neq 0$. By Theorem 2.6 and (2.22), $\phi_M \neq 0$ for $M \in \mathrm{Irr}(\mathcal{C})$. Hence also $(\phi_M)_G \neq 0$. By non-degeneracy of the pairings (4.4) (Corollary 4.7), there is an $f \in E$ such that $t_G((\phi_M)_G \circ f) \neq 0$. In particular, $z \neq 0$. □

As a corollary to the proof of Theorem 6.1, specifically (6.24) (using also Proposition 6.5), we obtain that, for $A, B \in \mathrm{Irr}(\mathcal{C})$,

$$t_{P_A}\big((\phi_B)_{P_A}\big) \;=\; \delta_{A,B} \cdot t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}}) \;. \tag{6.25}$$

7 Properties of the Reynolds and Higman ideals

In this section k, $\mathcal{C}$, t, G and E have the same meaning as in the beginning of Section 6. We give an alternative description of the Reynolds and Higman ideals defined in Section 5 in terms of the ϕ_U. Then we describe the multiplication in these ideals.

We start with the alternative description of $\mathrm{Rey}(\mathcal{C})$ and $\mathrm{Hig}(\mathcal{C})$.

Proposition 7.1. *We have*

$$\mathrm{Rey}(\mathcal{C}) = \mathrm{span}_k\big\{\, \phi_M \,\big|\, M \in \mathcal{C} \,\big\} \quad , \quad \mathrm{Hig}(\mathcal{C}) = \mathrm{span}_k\big\{\, \phi_P \,\big|\, P \in \mathcal{P}roj(\mathcal{C}) \,\big\} \;. \tag{7.1}$$

Proof. Abbreviate $S = \mathrm{span}_k\big\{\, (\phi_M)_G \,\big|\, M \in \mathcal{C} \,\big\}$ and $S' = \mathrm{span}_k\big\{\, (\phi_P)_G \,\big|\, P \in \mathcal{P}roj(\mathcal{C}) \,\big\}$. By Propositions 5.3 and 5.8 it is enough to show $\mathrm{Rey}(E) = S$ and $\mathrm{Hig}(E) = S'$.

The image of S under the map ζ from (5.2) (we take $\varepsilon = t_G$) is given by $\zeta(S) = \mathrm{span}_k\big\{\, t_G\big((\phi_M)_G \circ -\big) \,\big|\, M \in \mathcal{C} \,\big\}$. Theorem 6.1 allows us to rewrite this as

$$\zeta(S) = \mathrm{span}_k\big\{\, tr_{\mathcal{C}(G,M)}(-) \,\big|\, M \in \mathcal{C} \,\big\} = R(E) \;, \tag{7.2}$$

where for the last equality we used the definition of $R(E)$ below (5.4) and that every E-module is isomorphic to $\mathcal{C}(G, M)$ for some M (equivalence (5.8)). But by the isomorphisms in (5.4) the equality $\zeta(S) = R(E)$ implies $S = \mathrm{Rey}(E)$.

An analogous argument shows $S' = \mathrm{Hig}(E)$. □

The next proposition shows that the action of $\mathrm{End}(\mathcal{C})$ on $\mathrm{Rey}(\mathcal{C})$ is diagonal in the basis $\{\, \phi_U \,|\, U \in \mathrm{Irr}(\mathcal{C}) \,\}$.

Proposition 7.2. *For $\alpha \in \mathrm{End}(Id_{\mathcal{C}})$ and $U \in \mathrm{Irr}(\mathcal{C})$ we have*

$$\alpha \circ \phi_U \;=\; \langle \alpha_U \rangle \cdot \phi_U \;, \tag{7.3}$$

where $\langle \alpha_U \rangle \in k$ is determined by $\alpha_U = \langle \alpha_U \rangle\, id_U$. If $U, V \in \mathrm{Irr}(\mathcal{C})$ lie in the same block of $\mathcal{C}$, then $\langle \alpha_U \rangle = \langle \alpha_V \rangle$.

Proof. We pick a projective generator G and use the equivalence in (5.8). Let $e_1, \dots, e_b \in Z(E)$ be the primitive central idempotents. The e_j decompose E as an algebra into indecomposable direct summands, or, equivalently, via (5.8) they decompose $\mathcal{C}$ into blocks.

Abbreviate $z := \alpha_G \in Z(E)$. We can write $z = \sum_{j=1}^{b} y_j e_j + n$ for some $y_j \in k$, $n \in \mathrm{Jac}(E)$. Suppose U is in the j'th block of $\mathcal{C}$, that is, z acts on the simple E-module $\mathcal{C}(G, U)$ by $y_j\, id$. Then $\langle \alpha_U \rangle = y_j$, proving the second statement.

By Proposition 6.5, $(\phi_U)_G$ factors through the semisimple E-module $\mathcal{C}(G, U)^{\oplus m}$ for some m. Hence $z\,(\phi_U)_G = y_j\,(\phi_U)_G$, proving the first statement. □

Remark 7.3. Together with Proposition 7.1, Proposition 7.2 states that $\mathrm{End}(Id_{\mathcal{C}})$ acts diagonalisably on $\mathrm{Rey}(\mathcal{C})$, with eigenbasis given by $\{\phi_U \,|\, U \in \mathrm{Irr}(\mathcal{C})\}$. Since the value of $\alpha \in \mathrm{End}(Id_{\mathcal{C}})$ on ϕ_U only depends on the block of U, and since all composition factors of an indecomposable projective belong to the same block, we equally have

$$\alpha \circ \phi_{P_U} \;=\; \langle \alpha_U \rangle \cdot \phi_{P_U} \;, \tag{7.4}$$

with $\langle \alpha_U \rangle$ as in Proposition 7.2.

By Proposition 7.1, $\mathrm{Hig}(\mathcal{C})$ is spanned by the ϕ_P, $P \in \mathcal{P}roj(\mathcal{C})$. From this formulation, we see that $\mathrm{Hig}(\mathcal{C})$ contains a natural subspace, namely the span of all ϕ_Q for Q is both simple and projective.[9] For each simple projective object Q, by Proposition 6.5 and since $\phi_Q \neq 0$, there is a non-zero $b_Q \in k$ such that

$$(\phi_Q)_{P_V} = \begin{cases} 0 & , \quad P_V \ncong Q \\ b_Q \cdot id_{P_V} & , \quad P_V \cong Q \end{cases} \;. \tag{7.5}$$

This allows for another expression of the normalisation constant $t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}})$ in (6.1), namely, for Q a simple projective object,

$$t_{P_{\mathbf{1}}}\big((\phi_{\mathbf{1}})_{P_{\mathbf{1}}}\big) \overset{(6.24)}{=} t_Q\big((\phi_Q)_Q\big) \;=\; b_Q \cdot t_Q(id_Q) \;. \tag{7.6}$$

For later use we define

$$\mathrm{IrrProj}(\mathcal{C}) \;=\; \{\, U \in \mathrm{Irr}(\mathcal{C}) \,|\, U \text{ is projective} \,\} \;. \tag{7.7}$$

By Theorem 3.4, this set is not empty. We can use this notation to describe the product in $\mathrm{Rey}(\mathcal{C})$ and $\mathrm{Hig}(\mathcal{C})$:

Proposition 7.4. *We have, for $U, V \in \mathrm{Irr}(\mathcal{C})$,*

$$\phi_U \circ \phi_V \;=\; \delta_{U,V}\, \delta_{V \in \mathrm{IrrProj}(\mathcal{C})}\, b_V\, \phi_V \quad , \quad \phi_{P_U} \circ \phi_{P_V} \;=\; \delta_{U,V}\, \delta_{V \in \mathrm{IrrProj}(\mathcal{C})}\, b_V\, \phi_{P_V} \;. \tag{7.8}$$

Proof. By Proposition 7.2, $\phi_U \circ \phi_V = \langle (\phi_U)_V \rangle \cdot \phi_V$. By Corollary 6.6, $\langle (\phi_U)_V \rangle = 0$ unless $U = V$ and U is projective. The definition of b_Q in (7.5) now implies the first equality. For the second equality, expand $\phi_{P_U} = \sum_{W \in \mathrm{Irr}(\mathcal{C})} [P_U : W] \phi_W$, and similar for ϕ_{P_V}. Since P_U can only contain a composition factor from $\mathrm{IrrProj}(\mathcal{C})$ if already $U \in \mathrm{IrrProj}(\mathcal{C})$, the second equality follows from the first. □

[9] Via the equivalence (5.8) to symmetric algebras, this subspace corresponds to the subspace of the centre denoted by Z_0 in [CW].

8 S-invariance of projective characters

In this section k, $\mathcal{C}$, t, G and E have the same meaning as in the beginning of Section 6. Here we prove our third main result, namely that the Higman ideal of $\mathcal{C}$ gets mapped to itself under the action of $\mathcal{S}_{\mathcal{C}}$. As applications, we investigate the action of the Grothendieck ring on $\mathrm{Rey}(\mathcal{C})$ and $\mathrm{Hig}(\mathcal{C})$, then give a variant of the Verlinde formula for projective objects and we show how matrix coefficients of $\mathcal{S}_{\mathcal{C}}$ on $\mathrm{Hig}(\mathcal{C})$ are related to the modified trace.

Recall the definition of $\sigma : \mathrm{Gr}(\mathcal{C})_k \to \mathrm{End}(Id_{\mathcal{C}})$ from (2.13).

Lemma 8.1. *Let $P \in \mathcal{C}$ be projective and let $f \in \mathrm{Jac}(E)$. Then $\sigma([P])_G \circ f = 0$.*

Proof. Since t_G induces a non-degenerate pairing on E (Corollary 4.7), it is enough to show that for all $x \in E$, $t_G\big(\sigma([P])_G \circ f \circ x\big) = 0$. Combining the explicit expression of $\sigma([P])_G$ as a partial trace in (2.11) and Lemma 4.5 we see

$$t_G\big(\sigma([P])_G \circ f \circ x\big) = t_{P^*}\Big(tr^r_G\big(c_{G,P^*} \circ ((f \circ x) \otimes id_{P^*}) \circ c_{P^*,G}\big)\Big) . \tag{8.1}$$

As $f \in \mathrm{Jac}(E) = J_G$ (recall (5.11)), for all $U \in \mathcal{C}$ simple and all $u : G \to U$, we have $u \circ f = 0$, and hence also $u \circ f \circ x = 0$. Corollary 2.3 now shows that the right hand side of (8.1) is zero. □

Corollary 8.2. *Let $P \in \mathcal{C}$ be projective. Then $\sigma([P]) \in \mathrm{Rey}(\mathcal{C})$.*

For each $U \in \mathrm{Irr}(\mathcal{C})$ choose a primitive idempotent $e_U \in E$ corresponding to the projective cover of the irreducible representation $\mathcal{C}(G,U)$ of E. That is, $e_U E \cong \mathcal{C}(G, P_U)$ as right E-modules. We remark here that by (6.22) we have for $U, V \in \mathrm{Irr}(\mathcal{C})$,

$$t_G((\phi_U)_G \circ e_V) \;=\; z\,\delta_{U,V} \ , \tag{8.2}$$

where we set $z = t_{P_{\mathbf{1}}}((\phi_{\mathbf{1}})_{P_{\mathbf{1}}}) \neq 0$ as above. From this equality and Proposition 7.1 it follows that the pairing

$$\mathrm{Rey}(\mathcal{C}) \times \mathrm{span}_k\big\{e_V \,\big|\, V \in \mathrm{Irr}(\mathcal{C})\big\} \longrightarrow k \quad , \quad (r,u) \mapsto t_G(r \circ u) \tag{8.3}$$

is non-degenerate.

Let us abbreviate the image of the Cartan matrix $\mathsf{C}(\mathcal{C})$ in $\mathrm{Mat}_n(k)$ by $\widehat{\mathsf{C}}$, cf. Corollary 5.9. For $x \in \ker(\widehat{\mathsf{C}})$ write $\tilde{x} = \sum_{U \in \mathrm{Irr}(\mathcal{C})} x_U e_U \in E$. We define

$$K^\perp := \big\{\, r \in \mathrm{Rey}(\mathcal{C}) \,\big|\, t_G(r_G \circ \tilde{x}) = 0 \text{ for all } x \in \ker(\widehat{\mathsf{C}}) \big\} \ . \tag{8.4}$$

Lemma 8.3. $\mathrm{Hig}(\mathcal{C}) = K^\perp$.

Proof. We first show that $\mathrm{Hig}(\mathcal{C}) \subset K^\perp$. By Proposition 7.1 we need to show that for all $P \in \mathcal{P}roj(\mathcal{C})$ and $x \in \ker(\widehat{\mathsf{C}})$, $t_G\big((\phi_P)_G \circ \tilde{x}\big) = 0$. It is enough to consider the projective covers P_U, $U \in \mathrm{Irr}(\mathcal{C})$. By (5.23) we have $\phi_{P_U} = \sum_{V \in \mathrm{Irr}(\mathcal{C})} \widehat{\mathsf{C}}_{UV}\, \phi_V$. Using this, we compute

$$t_G\big((\phi_{P_U})_G \circ \tilde{x}\big) \;=\; \sum_{V,W \in \mathrm{Irr}(\mathcal{C})} \widehat{\mathsf{C}}_{UV}\, x_W\, t_G\big((\phi_V)_G \circ e_W\big)$$

$$\overset{(8.2)}{=} \sum_{V,W\in \mathrm{Irr}(\mathcal{C})} \widehat{\mathsf{C}}_{UV}\, x_W\, \delta_{V,W}\, z \overset{x\in\ker(\widehat{\mathsf{C}})}{=} 0 \ . \tag{8.5}$$

By non-degeneracy of the pairing (8.3), we have $\dim_k K^\perp = \mathrm{rank}(\widehat{\mathsf{C}})$, the rank of the Cartan matrix, seen as a matrix over k. Hence by Corollary 5.9, $\dim_k K^\perp = \dim_k \mathrm{Hig}(\mathcal{C})$. □

The following theorem generalises results of [La1, CW] as mentioned in the introduction.

Theorem 8.4. $\mathcal{S}_\mathcal{C}(\mathrm{Hig}(\mathcal{C})) = \mathrm{Hig}(\mathcal{C})$.

Proof. We will show $\mathcal{S}_\mathcal{C}(\mathrm{Hig}(\mathcal{C})) \subset K^\perp$. Let $P \in \mathcal{P}roj(\mathcal{C})$. By Corollary 8.2 and (2.25), $\mathcal{S}_\mathcal{C}(\phi_P) \in \mathrm{Rey}(\mathcal{C})$. It remains to check that $\mathcal{S}_\mathcal{C}(\phi_P) \in K^\perp$, i.e. that $t_G\big((\mathcal{S}_\mathcal{C}(\phi_P))_G \circ \tilde{x}\big) = 0$ for all $x \in \ker(\widehat{\mathsf{C}})$. We compute

$$\begin{aligned}
t_G\big((\mathcal{S}_\mathcal{C}(\phi_P))_G \circ \tilde{x}\big) &\overset{\text{as in (8.1)}}{=} t_{P^*}\big(tr^r_G(c_{G,P^*} \circ (\tilde{x}\otimes id_{P^*}) \circ c_{P^*,G})\big) \\
&\overset{(*)}{=} \sum_{U\in\mathrm{Irr}(\mathcal{C})} x_U\, t_{P^*}\big(\sigma([P_{U^*}])_{P^*}\big) \overset{(5.23)}{=} \sum_{U,V\in\mathrm{Irr}(\mathcal{C})} x_U\, \widehat{\mathsf{C}}_{U^*V}\, t_{P^*}\big(\sigma([V])_{P^*}\big) \\
&\overset{(**)}{=} \sum_{U,V\in\mathrm{Irr}(\mathcal{C})} \widehat{\mathsf{C}}_{V^*U}\, x_U\, t_{P^*}\big(\sigma([V])_{P^*}\big) \overset{x\in\ker(\widehat{\mathsf{C}})}{=} 0 \ .
\end{aligned} \tag{8.6}$$

In (*) we inserted $\tilde{x} = \sum_U x_U\, e_U$ and used that $e_U = a\circ b$, for $a\colon P_U \to G$, $b\colon G \to P_U$ realising the direct summand P_U in G, and moving b around the loop we get $b\circ a = id_{P_U}$. The reason that U^* appears instead of U is the extra dual in (2.11), together with $(P_U)^* \cong P_{U^*}$ (as follows from uni-modularity of $\mathcal{C}$) [10]. This isomorphism, combined with (5.24), shows $\mathsf{C}_{U,V} = \mathsf{C}_{V^*,U^*}$ (and hence $\widehat{\mathsf{C}}_{U,V} = \widehat{\mathsf{C}}_{V^*,U^*}$), which explains (**).

The inclusion $\mathcal{S}_\mathcal{C}(\mathrm{Hig}(\mathcal{C})) \subset K^\perp$ implies the theorem since $\mathcal{S}_\mathcal{C}$ is an isomorphism (Remark 2.11) and so together with Lemma 8.3 we have $\dim_k(\mathcal{S}_\mathcal{C}(\mathrm{Hig}(\mathcal{C}))) = \dim_k \mathrm{Hig}(\mathcal{C}) = \dim_k K^\perp$. Thus $\mathcal{S}_\mathcal{C}(\mathrm{Hig}(\mathcal{C})) = K^\perp = \mathrm{Hig}(\mathcal{C})$, once more appealing to Lemma 8.3. □

Recall the projective $SL(2,\mathbb{Z})$-action on $\mathrm{End}(Id_\mathcal{C})$ from Theorem 2.13 in case $\mathcal{C}$ is in addition ribbon.

Corollary 8.5. *Let $\mathcal{C}$ be in addition ribbon. Then* $\mathrm{Hig}(\mathcal{C})$ *is a submodule for the projective* $SL(2,\mathbb{Z})$*-action on* $\mathrm{End}(Id_\mathcal{C})$*.*

Proof. By Theorem 8.4, $\mathrm{Hig}(\mathcal{C})$ is stable under the action of the S-generator of $SL(2,\mathbb{Z})$. The action of T is given by $\theta\circ(-)$, and since $\mathrm{Hig}(\mathcal{C})$ is an ideal, it is stable under the T-action as well. □

We now give applications of Theorem 8.4 to the problem of diagonalisability of the Grothendieck ring of $\mathcal{C}$ (Proposition 8.6), to the Verlinde formula (Proposition 8.8) and to

[10] Applying (2.11) for $\sigma(V^*)_X$, then inserting $\delta_V\circ\delta_V^{-1}\colon V^{**}\to V\to V^{**}$ and moving δ_V^{-1} once around the loop, we get a loop coloured by V instead of V^{**}.

the modified trace (Proposition 8.10). Namely, we ask how much can be learned about the tensor product of projective objects from knowing the restriction of $\mathcal{S}_\mathcal{C}$ to $\mathrm{Hig}(\mathcal{C})$, and give a formula for the modified trace of the "open Hopf-link operator" $\sigma([P])$ in terms of the S-matrix elements. See e.g. [FHST, Fu, GR2, CG] for discussions of the Verlinde formula in the finitely non-semisimple setting and for more references.

We start with the application to the Grothendieck ring. Recall from Corollary 2.12 that $[M] \mapsto \sigma([M])$ is an injective ring homomorphism from $\mathrm{Gr}(\mathcal{C})$ to $\mathrm{End}(Id_\mathcal{C})$. For $M \in \mathcal{C}$ denote by

$$\rho^L([M])\colon\ \mathrm{End}(Id_\mathcal{C}) \to \mathrm{End}(Id_\mathcal{C}) \tag{8.7}$$

the left multiplication by $\sigma([M])$. Then $\rho^L([A])\rho^L([B]) = \rho^L([A]\cdot[B])$ and $\rho^L([\mathbf{1}]) = id$, i.e. $\rho^L\colon\ \mathrm{Gr}(\mathcal{C}) \to \mathrm{End}_k(\mathrm{End}(Id_\mathcal{C}))$ defines a faithful representation. Since $\mathrm{Rey}(\mathcal{C})$ and $\mathrm{Hig}(\mathcal{C})$ are ideals, they are submodules for the representation ρ^L.

We can now state the following result, part 1 of which is a generalisation of (part of) [CW, Thm. 3.14].

Proposition 8.6. *Consider the action $\rho^L : \mathrm{Gr}(\mathcal{C}) \to \mathrm{End}_k(\mathrm{End}(Id_\mathcal{C}))$.*

1. *The restriction of ρ^L to the submodule $\mathrm{Rey}(\mathcal{C})$, respectively $\mathrm{Hig}(\mathcal{C})$, is diagonalisable with action*

$$\rho^L([M])\,\phi_V \;=\; \ell_V^M\,\phi_V\ , \qquad \textit{resp.} \qquad \rho^L([M])\,\phi_{P_V} \;=\; \ell_V^M\,\phi_{P_V}\ , \tag{8.8}$$

   where $V \in \mathrm{Irr}(\mathcal{C})$ and $\ell_V^M := \big\langle \sigma([M])_V \big\rangle \in k$ (see Proposition 7.2).

2. *If $\mathcal{C}$ is semisimple, ρ^L can be diagonalised on $\mathrm{End}(Id_\mathcal{C})$.*

3. *If $\mathrm{char}(k) = 0$ and ρ^L can be diagonalised on $\mathrm{End}(Id_\mathcal{C})$, the $\mathcal{C}$ is semisimple.*

Proof. Part 1 is immediate from Proposition 7.2 and Remark 7.3. Part 2 follows from part 1 together with Proposition 5.10, which states that for $\mathcal{C}$ semisimple we have $\mathrm{Hig}(\mathcal{C}) = \mathrm{End}(Id_\mathcal{C})$.

Part 3 we prove by contradiction. Suppose that $\mathcal{C}$ is not semisimple, so that not every indecomposable projective object is simple. Let P be such a non-simple indecomposable projective object. Since $\mathrm{char}(k) = 0$, the map $\mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}_k(\mathcal{C})$ is injective, and hence by Theorem 2.6 (and the definition in (2.22)), $\phi_P \neq 0$. Furthermore, by Proposition 7.4, ϕ_P is nilpotent, in fact $\phi_P \circ \phi_P = 0$.

Since $\phi_P \in \mathrm{Hig}(\mathcal{C})$ (Proposition 7.1), by Theorem 8.4, we can write $\phi_P = \mathcal{S}_\mathcal{C}^{-1}(\sigma([P])) = \sum_{U\in\mathrm{Irr}(\mathcal{C})} x_U\,\sigma([P_U])$ and this linear combination is nilpotent because $\phi_P \circ \phi_P = 0$. Then, since σ is an injective algebra map (Corollary 2.12), $\mathrm{Gr}_k(\mathcal{C})$ contains a non-zero nilpotent element, and so ρ^L (which factors through $\mathrm{Gr}_k(\mathcal{C})$) cannot be diagonalised. □

In particular, we have the following consequence for the linearised Grothendieck ring:

Corollary 8.7. *If $\mathrm{char}(k) = 0$, the linearised Grothendieck ring $\mathrm{Gr}_k(\mathcal{C})$ is semisimple iff $\mathcal{C}$ is semisimple.*

Next we turn to an application of Theorem 8.4 to a Verlinde-type formula.

Write $\{M\}$ for the isomorphism class of an object $M \in \mathcal{C}$ (rather than its class $[M]$ in $\mathrm{Gr}(\mathcal{C})$). Define the ring $K_0(\mathcal{C})$ as the free abelian group spanned by the isomorphism classes of indecomposable projective objects (see e.g. [EGNO, Sec. 1.8]),

$$K_0(\mathcal{C}) := \bigoplus_{U \in \mathrm{Irr}(\mathcal{C})} \mathbb{Z}\{P_U\} . \tag{8.9}$$

The product of $\{P_U\}$ and $\{P_V\}$ is defined by the decomposition of $P_U \otimes P_V$ into indecomposable projectives. Thus $K_0(\mathcal{C})$ encodes the tensor product of projective objects in $\mathcal{C}$ up to isomorphism. We denote the structure constants of $K_0(\mathcal{C})$ by $M_{UV}^{\;W}$, that is,

$$\{P_U\}\{P_V\} = \sum_{W \in \mathrm{Irr}(\mathcal{C})} M_{UV}^{\;W}\, \{P_W\} , \tag{8.10}$$

or, equivalently,

$$P_U \otimes P_V \cong \sum_{W \in \mathrm{Irr}(\mathcal{C})} (P_W)^{\oplus M_{UV}^{\;W}} . \tag{8.11}$$

The map $K_0(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$, $\{P_U\} \mapsto [P_U]$ is a ring homomorphism. The matrix elements of this map are precisely the entries of the Cartan matrix of $\mathcal{C}$.

For the application to the Verlinde formula, we will need two bases of $\mathrm{Hig}(\mathcal{C})$. For the first basis, choose a subset $J \subset \mathrm{Irr}(\mathcal{C})$ such that

$$\big\{ \phi_{P_U} \,\big|\, U \in J \big\} \tag{8.12}$$

is a basis of $\mathrm{Hig}(\mathcal{C})$. Recall that $\widehat{\mathsf{C}}$ denotes the image of $\mathsf{C}(\mathcal{C})$ in $\mathrm{Mat}_n(k)$. By Corollary 5.9 we have $|J| = \mathrm{rank}(\widehat{\mathsf{C}})$. By Theorem 8.4, $\mathcal{S}_{\mathcal{C}}$ is an invertible endomorphism of $\mathrm{Hig}(\mathcal{C})$. The second basis then is

$$\big\{ \mathcal{S}_{\mathcal{C}}(\phi_{P_U}) \,\big|\, U \in J \big\} . \tag{8.13}$$

Let $\tilde{B}$ be the $\mathrm{Irr}(\mathcal{C}) \times J$-matrix expressing ϕ_{P_U} in the first basis,

$$\phi_{P_U} = \sum_{A \in J} \tilde{B}_{UA}\, \phi_{P_A} \qquad , \quad \text{for} \;\; U \in \mathrm{Irr}(\mathcal{C}) . \tag{8.14}$$

The change of basis map between (8.12) and (8.13) will be denoted by the $J \times J$-matrix $\tilde{S}$, that is

$$\mathcal{S}_{\mathcal{C}}(\phi_{P_A}) = \sum_{B \in J} \tilde{S}_{AB}\, \phi_{P_B} \qquad , \quad \text{for} \;\; A \in J . \tag{8.15}$$

Finally, let $\tilde{C}$ be the $J \times J$-matrix which describes the effect of taking duals,

$$\phi_{(P_A)^*} = \sum_{B \in J} \tilde{C}_{AB}\, \phi_{P_B} \qquad , \quad \text{for} \;\; A \in J . \tag{8.16}$$

By construction, $\tilde{S}$ and $\tilde{C}$ are invertible, while $\tilde{B}$ is invertible if and only if $\mathcal{C}$ is semisimple (Proposition 5.10).

Recall the definition of the constants $b_Q \in k^\times$ from (7.5) and of the subset $\mathrm{IrrProj}(\mathcal{C}) \subset \mathrm{Irr}(\mathcal{C})$ from (7.7). We remark that in order to get a basis in (8.12) one necessarily has $\mathrm{IrrProj}(\mathcal{C}) \subset J$.

Proposition 8.8. *For all $U, V \in \mathrm{Irr}(\mathcal{C})$ and $X \in J$ we have the following identity in k,*

$$\sum_{W\in\mathrm{Irr}(\mathcal{C})} M_{UV}^{\ W}\, \tilde{B}_{WX} \;=\; \sum_{Q\in\mathrm{IrrProj}(\mathcal{C})} b_Q\, (\tilde{B}\cdot\tilde{S})_{UQ}\, (\tilde{B}\cdot\tilde{S})_{VQ}\, (\tilde{S}\cdot\tilde{C})_{QX}\ . \tag{8.17}$$

Proof. From (8.10) we get the identity

$$\mathcal{S}_{\mathcal{C}}(\phi_{P_U}) \circ \mathcal{S}_{\mathcal{C}}(\phi_{P_V}) \;=\; \sum_{W\in\mathrm{Irr}(\mathcal{C})} M_{UV}^{\ W}\, \mathcal{S}_{\mathcal{C}}(\phi_{P_W})\ . \tag{8.18}$$

Substituting (8.14) and (8.15) gives

$$\begin{aligned} \sum_{A,B,X,Y\in J} \tilde{B}_{UA}\, \tilde{B}_{VB} \tilde{S}_{AX}\, \tilde{S}_{BY}\, \phi_{P_X} \circ \phi_{P_Y} \;&=\; \sum_{W\in\mathrm{Irr}(\mathcal{C})} \sum_{R,Z\in J} M_{UV}^{\ W}\, \tilde{B}_{WR}\, \tilde{S}_{RZ}\, \phi_{P_Z} \\ \overset{\text{Prop. 7.4}}{=} \sum_{Z\in\mathrm{IrrProj}(\mathcal{C})} (\tilde{B}\cdot\tilde{S})_{UZ} (\tilde{B}\cdot\tilde{S})_{VZ}\, b_Z\, \phi_{P_Z}\ . \end{aligned} \tag{8.19}$$

Next note that $\mathcal{S}_{\mathcal{C}}(\mathcal{S}_{\mathcal{C}}(\phi_M)) = \phi_{M^*}$ for all $M \in \mathcal{C}$ (as the square of $\mathcal{S}_{\mathcal{C}}$ results in the action of the antipode of $\mathcal{L}$ see [Ly1] and e.g. [FGR1, Lem. 4.19]). This results in the matrix identity $\tilde{S}\cdot\tilde{S} = \tilde{C}$. Hence $\tilde{S}$ and $\tilde{C}$ commute and $\tilde{S}^{-1} = \tilde{S}\cdot\tilde{C}$, as $\tilde{C}^2$ is the identity matrix. Multiplying both sides of (8.19) by $\tilde{S}^{-1}$ yields the claim. □

Note that (8.17) does not determine $M_{UV}^{\ W}$ uniquely unless $\mathcal{C}$ is semisimple and k is of characteristic zero. Indeed, otherwise $\tilde{B}$ does not have a right inverse and $M_{UV}^{\ W}$ is considered as an element of k, i.e. modulo $\mathrm{char}(k)$.

Remark 8.9.

1. Recall from Remark 1.6 (2) that one of the motivations for this paper comes from vertex operator algebras. Let V be a vertex operator algebra as in Remark 1.6 (2). In [GR2, Conj. 5.10], a precise conjecture is made on how the behaviour of so-called pseudo-trace functions of V under the modular S-transformation is related to $\mathcal{S}_{\mathcal{C}}$ for $\mathcal{C} = \mathrm{Rep}(V)$. In this relation, $\phi_M \in \mathrm{End}(Id_{\mathcal{C}})$ corresponds to the character of the V-module M (and not to a pseudo-trace function). Characters and their modular properties are much more accessible than those of pseudo-trace functions. That is, from a vertex operator algebra perspective, the following pieces of information are typically more accessible than the structure constants $M_{UV}^{\ W}$ themselves:
    - $\tilde{B}_{UA}$, $\tilde{C}_{AB}$ (find the projective modules of V and analyse their composition series as well as those of their contragredient modules),

- $\tilde{S}_{AB}$ (compute the modular S-transformation of characters of projective modules),
- b_Q (compute the modular S-transformation of the vacuum character and take the inverse of the coefficient of the character of Q).

Here the text in brackets indicates the computation one needs to do for V-modules, provided one assumes the conjectures in [GR2]. For the last point note that by (2.11) and (2.25) we have

$$\mathcal{S}_{\mathcal{C}}(\phi_{\mathbf{1}}) = id = b_Q^{-1}\phi_Q + (\text{terms that vanish when evaluated on } Q) \ . \qquad (8.20)$$

Thus one can expect all the data on the right hand side of (8.17) to be relatively accessible for a vertex operator algebra V (as compared to computing fusion of V-modules). We will see an example of this in Section 9.

2. If $\mathcal{C}$ is semisimple, then Proposition 8.8 is precisely the categorical Verlinde formula: Firstly, $K_0(\mathcal{C}) = \mathrm{Gr}(\mathcal{C})$, so that M_{UV}^{W} are the structure constants of $\mathrm{Gr}(\mathcal{C})$. Next, by [GR2, Rem. 3.10 (3)] we have $\mathcal{S}_{\mathcal{C}}(\phi_U) = (\mathrm{Dim}\,\mathcal{C})^{-\frac{1}{2}} \sum_{X\in\mathrm{Irr}(\mathcal{C})} s_{U^*,X}\,\phi_X$, where $\mathrm{Dim}\,\mathcal{C} = \sum_{X\in\mathrm{Irr}(\mathcal{C})} \dim(X)^2$ and $s_{A,B}\, id_{\mathbf{1}} = \mathrm{tr}(c_{B,A}\circ c_{A,B})$. Thus $\tilde{S}_{AB} = s_{A^*,B}/\sqrt{\mathrm{Dim}\,\mathcal{C}}$. Finally, $\tilde{B}_{AB} = \delta_{A,B}$, $\tilde{C}_{AB} = \delta_{A,B^*}$ and from (8.20) we see $b_Q = s_{\mathbf{1},Q}/\sqrt{\mathrm{Dim}\,\mathcal{C}}$. Altogether, (8.17) becomes (using $s_{X,Y} = s_{Y,X} = s_{X^*,Y^*}$)

$$M_{UV}^{W} \;=\; \frac{1}{\mathrm{Dim}\,\mathcal{C}} \sum_{Q\in\mathrm{Irr}(\mathcal{C})} \frac{s_{U^*,Q}\, s_{V^*,Q}\, s_{Q,W}}{s_{\mathbf{1},Q}} \;=\; \frac{1}{\mathrm{Dim}\,\mathcal{C}} \sum_{Q\in\mathrm{Irr}(\mathcal{C})} \frac{s_{U,Q}\, s_{V,Q}\, s_{W^*,Q}}{\dim(Q)} \ , \qquad (8.21)$$

where M_{UV}^{W} is considered as an element of k. This is the semisimple categorical Verlinde formula, see [Tu, Thm. 4.5.2].

3. A variant of the fusion algebra of projective characters was studied [La2, Cor. 4.4] in the context of the small quantum groups for simply laced simple Lie algebras. For factorisable ribbon Hopf algebras, a Verlinde-like formula for the action of the Grothendieck ring on the Higman ideal is given in [CW, Thm. 3.14].

It is an interesting problem to relate the modified trace on “open Hopf links” and properties of the modular S-transformation [CG]. The above results allow us to do this in the case the open Hopf link is coloured by projective objects:

Proposition 8.10. *For all $A \in J$, $X \in \mathrm{Irr}(\mathcal{C})$ and $Q \in \mathrm{IrrProj}(\mathcal{C})$ we have*

$$\frac{t_{P_X}\big(\sigma([P_A])_{P_X}\big)}{t_Q(id_Q)} \;=\; b_Q \sum_{B\in J} \tilde{S}_{AB}\, \widehat{\mathsf{C}}_{BX} \ . \qquad (8.22)$$

Proof. Apply $t_{P_X}(\cdots)$ to both sides of (8.15), expand ϕ_{P_B} as $\phi_{P_B} = \sum_{W\in\mathrm{Irr}(\mathcal{C})}[P_B : W]\phi_W$ and use (6.25) and (7.6). □

In particular, for $X = Q \in \mathrm{IrrProj}(\mathcal{C})$ we have

$$\frac{t_X(\sigma([P_A])_X)}{t_X(\sigma([\mathbf{1}])_X)} \;=\; \frac{\tilde{S}_{AX}}{b_X^{-1}} \ . \tag{8.23}$$

Moreover, by Lemma 4.5 we have $t_{P_B}(\sigma([P_A])_{P_B}) = t_{P_A^*}(\sigma([P_B^*])_{P_A^*})$, so that (8.22) implies that for $A, B \in J$,

$$\big(\tilde{S} \cdot \widehat{\mathsf{C}}\big)_{AB} \;=\; \big(\tilde{S} \cdot \widehat{\mathsf{C}}\big)_{B^*A^*} \ , \tag{8.24}$$

where by abuse of notation we restrict $\widehat{\mathsf{C}}$ to be a $J \times J$-matrix.

Remark 8.11. Equation (8.23) is a special case of a relation observed in the example of W_p-models in [CG, Sec. 3.3]. The notation used there is related to ours as $S^{\varpi;\mathcal{P}}_{M,P} = t_P(\sigma([M])_P)$, where $M \in \mathcal{C}$, $P \in \mathcal{P}roj(\mathcal{C})$. In [CG], the matrix of the modular S-transformation of pseudo-trace functions in some basis is denoted by S^χ. Using these notations, (8.23) reads $S^{\varpi;\mathcal{P}}_{P_A,X}/S^{\varpi;\mathcal{P}}_{\mathbf{1},X} = S^\chi_{P_A,X}/S^\chi_{\mathbf{1},X}$ for $A \in \mathrm{Irr}(\mathcal{C})$, $X \in \mathrm{IrrProj}(\mathcal{C})$ (assuming the conjectures in [GR2, Sec. 5] so that we can express $\tilde{S}_{AX}$ and b_X^{-1} in terms of modular properties of pseudo-trace functions).

9 Example: symplectic fermions

In this section we consider a family of examples of factorisable finite ribbon categories, namely the so-called symplectic fermion categories $\mathcal{SF}(\mathfrak{h}, \beta)$. The aim is to compute the modified trace for this class of examples and to illustrate the use of Proposition 8.8.

Let $\mathfrak{h}$ be a non-zero finite-dimensional symplectic vector space over $\mathbb{C}$ and let $\beta \in \mathbb{C}$ satisfy $\beta^4 = i^{\dim \mathfrak{h}}$. The $\mathbb{C}$-linear finite ribbon category $\mathcal{SF}(\mathfrak{h}, \beta)$ was introduced in [DR1, Ru]. Conjecturally, it is ribbon-equivalent to the representations of the even part of the symplectic fermion vertex operator super-algebra constructed from $\mathfrak{h}$ (hence the name), see [DR3, Conj. 7.3] for the precise statement and further references.

One can show that $\mathcal{SF}(\mathfrak{h}, \beta)$ is ribbon-equivalent to a certain finite-dimensional quasi-triangular quasi-Hopf algebra [GR1, FGR2], but we will not make use of this here. Instead we sketch the construction in [DR1]. We refer to [DR1, Sec. 5.2] or to e.g. [DR2, FGR2] for further details.

Denote the symmetric tensor category of finite-dimensional complex super-vector spaces by $s\mathcal{V}ect$. We define

$$\mathbb{\Lambda} = \Lambda(\mathfrak{h}) \ , \tag{9.1}$$

the exterior algebra of $\mathfrak{h}$, and consider it as an algebra in $s\mathcal{V}ect$ by taking the $\mathbb{Z}$-grading of $\Lambda(\mathfrak{h})$ modulo 2. In particular, $\dim(\mathbb{\Lambda}) = 2^{\dim \mathfrak{h}}$ and $\mathrm{sdim}(\mathbb{\Lambda}) = 0$, where $\mathrm{sdim}(-)$ denotes the super-dimension. $\mathbb{\Lambda}$ carries the structure of a commutative and cocommutative quasi-triangular Hopf algebra in $s\mathcal{V}ect$. We denote the multiplication, comultiplication, unit and counit by $\mu_{\mathbb{\Lambda}}$, $\Delta_{\mathbb{\Lambda}}$, $\eta_{\mathbb{\Lambda}}$, and $\varepsilon_{\mathbb{\Lambda}}$, respectively (see [DR1, Eqn. (5.4)] for the coalgebra structure).

We will make use of a specific cointegral $\Lambda^{co}_{\mathbb{\Lambda}} : \mathbb{\Lambda} \to \mathbb{C}$ for $\mathbb{\Lambda}$ (see [DR1, Eqn. (5.16)]). To describe it, let $d = \dim \mathfrak{h}$ and $a_1, \dots, a_d$ be a symplectic basis of $\mathfrak{h}$ such that $(a_1, a_2) = 1$, $(a_3, a_4) = 1$, etc. Then $\Lambda^{co}_{\mathbb{\Lambda}}$ is non-zero only on the top component of $\mathbb{\Lambda}$, where it takes the value

$$\Lambda^{co}_{\mathbb{\Lambda}}(a_1 a_2 \cdots a_d) \;=\; \beta^{-2} \,. \tag{9.2}$$

The category $\mathcal{SF}(\mathfrak{h}, \beta)$ is the direct sum of two full subcategories,

$$\mathcal{SF}(\mathfrak{h}, \beta) \;=\; \mathcal{SF}_0 \,\oplus\, \mathcal{SF}_1 \quad , \qquad \mathcal{SF}_0 = \mathrm{Rep}_{s\mathcal{V}ect}(\mathbb{\Lambda}) \;\;,\;\; \mathcal{SF}_1 = s\mathcal{V}ect \,, \tag{9.3}$$

where $\mathrm{Rep}_{s\mathcal{V}ect}(\mathbb{\Lambda})$ denotes the category of left $\mathbb{\Lambda}$-modules in $s\mathcal{V}ect$. $\mathcal{SF}_0$ and $\mathcal{SF}_1$ each contains two simple objects (up to isomorphism), which we denote as

$$\underbrace{\mathbf{1} = \mathbb{C}^{1|0} \,,\; \Pi\mathbf{1} = \mathbb{C}^{0|1}}_{\in \mathcal{SF}_0} \quad , \quad \underbrace{T = \mathbb{C}^{1|0} \,,\; \Pi T = \mathbb{C}^{0|1}}_{\in \mathcal{SF}_1} \,. \tag{9.4}$$

Here, Π is the parity-flip functor. The $\mathbb{\Lambda}$-action on $\mathbf{1}$ and $\Pi\mathbf{1}$ is trivial.

Since $\mathcal{SF}_1$ is semi-simple, T and ΠT are also projective. The projective cover of $\mathbf{1}$ is $\mathbb{\Lambda}$ itself, with $\pi_{\mathbf{1}} : \mathbb{\Lambda} \to \mathbf{1}$ being the projection on the top component of $\mathbb{\Lambda}$ (together with a choice of isomorphism to $\mathbb{C}^{1|0}$). Altogether, the projective covers are

$$P_{\mathbf{1}} = \mathbb{\Lambda} \quad , \quad P_{\Pi\mathbf{1}} = \Pi\mathbb{\Lambda} \quad , \quad P_T = T \quad , \quad P_{\Pi T} = \Pi T \,. \tag{9.5}$$

In particular, the Cartan matrix in (5.23) reads, in the ordering (9.4) of simple objects,

$$\mathsf{C}(\mathcal{SF}(\mathfrak{h}, \beta)) \;=\; \begin{pmatrix} 2^{2N-1} & 2^{2N-1} & 0 & 0 \\ 2^{2N-1} & 2^{2N-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \text{where} \quad N = \tfrac{1}{2} \dim \mathfrak{h} \,. \tag{9.6}$$

We write $* : \mathcal{SF}(\mathfrak{h}) \times \mathcal{SF}(\mathfrak{h}) \to \mathcal{SF}(\mathfrak{h})$ for the tensor product functor and will reserve the notation $\otimes$ for the tensor product in $s\mathcal{V}ect$ and for that of $\mathbb{\Lambda}$-modules in $s\mathcal{V}ect$. The tensor product $*$ is $\mathbb{Z}_2$-graded, and depending on which sector $X, Y \in \mathcal{SF}(\mathfrak{h})$ are chosen from, it is defined to be:

$$X * Y \;=\; \left\{ \begin{array}{cccl} X & Y & X * Y & \\ 0 & 0 & X \otimes Y & \in \mathcal{SF}_0 \\ 0 & 1 & X \otimes Y & \in \mathcal{SF}_1 \\ 1 & 0 & X \otimes Y & \in \mathcal{SF}_1 \\ 1 & 1 & \mathbb{\Lambda} \otimes X \otimes Y & \in \mathcal{SF}_0 \end{array} \right. \tag{9.7}$$

In sector 00, the tensor product is that of $\mathrm{Rep}_{s\mathcal{V}ect}(\mathbb{\Lambda})$, with $\mathbb{\Lambda}$-action given by the coproduct (and using the symmetric braiding of $s\mathcal{V}ect$). In sectors 01 and 10, the tensor product is that of the underlying super-vector spaces. In sector 11, the $\mathbb{\Lambda}$-action is given by the left regular action on $\mathbb{\Lambda}$.

We refer to [DR1, FGR2] for the associator and braiding isomorphisms in $\mathcal{SF}(\mathfrak{h}, \beta)$, as well as for the ribbon twist and for the rigid and the pivotal structures. The β-dependence enters into these structure maps. We have:

Proposition 9.1 ([DR1, FGR2])**.** *$\mathcal{SF}(\mathfrak{h},\beta)$ is factorisable.*

The embedding $s\mathcal{V}ect \hookrightarrow \mathcal{SF}_0$, $X \mapsto X$, where X is endowed with the trivial $\mathbb{\Lambda}$-action, is an embedding of ribbon categories. Every projective object in $\mathcal{SF}(\mathfrak{h},\beta)$ is isomorphic to $\mathbb{\Lambda} * X \oplus T * Y$ for some $X, Y \in s\mathcal{V}ect$, embedded in $\mathcal{SF}_0$. Note that by definition,

$$\mathrm{End}_{\mathcal{SF}}(\mathbb{\Lambda} * X) = \mathrm{End}_{s\mathcal{V}ect,\mathbb{\Lambda}}(\mathbb{\Lambda} \otimes X) \quad , \quad \mathrm{End}_{\mathcal{SF}}(T * X) = \mathrm{End}_{s\mathcal{V}ect}(X) \ , \tag{9.8}$$

where $\mathrm{End}_{s\mathcal{V}ect,\mathbb{\Lambda}}(\mathbb{\Lambda} \otimes X)$ are all even endomorphisms of the super-vector space $\mathbb{\Lambda} \otimes X$ which commute with the $\mathbb{\Lambda}$-action and $\mathrm{End}_{s\mathcal{V}ect}(X)$ are all even endomorphisms of the super-vector space X. Write $\mathrm{str}(-)$ for the super-trace.

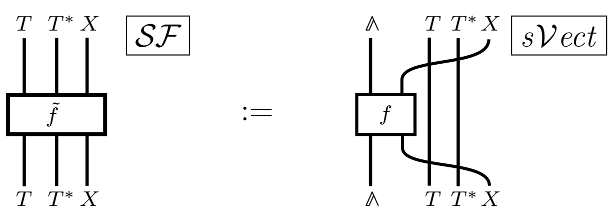
Proposition 9.2. *Let X be a finite-dimensional super-vector space and $f \in \mathrm{End}_{\mathcal{SF}}(\mathbb{\Lambda} * X)$, $g \in \mathrm{End}_{\mathcal{SF}}(T * X)$. Then, for each $t_0 \in \mathbb{C}^\times$,*

$$t_{\mathbb{\Lambda} * X}(f) = t_0 \, \mathrm{str}\big((\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}} \otimes id) \circ f \circ (\eta_{\mathbb{\Lambda}} \otimes id)\big) \quad , \quad t_{T*X}(g) = t_0 \, \mathrm{str}(g) \tag{9.9}$$

defines a modified trace on $\mathcal{P}roj(\mathcal{SF}(\mathfrak{h},\beta))$.

Proof. The proof makes use of the explicit associator and rigid structure of $\mathcal{SF}(\mathfrak{h},\beta)$. Since we only need them for this proof, we prefer to refer to [DR1, FGR2] for the explicit formulas, rather than repeat them here. We use the simple projective object $Q = T$ to start with as in (4.11) and set $t_T(id_T) = t_0$.

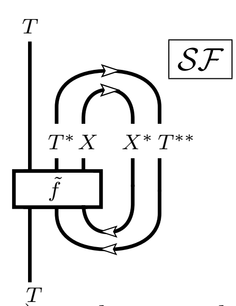
To compute $t_{\mathbb{\Lambda} * X}(f)$, we note that by (9.7), $\mathbb{\Lambda} = T * T^*$ and that the associator $\alpha_{T,T^*,X}$: $T * (T^* * X) \to (T * T^*) * X$ is the identity map for the trivial $\mathbb{\Lambda}$-module X. Here, the dual T^* is as in $s\mathcal{V}ect$. We define $\tilde{f} \in \mathrm{End}_{\mathcal{SF}}(T * (T^* * X))$ as



. (9.10)

Here, the boxes indicate in which category the string diagram is to be evaluated. In particular, the right hand side is given in $s\mathcal{V}ect$ and the crossings are the flips of super-vector spaces. We can now write

$$t_{\mathbb{\Lambda} * X}(f) \; = \; t_{T*(T^* * X)}(\tilde{f}) \; = \; t_T(F) \qquad \text{where} \quad F \; = $$



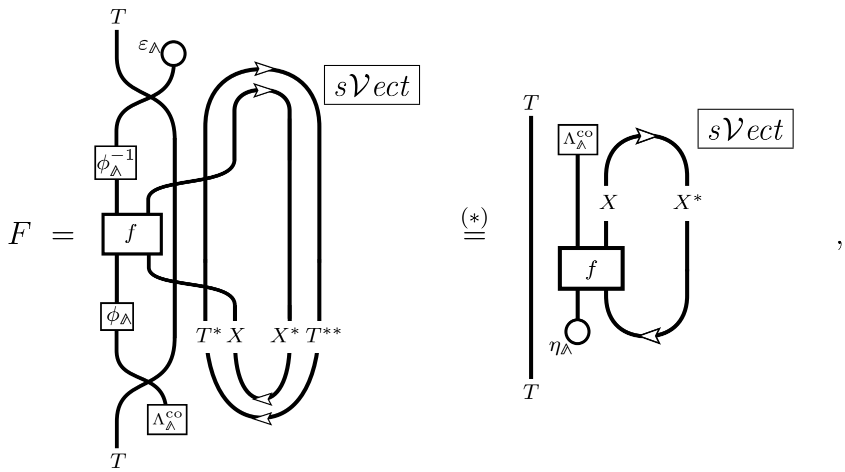
. (9.11)

In the last equality we used the partial trace property (4.1), and we implicitly used that $\alpha_{T,T^*,X}$ is the identity map. Next we write out F in terms of the structure morphisms (recall that we omit the tensor product '$*$' between objects for better readability):

$$F \;=\; \big[T \xrightarrow{\sim} T\mathbf{1} \xrightarrow{id \otimes \mathrm{coev}_{T^*X}} T((T^*X)(T^*X)^*) \xrightarrow{\alpha^{-1} \circ (\tilde{f} * id) \circ \alpha} T((T^*X)(T^*X)^*)$$

$$\xrightarrow{id\otimes\widetilde{\mathrm{ev}}_{T^*X}} T\mathbf{1} \xrightarrow{\sim} T\,\Big] \ , \tag{9.12}$$

where $\alpha^{-1}\circ(\tilde{f}*id)\circ\alpha$ stands for the composition $\alpha^{-1}_{T,T^*X,(T^*X)^*}\circ(\tilde{f}*id)\circ\alpha_{T,T^*X,(T^*X)^*}$. Finally, we compute F using the explicit associator in the 111-sector (see [DR1, Eqn. (2.27)]) and the duality maps from [DR1, Eqns. (3.59), (4.77), (4.78)]:



, (9.13)

where, $\phi_{\mathbb{A}}\colon \mathbb{A}\to\mathbb{A}$ appears in the associator in the 111-sector. In the equality (*) we used the identities $\varepsilon_{\mathbb{A}}\circ\phi_{\mathbb{A}}^{-1} = \Lambda^{\mathrm{co}}_{\mathbb{A}}$ and $\phi_{\mathbb{A}}\circ\Lambda^{\mathrm{co}}_{\mathbb{A}} = \eta_{\mathbb{A}}$ (for the first identity, use [DR1, Eqns. (3.45), (2.29)] and the fact that in the present case the distinguished group-like element g of [DR1] is just $\eta_{\mathbb{A}}$, for the second identity use [DR1, Eqns. (4.84), (3.45)]). Altogether, $t_{\mathbb{A}*X}(f) = t_T(F)$ gives the expression in (9.9).

The computation of $t_{T*X}(g)$ also uses the partial trace property (4.1): $t_{T*X}(g) = t_T(tr^r_X(g)) = t_0\,\mathrm{str}(g)$. □

Let us evaluate the modified trace from Proposition 9.2 for the four indecomposable projectives in (9.4). Note that all (left) $\mathbb{A}$-module maps $\mathbb{A}\to\mathbb{A}$ in $s\mathcal{V}ect$ are given by right multiplication R_a with an even element $a\in\mathbb{A}$. The same holds for $\mathbb{A}$-module maps $\Pi\mathbb{A}\to\Pi\mathbb{A}$.

$$t_{\mathbb{A}}(R_a) = t_0\,\Lambda^{\mathrm{co}}_{\mathbb{A}}(a) \ , \quad t_{\Pi\mathbb{A}}(R_a) = -t_0\,\Lambda^{\mathrm{co}}_{\mathbb{A}}(a) \ , \quad t_T(id_T) = t_0 \ , \quad t_{\Pi T}(id_{\Pi T}) = -t_0 \ . \tag{9.14}$$

After the computation of the modified trace, we now turn to an application of Proposition 8.8. Recall that $\mathrm{Irr}(\mathcal{SF}(\mathfrak{h},\beta)) = \{\mathbf{1},\Pi\mathbf{1},T,\Pi T\}$. From (9.5) one sees that $[P_{\mathbf{1}}] = [P_{\Pi\mathbf{1}}]$, so that we can choose $J = \{\mathbf{1},T,\Pi T\}$. For the matrices $\tilde{B}$, $\tilde{S}$ and $\tilde{C}$ from (8.14)–(8.16) one finds, for the above order of basis vectors,

$$\tilde{B} = \begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix} \ , \quad \tilde{S} = \begin{pmatrix}0 & 2^{N-1} & -2^{N-1}\\ 2^N & \frac{1}{2} & \frac{1}{2}\\ -2^N & \frac{1}{2} & \frac{1}{2}\end{pmatrix} \ , \quad \tilde{C} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \ , \tag{9.15}$$

where $\tilde{S}$ has been computed in [FGR2] (for $\beta = \mathrm{e}^{-\mathrm{i}N\pi/4}$). The constants b_Q are (using again the S-action from [FGR2] and (8.20))

$$b_T = 2^{-N-1} \quad , \quad b_{\Pi T} = -2^{-N-1} \ . \tag{9.16}$$

Remark 9.3.

1. The matrix $\tilde{S}$ can be read off directly from the modular S-transformation of characters for projective modules of the even part of the symplectic fermion vertex operator algebra (see [Ka, GK, Ab] and e.g. [DR3, Sec. 2.1] for the modular properties of the characters of simple modules).

2. Using Proposition 8.10, $\tilde{S}$ can be used to compute the modified trace of open Hopf link operators for projective objects. For example, (8.22) with $A = Q = T$, $X = \mathbf{1}$ gives

$$t_0^{-1}\, t_{P_\mathbf{1}}\big(\sigma([T])_{P_\mathbf{1}}\big) \;=\; b_T \sum_{B\in J} \tilde{S}_{TB}\, \mathsf{C}(\mathcal{SF})_{B\mathbf{1}} \;=\; 2^{-N-1}\cdot 2^N \cdot 2^{2N-1} \;=\; 4^{N-1}\ . \tag{9.17}$$

Starting from the data in (9.15), Equation (8.17) allows one to compute, for all $U, V \in \mathrm{Irr}(\mathcal{SF}(\mathfrak{h},\beta))$,

$$M_{UV}^{\mathbf{1}} + M_{UV}^{\Pi\mathbf{1}} \quad , \quad M_{UV}^{T} \quad , \quad M_{UV}^{\Pi T}\ . \tag{9.18}$$

Because $\tilde{B}$ is degenerate, we only obtain the sum $M_{UV}^{\mathbf{1}} + M_{UV}^{\Pi\mathbf{1}}$. However, in the present case we can do better. Namely, for finite tensor categories we have

$$M_{UV}^{\mathbf{1}} = \dim_k \mathcal{C}(P_U \otimes P_V, \mathbf{1}) = \dim_k \mathcal{C}(P_U, P_V^*) = [P_V^* : U] \overset{(5.24)}{=} \mathsf{C}(\mathcal{SF}(\mathfrak{h},\beta))_{V^*U}\ . \tag{9.19}$$

So in the present example, knowledge of the composition series of the P_U together with the data (9.15) actually fixes M_{UV}^{W} completely.

A Proofs of Lemma 2.1 and Corollary 2.3

Proof of Lemma 2.1. Consider the product complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^*C\otimes A & \xrightarrow{id\otimes f} & {}^*C\otimes B & \xrightarrow{id\otimes g} & {}^*C\otimes C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle {}^*g\otimes id} & & \downarrow{\scriptstyle {}^*g\otimes id} & & \downarrow{\scriptstyle {}^*g\otimes id} & & \\
0 & \longrightarrow & {}^*B\otimes A & \xrightarrow{id\otimes f} & {}^*B\otimes B & \xrightarrow{id\otimes g} & {}^*B\otimes C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle {}^*f\otimes id} & & \downarrow{\scriptstyle {}^*f\otimes id} & & \downarrow{\scriptstyle {}^*f\otimes id} & & \\
0 & \longrightarrow & {}^*A\otimes A & \xrightarrow{id\otimes f} & {}^*A\otimes B & \xrightarrow{id\otimes g} & {}^*A\otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{A.1}$$

Since the tensor product functor is biexact by assumption, this complex has exact rows and columns.

Let $K \xrightarrow{\kappa} {}^*B \otimes B$ be the kernel of ${}^*f \otimes g$. It is shown in the proof of [GKPM1, Lem. 2.5.1] that

1. the morphism $(id \otimes f) \circ \pi_1 + ({}^*g \otimes id) \circ \pi_2 : {}^*B \otimes A \oplus {}^*C \otimes B \to {}^*B \otimes B$, where $\pi_{1,2}$ are the projections on the first and second direct summand, factors through K as $\kappa \circ u$ with $u : {}^*B \otimes A \oplus {}^*C \otimes B \to K$ surjective.

2. there are (necessarily unique) morphisms $\alpha : K \to {}^*A \otimes A$ and $\gamma : K \to {}^*C \otimes C$ such that

$$(id \otimes f) \circ \alpha \;=\; ({}^*f \otimes id) \circ \kappa \quad , \quad ({}^*g \otimes id) \circ \gamma \;=\; (id \otimes g) \circ \kappa \; . \tag{A.2}$$

Write $\hat{b} := (id_{{}^*B} \otimes b) \circ \widetilde{\mathrm{coev}}_B$. Note that

$$({}^*f \otimes g) \circ \hat{b} \;=\; (id \otimes (g \circ b \circ f)) \circ \widetilde{\mathrm{coev}}_A \;=\; 0 \; , \tag{A.3}$$

where in the last step we used the commuting diagram (2.5). Thus $\hat{b} : \mathbf{1} \to {}^*B \otimes B$ factors through K, $\hat{b} = \big[\mathbf{1} \xrightarrow{\tilde{b}} K \xrightarrow{\kappa} {}^*B \otimes B\big]$. We have

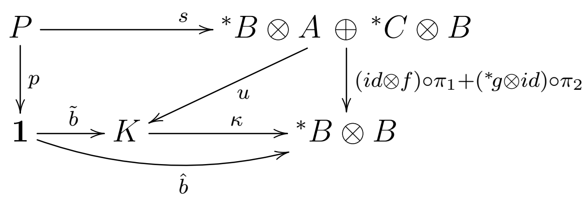
$$\begin{aligned}(id \otimes f) \circ \alpha \circ \tilde{b} &\overset{(A.2)}{=} ({}^*f \otimes id) \circ \kappa \circ \tilde{b} = ({}^*f \otimes id) \circ (id \otimes b) \circ \widetilde{\mathrm{coev}}_B \\ &= (id \otimes (b \circ f)) \circ \widetilde{\mathrm{coev}}_A \overset{(2.5)}{=} (id \otimes f) \circ (id \otimes a) \circ \widetilde{\mathrm{coev}}_A \; .\end{aligned} \tag{A.4}$$

Since $id \otimes f$ is injective, we get $\alpha \circ \tilde{b} = (id \otimes a) \circ \widetilde{\mathrm{coev}}_A$. Along similar lines – using injectivity of ${}^*g \otimes id$ – one shows that $\gamma \circ \tilde{b} = (id \otimes c) \circ \widetilde{\mathrm{coev}}_C$. Define the morphism $\Gamma : \mathbf{1} \to X$ as

$$\Gamma \;:=\; \eta_A \circ \alpha \circ \tilde{b} \;+\; \eta_C \circ \gamma \circ \tilde{b} \; . \tag{A.5}$$

By the above calculation we see that Γ is equal to the right hand side of (2.6). To see that Γ it also equal to the left hand side of (2.6), we make use of the surjection $p : P \to \mathbf{1}$, where $P \in \mathcal{M}$ is projective. We will now show that $\Gamma \circ p = \eta_B \circ \hat{b} \circ p$, which completes the proof.

Since by point 1 above, $u : {}^*B \otimes A \oplus {}^*C \otimes B \to K$ is surjective (and since P is projective), we can find s such that the left square in the following commuting diagram commutes:



(A.6)

Now compute

$$(id \otimes f) \circ \alpha \circ \tilde{b} \circ p \underset{(A.6)}{\overset{(A.2)}{=}} ({}^*f \otimes id) \circ \kappa \circ u \circ s \overset{(*)}{=} ({}^*f \otimes id) \circ (id \otimes f) \circ \pi_1 \circ s$$

$$= (id \otimes f) \circ ({}^*f \otimes id) \circ \pi_1 \circ s \ , \tag{A.7}$$

where in (*) we used the commuting diagram (A.6) and the fact that ${}^*f \circ {}^*g = 0$. Since $id \otimes f$ is injective, we can conclude that $\alpha \circ \tilde{b} \circ p = ({}^*f \otimes id) \circ \pi_1 \circ s$. Hence

$$\eta_A \circ \alpha \circ \tilde{b} \circ p \ = \ \eta_A \circ ({}^*f \otimes id) \circ \pi_1 \circ s \overset{\text{dinat.}}{=} \eta_B \circ (id \otimes f) \circ \pi_1 \circ s \ . \tag{A.8}$$

Similarly one can show $\gamma \circ \tilde{b} \circ p = (id \otimes g) \circ \pi_2 \circ s$ and

$$\eta_C \circ \gamma \circ \tilde{b} \circ p \ = \ \eta_B \circ ({}^*g \otimes id) \circ \pi_2 \circ s \ . \tag{A.9}$$

Combining (A.8) and (A.9) we finally find

$$\Gamma \circ p \ = \ \eta_B \circ (id \otimes f) \circ \pi_1 \circ s + \eta_B \circ ({}^*g \otimes id) \circ \pi_2 \circ s \overset{(A.6)}{=} \eta_B \circ \hat{b} \circ p \ . \tag{A.10}$$

□

Proof of Corollary 2.3. The proof of Part 1 follows that of [GKPM1, Cor. 2.5.2]. We give the details for the first equality, the second one is shown analogously using Remark 2.2 (1).

Let $A = \bigoplus_m A_m$ be a decomposition of A into indecomposable objects $A_m \in \mathcal{C}$, and write $j_m : A_m \to A$ and $p_m : A \to A_m$ for the corresponding embedding and projection maps. Writing $id_A = \sum_m j_m \circ p_m$ and applying dinaturality of η to j_m, we have

$$\begin{aligned} \eta_A \circ (id \otimes f) \circ \widetilde{\mathrm{coev}}_A \ &= \ \sum_m \eta_A \circ (id \otimes (j_m \circ p_m \circ f)) \circ \widetilde{\mathrm{coev}}_A \\ &= \ \sum_m \eta_{A_m} \circ (id \otimes (p_m \circ f \circ j_m)) \circ \widetilde{\mathrm{coev}}_{A_m} \ . \end{aligned} \tag{A.11}$$

Now for all simple $U \in \mathcal{C}$ and $u : A_m \to U$ we have $u \circ p_m \circ f \circ j_m = 0$ by assumption on f (as $u \circ p_m : A \to U$). It is therefore enough to prove the corollary for indecomposable A.

Let A be indecomposable. By Fitting's Lemma, every element of $\mathrm{End}(A)$ is either invertible or nilpotent. By assumption, f cannot be invertible, hence it is nilpotent, say $f^n = 0$. Let $K = \ker(f)$ and consider the commutative diagram

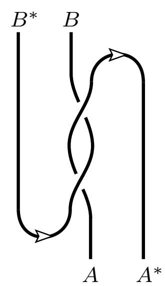
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\kappa} & A & \xrightarrow{\gamma} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \exists! c} & & \\ 0 & \longrightarrow & K & \xrightarrow{\kappa} & A & \xrightarrow{\gamma} & C & \longrightarrow & 0 \end{array} \tag{A.12}$$

with exact rows, i.e. (C, γ) is the cokernel of the kernel of f aka. the image of f. The map c exists and is unique by the universal property of the cokernel. By Lemma 2.1 we have

$$\eta_A \circ (id \otimes f) \circ \widetilde{\mathrm{coev}}_A \ = \ 0 + \eta_C \circ (id \otimes c) \circ \widetilde{\mathrm{coev}}_C \ . \tag{A.13}$$

We proceed by induction on the degree n of nilpotency. For $n = 1$ the claim of part 1 of the corollary is clear. Suppose now the claim holds for all maps whose degree of nilpotency is less or equal to $n-1$. Since $f \circ f^{n-1} = 0$, there is $h : A \to K$ such that $f^{n-1} = \kappa \circ h$. But then $c^{n-1} \circ \gamma = \gamma \circ f^{n-1} = \gamma \circ \kappa \circ h = 0$, and since γ is surjective, $c^{n-1} = 0$. By our induction assumption, (A.13) is zero.

For part 2, consider the dinatural transformation ξ from $(-) \otimes (-)^*$ to $B^* \otimes B$ with

$\xi_A \;=$ 

. (A.14)

To obtain (2.9), set $H := \xi_A \circ (f \otimes id_{A^*}) \circ \mathrm{coev}_A$ and use the second equality in part 1 to see $H = 0$. Then use $\widetilde{\mathrm{ev}}_B$ to turn $H : \mathbf{1} \to B^* \otimes B$ into an endomorphism of B, and after using the naturality of the braiding we get (2.9). □

B Proofs of Lemma 5.6 and Proposition 5.8

For this appendix, we fix

- $\mathcal{A}$: a finite abelian category over k,
- G: a projective generator of $\mathcal{A}$,
- $E = \mathrm{End}(G)$ the k-algebra of endomorphisms of G.

Write $Id_{\mathcal{P}roj(\mathcal{A})}$ for the identity functor on the projective ideal of $\mathcal{A}$, and $\mathrm{End}(Id_{\mathcal{P}roj(\mathcal{A})})$ for the algebra of its natural endomorphisms. The following lemma is standard.

Lemma B.1. *The map* $\mathrm{End}(Id_{\mathcal{A}}) \to \mathrm{End}(Id_{\mathcal{P}roj(\mathcal{A})})$, $\eta \mapsto \eta|_{\mathcal{P}roj(\mathcal{A})}$ *is a k-algebra isomorphism.*

Proof. Since $\eta \in \mathrm{End}(Id_{\mathcal{A}})$ is zero iff its values on all projective objects are zero, the map is injective. For surjectivity, let $A \in \mathcal{A}$ and pick an exact sequence $Q \xrightarrow{x} P \xrightarrow{y} A \to 0$ with $P, Q \in \mathcal{P}roj(\mathcal{A})$. Consider the diagram

$$\begin{array}{ccccccc} Q & \xrightarrow{x} & P & \xrightarrow{y} & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_Q} & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle \exists ! \eta_A} & & \\ Q & \xrightarrow{x} & P & \xrightarrow{y} & A & \longrightarrow & 0 \end{array} \tag{B.1}$$

The left square commutes by naturality of η on projectives. The dashed morphism η_A exists by the universal property of the cokernel (A, y) of $x : Q \to P$. It is straightforward to verify that η_A is independent of the choice of P, Q, x, y and that the family $\{\eta_A\}_{A \in \mathcal{A}}$ is a natural endomorphism of $Id_{\mathcal{A}}$ extending η from $\mathcal{P}roj(\mathcal{A})$ to all of $\mathcal{A}$. □

Proof of Lemma 5.6. Let $y \in \mathrm{End}(R)$ and define the family $(\eta_P)_{P\in\mathcal{P}roj(\mathcal{A})}$ as in (5.19):

$$\eta_P := \sum_{(\gamma_{PR})} \gamma'_{PR} \circ y \circ \gamma''_{PR} \ . \tag{B.2}$$

We will show that $\eta \in \mathrm{End}(Id_{\mathcal{P}roj(\mathcal{A})})$. The statement of the lemma then follows from Lemma B.1.

Let $P, Q \in \mathcal{P}roj(\mathcal{A})$ and $f : P \to Q$ be given. We need the following auxiliary result. Let X and Y in $\mathcal{A}(R,Q) \otimes \mathcal{A}(P,R)$ be defined as

$$X = \sum_{(\gamma_{QR})} \gamma'_{QR} \otimes (\gamma''_{QR} \circ f) \quad , \quad Y = \sum_{(\gamma_{PR})} (f \circ \gamma'_{PR}) \otimes \gamma''_{PR} \ . \tag{B.3}$$

Write $X = \sum_{(X)} X' \otimes X''$ and dito for Y. Then for all $z : Q \to R$ we have

$$\begin{aligned} \sum_{(X)} (z, X')_{QR}\, X'' &= \sum_{(\gamma_{QR})} (z, \gamma'_{QR})_{QR}\, \gamma''_{QR} \circ f \overset{(5.18)}{=} z \circ f \ , \\ \sum_{(Y)} (z, Y')_{QR}\, Y'' &= \sum_{(\gamma_{PR})} (z, f \circ \gamma'_{PR})_{QR}\, \gamma''_{PR} \overset{(5.17)}{=} \sum_{(\gamma_{PR})} (z \circ f, \gamma'_{PR})_{PR}\, \gamma''_{PR} \overset{(5.18)}{=} z \circ f \ . \end{aligned} \tag{B.4}$$

By non-degeneracy of the pairings we obtain $X = Y$. Using this identity, we get

$$f \circ \eta_P = \sum_{(Y)} Y' \circ y \circ Y'' = \sum_{(X)} X' \circ y \circ X'' = \eta_Q \circ f \ , \tag{B.5}$$

in other words, the family $(\eta_P)_{P\in\mathcal{P}roj(\mathcal{A})}$ is natural in $\mathcal{P}roj(\mathcal{A})$. □

To prepare the proof of Proposition 5.8, we need some further properties of the maps τ_R in (5.19), and we need a description of Calabi-Yau trace maps in terms of their restriction to indecomposable projectives.

Lemma B.2. *Let $R, S \in \mathcal{P}roj(\mathcal{A})$ and let $a : S \to R$, $b : R \to S$. Then $\tau_R(a \circ b) = \tau_S(b \circ a)$.*

Proof. By an argument similar to that showing $X = Y$ in (B.3) above, one obtains that for all $P, R, S \in \mathcal{P}roj(\mathcal{A})$ and $b : R \to S$,

$$\sum_{(\gamma_{PR})} \gamma'_{PR} \otimes (b \circ \gamma''_{PR}) \ = \ \sum_{(\gamma_{PS})} (\gamma'_{PS} \circ b) \otimes \gamma''_{PS} \ . \tag{B.6}$$

Then

$$\tau_R(a \circ b)_P \overset{(5.19)}{=} \sum_{(\gamma_{PR})} \gamma'_{PR} \circ a \circ (b \circ \gamma''_{PR}) \overset{(B.6)}{=} \sum_{(\gamma_{PS})} (\gamma'_{PS} \circ b) \circ a \circ \gamma''_{PS} \overset{(5.19)}{=} \tau_S(b \circ a)_P \ , \tag{B.7}$$

where we used (B.6) with both sides composed with $a \otimes id$. □

Corollary B.3. $\mathrm{Hig}(\mathcal{A}) = \mathrm{im}(\tau_G)$.

Proof. Let $R \in \mathcal{P}roj(\mathcal{A})$ and $h \in \mathrm{End}(R)$ be given. Pick an $m > 0$ such that there is a splitting $p : G^{\oplus m} \to R$, $i : R \to G^{\oplus m}$, $p \circ i = id_R$. Let $p_j : G \to R$, $i_j : R \to G$ be the components maps, that is, $id_R = \sum_{j=1}^m p_j \circ i_j$. Then

$$\tau_R(h) = \sum_{j=1}^m \tau_R(p_j \circ i_j \circ h) \overset{\text{Lem. B.2}}{=} \sum_{j=1}^m \tau_G(i_j \circ h \circ p_j) \ . \tag{B.8}$$

Thus $\tau_R(h)$ equals the image of $\sum_{j=1}^m i_j \circ h \circ p_j \in \mathrm{End}(G)$ under τ_G. □

For $T \in \prod_{U \in \mathrm{Irr}(\mathcal{A})}(\mathrm{End}(P_U) \to k)$ denote by T_U the component of T in $\mathrm{End}(P_U) \to k$. Write

$$\mathcal{T} \;=\; \Big\{ T \in \prod_{U \in \mathrm{Irr}(\mathcal{A})} (\mathrm{End}(P_U) \to k) \;\Big|\; T \text{ makes the full subcategory with objects } P_U,\, U \in \mathrm{Irr}(\mathcal{A}), \text{ Calabi-Yau} \Big\} \ . \tag{B.9}$$

In other words, $T \in \mathcal{T}$ iff for all $V, W \in \mathrm{Irr}(\mathcal{A})$, the pairings $(-,-)_{P_V P_W} : \mathcal{A}(P_V, P_W) \times \mathcal{A}(P_W, P_V) \to k$, $(f, g)_{P_V P_W} = T_W(f \circ g)$ are non-degenerate and symmetric in the sense that $(f, g)_{P_V P_W} = (g, f)_{P_W P_V}$.

Recall the notation $\mathrm{CY}(\mathcal{P}roj(\mathcal{A}))$ for the set of trace maps $(\mathsf{t}_P)_{P \in \mathcal{P}roj(\mathcal{A})}$ which turn $\mathcal{P}roj(\mathcal{A})$ into a Calabi-Yau category (Definition 5.4).

Lemma B.4. *The map*

$$\mathrm{CY}(\mathcal{P}roj(\mathcal{A})) \to \mathcal{T} \quad , \quad (\mathsf{t}_P)_{P \in \mathcal{P}roj(\mathcal{A})} \mapsto (\mathsf{t}_{P_U})_{U \in \mathrm{Irr}(\mathcal{A})} \tag{B.10}$$

is a k-linear isomorphism.

Proof.
Injectivity: Let $P \in \mathcal{P}roj(\mathcal{A})$ be given. Write $P = \bigoplus_{U \in \mathrm{Irr}(\mathcal{A})} P_U^{\oplus n_U}$ and let $j_{U\alpha} : P_U \to P$, $p_{U\alpha} : P \to P_U$, $U \in \mathrm{Irr}(\mathcal{A})$ and $\alpha = 1, \dots, n_U$, be the embedding and projection maps of the individual summands. By cyclicity and linearity of t_{P_U}, we have the decomposition

$$\mathsf{t}_P(f) \;=\; \sum_{U \in \mathrm{Irr}(\mathcal{A})} \sum_{\alpha=1}^{n_U} \mathsf{t}_{P_U}(p_{U\alpha} \circ f \circ j_{U\alpha}) \ , \tag{B.11}$$

where we used $\sum_{U,\alpha} e_{U\alpha} = id_P$ for the primitive idempotents $e_{U\alpha} = j_{U\alpha} \circ p_{U\alpha}$. Thus $(\mathsf{t}_P)_{P \in \mathcal{P}roj(\mathcal{A})}$ is uniquely determined by the values $(\mathsf{t}_{P_U})_{U \in \mathrm{Irr}(\mathcal{A})}$, showing that the map (B.10) is injective.

Surjectivity: Given $(\mathsf{t}_{P_U})_{U \in \mathrm{Irr}(\mathcal{A})} \in \mathcal{T}$, we can define all t_P via (B.11). One checks that t_P is independent of the choice of embedding and projection maps. It remains to verify that t_P is non-degenerate and symmetric. Let $P, Q \in \mathcal{P}roj(\mathcal{A})$ and let $j_{U\alpha}, p_{U\alpha}$ be a realisation of P as a direct sum of indecomposable projectives as above, and $j'_{U\alpha}, p'_{U\alpha}$ one of Q.

- *(Symmetry)* Let $f : P \to Q$, $g : Q \to P$. We compute

$$\begin{aligned} \mathsf{t}_Q(f \circ g) &= \sum_{U,V,\alpha,\beta} \mathsf{t}_{P_U}(p'_{U\alpha} \circ f \circ j_{V\beta} \circ p_{V\beta} \circ g \circ j'_{U\alpha}) \\ &\overset{\text{sym.}}{=} \sum_{U,V,\alpha,\beta} \mathsf{t}_{P_V}(p_{V\beta} \circ g \circ j'_{U\alpha} \circ p'_{U\alpha} \circ f \circ j_{V\beta}) = \mathsf{t}_P(g \circ f) \ . \end{aligned} \tag{B.12}$$

- *(Non-degeneracy)* Let $f : P \to Q$ be non-zero. Then there exist U, V, α, β such that $p'_{V\beta} \circ f \circ j_{U\alpha} \neq 0$. By non-degeneracy of the t_{P_U} there is $\tilde{g} : P_V \to P_U$ such that

$$0 \neq \mathsf{t}_{P_V}(p'_{V\beta} \circ f \circ j_{U\alpha} \circ \tilde{g}) \overset{\text{sym.}}{=} \mathsf{t}_Q(f \circ j_{U\alpha} \circ \tilde{g} \circ p'_{V\beta}) \tag{B.13}$$

  Thus for $g = j_{U\alpha} \circ \tilde{g} \circ p'_{V\beta}$ we get $\mathsf{t}_Q(f \circ g) \neq 0$.

This shows that the map (B.10) is surjective. □

Proof of Proposition 5.8. Recall that G denotes a choice of projective generator of $\mathcal{A}$ and $E = \text{End}(G)$. We will show the following statements, which are equivalent to points 1 and 2 in the proposition upon setting $\mathcal{A} = \text{mod-}A$ and $G = A_A$ as the right regular representation.

1. E admits a central form turning it into a symmetric algebra if and only if $\mathcal{P}roj(\mathcal{A})$ admits a Calabi-Yau structure.

2. If E is symmetric, the map ξ from (5.13) restricts to an isomorphism of algebras $\xi : \text{Hig}(\mathcal{A}) \to \text{Hig}(E)$.

Part 1: Clearly, if $\mathsf{t} \in \text{CY}(\mathcal{P}roj(\mathcal{A}))$, then $\mathsf{t}_G : E \to k$ turns E into a symmetric algebra with the central form $\varepsilon = \mathsf{t}_G$.

Suppose conversely that E is a symmetric algebra with respect to the central form $\varepsilon : E \to k$. Write $G = \bigoplus_{U \in \text{Irr}(\mathcal{A})} P_U^{\oplus n_U}$ and write $j_{U\alpha} : P_U \to G$, $p_{U\alpha} : G \to P_U$, $\alpha = 1, \dots, n_U$, for the embedding and projection maps of the individual summands. For $U \in \text{Irr}(\mathcal{A})$ define

$$T_U : \text{End}(P_U) \to k \quad , \quad f \mapsto \varepsilon(j_{U1} \circ f \circ p_{U1}) \ , \tag{B.14}$$

i.e. we only use the component with $\alpha = 1$ for each summand. We now claim that $(T_U)_{U \in \text{Irr}(\mathcal{A})} \in \mathcal{T}$ (recall the definition (B.9)). By Lemma B.4 this will prove part 1.

Non-degeneracy: Let $f : P_U \to P_V$ be non-zero. We need to find $g : P_V \to P_U$ such that $T_V(f \circ g) \neq 0$. Since f is non-zero, so is $j_{V1} \circ f \circ p_{U1}$. By non-degeneracy of ε, there is $\tilde{g} \in E$ such that $\varepsilon(j_{V1} \circ f \circ p_{U1} \circ \tilde{g}) \neq 0$. Set $g := p_{U1} \circ \tilde{g} \circ j_{V1}$ and compute

$$\begin{aligned} T_V(f \circ g) &= \varepsilon(j_{V1} \circ f \circ p_{U1} \circ \tilde{g} \circ j_{V1} \circ p_{V1}) \\ &\overset{\text{cycl.}}{=} \varepsilon(j_{V1} \circ p_{V1} \circ j_{V1} \circ f \circ p_{U1} \circ \tilde{g}) \\ &= \varepsilon(j_{V1} \circ f \circ p_{U1} \circ \tilde{g}) \neq 0 \ . \end{aligned} \tag{B.15}$$

Symmetry: Let $f : P_U \to P_V$, $g : P_V \to P_U$. We compute

$$\begin{aligned} T_V(f \circ g) &= \varepsilon(j_{V1} \circ f \circ g \circ p_{V1}) = \varepsilon(j_{V1} \circ f \circ p_{U1} \circ j_{U1} \circ g \circ p_{V1}) \\ &\overset{\text{cycl.}}{=} \varepsilon(j_{U1} \circ g \circ p_{V1} \circ j_{V1} \circ f \circ p_{U1}) = \varepsilon(j_{U1} \circ g \circ f \circ p_{U1}) \\ &= T_U(g \circ f) \ . \end{aligned} \tag{B.16}$$

Part 2: By the assumption that E is symmetric and by part 1, $\mathcal{P}roj(\mathcal{A})$ is Calabi-Yau via some $\mathsf{t} \in \mathrm{CY}(\mathcal{P}roj(\mathcal{A}))$. By Proposition 5.7 (and by a similar argument for $\mathrm{Hig}(E)$), neither $\mathrm{Hig}(\mathcal{A})$ nor $\mathrm{Hig}(E)$ depend on the choice of traces. In particular, we may replace whatever central form E was originally equipped with by t_G.

In terms of the central form t_G, the map $\tau : E \to Z(E)$ from (5.3) reads

$$\tau(f) = \sum_{(\gamma_{GG})} \gamma'_{GG} \circ f \circ \gamma''_{GG} \overset{(5.19)}{=} (\tau_G(f))_G \ . \tag{B.17}$$

Using this, we compute

$$\xi(\mathrm{Hig}(\mathcal{A})) \overset{\text{Cor. B.3}}{=} \xi(\tau_G(E)) \overset{(5.13)}{=} (\tau_G(E))_G \overset{(B.17)}{=} \tau(E) = \mathrm{Hig}(E) \ . \tag{B.18}$$

□

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