The unrolled quantum group inside
Lusztig's quantum group of divided powers

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ABSTRACT. In this letter we prove that the unrolled small quantum group, appearing in quantum topology, is a Hopf subalgebra of Lusztig's quantum group of divided powers. We do so by writing down non-obvious primitive elements with the right adjoint action.

We also construct a new larger Hopf algebra that contains the full unrolled quantum group. In fact this Hopf algebra contains both the enveloping of the Lie algebra and the ring of functions on the Lie group, and it should be interesting in its own right.

We finally explain how this gives a realization of the unrolled quantum group as operators on a conformal field theory and match some calculations on this side.

Our result extends to other Nichols algebras of diagonal type, including super Lie algebras.

CONTENTS

1. INTRODUCTION

Unrolled quantum groups are certain Hopf algebras that are important in quantum topology [GPT09][CGP15][GP16]. They are used to construct topological invariants from non-semisimple tensor categories, here the representation category of versions of quantum groups at an even root of unity. This approach is in particular able to recover the Reshetikhin-Turaev invariant [RT91]. The basic idea of unrolling is to decompose in every representation the eigenspaces V_ψ of the group elements in the quantum group into different weight space $\bigoplus_{q^\lambda=\psi} V_\lambda$. This is done by taking the semidirect product of the quantum group with the Cartan part of the corresponding Lie algebra. The theory is successfully worked out for $\mathfrak{sl}_2$, the concept of unrolling has been

generalized to higher rank Lie algebras in [GP16] and the unrolling construction has been categorically conceptualized and generalized to Nichols algebras of diagonal type in [AS17].

The main result of this letter is to realizes the unrolled small quantum group inside Lusztig's quantum group of divided powers $U_q^{\mathcal{L}}(\mathfrak{g})$. This is an infinite-dimensional Hopf algebras constructed by specialization. It fits into a Hopf algebra extension, see [Lus90][A96] for q odd and for G_2 not divisible by 3, which was generalized in [Len14] to divisible cases (as in this article) where the dual Lie algebra appears:

$$0 \longrightarrow u_q(\mathfrak{g}) \longrightarrow U_q^{\mathcal{L}}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}^{\vee}) \longrightarrow 0$$

From a mathematical perspective, realizing the unrolled small quantum group $u_q(\mathfrak{g}) \rtimes U(\mathfrak{h})$ inside $U_q^{\mathcal{L}}(\mathfrak{g})$ means to construct for this sequence a good section $U(\mathfrak{h})$ of the Cartan part $U(\mathfrak{g}^{\vee})^0$ i.e. to find (non-obvious) Cartan generator preimages $H_\alpha \in U_q^{\mathcal{L}}(\mathfrak{g})$ that are primitive elements and have the right adjoint action on $u_q(\mathfrak{g})$.

From a physical perspective, part of the Lusztig quantum group $U_q^{\mathcal{L}}(\mathfrak{g})$ is expected to act by screening charges on a free field theory, more precisely on the lattice vertex algebra associated to a rescaled root lattice Λ of $\mathfrak{g}$, see e.g. [FGST06, NT11, Len17]. Realizing the unrolled small quantum group inside $U_q^{\mathcal{L}}(\mathfrak{g})$ means to realize it as operators on this conformal field theory.
We can indeed present H_α as Λ^*-grading operators, whereas the group elements $K_\alpha \in u_q(\mathfrak{g})$ act as exponentiated grading operators. The grading operators H_α literally "unroll" any vertex algebra module into its Λ^*-eigenspaces, whereas K_α acts on each irreducible vertex algebra module as a single scalar. It is widely expected that a vertex subalgebra of this lattice vertex algebra (kernel of short screenings) should have a non-semisimple representation theory equivalent to $u_q(\mathfrak{g})$-representations, so the two unrolling constructions should be closely related.

Besides this main result we construct a curious Hopf algebra $U_q^{\mathcal{KL}}(\mathfrak{g})$ that acts as a "hybrid": It contains the Kac-DeConcini-Procesi quantum group $U_q^{\mathcal{K}}(\mathfrak{g})$ and surjects to the Lusztig quantum group $U_q^{\mathcal{L}}(\mathfrak{g})$. We can realize the full unrolled quantum group $U_q^{\mathcal{K}}(\mathfrak{g}) \rtimes U(\mathfrak{h})$ inside this new Hopf algebra. The representation category of $U_q^{\mathcal{K}}(\mathfrak{g})$ fibres over the points of the Lie group G (resp. $G^{\vee}$) and becomes non-semisimple over points of the subvariety of non-regular conjugacy classes of G. The fibre over the unit is the representation category of the small quantum group. Constructing topological invariants depending on the point by unrolling the entire $U_q^{\mathcal{K}}(\mathfrak{g})$ is very interesting.

Finally we mention how our results actually hold for quantum doubles of arbitrary Nichols algebras of diagonal type. As most prominent examples this allows us to cover also quantum groups associated to super-Lie algebras, where divided powers appear only for bosonic root vectors.

2. Preliminaries

Let $\mathfrak{g}$ a complex semisimple finite-dimensional Lie algebra with a choice of simple roots $\alpha_1, \ldots \alpha_{\mathrm{rank}}$ and positive roots Φ^+, with Cartan matrix a_{ij} and symmetrized Cartan matrix $d_{\alpha_i} a_{ij} = (\alpha_i, \alpha_j)$, where $d_\alpha = (\alpha, \alpha)/2 \in \{1, 2, 3\}$ for all $\alpha \in \Phi$. We will usually not denote the dependence on $\mathfrak{g}$.

Let q be a primitive ℓ-th root of unity and denote as usual $q_\alpha := q^{d_\alpha}$ and $\ell_\alpha := \mathrm{ord}(q_\alpha^2) = \ell/(\ell, 2d_\alpha)$. Let v be an indeterminant and denote again $v_\alpha := v^{d_\alpha}$. We use the notion of quantum numbers and quantum factorials dating back to Gauss

$$[n]_{v_\alpha} := \frac{v_\alpha^n - v_\alpha^{-n}}{v_\alpha - v_\alpha^{-1}} \qquad [n]_{v_\alpha}! := [n]_{v_\alpha} \cdots [1]_{v_\alpha} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{v_\alpha} := \frac{[n]_{v_\alpha}!}{[k]_{v_\alpha}![n-k]_{v_\alpha}!}$$

which after polynomial division all lay in $\mathbb{Z}[v, v^{-1}]$. The crucial property for us is $[\ell_\alpha]_{q_\alpha} = 0$.

Let us mention that for our application we usually require all $2d_\alpha \mid \ell$, while literature (in particular [Lus90] from Sec. 8.4 on) usually requires them to be relatively prime. Hence the orders ℓ_α become different from each other and from ℓ, which causes slightly unusual behaviour: Namely, the appearance of the dual root system $\mathfrak{g}^\vee$ in the exact sequence below and problems with the existence of the standard R-matrix. The algebra structure and the exact sequence for general ℓ are established in [Len14] and alternative R-matrices are determined in [LO16].

To avoid degeneracies in this article we exclude small values $\ell \neq 1, 2$, and also $\ell \neq 4$ if some $d_\alpha = 2$ and $\ell \neq 3, 4, 6$ if some $d_\alpha = 3$.

To this data the following Hopf algebras are associated by [Lus90]

- **Rational form** $U_q^{\mathbb{Q}(v)}(\mathfrak{g})$: An infinite-dimensional Hopf algebra over the field of rational functions in an indeterminant $\mathbb{Q}(v)$ defined by Drinfeld and Jimbo. It is generated by the root lattice Λ (or a different lattice between root- and weight-lattice) considered as an abelian group with group elements K_λ, $\lambda \in \Lambda$, together with elements $E_{\alpha_i}, F_{\alpha_i}$ for each simple root, and the following relations:

$$K_\lambda E_{\alpha_i} K_\lambda^{-1} = v^{(\lambda, \alpha_i)} E_{\alpha_i}, \ \forall \lambda \in \Lambda \qquad \textit{(group action)}$$

$$K_\lambda F_{\alpha_i} K_\lambda^{-1} = v^{-(\lambda, \alpha_i)} F_{\alpha_i}, \ \forall \lambda \in \Lambda \qquad \textit{(group action)}$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{i,j} \cdot \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{v_{\alpha_i} - v_{\alpha_i}^{-1}} \qquad \textit{(linking)}$$

and two sets of *quantum Serre-relations* for any $i \neq j \in I$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{v_{\alpha_i}} E_{\alpha_i}^{1-a_{ij}-r} E_{\alpha_j} E_{\alpha_i}^r = 0$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{v_{\alpha_i}^{-1}} F_{\alpha_i}^{1-a_{ij}-r} F_{\alpha_j} F_{\alpha_i}^r = 0$$

The coproduct, the counit and the antipode are on the group-algebra $\mathbb{C}[\Lambda]$ as usual and on the additional generators $E_{\alpha_i}, F_{\alpha_i}$ as follows:

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i} \qquad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}$$
$$S(E_{\alpha_i}) = -E_{\alpha_i} K_{\alpha_i}^{-1} \qquad S(F_{\alpha_i}) = -K_{\alpha_i} F_{\alpha_i}$$
$$\epsilon(E_{\alpha_i}) = 0 \qquad \epsilon(F_{\alpha_i}) = 0$$

Using Lusztig reflection operators, one can construct *root vectors* E_α, F_α for all $\alpha \in \Phi^+$ such that multiplication in the algebra gives a bijective linear map (PBW-basis):

$$\left(\bigotimes_{\alpha \in \Phi^+} \mathbb{C}[E_\alpha]) \right) \otimes \mathbb{C}[\Lambda] \otimes \left(\bigotimes_{\alpha \in \Phi^+} \mathbb{C}[F_\alpha] \right) \xrightarrow{\sim} U_q^{\mathbb{Q}(v)}$$

The three subalgebras generated by $E_{\alpha_i}, K_\lambda, F_{\alpha_i}$ respectively are called U_q^+, U_q^0, U_q^-.

- **Lusztig integral form of divided powers** $U_q^{\mathbb{Z}[v,v^{-1}], \mathcal{L}}(\mathfrak{g})$ ([Lus90] Thm 6.7): An infinite-dimensional Hopf algebra over the commutative ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$, generated by all

$$E_\alpha^{(t)} := \frac{E_\alpha^t}{[t]_{v_\alpha}!}, \quad F_\alpha^{(t)} := \frac{F_\alpha^t}{[t]_{v_\alpha}!}, \quad K_{\alpha_i}, \quad \begin{bmatrix} K_\alpha; 0 \\ t \end{bmatrix} := \prod_{s=1}^t \frac{K_\alpha v_\alpha^{1-s} - K_\alpha^{-1} v_\alpha^{-1+s}}{v_\alpha^s - v_\alpha^{-s}}, \quad t \in \mathbb{N}$$

such that multiplication in the algebra gives a bijective $\mathbb{Z}[v, v^{-1}]$-linear map (PBW-basis):

$$\left(\bigotimes_{t \in \mathbb{N}, \alpha \in \Phi^+} E_\alpha^{(t)} \mathbb{C} \right) \otimes \left(\bigotimes_{t \leq 1, i} K_{\alpha_i}^t \right) \otimes \left(\bigotimes_{t \in \mathbb{N}} \begin{bmatrix} K_\alpha; 0 \\ t \end{bmatrix} \mathbb{C} \right) \otimes \left(\bigotimes_{t \in \mathbb{N}, \alpha \in \Phi^+} F_\alpha^{(t)} \mathbb{C} \right) \xrightarrow{\sim} U_q^{\mathbb{Z}[v,v^{-1}], \mathcal{L}}$$

where we take $\{1, K_{\alpha_i}\}$ instead of full Λ because $K_{\alpha_i} - K_{\alpha_i}^{-1}$ is a multiple of $\begin{bmatrix} K_{\alpha_i}; 0 \\ 1 \end{bmatrix}$.

The Lusztig integral form is a Hopf subalgebra of the rational form and has the property that extension of scalars gives an isomorphism of Hopf algebras over $\mathbb{Q}(v)$:

$$U_q^{\mathbb{Q}(v)} \cong U_q^{\mathbb{Z}[v,v^{-1}], \mathcal{L}} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v)$$

- **Lusztig quantum group of divided powers** $U_q^{\mathcal{L}}(\mathfrak{g})$ (or restricted specialization): An infinite-dimensional Hopf algebra over $\mathbb{C}$ obtained for every choice of an element $q \in \mathbb{C}^\times$, defined by specialization

$$U_q^{\mathcal{L}} := U_q^{\mathbb{Z}[v,v^{-1}], \mathcal{L}} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}_q$$

where the indeterminant v acts on $\mathbb{C}_q$ by multiplication with the number q. So essentially we plug in $v = q$. This does not change the linear basis, but the algebra relations due to possible zeroes in $\mathbb{Z}[v, v^{-1}]$-coefficients , which depends very much on the choices made in the integral form. For q a primitive ℓ-th root of unity we get explicitly

$$E_\alpha^{\ell_\alpha} = [\ell_\alpha]! \cdot E_\alpha^{(\ell_\alpha)} = 0, \qquad K_{\alpha_i}^{2\ell_{\alpha_i}} = 1, \qquad F_\alpha^{\ell_\alpha} = 0$$

Multiplication in the algebra gives a bijective linear map (PBW-basis):

$$\left(\bigotimes_{\alpha\in\Phi^+}\mathbb{C}[E_\alpha]/(E_\alpha^{\ell_\alpha})\otimes\mathbb{C}[E_\alpha^{(\ell_\alpha)}])\right)\otimes\mathbb{C}[\Lambda/\Lambda']\otimes\left(\bigotimes_{\alpha\in\Phi^+}\mathbb{C}[\begin{bmatrix}K_\alpha;0\\\ell_\alpha\end{bmatrix}])\right)\otimes\left(\bigotimes_{\alpha\in\Phi^+}\mathbb{C}[F_\alpha]/(F_\alpha^{\ell_\alpha})\otimes\mathbb{C}[F_\alpha^{(\ell_\alpha)}])\right)\xrightarrow{\sim}U_q^{\mathcal{L}}$$

where Λ' is the sublattice of the root lattice Λ generated by all $K_{\alpha_i}^{2\ell_i}$.

- An **exact sequence of Hopf algebras**

$$0\longrightarrow u_q(\mathfrak{g})\longrightarrow U_q^{\mathcal{L}}(\mathfrak{g})\longrightarrow U(\mathfrak{g}^\vee)\longrightarrow 0$$

Here $U(\mathfrak{g}^\vee)$ is the universal enveloping algebra of the Lie algebra with the dual root system; a basis of the Lie algebra are the images of

$$E_\alpha^{(\ell_\alpha)},\ \begin{bmatrix}K_\alpha;0\\\ell_\alpha\end{bmatrix},\ F_\alpha^{(\ell_\alpha)}$$

The original result [Lus90] requires ℓ prime to all $2d_\alpha$ and has no $\mathfrak{g}^\vee$. The extended result [Len14] for arbitrary ℓ modifies the Lie algebras on the left and right hand side accordingly. Here we have written out the result under the assumptions we have in place for ℓ in this article (divisibility by $2d_\alpha$ and excluded small degenerate values).

The kernel in this exact sequence is:
- **Small quantum group** $u_q(\mathfrak{g})$: A finite-dimensional Hopf algebra over $\mathbb{C}$ generated by the elements $E_\alpha, K_\lambda, F_\alpha$, such that multiplication in the algebra gives a bijective linear map (PBW-basis):

$$\left(\bigotimes_{\alpha\in\Phi^+}\mathbb{C}[E_\alpha]/(E_\alpha^{\ell_\alpha})\right)\otimes\mathbb{C}[\Lambda/\Lambda']\otimes\left(\bigotimes_{\alpha\in\Phi^+}\mathbb{C}[F_\alpha]/(F_\alpha^{\ell_\alpha})\right)\xrightarrow{\sim}u_q$$

Similarly, the **Kac-Procesi-DeConcini integral form** $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}(\mathfrak{g})$ is be defined by elements

$$E_\alpha^t,\quad F_\alpha^t,\quad K_\lambda,\quad t\in\mathbb{N}$$

This (maybe more obvious) choice, which is properly contained in Lusztig's integral form, specializes to a Hopf algebra $U_q^{\mathcal{K}}$ (unrestricted specialization) with a large center properly containing $E_\alpha^{\ell_\alpha}, F_\alpha^{\ell_\alpha}, K_\alpha^{2\ell_\alpha}$. Conjecturally the following exact sequence holds:[1]

$$0\longleftarrow u_q(\mathfrak{g})\longleftarrow U_q^{\mathcal{L}}(\mathfrak{g})\longleftarrow \mathcal{O}(G^\vee)\longleftarrow 0$$

where $\mathcal{O}(G^\vee)$ is the commutative algebra of functions on the Lie group dual to the Lie group underlying $\mathfrak{g}$ (and the choice of Λ).

Note that similar to these two specializations one studies two versions of universal enveloping of a Lie algebra over a field in finite characteristic. In the obvious version one gets a large center with

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[1]This is proven if ℓ is relatively prime to the $2d_\alpha$ without $\mathfrak{g}^\vee$ appearing, but in the case of general ℓ one would need an analog result to [Len14] for this integral form and specialization

more primitive elements, in the non-obvious version one gets additional generators (divided powers) generating a new Lie algebra enveloping, which in this cases however again gets truncated etc.

The Kac-Procesi-DeConcini quantum group will not be used in the main construction of this article, but it is crucial for the application to quantum topology, see section 4.

3. MAIN RESULT

We already claimed that the images of the divided powers $E_\alpha^{(\ell_\alpha)}, F_\alpha^{(\ell_\alpha)}$ together with the rather unpleasant expression $\begin{bmatrix} K_\alpha; 0 \\ \ell_\alpha \end{bmatrix}$ map to a basis of the (dual) Lie algebra in the exact sequence

$$0 \longrightarrow u_q(\mathfrak{g}) \longrightarrow U_q^{\mathcal{L}}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}^\vee) \longrightarrow 0$$

A different (even more unpleasant) preimage of the Cartan generators in $\mathfrak{g}^\vee$ would be the commutator of the preimages $[E_\alpha^{(\ell_\alpha)}, F_\alpha^{(\ell_\alpha)}]$. We wish to find a nice preimage H_α of the Cartan generators, that has already appeared in our proof of [Len14] Thm. 4.1:

Theorem 3.1.

a) *The following elements[2] of the rational form are contained in the integral form*

$$H_\alpha := \frac{K_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)} \ \in U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}$$

where $\Phi_k(X)$ denotes the irreducible cyclotomic polynomial.
The H_α are $\begin{bmatrix} K_\alpha; 0 \\ \ell_\alpha \end{bmatrix}$ plus additional non-obvious K_α-terms made explicit in the proof.

b) *The elements H_α are skew-primitive in $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}$ and have a nice adjoint action*

$$\Delta(H_\alpha) = K_\alpha^{2\ell_\alpha} \otimes H_\alpha + H_\alpha \otimes 1$$

$$[H_\alpha, E_\beta] = E_\beta K_\alpha^{2\ell_\alpha} \cdot \frac{v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

Hence in Lusztig's quantum group $U_q^{\mathcal{L}}$ (after specialization) the elements are primitive and have the adjoint action we would expected from the image in $U(\mathfrak{g}^\vee)$:

$$\Delta(H_\alpha) = 1 \otimes H_\alpha + H_\alpha \otimes 1$$

$$[H_\alpha, E_\beta] = E_\beta \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \frac{q_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(q_\alpha^2)}$$

where after polynomial division the evaluation $\frac{q_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(q_\alpha^2)}$ is a well-defined nonzero complex number. It can be removed by rescaling H_α.

c) *We hence find the* unrolled small quantum group *inside $U_q^{\mathcal{L}}(\mathfrak{g})$:*

$$u_q(\mathfrak{g}) U_q^{\mathcal{L}}(\mathfrak{g})^0 \cong u_q(\mathfrak{g}) \rtimes U(\mathfrak{h})$$

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[2]Morally the reader should have in mind $\frac{K_\alpha^{2\ell} - 1}{v_\alpha^{2\ell_\alpha} - 1}$, which differs in the specialization by a nonzero scalar.

where the abelian Lie algebra $\mathfrak{h}$ is a preimage of the Cartan part $(\mathfrak{g}^\vee)^0$ under the exact sequence, spanned by the primitive elements $H_\alpha \in U_q^{\mathcal{L}^0}$ constructed above.

This unrolled small quantum group is the quantum group is the unrolled quantum group appearing in [GPT09, CGP15, GP16] modulo the relation $K_{\alpha_i}^{\ell_i} = 1$, i.e. it's representation category is the non-semisimple fibre $\mathcal{C}_0$. We will discuss this in section 4.
This unrolled small quantum group is the same as in [AS17]. We will discuss the suitable generalization of our results to arbitrary Nichols algebras of diagonal type in section 5.

Proof.

a) The following similar element from [Lus90] is by definition clearly contained in $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}$

$$K_\alpha^{+\ell_\alpha} \begin{bmatrix} K_\alpha; 0 \\ \ell_\alpha \end{bmatrix} = K_\alpha^{+\ell_\alpha} \prod_{s=1}^{\ell_\alpha} \frac{K_\alpha v_\alpha^{1-s} - K_\alpha^{-1} v_\alpha^{s-1}}{v_\alpha^s - v_\alpha^{-s}} = v_\alpha^{-\frac{\ell_\alpha(\ell_\alpha-1)}{2}} \cdot \frac{\prod_{s=1}^{\ell_\alpha} K_\alpha^2 - v_\alpha^{2(s-1)}}{\prod_{s=1}^{\ell_\alpha} v_\alpha^s - v_\alpha^{-s}},$$

Subtracting a suitable $\mathbb{Z}[v, v^{-1}]$-rescaling of this element from our element H_α gives

$$\frac{K_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)} - v_\alpha^{+\frac{\ell_\alpha(\ell_\alpha-1)}{2}} \frac{\prod_{s=1}^{\ell_\alpha} v_\alpha^s - v_\alpha^{-s}}{\Phi_{\ell_\alpha}(v_\alpha^2)} \cdot K_\alpha^{+\ell_\alpha} \begin{bmatrix} K_\alpha; 0 \\ \ell_\alpha \end{bmatrix} = \frac{(K_\alpha^{2\ell_\alpha} - 1) - (\prod_{s=1}^{\ell_\alpha} K_\alpha^2 - v_\alpha^{2(s-1)})}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

This is a polynomial in K_α^2, v_α^2, because the numerator has a zero at $v_\alpha^2 = q_\alpha^2$, so if we write the numerator as $\sum_n c_n(v_\alpha^2) K_\alpha^{2n}$ then each polynomial $c_n(v_\alpha^2)$ is divisible by $\Phi_{\ell_\alpha}(v_\alpha^2)$.

For example $\ell_\alpha = 2$ gives:

$$H_\alpha = v_\alpha(v_\alpha - v_\alpha^{-1})(1 - v_\alpha^{-2}) K_\alpha^2 \begin{bmatrix} K_\alpha; 0 \\ \ell_\alpha \end{bmatrix} + (K_\alpha^2 - 1)$$

b) The aspired relations are clearly true in the rational form, almost by construction:

$$\Delta(H_\alpha) = \frac{\Delta(K_\alpha^{2\ell_i} - 1)}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

$$= \frac{K_\alpha^{2\ell_\alpha} \otimes (K_\alpha^{2\ell_i} - 1) + (K_\alpha^{2\ell_i} - 1) \otimes 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

$$= K_\alpha^{2\ell_\alpha} \otimes H_\alpha + H_\alpha \otimes 1$$

$$[H_\alpha, E_\beta] = E_\beta \left(\frac{K_\alpha^{2\ell_\alpha} \left(v_\alpha^{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}} \right)^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)} - \frac{K_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^{2\ell_\alpha})} \right)$$

$$= E_\beta K_\alpha^{2\ell_\alpha} \cdot \frac{v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

The last q-factor may be rewritten as q-number or as a geometric series

$$\frac{v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)} = (v_\alpha^{\ell_\alpha})^{\frac{2(\alpha,\beta)}{(\alpha,\alpha)} - 1} \left[\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right]_{v_\alpha^{\ell_\alpha}} \frac{v_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

$$= \begin{cases} 0, & \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 0 \\ \left(\sum_{k=0}^{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}-1} (v_\alpha^2)^k\right) \frac{v_\alpha^{2\ell_\alpha}-1}{\Phi_{\ell_\alpha}(v_\alpha^2)}, & \frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0 \\ \left(-\sum_{k=0}^{-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}-1} (v_\alpha^2)^k\right) v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} \frac{v_\alpha^{2\ell_\alpha}-1}{\Phi_{\ell_\alpha}(v_\alpha^2)}, & \frac{2(\alpha,\beta)}{(\alpha,\alpha)} < 0 \end{cases}$$

so this obviously specializes for $q_\alpha^{2\ell_\alpha} = 1$ to

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \frac{v_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

c) The claim that u_q and the H_α generate all of $u_q U_q^{\mathcal{L}}$ follows from the PBW basis. The fact that the elements H_α commute is trivial. The structure as an algebra being a semidirect product $u_q \rtimes U(\mathfrak{h})$, and as a coalgebra being a tensor product, follows directly from the explicit results in b). .

$\square$

4. Unrolled quantum groups and categories

In this section we will show the applications of our results to the constructions in [GPT09, CGP15, GP16] by which it has been inspired and for which it has been intended:

In these articles, the authors construct quantum invariants for manifolds and knots from non-semisimple categories by "modified traces", which solves the problem that invariants obtained from nonsemisimple tensor categories tend to vanish. The first example in mind might be representations of a small quantum group u_q but for an even root of unity this does not have a braiding.

To get a modular tensor category, they instead consider the unrestricted quantum group $U_q^{\mathcal{K}}$, and then impose relations $E^r = F^r = 0$ (this is dropped in [GP16]) while K remains of infinite order; let us call this quotient $\bar{U}_q^{\mathcal{K}}$. Then they unroll this quantum group to $\bar{U}_q^{\mathcal{K}} \rtimes U(\mathfrak{h})$ and consider the category of modules $\mathcal{C}$ where K acts as e^H.

The author believes that there are two disjoint mechanisms at work here:

- The unrolling seems to be intended to get a modular tensor category, because it removes ambiguities in the weight λ associated to a K-eigenvalue q^λ. This should work and should be sufficient already in the case of the unrolled small quantum group treated above.

  Besides unrolling there are several attempts to construct modular tensor categories for small quantum groups at even roots of unity. The author would assume that they all can in principle be used for the same purpose:
  - Accept that the braiding is only up to an outer automorphism as in [Tan92, Res95].

- Consider larger lattices Λ and then accept that the monodromy matrix is only factorizable on a sub-Hopf algebra as in [RT91]. This seems like a small-scale unrolling, just enough to remove the $\mathbb{Z}_2$-obstruction

- Consider larger lattices Λ and modularize the larger category. This produces the representation category of a quasi-Hopf algebra with a 3-cocycle encapsulating the ambiguity in choices of λ in the braiding (the CFT side directly produces a 3-cocycle). This approach is currently pursued by the author, see [LO16].

- Taking the unrestricted quantum group $U_q^{\mathcal{K}}$ gives a much larger category fibred over $G^\vee$ according to the action of $\mathcal{O}(G^\vee)$ (e.g. according to the action of K^{2r}). The fibres are known (in the case ℓ prime to $2d_\alpha$) to be semisimple on elements in regular conjugacy classes. On the other hand there are singular points like $\mathcal{C}_0$ (all $K^{2r} = 1$) where we get the representations of the small quantum group. Studying quantum invariants attached to this situations is fascinating.

  The author's construction can also be applied to the unrolling of this situation. The authors suggestion would moreover be that this unrolled is again a subalgebra of a larger Hopf algebra, which is a curious hybrid of $U_q^{\mathcal{K}}$ and $U_q^{\mathcal{L}}$. Both points are made explicit in what follows.

The main result of this article realizes the unrolled small quantum group $u_q \rtimes U(\mathfrak{h})$ inside Lusztig's quantum group. Moreover, it makes the condition $K = e^H$ very natural and thus explains the category $\mathcal{C}_0$ from this point of view.

We wish to extend this to $U_q^{\mathcal{K}} \rtimes U(\mathfrak{h})$ and even further:

Lemma 4.1. *We regard the integral forms as (very large) complex Hopf algebra and define:*

$$U_q^{\mathcal{K}\mathcal{L}}(\mathfrak{g}) := U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}/(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}$$

This complex Hopf algebra depending on a choice $q \in \mathbb{C}^\times$ has the following properties:

a) There are Hopf algebra maps

$$U_q^{\mathcal{K}} \hookrightarrow U_q^{\mathcal{L}\mathcal{K}} \qquad U_q^{\mathcal{K}\mathcal{L}} \twoheadrightarrow U_q^{\mathcal{L}}$$

such that their composition sends $E_\alpha, F_\alpha, K_\lambda$ to themselves. Probably $U_q^{\mathcal{K}\mathcal{L}}$ is in some sense universal with this property.

b) There are elements such that multiplication induces a bijection of vectorspaces:

$$\left(\bigotimes_{\alpha \in \Phi^+} \mathbb{C}[E_\alpha] \otimes \mathbb{C}[E_\alpha^{(\ell_\alpha)}]\right) \otimes \mathbb{C}[\Lambda] \otimes \left(\bigotimes_{\alpha \in \Phi^+} \mathbb{C}[H_\alpha]\right) \otimes \left(\bigotimes_{\alpha \in \Phi^+} \mathbb{C}[F_\alpha] \otimes \mathbb{C}[F_\alpha^{(\ell_\alpha)}]\right) \xrightarrow{\sim} U_q^{\mathcal{K}\mathcal{L}}(\mathfrak{g})$$

,

Proof. While there is an injection between the integral forms $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}} \hookrightarrow U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}$, the respective maps between the specializations $U_q^{\mathcal{K}} \to U_q^{\mathcal{L}}$ is not injective (e.g. $E_\alpha^{\ell_\alpha} \mapsto 0$) because tensoring with the $\mathbb{Z}[v,v^{-1}]$-module $\mathbb{C}_q$ is not left-exact. Let us explicitly chose the obvious

minimal free resolution

$$0 \longrightarrow (v-q)\mathbb{Z}[v,v^{-1}] \overset{\subset}{\longrightarrow} \mathbb{Z}[v,v^{-1}] \overset{v \mapsto q}{\longrightarrow} \mathbb{C}_q$$

Then tensoring the integral forms with the free modules simply means regarding the integral forms as complex Hopf algebras with

$$U_q^{\mathcal{K}} \cong U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}/(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}} \qquad U_q^{\mathcal{L}} \cong U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}/(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}$$

so we can easily compute the Tor-Functors and get an exact sequence:

$$0 \longrightarrow U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}/(U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}} \cap (v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}) \longrightarrow U_q^{\mathcal{K}} \longrightarrow U_q^{\mathcal{L}}$$

Then it is natural to define the complex Hopf algebra

$$U_q^{\mathcal{KL}}(\mathfrak{g}) := U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}/(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}$$

There is an obvious injection from $U_q^{\mathcal{K}}$ and an obvious surjection to $U_q^{\mathcal{L}}$ which are the identity on the generators of the rational form.

The kernel of the surjection is $(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}}/(v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}$ and we explicitly know from the PBW basis of the integral forms that this is the ideal generated by the $(v-q)E_\alpha^{\ell_\alpha}$ and $K_\alpha^{\ell_\alpha} - K^{-\ell_\alpha}$, which are already in $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}$. Hence this kernel is the ideal generated by the kernel $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}/(U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}} \cap (v-q)U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{L}})$ of the surjection $U_q^{\mathcal{K}} \to U_q^{\mathcal{L}}$ $\qquad\qquad \square$

This hybrid Hopf algebra $U_q^{\mathcal{KL}}(\mathfrak{g})$ has the following applications:

- $U_q^{\mathcal{KL}}(\mathfrak{g})$ contains the unrolled $U_q^{\mathcal{K}}$ as Hopf subalgebra $U_q^{\mathcal{K}}(\mathfrak{g})U_q^{\mathcal{KL}}(\mathfrak{g})^0$.
  This is true because we have proven in the rational form in the Main Theorem 3.1:

$$[H_\alpha, E_\beta] = E_\beta K_\alpha^{2\ell_\alpha} \cdot \frac{v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} - 1}{v_\alpha^{2\ell_\alpha} - 1} \cdot \frac{v_\alpha^{2\ell_\alpha} - 1}{\Phi_{\ell_\alpha}(v_\alpha^2)}$$

  where the difference $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} - \frac{v_\alpha^{2\ell_\alpha \cdot \frac{2(\alpha,\beta)}{(\alpha,\alpha)}} - 1}{v_\alpha^{2\ell_\alpha} - 1}$ is clearly divisible by $(v-q)$, as this was equal in the specialization. Then the crucial observation is that the element $E_\beta K_\alpha^{2\ell_\alpha}$ is contained in $U_q^{\mathbb{Z}[v,v^{-1}],\mathcal{K}}$, so the difference still vanishes in $U_q^{\mathcal{KL}}(\mathfrak{g})$.
  Note that on the other hand there is no more relation $K_\alpha^{2\ell_\alpha} = 1$, so H_α is $(K_\alpha^{2\ell_\alpha}, 1)$-skew primitive in $U_q^{\mathcal{KL}}(\mathfrak{g})$.
- $U_q^{\mathcal{KL}}(\mathfrak{g})$ contains moreover a subalgebra $U(\mathfrak{g}^\vee)$, of which the Cartan-generator is responsible for unrolling. It also contains the commutative algebra of functions $\mathcal{O}(G^\vee)$ and we conjecture that the former acts on the latter as the Lie algebra acts on the function on the group by derivations. Is this action helpful to understand $U_q^{\mathcal{K}}$?

Example 4.2. *In $U_q^{\mathcal{KL}}(\mathfrak{sl}_2)$ with $\ell = 4$ the action of $E^{(2)} \in U(\mathfrak{sl}_2)$ on the algebra of functions is*

$$[E^{(2)}, K^4] = (1 - q^{16})E^{(2)}K^4$$

$$= \frac{1 - q^{16}}{(q - q^{-1})(q^2 - q^{-2})}E^2K^4 \neq 0$$

$$[E^{(2)}, E^2] = 0$$

$$[E^{(2)}, F^2] = \frac{q}{(q - q^{-1})^2}(K^2 - K^{-2})$$

5. Diagonal Nichols algebras

We argue, that the same construction can be done for arbitrary Nichols algebras of diagonal type.

Let (V, q_{ij}) be a diagonally braided vector space, then one associates a Nichols algebra $\mathfrak{B}(V)$, which is a braided Hopf algebra.

Example 5.1. *For some $q \in \mathbb{C}^\times$ the choices*

$$V = \bigoplus_{i=1}^{\text{rank}} E_{\alpha_i}, \qquad q_{ij} = q^{(\alpha_i, \alpha_j)}$$

give rise to the Nichols algebra $\mathfrak{B}(V) = U_q(\mathfrak{g})^+$ respectively $\mathfrak{B}(V) = u_q(\mathfrak{g})^+$ if q a root of unity.

Other examples include the super-Lie algebras with some $q_{ii} = -1$ (fermionic) and for other q_{ii} of small order algebras resembling additional Lie algebras in finite characteristic and a few unfamiliar algebras (called UFO's). A complete classification and a striking structure theory by arithmetic root systems and Weyl groupoids has been given by Heckenberger [Heck09].

Example 5.2 ($\mathfrak{sl}(2|1)$)**.** *For some $v \in \mathbb{C}^\times$ the choices*

$$V = E_1\mathbb{C} \oplus E_2\mathbb{C}, \qquad q_{ij} = \begin{pmatrix} -1 & v^{-1} \\ v^{-1} & -1 \end{pmatrix}$$

gives rise to the Nichols algebra $\mathfrak{B}(V) = U_v(\mathfrak{sl}(2|1))^+$. In fact for super-Lie algebras there are different non-equivalent Borel parts. Correspondingly the Weyl groupoid action (an odd reflection) changes the braiding matrix above to a different type of Weyl chamber:

$$V = F_1\mathbb{C} \oplus E_{12}\mathbb{C}, \qquad q_{ij} = \begin{pmatrix} -1 & v^{-1} \\ v^{-1} & v \end{pmatrix}$$

Let V be as usual realized as a Yetter-Drinfeld module over an abelian group G, say $G = \mathbb{Z}^{\text{rank}}$, so q_{ij} is given by a bicharacter $\chi(g_i, g_j)$. The quantum double construction can be used in the same way for an arbitrary Nichols algebra of diagonal type to define an analog of the quantum group $U(\chi)$, see [Heck10][AY13].

Now let us consider $U(\chi(v))$ with braiding matrix $q(v)_{ij} = \chi(v)(g_i, g_j)$ depending on a free parameter $v \in \mathbb{C}^\times$ as above, i.e. over the field $\mathbb{Q}(v)$. We can of course proceed precisely as in [Lus90]:

Definition 5.3. *Use the generalizations of Lusztig's automorphism to construct root vectors $E_\alpha, F_\alpha \in U(V, q_{ij})$ for all roots α as in [Heck10]. Then we can define the Hopf subalgebra and $\mathbb{Z}[v, v^{-1}]$-submodule A generated by all PBW monomials in the root vectors resp. by all PBW monomial in divided powers of the root generators.*
Then for a specific value $q \in \mathbb{C}^\times$, e.g. a root of unity, specialization $\otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{C}_q$ given then two complex Hopf algebras, which could be seen as generalized versions of the Kac-Procesi-DeConcini quantum group $U^{\mathcal{K}}(\chi)$ and the Lusztig quantum group of divided powers $U^{\mathcal{L}}(\chi)$.

Question 5.4. *It is not obvious that these algebras have the properties one would expect. Most severely, one would like to prove that indeed A is an integral form $A \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v)$, so that the PBW-monomials are a vector space basis of the specialization (a-priori the specialization could be trivial!). Apparently there are subtleties in the choice of prefactors for each Lusztig isomorphism.*

Once these questions are settled, the author conjectures that the Borel parts $U^{\mathcal{K}}(\chi)^\pm$ resp. $U^{\mathcal{L}}(\chi)^\pm$ are the distinguished pre- resp. post-Nichols algebra in [A15] resp. [AAR15]. But we expect moreover that again the Cartan part of the Lie algebra H_α (with the same formula as above) gives a realization of the unrolled small quantum group inside Lusztig's quantum group of divided powers. Moreover we would also again like to study the hybrid quantum group $U^{\mathcal{KL}}(\chi)$.

Note that the divided powers of interest are only over those root vectors E_α with no a-priori relations in place, e.g. in $\mathfrak{sl}(2|1)$ only for E_{12} but not for the fermionic generators with $E_1^2 = E_2^2 = 0$. These are the Cartan-like roots, see e.g. [A15]. The set of Cartan like roots forms an ordinary root system, typically of smaller rank than V, for example $\mathfrak{sl}_2$ for $\mathfrak{sl}(2|1)$ above. If we call this ordinary Lie algebra $\mathfrak{g}$ and the corresponding Lie group G, then we again expect exact sequences of Hopf algebras

$$0 \longrightarrow U_q(\chi(q)) \longrightarrow U^{\mathcal{L}}(\chi) \longrightarrow U(\mathfrak{g}) \longrightarrow 0$$

$$0 \longleftarrow U_q(\chi(q)) \longleftarrow U^{\mathcal{K}}(\chi) \longleftarrow \mathcal{O}(G) \longleftarrow 0$$

where $u_q(\chi) = U_q(\chi(q))$ is the (finite-dimensional) Hopf algebra associated to the braiding matrix with specific value $v = q$.

6. Conformal field theory

We also wish to point out the connection of our results on unrolling via Lusztig's quantum group to logarithmic conformal field theory [FGST06, NT11, FT10, Len17]:

Fix $\mathfrak{g}$ and ℓ such that all $2d_\alpha | \ell$ (i.e. $\ell_\alpha = \ell/2d_\alpha$) and define $p = \ell/2$. Let $\Lambda = \sqrt{p}\Lambda_{R^\vee}$ a rescaling of the root lattice of $\mathfrak{g}^\vee$. Then it has been conjectured that:

- There exists an action of parts of Lusztig quantum group $U_q^{\mathcal{L}}(\mathfrak{g})$ on the lattice vertex operator algebra $\mathcal{V}_\Lambda$ associated to Λ i.e. on the conformal field theory of a free boson on $\mathbb{C}^{\mathrm{rank}}/\Lambda$.
- The kernel $\mathcal{W} \subset \mathcal{V}_\Lambda$ of the action of the subalgebra $u_q(\mathfrak{g})^+$ has as category of representations a modular tensor category (this follows abstractly in the theory of vertex algebras), which is as an abelian category equivalent to $u_q(\mathfrak{g})$-modules (and as tensor category equivalent to a similar quasi-Hopf algebra)

The program has been proven for $\mathfrak{g} = \mathfrak{sl}_2$ in [FGST06, NT11], in the case $p = 2$ the quasi-Hopf algebra is obtained in [GR15]. The quantum group relations for the action of $u_q(\mathfrak{g})^+$ have been proven in general by the author in [Len17].

The action of F_α are given by *short screening charge operators* $\mathrm{Res}(\mathrm{Y}(e^{-\alpha/\sqrt{p}}))$, the action of $E_\alpha^{(\ell_\alpha)}$ by *long screening charge operators* $\mathrm{Res}(\mathrm{Y}(e^{+\alpha^\vee \sqrt{p}}))$, the action of H_α by a *scalar charge operator* $\mathrm{Res}(\mathrm{Y}(\partial\phi_{\alpha^\vee \sqrt{p}}))$ and the action of K_α by it's rescaled exponential $e^{\pi i\, \mathrm{Res}(\mathrm{Y}(\partial\phi_{\alpha/\sqrt{p}}))}$. The evaluation of the scalar charge operator H_α on a element in some module v_λ with degree $\lambda/\sqrt{p} \in \Lambda^*$ is

$$H_\alpha v_\lambda = (\alpha^\vee \sqrt{p}, \lambda/\sqrt{p}) = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v_\lambda$$

$$K_\alpha v_\lambda = e^{\pi i(\alpha/\sqrt{p}, \lambda/\sqrt{p})} v_\lambda = e^{\frac{2\pi i}{2p}(\alpha,\lambda)} v_\lambda$$

We see that this matches (up to a rescaling due to the dual) the condition $K = q^H$ on the category of the unrolled small quantum group. In this setting we can also recover our formula for H_α from our Main Theorem 3.1:

$$H_\alpha v_\lambda = \lim_{v \to q} \frac{K_\alpha^{2\ell_\alpha} - 1}{v_\alpha^{2\ell_\alpha} - 1} v_\lambda = \lim_{v \to q} \frac{v^{\frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\ell} - 1}{v^\ell - 1} v_\lambda = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v_\lambda$$

From a lattice vertex algebra perspective it is natural that a braiding in $\mathcal{V}_\Lambda$ comes not easy, because any module $\mathcal{V}_{[\lambda/\sqrt{p}]}$ contains an entire coset $[\lambda/\sqrt{p}] \in \Lambda^*/\Lambda$ but usually the braiding $q^{(\lambda,\mu)}$ does not factorize over this quotient, so one has to choose representatives, which is general causes a 3-cocycle to appear as associator. Note that a coset $[\lambda/\sqrt{p}]$ is precisely the set of degrees with the same action of all K_α. The effect of "unrolling" is to separate the different elements in the coset and hence to be able to define the braiding without ambiguities. This is the same effect we want for unrolling quantum group representations, and it is to expect the two versions are closely related.

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