ZMP-HH / 17-9
Hamburger Beiträge zur Mathematik Nr. 649
February 2017

Constructing new Borel subalgebras of quantum groups with a non-degeneracy property

Simon Lentner
Algebra and Number Theory, University Hamburg,
Bundesstraße 55, D-20146 Hamburg

Karolina Vocke
Phillips University Marburg,
Hans-Meerwein-Straße, D-35032 Marburg

Abstract. We construct explicit families of right coideal subalgebras of quantum groups, where all irreducible representations are one-dimensional and which are maximal with this property. We have previously called such a right coideal subalgebra a Borel subalgebra.

Conversely we can prove that any tringular Borel subalgebra fulfilling a certain non-degeneracy property is of the form we construct; this classification requires a key assertion about Weyl groups which we could only prove in type A_n. Borel subalgebras are interesting for structural reasons, but also because the induced representations give interesting unfamiliar analoga of category $\mathcal{O}$.

Contents

1. Introduction 2
2. Preliminaries 3
3. Counter indicators for ede property 6
4. A theorem on Weyl group elements 9
5. Triangular Borel subalgebras with non-degenerate character shifts 16
5.1. Main construction theorem 17
5.2. Generalizations of r_α 17
5.3. Commutator relations of character-shifted root vectors 23
5.4. On representation theory of triangular coideal subalgebras 25
5.5. Proof of the main construction theorem 26
5.6. Classification theorem 27
5.7. Induction of one-dimensional modules 30
6. Triangular Borel subalgebras with degenerate character shifts 30
6.1. Some classification results on Borel subalgebras of degenerate type 31
6.2. Construction of degenerate Borel subalgebras of height 1 32
7. Examples 34
7.1. Borel subalgebras of $U_q(\mathfrak{sl}_2)$ 34
7.2. Borel subalgebras of $U_q(\mathfrak{sl}_3)$ 34
7.3. Triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$ 36
References 41

1. Introduction

The quantum group $U_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra. For Lie algebras, a theorem by Sophus Lie states that the Borel subalgebras are the maximal solvable Lie-subalgebras and that they are all conjugate.

In [HLV17] we ask for quantum groups the same question: Which are the maximal right coideal subalgebras with the representation theoretic property, that all irreducible finite-dimensional representations are one-dimensional. We call this a *Borel subalgebra* of $U_q(\mathfrak{g})$. Besides the standard Borel subalgebra $U_q(\mathfrak{g})^+$ and its reflections, there is already for $\mathfrak{g} = \mathfrak{sl}_2$ a new family of Borel subalgebras, generated by elements

$$EK^{-1} + \lambda K^{-1}, \quad F + \lambda' K^{-1}, \qquad \lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$$

which are isomorphic to the quantized Weyl algebra. Note that they are not defined for $q = 1$ and interpolate between U_q^+ and U_q^-. We have conjectured (and proven for type A_n) a main structural result on the graded algebra associated to any right coideal subalgebra that has a triangular decomposition. As a second conjecture, this suggests an explicit description of all triangular Borel subalgebras.

The goal of the present article is to directly construct and classify triangular Borel subalgebras fulfilling an additional non-degeneracy condition (5). The algebras we construct consist of copies of the quantized Weyl algebra, as well as a subset of the remaining standard Borel subalgebra. In type A_n we can prove that all triangular non-degenerate Borel subalgebras are of this form. We also discuss some degenerate examples for type A_n.

Our main reason for studying Borel subalgebras is to gain more knowledge on the theory of arbitrary right coideal subalgebras. Another curious application is that for a given Borel subalgebra one may look at induced modules from one-dimensional representations. For the new Borel subalgebras of $U_q(\mathfrak{sl}_2)$ this yields infinite-dimensional reprentations isomophic as a vector space to $\mathbb{C}[K, K^{-1}]$, on which K acts by multiplication. We could prove that most of these induced modules are irreducible, while a discrete family has as quotients the irreducible finite-dimensional representations of $U_q(\mathfrak{sl}_2)$. It would be interesting to study an analogue of category $\mathcal{O}$ for these Borel subalgebras.

After preliminaries on right coideal subalgebras in Section 2 we proceed as follows:

In Section 3 we prepare a strategy to construct non-one-dimensional irreducible representations of right coideal subalgebras by restricting suitable $U_q(\mathfrak{g})$-modules and detecting non-one-dimensional composition factors.

In Section 4 we study a curious questions about Weyl groups, namely about "filling up" two given Weyl group elements to maximal Weyl group elements that retain the same property. This theorem later-on precisely tells us how far an interesting coideal subalgebra can be filled up until it is maximal. In this paper we were only able to prove this Weyl group assertion for type A_n.

In Section 5 we construct Borel subalgebras that consist of several quantized Weyl algebras associated to a set of pairwise orthogonal simple roots, together with a suitable subset of the remaining positive roots, see Main Theorem 5.1. Then we can prove that under the additional nondegeneracy condition these are in fact all Borel subalgebras for type A_n. Finally we determine the induced representations of one-dimensional representations of the Borel subalgebra; these are infinite-dimensional representations with a non-diagonal action of the Cartan part of $U_q(\mathfrak{g})$.

In Section 6 we also treat the next-difficult step, which are Borel subalgebras, where the degeneracy height is 1 in some sense. The degeneracy causes extensions of the Weyl algebras. In this case we first derive necessary conditions for the Weyl group elements and then again construct for type A_n right coideal subalgebras where all irreducible representations are one-dimensional.

In Section 7 we discuss as examples all Borel subalgebras for $\mathfrak{sl}_2, \mathfrak{sl}_3$ by-hand to compare this with our conjectures and results. Moreover we determine all triangular Borel subalgebras for $\mathfrak{sl}_4$, which is the first case where a degeneracy of height 2 appears. This is worked out thoroughly by the second author in [Vocke16], using a generating system for an arbitrary right coideal subalgebra.

2. Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of rank n over the field of complex numbers $\mathbb{K} = \mathbb{C}$.

We denote by $\Pi = \{\alpha_1, \dots \alpha_n\}$ a set of positive simple roots, by Q the root lattice, and by $\Phi^+ \subset Q$ the set of all positive roots. We denote by $(,)$ the symmetric bilinear form on $\mathbb{R}^\Pi$ with the Cartan matrix $c_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.

Our article is concerned with the quantum group $U_q(\mathfrak{g})$ where q is not a root of unity. There are algebra automorphisms due to Lusztig T_w for each Weyl group element $w \in W$, and with these one constructs root vectors E_μ for all $\mu \in \Phi^+$, see [Jan96] Chapter 8.

A subalgebra C of a Hopf algebra H is called a right coideal subalgebra (RCS) if $\Delta(C) \subset C \otimes H$. Three essential results in the theory of coideal subalgebras of quantum groups are:

Call a right coideal subalgebra $C \subset U_q(\mathfrak{g})$ homogeneous iff $U^0 \subset C$ (in particular C is then homogeneous with respect to the Q-grading).

Theorem 2.1 ([HS09] Theorem 7.3)**.** *For every $w \in W$ there is an RCS $U^+[w]U^0$, where $U^+[w]$ is generated by the root vectors E_{β_i} for all β_i in the subset of roots*

$$\Phi^+(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \prec 0\} = \{\beta_i \mid i \in \{1, \dots, \ell(w)\}\}$$

In particular $|\Phi^+(w)| = \ell(w)$, and

$$v < w \text{ iff } \Phi^+(v) \subseteq \Phi^+(w) \tag{1}$$

and for the longest element $\Phi^+(w_0) = \Phi^+$.

Conversely, every homogeneous RCS $C \subset U_q^+(\mathfrak{g})U^0$ is of this form for some w.

Theorem 2.2 ([HK11a] Theorem 3.8)**.** *The homogeneous RCS $C \subset U_q(\mathfrak{g})$ are of the form*

$$C = U^+[w]U^0U^-[v]$$

for a certain subset of pairs $v, w \in W$.

Non-homogeneous RCS are only classified on $U_q^- U^0$ (or $U_q^+ U^0$):

Theorem 2.3 ([HK11b] Theorem 2.15)**.** *For $w \in W$, let $\phi : U_q^-[w] \to \mathbb{K}$ be a character and define*

$$\texttt{supp}(\phi) := \{\beta \in Q \mid \exists x_\beta \in U_q^-[w] \text{ with } \phi(x_\beta) \neq 0\}$$

Take any subgroup $L \subset \texttt{supp}(\phi)^\perp$, then there exists a character-shifted RCS

$$U_q^-[w]_\phi := \{\phi(x^{(1)})x^{(2)} \mid \forall x \in U_q^-[w]\}$$

and an RCS $U_q^-[w]_\phi T_L$ with group ring $T_L = \mathbb{K}[L] \subset U^0$.

Conversely, every RCS $C \subset U_q^-(\mathfrak{g})U^0$ if of this form

To construct non-homogeneous RCS $C \subset U_q(\mathfrak{g})$ we shall in the following restrict our attention to:

Definition 2.4. *We call a right coideal subalgebra* triangular*, if each element splits into elements in $C^{\geq 0}$ and $C^{\leq 0}$:*

$$C = (C \cap U^{\geq 0})(C \cap U^{\leq 0})$$

We denote $C^{\geq 0} := C \cap U^{\geq 0}$, $C^{\geq 0} := C \cap U^{\leq 0}$ and $C^+ := C \cap U^+$ and $C^- := C \cap U^-$.

Our main interest is

Definition 2.5. *We call an algebra* ede *iff every finite-dimensional irreducible representation is one-dimensional. We call an RCS of $U_q(\mathfrak{g})$ a* Borel subalgebra *iff it is ede and it is a maximal RCS with this property.*

Example 2.6. *All homogeneous Borel subalgebras are isomorphic (via some T_w) to the* Standard Borel subalgebra *U^+U^0. The fact that this is a Borel subalgebra is not entirely trivial, see* [HLV17].

Example 2.7. *There is a family of non-homogeneous Borel subalgebras of $U_q(\mathfrak{sl}_2)$ generated by*

$$\{EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1}\}, \qquad \lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$$

Different choices of λ, λ' are isomophic via some Hopf automorphism $E \mapsto tE$, $F \mapsto t^{-1}F$.

As an algebra, this is isomorphic to a quantized Weyl algebra

$$k\langle X, Y\rangle/(XY - qYX - 1)$$

Proof. We want to prove the ede property of the quantized Weyl algebra: Let V be a finite dimensional irreducible representation. Consider the eigenvector v of the element $T := YX$ with eigenvalue t. One can easily see, that Yv is an eigenvector with the eigenvalue $qt + 1$:

$$YX(Yv) = Y(qYX + 1)v = (qt + 1)Yv$$

Similarly, one can show that Xv is an eigenvector of T with eigenvalue $\frac{1}{q}(t-1)$:

$$YX(Xv) = \frac{1}{q}(XYX - X)v = \frac{1}{q}(t-1)Xv$$

Thus the eigenvectors of T are a basis of V, as V is irreducible. On the other hand for each i there are eigenvectors $Y^i v$ of T. As V is finite dimensional, they cannot have pairwise distinct eigenvalues. If T has eigenvalue 0, 1 would thus be an eigenvalue too and there would be an infinite number of different eigenvalues, as q is not a root of unity. Thus the eigenvalue 0 is not possible, and so each two eigenvectors must be equal and thus each two eigenvalues must be equal. Then follows $t = \frac{1}{1-q}$. With this t we get:

$$XYv = (qYX + 1)v = \left(\frac{1}{1-q}q + 1\right)v = \frac{(q+1-q)}{1-q}v = tv$$

As T has only the eigenvalue t, it acts as a scalar on V, the same is true for XY. Thus X and Y commute on all of V and then each finite dimensional irreducible representation is 1- dimensional. □

A main structure theory of general Borel subalgebras stems from:

Conjecture 2.8 ([HLV17])**.** *The map $f : \mathtt{gr}(U^-[w]_\phi) \to U^{\leq 0}$, sending all elements to their leading terms is an injektive homomorphism of $\mathbb{Z}$-graded right coideal subalgebras. The image D of f has the following form:*

- *$D_0 =: M$ is the monoid (!) $M := \langle K_\mu^{-1} \mid \mu \in \mathtt{supp}(\phi)\rangle$*
- *for the quotient group L of M we get $DT_L = U^-[w']T_L$ for the following $w' \in W$:*

$$w' := (\prod_{\beta \in \mathtt{supp}(\phi)} s_\beta)w, \qquad \text{where } w = s_{\alpha_{k_1}} s_{\alpha_{k_2}} \ldots s_{\alpha_{k_m}} \in W$$

As all elements in $\mathtt{supp}(\phi)$ are pairwise orthogonal, w' is the element, which arises from w by deleting all factors s_{α_i}, for all i with $\beta_i \in \mathtt{supp}(\phi)$.

Theorem 2.9. *[*[HLV17]*] Conjecture 2.8 holds for A_n.*

This conjecture has implications to the representation theory of the RCS:

Conjecture 2.10. *[*[HLV17]*] Let $C = U^-[w]_{\phi^-} T_L \psi(U^+[w^+])_{\phi^+}$ be a triangular right coideal subalgebra with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-)$ and $L = \mathtt{supp}(\phi^+)^\perp$, where $\psi(x_\beta) = q^{-(\beta,\beta)/2} x_\beta K_\beta^{-1}$ for all elements x_β of degree β, which is essentially the antipode. Then C is a Borel subalgebra, if and only if $w^+ w'^{-1} = w_0$.*

At least we could prove:

Lemma 2.11. *[*[HLV17]*] In the case A_n holds: Given w and w' as in conjecture 2.8 and given a triangular right coideal subalgebra $C = U^-[w]_{\phi^-} T_L \psi(U^+[w^+])_{\phi^+}$ with $L \subset (\mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-))^\perp$. If $\ell(w'^{-1}w^+) < \ell(w') + \ell(w^+)$ then C is not ede.*

3. Counter indicators for ede property

Corollary 3.1. *For all one-dimensional representations V of $U_q(\mathfrak{sl}_2)$ follows in particular $E(V) = F(V) = 0$ and $K(V) = \pm 1$. This can be seen directly from the relations: Due to the relation $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$ we get that in any one dimensional representation K has to act as ± 1. From $EK = q^2KE$ and $FK = q^{-2}KF$ thus follows, that E and F have to act trivial.*

This gives a first method to detect non-one-dimensional composition factors in representations restricted to RCS. A direct application is that RCS containing the semisimple subalgebra $U_q(\mathfrak{sl}_2)$ are not ede as follows. Later we will deal with restrictions to non-semisimple subalgebras:

Lemma 3.2. *Let $B \subset U$ be a Borelsubalgebra. If for $\beta \in Q$ the element K_β lies in B, then the element K_β^{-1} lies in B too.*

Proof. Assume $K_\beta^{-1} \notin B$, consider $A := \langle K_\beta^{-1}, B\rangle \subset U$: This is an RCS as both B and $k[K_\beta^{-1}]$ are RCS. Due to the assumed maximality of $B \subsetneq A$ there exists an irreducible representation V of A with $dim(V) > 1$. As B is ede, the representation $V|_B$ has a one dimensional subrepresentation $\langle v\rangle$ and with $K_\beta \in B$ we get $K_\beta v = \lambda v$ for some $\lambda \in k$. But this already gives a one-dimensional subrepresentation of V which contradicts the maximality of B. □

Theorem 3.3. *Let A resp. A' be subalgebras of U with $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} \in A$ resp. $E_\alpha K_\alpha^{-1}, F_\alpha, K_\alpha^2, K_\alpha^{-2} \in A'$ for some $\alpha \in \Pi$. Then A resp. A' has a multidimensional irreducible representation.*

Proof. Let M be a representation of A. Consider first the restriction of A on $U_q(\mathfrak{sl}_2) \subset A$ generated by E_α, F_α and K_α, K_α^{-1}. Due to Corollary 3.1 we know, that in each finite dimensional representation of $U_q(\mathfrak{sl}_2)$ the element K_α^2 acts as 1. We know that restricted to the semisimple algebra $U_q(\mathfrak{sl}_2)$ the representation M decomposes into irreducible representations. Assume each finite dimensional irreducible representation of A is one dimensional, then $M|_{U_q(\mathfrak{sl}_2)}$ decomposes in one-dimensional representations, thus K_α^2 acts as 1 on M. It suffices to prove that there is a finite-dimensional representation on U, for which K_α^2 does not act as 1. We know, that for all dominant weights $\lambda \in \Lambda$ the U-module $L(\lambda, \sigma)$ is one dimensional. With the fundamental dominant weights we can find for all α a λ with $(\lambda, \alpha) \neq 0$, and for this λ the element K_α^2 does not act as 1 on $L(\lambda, \sigma)$. This proves the assertion.

In the second case: $E_\alpha K_\alpha^{-1}, F_\alpha, K_\alpha^2, K_\alpha^{-2} \in A'$ the proof is completely analogous: We would suspect that A'-modules decomposing into their K_α^2-eigenspaces are semisimple with irreducible modules $L(\lambda)$. This is however not necessary for the proof: The restriction of an U-module $M = L(\lambda, \sigma)$ decomposes in $U_q(\mathfrak{sl}_2)$ semisimple into $L_{\mathfrak{sl}_2}(\lambda_i, \sigma_i)$. Apparently, the restriction of any irreducible $U_q(\mathfrak{sl}_2)$-module $L_{\mathfrak{sl}_2}(\lambda_i, \sigma_i)$ on A' is still irreducible. $M|_{A'}$ still decomposes into its irreducible representations and as the element K_α^2 does not act as 1, by choice of λ, again we can find a multidimensional representation. □

Corollary 3.4. *Let $A \subset U$ be an RCS with $E_\alpha K_\alpha^{-1}, F_\alpha \in A$ for $\alpha \in \Pi$, then A is not a Borel subalgebra.*

Proof. This follows from the previous theorem, but we need to get the K_α, K_α^{-1} inside A: Assume A was a Borelsubalgebra. Consider the q-commutator of $E_\alpha K_\alpha^{-1}$ with F_α:

$$[E_\alpha K_\alpha^{-1}, F_\alpha]_{q_\alpha^2} = (1 - K_\alpha^{-2}) \frac{q_\alpha^2}{q_\alpha - q_\alpha^{-1}}$$

Thus K_α^{-2} and due to Lemma 3.2 also K_α^2 lie in A. Then applying Theorem 3.3 proves the claim. □

Corollary 3.5. *Let $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-}$ be an arbitrary triangular RCS. If there exists an $\alpha \in \Pi \cap \Phi^+(w^+) \cap \Phi^+(w^-)$ with $\alpha \notin \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$, then C is not a Borel subalgebra.*

Lemma 3.6. *If for an $\alpha \in \Pi$ the character-shifted elements $E_\alpha K_\alpha^{-1} + \lambda_\alpha K_\alpha^{-1}, F_\alpha + \lambda'_\alpha K_\alpha^{-1}$ with $\lambda_\alpha \lambda'_\alpha \neq \frac{q_\alpha^2}{(1-q_\alpha^2)(q_\alpha - q_\alpha^{-1})}$ lie in a subalgebra $A \subset U$, then A is not a Borel subalgebra.*

If however for two elements $E_\alpha K_\alpha^{-1} + \lambda_\alpha K_\alpha^{-1}, F_\alpha + \lambda'_\alpha K_\alpha^{-1} \in A$ holds $\lambda_\alpha \lambda'_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^2)(q_\alpha - q_\alpha^{-1})}$ they form a Weyl algebra as in Example 2.7, thus they can be part of a Borel subalgebra.

Proof. Considering the q_α^2-commutator of the two elements yields:

$$[E_\alpha K_\alpha^{-1} + \lambda_\alpha K_\alpha^{-1}, F_\alpha + \lambda'_\alpha K_\alpha^{-1}]_{q_\alpha^2} = \frac{q_\alpha^2}{q_\alpha - q_\alpha^{-1}}(1 - K_\alpha^{-2}) + (1 - q_\alpha^2)\lambda_\alpha \lambda'_\alpha K_\alpha^{-2}$$

We can see, that for $\lambda_\alpha \lambda'_\alpha \neq \frac{q_\alpha^2}{(1-q_\alpha^2)(q_\alpha - q_\alpha^{-1})}$ again K_α^{-2} lies in A.

Assume A was a Borel subalgebra, then due to Lemma 3.2 also $K_\alpha^2 \in A$. Then similar to Corollary 3.5 also $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} \in A$ follows, which is a contradiction by Theorem 3.3. □

Even without the ede property follows quickly:

Lemma 3.7. *In a triangular right coideal subalgebra $C = \psi(U^+[w_1])_{\phi^+} T_L U^-[w_2]_{\phi^-}$ with $L \subset (\mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-))^\perp$ holds for all $\alpha \in \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-) \cap \Pi$ the relation*

$$\phi^+(E_\alpha)\phi^-(F_\alpha) = \frac{q_\alpha^2}{(1 - q_\alpha^2)(q_\alpha - q_\alpha^{-1})}$$

Proof. From $\phi^+(E_\alpha)\phi^-(F_\alpha) = \frac{q_\alpha^2}{(1-q_\alpha^2)(q_\alpha - q_\alpha^{-1})}$ we can follow as in Lemma 3.6 again $K_\alpha^{-2} \in C$ which is a contradiction to $L \subset (\mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-))^\perp$. □

We now consider the restriction of $U_q(\mathfrak{g})$-representations to C (which is in general nonsemisimple) in order to construct non-one-dimensional irreducible C-representations as composition factors. This works particularly smooth for minuscule representations, which is sufficient for type A_n.

Notation 3.8. *For the roots μ in A_n we use the notation $\mu = [\mu_1, \mu_2]$ for $1 \leq \mu_1 \leq \mu_2 \leq n$ if $\mu = \sum_{m=\mu_1}^{\mu_2} \alpha_i$. All roots in A_n are of this form.*

Lemma 3.9. *Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\bar{E}_\mu$, $\bar{F}_\mu$ be two character-shifted root vectors in $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-} \subset U_q(\mathfrak{g})$ for a root $\mu = [\mu_1, \mu_2] \in \Phi^+(w^+) \cap \Phi^+(w^-)$ such that for all $\nu \in \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$ with $\nu \prec \mu$ holds $(\nu, \mu) = 0$, then C is not ede.*

Proof. Our general strategy for constructing irreducible representations of dimension greater 1 is the following: We consider the action of the commutator $[E, F]_1 \in C$ on a suitable finite-dimensional $U_q(\mathfrak{g})$-representation $V = L(\lambda)$, which we restrict to C. The commutator acts trivial on any finite dimensional irreducible representation of C and thus nilpotent on all representations of C, whose series of compositions contains only one-dimensional representations.

In our case we get from $(\nu, \mu) = 0$ for all $k' < \mu_2$ and $j' > \mu_1$ that $[j', \mu_2], [\mu_1, k'] \notin (\mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-))$. Due to this restriction for $\mathtt{supp}(\phi^+)$ the element $\bar{E}_\mu$ acts on the minuscule representation $M(\lambda_1)$ with highest weight λ_1 equal to $E_\mu K_\mu^{-1}$, and $\bar{F}_\mu$ acts equal to F_μ.

If for a given $\mathfrak{g}, \mu$ we can find a $\lambda, v \in V(\lambda)$, such that $[E_\mu K_\mu^{-1}, F_\mu]_1.v = cv$ with $c \neq 0$, the commutator cannot act nilpotent and thus $V|_C$ must have an irreducible composition factor of dimension greater 1, so C is not ede. For Example this condition is for $\mu \in \Pi$ fulfilled for the highest weight vector v, but in general we don't know $[E_\mu K_\mu^{-1}, F_\mu]_1$.

In the case of A_n we can calculate the condition explicitly on the minuscule representation $V = V(\lambda_1)$. Let $\mu = [\mu_1, \mu_2] = \sum_{k=\mu_1}^{\mu_2} \alpha_k$. Then the action of the simple root vectors is:

$$F_{\alpha_{k+1}}.v_{\lambda_1 - \alpha_1 - \ldots - \alpha_k} \sim v_{\lambda_1 - \alpha_1 - \ldots - \alpha_{k+1}} \tag{2}$$

$$F_{\alpha_j}.v_{\lambda_1 - \alpha_1 - \ldots - \alpha_k} = 0, \quad j \neq k+1 \tag{3}$$

and all weight spaces are one dimensional. Consider $v := v_{\lambda_1 - \alpha_1 - \ldots - \alpha_{\mu_1 - 1}}$, then $E_\mu K_\mu^{-1}.v$ is trivial, $F_\mu.v$ is a non trivial multiplier of $v_{\lambda_1 - \alpha_1 - \ldots - \alpha_{\mu_2}}$, because only the summand $F_{\alpha_{\mu_2}} \cdots F_{\alpha_{\mu_1}}$ can act nontrivial, and with the same argumentation also $E_\mu K_\mu^{-1} F_\mu.v$ is a nontrivial multiplier of v. Thus v is a nontrivial eigenvector of the commutator. Thus, according to the above explanation, the assertion follows. □

Corollary 3.10. *Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\bar{E}_\mu$, $\bar{F}_\mu$ be two character-shifted root vectors in $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-} \subset U_q(\mathfrak{g})$ for a root $\mu = [\mu_1, \mu_2] \in \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$ such that for all $\mu \neq \nu \in \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$ with $\nu \prec \mu$ holds $(\nu, \mu) = 0$, and $\phi^+(E_\mu K_\mu^{-1}) = \lambda$, $\phi^-(F_\mu) = \lambda'$ with $\lambda\lambda' \neq \frac{q^2}{(1-q^2)(q-q^{-1})}$, then C is not ede.*

Proof. As in the proof of Lemma 3.9 we can see that, on the minuscule representation with highest weight λ_{μ_1} the element $\bar{E}_\mu$ acts equal to $E_\mu K_\mu^{-1} + (1 - q^2)\lambda K_\mu^{-1}$ and $\bar{F}_\mu$ equal to $F_\mu + (q^{-1} - q)\lambda' K_\mu^{-1}$. Thus the representation has a multidimensional irreducible subrepresentation. □

Lemma 3.11. *Let $C = \psi(U^+[w])_{\phi^+} T_L U^-[w]_{\phi^-} \subset U_q(\mathfrak{sl}_n)$ such that $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) =: \mathtt{supp}(\phi)$ and given two character-shifted root vectors $\bar{E}_\mu, \bar{F}_\mu$ for a root $\mu = [\mu_1, \mu_2] \notin \Pi$ with $\mu \notin \mathtt{supp}(\phi)$, but $\alpha_r \neq \alpha_s \in \mathtt{supp}(\phi) \cap \Pi$. If there are two different reduced expressions w_1 and w_2 of w with $E_\mu = T_{w_1}(E_\alpha)$ and $F_\mu = T_{w_2}^{-1}(F_\beta)$,such that*

$$\bar{E}_\mu = E_\mu K_\mu^{-1} + X_{\mu-\alpha_r} K_\mu^{-1} \quad \bar{F}_\mu = F_\mu + Y_{\mu-\alpha_s} K_{\alpha_s}^{-1}$$

for two elements $X_{\mu-\alpha_s} \in U^+_{\mu-\alpha_s}$ and $Y_{\mu-\alpha_r} \in U^-_{\mu-\alpha_r}$, then C is not ede.

Proof. If $(\alpha_r, \mu) = 0$ and $(\alpha_s, \mu) = 0$ then the situation is a special case of the preceding Lemma 3.9. Thus it is enough to consider the case $r = \mu_1$ (resp. $r = \mu_2$): Then $\bar{E}_\mu$ acts on the minuscule representation with highest weight λ_1 equal to $E_\mu K_\mu^{-1}$. Thus the requirements of the proof of Lemma 3.9 are fulfilled and we can find a multidimensional irreducible representation as factor of the decomposition of $V(\lambda_1)|_C$. □

On the other hand, a method of finding one-dimensional subrepresentations and thereby proving the ede property is:

Theorem 3.12. *Let A be an algebra with a generating system Z and with the following property: There is an element $X \in A$ such that for all generators $Y \in Z$ the commutator $[X, Y]_c = 0$ for some constant $c \in \mathbb{K}^*$, and there is an element $K \in A$ with $[K, X]_{c'} = 0$ for a constant $c' \in \mathbb{K}^*$, which is not a root of unity, and K has a non-trivial eigenvector.*

Then $X.V = 0$ for any finite dimensional irreducible representation V of A.

Proof. Assume there is an eigenvector of X with eigenvalue 0, then all elements $w \in V$ with $X.w = 0$ span a subrepresentation V, because by assumption after acting with any generator Y:

$$X.(Y.w) = cY.(X.w) = 0$$

Because V is assumed irreducible, the existence of an eigenvalue 0 thus implies $X.v = 0$ for all $v \in V$.

Assume now there is no eigenvector of X with eigenvalue 0, then let us consider the assumed eigenvector $w \neq 0$ of K, with associated eigenvalue some λ. As $X.w \neq 0$ we get $X.w$ is again an eigenvector of K:

$$K(X.w) = c'X(K.w) = c'\lambda X.w$$

As c' is assumed to be not a root of unity and as $X^j.w \neq 0$ for all $j > 0$, the elements $X^j.w$ are eigenvectors of K with pairwise distinct eigenvalues. As V is finite dimensional, only a finite number of these values can exist, this yields a contradiction. □

4. A theorem on Weyl group elements

For $w \in W$ consider as in Theorem 2.1 the subset of roots:

$$\Phi^+(w) = \{\beta \in \Phi^+ \mid w^{-1}(\mu) \prec 0\} \mu \in \Phi^+$$

The goal of this section is to prove, how to supplement two Weyl group elements $w_1, w_2 \in W$ in a special way to two elements w_1', w_2', such that $\Phi^+(w_1') \cup \Phi^+(w_2') = \Phi^+$.

Theorem 4.1. *Fix $w_1, w_2 \in W$ and assume $\mathtt{B} := \Phi^+(w_1) \cap \Phi^+(w_2)$ is not empty and consists of pairwise orthogonal roots. Then:*

(1) *There exist elements* $w_1', w_2' \in W$ *with* $\Phi^+(w_1) \subseteq \Phi^+(w_1')$, $\Phi^+(w_2) \subseteq \Phi^+(w_2')$, *such that the following relations hold:*

$$\Phi^+(w_1') \cap \Phi^+(w_2') = \mathtt{B} \text{ and } \Phi^+(w_1') \cup \Phi^+(w_2') = \Phi^+$$

(2) *There exists an element* $w_1'' \in W$ *such that* $\Phi^+(w_1) \subseteq \Phi^+(w_1'')$ *and* w_1'' *has a reduced expression* $w_1'' = s_{\alpha_1} \dots s_{\alpha_{\ell(w_1'')}}$ *with* $\mathtt{B} = \{\beta_{\ell(w_1'')-|B|+1}, \dots, \beta_{\ell(w_1'')}\}$.

We can prove this Theorem for type A_n.

The rest of this section is devoted to prove Theorem 4.1. In fact most steps are always true, but a key property of root systems 4.12 we did only prove for A_n and for rank 2.

We consider two orderings on the root lattice:

Definition 4.2. *For* $a := a_1\alpha_1 + \ldots + a_{|\Pi|}\alpha_{|\Pi|}$, *and* $b := b_1\alpha_1 + \ldots + b_{|\Pi|}\alpha_{|\Pi|} \in Q$ *with* $a_i, b_i \in \mathbb{N}$ *let* $a \prec b$ *if and only if* $a_i < b_i$ *for all* $i \leq n$.

We define, depending on the choice of the reduced expression of w, *a total ordering* $<$ *on* $\Phi^+(w)$ *by*

$$\beta_i < \beta_j \in \Phi^+(w) \Leftrightarrow i < j$$

This ordering is convex, i.e. for $\mu < \nu \in \Phi^+(w)$ *we get* $\mu < \mu + \nu < \nu$, *if* $\mu + \nu \in \Phi^+(w)$ *see therefore* [PA] *s.662.*

On W we use the weak ordering. From the strong exchange property follows:

Corollary 4.3. *Let* $w \in W$ *and* $\alpha \in \Phi^+(w) \cap \Pi$, *then there exists a reduced expression of* w *of the form* $w = s_\alpha x$ *for a* $x \in W$ *with* $\ell(x) = \ell(w) - 1$.

Definition 4.4.

$$T^w := \{\Theta \subset \Phi^+(w) \mid \Theta \text{ pairwise orthogonal roots and } \ell(\underbrace{\prod_{\beta\in\Theta} s_\beta w}_{w_\Theta}) = \ell(w) - |\Theta|\}$$

Remark 4.5 ([HK11b] s.12)**.** *In explicit examples of small rank the set* T^w *is easy to calculate: If* $J \subseteq \{1, \dots, \ell(w)\}$ *is a subset, such that the elements in* $\Theta := \{\beta_i \mid i \in J\}$ *are pairwise orthogonal, then* Θ *belongs to* T^w *if and only if after deleting all reflections* s_{α_i} *for* $i \in J$ *the resulting expression* $w = s_{\alpha_1} \dots s_{\alpha_{\ell(w)}}$ *is still a reduced expression.*

Lemma 4.6. *Each* k *roots in* Φ *can be reflected to a rank* k *root system, i.e. for any set of roots* $\mathtt{B} = \{\mu_1, \dots, \mu_{|\mathtt{B}|}\} \subset \Phi$ *there is a* $x \in W$ *and a rank* $|\mathtt{B}|$ *root system* $\Phi_{|\mathtt{B}|}$ *with* $x(\mu_i) \in \Phi_{|\mathtt{B}|}$ *for all* i.

A proof can be found e.g. in [Len14] Theorem 6.3 a).

Lemma 4.7. *For a* $\mu \in \Phi^+$ *not simple, there exists a* $\beta \in \Pi$ *such that* $\mu - \beta \in \Phi^+$.

The proof is found in [Hum70] Lemma 10.2 A.

Corollary 4.8. *Given a* $w \in W$. *For all* $\mu \in \Phi^+(w)$ *there exists a* $\beta \in \Pi \cap \Phi^+(w)$ *such that* $\beta \prec \mu$.

Proof. We prove this by induction on the height $ht(\mu)$:

For $ht(\mu) = 1$ follows $\mu \in \Pi$ and the claim is trivial. Let $ht(\mu) > 1$. Due to Lemma 4.7 there exists a $\beta \in \Pi$ such that $\mu - \beta \in \Phi^+$. Due to the convexity of the ordering $<$ on $\Phi^+(w)$ we get either $\beta \in \Pi \cap \Phi^+(w)$ or $\mu - \beta \in \Phi^+(w)$. In the first case the claim is already proven. In the second case there exists from the induction assumption an $\alpha \in \Pi \cap \Phi^+(w)$ with $\alpha \prec \mu - \beta \prec \mu$, thus the claim is true. □

Definition 4.9. *Let $\mu, \nu \in \Phi$ be orthogonal ($\mu \perp \nu$), that is $(\mu, \nu) = 0$. We call μ and ν* strongly orthogonal, *if there exists a $x \in W$ such that $x(\mu), x(\nu) \in \Pi$.*

Using Lemma 4.6 we can reduce to the rank 2 situation. Then clearly

Remark 4.10. *Two orthogonal roots $\mu, \nu \in \Phi$ are strongly orthogonal, if $m\mu + n\nu \notin \Phi$ for all $m, n \in \mathbb{Q}\backslash\{0\}$.*

In the following we study for two Weyl group elements $w_1, w_2 \in W$ the intersection $\mathtt{B} := \Phi^+(w_1) \cap \Phi^+(w_2)$ more closely. In particular we want to make some statements on possible supplements of w_1 and w_2 for $\mathtt{B}$ consisting of orthogonal elements.

Lemma 4.11. *Assume all elements in $\mathtt{B}$ are pairwise orthogonal. Then all the elements in $\mathtt{B}$ are pairwise strongly orthogonal.*

Proof. Given two arbitrary elements $\mu, \nu \in \mathtt{B}$. Assume $\mu + \nu \in \Phi$, we get from the convexity of $<$ on $\Phi^+(w_1)$ resp. $\Phi^+(w_2)$ already $\mu+\nu \in \Phi^+(w_1) \cap \Phi^+(w_2)$, so in particular $(\mu+\nu, \mu) = 0$. That is a contradiction to the pairwise orthogonality of the elements in $\mathtt{B}$. Now we use a projection $x \in W$ from μ and ν in a rank 2 root system. From $\mu + \nu \notin \Phi^+$, i.e. $x(\mu) + x(\nu) \notin \Phi$ the only options for the root systems are $A_1 \times A_1$ and G_2, in the first case the claim is proven. In the second case $\mathfrak{g} = G_2$ with a long root μ and a short root ν orthogonal to each other, also the roots between μ and ν with respect to the total ordering for an arbitrary reduced expression of w_0 lie in $\Phi^+(w_1) \cap \Phi^+(w_2)$ as is easily shown, these are however not orthogonal on μ, which leads us to a contradiction. □

Vermutung 4.12. *Assume all elements in $\mathtt{B}$ are pairwise orthogonal. Then there exists either an element $\alpha \in \Pi \cap \Phi^+(w_1)$ with $\alpha \notin \mathtt{B}$ or $|\Pi \cap \Phi^+(w_1)| = |\mathtt{B}|$.*

Example 4.13. *In the small cases of rank 2 the assumption 4.12 is true, as we know from Lemma 4.11, that the elements in $\mathtt{B}$ are even pairwise strongly orthogonal. Thus in the case of rank 2 only $|B| = 1$ is possible and the claim follows directly.*

In the following we will restrict our considerations to A_n, but we conjecture, that the theorems are also true in general.

Lemma 4.14. *In type A_n conjecture 4.12 holds.*

Proof. Assume there exists no element $\alpha \in \Pi \cap \Phi^+(w)$ with $\alpha \notin \mathtt{B}$, i.e.:

$$\Phi^+(w_1) \cap \Pi \subset \mathtt{B} \tag{4}$$

We want to show $\mathtt{B} \subset \Pi$:

A) For all $\mu \in \Phi^+(w_1)$ we get from Lemma 4.8 a $\beta \in \Phi^+(w_1) \cap \Pi$ with $\beta \prec \mu$. Due to (4) even $\beta \in \mathtt{B}$ holds.

B) Assume there is a $\mu \in \mathtt{B}$ with $\mu \notin \Pi$. We look at the minimal root $\mu = [\mu_1, \mu_2]$ of this type with respect to the ordering $\prec$. Let $\alpha_{b_1}, \ldots \alpha_{b_l} \in \mathtt{B} \cap \Pi$ with $\mu_1 \leq b_1 \leq \ldots \leq b_l \leq \mu_2$ be the elements $\prec \mu$ in $\mathtt{B}$. Due to A) there exists at least one such element and from the pairwise orthogonality of the elements in $\mathtt{B}$ we even get $\mu_1 < b_1 < \ldots < b_l < \mu_2$. We define the following types of roots for $0 \leq j \leq l-1$:

$$X_j = [\mu_1, b_{l-j}]$$

$$Y_j = [\mu_1, b_{l-j} - 1]$$

Here for all $0 \leq j \leq l-1$ holds $X_j, Y_j, \mu - X_j, \mu - Y_j \notin \mathtt{B}$ because $(X_j, \alpha_{b_{l-j}}) = 1$ and $(Y_j, \alpha_{b_{l-j}}) = -1$ and because the elements in $\mathtt{B}$ are pairwise orthogonal.

We want to show now for all $0 \leq j \leq l-1$ holds $Y_j \in \Phi^+(w_1)$. For this purpose, we use the following considerations:

(1) $Y_j \in \Phi^+(w_1) \Rightarrow X_j \in \Phi^+(w_1)$, as $\alpha_{b_{j-l}} \in \Phi^+(w_1)$. As well for w_2.
$\mu - X_j \in \Phi^+(w_1) \Rightarrow \mu - Y_j \in \Phi^+(w_1)$, as $\alpha_{b_{j-l}} \in \Phi^+(w_1)$. As well for w_2.

(2) $X_j \notin \Phi^+(w_1) \Rightarrow \mu - X_j \in \Phi^+(w_1)$, this is due to the convexity of $<$. As well for w_2 and Y_j.
$\mu - X_j \notin \Phi^+(w_1) \Rightarrow X_j \in \Phi^+(w_1)$, this is due to the convexity of $<$. As well for w_2 and Y_j.

(3) $X_j \in \Phi^+(w_1) \Rightarrow X_j \notin \Phi^+(w_2)$, as $X_j, Y_j \notin \mathtt{B}$. As well for Y_j.
$\mu - X_j \in \Phi^+(w_1) \Rightarrow \mu - X_j \notin \Phi^+(w_2)$, as $\mu - X_j, \mu - Y_j \notin \mathtt{B}$. As well for Y_j.

With these considerations we can prove the claim inductively on $0 \leq j \leq l-1$:

Y_0 lies in $\Phi^+(w_1)$, because:

$$\overset{A}{\Rightarrow} \mu - X_0 \notin \Phi^+(w_1) \overset{2}{\Rightarrow} X_0 \in \Phi^+(w_1) \overset{3}{\Rightarrow} X_0 \notin \Phi^+(w_2) \overset{2}{\Rightarrow} \mu - X_0 \in \Phi^+(w_2)$$
$$\overset{1}{\Rightarrow} \mu - Y_0 \in \Phi^+(w_2) \overset{3}{\Rightarrow} \mu - Y_0 \notin \Phi^+(w_1) \overset{2}{\Rightarrow} Y_0 \in \Phi^+(w_1)$$

Let $Y_j \in \Phi^+(w_1)$. Assume $Y_{j+1} \notin \Phi^+(w_1)$:

$$Y_{j+1} \notin \Phi^+(w_1) \overset{2}{\Rightarrow} \mu - Y_{j+1} \in \Phi^+(w_1) \overset{3}{\Rightarrow} \mu - Y_{j+1} \notin \Phi^+(w_2)$$
$$\overset{1}{\Rightarrow} \mu - X_{j+1} \notin \Phi^+(w_2) \overset{2}{\Rightarrow} X_{j+1} \in \Phi^+(w_2) \overset{3}{\Rightarrow} X_{j+1} \notin \Phi^+(w_1)$$
$$\overset{2}{\Rightarrow} \mu - X_{j+1} \in \Phi^+(w_1) \overset{A}{\Rightarrow} \mu - Y_j \in \Phi^+(w_1) \overset{3}{\Rightarrow} \mu - Y_j \notin \Phi^+(w_2)$$
$$\overset{2}{\Rightarrow} Y_j \in \Phi^+(w_2) \lightning \text{ as } Y_j \in \Phi^+(w_1) \text{ and } Y_j \notin \mathtt{B}$$

The induction yields $Y_j \in \Phi^+(w_1)$ for all $0 \leq j \leq l-1$. This is a contradiction to A) for $j = l-1$. This proves the assertion.

□

Corollary 4.15. *Let* $\mathtt{B}$ *consist of pairwise orthogonal roots and let* $\mathtt{B} \neq \emptyset$. *Then* w_1 *has a reduced expression of the form* $w_1 = s_{\alpha_1} \ldots s_{\alpha_{\ell(w_1)}}$ *and* $1 \leq i \leq i + |B| - 1 \leq \ell(w_1)$ *with* $\mathtt{B} = \{\beta_i, \ldots \beta_{i+|B|-1}\}$. *As well for* w_2.

Proof. We prove this claim with an induction on the length $\ell(w_1)$:

For $\ell(w_1) = |\mathtt{B}|$ we get $\Phi^+(w_1) \subset \Pi$ and the claim is trivial. Let now w be with $\ell(w_1) > |\mathtt{B}|$. Assume $\mathtt{B} \subset \Pi$ then, due to Corollary 4.3 and as the elements in $\mathtt{B}$ are pairwise orthogonal, we can write all elements in $\mathtt{B}$ in the beginning of w. Then the claim is true for $i = 1$ and $j = |\mathtt{B}|$.

If an element $\mu \in \mathtt{B}$ with $\mu \notin \Pi$ exists, then due to Lemma 4.14 there is a $\beta \in \Phi^+(w_1) \cap \Pi$ with $\beta \prec \mu$ and $\beta \notin \mathtt{B}$. So there is a reduced expression of w_1 with s_β in the beginning, due to Corollary 4.3. Then we look at the two Weyl group elements $w_2' = s_\beta w_2$ and $w_1' = s_\beta w_1$. Here the claim is already proven, as $\ell(s_\beta w_1) < \ell(w_1)$ and for $\mathtt{B}' := s_\beta(\mathtt{B})$ we can bring the element w_1' in the desired form. With this reduced expression of w_1', as $\beta \notin \mathtt{B}$ also $w_1 = s_\beta w_1'$ is in the right form. ☐

Now we want to prove our main Theorem 4.1 which gives us an supplement of w_1, w_2 to w_1', w_2'. We will do this in several steps.

Lemma 4.16. *Under the assumptions of Theorem 4.1 the two assertions 1) and 2) are equivalent.*

Proof. "'$\Rightarrow$"' Let w_2', w_1', fulfilling 1) be given i.e. $\Phi^+(w_2') \cap \Phi^+(w_1') = \mathtt{B}$ and $\Phi^+(w_2') \cup \Phi^+(w_1') = \Phi^+$. Now we construct for $w_1'' := w_1'$ a reduced expression of the required form.

For this we look at $\bar{w} := w_2' w_0$:

$$\Phi^+(\bar{w}) \subset \Phi^+(w_1')$$

Because for $\mu \in \Phi^+(\bar{w})$ already $\bar{w}^{-1}(\mu) \prec 0 \Rightarrow w_0^{-1} w_2'^{-1}(\mu) \prec 0 \Rightarrow w_2'^{-1}(\mu) \succ 0$ is true, so $\mu \notin \Phi^+(w_2')$ and from $\Phi^+(w_2') \cup \Phi^+(w_1') = \Phi^+$ it follows, that $\mu \in \Phi^+(w_1')$. Due to the relation 1 we can choose now the following reduced expression of w_1': $w_1' = \bar{w}x$ for some $x \in W$ with $\ell(x) = \ell(w_1') - \ell(\bar{w})$. It only remains to prove:

$$\bar{w}(\Phi^+(x)) = \mathtt{B}$$

"$\subset$": This is true, because the elements in $\mathtt{B}$ are exactly the elements which lie in both $\Phi^+(w_2')$ and $\Phi^+(w_1')$. The elements on the left hand side lie in $\Phi^+(w_1')$ by construction and for $\nu \in \Phi^+(x)$ holds $w_2'^{-1}\bar{w}(\nu) = w_0(\nu) \prec 0$, so they also lie in $\Phi^+(w_2')$. Thus we have found a suitable reduced expression and 2) holds.

"$\supset$": This is true, because $\Phi^+(\bar{w}x) = \Phi^+(w_1') \supset \mathtt{B}$, on the other hand for all elements $\mu \in \Phi^+(w_2')$, so in particular for all elements $\mu \in \mathtt{B}$ we have $\mu \notin \Phi^+(\bar{w})$, i.e. $\mathtt{B} \cap \Phi^+(\bar{w}) = \emptyset$. So we get $\Phi^+(\bar{w}x) \supset \mathtt{B}$, but $\mathtt{B} \cap \Phi^+(\bar{w}) = \emptyset$, so $\bar{w}(\Phi^+(x)) \supset \mathtt{B}$.

"'$\Leftarrow$"' Assume there exists a $w_1'' = s_{\alpha_1} \dots s_{\alpha_k}$ with $\mathtt{B} = \{\beta_i, \dots, \beta_k\}$ for an i.

Then we choose $w_1' := w_1''$ and $w_2' := s_{\alpha_1} \dots s_{\alpha_{i-1}} w_0$ and show that these elements fulfil 1): We know

$$\Phi^+(w_1') \cup \Phi^+(w_2') = \Phi^+$$

as for $\mu \notin \Phi^+(w_2')$ we have $w_0^{-1}s_{\alpha_{i-1}}\dots s_{\alpha_1}(\mu) \succ 0 \Rightarrow s_{\alpha_{i-1}}\dots s_{\alpha_1}(\mu) \prec 0$, so $\mu \in \Phi^+(w_1')$. Moreover we know

$$\Phi^+(w_1') \cap \Phi^+(w_2') = \mathtt{B}$$

as for all $\beta_j \in \Phi^+(w_1')$ is $\beta_j \in \Phi^+(w_2') \Leftrightarrow j > i \Leftrightarrow \beta_j \in \mathtt{B}$. □

To prove Theorem 4.1 we look at some special cases for $\mathtt{B}$ and will then combine the methods of the proofs.

Corollary 4.17 ($\mathtt{B} = \{\alpha\} \subset \Pi$)**.** *For* $\mathtt{B} = \{\alpha\} \subset \Pi$ *Theorem 4.1 is true.*

Proof. If $\Phi^+(w_1) \cup \Phi^+(w_2) = \Phi^+$, then the claim is trivial. Assume $\Phi^+(w_1) \cup \Phi^+(w_2) \neq \Phi^+$. Then we construct elements w_1' and w_2' with $\Phi^+(w_1) \subseteq \Phi^+(w_1')$, $\Phi^+(w_2) \subseteq \Phi^+(w_2')$ inductively as follows: We enlarge w_1 until for w_1^{neu} and w_2 either 1) or 2) in the Theorem 4.1 is fulfilled: Let w_1^i be the following sequence of elements in W with associated reduced expressions, such that $w_1 =: w_1^1 < w_1^2 < \dots w_1^i$ is true for all i, that is $\Phi^+(w_1) \subset \Phi^+(w_1^i)$ for all i:

$$w_1^i = s_\alpha x_i \quad w_2 = s_\alpha y$$

for elements $x_i, y \in W$ with $\ell(w_1^i) = \ell(x_i) + 1$ and $\ell(w_2) = \ell(y) + 1$. These reduced expressions exist due to Corollary 4.3. Then in particular $w_2^{-1}w_1^i = y^{-1}x_i$ with $\ell(y^{-1}x_i) = \ell(y) + \ell(x_i)$. As $\Phi^+(w_1) \cup \Phi^+(w_2) \neq \Phi^+$ there exists at least a $\gamma \in \Pi$ with $y^{-1}x_i(\gamma) \succ 0$, so with $\ell(y^{-1}x_i) = \ell(y) + \ell(x_i)$ also $x_i(\gamma) \succ 0$ is true. Now we distinguish two cases for $w_1^i(\gamma)$:

1. Case $w_1^i(\gamma) \succ 0$: Then we define $w_1^{i+1} := w_1^i s_\gamma$ and we get $\Phi^+(w_1^{i+1}) \cap \Phi^+(w_2) = \alpha$. By induction we can enlarge w_1, until either $\Phi^+(w_1^n) \cup \Phi^+(w_2) = \Phi^+$, i.e. 1) is fulfilled, or for some i the 2. case occurs.

2. Case $w_1^i(\gamma) \prec 0$: Then we get from $x_i(\gamma) \succ 0$ already $x_i(\gamma) = \alpha$. In this case there is a reduced expression of w_1^i of the form:

$$w_1^i = s_{\alpha_1} \dots s_{\alpha_{\ell(w_1^i)}} \text{ with } \beta_{\ell(w_1^i)} = \alpha$$

and 2) is fulfilled for $w_2' := w_2$ and $w_1' := w_1^i$, because $w_1^i(\gamma) = s_\alpha x_i(\gamma) = s_\alpha(\alpha) \prec 0$, i.e. we know from Lemma 4.3 applied to $(w_1^i)^{-1}$, that there exists a reduced expression of w_1^i of the form $w_1^i = s_{\alpha_1} \dots s_{\alpha_{\ell(w_1^i)-1}} s_\gamma$. In this reduced expression we can see, that in particular holds: $w_1^i(\gamma) = s_{\alpha_1} \dots s_{\alpha_{\ell(w_1^i)-1}} s_\gamma(\gamma) = -\beta_{\ell(w_1^i)} = -\alpha$, i.e. $\beta_{\ell(w_1^i)} = \alpha$. Thus the claim is proven. □

Corollary 4.18 ($\mathtt{B} \subset \Pi$)**.** *For* $\mathtt{B} \subset \Pi$ *the Theorem 4.1 is true.*

Proof. We want to prove this Corollary analogous to the proof of Corollary 4.17. From $\mathtt{B} = \{\alpha_1, \dots, \alpha_{|\mathtt{B}|}\}$ and Lemma 4.3 we know, that there exists a reduced expression of w_1 of the form: $w_1 = \prod_{j=1}^{|\mathtt{B}|} s_{\alpha_j} x$ for a $x \in W$ with $\ell(x) = \ell(w_1) - |\mathtt{B}|$, the same for w_2. As above we construct again elements w_1', w_2' inductively: Let therefore w_1^i be the following sequence of elements in W with associated reduced expression, such that $w_1 =: w_1^1 < w_1^2 < \dots w_1^i$ holds for all i, so $\Phi^+(w_1) \subset \Phi^+(w_1^i)$ for all i:

$$w_1^i = \prod_{j=1}^{|\mathtt{B}|} s_{\alpha_j} x_i \quad w_2 = \prod_{j=1}^{|\mathtt{B}|} s_{\alpha_j} y \quad w_{\mathtt{B}} = \prod_{j=1}^{|\mathtt{B}|} s_{\alpha_j}$$

for elements $x_i, y \in W$ with $\ell(x_i) = \ell(w_1^i) - |\mathtt{B}|$ and with $\ell(y) = \ell(w_2) - |\mathtt{B}|$. Again we get $w_2^{-1} w_1^i = y^{-1} x_i$ with $\ell(y^{-1}x_i) = \ell(y) + \ell(x_i)$. As in the proof above we search for elements $\gamma_i \in \Pi$ with $0 \prec w_1^i(\gamma_i) \notin \Phi^+(w_2)$. I.e. with $y^{-1}x_i(\gamma_i) \succ 0$. For these elements we get $x_i(\gamma_i) \succ 0$, from $\ell(y^{-1}x_i) = \ell(y) + \ell(x_i)$. If we find such elements we define $w_1^{i+1} := w_1^i s_{\gamma_i}$ until $\Phi^+(w_1^i) \cup \Phi^+(w_2) = \mathtt{B}$ or there exists an i, for which w_1^i fulfils the property 2) from the Theorem 4.1. This works as follows:

1. Case: There exists a γ_i with $w_1^i(\gamma_i) \succ 0$ and $y^{-1}x_i(\gamma_i) \succ 0$.

Then the requirements for the induction are fulfilled, as from $y^{-1}x_i(\gamma_i) \succ 0$ follows $w_2^{-1}w_1^i(\gamma_i) \succ 0$ and therefore $w_1^i(\gamma_i) \notin \Phi^+(w_2)$. Then we define again $w_1^{i+1} := w_1^i s_{\gamma_i}$. We get $\Phi^+(w_1^{i+1}) \cap \Phi^+(w_2) = \mathtt{B}$. So we can continue the induction with this γ_i until either in the Theorem 4.1 assertion 1) is fulfilled or the second case occurs.

2. Case: For all γ_j with $y^{-1}x_i(\gamma_j) \succ 0$ holds $w_1^i(\gamma_j) \prec 0$.

Then we take a closer look at the following elements:

$$\bar{w}_i := x_i^{-1} y w_0 \quad \text{and} \quad x_i^{-1} w_B$$

For these elements the requirements from Lemma 4.14 are fulfilled, because:

- $\mathtt{B}' := \Phi^+(\bar{w}_i) \cap \Phi^+(x_i^{-1} w_{\mathtt{B}}) = x_i^{-1}(\mathtt{B})$. This can be seen as follows:
  - $x_i^{-1}(\mathtt{B}) \subset \Phi^+(\bar{w}_i)$, because For all $x_i^{-1}(b) \in x_i^{-1}(\mathtt{B})$ we get
  $$\bar{w}_i^{-1}(x_i^{-1}(b)) = w_0^{-1} y^{-1} x_i x_i^{-1}(b) = w_0^{-1} \underbrace{y^{-1}(b)}_{\succ 0} \prec 0$$
  - For all $\mu \in \Phi^+(x_i^{-1})$ holds $\mu \notin \Phi^+(\bar{w}_i)$, because:
  Assume $\mu \in \Phi^+(\bar{w}_i) \Rightarrow \bar{w}_i^{-1}(\mu) \prec 0 \Rightarrow w_0^{-1} y^{-1} x_i(\mu) \prec 0 \Rightarrow y^{-1}x_i(\mu) \succ 0 \Rightarrow x_i(\mu) \succ 0 \Rightarrow \mu \notin \Phi^+(x_i^{-1})$
- $\mathtt{B}' = x_i^{-1}(\mathtt{B})$ consists of pairwise strongly orthogonal roots, as $\mathtt{B}$ consists of pairwise strongly orthogonal roots
- For all $\gamma_j \in \Phi^+(\bar{w}_i) \cap \Pi$ we know that γ_j lies in $\mathtt{x_i^{-1}}(\mathtt{B})$, because we get $w_1^i(\gamma_j) \prec 0$ and $x_i(\gamma_j) \succ 0$, so $\gamma_j \in \Phi^+(x_i^{-1} w_{\mathtt{B}})$ and $\gamma_j \notin \Phi^+(x_i^{-1})$.

Thus we can apply Lemma 4.14, this provides that there exist exactly $|\mathtt{B}|$ elements in $\Phi^+(\bar{w}_i) \cap \Pi$. We call these elements $\alpha'_1, \dots \alpha'_{|\mathtt{B}|}$, they are pairwise strongly orthogonal and we define:

$$w_{\mathtt{B}'} := \prod_{i=1}^{|\mathtt{B}|} \alpha'_i$$

For all $1 \leq j \leq |\mathtt{B}|$ we get $w_1^i(\alpha'_j) \in -\mathtt{B}$, because $w_1^i(\alpha'_j) = w_{\mathtt{B}} x_i(x_i^{-1}(b))$ for some $b \in \mathtt{B}$ with $\alpha'_j = x_i^{-1}(b)$, then $w_{\mathtt{B}} x_i(x_i^{-1}(b)) = w_{\mathtt{B}}(b) = -b \in -\mathtt{B}$.

Due to $\alpha'_j \in \Phi^+((w_1^i)^{-1})$ and as the α'_j are pairwise orthogonal there exist by Lemma 4.3 a reduced expression of $(w_1^i)^{-1}$ beginning with $w_{\mathtt{B}'}$. The associated reduced expression of w_1^i fulfils the requirements of Theorem 4.1. This follows analogously to Corollary 4.17 and the claim is proven. □

Proof of Theorem 4.1. The general case of arbitrary $\mathtt{B} \subset \Phi^+$ can be reduced to the special case in Corollary 4.18 with Corollary 4.15. Let therefore be a reduced expression of w_1 given as in Corollary 4.15: $w_1 = s_{\alpha_1} \dots s_{\alpha_m}$ and $i \leq j \leq m$ with $\mathtt{B} : \{\beta_i, \dots \beta_j\}$. Then we look at the elements $v_1 = s_{\alpha_{i-1}} \dots s_{\alpha_1} w_1$ and $v_2 = s_{\alpha_{i-1}} \dots s_{\alpha_1} w_2$. For these elements $\mathtt{B}' := \Phi^+(v_1) \cap \Phi^+(v_2) \subset \Pi$, i.e. the requirements of Corollary 4.18 are fulfilled and we can apply the Theorem 4.1. Now we choose for w_1 and w_2 the following elements: $w'_1 = s_{\alpha_1} \dots s_{\alpha_{i-1}} v'_1$ and $w'_2 = s_{\alpha_1} \dots s_{\alpha_{i-1}} v'_2$, we get for the intersection:

$$\Phi^+(w'_1) \cap \Phi^+(w'_2) = s_{\alpha_1} \dots s_{\alpha_{i-1}}(\mathtt{B}')$$

because $\Phi^+(v'_1) \cup \Phi^+(v'_2) = \Phi^+$ and further we get:

$$\Phi^+(w'_1) \cap \Phi^+(w'_2) = s_{\alpha_1} \dots s_{\alpha_{i-1}}(s_{\alpha_{i-1}} \dots s_{\alpha_1} \mathtt{B}) = \mathtt{B}$$

Moreover by induction on the length i, we get for the union of $\Phi^+(w'_1)$ and $\Phi^+(w'_2)$:

$$\Phi^+(w'_1) \cup \Phi^+(w'_2) = \Phi^+$$

Thus the Theorem 4.1 is proven for an arbitrary $\mathtt{B}$.

□

5. Triangular Borel subalgebras with non-degenerate character shifts

In this section we construct a large family of Borel subalgebras associated to a choice (c, ϕ) of an orthogonal subset of simple roots c (i.e. a coclique in a Dynkin diagram) and an associated character ϕ, which is uniquely given by a family of scalars $\lambda : c \to \mathbb{K}^\times$. These Borel subalgebras look like a disjoint family of Weyl algebras for each element in c, filled up with a suitable maximal set of remaining positive roots.

The Borel subalgebras we construct are by construction triangular and fulfil an additional non-degeneracy property:

$$\Phi^+(w^+) \cap \Phi^+(w^-) = \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-) \tag{5}$$

where for an arbitrary triangular RCS the relation $\supseteq$ holds by construction. In the next section we will then prove that all triangular nondegenerate Borel subalgebras are of the type constructed here (assumed type A_n).

5.1. **Main construction theorem.** Consider now the right coideal subalgebra

$$C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$$

where $w_0 \in W$ is the longest element of the Weyl group, $x \in W$ is a choice of a Weyl group element such that $c = \Phi^+(x) \subset \Pi$ consists of pairwise orthogonal roots; i.e. $x = \prod_{\alpha \in c} s_\alpha$. Furthermore let ϕ^+, ϕ^- be characters with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \Phi^+(x)$. Any such character is defined by: $\phi^+(\psi(E_\alpha)) = \lambda_\alpha \in k^*$ for $\alpha \in \mathtt{supp}(\phi^+)$ and 0 otherwise and $\phi^-(F_\alpha) = \lambda'_\alpha \in k^*$ and 0 otherwise with the condition

$$\lambda'_\alpha \lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$$

Moreover let $L = \mathtt{supp}(\phi)^\perp$; this choice is for maximality, there is an ede RCS for any choice $L \subseteq \mathtt{supp}(\phi)^\perp$.

Using the commutation relations in C we will show first, that C is a triangular right coideal subalgebra, that means closed under multiplication.

Afterwards we will show, that right coideal subalgebras of this form are ede, that means they have the property, that each finite dimensional irreducible representation is one dimensional, and if furthermore $L = \mathtt{supp}(\phi)^\perp$ we show, that C is a Borel subalgebra.

Main Theorem 5.1. *The right coideal subalgebra*

$$C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$$

with data x, ϕ^+, ϕ^-, L as discussed above, is a Borel subalgebra of $U_q(\mathfrak{g})$.

In the rest of the section we will work out the proof of this theorem.

First we take a closer look at the commutation relations in C. For this purpose we first generalize known results on the description of first terms in the comupltiplication via technical maps r_α, r'_α, that are of utmost importance. Then we apply this knowledge to show that for all $\alpha \in \Phi^+(x)$ the character-shifted root vectors $\bar{F}_\alpha \in U^-[x]_{\phi^-}$ q-commute with all character-shifted root vectors in $\psi(U^+[w_0])_{\phi^+}$ except $\bar{E}_\alpha$.

5.2. **Generalizations of r_α.** For calculating the commutators we utilize the maps r_α and r'_α due to Lusztig and the following definition and Lemmata can be found in [Jan96] chapter 6. The aim of the subsection is to generalize Lemma 5.3, which describes the first nontrivial term (containing a simple root vector) in the coproduct of any element, stepwise until Lemma 5.10, which gives the same information for certain non-simple root vectors. Last we apply Lemma 5.5, which describes $T_\alpha(U^+[s_\alpha w_0])$ as zeroes of r_α to the specific situation of a non-simple root vector in Lemma 5.11.

Definition 5.2. *Let $x \in U_\mu^+$. For all $\alpha \in \Pi$ there exist elements $r_\alpha(x)$ and $r'_\alpha(x)$ in $U_{\mu-\alpha}^+$ such that:*

$$\Delta(x) = x \otimes 1 + \sum_{\alpha \in \Pi} r_\alpha(x) K_\alpha \otimes E_\alpha + (rest) \tag{6}$$

$$\Delta(x) = K_\mu \otimes x + \sum_{\alpha \in \Pi} E_\alpha K_{\mu-\alpha} \otimes r'_\alpha(x) + (rest) \tag{7}$$

where (rest) contains terms in $U_{\mu-\nu} K_\nu \otimes U_\nu^+$ for $\nu \succ 0$ and $\nu \notin \Pi$ in (6) resp. $\mu - \nu \succ 0$ and $\mu - \nu \notin \Pi$ in (7). In particular $r_\alpha(1) = 0 = r'_\alpha(1)$ and $r_\alpha(E_\beta) = r'_\alpha(E_\beta) = \delta_{\alpha\beta}$ for all $\beta \in \Pi$.

Lemma 5.3. *For these r_α and r'_α the following relations hold:*
a) For all $x \in U_\mu^+$ and $x' \in U_{\mu'}^+$:

$$r_\alpha(xx') = x r_\alpha(x') + q^{(\alpha,\mu')} r_\alpha(x) x' \text{ and } r'_\alpha(xx') = q^{(\alpha,\mu)} x r'_\alpha(x') + r'_\alpha(x) x'$$

b) For all $x \in U_\mu^+$ and $y \in U^-$:

$$(F_\alpha y, x) = (F_\alpha, E_\alpha)(y, r'_\alpha(x)) \text{ and } (y F_\alpha, x) = (F_\alpha, E_\alpha)(y, r_\alpha(x))$$

c) We have $r'_\alpha(x) = \tau r_\alpha \tau(x)$ for all $x \in U_\mu^+$ with τ the Cartan involution

Lemma 5.4. *Let $\alpha \in \Pi$ and $\mu \in Q_+$. Then for all $y \in U_{-\mu}^-$ and $x \in U_\mu^+$:*

$$E_\alpha y - y E_\alpha = (q_\alpha - q_\alpha^{-1})^{-1} (K_\alpha r_\alpha(y) - r'_\alpha(y) K_\alpha^{-1}),$$
$$x F_\alpha - F_\alpha x = (q_\alpha - q_\alpha^{-1})^{-1} (r_\alpha(x) K_\alpha - K_\alpha^{-1} r'_\alpha(x))$$

See [Jan96] chapter 6.

Lemma 5.5 ([Jan96] p.166)**.** *For $\alpha \in \Pi$ and with w_0 a longest element:*

$$T_\alpha(U^+[s_\alpha w_0]) = \{x \in U^+ \mid r_\alpha(x) = 0\}$$
$$U^+[s_\alpha w_0] = \{x \in U^+ \mid r'_\alpha(x) = 0\}$$

where T_α are the Lusztig automorphisms.

First we prove some generalizations of the Lemmata 5.3 and 5.4:

Lemma 5.6. *Let $\alpha \in \Pi$ and r'_α as in Definition 5.2, then we get for any $X_\mu \in U_\mu^+$ in degree $\mu \in Q_+$ and any $\beta \in \Pi$:*

$$r'^i_\alpha(X_\mu E_\beta) = c^i_\alpha q^{(\mu,\alpha)} r'^{i-1}_\alpha(X_\mu) r_\alpha(E_\beta) + r'^i_\alpha(X_\mu) E_\beta$$

for the constant $c^i_\alpha = q_\alpha^{1-i}[i]_\alpha \in k$.

Proof. We prove the claim by induction over i. From Lemma 5.3 we get $r'_\alpha(xx') = q^{(\alpha,\mu)} x r'_\alpha(x') + r'_\alpha(x) x'$ for $x \in U_\mu^+$ and $r'_\alpha(E_\beta) = \delta_{\alpha\beta}$. This proves the claim for $i = 1$.

From the induction assertion follows easily:

$$\begin{aligned}
r'^{i}_\alpha(X_\mu E_\beta) =& r'^{i-1}_\alpha r'_\alpha(X_\mu E_\beta)\\
=& r'^{i-1}_\alpha(q^{(\alpha,\mu)}X_\mu r'_\alpha(E_\beta) + r'_\alpha(X_\mu)E_\beta)\\
=& r'^{i-1}_\alpha(q^{(\alpha,\mu)}X_\mu \delta_{\alpha\beta} + r'_\alpha(X_\mu)E_\beta)\\
\overset{I.A.}{=}& q^{(\alpha,\mu)}r'^{i-1}_\alpha(X_\mu)\delta_{\alpha\beta} + c^{i-1}_\alpha q^{(\alpha,\mu-\alpha)}r'^{i-1}_\alpha(X_\mu)r'_\alpha(E_\beta) + r'^{i}_\alpha(X_\mu)E_\beta\\
=& r'^{i-1}_\alpha(X_\mu)r'_\alpha(E_\beta)(q^{(\alpha,\mu)} + c^{i-1}_\alpha q^{(\alpha,\mu-\alpha)}) + r'^{i}_\alpha(X_\mu)E_\beta
\end{aligned}$$

That is, it is sufficient to show that: $q^{(\alpha,\mu)} + c^{i-1}_\alpha q^{(\alpha,\mu-\alpha)} = q^{(\alpha,\mu)}c^i_\alpha$. This holds if and only if $1 + c^{i-1}_\alpha q^{-(\alpha,\alpha)} = c^i_\alpha$. So it only remains to prove: $q^{1-i}_\alpha[i]_\alpha = q^{1-(i-1)}_\alpha[i-1]_\alpha q^{-2}_\alpha + 1$ which works as follows:

$$\begin{aligned}
& q^{1-i}_\alpha[i]_\alpha - 1 - q^{1-(i-1)}_\alpha[i-1]_\alpha q^{-2}_\alpha\\
=& q^{1-i}_\alpha \frac{q^i_\alpha - q^{-i}_\alpha}{q_\alpha - q^{-1}_\alpha} - 1 - q^{1-(i-1)}_\alpha \frac{q^{i-1}_\alpha - q^{1-i}_\alpha}{q_\alpha - q^{-1}_\alpha} q^{-2}_\alpha\\
=& (q^{1-i}_\alpha(q^i_\alpha - q^{-i}_\alpha) - q_\alpha + q^{-1}_\alpha - q^{-i}_\alpha(q^{i-1}_\alpha - q^{1-i}_\alpha))\frac{1}{q_\alpha - q^{-1}_\alpha}\\
=& (q_\alpha - q^{1-2i}_\alpha - q_\alpha + q^{-1}_\alpha - q^{-1}_\alpha + q^{1-2i}_\alpha)\frac{1}{q_\alpha - q^{-1}_\alpha}\\
=& 0
\end{aligned}$$

This proves the Lemma. □

Lemma 5.7. *For $\alpha \in \Pi$ let r'_α be the element as in Definition 5.2 and $X_\mu \in U^+_\mu$ in degree $\mu \in Q_+$ then:*

$$\Delta(X_\mu) = K_\mu \otimes X_\mu + \sum_{\alpha\in\Pi}\sum_i E^i_\alpha K_{\mu-i\alpha} \otimes s^i_\alpha(X_\mu) + (rest)$$

where $(rest)$ contains terms in $U_\nu \otimes U$ for any $\nu \in Q_+$ which is not a multiple of α. Here $s^i_\alpha(x) = z^i_\alpha \cdot r'^{i}_\alpha(x) \in U^+$ for the constant $z^i_\alpha = \frac{z^{i-1}_\alpha}{q^{2(i-1)}_\alpha c^i_\alpha} = \frac{z^{i-1}_\alpha}{q^{i-1}_\alpha [i]} \in k$.

Proof. Both sides are k-linear. We prove the statement inductively over the height $ht(\mu)$ of X_μ. We look at words in the generators $E_\alpha, \alpha \in \Pi$. For $ht(\mu) = 1$, i.e. $X_\mu = E_\alpha$ for an $\alpha \in \Pi$, the claim is trivial. Assume the claim is true for elements $X_\mu \in U_\mu$, then we prove the claim for $X_\mu E_\beta$ for an arbitrary $\beta \in \Pi$:

$$\begin{aligned}
\Delta(X_\mu E_\beta) =& \Delta(X_\mu)\Delta(E_\beta)\\
=& (K_\mu \otimes X_\mu + \sum_{\alpha\in\Pi}\sum_i E^i_\alpha K_{\mu-i\alpha} \otimes s^i_\alpha(X_\mu) + (rest))(K_\beta \otimes E_\beta + E_\beta \otimes 1)\\
=& K_\mu E_\beta \otimes X_\mu + K_\mu K_\beta \otimes X_\mu E_\beta + \sum_{\alpha\in\Pi}\sum_i E^i_\alpha K_{\mu-i\alpha}E_\beta \otimes s^i_\alpha(X_\mu)\\
&+ \sum_{\alpha\in\Pi}\sum_i E^i_\alpha K_{\mu-i\alpha}K_\beta \otimes s^i_\alpha(X_\mu)E_\beta + (rest)'
\end{aligned}$$

Where $(rest)'$ contains terms of $(rest)$ and $(rest)E_\beta$. We want to prove, that:

$$K_\mu E_\beta \otimes X_\mu + \sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu-i\alpha}E_\beta \otimes s_\alpha^i(X_\mu) \tag{8}$$

$$+\sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu-i\alpha}K_\beta \otimes s_\alpha^i(X_\mu)E_\beta + (rest)' \tag{9}$$

$$= \sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu+\beta-i\alpha} \otimes s_\alpha^i(X_\mu E_\beta) + (rest)_{new} \tag{10}$$

Let us first consider the case $\alpha \neq \beta$, here

$$K_\mu E_\beta \otimes X_\mu + \sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu-i\alpha}E_\beta \otimes s_\alpha^i(X_\mu) + (rest) \in (rest)_{new}$$

It remains to prove that

$$\sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu-i\alpha}K_\beta \otimes s_\alpha^i(X_\mu)E_\beta = \sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu+\beta-i\alpha} \otimes s_\alpha^i(X_\mu E_\beta)$$

This holds if and only if $s_\alpha^i(X_\mu)E_\beta = s_\alpha^i(X_\mu E_\beta)$. The latter, however, follows from the corresponding property of r'^i_α.

Now we consider the case $\beta = \alpha$. Here: $X_\mu = X_\mu s_\alpha(E_\alpha)$. So we can rewrite the left hand side of equation (9) as:

$$\begin{aligned}
K_\mu E_\alpha \otimes X_\mu s_\alpha(E_\alpha) + &\sum_{\alpha\in\Pi}\sum_{j\geq 2} q^{(\mu-(j-1)\alpha,\alpha)} E_\alpha^j K_{\mu+\alpha-j\alpha} \otimes s_\alpha^{j-1}(X_\mu)s_\alpha(E_\alpha)\\
&+\sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu-i\alpha}K_\alpha \otimes s_\alpha^i(X_\mu)E_\beta + (rest)'\\
=&\sum_{\alpha\in\Pi}\sum_{j\geq 1} q^{(\mu-(j-1)\alpha,\alpha)} E_\alpha^j K_{\mu+\alpha-j\alpha} \otimes s_\alpha^{j-1}(X_\mu)s_\alpha(E_\alpha)\\
&+\sum_{\alpha\in\Pi}\sum_i E_\alpha^i K_{\mu+\alpha-i\alpha} \otimes s_\alpha^i(X_\mu)E_\beta + (rest)'
\end{aligned}$$

Comparing this to the right hand side of equation (9), it remains to prove that for all i:

$$s_\alpha^i(X_\mu)E_\beta + q^{(\mu-(i-1)\alpha,\alpha)} s_\alpha^{i-1}(X_\mu)s_\alpha(E_\alpha) = s_\alpha^i(X_\mu E_\alpha)$$

This in turn follows directly from the definition of s_α and Lemma 5.6. □

Lemma 5.8. *For all r'_α and r_α as in Definition 5.2 the following equations hold:*
a) For all $\alpha, \beta \in \Pi$:

$$r_\alpha r'_\beta = r'_\beta r_\alpha$$

b) For $\alpha, \beta \in \Pi$ with either $\alpha = \beta$ or $\alpha \perp \beta$:

$$r_\alpha r_\beta = r_\beta r_\alpha$$

c) For $\alpha, \beta \in \Pi$ with either $\alpha = \beta$ or $\alpha \perp \beta$:

$$r'_\alpha r'_\beta = r'_\beta r'_\alpha$$

Proof. We prove the statements again by induction on $ht(\mu)$ for elements $X_\mu \in U^+$ for $\mu \in Q_+$:
a) For $\mu \in \Pi$ both sides are 0. Consider now for an arbitrary $\gamma \in \Pi$, $\mu \in Q_+$ the product $X_\mu E_\gamma$:

$$\begin{aligned} r_\alpha r'_\beta(X_\mu E_\gamma) =& r_\alpha(q^{(\beta,\mu)} X_\mu \delta_{\beta\gamma} + r'_\beta(X_\mu)E_\gamma) \\ =& q^{(\beta,\mu)} r_\alpha(X_\mu)\delta_{\beta\gamma} + r'_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\gamma,\alpha)} r_\alpha r'_\beta(X_\mu)E_\gamma \end{aligned}$$

On the other side:

$$\begin{aligned} r'_\beta r_\alpha(X_\mu E_\gamma) =& r'_\beta(X_\mu \delta_{\alpha\gamma} + q^{(\alpha,\gamma)} r_\alpha(X_\mu)E_\gamma) \\ =& r'_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\beta,\alpha)} q^{(\beta,\mu-\alpha)} r_\alpha(X_\mu)\delta_{\beta\gamma} + q^{(\gamma,\alpha)} r'_\beta r_\alpha(X_\mu)E_\gamma \\ \overset{I.A.}{=}& r'_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\beta,\alpha)} q^{(\beta,\mu-\alpha)} r_\alpha(X_\mu)\delta_{\beta\gamma} + q^{(\gamma,\alpha)} r_\alpha r'_\beta(X_\mu)E_\gamma \end{aligned}$$

b) Here for $\mu \in \Pi$ both sides are 0. For $\mu \in Q_+, \gamma \in \Pi$ and arbitrary $\alpha, \beta \in \Pi$ we consider first $r_\alpha r_\beta$:

$$\begin{aligned} r_\alpha r_\beta(X_\mu E_\gamma) =& r_\alpha(X_\mu \delta_{\beta\gamma} + q^{(\beta,\gamma)} r_\beta(X_\mu)E_\gamma) \\ =& r_\alpha(X_\mu)\delta_{\beta\gamma} + q^{(\beta,\gamma)} r_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\beta,\gamma)} q^{(\gamma,\alpha)} r_\alpha r_\beta(X_\mu)E_\gamma \end{aligned}$$

On the other side we get for $r_\beta r_\alpha$:

$$r_\beta r_\alpha(X_\mu E_\gamma) = r_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\gamma,\alpha)} r_\alpha(X_\mu)\delta_{\beta\gamma} + q^{(\beta,\gamma)} q^{(\gamma,\alpha)} r_\beta r_\alpha(X_\mu)E_\gamma$$

By the induction hypothesis, the two sides are equal, if:

$$r_\alpha(X_\mu)\delta_{\beta\gamma} + q^{(\beta,\gamma)} r_\beta(X_\mu)\delta_{\alpha\gamma} = r_\beta(X_\mu)\delta_{\alpha\gamma} + q^{(\gamma,\alpha)} r_\alpha(X_\mu)\delta_{\beta\gamma}$$

This is true if and only if $\alpha = \beta$ or $(\alpha, \beta) = 0$. c) The proof is analogous to b). □

Lemma 5.9. *We expand the definition of r_α to $r_{\bar\alpha}$ for elements: $\bar\alpha = \sum_k i_k \alpha_k \in Q$ with $i_k \in \mathbb{N}$ and pairwise orthogonal roots $\alpha_k \in \Pi$ as follows: Let x be a homogeneous element in U^+. Then we define:*

$$r_{\bar\alpha}(x) := \prod_k r_{\alpha_k}^{i_k}(x) \in U^+$$

Due to Lemma 5.8 this is well defined. Moreover let $\mathtt{supp}(\bar\alpha) := \{k \in \mathbb{N} \mid i_k \neq 0\}$. Now we get for arbitrary $X_\mu \in U^+_\mu$ for $\mu \in Q$ and $E_\gamma \in U^+_\gamma$ for $\gamma \in \Pi$:

$$r_{\bar\alpha}(X_\mu E_\gamma) = \sum_{k \in \mathtt{supp}(\bar\alpha)} c_{\alpha_k}^{i_k} q^{(\mu,\alpha_k)} r_{\bar\alpha-\alpha_k}(X_\mu) r_{\alpha_k}(E_\beta) + r_{\bar\alpha}(X_\mu)E_\beta$$

Proof. We prove the statement with induction on $|\mathtt{supp}(\bar\alpha)|$. For $n = 1$ the claim follows from Lemma 5.6. Consider a root as above $\bar\alpha = \sum_k i_k \alpha_k$, then for an arbitrary j with $i_j \neq 0$ in particular we get $r_{\bar\alpha} = r_{\alpha_j}^{i_j} \prod_{k \neq j} r_{\alpha_k}^{i_k}$. We call $\sum_{k \neq j} i_k \alpha_k = \bar\alpha'$, so $\bar\alpha = \bar\alpha' + i_j \alpha_j$

and for the root $\bar{\alpha}'$ the claim follows from the induction hypothesis.

$$\begin{aligned}
r_{\bar{\alpha}}(X_\mu E_\gamma) =& r_{\alpha_j}^{i_j} r_{\bar{\alpha}'}(X_\mu E_\gamma) \\
=& r_{\alpha_j}^{i_j} \Big(\sum_{k \in \mathrm{supp}(\bar{\alpha}')} c_{\alpha_k}^{i_k} q^{(\mu,\alpha_k)} r_{\bar{\alpha}'-\alpha_k}(X_\mu) r_{\alpha_k}(E_\beta) + r_{\bar{\alpha}'}(X_\mu) E_\beta \Big) \\
=& \sum_{k \in \mathrm{supp}(\bar{\alpha}')} c_{\alpha_k}^{i_k} q^{(\mu,\alpha_k)} \underbrace{r_{\bar{\alpha}'+i_j\alpha_j-\alpha_k}}_{r_{\bar{\alpha}-\alpha_k}}(X_\mu)\delta_{\alpha_k\beta} + c_{\alpha_j}^{i_j} q^{(\alpha_j,\mu)} \underbrace{r_{\bar{\alpha}'} r_{\alpha_j}^{(i_j-1)}}_{r_{\bar{\alpha}-\alpha_j}}(X_\mu) r_{\alpha_j}(E_\beta) \\
& + r_{\bar{\alpha}'+i_j\alpha_j}(X_\mu) E_\beta \\
=& \sum_{k \in \mathrm{supp}(\bar{\alpha})} c_{\alpha_k}^{i_k} q^{(\mu,\alpha_k)} r_{\bar{\alpha}-\alpha_k}(X_\mu)\delta_{\alpha_k\beta} + r_{\bar{\alpha}}(X_\mu) E_\beta
\end{aligned}$$

Thus the claim is proven. □

Now we can prove the strongest generalization of the technical Lemma 5.3.

Lemma 5.10. *Let $\bar{\alpha} = \sum_k i_k \alpha_k \in Q$ and $r_{\bar{\alpha}} = \prod_k r_{\alpha_k}^{i_k}$ as above and $z_{\bar{\alpha}} = \prod_k z_{\alpha_k}^{i_k} \in k$ for the elements z_{α_k} from Lemma 5.7. Then we get for $X_\mu \in U_\mu$ for a root $\mu \in Q_+$:*

$$\Delta(X_\mu) = K_\mu \otimes X_\mu + \sum_{\bar{\alpha}} E_{\bar{\alpha}} K_{\mu-\bar{\alpha}} \otimes s_{\bar{\alpha}}(X_\mu) + (rest)$$

Where $(rest)$ contains terms in $U_\nu \otimes U_{\mu-\nu}$, such that $\nu \in Q$ is not a linear combination of pairwise orthogonal roots $\alpha_k \in \Pi$.

Proof. The proof works similar to the proof of Lemma 5.7 together with Lemma 5.9. □

Having finished this we close this technical subsection by applying Lemma 5.5:

Lemma 5.11. *Let $w_0 \in W$ be the longest element of the Weyl group with a reduced expression $w_0 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{\ell(w_0)}}$. Let $\Phi^+(w) = \{\beta_1, \dots, \beta_{\ell(w_0)}\}$ with respect to this reduced expression. For an $\alpha \in \Pi$ let $\beta_j = \alpha$. For the root vectors E_{β_i} the following relations hold:*
a) For $i < j$:

$$r'_\alpha(E_{\beta_i}) = 0$$

b) For $i > j$:

$$r_\alpha(E_{\beta_i}) = 0$$

Proof. According to Lemma 5.5 we know:

$$\begin{aligned}
T_\alpha(U^+[s_\alpha w_0]) &= \{x \in U^+ \mid r_\alpha(x) = 0\} \\
U^+[s_\alpha w_0] &= \{x \in U^+ \mid r'_\alpha(x) = 0\}
\end{aligned}$$

a) For $i < j$ in particular E_{β_i} lies in $U^+[s_\alpha w_0]$, thus $r'_\alpha(E_{\beta_i}) = 0$.
b) For $i > j$ in particular E_{β_i} lies in $T_\alpha(U^+[s_\alpha w_0])$, thus $r_\alpha(E_{\beta_i}) = 0$. □

5.3. **Commutator relations of character-shifted root vectors.** We use the results of the preceding subsection to calculate commutator relations of character-shifted root vectors.

Lemma 5.12. *Let ϕ be a character on a right coideal subalgebra A of U^+ and A_ϕ be the character-shifted subalgebra. We define the character-shifted elements of A_ϕ as follows:*

$$\bar{X}_\mu := (\phi \otimes id)\Delta(X_\mu) \text{ for } X_\mu \in A_\mu$$

Let ϕ be a character whose support contains only pairwise orthogonal roots in Π, i.e. $\mathtt{supp}(\phi) = \{\alpha_1, \dots, \alpha_k\} \subset \Pi$ with $\phi(\alpha_i) = \lambda_i \in k$. We define $\bar{\alpha} = \sum_k i_k \alpha_k$ as above and $\lambda_{\bar{\alpha}} = \prod_k \lambda_k^{i_k}$. Then:

$$\bar{X}_\mu = X_\mu + \sum_{0 \prec \bar{\alpha} \prec \mu} \lambda_{\bar{\alpha}} s_{\bar{\alpha}}(X_\mu)$$

Proof. From Lemma 5.10 this follows directl, because $(rest)$ is zero. □

Lemma 5.13. *Given $C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-} \subset U$ as from in the beginning, i.e. for $x \in W$ a Weyl group element such that $\Phi^+(x) \subset \Pi$ consists of pairwise orthogonal roots. Further let ϕ^+, ϕ^- be the characters with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \Phi^+(x)$ defined as follows: $\phi^+(\psi(E_\alpha)) = \lambda_\alpha \in k^*$ for $\alpha \in \mathtt{supp}(\phi^+)$ and 0 otherwise and $\phi^-(F_\alpha) = \lambda'_\alpha \in k^*$ and 0 otherwise, with $\lambda'_\alpha \lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$.*

Then for any $\alpha \in \mathtt{supp}(\phi^+)$ and $\mu \neq \alpha$ for elements $\bar{X}_\mu \in (\psi(U^+[w_0])_{\phi^+})_\mu$ and $\bar{F}_\alpha \in (U^-[x]_{\phi^-})_\alpha$ the following q-commutator relation holds:

$$[F_\alpha + \lambda'_\alpha K_\alpha^{-1}, \bar{X}_\mu]_{q^{-(\mu,\alpha)}} = 0 \tag{11}$$

Proof. Consider an element $r'_\alpha(s_{\bar{\alpha}}(X_\mu))$. Then:

$$r'_\alpha(s_{\bar{\alpha}}(X_\mu)) = \frac{x_\alpha^{i-1}}{x_\alpha^i} s_{\bar{\alpha}+\alpha}(X_\mu)$$

for $i = \frac{(\bar{\alpha},\alpha)}{(\alpha,\alpha)} + 1$. That is:

$$r'_\alpha(s_{\bar{\alpha}}(X_\mu)) = s_{\bar{\alpha}+\alpha}(X_\mu) q_\alpha^{-1}[i] = s_{\bar{\alpha}+\alpha}(X_\mu)(q_\alpha^{-i-1} \frac{q_\alpha^{2i} - 1}{q_\alpha - q_\alpha^{-1}})$$

Inserting 5.12 in (11), yields:

$$[F_\alpha + \lambda'_\alpha K_\alpha^{-1}, \bar{X}_\mu]_{q^{-(\mu,\alpha)}} = \sum_{0 \preccurlyeq \bar{\alpha}} \lambda_{\bar{\alpha}}([F_\alpha, s_{\bar{\alpha}}(X_\mu) K_\mu^{-1}]_{q^{-(\mu,\alpha)}} + \lambda'_\alpha [K_\alpha^{-1}, s_{\bar{\alpha}}(X_\mu) K_\mu^{-1}]_{q^{-(\mu,\alpha)}})$$

For (11) it remains to prove, that:

$$\sum_{0 \preccurlyeq \bar{\alpha}} \lambda_{\bar{\alpha}} [F_\alpha, s_{\bar{\alpha}}(X_\mu) K_\mu^{-1}]_{q^{-(\mu,\alpha)}} = -\sum_{0 \preccurlyeq \bar{\alpha}} \lambda_{\bar{\alpha}} \lambda'_\alpha [K_\alpha^{-1}, s_{\bar{\alpha}}(X_\mu) K_\mu^{-1}]_{q^{-(\mu,\alpha)}} \tag{12}$$

To see this we use Lemma 5.4, this yields for $\alpha \in \Pi$ and $x \in U_\mu^+$:

$$x F_\alpha - F_\alpha x = (q_\alpha - q_\alpha^{-1})^{-1}(r_\alpha(x) K_\alpha - K_\alpha^{-1} r'_\alpha(x)) \tag{13}$$

Consider first the left hand side of (12):

$$\begin{aligned}
\sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}[F_\alpha, s_{\bar\alpha}(X_\mu)K_\mu^{-1}]_{q^{-(\mu,\alpha)}} &= \sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}(F_\alpha s_{\bar\alpha}(X_\mu)K_\mu^{-1} - q^{-(\mu,\alpha)}s_{\bar\alpha}(X_\mu)K_\mu^{-1}F_\alpha)\\
&= \sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}(F_\alpha s_{\bar\alpha}(X_\mu) - s_{\bar\alpha}(X_\mu)F_\alpha)K_\mu^{-1}\\
&\overset{(13)}{=} -\sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}(q_\alpha - q_\alpha^{-1})^{-1}(r_\alpha(s_{\bar\alpha}(X_\mu))K_\alpha - K_\alpha^{-1}r'_\alpha(s_{\bar\alpha}(X_\mu)))K_\mu^{-1}\\
&\overset{Lm.5.11}{=} \sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}(q_\alpha - q_\alpha^{-1})^{-1}q^{-(\alpha,\mu-\bar\alpha-\alpha)}r'_\alpha(s_{\bar\alpha}(X_\mu))K_\mu^{-1}K_\alpha^{-1}\\
&= \sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}s_{\bar\alpha+\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}q_\alpha^{-i-1}\frac{q_\alpha^{2i}-1}{q_\alpha - q_\alpha^{-1}}(q_\alpha - q_\alpha^{-1})^{-1}q^{-(\alpha,\mu)}q_\alpha^{i}\\
&= \sum_{\alpha\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda_\alpha^{-1}s_{\bar\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}q_\alpha^{-1}\frac{q_\alpha^{2i-2}-1}{q_\alpha - q_\alpha^{-1}}(q_\alpha - q_\alpha^{-1})^{-1}q^{-(\alpha,\mu)}
\end{aligned}$$

On the right hand side of the equation (12) we have:

$$\begin{aligned}
-\sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda'_\alpha[K_\alpha^{-1}, s_{\bar\alpha}(X_\mu)K_\mu^{-1}]_{q^{-(\mu,\alpha)}} &= -\sum_{0\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda'_\alpha(K_\alpha^{-1}s_{\bar\alpha}(X_\mu)K_\mu^{-1} - q^{-(\mu,\alpha)}s_{\bar\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}\\
&= -\sum_{\alpha\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda'_\alpha s_{\bar\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}q^{-(\mu,\alpha)}(q_\alpha^{2i-2}-1)
\end{aligned}$$

Comparing both sides we get:

$$\begin{aligned}
&\sum_{\alpha\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda_\alpha^{-1}s_{\bar\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}q_\alpha^{-1}\frac{q_\alpha^{2i-2}-1}{q_\alpha - q_\alpha^{-1}}(q_\alpha - q_\alpha^{-1})^{-1}q^{-(\alpha,\mu)}q_\alpha^2\\
&= -\sum_{\alpha\preccurlyeq\bar\alpha}\lambda_{\bar\alpha}\lambda'_\alpha s_{\bar\alpha}(X_\mu)K_\mu^{-1}K_\alpha^{-1}q^{-(\mu,\alpha)}(q_\alpha^{-2i}-1)
\end{aligned}$$

It remains to prove that:

$$\lambda_\alpha^{-1}q_\alpha^{-1}\frac{q_\alpha^{2i-2}-1}{q_\alpha - q_\alpha^{-1}}(q_\alpha - q_\alpha^{-1})^{-1}q_\alpha^2 = -\lambda'_\alpha(q_\alpha^{2i-2}-1)$$

Rearranging yields:

$$\lambda'_\alpha\lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$$

Thus the claim is proven. □

Corollary 5.14. *Given $C = \psi(U^+[w_0])_{\phi^+}T_L U^-[x]_{\phi^-} \subset U$ as from the beginning, i.e. for $x \in W$ a Weyl group element such that $\Phi^+(x) \subset \Pi$ consists of pairwise orthogonal roots. Further let ϕ^+, ϕ^- be characters with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \Phi^+(x)$ defined as follows: $\phi^+(\psi(E_\alpha)) = \lambda_\alpha \in k^*$ for $\alpha \in \mathtt{supp}(\phi^+)$ and 0 otherwise and $\phi^-(F_\alpha) = \lambda'_\alpha \in k^*$ and 0 otherwise, with $\lambda'_\alpha\lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha-q_\alpha^{-1})}$ and let $L = \Phi^+(x)^{perp}$.*

Then C is a triangular right coideal subalgebra, i.e. closed under multiplication.

Proof. This follows directly from the commutator relation of Lemma 5.13:

$$[\bar{E}_\beta, \bar{F}_\alpha] = 0$$

for all $\alpha \in \mathtt{supp}(\phi^+)$, $\beta \in \Phi^+ \backslash \mathtt{supp}(\phi^+)$. On the other hand we know:

$$[\bar{E}_\alpha, \bar{E}_\beta] = [\bar{E}_\alpha, \bar{F}_\beta] = [\bar{F}_\alpha, \bar{F}_\beta] = 0,$$

$$[\bar{E}_\alpha, \bar{F}_\alpha] = 1\frac{q_\alpha^2}{q_\alpha - q_\alpha^{-1}}$$

for all $\alpha \neq \beta \in \mathtt{supp}(\phi^+)$ and of course

$$[\bar{E}_\mu, \bar{E}_\nu] \subset \psi(U^+[w_0])_{\phi^+} \qquad [\bar{E}_\mu, K_\gamma] = 0$$

for all $\mu, \nu \in \Phi^+$ and $\gamma \in \Phi^+$ with $K_\gamma \in C$. So the ordered product of subalgebras C is closed under multiplication and therefore a subalgebra. As a product of right coideals, it is in particular also a right coideal subalgebra. □

Remark 5.15. *Let $C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$ be the right coideal subalgebra as from the beginning. If $C \subset U_q(\mathfrak{sl}_n)$, then for a maximal root $\mu \in \Phi^+$ the root vector $\bar{E}_\mu \in \psi(U^+[w_0])_{\phi^+}$ we obtained in Lemma* [Vocke16] *Lemma 3.36 q-commutes with all root vectors in C.*

Due to Lemma 5.13 $\bar{E}_\mu$ q-commutes with all root vectors in $T_L U^-[x]_{\phi^-}$. From Lemma [Vocke16] Lemma 3.36 we know, that E_μ q-commutes with all root vectors in U^+. As $U^+[w_0] \cong U^+[w_0]_\phi$ for characters ϕ on $U^+[w_0]$, we already know, that all root vectors in $\psi(U^+[w_0])_{\phi^+}$ q-commute with $\bar{E}_\mu$ and thus the claim is proven.

5.4. **On representation theory of triangular coideal subalgebras.** Recall how we have proven in [HLV17] that the standard Borel subalgebras are indeed ede:

Lemma 5.16. *Given a Weyl group element $w \in W$ and the corresponding right coideal subalgebra $U^-[w]$. Each finite dimensional irreducible representation V, on which all elements in $U^-[w] \cap ker\epsilon$ act nilpotent is one dimensional.*

In particular $U^-[w]$ is always weak ede. The same is true for $U^+[w]$.

Corollary 5.17. *Given a Weyl group element $w \in W$, L a subgroup of Q and the corresponding right coideal subalgebra $C = T_L U^-[w]$ with the property, that for all $\mu \in \Phi^+(w)$ there exists a $\nu \in L$ such that $(\mu, \nu) \neq 0$, then C is ede.*

In particular $U^0 U^-[w]$ is ede, as well as the right coideal subalgebra $U^0 U^+[w]$.

From now on let $C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$ be as in the previous subsections, i.e. let $x \in W$ be the Weyl group element such that $\Phi^+(x) \subset \Pi$ consists of pairwise orthogonal roots. Further let ϕ^+, ϕ^- be characters with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \Phi^+(x)$, and $\phi^+(E_\alpha)\phi^-(F_\alpha) = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$ and $L \subseteq \Phi^+(x)^\perp$.

Lemma 5.18. *For $C = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$ any finite-dimensional irreducible representation is one-dimensional.*

Proof. Let V be a finite irreducible representation of C. As the roots in $\mathtt{supp}(\phi^+)$ are pairwise orthogonal and as $L = \mathtt{supp}(\phi^+)^\perp$, then due to Lemma 5.13 the following commutator relations hold:

$$[\bar{E}_\alpha, \bar{E}_\beta] = [\bar{E}_\alpha, K_\mu] = [\bar{E}_\alpha, \bar{F}_\beta] = [\bar{F}_\alpha, \bar{F}_\beta] = 0$$

for all $\alpha \neq \beta \in \mathtt{supp}(\phi^+)$ and $\mu \in \Phi^+$ with $K_\mu \in C$. We consider the reduced expression of the longest element:

$$w_0 = (\prod_{\alpha_j \in \mathtt{supp}(\phi^+)} s_{\alpha_j})v$$

Let C' be the subalgebra generated by the elements E_{β_i} for $i > |\mathtt{supp}(\phi^+)|$. From the maximal choice $L = \mathtt{supp}(\phi^+)^\perp$ we get for any $E_{\beta_i} \in C'$ an element $K_\mu \in T_L$ with $q^{(\beta_i,\mu)} \neq 1$. C' is as algebra isomorphic to $U^+[v]$, so we can apply Corollary 5.17 and show, that C' is ede and we can thus prove the existence of a vector $v \in V$ which is annihilated by C'.

Let $V_{C'}$ be the annihilator of C' in V, i.e. $V_{C'} := \{v \in V | \forall x \in C' : xv = 0\}$. We claim that $V_{C'}$ is a C-submodul of V: We know from Theorem 5.13 that all elements $X_\mu \in C'$ q-commute with all elements in $T_L U^-[x]_{\phi^-}$.

For degree reasons we know that the q-commutator of the elements in C' and $\psi(U^+[x])_{\phi^+}$ lies in C', i.e. vanishes on the representation $V_{C'}$. Thus with Theorem 3.12 $V_{C'}$ is a C-subrepresentation, and as V is irreducible we get $V_{C'} = V$.

Thus we get for the irreducible representation V of C:

$$C|_V \cong T_L \otimes \bigotimes_{\alpha \in \mathtt{supp}(\phi^+)} \langle \bar{E}_\alpha, \bar{F}_\alpha \rangle$$

So the ede property follows from the ede property of the components of the tensor product. T_L is abelian, and for any $\alpha \in \mathtt{supp}(\phi^+)$ the subalgebras $\langle \bar{E}_\alpha, \bar{F}_\alpha \rangle$ are quantised Weyl algebras, which are ede by Example 2.7. □

5.5. **Proof of the main construction theorem.**

Proof. From Lemma 5.18 we already know, that C is ede, it remains to prove the maximality of C. Let C' be an ede right coideal subalgebra with $C \subset C'$.

Due to the second authors generating system for any RCS in [Vocke16] Lemma 4.11 or [Vocke17] we can choose a generating system of C' consisting of elements, whose E-Leitterms lie in $U^{\geq 0}$.As we know that $\psi(U^+[w_0])_{\phi^+} \subset C \subset C'$, all elements with non-trivial E-Leitterm lie in $U^{\geq 0}$: That is, to each additional element X in C' we find an element Y in $C \cap U^{\geq 0}$, such that X and Y have the same E-Leitterms. Thus the difference has a smaller degree and can be generated as in [Vocke16] Lemma 4.7 by elements with E-Leitterms in $U^{\geq 0}$, which again have smaller degree as before. By induction it follows that all generators as in [Vocke16] lie either in $U^{\geq 0}$ or $U^{\leq 0}$ and the right coideal subalgebra C' is triangular generated, i.e. of the form $C' = \psi(U^+[w_0])_{\phi^+_{neu}} T_{L'} U^-[xv]_{\phi^-_{neu}}$ for a $v \in W$ with $\ell(xv) = \ell(x) + \ell(v)$ and characters ϕ^+_{neu} and ϕ^-_{neu}.

As $T_L \subset C'$, we can consider ϕ^+_{neu}, ϕ^-_{neu} such that all elements in C' are ad-T_L-stable. As C' is ede we can conclude: $\mathtt{supp}(\phi^+_{neu}), \mathtt{supp}(\phi^-_{neu}) \subset (T_L)^\perp = \mathtt{supp}(\phi^+)$. On the other hand Theorem 3.3 yields, that for all elements $\mu \in \Phi^+(w_0) \cap \Phi^+(xv) \supset \Phi^+(x)$ holds $\mu \in \mathtt{supp}(\phi^+_{neu}) \cap \mathtt{supp}(\phi^-_{neu})$. Thus we get $\mathtt{supp}(\phi^+_{neu}) = \mathtt{supp}(\phi^-_{neu}) = \mathtt{supp}(\phi^+)$. As $T_{L'} \subset (\mathtt{supp}(\phi^+_{neu}) \cap \mathtt{supp}(\phi^-_{neu}))^\perp = (\mathtt{supp}(\phi^+))^\perp$ and of course $L \subset L'$, so $L' = L$ follows directly.

In the case $\mathfrak{g} = A_n$ the claim follows from Lemma 2.11, in general we can prove it as follows: Assume $v \neq 1$, then there exists $\alpha \in \Pi \cap \Phi^+(v)$. Is α orthogonal to all elements in $\mathtt{supp}(\phi^+)$, then $E_\alpha K_\alpha^{-1}, F_\alpha \in C'$ and Corollary 3.4 yields a contradiction to C' ede. Is α not orthogonal on all elements in $\mathtt{supp}(\phi^+)$, then there exists a root $\mu \in \Phi^+(xv)$ and a term $\bar{F}_\mu = cF_\alpha K_{\mu-\alpha}^{-1} + \sum_{\nu \succ \alpha} X_\nu K_{\mu-\nu}^{-1} \in C'$ for a constant $c \in k^*$ and terms $X_\nu \in U_\nu^{\leq 0}$ with $\alpha \prec \nu$: This claim follows from the fact that the elements in $\mathtt{supp}(\phi^+)$ are pairwise orthogonal and simple and in the relevant cases A_3, C_3 resp. D_4 resp. B_3 one can calculate the coproduct of F_α directly.

On the other hand we know $E_\alpha K_\alpha^{-1} \in C'$. Considering the action of the 1-commutator on the lowest weight representation $L(\lambda)$ to a lowest weight λ with $q^{(\alpha,\lambda)} \neq \pm 1$, we find:

$$\begin{aligned} [\bar{F}_\mu, E_\alpha K_\alpha^{-1}].v_\lambda &= \bar{F}_\mu E_\alpha K_\alpha^{-1}.v_\lambda \\ &= cF_\alpha K_{\mu-\alpha}^{-1} E_\alpha K_\alpha^{-1}.v_\lambda + \sum_{\nu \succ \alpha} X_\nu E_\alpha K_\alpha^{-1}.v_\lambda \\ &= \frac{cq^{-(\mu-\alpha,\alpha)}}{q_\alpha - q_\alpha^{-1}} K_\mu^{-1}(K_\alpha - K_\alpha^{-1}).v_\lambda = c' v_\lambda \end{aligned}$$

for a $c' \in k^*$ by choice of the smallest weight λ. It follows that the commutator has a non-trivial eigenvalue in the finite-dimensional representation $L(\lambda)$ and thus not any finite-dimensional irreducible representation of C' can be one-dimensional. So $v = 1$ and $C' = C$, i.e. C is maximal and thus a Borel subalgebra.

□

5.6. **Classification theorem.** A triangular right coideal subalgebra is always of the form: $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-}$, where $w^-, w^+ \in W$ are elements in the Weyl group; be advised that not any choice of w^+, w^- conversely leads to a well defined algebra. By definition of the character-shift it makes only sense to consider $L \subset (\mathtt{supp}(\phi^+) \cup \mathtt{supp}(\phi^-))^\perp$, see Theorem 2.3. In the previous subsection we have constructed Borel subalgebras with an additional non-degeneracy property $\mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-) = \Phi^+(w^+) \cap \Phi^+(w^-)$ and $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-)$. Now we conversely prove in the case of A_n that we have found indeed all Borel subalgebras with these properties. We use in particular Theorem 4.1 on Weyl groups, which tells us that an arbitrary triangular ede RCS is contained in a larger triangular ede RCS which is then isomorphic via some T_w to one of the Borel subalgebras we constructed. It is this theorem on Weyl groups which we generally conjecture, but here could only be proven for A_n.

Main Theorem 5.19. *Assume type A_n. Every triangular Borel subalgebra of $U_q(\mathfrak{g})$*

$$B = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-}$$

with the additional non-degeneracy property $\Phi^+(w^+) \cap \Phi^+(w^-) = \mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-)$ is isomorphic as algebra to a Borel subalgebra of the form constructed in our Main Theorem 5.1.

That is: $\psi(U^+[w_0])_{\phi^+} T_L U^-[\prod_{\alpha\in c} s_\alpha]_{\phi^-}$ with $L = c^\perp$, where c is a set of pairwise orthogonal simple roots and ϕ^+, ϕ^- are characters on $\psi(U^+[\prod_{\alpha\in c} s_\alpha])$ resp. $U^-[\prod_{\alpha\in c} s_\alpha]$, given by an arbitrary choice of the values $\lambda_\alpha = \phi^+(E_\alpha K_\alpha^{-1})$, $\lambda'_\alpha = \phi^-(F_\alpha)$ for $\alpha \in c$ with $\lambda'_\alpha \lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$.

In the rest of the section we will give a proof of this classification result.

Lemma 5.20. *Let C be a triangular ede right coideal subalgebra of the form $C = \psi(U^+[w_1])_{\phi^+} T_L U^-[w_2]_{\phi^-}$ with $\Phi^+(w_1) \cap \Phi^+(w_2) = \mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) =: \mathtt{B}$. Then there are elements $w'_1, w'_2 \in W$ with $\Phi^+(w_1) \subseteq \Phi^+(w'_1)$, $\Phi^+(w_2) \subseteq \Phi^+(w'_2)$, such that $\Phi^+(w'_1) \cap \Phi^+(w'_2) = \mathtt{B}$ and $\Phi^+(w'_1) \cup \Phi^+(w'_2) = \Phi^+$. For these elements w'_1, w'_2 and $L' := \mathtt{B}^\perp$ we get:*[1]

$$C \subseteq \psi(U^+[w'_1])_{\phi^+} T_{L'} U^-[w'_2]_{\phi^-} =: C'$$

Proof. The existence of such Weyl group elements w'_1, w'_2 is the purpose of Theorem 4.1. Of course we have $U^+[w_1] \subset U^+[w'_1]$ and $U^+[w_2] \subset U^+[w'_2]$. So it remains to prove, that ϕ^+ is a character on $U^+[w'_1]$. From Theorem 4.1 we already know, that w'_1 has a reduced expression of the form $w'_1 = s_{\alpha_{i_1}} \dots s_{\alpha_{i_k}}$, such that there is a $j \leq k$ with $\mathtt{B} = \{\beta_{i_j}, \dots, \beta_{i_k}\}$. Due to Remark 4.5 ϕ^+ is thus a character on $U^+[w'_1]$. The same holds for $U^-[w'_2]$. Thus the inclusion $C \subseteq \psi(U^+[w'_1])_{\phi^+} T_{L'} U^-[w'_2]_{\phi^-}$ holds. □

Now we take a closer look at $C' = \psi(U^+[w'_1])_{\phi^+} T_{L'} U^-[w'_2]_{\phi^-}$. In particular we want to show, that it is a triangular right coideal subalgebra and ede. For this in particular we use Theorem 4.1 and the Lusztig automorphisms. We already know that for w'_2 there is a reduced expression $w'_2 = s_{\alpha_{i_1}} \dots s_{\alpha_{i_{\ell(w'_2)}}}$, such that $\mathtt{B} = \{\beta_{i_{\ell(w'_2)-|B|+1}}, \dots, \beta_{i_{\ell(w'_2)}}\}$. We define: $v^{-1} := s_{\alpha_{i_1}} \dots s_{\alpha_{i_{\ell(w'_2)-|\mathtt{B}|}}}$ and $x := s_{\alpha_{i_{\ell(w'_2)-|\mathtt{B}|+1}}} \dots s_{\alpha_{i_{\ell(w'_2)}}}$, such that $w'_2 = v^{-1}x$ and from the choice of w'_1 we get $vw'_1 = w_0$, as $\Phi^+(w'_1) \cup \Phi^+(w'_2) = \Phi^+$, moreover $\Phi^+(x)$ consists of pairwise orthogonal simple roots, by Definition of $\mathtt{B}$.

In the following Lemma we now prove that in this situation we can apply Lusztigs automorphism T_v to the chracter-shifted RCS (in general this does not give a character-shifted RCS!) and show that it has the intended result claimed in the classification:

Lemma 5.21. *Let C be a triangular ede right coideal subalgebra of the form $C = \psi(U^+[w_1])_{\phi^+} T_L U^-[w_2]_{\phi^-}$ with $\Phi^+(w_1) \cap \Phi^+(w_2) = \mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) =: \mathtt{B}$. Given the corresponding w'_1, $w'_2 = v^{-1}x$ with v^{-1} and x as above, then the Lusztigautomorphism T_v maps the coideal C' constructed in the previous Lemma :*

$$T_v(\underbrace{\psi(U^+[w'_1])_{\phi^+} T_{L'} U^-[v^{-1}x]_{\phi^-}}_{C'}) = \psi(U^+[vw'_1])_{T_v(\phi^+)} T_v(T_{L'})(U^-[x])_{T_v(\phi^-)} \tag{14}$$

[1]The reader be again advised that C' is a product of coideals, but at this point not necessarily an algebra. In our specific situation this follows from the next Lemma

where $T_v(\phi^{\pm})$ is defined as $\phi^{\pm} \circ T_v^{-1}$.

Proof. Let us consider the PBW generators of C: As C is triangular, all of them lie in $\psi(U^+[w_1'])_{\phi^+}, U^-[v^{-1}x]_{\phi^-}$ and T_L. Let's consider first $U^-[v^{-1}x]_{\phi^-}$. Due to the choice of the reduced expression of $v^{-1}x$, the basis elements of $U^-[v^{-1}x]_{\phi^-}$ have the following form:

$$\bar{F}_\mu = \begin{cases} F_\mu + \phi^-(F_\mu)K_\mu^{-1} & \text{for } \mu \in \mathtt{supp}(\phi^-) \\ F_\mu & \text{otherwise} \end{cases}$$

Thus we get for the Lusztig automorphism T_v:

$$T_v(\bar{F}_\mu) = \begin{cases} T_v(F_\mu) + \phi^-(F_\mu)T_v(K_\mu^{-1}) = & \text{for } \mu \in \mathtt{supp}(\phi^-) \\ T_v(F_\mu) + T_v(\phi^-)(T_v(F_\mu))T_v(K_\mu^{-1}) = \overline{T_v(F_\mu)} & \\ & \\ T_v(F_\mu) = \overline{T_v(F_\mu)} & \text{otherwise} \end{cases}$$

The same is true for the basis elements of $U^+[w_1']_{\phi^+}$.

With these considerations, we can argue analogously to non-character-shifted right coideal subalgebras in [HK11a] and obtain the assertion: As $\Phi^+(w_1') \cup \Phi^+(w_2') = \Phi^+$ and $\Phi^+(w_1') \cap \Phi^+(w_2') = \mathtt{B}$ we get $\ell(xw_1') = \ell(w_0)$, so:

$$\begin{aligned} T_v(\psi(U^+[w_1'])_{\phi^+}T_{L'}U^-[v^{-1}x]_{\phi^-}) =& T_v(\psi(U^+[w_1'])_{\phi^+})T_v(T_{L'})T_v(U^-[v^{-1}]_{\phi^-}T_v^{-1}(U^-[x]_{T_v(\phi^-)})) \\ =& \psi(U^+[v])_{T_v(\phi^-)}T_v(\psi(U^+[w_1'])_{\phi^+}T_v(T_{L'})T_vU^-[x]_{T_v(\phi^-)} \\ =& \psi(U^+[vw_1'])_{T_v(\phi^+)}T_v(T_{L'})(U^-[x])_{T_v(\phi^-)} \\ =& \psi(U^+[w_0])_{T_v(\phi^+)}T_v(T_{L'})(U^-[x])_{T_v(\phi^-)} \end{aligned}$$

The Lusztig autmorphism T_w is in general not an coalgebra homomorphism, but in this special case T_v is an algebra homomorphism, sending a right coideal to a right coideal.

□

The image $T_v(C')$ in the previous Lemma is one of the Borel subalgebras we constructed in our Main Theorem 5.1; regarding the characters: The relation between ϕ^+ and ϕ^- has to be as asserted in order for C' and even C to be ede by Lemma 3.7. Now also C' is an ede right coideal subalgebra, because T_v is an algebra homomorphism, so in order for C to be maximal we have at least $C = C'$.

The remaining problem is: A triangular ede right coideal subalgebra of the form $C = \psi(U^+[w_1])_{\phi^+}T_{L'}U^-[w_2]_{\phi^-}$ which is via T_v isomorphic to the above-constructed triangular Borel subalgebra, a-priori itself does not have to be maximal, as C could lie in a bigger ede right coideal subalgebra $C'' \supset C' = C$, whose reflections to $T_v(C'')$ is not a right coideal subalgebra any more.

However in A_n we know from Lemma 2.11 and the maximality of L, that C cannot lie in a bigger triangular ede right coideal subalgebra.

This concludes the proof of the Main Classification Theorem 5.19

5.7. **Induction of one-dimensional modules.** In the Main Theorem 5.1 we have constructed a large family of Borel algebras

$$B = \psi(U^+[w_0])_{\phi^+} T_L U^-[x]_{\phi^-}$$

where $x = \prod_{\alpha\in c} s_\alpha$ such that $c = \Phi^+(x) \subset \Pi$ consists of pairwise orthogonal simple roots, and $L = \mathtt{supp}^\perp$, and the characters ϕ^+, ϕ^- on $\mathtt{supp} := \mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \Phi^+(x)$ are given as usual by suitable values for $\alpha \in \mathtt{supp}$.

$$\phi^+(\psi(E_\alpha)) = \lambda_\alpha \in k^*, \quad \phi^-(F_\alpha) = \lambda'_\alpha \in k^*, \qquad \lambda'_\alpha\lambda_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$$

Now let $\mathbb{C}_\chi$ be a one-dimensional representation of B. Since there is a quotient algebra where all $E_\beta \mapsto 0$ for all $\beta \notin \mathtt{supp}$, the one-dimensional representations are in bijection to one-dimensional representations of the $|\mathtt{supp}|$-fold quantum Weyl algebra generated by the character-shifted simple root vectors $\bar{E}_\alpha, \bar{F}_\alpha$ for all $\alpha \in \mathtt{supp}$. By Example 2.7 hence any one-dimensional representation $\mathbb{C}_\chi$ is given though scalars e_α, f_α for all $\alpha \in \mathtt{supp}$ with again $e_\alpha f_\alpha = \frac{q_\alpha^2}{(1-q_\alpha^{-2})(q_\alpha - q_\alpha^{-1})}$. It is then clear that:

Lemma 5.22. *Let B be a Borel algebra and V a one-dimensional representation as above. Then the induced representation*

$$V(B,\chi) := U_q(\mathfrak{g}) \otimes_B \mathbb{C}_\chi$$

is as a graded vectorspace isomorphic to

$$\left(\bigotimes_{\beta\in\Phi^+\setminus\mathtt{supp}} \mathbb{C}[E_\beta]\right) \otimes \left(\bigotimes_{\alpha\in\mathtt{supp}} \mathbb{C}[K_\alpha, K_\alpha^{-1}]\right)$$

The first factor is the space of coinvariants under projection to the Borel part $U_q(\mathfrak{g}_{\mathtt{supp}})^+$ where here $\mathfrak{g}_{\mathtt{supp}} \subset \mathfrak{g}$ is of type $A_1 \times A_1 \times \cdots$; in fact this is a Nichols algebra in the non-semisimple category of $U_q(\mathfrak{g}_{\mathtt{supp}})$-modules. The second factor makes the induced module for $\mathtt{supp} \neq 0$ non-diagonal in $U_q(\mathfrak{g}_{\mathtt{supp}})^0 \subset U_q(\mathfrak{g})^0$, which acts simply by left-multiplication.

Question 5.23. *Similar to the $\mathfrak{sl}_2$-case in* [HLV17] *we may ask for the decomposition behaviour of these modules. We expect that they are again largely irreducible up to discrete series' in some e_α, f_α. In particular if all e_α, f_α are of this form, we expect again all finite-dimensional irreducible modules of $U_q(\mathfrak{g})$ as unique quotients.*

6. Triangular Borel subalgebras with degenerate character shifts

Now we construct and study Borel subalgebras without the non-degeneracy property, so:

$$\Phi^+(w^+) \cap \Phi^+(w^-) \supsetneq \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$$

These are not isomorphic as algebra to those of the non degenerated types.

These Borel subalgebras can contain non trivial character-shifted root vectors $\bar{E}_\mu$, $\bar{F}_\mu$ even though $\mu \notin \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$, if μ is not simple and there is a smaller $\nu \prec \mu$ with

$\nu \in\in \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$. The properly character-shifted root vectors $\bar{E}_\nu, \bar{F}_\nu$ generate again a quantized Weyl algebra and the entire Borel subalgebra contains extensions of Weyl algebras.
In this section we restrict to the case A_n.

6.1. **Some classification results on Borel subalgebras of degenerate type.** We want to give some preliminary classification results for general Borel subalgebras $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w^-]_{\phi^-}$ of the quantum group of type A_n with $\Phi^+(w^+) \cap \Phi^+(w^-) \supsetneq \mathtt{supp}(\phi^+) \cap \mathtt{supp}(\phi^-)$. To prove the expected result we use two technical restrictions, of which we conjecture that they are true for general Borel subalgebras:

$$\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-)$$

$$\text{w.l.o.g.} \quad \Phi^+(w^-) \subset \Phi^+(w^+)$$

Under these assumptions, we can show that every triangular Borel subalgebra is of the following type.

Definition 6.1. *In A_n we consider the following types of Weyl group elements:*

- *For $1 \le i \le j \le n$ a* ladder *$w(i,j)$ is the following Weyl group element together with a reduced expression:*

$$w(i,j) := s_{\alpha_i} s_{\alpha_{i+1}} \dots s_{\alpha_j}$$

For the associated roots in $\Phi^+(w(i,j))$ we use the notation

$$\beta_k(i,j) := s_{\alpha_i} s_{\alpha_{i+1}} \dots s_{\alpha_{i+k-2}}(\alpha_{i+k-1}) \quad 0 < k \le j-i+1$$

- *For $1 \le i \le n$ and $0 \le l \le n-i$, $0 \le k \le i-1$ a $\mathtt{V}_i^{lk}$ is the following Weyl group element together with a reduced expression:*

$$\mathtt{V}_i^{lk} := s_{\alpha_i} s_{\alpha_{i+1}} s_{\alpha_{i+2}} \dots s_{\alpha_{i+l}} s_{\alpha_{i-1}} s_{\alpha_{i-2}} \dots s_{\alpha_{i-k}}$$

Of course $\Phi^+(\mathtt{V}_i^{lk}) := \{\sum_{j=0}^r \alpha_{i+j}, \sum_{j=-s}^0 \alpha_{i+j} \mid 0 \le r \le l, 0 \le s \le k\}$.

- *For $1 \le i \le n$ and $j \le \min\{i-1, n-i\}$ a* diamond *$\diamond_{i,j}$ is the following Weyl group element together with a reduced expression:*

$$\diamond_{i,j} := \mathtt{V}_i^{jj} \mathtt{V}_i^{j-1j-1} \dots \mathtt{V}_i^{00}$$

- *For $1 \le i \le n$ a* palm *$\mathbin{⩔}_i$ is the following Weyl group element together with a reduced expression:*

$$\mathbin{⩔}_i := \mathtt{V}_i^{l_1 k_1} \mathtt{V}_i^{l_2 k_2} \dots$$

With $0 \le l_j \le n-i$, $0 \le k_j \le i-1$ and the property $l_j > l_{j+1}$ and $k_j > k_{j+1}$.

Theorem 6.2. *For $C = \psi(U^+[w^+])_{\phi^+} T_L U^-[w]_{\phi^-}$ ede with $\Phi^+(w) \subset \Phi^+(w^+)$ and if $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) =: \mathtt{supp}(\phi)$ only the following choices of w are possible:*

(1) In the case $\mathtt{supp}(\phi) = \{\alpha_i\} \subset \Pi$:

$$w = \mathtt{V}_i^{lk} := s_{\alpha_i} s_{\alpha_{i+1}} s_{\alpha_{i+2}} \dots s_{\alpha_{i+l}} s_{\alpha_{i-1}} s_{\alpha_{i-2}} \dots s_{\alpha_{i-k}}$$

for some $0 \le i, l, k \le n$, as in Definition 6.1.

(2) In the more general case $\mathtt{supp}(\phi) \cap \Pi = \{\alpha_i\}$*:*

$$w = \mathbb{V}_i := \mathtt{V}_i^{l_1 k_1} \mathtt{V}_i^{l_2 k_2} \dots$$

for $1 \le i \le n$ *with the property* $l_j > l_{j+1}$ *and* $k_j > k_{j+1}$*.*

This w *is the inverse of a special element in the following sense*

Lemma 6.3. *[*[HLV17]*] For a Weyl group element* $w \in W$ *and a root* $\alpha_i \in \Pi$ *the following statements are equivalent:*

- *the set* $\Phi^+(w^{-1})$ *contains exactly one simple root, which is* α_i
- *for all* $j \ne i$ *holds* $\ell(s_{\alpha_j} w^{-1}) = \ell(w) + 1$*, but* $\ell(s_{\alpha_i} w^{-1}) = \ell(w) - 1$
- *In each reduced expression of* w *the last factor is* s_{α_i}

Let W_i *be the (so called parabolic) subgroup of* W*, which is generated by the* s_{α_j} *with* j *different to* i*. Then in the case above* w *is the unique representative of the left coset* wW_i *with minimal length.*

If $\diamond_{i,m}$ *is the maximal diamond with* $\Phi^+(\diamond_{i,j}) \subset \Phi^+(\mathbb{V}_i)$*, then:*

$$\mathtt{supp}(\phi) = \left\{ \sum_{k=i-l}^{i+l} \alpha_k \mid 0 \le l \le m \right\}$$

(3) In the general case $\mathtt{supp}(\phi) \cap \Pi = \{\alpha_{i_1}, \alpha_{i_2} \dots\} =: J$ *(pairwise orthogonal) holds:* w *is the special element with* $\Phi^+(w) \cap \Pi = J$*. That is* w *is the generalisation of the inverse of a special element in the sense of Lemma 6.3.*

If C is even a Borel subalgebra, then we conjecture due to the maximality:

$$w = \prod_{i \in J} \mathbb{V}_i$$

For special elements $\mathbb{V}_i$, which commute. The support of the character $\mathtt{supp}(\phi)$ is the union of the support of the respective palms.

Proof. As the elements in $\mathtt{supp}(\phi)$ are pairwise orthogonal the claims follows directly from 3.3. The form of the support $\mathtt{supp}(\phi)$ in 2. follows directly from Corollary 3.10.

If C is a Borel subalgebra we get from Lemma 3.11, that only those combinations of $\mathbb{V}_i$ are possible which are disjoint, i.e. which commute. □

6.2. **Construction of degenerate Borel subalgebras of height** 1**.** In the following we want to construct non-degenerate ede right coideal subalgebras which correspond to palms of height 1 in the previous subsection,

$$\mathtt{supp}(\phi) = \{\alpha_i\} \in \Pi \qquad w = \mathbb{V}_i = \mathtt{V}_i^{lk}$$

for some $l, k \in \mathbb{N}_0$. Due to Theorem 3.3 the ede property of C implies $\lambda_{\alpha_i}\lambda_{\alpha_i'} = \frac{q^2}{(q-q^{-1})(1-q^2)}$ for $\phi^+(\psi(E_{\alpha_i})) = \lambda_{\alpha_i}$ and 0 otherwise and $\phi^-(F_{\alpha_i}) = \lambda'_{\alpha_i}$ and 0 otherwise.

Lemma 6.4. *For type A_n the following is an ede right coideal subalgebra: Let $1 \leq i \leq n$ and $0 \leq l \leq n-i$, $0 \leq k \leq i$ be arbitrarily chosen, then we consider an arbitrary palm of height 1 $w^+, w^- := \mathtt{V}_i^{lk}$ which means:*

$$\Phi^+(w^+) = \Phi^+(w^-) := \{\sum_{j=0}^{r} \alpha_{i+j}, \sum_{j=-s}^{0} \alpha_{i+j} \mid 0 \leq r \leq l, 0 \leq s \leq k\}$$

as above

$$C := \psi(U^+[\mathtt{V}_i^{lk}])_{\phi^+} T_L U^-[\mathtt{V}_i^{lk}]_{\phi^-}$$

for $L = \{\mu \mid \mu \perp \alpha_i\}$ and characters ϕ^+ and ϕ^- with $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_i\}$.

More precisely C has the following relations between two character-shifted root vectors for roots $\mu, \nu, \mu' \in \Phi^+(\mathtt{V}_i^{lk})$:

$$[\bar{E}_{\alpha_i}, \bar{F}_{\alpha_i}] = \frac{q^2}{q - q^{-1}}$$

$$[\bar{E}_\mu, \bar{E}_\nu] = 0 \quad [\bar{F}_\mu, \bar{F}_\nu] = 0$$

$$[\bar{E}_\mu, \bar{F}_\nu] = 0 \quad \text{if } \mu \neq \nu$$

$$[\bar{E}_\mu, \bar{F}_\mu] = q^2[\bar{E}_{\mu'}, \bar{F}_{\mu'}]_1 \quad \text{if } \mu \neq \alpha_i$$

$$\text{for } \mu' = \begin{cases} \mu - \alpha_{i+r} & \text{if } \mu = \sum_{j=0}^{r} \alpha_{i+j} \\ \mu - \alpha_{i-s} & \text{if } \mu = \sum_{j=-s}^{0} \alpha_{i+j} \end{cases}$$

Proof. The commutator relations follow from the explicit calculations in [Vocke16] chapter 3.

Now we prove the ede property: Let V be an arbitrary finite dimensional representation of C. We consider the restriction to the right coideal subalgebra $\langle \bar{E}_{\alpha_i}, \bar{F}_{\alpha_i} \rangle$, which is a quantized Weyl algebra as usual by the choice of characters. We know from Example 2.7, that on any finite dimensional representation of the Weyl algebra the commutator $[\bar{E}_{\alpha_i}, \bar{F}_{\alpha_i}]_1$ vanishes. In particular, each finite-dimensional representation of the Weyl algebra factorizes to a representation of the commutative algebra $\mathbb{C}[e,f]/(ef - \frac{q^2}{(q-q^{-1})(1-q^2)})$.

We now consider the next-largest subalgebra, which is generated by the Weyl algebra and all $\bar{E}_\mu, \bar{F}_\mu$ with $\mu' = \alpha_i$. Due to the commutator relation $[\bar{E}_\mu, \bar{F}_\mu] = [\bar{E}_{\mu'}, \bar{F}_{\mu'}]_1$ we get that $[\bar{E}_\mu, \bar{F}_\mu]_{q^{(\mu,\mu)}}$ acts trivially on V. Due to Theorem 3.12 (and as $q^{(\mu,\mu)} \neq 1$) this implies, that the elements $\bar{E}_\mu, \bar{F}_\mu$, which q-commute with all elements on V, act trivial.

Consider now inductively the next larger subalgebra with $\bar{E}_\mu, \bar{F}_\mu$ for $(\mu')' = \alpha_i$. From the relation $[\bar{E}_\mu, \bar{F}_\mu] = [\bar{E}_{\mu'}, \bar{F}_{\mu'}]_1$ and the just proven trivial action of $\bar{E}_{\mu'}, \bar{F}_{\mu'}$ it follows that the q-commutator acts trivial on V. Inductively we know that all $\bar{E}_\mu, \bar{F}_\mu$ with $\mu \neq \alpha_i$ act trivial on V.

Thus, the category of finite-dimensional representations of C is equivalent to the category of finite-dimensional representation of the commutative algebra $\mathbb{C}[e,f]/(ef - \frac{q^2}{(q-q^{-1})(1-q^2)})$ and, in particular, all irreducible finite-dimensional representations of C are one-dimensional.

□

7. Examples

We now present all Borel subalgebras of the examples $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$ and all triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$, as discussed in the second authors work [Vocke16] chapter 9-11 or [Vocke17]. We do this for one to illustrate the results of our present article. On the other hand it is interesting to know all (potentially non-triangular) examples with a different method, this is how we arrive at our conjecture that all Borel subalgebras are triangular.

The strategy to find all maximal ede RCS requires first to get a hold on all RCS: This is the theorem providing a generating system for an arbitrary RCS in [Vocke16] chapter 4. Then we disprove the ede property using restrictions of suitable minuscule Vermamodules as in section 3.

7.1. **Borel subalgebras of $U_q(\mathfrak{sl}_2)$.** The standard Borel subalgebras $U^{\geq 0}$ and $U^{\leq 0}$ and U^0 are the only homogeneous right coideal subalgebras of $U_q(\mathfrak{sl}_2)$. Furthermore there are the right coideal subalgebras $\langle EK^{-1}\rangle$ in $U^{\geq 0}$ and $\langle F\rangle$ in $U^{\leq 0}$, and families of character-shifted right coideal subalgebras $\langle EK^{-1}\rangle_{\phi^+}$ resp. $\langle F\rangle_{\phi^-}$ for characters on $\langle EK^{-1}\rangle$ resp. $\langle F\rangle$ given by $\phi^+(EK^{-1}) = \lambda$ and $\phi^-(F) = \lambda'$. They have the form $\langle EK^{-1}\rangle_{\phi^+} = \langle EK^{-1} + \lambda K^{-1}\rangle$ and $\langle F\rangle_{\phi^-} = \langle F + \lambda' K^{-1}\rangle$.

As in example 2.7 the right coideal subalgebra

$$\langle EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1}\rangle$$

is ede and hence a Borel subalgebra, iff

$$\lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$$

Note, that different Borel subalgebras B for different choices of λ, λ' are mapped onto each other via the Hopf-automorphism $E \mapsto tE,\ F \mapsto t^{-1}F$.

Together with the standard Borel subalgebras these are all Borel subalgebras of $U_q(\mathfrak{sl}_2)$.

7.2. **Borel subalgebras of $U_q(\mathfrak{sl}_3)$.** For $U_q(\mathfrak{sl}_3)$ we found that all Borel subalgebras are triangular. Besides the standard Borel subalgebra there are two isomorphism classes of non-degenerate Borel subalgebras as discussed in section 5 and one degenerate of height 1 as discussed in section 6.

Standard Borel subalgebras. Due to Theorem 2.6 $U^{\geq 0}$ and $U^{\leq 0}$ are Borel subalgebras.

Non-degenerate Borel subalgebras. The Borel subalgebras $\psi(U^+[w^+])_{\phi^+}T_LU^-[w^-]_{\phi^-}$ with $\Phi^+(w^+)\cap\Phi^+(w^-)=\mathtt{supp}(\phi^+)\cap\mathtt{supp}(\phi^-)$ are due to Theorem 5.19 up to symmetry isomorphic as algebra to $\psi(U^+[w_0])_{\phi^+}\langle K_{2\beta+\alpha},K_{2\beta+\alpha}^{-1}\rangle U^-[s_\alpha]_{\phi^-}$ with $\phi^+(E_\alpha K_\alpha^{-1})=\lambda_\alpha^+$ and 0 otherwise, and $\phi^-(F_\alpha)=\lambda_\alpha^-$, such that $\lambda_\alpha^+\lambda_\alpha^-=\frac{q^2}{(1-q^2)(q-q^{-1})}$. More precisely there are up to symmetry exactly two such Borel subalgebras. These are

$$\psi(U^+[w_0])_{\phi^+}\langle K_{2\beta+\alpha},K_{2\beta+\alpha}^{-1}\rangle U^-[s_\alpha]_{\phi^-}$$

with characters as above and

$$\psi(U^+[s_\alpha s_\beta])_{\phi^+}\langle K_{\alpha-\beta},K_{\alpha-\beta}^{-1}\rangle U^-[s_\beta s_\alpha]_{\phi^-}$$

with $\phi^+(E_{\alpha\beta}K_{\alpha+\beta}^{-1})=\lambda_{\alpha\beta}^+$ and $\phi^-(F_{\alpha\beta})=\lambda_{\alpha\beta}^-$, such that $\lambda_{\alpha\beta}^+\lambda_{\alpha\beta}^-=\frac{q^2}{(1-q^2)(q-q^{-1})}$ holds.

A degenerate Borel subalgebra $\psi(U^+[s_\alpha s_\beta])_{\phi^+}\langle K_{2\beta+\alpha},K_{2\beta+\alpha}^{-1}\rangle U^-[s_\alpha s_\beta]_{\phi^-}$. The third type of triangular Borel subalgebras in $U_q(\mathfrak{sl}_3)$ is of the form

$$\psi(U^+[s_\alpha s_\beta])_{\phi^+}\langle K_{2\beta+\alpha},K_{2\beta+\alpha}^{-1}\rangle U^-[s_\alpha s_\beta]_{\phi^-}$$

with $\phi^+(E_\alpha K_\alpha^{-1})=\lambda_\alpha^+$ and 0 otherwise, and $\phi^-(F_\alpha)=\lambda_\alpha^-$ and 0 otherwise, such that $\lambda_\alpha^+\lambda_\alpha^-=\frac{q^2}{(1-q^2)(q-q^{-1})}$ as above. The generating elements of this right coideal subalgebra are the following:

$$\begin{aligned}\bar{E}_\alpha &:= E_\alpha K_\alpha^{-1}+\lambda_\alpha^+K_\alpha^{-1}\\ \bar{F}_\alpha &:= F_\alpha+\lambda_\alpha^-K_\alpha^{-1}\\ K:=K_{2\beta+\alpha} &\text{ und } K^{-1}:=K_{2\beta+\alpha}^{-1}\\ \bar{E}_{\alpha\beta} &:= E_{\alpha\beta}K_{\alpha+\beta}^{-1}+(1-q^{-2})\lambda_\alpha^+E_\beta K_{\alpha+\beta}^{-1}\\ \bar{F}_{\beta\alpha} &:= F_{\beta\alpha}+(q^{-1}-q)\lambda_\alpha^-F_\beta K_\alpha^{-1}\end{aligned}$$

The product of the constants $c_1:=(1-q^{-2})\lambda_\alpha^+$ resp. $c_2:=(q^{-1}-q)\lambda_\alpha^-$ in the terms $\bar{E}_{\alpha\beta}$ resp. $\bar{F}_{\beta\alpha}$ is:

$$c_1c_2=(1-q^{-2})\lambda_\alpha^+(q^{-1}-q)\lambda_\alpha^-=(1-q^{-2})(q^{-1}-q)\frac{q^2}{(1-q^2)(q-q^{-1})}=1$$

For these c_1 and c_2 consider the commutator of the both elements:

$$\begin{aligned}&[E_{\alpha\beta}K_{\alpha+\beta}^{-1}+(1-q^{-2})\lambda_\alpha^+E_\beta K_{\alpha+\beta}^{-1},F_{\beta\alpha}+(q^{-1}-q)\lambda_\alpha^-F_\beta K_\alpha^{-1}]_{q^2}\\ &=(F_\alpha+\lambda_\alpha^-K_\alpha^{-1})(E_\alpha K_\alpha^{-1}+\lambda_\alpha^+K_\alpha^{-1})(q^4-q^2)+\frac{q^4}{q-q^{-1}}1\end{aligned}$$

Thus the commutators of the generating elements in B are given by:

$$\begin{aligned}[K,\bar{E}_\alpha]_1=[K,\bar{F}_\alpha]_1=[K,\bar{E}_{\alpha\beta}]_1=[K,\bar{F}_{\beta\alpha}]_1&=0\\ [\bar{E}_\alpha,\bar{F}_\alpha]_{q^2}=\frac{q^2}{q-q^{-1}}1\quad[\bar{E}_\alpha,\bar{E}_{\alpha\beta}]_q=[\bar{E}_\alpha,\bar{F}_{\beta\alpha}]_q&=0\\ [\bar{F}_\alpha,\bar{E}_{\alpha\beta}]_{q^{-1}}=[\bar{F}_\alpha,\bar{F}_{\beta\alpha}]_{q^{-1}}&=0\\ [\bar{E}_{\alpha\beta},\bar{F}_{\beta\alpha}]_{q^2}=\bar{F}_\alpha\bar{E}_\alpha(q^4-q^2)+\frac{q^4}{q-q^{-1}}&1\end{aligned}$$

Consider now a representation V of B. We already know, that $\langle \bar{E}_\alpha, \bar{F}_\alpha\rangle$ is ede. As in the general case of Lemma 6.4 $[\bar{E}_\alpha, \bar{F}_\alpha]_1$ acts trivial on every finite dimensional V. So the term $\bar{F}_\alpha\bar{E}_\alpha$ acts as $\frac{q^2}{(q-q^{-1})(1-q^2)}1$ on V. Inserting this in the q-commutator of $\bar{E}_{\alpha\beta}$ and $\bar{F}_{\beta\alpha}$ we see that $[\bar{E}_{\alpha\beta}, \bar{F}_{\beta\alpha}]_{q^2}$ acts as 0 on all of V.

On the other hand $\bar{E}_\alpha$ has an eigenvector v to the eigenvalue $\neq 0$ and due to the coproduct the elements $\bar{E}^n_{\alpha\beta}.v$ are eigenvectors of $\bar{E}_\alpha$ to either eigenvalue 0 or pairwise distinct eigenvalues. As V is finite, we can find a vector w with $\bar{E}_{\alpha\beta}.w = 0$. As on V all elements q-commute with $\bar{E}_{\alpha\beta}$ we can apply Theorem 3.12 to show, that $\bar{E}_{\alpha\beta}$ acts as 0 on any irreducible representation. The same we can show for $\bar{F}_{\beta\alpha}$ with the same argument. Then $B|_V \cong \langle K, K^{-1}\rangle \otimes \langle \bar{E}_\alpha, \bar{F}_\alpha\rangle$ and B is ede.

The maximality of this Borel subalgebra is proven in [Vocke17] by considering any extension of this RCS. Thus we have found all Borel subalgebras of $U_q(\mathfrak{sl}_3)$.

7.3. **Triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$.** We now give all possible triangular Borel subalgebras of $U_q(\mathfrak{sl}_4)$. We do not consider non-triangular RCS so we can also not prove the maximality of the degenerate examples we give below. However we do prove that these are ede RCS, which are maximal among all triangular RCS, and there exist no other triangular ede Borel subalgebras. We conjecture that these are in fact all Borel subalgebras of $U_q(\mathfrak{sl}_4)$.

Standard Borel subalgebras. Due to Theorem 2.6 all reflections of the standard Borel $U^{\geq 0}$ and $U^{\leq 0}$ are Borel subalgebras.

Non-degenerate Borelsubalgebras. The triangular Borel subalgebras $\psi(U^+[w^+])_{\phi^+}T_LU^-[w^-]_{\phi^-}$ with $\Phi^+(w^+)\cap\Phi^+(w^-) = \mathtt{supp}(\phi^+)\cup\mathtt{supp}(\phi^-)$ are due to Theorem 5.19 all isomorphic to a Borel subalgebra of the following form:

- $\psi(U^+[w_0])_{\phi^+}\langle K_{2\alpha_2+\alpha_1}, K^{-1}_{2\alpha_2+\alpha_1}\rangle U^-[s_{\alpha_i}]_{\phi^-}$ for $i\in\{1,2,3\}$ with $\phi^+(E_{\alpha_i}K^{-1}_{\alpha_i}) = \lambda^+_{\alpha_i}$ and 0 otherwise and $\phi^-(F_{\alpha_i}) = \lambda^-_{\alpha_i}$, such that $\lambda^+_{\alpha_i}\lambda^-_{\alpha_i} = \frac{q^2}{(1-q^2)(q-q^{-1})}$.
- $\psi(U^+[w_0])_{\phi^+}\langle K_{2\alpha_2+\alpha_1}, K^{-1}_{2\alpha_2+\alpha_1}\rangle U^-[s_{\alpha_1}s_{\alpha_3}]_{\phi^-}$ with $\phi^+(E_{\alpha_1}K^{-1}_{\alpha_1}) = \lambda^+_{\alpha_1}$, $\phi^+(E_{\alpha_3}K^{-1}_{\alpha_3}) = \lambda^+_{\alpha_3}$ and 0 otherwise, and $\phi^-(F_{\alpha_1}) = \lambda^-_{\alpha_1}$, such that $\lambda^+_{\alpha_1}\lambda^-_{\alpha_1} = \frac{q^2}{(1-q^2)(q-q^{-1})}$, and $\phi^-(F_{\alpha_3}) = \lambda^-_{\alpha_3}$, such that $\lambda^+_{\alpha_3}\lambda^-_{\alpha_3} = \frac{q^2}{(1-q^2)(q-q^{-1})}$.

Degenerate Borelsubalgebras. We list all possibilities for $w^+, w^-, \mathtt{supp}(\phi^+), \mathtt{supp}(\phi^-)$ from Theorem 6.2. They all contain the degenerate Borel subalgebra from $U_q(\mathfrak{sl}_3)$ in the previous subsection. Due to Lemma 2.11 we know,that for any triangular ede right coideal subalgebra

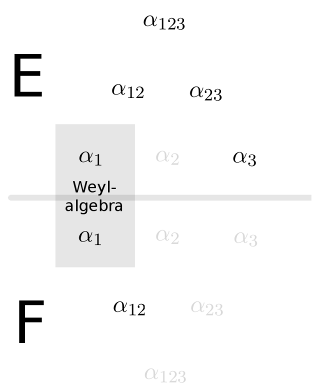
$$\ell((w^-)'^{-1}w^+) = \ell(w^-) + \ell(w^+) \quad \text{or} \quad \ell((w^+)'^{-1}w^-) = \ell(w^+) + \ell(w^-)$$

where w' is calculated via w and $\mathtt{supp}(\phi^+)$.

We want to prove now that the possible non-degenerate triangular right coideal subalgebras in $\mathfrak{sl}_4$ are in fact all ede, i.e. $C = \psi(U^+[w^+])_{\phi^+}T_LU^-[w^-]_{\phi^-}$ with $\Phi^+(w^+)\cap\Phi^+(w^-)\not\subset\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-)$ and with $w^+w'^{-1} = w_0$ and $L = \mathtt{supp}(\phi^+)^\perp$. There are up to isomorphism and symmetry three such right coideal subalgebras with $|\mathtt{supp}(\phi^+)| = 1$ and two with $|\mathtt{supp}(\phi^+)| = 2$.

In each of these cases, we manually compute the q-commutator relations of the generators of B and show, as the case may be, that an arbitrary finite-dimensional irreducible representation V of the algebra B is one-dimensional. The strategy for this is as in Lemma 6.4 (General palm of height 1): We find a contained quantized Weyl algebra $\langle X, Y\rangle$, of which we already know that it acts on finite-dimensional representations commutative, i.e. the commutator $[X, Y]_1$ acts trivial. Thus, by means of the respective explicit list of commutators, we find an element $Z \in C$ which q-commutes (on V) with all generators of C. By Theorem 3.12 then Z must act on every irreducible representation V trivial, which inductively induces further commutators to vanish until all elements except the Weyl algebras act trivial. The details of the calculation differ from case to case, but the argument is in any case the same.

1. $|\mathtt{supp}(\phi^+)| = 1$*:*
1.1. $|\Phi^+(w^+) \cap \Phi^+(w^-)| = |\mathtt{supp}(\phi^+)| + 1$. Here, apart from isomorphism and symmetry, there is exactly one type of maximal triangular and right-coideal subalgebra given by $B = \psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}, K_{\alpha_2}^2K_{\alpha_1}, K_{\alpha_3}^{-1}, K_{\alpha_2}^{-2}K_{\alpha_1}^{-1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}]_{\phi^-}$ with $\phi^+(E_{\alpha_1}K_{\alpha_1}^{-1}) = \lambda$ otherwise 0 and $\phi^-(F_{\alpha_1}) = \lambda'$, otherwise 0, such that $\lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$.



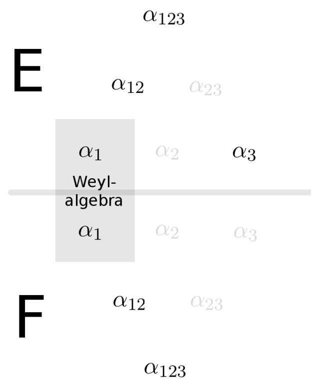
Consider the q-commutator relations and denote the character-shifted root vectors by $\bar{E}_{\alpha_1}$, $\bar{E}_{\alpha_1\alpha_2}$ etc., then:

	$\bar{F}_{\alpha_1}$	$F_{\alpha_1\alpha_2}$
$\bar{E}_{\alpha_1}$	$\frac{q^2}{q-q^{-1}}$	0
$E_{\alpha_1\alpha_2}$	0	$q^2[E_{\alpha_1}, \bar{F}_{\alpha_1}]_1$
$\bar{E}_{\alpha_1\alpha_2\alpha_3}$	0	0
$\bar{E}_{\alpha_3}$	0	0
$\bar{E}_{\alpha_3\alpha_2}$	0	0

The quantized Weyl algebra is here $\langle \bar{E}_{\alpha_1}, \bar{F}_{\alpha_1}\rangle$, and the commutator vanishes on each finite dimensional representation. The elements Z which thus q-commute on V with all generators of C are $\bar{F}_{\alpha_1\alpha_2}$ resp. $\bar{E}_{\alpha_1\alpha_2}$, resp. $\bar{E}_{\alpha_3\alpha_2}$, whereas the other generators $\bar{E}_{\alpha_3}$ and $\bar{E}_{\alpha_1\alpha_2\alpha_3}$ q-commute anyhow with all the other generators.

1.2. $|\Phi^+(w^+) \cap \Phi^+(w^-)| = |\mathtt{supp}(\phi^+)| + 2$.

1.2.1. $B = \psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}, K_{\alpha_2}^2 K_{\alpha_1}, K_{\alpha_3}^{-1}, K_{\alpha_2}^{-2}K_{\alpha_1}^{-1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}]_{\phi^-}$ with ϕ^+ and ϕ^- as above.

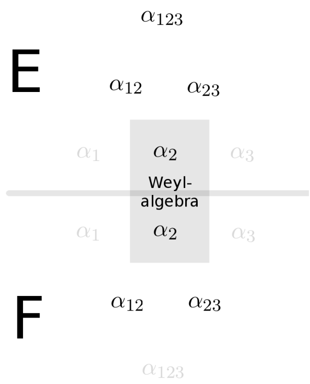


Moreover the following q-commutator relations hold:

	$\bar{F}_{\alpha_1}$	$\bar{F}_{\alpha_1\alpha_2}$	$\bar{F}_{\alpha_1\alpha_2\alpha_3}$
$\bar{E}_{\alpha_1}$	$\frac{q^2}{q-q^{-1}}$	0	0
$\bar{E}_{\alpha_1\alpha_2}$	0	$q^2[\bar{E}_{\alpha_1}, \bar{F}_{\alpha_1}]_1$	0
$\bar{E}_{\alpha_1\alpha_2\alpha_3}$	0	0	$q^2[\bar{E}_{\alpha_1\alpha_2}, \bar{F}_{\alpha_1\alpha_2}]_1$
$\bar{E}_{\alpha_3}$	0	0	$q^2\bar{F}_{\alpha_1\alpha_2}$

The quantized Weyl algebra is here $\langle \bar{E}_{\alpha_1}, \bar{F}_{\alpha_1}\rangle$, and the commutator vanishes on each finite dimensional representation. The elements Z which thus q-commute on V with all generators of C are $\bar{F}_{\alpha_1\alpha_2}$ resp. $\bar{E}_{\alpha_1\alpha_2}$. If these elements vanish, the remaining elements q-commutate in the next step of the induction with all other generators.

1.2.2. $B = \psi(U^+[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}])_{\phi^+}\langle K_{\alpha_1}^{-1}K_{\alpha_3}, K_{\alpha_1+\alpha_2+\alpha_3}, K_{\alpha_1}K_{\alpha_3}^{-1}, K_{\alpha_1+\alpha_2+\alpha_3}^{-1}\rangle U^-[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}]_{\phi^-}$ with the characters $\phi^+(E_{\alpha_2}K_{\alpha_2}^{-1}) = \lambda$, otherwise 0 and $\phi^-(F_{\alpha_2}) = \lambda'$, otherwise 0, such that $\lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$.



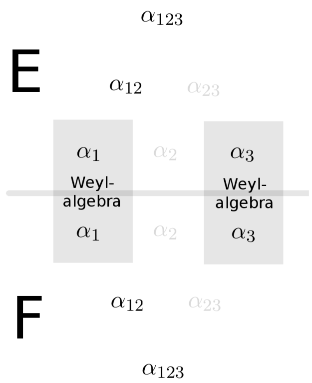
Here the following q-commutator relations hold:

	$\bar{F}_{\alpha_2}$	$\bar{F}_{\alpha_2\alpha_1}$	$\bar{F}_{\alpha_2\alpha_3}$
$\bar{E}_{\alpha_2}$	$\frac{q^2}{q-q^{-1}}$	0	0
$\bar{E}_{\alpha_2\alpha_1}$	0	$q^2[E_{\alpha_2}, F_{\alpha_2}]_1$	0
$\bar{E}_{\alpha_2\alpha_3}$	0	0	$q^2[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1$
$\bar{E}_{\alpha_2\alpha_1\alpha_3\alpha_2}$	0	$-q^{-1}[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{E}_{\alpha_3}]_1$	$-q^{-1}[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{E}_{\alpha_1}]_1$

The quantized Weyl algebra is here $\langle \bar{E}_{\alpha_2}, \bar{F}_{\alpha_2} \rangle$. All other elements vanish again by induction on all irreducible finite dimensional representations V.

2. $|\mathtt{supp}(\phi^+)| = 2$*:* Consider now the case that the support of the character contains two elements. In type A_3 there are two possibilities for this case and we get either $|\Phi^+(w^+) \cap \Phi^+(w^-)| = |\mathtt{supp}(\phi^+)|$, so B is triangular and non-degenerate or $|\Phi^+(w^+) \cap \Phi^+(w^-)| = |\mathtt{supp}(\phi^+)| + 2$

2.1. $B = \psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}K_{\alpha_2}^2K_{\alpha_1}, K_{\alpha_3}^{-1}K_{\alpha_2}^{-2}K_{\alpha_1}^{-1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}]_{\phi^-}$. with characters $\phi^+(E_{\alpha_1}K_{\alpha_1}^{-1}) = \lambda_{\alpha_1}$ and $\phi^-(F_{\alpha_1}) = \lambda'_{\alpha_1}$ such that $\lambda_{\alpha_1}\lambda'_{\alpha_1} = \frac{q^2}{(1-q^2)(q-q^{-1})}$, and $\phi^+(E_{\alpha_3}K_{\alpha_3}^{-1}) = \lambda_{\alpha_3}$ and $\phi^-(F_{\alpha_3}) = \lambda'_{\alpha_3}$ such that $\lambda_{\alpha_3}\lambda'_{\alpha_3} = \frac{q^2}{(1-q^2)(q-q^{-1})}$ otherwise 0.

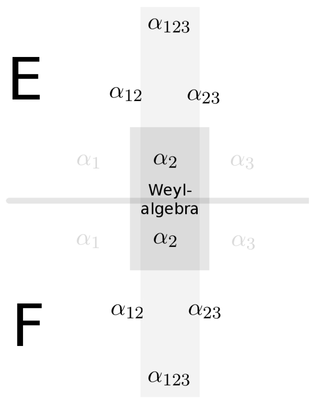


Here the following commutator relations hold:

	$\bar{F}_{\alpha_1}$	$\bar{F}_{\alpha_1\alpha_2}$	$\bar{F}_{\alpha_1\alpha_2\alpha_3}$	$\bar{F}_{\alpha_3}$
$\bar{E}_{\alpha_1}$	$\frac{q^2}{q-q^{-1}}$	0	0	0
$\bar{E}_{\alpha_1\alpha_2}$	0	$q^2[\bar{E}_{\alpha_1}, \bar{F}_{\alpha_1}]_{\alpha_1}$	0	0
$\bar{E}_{\alpha_1\alpha_2\alpha_3}$	0	0	$q^2[\bar{E}_{\alpha_1\alpha_2}, \bar{F}_{\alpha_1\alpha_2}]_1$	$\bar{E}_{\alpha_1\alpha_2}$
$\bar{E}_{\alpha_3}$	0	0	$q^2\bar{F}_{\alpha_1\alpha_2}$	$\frac{q^2}{q-q^{-1}}$

Here there are two Weyl algebras $\langle \bar{E}_{\alpha_1}, \bar{F}_{\alpha_1}\rangle$ and $\langle \bar{E}_{\alpha_3}, \bar{F}_{\alpha_3}\rangle$, commuting with each other. The remaining elements vanish again by induction on any irreducible finite dimensional representation.

2.2. $B = \psi(U^+[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}])_{\phi^+}\langle K_{\alpha_1}^{-1}K_{\alpha_3}, K_{\alpha_1}K_{\alpha_3}^{-1}\rangle U^-[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}]_{\phi^-}$. with characters $\phi^+(E_{\alpha_2}K_{\alpha_2}^{-1}) = \lambda_{\alpha_2}$ and $\phi^-(F_{\alpha_2}) = \lambda'_{\alpha_2}$ such that $\lambda_{\alpha_2}\lambda'_{\alpha_2} = \frac{q^2}{(1-q^2)(q-q^{-1})}$, and $\phi^+(E_{\alpha_2\alpha_1\alpha_3\alpha_2}K_{\alpha_2\alpha_1\alpha_3\alpha_2}^{-1}) = \lambda_{\alpha_2\alpha_1\alpha_3\alpha_2}$ and $\phi^-(F_{\alpha_2\alpha_1\alpha_3\alpha_2}) = \lambda'_{\alpha_2\alpha_1\alpha_3\alpha_2}$ such that $\lambda_{\alpha_2\alpha_1\alpha_3\alpha_2}\lambda'_{\alpha_2\alpha_1\alpha_3\alpha_2} = \frac{q^2}{(1-q^2)(q-q^{-1})}$ otherwise 0.



Here the following commutating relations hold:

	$\bar{F}_{\alpha_2}$	$\bar{F}_{\alpha_2\alpha_1}$	$\bar{F}_{\alpha_2\alpha_3}$	$\bar{F}_{\alpha_2\alpha_1\alpha_3\alpha_2}$
$\bar{E}_{\alpha_2}$	$\frac{q^2}{q-q^{-1}}$	0	0	0
$\bar{E}_{\alpha_2\alpha_1}$	0	$q^2[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1$	0	$[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{F}_{\alpha_3}]_1$
$\bar{E}_{\alpha_2\alpha_3}$	0	0	$q^2[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1$	$[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{F}_{\alpha_1}]_1$
$\bar{E}_{\alpha_2\alpha_1\alpha_3\alpha_2}$	0	$-q^{-1}[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{E}_{\alpha_3}]_1$	$-q^{-1}[[\bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}]_1, \bar{E}_{\alpha_1}]_1$	$-[\bar{E}_{\alpha_2\alpha_1}, \bar{F}_{\alpha_2\alpha_1}]_1$ $-[\bar{E}_{\alpha_2\alpha_3}, \bar{F}_{\alpha_2\alpha_3}]_1$ $+c(1-K^{-2}_{\alpha_1+\alpha_2+\alpha_3})$

The first quantized Weyl algebra is here $\langle \bar{E}_{\alpha_2}, \bar{F}_{\alpha_2}\rangle$, whose commutator again disappears on each finite dimensional representation. Then the four elements of height 2 q-commute with all generators and thus act trivially. Modulo these relations $\langle \bar{E}_{\alpha_2\alpha_1\alpha_3\alpha_2}, \bar{F}_{\alpha_2\alpha_1\alpha_3\alpha_2}\rangle$ is another quantized Weyl algebra commuting with the first quantized Weyl algebra.

Theorem 7.1. *Each triangular Borel subalgebra of $U_q(\mathfrak{sl}_4)$ is up to reflections and algebra isomorphism of the following form:*

- *The standard Borel subalgebra $U^{\geq 0}$*
- *A Borel subalgebra of non-degenerate type:*
  - $\psi(U^+[w_0])_{\phi^+}\langle K_{2\alpha_2+\alpha_1}, K^{-1}_{2\alpha_2+\alpha_1}, K_{\alpha_3}, K^{-1}_{\alpha_3}\rangle U^-[s_{\alpha_1}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_1\}$
  - $\psi(U^+[w_0])_{\phi^+}\langle K_{2\alpha_2+\alpha_1}, K^{-1}_{2\alpha_2+\alpha_1}\rangle U^-[s_{\alpha_1}s_{\alpha_3}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_1, \alpha_3\}$
- *A Borel subalgebra of degenerate type, i.e.:*
  - $\psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}, K^2_{\alpha_2}K_{\alpha_1}, K^{-1}_{\alpha_3}, K^{-2}_{\alpha_2}K^{-1}_{\alpha_1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_1\}$
  - $\psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}, K^2_{\alpha_2}K_{\alpha_1}, K^{-1}_{\alpha_3}, K^{-2}_{\alpha_2}K^{-1}_{\alpha_1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_1\}$
  - $\psi(U^+[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}])_{\phi^+}\langle K_{\alpha_3}K^2_{\alpha_2}K_{\alpha_1}, K^{-1}_{\alpha_3}K^{-2}_{\alpha_2}K^{-1}_{\alpha_1}\rangle U^-[s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_1, \alpha_3\}$
  - $\psi(U^+[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}])_{\phi^+}\langle K^{-1}_{\alpha_1}K_{\alpha_3}, K_{\alpha_1+\alpha_2+\alpha_3}, K_{\alpha_1}K^{-1}_{\alpha_3}, K^{-1}_{\alpha_1+\alpha_2+\alpha_3}\rangle U^-[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_2\}$
  - $\psi(U^+[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}])_{\phi^+}\langle K^{-1}_{\alpha_1}K_{\alpha_3}, K_{\alpha_1}K^{-1}_{\alpha_3}\rangle U^-[s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}]_{\phi^-}$ *with* $\mathtt{supp}(\phi^+) = \mathtt{supp}(\phi^-) = \{\alpha_2, \alpha_1+\alpha_2+\alpha_3\}$

Acknowledgement. *Both authors thank Istvan Heckenberger for answering questions and thank the RTG 1670 of the University of Hamburg for support and hospitality, in particular for a MicroWorkshop on Quantum Symmetric Pairs (Hamburg, February 2017), where most of this work was completed.*

References

[BB05] A. Bjrner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer-Verlag, 2005.

[HK11a] I. Heckenberger and S. Kolb, *Homogeneous right coideal subalgebras of quantized enveloping algebras*, Bull. London Math. Soc. 44, 837-848, 2012.

[HK11b] I. Heckenberger and S. Kolb, *Right coideal subalgebras of the Borel part of a quantized enveloping algebra*, Int. Math. Res. Not. IMRN, no. 2, pp 419-451, 2011.

[HLV17] I. Heckenberger and S. Lentner and K. Vocke, *On Borel subalgebras of quantum groups*, to appear soon.

[HS09] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Israel Journal of Mathematics 197, 139-187, 2013.

[HS10] I. Heckenberger, H.-J. Schneider: *Root systems and Weyl groupoids for Nichols algebras*, Proc. Lond. Math. Soc. 101 (2010) 623-654

[Hum70] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, Library of Congress Catalog, 1980.

[Jan96] J. Jantzen, *Lectures in Quantum Groups*, Graduate Studies in Mathematics, Volume 6, American Mathematical Society, 1996.

[Kolb14] S. Kolb: *Quantum symmetric Kac-Moody pairs*, Adv. Math. 267 (2014), 395-469.

[Len14] S. Lentner, *A Frobenius homomorphism for Lusztig's quantum groups for arbitrary roots of unity* Commun. Contemp. Math., Volume 18, 2016.

[Let97] G. Letzter: *Subalgebras which appear in quantum Iwasawa decompositions*, Canadian Journal of Mathematics 49 (1997), 1206–1223.

[Lusz90] G. Lusztig: *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 257-296.

[PA] P. Papi, *A characterization of a special ordering in a root system* Proceedings of the American Mathematical Society, Volume 120, pp 661-665, 1994.

[Vocke16] K. Vocke, *Ueber Borelunteralgebren von Quantengruppen* Dissertation University of Marburg (2016).

[Vocke17] K. Vocke, *On right coideal subalgebras of quantum groups*, in preparation.