ZMP-HH / 17-6
Hamburger Beiträge zur Mathematik Nr. 647

Quantum groups and Nichols algebras acting on conformal field theories

Simon Lentner
Algebra and Number Theory, University Hamburg,
Bundesstraße 55, D-20146 Hamburg
simon.lentner@uni-hamburg.de

Abstract. We prove a long-standing conjecture by B. Feigin et al. that certain screening operators on a conformal field theory obey the algebra relations of the Borel part of a quantum group (and more generally a diagonal Nichols algebra). Up to now this has been proven only for the quantum group $u_q(\mathfrak{sl}_2)$.

The proof is based on a novel, intimate relation between Hopf algebras, Vertex algebras and a class of analytic functions in several variables, which are generalizations of Selberg integrals. These special functions have zeroes wherever the associated diagonal Nichols algebra has a relation, because we can prove analytically a quantum symmetrizer formula for them. Morevover, we can use the poles of these functions to construct a crucial Weyl group action.

Our result produces an infinite-dimensional graded representation of any quantum group or Nichols algebra. We discuss applications of this representation to Kazhdan-Lusztig theory.

Contents

1. Introduction

2-dimensional conformal quantum field theory can be described in terms of vertex operator algebras (VOA). A VOA is a graded infinite-dimensional algebraic structure with an extra layer of analysis. Interesting first examples come from affine Lie algebras $\hat{\mathfrak{g}}_\ell$, moreover for any (even) lattice Λ there is the *lattice VOA* $\mathcal{V}_\Lambda$, which physically describes a free boson on a torus $\mathbb{C}^n/\Lambda$.

Under some finiteness assumptions a VOA has an intriguing representation theory: Assume first in addition semisimplicity, then we call the VOA *rational* and it turns out the representations category is always a modular tensor category. As such it gives already at a categorical level rise to 3-dimensional topological invariants and mapping class group actions. In addition, to each module is attached an analytic function (graded character) and under the action of the torus mapping class group $\mathrm{SL}_2(\mathbb{Z})$ these analytic functions piece together to a vector-valued modular form. For example, the representation category of $\mathcal{V}_\Lambda$ is equivalent to representations of the abelian group Λ^*/Λ and the graded characters associated to each module is essentially a theta function. If Λ is unimodular, the category has only one object and the graded character is a single modular form.

A similar behaviour is widely expected in the non-semisimple setting. We call a VOA *logarithmic*[1] if it is non-semisimple but still fulfills the finiteness-conditions. Only few examples are known, and understanding and construction logarithmic VOA's has been called one of the fundamental questions in 2D quantum field theory [Huang16]. In this case the representation category is non-semisimple modular in the sense of [KL01] and as such still produces 3-dimensional topological invariants and mapping class group actions [FRS02][Shim16]. However the space of characters of irreducible representations are only some components of a vector-valued modular form, the others should come from so-called pseudocharacters associated to projective covers; see the survey article [CG16].

In this article we are interested to construct and study a logarithmic VOA $\mathcal{W}$ that should have the same representation theory as a given small quantum groups $u_q(\mathfrak{g})$ at a given ℓ-th root of unity q, or as a more general finite-dimensional pointed Hopf algebra.

Small quantum groups are non-semisimple quotients of the Drinfeld-Jimbo deformations of the universal enveloping of a Lie algebra $U_q(\mathfrak{g})$ at roots of unity. Lusztig has constructed the small quantum groups in [Lusz90a] with the conjecture that their representation theory is related to representations of $\mathfrak{g}$ in finite characteristic ℓ (if prime) and related to representations of the affine Lie algebra $\hat{\mathfrak{g}}_\ell$ at level ℓ. The former was proven in [AJW94], the latter has been developed by Kazhdan and Lusztig in a series of papers.

[1]The name comes from the fact, that for modules with non-diagonalizable L_0 the correlator functions have logarithmic simgularities, while in the semisimple case they are rational functions.

A strategy to construct the VOA is as *free-field realization*, which means as subalgebras (or subquotient) of a lattice VOA $\mathcal{V}_\Lambda$. A first instance of constructing this way a semisimple VOA $\mathcal{R}$ goes back to Wakimoto realization of $\hat{\mathfrak{sl}_2}$ [Wak86] and the ingenious realization of any affine Lie algebra $\hat{\mathfrak{g}}$ by Feigin and Frenkel [FF88], which is nowadays a cornerstone of the Langland's correspondence [F95].

The idea is to start with a lattice VOA $\mathcal{V}_\Lambda$, where Λ is the root lattice of the given finite-dimensional semisimple Lie algebra $\mathfrak{g}$ rescaled by the given number ℓ. Then consider *screening charge operators* acting on $\mathcal{V}_\Lambda$, which produces a differential complex, the Felder complex [Fel89]. Taking cohomology is expected to produce a rational VOA $\mathcal{R}$ equivalent to certain representations of the affine Lie algebra $\hat{\mathfrak{g}}$. In the algebraic-geometry view, the cohomology is equal to a sheaf cohmology on the flag variety of $\mathfrak{g}$. All these claims are proven in [FF88] for generic q, they are open problems in our setting.

In [FHST02] the authors consider instead the kernel of the screening operators (i.e. cocycles instead of cohomology classes), which produces a non-semisimple VOA $\mathrm{W} \subset V_\Lambda$, conjecturally the Quantum-Drinfel'd-Sokolov reduction of the affine Lie algebra $\hat{\mathfrak{g}}$. In the example $\mathfrak{g} = \mathfrak{sl}_2$ we get for W the minimal models of the Virasoro algebra.

Then together with Feigin they went one step further, conjecturing a free-field theory for a much larger VOA $\mathcal{W} \supset W$, which carries an action of $\mathfrak{g}$ with kernel W, and this new $\mathcal{W}$ should have a *finite, non-semisimple* representation theory equivalent to the small quantum group $u_q(\mathfrak{g})$. In the example $\mathfrak{g} = \mathfrak{sl}_2$ the minimal model is extended by three generators W^+, W^0, W^- forming an adjoint representation of $\mathfrak{sl}_2$, to give the so-called triplet algebra. By this extension the infinitely many Virasoro representations are combined to finitely many representations, which can be identified with representations of the small quantum group $u_q(\mathfrak{sl}_2)$.

Conjecturally, and this is what we prove in this paper, the (short) screening operators facititate in fact an action of the small quantum group $u_q(\mathfrak{g})^+$ (and together with long screenings operators the infinite-dimensional Lusztig quantum group of divided powers). Thus the previous constructions in fact realize Kazhdan-Lusztig correspondence by providing infinite-dimensional graded representations, which have commuting action by the kernel of the screenings (related to the affine Lie algebra) and the screenings (the quantum group). So this action should be a first step to tackle these conjectures in general.

It is widely believed among experts that the program outlined above is true and produces logarithmic CFT's with a representation category equivalent to a quantum group $u_q(\mathfrak{g})$ at a $2p$-th root of unity. But carrying out this program has proven very difficult despite considerable effort; partly because CFT's seem to not have enough algebraic structure [Borch99], and non-integral lattices are involved.

Major achievements were for the case $\mathfrak{sl}_2$ the construction of $\mathcal{W}$ in [FGST06a] and the action of $\mathrm{SL}_2(\mathbb{Z})$ in [FGST06b]. The best understood case is $p = 2$ where the free CFT is isomorphic to so-called symplectic fermions; in this case the proof of the equivalence of abelian categories $\mathcal{W}$-mod $\cong u_i(\mathfrak{sl}_2)$-mod was carried out in [FGST06a] and the equivalence of modular tensor categories $\mathcal{W}$-mod $\cong \tilde{u}_i(\mathfrak{sl}_2)$-mod in [GR15]. For $\mathfrak{sl}_2$ with p arbitrary the category of representations $\mathcal{W}$-mod was determined in [AM08], the action of $u_q(\mathfrak{sl}_2)^+$ on the free field realization via screening operators in [TW13], and the equivalence of abelian categories in [NT11].

For $\mathfrak{g}$ simply-laced there is an intriguing paper by Feigin and Tipunin [FT10] in terms of sheaf cohomology that sketches among others how the graded character can be obtained explicitly as Euler characteristic of the cohomology ring (compare this to Deligne-Lusztig theory). On the other hand there has been a remarkable attempt by Semikhatov and Tipunin to generalize the program from quantum groups to diagonal Nichols algebras [ST12], e.g. calculations for the super-Lie algebra $\mathfrak{sl}(2|1)$ in [ST13] and conjectural central charges for each rank-2 Nichols algebra in [S14].

From a physical perspective, the question of realizing super-Lie algebras [FT10] is the most interesing reason to consider general Nichols algebras. From the algebraic perspective, is is fascinating to discover general Nichols algebras as the natural symmetry structure of a conformal field theory. But most of all, the structure theory of Nichols algebras may show us the natural framework in which to proceed and prove our claims, even for $u_q(\mathfrak{g})$. This is the approach taken in the present article.

In this article we achieve the first step to prove the program outlined about in great generality: We prove that any quantum group and more generally any diagonal Nichols algebra acts via screening charge operators on the free field theory. This means, we prove that screening charge operators obey Nichols algebra relations. We do so by developing a purely algebraic theory to calculate screening operators of VOA's and of generalized VOA's with fractional powers. Our approach, partly going back until [Len07], separates calculations in generalized VOA's into a Hopf algebraic part, and into structure constants which are generalized hypergeometric functions. The fact that screening operators obey Nichols algebra relations can then be proven from the astonishing fact, that these hypergemetric functions have zeroes according to Nichols algebra relations, as we prove.

Besides proving the CFT conjecture and it's generalizations to Nichols algebras, we thus discover a complex-analysis appearance of Quantum groups and Nichols algebras. Moreover, we show that in fact that *any* generalized VOA underlies a Nichols algebra (instead of a Lie algebra) due to the multivaluedness of the fractional powers. We would hope that our approach provides a big step in further algebraization of VOA theory and makes it more appealing to scientists working in Hopf algebras. The author is e.g. currently working with Feigin and Semikhatov to extend the program to Liouville theory, and again the algebraic approach to screening charge operators is the main tool of proof.

In **Section 2** we very briefly review the construction of Nichols algebras as quotient of a free algebra by the kernel of the *quantum symmetrizer* due to Woronowicz [Wor89]. The most important message is the modern view, that the relations of a quantum group follow completely from this construction (Example 2.3), as well as generalized root systems e.g. super-Lie algebras. So our goal is to prove that expressions in screening operators vanish, whenever they lie in the kernel of the quantum symmetrizer.

In **Section 3** we present for this purpose a purely algebraic approach to these conformal field theories resp. vertex algebras (VOA). The author has already in [Len07] constructed vertex algebras purely out of Hopf algebra data, in particular free field theories $\mathcal{V}_\Lambda$ associated to a torus $\mathbb{C}^{\mathrm{rank}}/\Lambda$ with an integral lattice Λ. For more general choices this thesis developes an algebraic formalism for a generalized VOA structure and proves generalizations of locality and associativity.

Some reasons to go into this algebraic framework are: First, avoids the usual infinite-sum and delta-function calculations, second it makes the theory purely algebraic and thus hopefully much more accessible to non-physicists, and third it encloses many tedious calculations into Hopf algebra expressions, that would otherwise be hopeless.

The much more substantial reason is that the relevant screening operators are not actually living in an integral lattice VOA $\mathcal{V}_\Lambda$, but in a rescaled non-integral analog. There is however, to the authors knowledge, no VOA theory available for this scenario, probably because due to the rational exponents z^m there are monodromy terms that destroy the main tool (OPE-associativity). But it is precisely this effect where the Nichols algebra comes from, and the failure of OPE-associativity means precisely that we are not dealing with a Lie algebra any more. The fact that our framwork allows to naturally work with these fractional lattice VOA's $\mathcal{V}_\Lambda$ seems to be the first main idea of this approach.

Subsection 3.1 is a self-contained description of the *fractional lattice VOA* $\mathcal{V}_\Lambda$, which is an infinite-dimensional graded vectorspace associated to a (non-integral) lattice Λ. The reader who wishes to ignore the physics background may simply accept that the Hopf algebraic map Y in Definition 3.9 is a source for interesting endomorphismsm on a commutative, cocommutative Hopf algebra $\mathcal{V}_\Lambda$.

The reader familiar with VOA's should be warned, that on $\mathcal{V}_\Lambda$ there is no (familiar) locality-property and no choice of Virasoro-action; both are not necessary for the present proofs. Locality will be replaced by Nichols algebra relations, and later there will be different choices of Virasoro-actions (one for each Weyl chamber).

Subsection 3.2 defines how to turn for each element $a \in \mathcal{V}_\Lambda$ the vertex operator $\mathrm{Y}(a)$ into an endomorphism of $\mathcal{V}_\Lambda$ by taking a formal residue; these endomorphisms $\mathrm{ResY}(a)$ are called *charge operators*. For any fractional power z^m this residue is non-zero, because we defined it by lifting a circle to the multivalued covering. This makes everything much more involved than in the integral case; in effect hypergeometric functions replace binomial coefficients in the integral case.

Subsection 3.3 turns the attention to charge operators $\mathrm{ResY}(a)$ for two specific families of elements $\partial\phi_\alpha, \mathrm{e}^{\phi_\alpha} \in \mathcal{V}_\Lambda$, leading to endomorphisms $Ъ_\alpha$ and $З_\alpha$ for $\alpha \in \Lambda$. While $Ъ_\alpha$ is a derivation, essentially the Λ-grading of $\mathcal{V}_\Lambda$, the endomorphisms $З_\alpha$ are the much more complicated *screening operators* in question. We use our machinery to prove some easier relations between these endomorphisms.

Subsection 3.4 contains a trivial example of the program outlined above as an instructive nutshell: If we take indeed Λ to be the (integral) root lattice of any semisimple Lie algebra $\mathfrak{g}$ with simple roots α_i, then the results from the previous subsection suffice to prove that $Ъ_{\alpha_i}, З_{\alpha_i}$ give an action of $H_{\alpha_i}, E_{\alpha_i} \in U(\mathfrak{g})^{\geq}$ on $\mathcal{V}_\Lambda$. In fact in this case $\mathcal{V}_\Lambda$ is isomorphic to the affine Lie algebra $\hat{\mathfrak{g}}$ at trivial level.

In **Section 4** we apply our Hopf algebra machinery to simplify products of charge operators $\prod_{i=1}^n \mathrm{ResY}(b_i)v$ such as $\prod_{i=1}^n З_{\beta_i} v_\lambda$. In Theorem 4.3 we derive an expression of the symbolic form[2]

$$\prod_{i=1}^n \mathrm{ResY}(b_i)v = \sum_{k_1,\ldots k_n \in \mathbb{N}_0} v_{(k_i)_i} \cdot \mathrm{F}_-((m_i + k_i, m_{ij})_{i<j})$$

where $v_{(k_i)_i}$ are Hopf algebra elements which are essentially invariant under permutation of the b_i, while the *Quantum Monodromy Numbers* F_- are *not* permutation invariant. These complex numbers depend only on the scalar products $m_{ij} := (\beta_i, \beta_j), m_i := (\beta_i, \lambda)$

$$\mathrm{F}_\pm((m_i, m_{ij})_{ij}) = \sum_{(k_{ij})_{ij} \in \mathbb{N}_0^{\binom{n}{2}}} \prod_i \frac{(e^{2\pi\mathrm{i}(m_i + \sum_{i<j} m_{ij})} - 1)/2\pi\mathrm{i}}{1 + m_i + \sum_{i<j}(m_{ij} - k_{ij}) + \sum_{j<i} k_{ji}} \prod_{i<j} (\pm 1)^{k_{ij}} \binom{m_{ij}}{k_{ij}}$$

Thus we have to prove that a formal linear combination of F_- vanishes, whenever it is in the kernel of the quantum symmetrizer.

[2] As a remark, from a VOA perspective this formula generalizes OPE-associativitly and -locality, because it allows to rearrange products and the effect is only noticed inside F_-, which is in the integral case simply a binomial coefficient causing delta-function differences.
Our result in fact shows that any VOA with fractional powers is controlled by these F_- and any of these will show Nichols algebras instead of Lie algebras.

In **Section 5** we study the analytic functions $\mathrm{F}_\pm((m_i,m_{ij})_{ij})$. These are some generalized hypergeometric functions on the boundary of convergence $z=\mp 1$, and we shall not attempt to simplify them except the first two (Example 5.13)

$$\begin{aligned}
\mathrm{F}_-(m_1) &= \frac{(e^{2\pi\mathrm{i}m_1}-1)/2\pi\mathrm{i}}{m_1+1}\\
\mathrm{F}_-(m_1,m_2;m_{12}) &= \frac{e^{2\pi\mathrm{i}m_2}-1}{2\pi\mathrm{i}}\,\frac{e^{2\pi\mathrm{i}m_1+2\pi\mathrm{i}m_{12}}-1}{2\pi\mathrm{i}}\;\frac{1}{m_1+m_2+m_{12}+2}\cdot\\
&\quad\cdot\left(\mathrm{B}(m_2+1,m_{12}+1)+\frac{\sin\pi m_1}{\sin\pi(m_1+m_{12})}\mathrm{B}(m_1+1,m_{12}+1)\right)
\end{aligned}$$

Subsection 5.1 studies these quantum monodromy numbers, in particular small cases, degenerate cases (where the sum has poles, which are then suppressed by the numerator), and most importantly convergence, which is rather subtle. E.g., when the lattice is positive-definite and all $|\beta_i|\leq 1$, then the series converges.

Subsection 5.2 contains the main result of this article, Theorem 5.11:

Quantum Symmetrizer Formula

$$\boxed{\mathrm{F}_-((m_i,m_{ij})_{ij})=\textstyle\sum_{\sigma\in\mathbb{S}_n}q(\sigma)\,\tilde{\mathrm{F}}_-((m_{\sigma^{-1}(i)},m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij})=:\text{Ш}_q\,\tilde{\mathrm{F}}_-((m_i,m_{ij})_{ij})}$$

with respect to the braiding matrix $q_{ij}=e^{\pi\mathrm{i}\,m_{ij}}$ and braiding factor $q(\sigma)$, in terms of generalized Selberg-integrals $\tilde{\mathrm{F}}_-((m_i,m_{ij})_{ij})$. We also give numerical examples.

This formula (independent of the expression for $\tilde{\mathrm{F}}_-$) shows that indeed linear combinations of F_- vanish whenever they are in the kernel of the quantum symmetrizer, *because* we can write them as quantum symmetrizer of something!

Subsection 5.2 contains the proof of the Quantum Symmetrizer Formula using complex analysis. Essentially we write F_- (before passing to a limit on the boundary of the convergence disc) as an residue integral

$$\mathrm{F}_-((m_i,m_{ij})_{ij})=\int\cdots\int_{[e^0,e^{2\pi}]^n}\mathrm{d}z_1\ldots\mathrm{d}z_n\prod_i z_i^{m_i}\prod_{i<j}(z_i-z_j)^{m_{ij}}$$

but where $[e^0,e^{2\pi}]^n$ is a specific lift of an n-torus to an n-cube in the multivalued covering. Then we partition the n-cube into $n!$ simplices $t_{\sigma^{-1}(1)}<\cdots<t_{\sigma^{-1}(1)}$ for all $\sigma\in\mathbb{S}_n$ which get transported back catching monodromy $q(\sigma)$. Finally we deform the lifted simplex in the covering to a real simplex on $\mathbb{R}^n$ to write it as a generalized Selberg integral.

In **Section 6** we come back to the program outlined at the beginning.

In **Subsection 6.1** we use the quantum symmetrizer formula to prove as intended in Theorem 6.1, that for elements $\alpha_1,\ldots,\alpha_{\mathrm{rank}}$ in a positive definite lattice satisfying $|\alpha_i|<1$ the screenings $\mathfrak{Z}_{\alpha_i}$ fulfill the relations of the diagonal Nichols algebra with braiding matrix $q_{ij}=e^{\pi\mathrm{i}(\alpha_i,\alpha_j)}$.

The restriction is no formality: For other cases, e.g. ordinary Lie algebras $|\beta_i|^2 = 2$ resp. Liouville models $|\beta_i|^2 < 0$) we will indeed get additional Lie-algebra terms to our Nichols algebra relations, generating e.g. the Lie algebra in question resp. the Lie algebra part of the Kac-Procesi-DeConcini quantum group!

In **Subsection 6.2** we discuss the program and try to separate and formulate thoroughly the conjectures that are implicitly assumed from the experts. The author also tries to put the conjectures in a formulation as general as possible (including more general cases than diagonal Nichols algebras), and closes by an outlook of some own conjectures that give a roadmap to the main category equivalence. Our article clarifies two of these points in term of Nichols algebras: Mainly, the action of the Nichols algebra on the the free field theory, and secondly, products of screenings can be used to construct Weyl reflections, which are exceptionally $\mathcal{V}ir$-homomorphisms (these are again exceptional situations where the Nichols part vanishes, but due to poles in F_- there is still non-zero contributions).

In **Subsection 6.3** we close by thoroughly discussing the example $u_q(\mathfrak{sl}_2)$ in light of this program.

2. Quantum Symmetrizers in Quantum Groups and Nichols algebras

Nichols algebras are certain universal algebras associated to a vector space with braiding. They appear most prominently as positive Borel part of the small quantum group $u_q(\mathfrak{g})^+$, a deformation of the universal enveloping of a Lie algebra $U(\mathfrak{g})$.

Let $M = \langle x_1, \ldots, x_{\mathrm{rank}} \rangle_{\mathbb{C}}$ be a complex vector space and let $(q_{ij})_{i,j=1,\ldots \mathrm{rank}}$ be an arbitrary matrix with $q_{ij} \in \mathbb{C}^\times$. This defines a *braiding of diagonal type* on M via:

$$c: \ c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

Hence we get an action ρ_n of the *braid group* $\mathbb{B}_n$ on $M^{\otimes n}$ via

$$c_{i,i+1} := id \otimes \cdots \otimes c \otimes \cdots \otimes id.$$

The braided vector space (M, c) gives a special type of algebra, which is called the Nichols algebra $\mathcal{B}(M)$ generated by (M, c).

Definition 2.1. *Let (M, c) be a braided vector space. We consider the canonical projections $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$ sending the braiding $c_{i,i+1}$ to the transposition $(i, i+1)$. There exists the* Matsumoto section *of sets $s : \mathbb{S}_n \to \mathbb{B}_n$ given by $(i, i+1) \mapsto c_{i,i+1}$ which has the property $s(xy) = s(x)s(y)$ whenever* $\mathrm{length}(xy) = \mathrm{length}(x) + \mathrm{length}(y)$. *Then we define the* quantum symmetrizer *by*

$$\text{Ш}_{q,n} := \sum_{\tau \in \mathbb{S}_n} \rho_n(s(\tau)) \tag{1}$$

where ρ_n is the representation of $\mathbb{B}_n$ on $M^{\otimes n}$ induced by the braiding c. Then the Nichols algebra *generated by (M, c) is*

$$\mathcal{B}(M) = \bigoplus_n M^{\otimes n} / \ker(\text{Ш}_{q,n}).$$

In general the kernels of the map $\text{Ш}_{q,n}$ is hard to calculate in explicit terms. So that the description relations of $\mathcal{B}(M)$ does not mean the relations are known.

In fact a Nichols algebra is a Hopf algebra in a braided category fulfilling several equivalent universal properties.

Example 2.2 (Rank 1)**.** *Let $M = x\mathbb{C}$ be a 1-dimensional vector space with braiding given by $q_{11} = q \in \mathbb{C}^\times$, then*

$$\mathbb{C} \ni \text{Ш}_{q,n} = \sum_{\tau \in \mathbb{S}_n} q_{11}^{|\tau|} = \prod_{k=1}^{n} \frac{1-q^k}{1-q} =: [n]_q!$$

Because this polynomial has zeroes all $q \neq 1$ of order $\leq n$ the Nichols algebra is

$$\mathcal{B}(M) = \begin{cases} \mathbb{C}[x]/(x^\ell), & q_{11} \text{ primitive } \ell\text{-th root of unity} \\ \mathbb{C}[x], & \text{else} \end{cases}$$

Example 2.3 (Quantum group)**.** *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank n with root system Φ and simple roots $\alpha_1, \ldots, \alpha_n$ and Killing form (α_i, α_j). Let q be a primitive ℓ-th root of unity. Consider the n-dimensional vector space M with diagonal braiding $q_{ij} := q^{(\alpha_i, \alpha_j)}$ Then the Nichols algebra $\mathcal{B}(M)$ is isomorphic to the positive part of the small quantum group $u_q(\mathfrak{g})^+$.*

In fact every Nichols algebra (e.g. from diagonal braiding) comes with (generalized) *root system*, a *Weyl groupoid* and a *PBW-type basis*, see [Heck09, HS10]. The Weyl groupoid plays a similar role as the Weyl group does for ordinary root systems in Lie algebras and quantum gropus, but in the general case not all Weyl chambers look the same. This behaviour is already familiar from super-Lie algebras, which are a special case of Nichols algebras.

Example 2.4 (Super Lie algebra $\mathfrak{sl}(2|1)$)**.** *Let q be a primitive ℓ-th root of unity. Consider the following two braiding matrices:*

$$q'_{ij} = \begin{pmatrix} -1 & q^{-1} \\ q^{-1} & -1 \end{pmatrix} \qquad q''_{ij} = \begin{pmatrix} -1 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}$$

Then the associated Nichols algebras $\mathcal{B}(M'), \mathcal{B}(M'')$ are not isomorphic, but of the same dimension $2 \cdot 2 \cdot \mathrm{ord}(q^2)$. Reflection on the first root of the Nichols algebra $\mathcal{B}(M')$ returns $\mathcal{B}(M'')$ and vice versa (this is the so-called "odd reflection" in super-Lie algebra theory), while reflection on the second root of the Nichols algebra $\mathcal{B}(M'')$ maps $\mathcal{B}(M'')$ to itself.

Finite dimensional Nichols algebra of diagonal type over fields of characteristic 0 were classified by I. Heckenberger [Heck09].

3. Calculating in Vertex Algebras using Hopf Algebras

3.1. **Fractional Lattice VOA.**

In this section we construct the fractional lattice VOA $\mathcal{V}_\Lambda$.

Definition 3.1. *Fix once and for all the Hopf algebra and module algebra*

$$H := \mathbb{C}[\partial], \qquad R := \mathbb{C}[z, z^{-1}] \cong \mathbb{C}[\mathbb{Z}]$$

where the primitive generator ∂ acts on the Laurent polynomials by $-\frac{\partial}{\partial z}$

Definition 3.2. *Let $\Lambda \subset \mathbb{C}^{\mathrm{rank}}$ be a lattice with inner product $(,) : \Lambda \times \Lambda \to \frac{1}{N}\mathbb{Z}$ for some $N \in \mathbb{N}$ (non-rational powers are also possible, but not desirable at this point).*

Let $\mathcal{V}_\Lambda$ be commutative, cocommutative, infinite-dimensional $\mathbb{N}_0$-graded Hopf algebra generated by formal symbols

$$\mathrm{e}^{\phi_\alpha},\ \partial^k \phi_\alpha \qquad \forall \alpha \in \Lambda, 0 \neq k \in \mathbb{N}$$

subject to the following relations (besides commutativity) suggested by notation:

$$\mathrm{e}^{\phi_\alpha}\mathrm{e}^{\phi_\beta} = \mathrm{e}^{\phi_{\alpha+\beta}}$$

$$\partial^k \phi_\alpha + \partial^k \phi_\beta = \partial^k \phi_{\alpha+\beta}$$

where the e^{ϕ_α} are grouplike and in $\mathbb{N}_0$-degee 0 and where $\partial^{1+k}\phi_\alpha$ are primitve and in $\mathbb{N}_0$-degree k. Differently spoken: $\mathcal{V}_\Lambda = \mathbb{C}[\Lambda] \otimes U(H \otimes_{\mathbb{Z}} \Lambda)$ with $\partial^{1+k}\phi_\alpha = \partial^k \otimes \alpha$.

Definition 3.3 (Deformations, optional)**.** *We also introduce deformations, which may be used later on to forcefully remove naturally appearing signs:*

Given an additional group cohomology class $[\kappa] \in \mathrm{Z}^2(\Lambda, \mathbb{C}^\times)$ with associated alternating bicharacter Ω, then the twisted groupring (defined up to isomorphy) is a comodule algebra *(a so-called Galois object) over the group ring $\mathcal{V}_\Lambda = \mathbb{C}[\Lambda]$*

$$\mathcal{V}_\Lambda^{(\Omega,1)} := \mathbb{C}_\kappa[\Lambda] \qquad \lambda\lambda' = \Omega(\lambda, \lambda')\lambda'\lambda$$

As a remark, we would consider more generally: For every 3*-cocycle ω there is a coquasi-Hopf algebra $\mathcal{V}_{\Lambda,\omega} = \mathbb{C}^\omega[\Lambda]$ with category of comodules* $\mathrm{Vect}_\Lambda^\omega$*. For every* abelian 3-cocycle (Ω, ω) [MacLane50, FRS04] *we have in this category* $\mathrm{Vect}_\Lambda^\omega$ *a twisted group ring $\mathcal{V}_\Lambda^{(\Omega,\omega)} := \mathbb{C}_\Omega^\omega[\Lambda]$. It is known that abelian cohomology classes are parametrized by the quadratic* *form, here* $\Omega(\lambda,\lambda) = e^{\pi \mathrm{i}(\lambda,\lambda)_\Lambda}$*, and we remark that our later-on Nichols algebra depends essentially only on this form, i.e.* $q_{ii}, q_{ij}q_{ji}$*.*

This freedom of deformations explains subtile sign choices in [FT10] *and on the other side supposably the appearance of respective quasi-Hopf algebras in the* $\mathfrak{sl}_2$*-case* [GR15].

The notation should suggests how to turn $\mathcal{V}_\Lambda$ into an H-module algebra:

Lemma 3.4. *The following assignment endows* $\mathcal{V}_\Lambda$ *with the unique structure of an* H*-module algebra:*

$$\partial.\mathrm{e}^{\phi_\alpha} := \partial\phi_\alpha \cdot \mathrm{e}^{\phi_\alpha}$$
$$\partial.\partial^k\phi_\alpha := \partial^{k+1}\phi_\alpha$$

where ∂ *raises the* $\mathbb{N}_0$*-degree by* 1*.*

Note that by the Leibniz rule the expression $\partial^k.\mathrm{e}^{\phi_\alpha}$ becomes soon complicated:

Definition 3.5. *We define the differential polynomial* $P_{\alpha,k} \in U(\Lambda \otimes_\mathbb{Z} H)$ *by*

$$\frac{1}{k!}\partial^k.\mathrm{e}^{\phi_\alpha} = P_{\alpha,k}\,\mathrm{e}^{\phi_\alpha}$$

For later convenience we give the first couple $P_{\alpha,k} \in U(H)$*:*

$$1,\ \ \partial\phi_\alpha,\ \ \frac{1}{2!}\left(\partial\phi_\alpha\partial\phi_\alpha + \partial^2\phi_\alpha\right),\ \ \frac{1}{3!}\left(\partial\phi_\alpha\partial\phi_\alpha\partial\phi_\alpha + 3\,\partial\phi_\alpha\partial^2\phi_\alpha + \partial^3\phi_\alpha\right),\ \ \ldots$$

Notice the obvious similarity to Hermite polynomials.

Corollary 3.6. *By definition via derivations and grouplikes*

$$\Delta(P_{k,\alpha}) = \sum_{k_1+k_2=k} P_{k_1,\alpha} \otimes P_{k_2,\alpha}$$

We repeat an essential structure of the previous section and give the example relevant to us:

Definition 3.7. *Let* $\mathcal{V}$ *be an* H*-module Hopf algebra, then a* Hopf pairing with coefficients *in the* H*-module algebra* R *is a map* $\mathcal{V} \otimes \mathcal{V} \to R$ *fulfilling*

$$\langle a, bc\rangle = \langle a^{(1)}, b\rangle\langle a^{(2)}, c\rangle$$
$$\langle ab, c\rangle = \langle a, c^{(1)}\rangle\langle a, c^{(2)}\rangle$$
$$\langle a, \partial.b\rangle = -\frac{\partial}{\partial z}\langle a, b\rangle$$
$$\langle \partial.a, b\rangle = \frac{\partial}{\partial z}\langle a, b\rangle$$

Lemma 3.8. *Let Λ be a lattice and $\mathcal{V}_\Lambda$ choosen as above, the following assignment defines a Hopf pairing with coefficients $\mathcal{V}_\Lambda \otimes \mathcal{V}_\Lambda \to R$:*

$$\begin{aligned}
\langle \mathrm{e}^{\phi_\alpha}, \mathrm{e}^{\phi_\beta} \rangle &= z^{(\alpha,\beta)} \\
\langle \mathrm{e}^{\phi_\alpha}, \partial\phi_\beta \rangle &= -(\alpha,\beta) z^{-1} \\
\langle \partial\phi_\alpha, \mathrm{e}^{\phi_\beta} \rangle &= (\alpha,\beta) z^{-1} \\
\langle \partial\phi_\alpha, \partial\phi_\beta \rangle &= (\alpha,\beta) z^{-2}
\end{aligned}$$

We also introduce the structure of a diagonal Λ-Yetter-Drinfel'd module on $\mathcal{V}_\Lambda$ by

$$P\ \mathrm{e}^{\phi_\alpha} \longmapsto \alpha \otimes P\ \mathrm{e}^{\phi_\alpha}$$

$$\alpha \otimes P\ \mathrm{e}^{\phi_\beta} \longmapsto e^{\pi \mathrm{i}(\alpha,\beta)}\ P\ \mathrm{e}^{\phi_\beta}$$

were $P \in U(\Lambda \otimes_{\mathbb{Z}} H)$.
The following is the central notion for the following work:

Definition 3.9. *Let $\mathcal{V}'$ be a comodule algebra over a Hopf algebra $\mathcal{V}$ (for us usually $\mathcal{V}' = \mathcal{V}$) with a Hopf pairing with R-coefficients on $\mathcal{V}$ inside the category of H-modules. Then the following map* Y *shall be called* vertex operator *on $\mathcal{V}'$*

$$\begin{aligned}
\mathrm{Y}:\ \mathcal{V}' &\longrightarrow \mathrm{End}(\mathcal{V})[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]] \\
a &\longmapsto \left(b \mapsto \sum_{k\geq 0} \langle a^{(-1)}, b^{(-1)} \rangle \cdot b^{(0)} \cdot \frac{z^k}{k!} \partial^k a^{(0)} \right)
\end{aligned}$$

In this article we shall work with the *(undeformed) fractional Lattice VOA* $\mathcal{V}' = \mathcal{V} = \mathcal{V}_\Lambda$, as defined above for any given lattice Λ. In later considerations one would wants wo work with the *(deformed, local) fractional Lattice VOA* $\mathcal{V}_\Lambda^{(\Omega,\omega)}$.

In [Len07] the author has proven that this definition satisfies for general Hopf algebra data a generalized form of vertex algebra associativity and locality. For $N = 1$ these properties reduce by [Len07] Sec. 5.1 to the familiar vertex algebra axioms (without $\mathbb{Z}$-grading and conformal structure).
In particular for $N = 1$ (i.e. integral lattice) the VOA $\mathcal{V}_\Lambda^{(\Omega,1)}$ for the familiar 2-cocycle Ω is by [Len07] Sec. 5.3 isomorphic to the familiar Lattice (super-)VOA.

Remark 3.10. *From a physics perspective, the vector space $\mathcal{V}$ is the vector space of* states *of a quantum mechanical system, e.g. 2-dimensional for a single up/down state, but in our case usually infinite-dimensional, e.g. wave function (probability distribution) of a single particle. The vertex operator describes the interaction of two states at a specific point in space-time $\mathbb{C}$; and hence a conformal quantum* field *theory. The endomorphisms*

$Y(a)$ *of* $\mathcal{V}$ *depending on* $z \in \mathbb{C}$ *is called a* field*, depending on the location* z.

For example, the easiest vertex algebra is the vertex algebra associated to $\mathcal{V} = U(H)$*, the* Heiseberg algebra*. It describes a single free oscillators / free bosonic with* $\partial^{1+n}\phi$ *the different excitation modes - so an expression like* $\partial\phi\partial\phi\partial^2\phi$ *describes two quanta in the ground state and one in the first excitation state. Analytically, it describes the superposition of waves with amplitudes of the fist two frequencies* $2:1$*. The vertex algebra* $\mathcal{V}_\Lambda$ *with additional fields describes a* σ*-model in string theory, one would say: A free boson (in* 2*-dim) compactified on the torus* $\mathbb{C}^{\text{rank}}/\Lambda$*. The* e^{ϕ_α} *describe the momenta around the homologies of this torus.*

As an example for Definition 3.9 we calculate some vertex operators in $\mathcal{V}_\Lambda$:

$$
\begin{aligned}
\mathrm{Y}(a)\,1_{\mathcal{V}} &= \underbrace{\langle a^{(-1)}, 1\rangle}_{=\epsilon(a^{(-1)})} \cdot 1 \cdot \sum_{k\geq 0} \frac{z^k}{k!}\,\partial^k.a^{(0)} = \sum_{k\geq 0}\frac{z^k}{k!}\partial^k.a \qquad \text{(field acting on vacuum)}\\
&\approx z^0\cdot a + z^1\cdot\partial.a + \frac{z^2}{2}\cdot\partial^2.a\cdots\\
\mathrm{Y}(\partial\phi_\alpha)\partial\phi_\beta &= \langle(\partial\phi_\alpha)^{(-1)}, (\partial\phi_\beta)^{(-1)}\rangle\cdot(\partial_\phi\beta)^{(0)}\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.(\partial_\phi\alpha)^{(0)}\\
&= \underbrace{\langle\partial\phi_\alpha, \partial\phi_\beta\rangle}_{=(\alpha,\beta)z^{-2}}\cdot 1\cdot\underbrace{\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.1}_{=z^0\cdot 1} + \underbrace{\langle\partial\phi_\alpha, 1\rangle}_{=0}\cdot\partial\phi_\beta\cdot\underbrace{\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.1}_{=z^0\cdot 1}\\
&+ \underbrace{\langle 1, \partial\phi_\beta\rangle}_{=0}\cdot 1\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\partial\phi_\alpha + \underbrace{\langle 1, 1\rangle}_{=z^0}\cdot\partial\phi_\beta\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\partial\phi_\alpha\\
&= (\alpha,\beta)z^{-2}\cdot 1 + \sum_{k\geq 0}\frac{z^k}{k!}\cdot\partial\phi_\beta\cdot\partial^k.\partial\phi_\alpha\\
&\approx (\alpha,\beta)z^{-2}\cdot 1 + z^0\cdot\partial\phi_\beta\partial\phi_\alpha + z^1\cdot\partial\phi_\beta\partial^2\phi_\alpha + \frac{z^2}{2}\cdot\partial\phi_\beta\partial^3\phi_\alpha + \cdots\\
\mathrm{Y}(\mathrm{e}^{\phi_\alpha})\partial\phi_\beta &= \langle\left(\mathrm{e}^{\phi_\alpha}\right)^{(-1)}, (\partial\phi_\beta)^{(-1)}\rangle\cdot(\partial\phi_\beta)^{(0)}\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\left(\mathrm{e}^{\phi_\alpha}\right)^{(0)}\\
&= \langle\mathrm{e}^{\phi_\alpha}, \partial\phi_\beta\rangle\cdot 1\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\mathrm{e}^{\phi_\alpha} + \langle\mathrm{e}^{\phi_\alpha}, 1\rangle\cdot\partial\phi_\beta\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\mathrm{e}^{\phi_\alpha}\\
&= \left(-(\alpha,\beta)z^{-1}\cdot 1 + z^0\cdot\partial\phi_\beta\right)\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\mathrm{e}^{\phi_\alpha}
\end{aligned}
$$

$$\approx -(\alpha,\beta)z^{-1}\cdot \mathrm{e}^{\phi_\alpha} - (\alpha,\beta)z^0\cdot\partial\phi_\alpha\,\mathrm{e}^{\phi_\alpha} - (\alpha,\beta)\frac{z}{2}\partial\phi_\alpha\partial\phi_\alpha\,\mathrm{e}^{\phi_\alpha} - (\alpha,\beta)\frac{z}{2}\partial^2\phi_\alpha\,\mathrm{e}^{\phi_\alpha} - \cdots$$
$$+ z^0\cdot\partial\phi_\beta\,\mathrm{e}^{\phi_\alpha} + z^1\cdot\partial\phi_\beta\partial\phi_\alpha\,\mathrm{e}^{\phi_\alpha} + \frac{z^2}{2}\partial\phi_\beta\partial\phi_\alpha\partial\phi_\alpha\,\mathrm{e}^{\phi_\alpha} + \frac{z^2}{2}\partial\phi_\beta\partial^2\phi_\alpha\,\mathrm{e}^{\phi_\alpha} + \cdots$$

Here we gray out all non-singular terms $z^k, k\in\mathbb{N}_0$ - in physics literature these are usually omitted.

The vertex operator in the next example may have fractional z-powers and singular terms depending on $(\alpha,\beta)\in\frac{1}{N}\mathbb{Z}$

$$\begin{aligned}
\mathrm{Y}(\mathrm{e}^{\phi_\alpha})\mathrm{e}^{\phi_\beta} &= \langle\left(\mathrm{e}^{\phi_\alpha}\right)^{(-1)},\left(\mathrm{e}^{\phi_\beta}\right)^{(-1)}\rangle\cdot\left(\mathrm{e}^{\phi_\beta}\right)^{(0)}\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\left(\mathrm{e}^{\phi_\alpha}\right)^{(0)}\\
&= \langle\mathrm{e}^{\phi_\alpha},\mathrm{e}^{\phi_\beta}\rangle\cdot\mathrm{e}^{\phi_\beta}\cdot\sum_{k\geq 0}\frac{z^k}{k!}\partial^k.\mathrm{e}^{\phi_\alpha}\\
&= z^{(\alpha,\beta)}\cdot\mathrm{e}^{\phi_{\alpha+\beta}}\sum_{k\geq 0}z^k\;P_{k,\alpha}\\
&\approx z^{(\alpha,\beta)}\cdot\mathrm{e}^{\phi_{\alpha+\beta}} + z^{(\alpha,\beta)+1}\cdot\partial\phi_\alpha\,\mathrm{e}^{\phi_{\alpha+\beta}}+\\
&+\frac{z^{(\alpha,\beta)+2}}{2}\partial\phi_\alpha\partial\phi_\alpha\,\mathrm{e}^{\phi_{\alpha+\beta}} + \frac{z^{(\alpha,\beta)+2}}{2}\partial^2\phi_\alpha\,\mathrm{e}^{\phi_{\alpha+\beta}}+\cdots
\end{aligned}$$

3.2. **Mode- and Residue-Operators.**

For a given state $a\in\mathcal{V}$ and $m\in\frac{1}{N}\mathbb{Z}$ we may consider the z^m-term[3] in $\mathrm{Y}(a)b$ and thus produce an endomorphism, the *mode operator* $\mathrm{Y}(a)_m:\mathcal{V}\to\mathcal{V}$:

$$\mathrm{Y}(a)_m:\;b\longmapsto\sum_{k\geq 0}\langle a^{(0)},b^{(0)}\rangle_{-k+m}\;b^{(2)}\;\frac{1}{k!}\partial^k.a^{(2)}\qquad\qquad \mathrm{Y}(a)=\sum_{m\in\frac{1}{N}}\mathrm{Y}(a)_m z^m$$

where $\langle a,b\rangle_m$ denotes the z^m-coefficient.

Definition 3.11. *The formal* residue *of fractional polynomials* $f\in\mathbb{C}[z^{\frac{1}{N}},z^{-\frac{1}{N}}]$ *be*

$$\mathrm{Res}_\hbar(z^m)\;:=\;\begin{cases}0 & m\in\mathbb{Z}\backslash\{-1\}\\ 1 & m=-1\\ \frac{\hbar^{m+1}}{2\pi\mathrm{i}\,(m+1)}\left(e^{2\pi\mathrm{i}\,(m+1)}-1\right), & m\notin\mathbb{Z}\end{cases}$$

[3]Caution: We do not use the commen convention z^{-h-n}, because there is no a-priori Virasoro action and later we will deal with multiple choices.

depending on a formal parameter[4] $\hbar$ *which we will usually set to* $\hbar = 1$.

Geometrically, *this it the integral along the unique lift (starting in the principal branch) of the circle of radius* $\hbar$ *to the multivalued covering on which* f *is defined.*

Definition 3.12. *For every* $a \in \mathcal{V}$ *we define the* ResY-*operator*

$$\mathrm{ResY}(a) : b \longmapsto \mathrm{Res}\,(\mathrm{Y}(a)b)$$

If $a \in \mathcal{V}_\alpha, b \in \mathcal{V}_\beta$ *and* $m := (\alpha, \beta)$ *then*

$$\mathrm{ResY}(a)\; b = \begin{cases} \mathrm{Y}(a)_{-1} & \text{for } m \in \mathbb{Z} \\ \dfrac{e^{2\pi \mathrm{i} m} - 1}{2\pi \mathrm{i}} \displaystyle\sum_{k \in \mathbb{Z}} \frac{1}{m+k+1} \mathrm{Y}(a)_{m+k} & \text{for } m \notin \mathbb{Z} \end{cases}$$

Technically, this operator maps $\mathcal{V} \to \bar{\mathcal{V}}$ where $\bar{\mathcal{V}}$ consist of infinite linear combinations. This means that for successive application of $\mathrm{ResY}(a)$-operators one has to check convergence of the explicit infinite series' at hand, depending on some norm of a, as we shall do in the following.

A cleaner course of action would be to introduce a well-behaved subset of the infinite linear combinations involving an L^2-condition, and then prove that $\mathrm{ResY}(a)$ for small a is well-defined and lands again in this space.

3.3. Screening Charge Operators.

We now introduce residue operators associated to certain elements in $\mathcal{V}_\Lambda$:

Definition 3.13. *For* $\alpha \in \Lambda$ *the the* scalar charge operator [5] $Ъ_\alpha$ *is*

$$Ъ_\alpha v := \mathrm{ResY}(\partial\phi_\alpha) v = \mathrm{Y}(\partial\phi_\alpha)_{-1} v = \langle \partial\phi_\alpha, v^{(-1)} \rangle_{-1} v^{(0)}$$

Lemma 3.14. *The following properties clearly hold by definition:*

a) The scalar charge operator is a derivation, *explicitly given by*

$$Ъ_\alpha\; u\mathrm{e}^{\phi_\beta} = (\alpha, \beta)\; u\mathrm{e}^{\phi_\beta} \qquad\qquad u \in U(\Lambda \otimes_{\mathbb{Z}} H)$$

b) It is an action of the additive group Λ*, even of* $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$*, on the VOA* $\mathcal{V}_\Lambda$ *dual to the* Λ*-Yetter-Drinfel'd grading* $\mathcal{V}_\beta$.

$$Ъ_\alpha Ъ_\beta = Ъ_{\alpha+\beta} \qquad Ъ_0 = \mathrm{id}$$

Note that $e^{\pi \mathrm{i}\, Ъ_\alpha}$ *is the* Λ*-Yetter-Drinfel'd action of the group ring on* $\mathcal{V}_\Lambda$.

[4] It is however often instructive to leave the $\hbar$-dependence and in our main proof this is one of the key ideas to regularize the situation. The notion $\hbar$ came up with B. Feigin and the reason becomes more clear when we compute regularized screenings depending on $\hbar$.

[5] pronounce yer-alpha

Definition 3.15. *For* $\alpha \in \Lambda$ *the* momentum charge operator[6] $\mathfrak{Z}_\alpha : \mathcal{V}_\Lambda \to \mathcal{V}_\Lambda$ *is*

$$\mathfrak{Z}_\alpha v := \mathrm{Res}\mathrm{Y}(\mathrm{e}^{\phi_\alpha})v$$

If $(\alpha, \beta) \in \mathbb{Z}$ *then this simplifies to*

$$\mathfrak{Z}_\alpha\ u\mathrm{e}^{\phi_\beta} = \sum_{k\geq 0}\langle \mathrm{e}^{\phi_\alpha}, u^{(0)}\rangle_{-k-(\alpha,\beta)-1}\ u^{(-1)}\ \underbrace{\mathrm{e}^{\phi_\beta}\ \frac{1}{k!}\partial^k \mathrm{e}^{\phi_\alpha}}_{=P_{\alpha,k}\mathrm{e}^{\phi_{\alpha+\beta}}}$$

This is what is commonly known as the screening operator*, although some authors use the term for any* $\mathrm{Res}\mathrm{Y}(a)$ *while others require screening operators to additionally preserve a given choice of a Virasoro action.*

Example 3.16. *By using the explicit series in the preceding section or directly applying the above formula we easily yield:*

$$\begin{aligned}
\mathfrak{Z}_0\ v &= 0\\
\mathfrak{Z}_\alpha\ 1 &= 0\\
\mathfrak{Z}_\alpha\ \partial\phi_\beta &= -(\alpha,\beta)\mathrm{e}^{\phi_\alpha}\\
\mathfrak{Z}_\alpha\ \partial\phi_\beta\partial\phi_\gamma &= \big(-(\alpha,\beta)\partial\phi_\gamma - \partial\phi_\beta(\alpha,\gamma) + (\alpha,\beta)(\alpha,\gamma)\cdot\partial\phi_\alpha\big)\,\mathrm{e}^{\phi_\alpha}\\
\mathfrak{Z}_\alpha\ \mathrm{e}^{\phi_\beta} &= \begin{cases} 0, & \text{if } (\alpha,\beta)\in\mathbb{N}_0\\ \frac{1}{k!}\ \mathrm{e}^{\phi_\beta}\cdot\partial^k.\mathrm{e}^{\phi_\alpha} \quad = P_{\alpha,k}\ \mathrm{e}^{\phi_{\alpha+\beta}} & \text{if } (\alpha,\beta)\in -\mathbb{N}, \quad k := -(\alpha,\beta)-1\\ \sum_{k\geq 0}\frac{e^{2\pi\mathrm{i}(k+(\alpha,\beta)+1)}-1}{2\pi\mathrm{i}(k+(\alpha,\beta)+1)}\cdot P_{\alpha,k}\ \mathrm{e}^{\phi_{\alpha+\beta}} & \text{if } (\alpha,\beta)\notin\mathbb{Z}\end{cases}
\end{aligned}$$

In particular the third equation shows to which extend $\mathfrak{Z}_\alpha$ fails *to be a derivation.*

Note that it is not a-priori true that the series converges, this will be checked below in the concrete cases at hand.

Lemma 3.17. *The following properties hold:*

a) The screening $\mathfrak{Z}_\alpha$ *maps* $\mathcal{V}_\beta \to \mathcal{V}_{\alpha+\beta}$*.*
b) For the commutation rule with the operator ∂ *we have*

$$(\partial\mathfrak{Z}_\alpha)v\ -\ (\mathfrak{Z}_\alpha\partial)v\ =\ \frac{e^{2\pi\mathrm{i}m}-1}{2\pi\mathrm{i}}\left(\sum_m\langle\mathrm{e}^{\phi_\alpha}, u^{(0)}\rangle_m\ u^{(1)}\right)\mathrm{e}^{\phi_\beta}\ \underbrace{\sum_{k\geq 0}\frac{\partial^k}{k!}.\mathrm{e}^{\phi_\alpha}}_{\mathrm{e}^{\phi_\alpha(z+1)}}$$

where $v \in \mathcal{V}_\beta$ *with* $(\alpha,\beta)\mathbb{Z} = [m] \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ *.*
In particular for $(\alpha,\beta)\in\mathbb{Z}$ *the screening* $\mathfrak{Z}_\alpha :\ \mathcal{V}_\beta \to \mathcal{V}_{\alpha+\beta}$ *is* H*-linear.*

---

[6]pronounce zemljá-alpha, where z is a voiced s

c) *For $(\alpha,\beta)\in\mathbb{Z}$ we have the following commutation rules*

$$\begin{aligned}[\text{З}_\alpha,\text{З}_\beta]_q &= \text{З}_\alpha\text{З}_\beta - e^{\pi\mathrm{i}\,(\alpha,\beta)}\,\text{З}_\beta\text{З}_\alpha \\ &= \begin{cases} 0, & (\alpha,\beta)\in\mathbb{N}_0 \\ \mathrm{Res}\mathrm{Y}\big(P_{\alpha,k}\mathrm{e}^{\phi_{\alpha+\beta}}\big), \quad k:=-(\alpha,\beta)-1 & (\alpha,\beta)\in-\mathbb{N}\end{cases}\end{aligned}$$

In particular for $(\alpha,\beta)=-1$ resp. $(\alpha,\alpha)=2$ we obtain[7]

$$\begin{aligned}[\text{З}_\alpha,\text{З}_\beta]_+ &= \text{З}_{\alpha+\beta} \\ [\text{З}_\alpha,\text{З}_{-\alpha}] &= \text{Ъ}_\alpha\end{aligned}$$

d) *We have for arbitrary α and $v\in\mathcal{V}_\beta$:*

$$[\text{Ъ}_\alpha,\mathrm{Y}_m(v)] = (\alpha,\beta)\,\mathrm{Y}_m(v)$$

In particular for arbitrary α,β:

$$[\text{Ъ}_\alpha,\text{З}_\beta] = (\alpha,\beta)\,\text{З}_\beta$$

e) *Let $u\in U(\Lambda\otimes_{\mathbb{Z}}H)$ be any differential polynomial, then $(\alpha,\beta)\in|u|+\mathbb{N}_0$ implies*

$$\text{З}_\alpha\ u\mathrm{e}^{\phi_\beta} = 0$$

For general $(\alpha,\beta)\in\mathbb{Z}$ the screening З_α decreases *the degree of u by $(\alpha,\beta)+1$, in particular the extremal case $(\alpha,\beta)=|u|-1$ yields*

$$\text{З}_\alpha\ u\mathrm{e}^{\phi_\beta} = \underbrace{\langle\mathrm{e}^{\phi_\alpha},u\rangle_{-|u|}}_{\eta_\alpha(u)}\cdot\mathrm{e}^{\phi_{\alpha+\beta}}$$

Proof. We first calculate in general

$$\mathrm{Y}(\mathrm{e}^{\phi_\alpha})\mathrm{e}^{\phi_\beta} = \langle\mathrm{e}^{\phi_\alpha,\phi_\beta}\rangle\cdot\sum_{k\geq 0}\frac{z^k}{k!}\mathrm{e}^{\phi_\beta}\partial^k\mathrm{e}^{\phi_\alpha} = \sum_{k\geq 0} z^{(\alpha,\beta)+k}\cdot P_{k,\alpha}\mathrm{e}^{\phi_{\alpha+\beta}}$$

a) This follows clearly from the general formula.

b) The commutator with ∂ can be calculated quite general; this is the VOA *translation axiom*:

$$\begin{aligned}\mathrm{Y}(a)\ \partial.b &= \sum_{k\geq 0}\langle a^{(0)},\partial^{(2)}.b^{(0)}\rangle\ \partial^{(2)}.b^{(2)}\ \frac{z^k}{k!}\partial^k.a^{(2)} \\ &= \sum_{k\geq 0}\partial^{(3)}.\langle a^{(0)},b^{(0)}\rangle\frac{z^k}{k!}\ \partial^{(0)}.\left(b^{(2)}\ S(\partial^{(2)}).\partial^k.a^{(2)}\right) \\ &= \sum_{k\geq 0}\langle a^{(0)},b^{(0)}\rangle\frac{z^k}{k!}\ \partial.\left(b^{(2)}\ \partial^k.a^{(2)}\right)\end{aligned}$$

[7] The anticommutator comes naturally from $e^{\pi\mathrm{i}(\alpha_i,\alpha_j)}=\pm 1$. To turn it to a commutator we have to deform $\mathcal{V}_\Lambda$ by a 2-cocycle, which yield a truely local super-VOA, in which case the sign were $(-1)^{(\alpha,\alpha)(\beta,\beta)}$

$$-\frac{\partial}{\partial z}\sum_{k\geq 0}\langle a^{(0)},b^{(0)}\rangle\frac{z^k}{k!}\,\left(b^{(2)}\,\partial^k.a^{(2)}\right)$$

The first summand is the asserted $\partial.\mathrm{Y}(a)b$. The residue of the second term is zero for integral powers, but for fractional powers

$$\mathrm{Res}\left(\frac{\partial}{\partial z}z^m\right)=m\mathrm{Res}\left(\frac{\partial}{\partial z}z^{m-1}\right)\qquad\qquad=\frac{e^{2\pi\mathrm{i}m}-1}{2\pi\mathrm{i}}$$

So the residue in question is

$$\begin{aligned}(\partial\text{З}_\alpha)u\mathrm{e}^{\phi_\beta}\;-\;(\text{З}_\alpha\partial)u\mathrm{e}^{\phi_\beta}&\sum_{k\geq 0}\sum_m\langle\mathrm{e}^{\phi_\alpha},u^{(0)}\mathrm{e}^{\phi_\beta}\rangle_m\frac{1}{k!}\,\left(u^{(1)}\mathrm{e}^{\phi_\beta}\,\partial^k.\mathrm{e}^{\phi_\alpha}\right)\\&=\frac{e^{2\pi\mathrm{i}m}-1}{2\pi\mathrm{i}}\left(\sum_m\langle\mathrm{e}^{\phi_\alpha},u^{(0)}\rangle_m\,u^{(1)}\right)\mathrm{e}^{\phi_\beta}\,\sum_{k\geq 0}\frac{\partial^k}{k!}.\mathrm{e}^{\phi_\alpha}\end{aligned}$$

Note that for $\hbar\neq 1$ the last factor is a translation by $+\hbar$.

c) It follows from the usual (integral!) VOA associativity

$$\text{З}_\alpha\text{З}_\beta=\text{З}_\beta\text{З}_\alpha\;+\;\mathrm{Y}(\mathrm{Y}(\mathrm{e}^{\phi_\alpha})_{-1}\mathrm{e}^{\phi_\beta})_{-1}$$

and the preceding formula for $\text{З}_\alpha\mathrm{e}^{\phi_\beta}$.

This implies in particular for $(\alpha,\beta)\notin-\mathbb{N}$ that $\mathrm{Y}(\mathrm{e}^{\phi_\alpha})_{-1}\mathrm{e}^{\phi_\beta}=0$, and for $(\alpha,\beta)=-1$ that $\mathrm{Y}(\mathrm{e}^{\phi_\alpha})_{-1}\mathrm{e}^{\phi_\beta}=\mathrm{e}^{\phi_{\alpha+\beta}}$, which yields in the first formula again a screening operator $\text{З}_{\alpha+\beta}$.

Note that one may derive similar formulae for other values $(\alpha,\beta)\in-\mathbb{N}$; they involve ∂ and the screening operator $\text{З}_{\alpha+\beta}$.

d) We proceed similarly. Note that $\phi_\alpha\in\mathcal{V}_0$ so $(0,\beta)$ is integral and even (so the commutators are usual commutators). Then using the explicit grading action of Ъ_α we get:

$$\begin{aligned}\mathrm{Y}(\phi_\alpha)_{-1}\mathrm{Y}_m(v)-\mathrm{Y}_m(v)\mathrm{Y}(\phi_\alpha)_{-1}&=\mathrm{Y}\left(\mathrm{Y}(\phi_\alpha)_{-1}v\right)_m\\&=Y\left((\alpha,\beta)v\right)_m\end{aligned}$$

e) Since e^{ϕ_α} is grouplike the expression $\langle\mathrm{e}^{\phi_\alpha},u\rangle$ is multiplicative in u, and it is anyways H-linear, thus $\sim z^{-|u|}$ (with coefficients product of factorials). Applying the definition of З_α and $\langle\mathrm{e}^{\phi_\alpha},\mathrm{e}^{\phi_\beta}\rangle=z^{(\alpha,\beta)}$ yields:

$$\text{З}_\alpha\,u\mathrm{e}^{\phi_\beta}=\sum_{k\geq 0}\langle\mathrm{e}^{\phi_\alpha},u^{(0)}\mathrm{e}^{\phi_\beta}\rangle_{-k-1}\,u^{(2)}\mathrm{e}^{\phi_\beta}\,\frac{1}{k!}\partial^k\mathrm{e}^{\phi_\alpha}$$

$$= \sum_{k\geq 0} \langle \mathrm{e}^{\phi_\alpha}, u^{(0)} \rangle_{-k-1-(\alpha,\beta)}\; u^{(2)} \mathrm{e}^{\phi_\beta}\; \frac{1}{k!} \partial^k \mathrm{e}^{\phi_\alpha}$$

So for non-vanishing terms we need $|u^{(0)}| = -k-1-(\alpha,\beta)$ together with $k \geq 0$, which is only possible for $(\alpha,\beta) \notin |u| - \mathbb{N}$. In the extremal case $(\alpha,\beta) = |u| - 1$ it can only be achieved for $k = 0$ and $|u^{(0)}| = |u|$, which means the summand $u^{(0)} \otimes u^{(2)} = u \otimes 1$.

$\square$

3.4. **Example: Lie algebra at Trivial Level.**

The following well-known example only requires the non-fractional relations above, but the statement and proof contain in a nutshell part of what's ahead:

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of rank n with Cartan subalgebra $\mathfrak{h}$ and $\Lambda = \Lambda_R(\mathfrak{g})$ be its root lattice and Φ its set of roots. Choose a basis of positive simple roots $\alpha_1, ..., \alpha_n$ and hence a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}^+$ and a set of positive roots Φ^+ in $\mathfrak{h}^*$.

Proposition 3.18.
The screening operators and the additive representation of $\Lambda_R \subset \mathfrak{h}^*$

$$\text{З}_\alpha,\ \alpha \in \Phi^+ \qquad\qquad \text{Ъ}_\lambda,\ \lambda \in \Lambda_R$$

generate the universal enveloping algebra of the Borel part $U(\mathfrak{b})$ *inside* $\mathrm{End}(\mathcal{V}_\Lambda)$.

For $\mathfrak{g}$ *simply-laced we can even realize the entire universal enveloping algebra* $U(\mathfrak{g})$*. Moreover* $\mathcal{V}_\Lambda$ *is isomorphic to the universal enveloping algebra of the loop Lie algebra* $U(\mathfrak{g}[z, z^{-1}])$*, which is the affine Lie algebra at trivial level.*

If $\mathfrak{g}$ is *simply laced* the proof is almost trivial when using all our basic properties in Lemma 3.17. We identify

$$E_\alpha := \text{З}_\alpha,\ \alpha \in \Phi^+ \qquad F_\alpha := \text{З}_\alpha,\ \alpha \in \Phi^- \qquad H_\lambda := \text{Ъ}_\lambda,\ \lambda \in \mathfrak{h}^*$$

Then the important cases in Lemma 3.17 are:

- $(\alpha,\beta) = -1$, then $[\text{З}_\alpha, \text{З}_\beta]_+ = \text{З}_{\alpha+\beta}$
- $(\alpha,\beta) = 0, 1$, then $[\text{З}_\alpha, \text{З}_\beta]_\pm = 0$
- $(\alpha,-\alpha) = -2$, then $[\text{З}_\alpha, \text{З}_{-\alpha}] = \text{Ъ}_\alpha$
- Moreover we have $[\text{Ъ}_\lambda, \text{З}_\alpha] = (\lambda,\alpha)\, \text{З}_\alpha$

If $\mathfrak{g}$ is *not-simply laced* we encounter more cases and the relation between E, F gets destroyed, so one has to decide for one Borel part. This is the usual case as we proceed further to the fractional case. .

4. Fractional Associativity Formula

We first recall how *associativity* and from this *locality* were proven in our general approach [Len07] Sec 4.2 and Sec. 4.3.4 (although technically we shall keep independent of these results):

We try to simplify the concatenation of two vertex operators with our formulae:

$$Y(a)_z Y(b)_w v = Y(a)_z \sum_{k\in\mathbb{N}_0} \langle b^{(-1)}, v^{(-1)}\rangle_w \cdot v^{(0)} \cdot w^{k_b}\,\frac{\partial^k}{k!}.b^{(0)}$$

$$= \sum_{k_a\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \langle b^{(-1)}, v^{(-1)}\rangle_w w^k \cdot \langle a^{(-1)}, \left(v^{(0)}\frac{\partial^k}{k!}.b^{(0)}\right)^{(-1)}\rangle_z \cdot \left(v^{(0)}\frac{\partial^k}{k!}.b^{(0)}\right)^{(0)} \cdot z^{k_a}\frac{\partial^{k_a}}{k_a!}.a^{(0)}$$

$$= \sum_{k_a\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \langle b^{(-1)}, v^{(-2)}\rangle_w w^k \cdot \sum_{k_{ab}+k_b=k} \langle a^{(-2)}, v^{(0)}\rangle_z \langle a^{(-1)}, \frac{\partial^{k_{ab}}}{k_{ab}!}.b^{(-1)}\rangle_z z^{k_a} \cdot v^{(0)} \cdot \frac{\partial^{k_b}}{k_b!}.b^{(0)} \cdot \frac{\partial^{k_a}}{k_a!}.a^{(0)}$$

$$= \sum_{k_a,k_b\in\mathbb{N}_0} \langle b^{(-1)}, v^{(-2)}\rangle_w w^{k_b} \cdot \langle a^{(-2)}, v^{(-1)}\rangle_z z^{k_a} \cdot v^{(0)} \cdot \frac{\partial^{k_b}}{k_b!}.b^{(0)} \cdot \frac{\partial^{k_a}}{k_a!}.a^{(0)}$$

$$\cdot \left(\sum_{k_{ab}\in\mathbb{N}_0} \frac{w^{k_{ab}}}{k_{ab}!}(-\frac{\partial}{\partial z})^{k_{ab}} \langle a^{(-1)}, b^{(-1)}\rangle_z\right)$$

We see two independent vertex operators in the second-last line and an interaction term between a, b and z, w in the last line. If we assume for simplicity commutativity and co-commutativity, then it is this last term, that will prevent the endomorphismsm $Y(a), Y(b)$ from commuting.

Remark 4.1. *On the level of meromorphic functions we can use a relation like*

$$\sum_{k_{ab}\in\mathbb{N}_0} \frac{w^{k_{ab}}}{k_{ab}!}(-\frac{\partial}{\partial z})^{k_{ab}}\ f(z) = e^{-w\frac{\partial}{\partial z}} f(z) = f(z-w)$$

Then locality holds i.e. $Y(a), Y(b)$ *commute iff* $f(t) = \langle a, b\rangle_t = \langle b, a\rangle_{-t}$.

However, on the level of series there is no equality and the difference is measured by derivations of the omnipresent delta-function, e.g. for $f(t) = t^{-1}$*:*

$$\sum_{k_{ab}\in\mathbb{N}_0} \frac{w^{k_{ab}}}{k_{ab}!}(-\frac{\partial}{\partial z})^{k_{ab}} z^{-1} - \sum_{k_{ba}\in\mathbb{N}_0} \frac{z^{k_{ba}}}{k_{ba}!}(-\frac{\partial}{\partial w})^{k_{ba}}(-w^{-1})$$
$$= \sum_{k_{ab}\in\mathbb{N}_0} w^{k_{ab}} z^{-1-k_{ab}} + \sum_{k_{ba}\in\mathbb{N}_0} z^{k_{ba}} w^{-1-k_{ba}} = \delta(z-w)$$

So one could say, "up to delta-function" means being allowed to identify translations

$$e^{w\frac{\partial}{\partial z}}f(z) = f(z+w) = e^{z\frac{\partial}{\partial w}}f(w)$$

All of these considerations work equally well in our generalized setting in [Len07], *including fractional powers.*

In VOA theory and our upcoming calculations we need moreover to express say the residue of the right-hand-side. For *integral powers* only one value of k_{ab} resp. k_{ba} contribute, but for *fractional powers* all powers contribute to the residue (see Definition 3.12), so things become much more involved and we have an infinite series to calculate. We first record our result so far using the notation $\langle -,- \rangle_m$ for the coefficient of z^m (only finitely many m contribute).

Lemma 4.2. *The associativity formula above for general vertex algebras in our Hopf algebra VOA framework implies the following* OPE associativity formula *for general residue operators: We have*

$$\begin{aligned}
&\mathrm{Res}\mathrm{Y}(a)\mathrm{Res}\mathrm{Y}(b)v \\
&= \sum_{\substack{k_b,k_a,k_{ab}\in\mathbb{N}_0\\ m_b,m_a,m_{ab}}} \langle b^{(-2)}, v^{(-2)}\rangle_{m_b} \langle a^{(-2)}, v^{(-1)}\rangle_{m_a} \langle a^{(-1)}, b^{(-1)}\rangle_{m_{ab}} \cdot v^{(0)} \frac{\partial.^{k_b}}{k_b!} b^{(0)} \frac{\partial.^{k_a}}{k_a!} a^{(0)} \\
&\cdot \underbrace{\sum_{k_{ab}\in\mathbb{N}_0} \mathrm{Res}\left(w^{m_b+k_b}\cdot w^{k_{ab}}\right) \mathrm{Res}\left(z^{m_a+k_a}\cdot \frac{1}{k_{ab}!}(-\frac{\partial}{\partial z})^{k_{ab}} z^{m_{ab}}\right)}_{=:\,\mathrm{F}_-(m_a+k_a,m_b+k_b;m_{ab})\,\in\mathbb{C}}
\end{aligned}$$

and the similar expression for the other bracketing order:

$$\begin{aligned}
&\mathrm{Res}\left(\mathrm{Y}\left(\mathrm{Res}\left(\mathrm{Y}(a)b\right)\right)\right)v \\
&= \sum_{\substack{k_b,k_a\in\mathbb{N}_0\\ m_b,m_a,m_{ab}}} \langle a^{(-2)}, b^{(-2)}\rangle_{m_{ab}} \langle a^{(-1)}, v^{(-2)}\rangle_{m_a} \langle b^{(-1)}, v^{(-1)}\rangle_{m_b} \cdot v^{(0)} \frac{\partial.^{k_b}}{k_b!} b^{(0)} \frac{\partial.^{k_a}}{k_a!} a^{(0)} \\
&\cdot \underbrace{\sum_{k_{av}\in\mathbb{N}_0} \mathrm{Res}\left(t^{m_{ab}}\cdot t^{k_{av}}\right) \mathrm{Res}\left(w^{m_b+b_k}\cdot \frac{1}{k_{av}!}(+\frac{\partial}{\partial w})^{k_{av}} w^{m_a+k_a}\right)}_{=:\,\mathrm{F}_+(m_a+k_a,m_b+k_b;m_{ab})\,\in\mathbb{C}}
\end{aligned}$$

In the next section we will study the structure constants $\mathrm{F}_\pm(m_a,m_b;m_{ab})$ *further.*

The obvious generalization to n-fold expressions is:

Theorem 4.3.

$$
\begin{aligned}
&\left(\prod_{i=1}^{n} \mathrm{ResY}(a_i)\right)\ v \\
&= \sum_{(k_i)_i\in\mathbb{N}_0^n} \sum_{(m_i,m_{ij})_{i,j}} \prod_{1\leq i\leq n} \langle a_i^{(-n)}, v^{(-i)}\rangle_{m_i} \prod_{1\leq i<j\leq n} \langle a_i^{-(j-1)}, a_j^{(-i)}\rangle_{m_{ij}} \cdot v^{(0)} \prod_{i=n}^{1} \frac{\partial^{k_i}}{k_i!} a_i^{(0)} \\
&\underbrace{\cdot \prod_i \mathrm{Res}\left(z_i^{(m_i+k_i)+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\right) \prod_{i<j} (\pm 1)^{k_{ij}} \binom{m_{ij}}{k_{ij}}}_{=:\ \mathrm{F}_-((m_i+k_i,m_{ij})_{ij})\ \in\mathbb{C}}
\end{aligned}
$$

where the product $\prod_{i=n}^{1}$ means to be taken in reversed order.

5. Quantum Symmetrizer Formula

5.1. **Quantum Monodromy Numbers.** Having discussed the Hopf algebra part, we now turn to the structure constants $\mathrm{F}_{\pm}((m_i, m_{ij})_{ij})$ in question. The following section consists purely of analysis. Let us begin with the case $n = 2$:

Definition 5.1. *We shall call the complex numbers* $\mathrm{F}_{\pm}(m_a, m_b; m_{ab})$ *the* quantum monodromy numbers*:*

$$
\begin{aligned}
\mathrm{F}_-(m_a, m_b; m_{ab}) &:= \sum_{k\in\mathbb{N}_0} \mathrm{Res}\left(w^{m_b}\cdot w^k\right) \mathrm{Res}\left(z^{m_a}\cdot \frac{1}{k!}(-\frac{\partial}{\partial z})^k z^{m_{ab}}\right) \\
&= \sum_{k\in\mathbb{N}_0} \mathrm{Res}\left(w^{m_b+k}\right) \mathrm{Res}\left(z^{m_a+m_{ab}-k}\right) (-1)^k \binom{m_{ab}}{k} \\
\mathrm{F}_+(m_a, m_b; m_{ab}) &:= \sum_{k\in\mathbb{N}_0} \mathrm{Res}\left(t^{m_{ab}}\cdot t^k\right) \mathrm{Res}\left(w^{m_b}\cdot \frac{1}{k!}(+\frac{\partial}{\partial w})^k w^{m_a}\right) \\
&= \sum_{k\in\mathbb{N}_0} \mathrm{Res}\left(t^{m_{ab}+k}\right) \mathrm{Res}\left(w^{m_b+m_a-k}\right) \binom{m_a}{k}
\end{aligned}
$$

They can be studied (as is done frequently) for $m_a, m_b, m_b \in \mathbb{Z}$ by contour integration of the function $f := z^{m_a}w^{m_b}t^{m_{ab}},\ t = z - w$ over a suitable 2-cycles in $\mathbb{C}^2$ (a torus). For fractional values however, the contour path has to be suitably lifted to the covering has to be chosen, which produces very interesting and subtile effects. We will make this precise in the next section.

Example 5.2. *The following are (partly)* integral cases $m_b,\ m_a+m_{ab}\in\mathbb{Z}$*, which we'll usually exclude later*[8]*. The first formula leads to the fundamental identity in ordinary (i.e. non-fraction) VOA theory, in particular the familiar associativity.*

For $m_b,\ m_a+m_{ab}\in\mathbb{Z}$ *we get*

$$\mathrm{F}_\pm(m_a,m_b;m_{ab})=\begin{cases}(\pm1)^{m_b+1}\binom{m_{ab}}{-m_b-1}\\ =(\pm1)^{m_a+m_{ab}+1}\binom{m_{ab}}{m_a+m_{ab}+1}, & \text{if } m_a+m_b+m_{ab}+2=0 \text{ and } -m_b-1\geq 0\\ 0, & \text{else}\end{cases}$$

For $m_b\in\mathbb{Z},\ m_a+m_{ab}\notin\mathbb{Z}$ *we get*

$$\mathrm{F}_\pm(m_a,m_b;m_{ab})=\begin{cases}\frac{(e^{2\pi\mathrm{i}\,(m_a+m_{ab})}-1)/2\pi\mathrm{i}}{m_a+m_b+m_{ab}+2}(\pm1)^{m_b+1}\binom{m_{ab}}{-m_b-1}, & \text{if } -m_b-1\geq 0\\ 0, & \text{else}\end{cases}$$

For $m_b\notin\mathbb{Z},\ m_a+m_{ab}\in\mathbb{Z}$ *we get similarly*

$$\mathrm{F}_\pm(m_a,m_b;m_{ab})=\begin{cases}\frac{(e^{2\pi\mathrm{i}\,m_b}-1)/2\pi\mathrm{i}}{m_a+m_b+m_{ab}+2}(\pm1)^{m_a+m_{ab}+1}\binom{m_{ab}}{m_a+m_{ab}+1}, & \text{if } m_a+m_b+m_{ab}+2\geq 0\\ 0, & \text{else}\end{cases}$$

The nontrivial case for arguments is full *fracturedness* $m_b,\ m_a+m_{ab}\notin\mathbb{Z}$:

$$\mathrm{F}_\pm(m_a,m_b;m_{ab})=\frac{e^{2\pi\mathrm{i}\,m_b}-1}{2\pi\mathrm{i}}\cdot\frac{e^{2\pi\mathrm{i}\,(m_a+m_{ab})}-1}{2\pi\mathrm{i}}\sum_{k\in\mathbb{N}_0}\frac{(\pm1)^k\binom{m_{ab}}{k}}{(m_b+k+1)(m_a+m_{ab}-k+1)}$$

In general the relevant structure constants were:

Definition 5.3. *We define the* quantum monodromy number $\mathrm{F}_\pm((m_i,m_{ij})_{ij})$ *for real-valued arguments* $(m_i,m_{ij})_{1\leq i<j\leq n}$

$$\mathrm{F}_\pm((m_i,m_{ij})_{ij}):=\sum_{(k_{ij})_{ij}\in\mathbb{N}_0^{\binom{n}{2}}}\prod_i\mathrm{Res}\left(z_i^{m_i+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\right)\prod_{i<j}(\pm1)^{k_{ij}}\binom{m_{ij}}{k_{ij}}$$

Assuming fracturedness $m_i+\sum_{i<j}m_{ij}\notin\mathbb{Z}$ *this is explicitly the series:*

$$\mathrm{F}_\pm((m_i,m_{ij})_{ij})=\sum_{(k_{ij})_{ij}\in\mathbb{N}_0^{\binom{n}{2}}}\prod_i\frac{(e^{2\pi\mathrm{i}(m_i+\sum_{i<j}m_{ij})}-1)/2\pi\mathrm{i}}{1+m_i+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\prod_{i<j}(\pm1)^{k_{ij}}\binom{m_{ij}}{k_{ij}}$$

[8] This is correspondingly the case where one of the vertex operators $b\otimes v$ resp. $a\otimes\mathrm{ResY}(b)v$ has a symmetric braiding, i.e. bosonic or fermionic. In contrast the case $m_{ab}\in\mathbb{Z}$ may be trivial in the sense that it has only finitely many nontrivial terms, but will not be discussed seperately - it means that $a\otimes b$ has symmetric braiding.

This serie expresses a generalized hypergeometric functions at the boundary of the convergence disc $z = \mp 1$. For $n = 1$ we have $\mathrm{F}_{\pm}(m_1) = \frac{(e^{2\pi \mathrm{i} m_1}-1)/2\pi\mathrm{i}}{1+m_1}$. For $n = 2$ this view gives a very useful formula in terms of two Beta functions (see Example 5.13), but for $n > 2$ the author was unable to derive a simpler expression from this.

Convergence of this series in terms of the m_{ij} is rather subtile. The series is absolutely convergent e.g. if all $m_{ij} \geq 0$ (and there for any m_i), but this is too restrictive for our purposes. We can derive a sufficient condition for conditional convergence after our main theorem in Lemma 5.14.

From a physical perspective it is not tragic if the series does converge, we should rather be interested in the full analytic continuation of these structure constants (e.g. Fadeev); although then we can not strictly speak of an algebra of screening operators acting. The physically interesting information is the maximal domain of definition and the location of the poles.

Problem 5.4. *Analytically continue* $\mathrm{F}_{\pm}(m_i, m_{ij})$ *as function in all parameters! The proof of our main theorem provides an analytic continuation to negative* m_{ij} *bounded from below, see Corollary 5.12, which is sufficient for the present article.*

5.2. **Generalized Selberg Integrals.**

Definition 5.5. *The* generalized n-fold Selberg integral *for real parameters be*

$$\mathrm{Sel}((m_i;\bar{m}_i;m_{ij})_{i<j}) := \int \cdots \int_{1 \geq z_1 > \ldots > z_n \geq 0} \mathrm{d}z_1 \cdots \mathrm{d}z_n \prod_i z_i^{m_i} \prod_i (1-z_i)^{\bar{m}_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}$$

This integral converges if the following three conditions are fulfilled:

- *All* $\sum_{r \leq i < j \leq s} m_{ij} > -(s-r)$.
- *All* $\sum_{i \leq r} \bar{m}_i + \sum_{j < i \leq r} m_{ji} > -r$
- *All* $\sum_{r \leq i} m_i + \sum_{r \leq i < j} m_{ij} > -(n-r+1)$.

We will only require cases $\bar{m}_i = 0$. Substituting z_i/z_1 and integrating out z_1 reduces this to:

$$\mathrm{Sel}((m_i;0;m_{ij})_{i<j}) = \frac{1}{n + \sum_i m_i + \sum_{i<j} m_{ij}} \mathrm{Sel}((m_i;\bar{m}_{1i};m_{ij})_{1 \neq i<j})$$

There are some evident recursions by expanding some $(z_a - z_b)^1$ resp. by partial integration, which we could not put to much use.

Example 5.6. *For* $k = 1$ *this is Euler's Beta integral*

$$\mathrm{Sel}(m_1;\bar{m}_1) = \mathrm{B}(m_1+1, \bar{m}_1+1)$$

$$\mathrm{Sel}(m_1, m_2; 0, m_{12}) = \frac{1}{2 + m_1 + m_2 + m_{12}} \mathrm{B}(m_2 + 1, m_{12} + 1)$$

All equal $m_i = a - 1,\ \bar{m}_i = b - 1,\ m_{ij} = 2c$ *gives the famous Selberg integral* [Sel44]

$$\mathrm{Sel}(a-1; b-1; 2c) = \frac{1}{k!} \prod_{j=0}^{k-1} \frac{\Gamma(a + jc)\Gamma(b + jc)\Gamma(1 + (j+1)c)}{\Gamma(a + b + (k + j - 1)c)\Gamma(1 + c)}$$

For equal m_{ij} *and equal* $m_i, \bar{m}_i \bmod \mathbb{Z}$ *additional Jack polynomials appear*[9].

Problem 5.7. *Again, find a full analytic continuation!*

We remark that the poles in m_i (which we will later find) will cause exceptionally non-zero terms that make up the Weyl reflections in Section 6.2. Analytically continuing in the variables m_{ij} should on the other hand reveal poles that cause the Nichols algebra relations below to fail and catch additional terms in a Lie algebra. This makes up the non-trivial Nichols algebra extensions present in Liouville models (and affine Lie algebras).

Accordingly, we will overcome the two latter convergence conditions for the integral depending on m_i in Corollary 5.12, but the first condition on the m_{ij} is severe and characterizes the situation we are dealing with in the present article:

Definition 5.8. *We say real parameters* $(m_{ij})_{i<j} \in \mathbb{R}^{\binom{n}{2}}$ *fulfill* smallness, *if for any subset* J *of the index set the following unequality holds*

$$\sum_{i<j,\ i,j \in J} m_{ij} > -|J| + 1$$

In particular all $m_{ij} > -1$.

Lemma 5.9. *If* $\alpha_1, \ldots \alpha_n$ *are in a positive-definite euclidean vectorspace with* $||\alpha_i|| < 2$, *then* $m_{ij} := (\alpha_i, \alpha_j)$ *fulfills smallness.*

Proof. Without loss of generality we assume $J = \{1, \ldots, n\}$. Then

$$\sum_{i<j} m_{ij} = \frac{1}{2}\sum_{i,j} m_{ij} - \frac{1}{2}\sum_{i} m_{ii} = \frac{1}{2}||\sum_{i} \alpha_i|| - \frac{1}{2}\sum_{i} ||\alpha_i|| > 0 - n$$

□

[9]This explains the appearing of Selberg integrals and Jack polynomials in [TW13]

5.3. **Main Theorem.** We have defined the quantum monodromy numbers $\mathrm{F}_\pm$ by a series, related to generalized hypergeometric function. In addition, from their interpretation as a residue they inherit a representation as an n-fold integral of a multivalued n-variable function along a specific suitable lift $[0,2\pi]^n \to (S^1)^n$ of an n-torus to the multivalued covering, *morally*:

$$\mathrm{F}_\pm((m_i,m_{ij})_{ij}) = \int \cdots \int_{[e^0,e^{2\pi}]^n} \mathrm{d}z_1 \ldots \mathrm{d}z_n \prod_i z_i^{m_i} \prod_{i<j} (z_i \pm z_j)^{m_{ij}}$$

We will make this precise below. In contrast to F_+, the expression for F_- is *not* symmetric in the variables z_i, not even up to factors $e^{\pi \mathrm{i} m_{ij}}$ (due to the lift). We will however now prove a substantially more subtile and interesting algebraic symmetry property that holds for quantum monodromy numbers F_-. Recall Definition 2.1:

Definition 5.10. *Given a set* $|I| = \mathrm{rank}$ *and real numbers* $m_{ij}, i,j \in I$*, associate to it the following* diagonal braiding matrix

$$q_{ij} := e^{\pi \mathrm{i} m_{ij}}$$

Given a set $X = \{1,\ldots,n\}$ *and each number in* X *somehow colored by an element in* I*, then we may write* q_{ij} *for* $i,j \in X$*. Now for each* X*-permuation* $\sigma \in \mathbb{S}_n$ *we define a scalar called* braiding factor $q(\sigma) \in \mathbb{C}^\times$ *inductively from transpositions:*

$$q(\mathrm{id}) = 1, \qquad q((x,x+1)) = q_{x,x+1}, \qquad q(\sigma\sigma') = q(\sigma)q(\sigma')$$

whenever the product has full length $\ell(\sigma\sigma') = \ell(\sigma) + \ell(\sigma')$*; in general* q *is* not *a group homomorphism from* $\mathbb{S}_n$*. Define the* quantum symmetrizer *in the groupring*

$$Ш_q := \sum_{\sigma \in \mathbb{S}_n} q(\sigma)\ \sigma \ \in \mathbb{C}[\mathbb{S}_n]$$

Our main result states: Quantum monodromy numbers are equal to the Quantum symmetrizers of Selberg integrals:

Theorem 5.11. *Assume real parameters* m_i, m_{ij} *for which that the series* F_- *and the integral* Sel *converges (e.g. all* $m_i, m_{ij} \geq 0$*). Then the following* quantum symmetrizer formula *holds.*

$$\boxed{\mathrm{F}_-((m_i,m_{ij})_{ij}) = \textstyle\sum_{\sigma \in \mathbb{S}_n} q(\sigma)\ \tilde{\mathrm{F}}_-((m_{\sigma^{-1}(i)}, m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij}) =: Ш_q\ \tilde{\mathrm{F}}_-((m_i,m_{ij})_{ij})}$$

with respect to the braiding matrix $q_{ij} = e^{\pi \mathrm{i}\, m_{ij}}$ *and the braiding factor* $q(\sigma)$ *above.*

Here the reduced quantum monodromy numbers $\tilde{\mathrm{F}}_-((m_i, m_{ij})_{ij})$ *can be expressed explicitly in terms of* 2^n *generalized Selberg integrals (see above) as follows*

$$\tilde{\mathrm{F}}_-((m_i;m_{ij})_{ij}) := \frac{1}{(2\pi\mathrm{i})^n}\sum_{k=0}^{n}(-1)^k\left(\prod_{i=k+1}^{n} e^{2\pi\mathrm{i}\, m_i}\right)\sum_{\eta\in\mathbb{S}_{k,\overline{n-k}}}\left(\prod_{i<j,\ \eta(i)>\eta(j)} e^{\pi\mathrm{i}\, m_{ij}}\right)$$
$$\cdot\, \mathrm{Sel}((m_{\eta^{-1}(i)};0;m_{\eta^{-1}(i)\eta^{-1}(j)})_{ij})$$

where we define in slight variation to familiar $(k, n-k)$-shuffles

$$\mathbb{S}_{k,\overline{n-k}} := \{\eta\in\mathbb{S}_n \mid \forall_{i<j\le k}\ \eta(i)<\eta(j)\ \textit{and}\ \forall_{k<i<j}\ \eta(i)>\eta(j)\}\qquad |\mathbb{S}_{k,\overline{n-k}}| = \binom{n}{k}$$

Corollary 5.12 (Analytic continuation)**.** *The n-fold complex integral above (made precise in the proof) is clearly well-defined whenever the m_{ij} fulfill smallness (Definition 5.8) and for all m_i. This is the maximal analytic continuation we achieve in this article for* F_- *and* $\tilde{\mathrm{F}}_-$ *(the function* Sel *has additional poles inside this domain).*

Then the quantum symmetrizer formula holds for these analytic continuations.

The proof will require the next subsection. Before, let us discuss the result:

Example 5.13. *For $n = 2$ we get:*

$$\begin{aligned}
\tilde{\mathrm{F}}_-(m_1,m_2,m_{12}) &= \frac{1}{(2\pi\mathrm{i})^2}\Big(e^{2\pi\mathrm{i}(m_1+m_2)+\pi\mathrm{i}m_{12}}\,\mathrm{Sel}(m_2,m_1;0;m_{12}) - e^{2\pi\mathrm{i}m_2}\,\mathrm{Sel}(m_1,m_2;0;m_{12})\\
&\qquad - e^{2\pi\mathrm{i}m_2+\pi\mathrm{i}m_{12}}\,\mathrm{Sel}(m_2,m_1;0;m_{12}) + e^0\,\mathrm{Sel}(m_1,m_2;0;m_{12})\Big)\\
&= \frac{1}{(2\pi\mathrm{i})^2}\left(1-e^{2\pi\mathrm{i}m_2}\right)\frac{\mathrm{B}(m_2+1,m_{12}+1)}{m_1+m_2+m_{12}+2}\\
&- \frac{1}{(2\pi\mathrm{i})^2}e^{2\pi\mathrm{i}m_2+\pi\mathrm{i}m_{12}}\left(1-e^{2\pi\mathrm{i}m_1}\right)\frac{\mathrm{B}(m_1+1,m_{12}+1)}{m_1+m_2+m_{12}+2}\\
\mathrm{F}_-(m_1,m_2,m_{12}) &= \tilde{\mathrm{F}}_-(m_1,m_2,m_{12}) + e^{\pi\mathrm{i}m_{12}}\cdot\tilde{\mathrm{F}}_-(m_2,m_1,m_{12})\\
&= \frac{1}{(2\pi\mathrm{i})^2}\left(\left(1-e^{2\pi\mathrm{i}m_2}\right) - e^{\pi\mathrm{i}m_{12}}\cdot e^{2\pi\mathrm{i}m_1+\pi\mathrm{i}m_{12}}\left(1-e^{2\pi\mathrm{i}m_2}\right)\right)\frac{\mathrm{B}(m_2+1,m_{12}+1)}{m_1+m_2+m_{12}+2}\\
&+ \frac{1}{(2\pi\mathrm{i})^2}\left(-e^{2\pi\mathrm{i}m_2+\pi\mathrm{i}m_{12}}\left(1-e^{2\pi\mathrm{i}m_1}\right) + e^{\pi\mathrm{i}m_{12}}\cdot\left(1-e^{2\pi\mathrm{i}m_1}\right)\right)\frac{\mathrm{B}(m_1+1,m_{12}+1)}{m_1+m_2+m_{12}+2}\\
&= \frac{e^{2\pi\mathrm{i}m_2}-1}{2\pi\mathrm{i}}\,\frac{e^{2\pi\mathrm{i}m_1+2\pi\mathrm{i}m_{12}}-1}{2\pi\mathrm{i}}\,\frac{1}{m_1+m_2+m_{12}+2}\cdot\\
&\cdot\left(\mathrm{B}(m_2+1,m_{12}+1) + \frac{\sin\pi m_1}{\sin\pi(m_1+m_{12})}\mathrm{B}(m_1+1,m_{12}+1)\right)
\end{aligned}$$

This particular result can be obtained by-hand from the series for F_- *by partial fraction decomposition and using twice Eulers reflection formula on the second Beta-function summand, which produces the sine-terms. This was the author's first clue.*

Some quick numerical examples using SAGE:

$$
\begin{array}{rcl}
\hline
\mathrm{F}_-(+\frac{1}{3},+\frac{1}{5},+\frac{1}{7}) &=& -0.0148+\mathrm{i}\,0.0240\\
&=& (-0.0007+\mathrm{i}\,0.0161)+e^{\pi \mathrm{i}\,\frac{1}{7}}\,(-0.0093+\mathrm{i}\,0.0132)\\
&=& \tilde{\mathrm{F}}_-(+\frac{1}{3},+\frac{1}{5},+\frac{1}{7})+e^{\pi \mathrm{i}\,\frac{1}{7}}\,\tilde{\mathrm{F}}_-(+\frac{1}{5},+\frac{1}{3},+\frac{1}{7})\\
\hline
\mathrm{F}_-(+\frac{1}{7},+\frac{1}{7},+1) &=& 0\\
&=& (-0.0038+\mathrm{i}\,0.0030)+e^{\pi \mathrm{i}}\,(0.0038+\mathrm{i}\,0.0030)\\
&=& \tilde{\mathrm{F}}_-(+\frac{1}{7},+\frac{1}{7},+1)+e^{\pi \mathrm{i}}\,\tilde{\mathrm{F}}_-(+\frac{1}{7},+\frac{1}{7},+1)\\
\mathrm{F}_-(+\frac{8}{7},+\frac{1}{7},+1) &=& 0.0007+\mathrm{i}\,0.0009\\
&=& (-0.0016+\mathrm{i}\,0.0020)+e^{\pi \mathrm{i}}\,(-0.0023+\mathrm{i}\,0.0011)\\
&=& \tilde{\mathrm{F}}_-(+\frac{8}{7},+\frac{1}{7},+1)+e^{\pi \mathrm{i}}\,\tilde{\mathrm{F}}_-(+\frac{1}{7},+\frac{8}{7},+1)\\
\mathrm{F}_-(+\frac{1}{7},+\frac{8}{7},+1) &=& -0.0007-\mathrm{i}\,0.0009\\
&=& (-0.0023+\mathrm{i}\,0.0011)+e^{\pi \mathrm{i}}\,(-0.0016+\mathrm{i}\,0.0020)\\
&=& \tilde{\mathrm{F}}_-(+\frac{1}{7},+\frac{8}{7},+1)+e^{\pi \mathrm{i}}\,\tilde{\mathrm{F}}_-(+\frac{8}{7},+\frac{1}{7},+1)\\
\hline
\mathrm{F}_-(-\frac{1}{3},-\frac{1}{3},+\frac{2}{3}) &=& 0\\
&=& 0+e^{\pi \mathrm{i}\,\frac{2}{3}}\,0\\
&=& \tilde{\mathrm{F}}_-(-\frac{1}{3},-\frac{1}{3},+\frac{2}{3})+e^{\pi \mathrm{i}\,\frac{2}{3}}\,\tilde{\mathrm{F}}_-(-\frac{1}{3},-\frac{1}{3},+\frac{2}{3})\\
\mathrm{F}_-(+\frac{2}{3},-\frac{1}{3},+\frac{2}{3}) &=& -0.0185\\
&=& (-0.0092-\mathrm{i}\,0.0053)+e^{\pi \mathrm{i}\,\frac{2}{3}}\,(0.0092+\mathrm{i}\,0.0053)\\
&=& \tilde{\mathrm{F}}_-(+\frac{2}{3},-\frac{1}{3},+\frac{2}{3})+e^{\pi \mathrm{i}\,\frac{2}{3}}\,\tilde{\mathrm{F}}_-(-\frac{1}{3},+\frac{2}{3},+\frac{2}{3})\\
\mathrm{F}_-(-\frac{1}{3},+\frac{2}{3},+\frac{2}{3}) &=& 0.0185\\
&=& (-0.0092-\mathrm{i}\,0.0053)+e^{\pi \mathrm{i}\,\frac{2}{3}}\,(0.0092+\mathrm{i}\,0.0053)\\
&=& \tilde{\mathrm{F}}_-(-\frac{1}{3},+\frac{2}{3},+\frac{2}{3})+e^{\pi \mathrm{i}\,\frac{2}{3}}\,\tilde{\mathrm{F}}_-(+\frac{2}{3},-\frac{1}{3},+\frac{2}{3})\\
\hline
\end{array}
$$

The main implication is the mere *existence* of such a quantum symmetrizer formula (not the expression for $\tilde{\mathrm{F}}_-$). It implies: Wherever we have a formal linear combination of $(m_i, m_{ij})_{ij}$ such that their quantum symmetrizer vanishes, then the resp. linear combination of $\mathrm{F}_-((m_i, m_{ij})_{ij})$ vanishes. 6.1, this later implies the main result of this article in Theorem 6.1:

> "Quantum monodromy numbers fulfill the Nichols algebra relations."
> "As does any ordinary VOA associativity generate a Lie algebra,
> so does any fractional VOA associativity generate a Nichols algebra."

We remark that the result is indeed *wrong* for $|\alpha_i| \geq 2$, where the smallness condition in the theorem is not met. E.g. for the trivial Lie algebra case $|\alpha_i| = 2$ the Nichols algebra would be trivial, but nevertheless the $\mathfrak{Z}_{\alpha_i}$ generate the Lie algebra, see Section 3.4.

More technically, our main theorem enables us to give bounds and thus proving conditional convergence for F_-:

Lemma 5.14. *The generalized Selberg integral has bounds for large m_i:*

$$\mathrm{Sel}((m_i,0,m_{ij})_{ij}) \leq C \prod_k \left(\sum_{k\leq i} m_i\right)^{-1-\sum_{k<j} m_{kj}}$$

Then, the series F_- *converges conditionally if for all $J \subset \{1,\dots n\}$ holds $\sum_{i<j,\, i,j\in J} m_{ij} > -|J|$ which is even a slightly larger domain then the domain of definition for the integral presentation of this function (Def. 5.8).*

Proof. We first prove the bound for the Selberg integrals. With variables $t_i = z_i - z_{i+1}, t_n = z_n$ we have

$$\mathrm{Sel}((m_i,0,m_{ij})_{ij}) = \int\cdots\int_{t_i\geq 0,\,\sum_i t_i\leq 1} \prod_i (t_i+\dots+t_n)^{m_i} \prod_{ij}(t_i+\dots+t_{j-1})^{m_{ij}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

Since depending on the sign of m_{ij} we have a bound $(a+b)^{m_{ij}} \leq a^{m_{ij}}$ or $\leq 2^{m_{ij}}a^{m_{ij}}$ there is a constant C depending on m_{ij} but not on m_i, such that

$$\begin{aligned}
&\leq C\int\cdots\int_{t_i\geq 0,\,\sum_i t_i\leq 1} \prod_i (\underbrace{t_i+\dots+t_n}_{=:s_i})^{m_i} \prod_{ij} t_i^{m_{ij}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&= C\int_0^1 \mathrm{d}s_1\, s_1^{m_1}\int_0^{s_1}\mathrm{d}t_1\, t_1^{\sum_{1<j} m_{1j}}(s_1-t_1)^{m_2}\int_0^{s_2}\mathrm{d}t_2\, t_2^{\sum_{2<j} m_{2j}}(s_2-t_2)^{m_3}\cdots
\end{aligned}$$

where $s_i = s_{i-1} - t_i$. This iterated integral can be decoupled to a product of integrals if we successively rescale $t_i := s_i\tau_i$, which yields a product of Beta-functions

$$\begin{aligned}
&= C\int_0^1 \mathrm{d}s_1\, s_1^{(n-1)+\sum_{1\leq i} m_i + \sum_{1\leq i<j} m_{ij}} \cdot \int_0^1 \mathrm{d}\tau_1\, \tau_1^{\sum_{1<j} m_{1j}}(1-\tau_1)^{(n-2)+\sum_{2\leq i} m_i+\sum_{2\leq i<j} m_{ij}}\dots \\
&= \left((n-1)+\sum_{1\leq i} m_i + \sum_{1\leq i<j} m_{ij}\right)^{-1}\cdot \mathrm{B}\left(1+\sum_{1<j} m_{1j}, 1+(n-2)+\sum_{2\leq i} m_i + \sum_{2\leq i<j} m_{ij}\right)\cdots
\end{aligned}$$

Since for large x we have the asymptotics $\mathrm{B}(1+a,1+x)\approx x^{-1-a}$, we get for large m_i

$$\mathrm{Sel}((m_i,0,m_{ij})_{ij}) \leq C' \prod_k \left(\sum_{k\leq i} m_i\right)^{-1-\sum_{k<j} m_{kj}}$$

Regardless of the order, this bound is with total exponent $-n-\sum_{i<j} m_{ij}$, so by the main theorem, such a bound also holds for $\mathrm{F}_-((m_i+k_{0i};m_{ij})_{1\leq i<j\leq n})$. Then we use this

bound for the series of F_- in $n+1$ variable

$$
\begin{aligned}
&\mathrm{F}_-((m_i; m_{ij})_{0\le i<j\le n}) \\
&= \sum_k \frac{1}{1+m_0+\sum_{0<j} m_{0j} - k} \sum_{\sum_i k_{0i}=k} \mathrm{F}_-((m_i+k_{0i}; m_{ij})_{1\le i<j\le n}) \\
&\approx \sum_k \frac{1}{1+m_0+\sum_{0<j} m_{0j} - k} \int_{\sum_i k_{0i}=k} \mathrm{F}_-((m_i+k_{0i}; m_{ij})_{1\le i<j\le n}) \\
&\approx \sum_k \frac{1}{1+m_0+\sum_{0<j} m_{0j} - k} k^{(n-1)-n-\sum_{i<j} m_{ij}}
\end{aligned}
$$

This converges whenever $\sum_{i<j} m_{ij} > -1$ which shows the second claim. □

5.4. **Proof of Main Theorem.**

Proof of Theorem 5.11. The reader is adviced to consider the case $n=2$, which contains the essential picture. The proof will proceed in the following steps:

$$
\begin{aligned}
\mathrm{F}_-((m_i, m_{ij})_{ij}) &\overset{2}{=} \lim_{(\hbar_i)_i \to 1} F^{\hbar_1,\ldots\hbar_n}((m_i, m_{ij})_{ij}) \\
&\overset{1}{=} \lim_{(\hbar_i)_i \to 1} \mathrm{I}^{\hbar_1,\ldots\hbar_n}((m_i, m_{ij})_{ij}) \\
&\overset{3}{=} \lim_{(\hbar_i)_i \to 1} \sum_{\sigma\in\mathbb{S}_n} \tilde{\mathrm{I}}^{\hbar_1,\ldots,\hbar_n}_{\Delta^\sigma}((m_i, m_{ij})_{ij}) \\
&\overset{4}{=} \sum_{\sigma\in\mathbb{S}_n} q'(\sigma)\, \tilde{\mathrm{I}}((m_{\sigma^{-1}(i)}, m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij}) \\
&\overset{5}{=} \sum_{\sigma\in\mathbb{S}_n} q(\sigma)\, \tilde{\mathrm{I}}((m_{\sigma^{-1}(i)}, m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij}) \\
&\overset{6}{=} \sum_{\sigma\in\mathbb{S}_n} q(\sigma)\, \tilde{\mathrm{F}}_-((m_{\sigma^{-1}(i)}, m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij})
\end{aligned}
$$

We may assume fracuredness $m_i + \sum_{i<j} m_{ij} \notin \mathbb{Z}$ allowing for the comforable sum presenatation for F on this dense subset.

Step 1): Relating Series and Integral for $\hbar_1 > \hbar_2 > \cdots$ The rather obvious first idea of this proof is to consider a more general series and integral expression and prove their equality on a generic subset. With the integrand $\prod_i z_i^{m_i} \prod_{i<j}(z_i - z_j)^{m_{ij}}$ in mind we consider for fixed positive real parameters $\hbar_1 > \cdots > \hbar_n$ the following contour integral

$$
I^{\hbar_1,\ldots,\hbar_n}((m_i, m_{ij})_{ij}) := \frac{1}{(2\pi \mathrm{i})^n} \int_{S^1_{\hbar_1}\times\cdots\times S^1_{\hbar_1}} \prod_i \mathrm{d}z_i \prod_i z_i^{m_i+\sum_{i<j} m_{ij}} \prod_{i<j} (1 - z_j/z_i)^{m_{ij}}
$$

where the integration is performed over a torus with radii $\hbar_1, \dots \hbar_n$, which is uniquely lifted to an n-cube in the multivalued covering, which contains the point $(0, \dots, 0)$. Note that since $|z_i| = \hbar_i > \hbar_j = |z_j|$ for $i < j$ we always have $\mathfrak{Re}(1 - z_j/z_i) \geq 0$ and thus may choose a consistent phase $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for $1 - z_j/z_i$.

To be clear and explicit, we mean the integral of the following parametrization

$$
\begin{aligned}
\mathrm{I}^{\hbar_1,\dots,\hbar_n}((m_i, m_{ij})_{ij}) &= \frac{1}{(2\pi \mathrm{i})^n} \int_{[0,2\pi]^n} \prod_i \mathrm{i} e^{\mathrm{i} t_i} \mathrm{d}t_i \prod_i e^{\mathrm{i} t_i (m_i + \sum_{i<j} m_{ij})} \hbar_i^{m_i + \sum_{i<j} m_{ij}} \prod_{i<j} e^{\mathrm{i}\theta_{ij} m_{ij}} r_{ij}^{m_{ij}} \\
&= \frac{\prod_i \hbar_i^{m_i + \sum_{i<j} m_{ij}}}{(2\pi)^n} \int_{[0,2\pi]^n} \prod_i \mathrm{d}t_i \; e^{\mathrm{i}(\sum_i t_i (1+m_i) + \sum_{i<j} (t_i + \theta_{ij}) m_{ij})} \prod_{i<j} r_{ij}^{m_{ij}}
\end{aligned}
$$

with the obvious geometric functions

$$
\theta_{ij}(t_i, t_j) := \tan^{-1}\left(\frac{1 \sin(0) - \frac{\hbar_j}{\hbar_i} \sin(t_j - t_i))}{1 \cos(0) - \frac{\hbar_j}{\hbar_i} \cos(t_j - t_i))} \right) = -\tan^{-1}\left(\frac{\sin(t_j - t_i))}{\frac{\hbar_i}{\hbar_j} - \cos(t_j - t_i))} \right) \quad \in [-\frac{\pi}{2}, \frac{\pi}{2}]
$$

$$
r_{ij}(t_i, t_j) := \sqrt{1 + \left(\frac{\hbar_j}{\hbar_i}\right)^2 - 2\frac{\hbar_j}{\hbar_i} \cos(t_j - t_i)} \; \in \mathbb{R}^+
$$

We now use $|z_i| = \hbar_i > \hbar_j = |z_j|$ to expand $(1 - z_j/z_i)^{m_{ij}}$ into a power series inside its convergence radius and hence we may integrate summand-wise

$$
\begin{aligned}
&\mathrm{I}^{\hbar_1,\dots,\hbar_n}((m_i, m_{ij})_{ij}) \\
&:= \frac{1}{(2\pi \mathrm{i})^n} \int_{S^1_{\hbar_1} \times \dots \times S^1_{\hbar_1}} \prod_i \mathrm{d}z_i \prod_i z_i^{m_i + \sum_{i<j} m_{ij}} \prod_{i<j} (1 - z_j/z_i)^{m_{ij}} \\
&= \frac{1}{(2\pi \mathrm{i})^n} \int_{S^1_{\hbar_1} \times \dots \times S^1_{\hbar_1}} \prod_i \mathrm{d}z_i \prod_i z_i^{m_i + \sum_{i<j} m_{ij}} \sum_{(k_{ij})_{i<j} \in \mathbb{N}_0^{\binom{n}{2}}} \prod_{i<j} (-1)^{k_{ij}} \binom{m_{ij}}{k_{ij}} (z_j/z_i)^{k_{ij}} \\
&= \frac{1}{(2\pi \mathrm{i})^n} \sum_{(k_{ij})_{i<j} \in \mathbb{N}_0^{\binom{n}{2}}} \int_{S^1_{\hbar_1} \times \dots \times S^1_{\hbar_1}} \prod_i \mathrm{d}z_i \prod_i z_i^{m_i + \sum_{i<j}(m_{ij} - k_{ij}) + \sum_{j<i} k_{ji}} \prod_{i<j} (-1)^{k_{ij}} \binom{m_{ij}}{k_{ij}} \\
&= \sum_{(k_{ij})_{i<j} \in \mathbb{N}_0^{\binom{n}{2}}} \prod_i \hbar_i^{m_i + \sum_{i<j}(m_{ij} - k_{ij}) + \sum_{j<i} k_{ji}} \frac{(e^{2\pi \mathrm{i}(m_i + \sum_{i<j} m_{ij})} - 1)/2\pi \mathrm{i}}{1 + \sum_{i<j}(m_{ij} - k_{ij}) + \sum_{j<i} k_{ji}} \prod_{i<j} (-1)^{k_{ij}} \binom{m_{ij}}{k_{ij}} \\
&=: F^{\hbar_1, \dots \hbar_n}((m_i, m_{ij})_{ij})
\end{aligned}
$$

As a remark, for $n = 2$ this function in the variable $\hbar_2/\hbar_1$ is the hypergeometric function ${}_3F_2$ which gives for $z = \mp$ an explicit expression for $\mathrm{F}_\pm(m_1, m_2, m_{12})$.

Step 2): The series in the limit of equal $\hbar_i$. We convince ourselves that the newly introduced series $F^{\hbar_1,\ldots\hbar_n}((m_i,m_{ij})_{ij})$ has as limit case for equal $\hbar_i$ the actual quantum mondromy numbers in question: It is trivial that specializing of the power series (in $\hbar_1,\ldots,\hbar_n$) to some fixed value $\hbar_i=\hbar$ for all i returns

$$\sum_{(k_{ij})_{i<j}\in\mathbb{N}_0^{\binom{n}{2}}}\prod_i \hbar^{m_i+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\frac{(e^{2\pi\mathrm{i}(m_i+\sum_{i<j}m_{ij})}-1)/2\pi\mathrm{i}}{1+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\prod_{i<j}(-1)^{k_{ij}}\binom{m_{ij}}{k_{ij}}$$
$$=\sum_{(k_{ij})_{i<j}\in\mathbb{N}_0^{\binom{n}{2}}}\hbar^{\sum_i m_i+\sum_{i<j}m_{ij}}\cdot\prod_i\frac{(e^{2\pi\mathrm{i}(m_i+\sum_{i<j}m_{ij})}-1)/2\pi\mathrm{i}}{1+\sum_{i<j}(m_{ij}-k_{ij})+\sum_{j<i}k_{ji}}\prod_{i<j}(-1)^{k_{ij}}\binom{m_{ij}}{k_{ij}}$$
$$=\hbar^{\sum_i m_i+\sum_{i<j}m_{ij}}\cdot\mathrm{F}_-((m_i,m_{ij})_{ij})$$

From our assumptions on m_i,m_{ij} the defining series for $\mathrm{F}_-((m_i,m_{ij})_{ij})$ converges. So we invoke Abel's theorem and see

$$\lim_{(\hbar_i)_i\to\hbar}F^{\hbar_1,\ldots\hbar_n}((m_i,m_{ij})_{ij})=\hbar^{\sum_i m_i+\sum_{i<j}m_{ij}}\cdot\mathrm{F}_-((m_i,m_{ij})_{ij})$$

Having made connection with the left side of our assertion, we contiunue to decompose and determine the respective limit of the integral $\mathrm{I}^{\hbar_1,\ldots\hbar_n}((m_i,m_{ij})_{ij})$.

Step 3): Decomposition of the integral for $\hbar_1>\hbar_2>\cdots$. The main idea of proving the symmetrizer formula is as follows: Consider in $[0,1]^n$ the simplex $\Delta:=\{(t_i)_i\mathrm{mod}t_1<t_2<\cdots\}$ and for each permutation $\sigma\in\mathbb{S}_n$ the simplex $\Delta^\sigma:=\{(t_i)_i\mid t_{\sigma^{-1}(1)}<t_{\sigma^{-1}(2)}<\cdots\}$, then up to the zero-set consisting of hyperplanes $t_i=t_j$ this gives a decomposition of the integration domain $[0,2\pi]^n$:

$$\mathrm{I}^{\hbar_1,\ldots,\hbar_n}((m_i,m_{ij})_{ij})=\sum_{\sigma\in\mathbb{S}_n}\underbrace{\frac{\prod_i\hbar_i^{m_i+\sum_{i<j}m_{ij}}}{(2\pi)^n}\int_{\Delta^\sigma}\prod_i\mathrm{d}t_i\;e^{\mathrm{i}(\sum_i t_i(1+m_i)+\sum_{i<j}(t_i+\theta_{ij})m_{ij})}\prod_{i<j}r_{ij}^{m_{ij}}}_{=:\,\tilde{\mathrm{I}}^{\hbar_1,\ldots,\hbar_n}_{\Delta^\sigma}((m_i,m_{ij})_{ij})}$$

The ideal occurrence would be that $\tilde{\mathrm{I}}_{\Delta^\sigma}$ is equal to $\tilde{\mathrm{I}}_\Delta$ up to a permutation of the m_i,m_{ij} - this is obviousely true for $m_i,m_{ij}\in\mathbb{Z}$. In the fractional case, the next ideal occurrence would be, that $\tilde{\mathrm{I}}_{\Delta^\sigma}$ is equal to $\tilde{\mathrm{I}}_\Delta$ up to a braiding factor and a permutation, but this seems again not to be true. Hoewever the latter becomes provable in the limit where all $\hbar_i$ are equal, as we shall see next:

Step 4): Relating the reduced integrals in the limit of equal $\hbar_i$. We now consider the integrals $\tilde{\mathrm{I}}_{\Delta^\sigma}$ in the limit where all $\hbar_i\to\hbar$.

We first have to argue by Lebesgue's theorem of dominated convergence, that we may switch limit and integration: All terms are bound except $r_{ij}^{m_{ij}}$, which is bound for $m_{ij} \geq 0$.

Let us now calculate this limit (and check that it exists everywhere except the hyperplanes $t_i = t_j$): We have

$$\lim_{(\hbar_i)_i \to \hbar} r_{ij} = \sqrt{2 - 2\cos(t_j - t_i)}$$

$$\lim_{(\hbar_i)_i \to \hbar} \theta_{ij} = -\tan^{-1}\left(1 - \frac{\sin(t_j - t_i)}{\cos(t_j - t_i)}\right) = \begin{cases} \frac{t_j - t_i}{2} - \frac{\pi}{2}, & \text{for } t_j - t_i > 0 \\ \frac{t_j - t_i}{2} + \frac{\pi}{2}, & \text{for } t_j - t_i < 0 \end{cases}$$

The cases $\mp\frac{\pi}{2}$ in the limit formula for θ_{ij} is *the* crucial feature! Thus we get

$$\lim_{(\hbar_i)_i \to \hbar} \tilde{\mathrm{I}}_{\Delta^\sigma}^{\hbar_1,\ldots,\hbar_n} = \frac{\prod_i \hbar^{m_i + \sum_{i<j} m_{ij}}}{(2\pi)^n} \int_{\Delta^\sigma} \prod_i \mathrm{d}t_i \; e^{\mathrm{i}(\sum_i t_i(1+m_i) + \sum_{i<j} \frac{t_i+t_j}{2} m_{ij})} \prod_{i<j} \sqrt{2 - 2\cos(t_j - t_i)}^{\,m_{ij}}$$
$$\cdot \underline{\underline{e^{-\mathrm{i}\frac{\pi}{2} \sum_{i<j} \mathrm{sgn}(t_j - t_i) m_{ij}}}}$$

where we introduced the sign-function $\mathrm{sgn}(t_2 - t_1) = \pm 1$ and the underlined factor does only depend on $\sigma \in \mathbb{S}_n$. Since the remaining integral is independent of the order of i's (except the m_i, m_{ij}) we have hence successfully decomposed

$$\lim_{(\hbar_i)_i \to \hbar} \mathrm{I}_{\Delta^\sigma}^{\hbar_1,\ldots,\hbar_n}((m_i, m_{ij})_{ij}) = \hbar^{\sum_i m_i + \sum_{i<j} m_{ij}} \cdot q'(\sigma)\; \tilde{\mathrm{I}}((m_{\sigma^{-1}(i)}, m_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij})$$

with the *braiding factor* $q'(\sigma)$ and the *reduced quantum monodromy numbers* $\tilde{\mathrm{I}}$

$$q'(\sigma) := e^{-\mathrm{i}\frac{\pi}{2} \sum_{i<j} \mathrm{sgn}(t_j - t_i) m_{ij}} \cdot e^{+\mathrm{i}\frac{\pi}{2} \sum_{i<j} m_{ij}}$$

$$\tilde{\mathrm{I}}((m_i, m_{ij})_{ij}) := \tilde{\mathrm{I}}_\Delta^{1,\ldots,1}((m_i, m_{ij})_{ij})$$
$$= e^{-\mathrm{i}\frac{\pi}{2} \sum_{i<j} m_{ij}} \frac{1}{(2\pi)^n} \int_\Delta \prod_i \mathrm{d}t_i \; e^{\mathrm{i}(\sum_i t_i(1+m_i) + \sum_{i<j} \frac{t_i+t_j}{2} m_{ij})} \prod_{i<j} \sqrt{2 - 2\cos(t_j - t_i)}^{\,m_{ij}}$$

Step 5): The braiding factor. Consider the factor in step 4)

$$q'(\sigma) = e^{-\mathrm{i}\frac{\pi}{2} \sum_{i<j} \mathrm{sgn}(t_j - t_i) m_{ij}} \cdot e^{+\mathrm{i}\frac{\pi}{2} \sum_{i<j} m_{ij}} \qquad t_{\sigma^{-1}(1)} < t_{\sigma^{-1}(2)} < \cdots$$

for a permutation $\sigma \in \mathbb{S}_n$. By assumption $\mathrm{sgn}(t_j - t_i) = +1$ iff $t_i < t_j$ iff $\sigma(i) < \sigma(j)$, then the exponentials cancel, and $\mathrm{sgn}(t_j - t_i) = -1$ iff $\sigma(i) > \sigma(j)$. Thus

$$q'(\sigma) = e^{+\mathrm{i}\pi \sum_{i<j,\; \sigma(j)>\sigma(i)} m_{ij}}$$

The number of summands is called the *inversion number* of σ and it is equal to length $\ell(\sigma)$, which is the minimal number of neighbouring transpositions $(k, k+1)$ needed to generate σ.

We claim that this factor $q'(\sigma)$ is precisely the familiar *braiding factor* $q(\sigma)$ introduced above for the braiding matrix $q_{ij} := e^{\pi \mathrm{i} m_{ij}}$ which is inductively defined

$$q(\sigma') = q_{\sigma^{-1}(k),\sigma^{-1}(k+1)} \cdot q(\sigma) \qquad \text{for } \sigma' = (k,k+1)\sigma, \quad \ell(\sigma') = \ell(\sigma) + 1$$

and $q(\mathrm{id}) = 1$. We prove this claim inductively: For $\sigma = \mathrm{id}$ both terms are trivial. For $\sigma' = (k,k+1)\sigma$ the assumption of higher length means that the set of inversions (i,j) is increased by precisely this one element

$$\{i < j,\ \sigma'(i) > \sigma'(j)\} = \{\sigma(i) > \sigma(j)\} \cup \{(\sigma^{-1}(k), \sigma^{-1}(k+1))\}$$

(in particular $\sigma^{-1}(k) < \sigma^{-1}(k+1)$) which implies

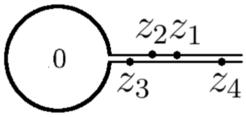
$$q'(\sigma') = q'(\sigma) \cdot e^{+\mathrm{i}\pi m_{\sigma^{-1}(k),\sigma^{-1}(k+1)}}$$

and since this is the defining property of $q(\sigma)$ this concludes the induction.

Step 6): Expressing $\tilde{I}$ as Selberg integrals. We finally rewrite $\tilde{I}$ as an ordinary integral, more precisely as generalized Selberg integrals. In particular for $n = 2$ this will lead to an expression as Beta-Functions. Recall that former definition is a contour integral over a specific lift of the simplex $\mathcal{C}$ given by $z_1 = e^{2\pi \mathrm{i} t_1}, \ldots, z_n = e^{2\pi \mathrm{i} t_1}$, where the phases $t_1 < \cdots < t_n$:

$$\tilde{I}^{\hbar_1,\ldots \hbar_n}((m_i, m_{ij})_{ij}) = \frac{1}{(2\pi \mathrm{i})^n} \int \cdots \int_{\mathcal{C}} \mathrm{d}z_1 \cdots \mathrm{d}z_n \prod_i z_i^{m_i} (z_i - z_j)^{m_{ij}}$$

We deform the integration path as one might expect[10]; Since by our assumption $-n > \sum_i m_i + \sum_{ij} m_{ij}$ the integration over the small circle vanishes in the limit:



e.g. $k = 2, \eta = (1234) \in \mathbb{S}_{2,\overline{2}}$

Thereby our integration domain decomposes into $n! = \sum_{k=0}^{n} \binom{n}{k}$ pieces, namely $n{+}1$ cases respectively with $t_1, \ldots t_k \approx 0$ and $t_{k+1}, \ldots t_n \approx 2\pi$, so $z_1 > \cdots > z_k$ and $z_{k+1} < \cdots < z_n$ and in each case all $\binom{n}{k}$ subcases how the two subsets are shuffled against each other. We define in slight variation to familiar $(k, n-k)$-*shuffles*

$$\mathbb{S}_{k,\overline{n-k}} := \{\eta \in \mathbb{S}_n \mid \forall_{i<j\leq k}\ \eta(i) < \eta(j) \text{ and } \forall_{k<i<j}\ \eta(i) > \eta(j)\} \qquad |\mathbb{S}_{k,\overline{n-k}}| = \binom{n}{k}$$

So we have the formula:

$\tilde{I}((m_i, m_{ij})_{ij})$

[10]It might have been briefer to do this deformation already before Step 4 or earlier, but we found it more clear this way.

$$= \sum_{k=0}^{n} (-1)^k s'(k) \sum_{\eta \in \mathbb{S}_{k,\overline{n-k}}} s''(k) \cdot \frac{1}{(2\pi \mathrm{i})^n} \int \cdots \int_{z_{\eta^{-1}(1)} > \cdots > z_{\eta^{-1}(n)}} \mathrm{d}z_1 \cdots \mathrm{d}z_n \prod_i z_i^{m_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \sum_{k=0}^{n} (-1)^k s'(k) \sum_{\eta \in \mathbb{S}_{k,\overline{n-k}}} s''(k) \cdot \int \cdots \int_{z_1 > \cdots > z_n} \mathrm{d}z_1 \cdots \mathrm{d}z_n \prod_i z_i^{m_{\eta^{-1}(i)}} \prod_{i<j} (z_i - z_j)^{m_{\eta^{-1}(i)\eta^{-1}(j)}}$$

Where the $(-1)^k$ comes from reversing integration direction and the chosen lift determines the first factors to be $s'(k) = \prod_{i=k+1}^{n} e^{2\pi \mathrm{i}\, m_i}$ and, after considering all four possibilities for the argument of $z_i - z_j$ to yield 0 or π, determines the second factor to be $s''(\eta) = \prod_{i<j,\, \eta(i)>\eta(j)} e^{\pi \mathrm{i}\, m_{ij}}$. This is the definition of $\tilde{\mathrm{F}}_-((m_i, m_{ij})_{ij})$ in our assertion.

We have thus altogether proven our assertion (say, for $\hbar = 1$). $\square$

6. Applications to Kazhdan-Lusztig Correspondence

6.1. **Nichols Algebra Action.** As a corollary of the previous two sections, we now prove the conjecture in a precise and more general form.

Theorem 6.1. *Let $\Lambda \subset \mathbb{C}^{\mathrm{rank}}$ be a positive-definite lattice and $\alpha_1, \ldots, \alpha_{\mathrm{rank}}$ be a fixed basis*[11] *that fulfills $|\alpha_i| \leq 1$*

Then the endomorphisms $\mathfrak{Z}_{\alpha_i}$ in Definition 3.15 on the fractional lattice VOA $\mathcal{V}_\Lambda$ and on any deformation $\mathcal{V}_\Lambda^{(\Omega,1)}$ constitute an action of the diagonal Nichols algebra generated by $\mathfrak{Z}_{\alpha_i}$ with braiding matrix

$$q_{ij} = e^{\pi \mathrm{i}\, (\alpha_i, \alpha_j)} \cdot \Omega(\alpha_i, \alpha_j)$$

In particular the crucial quantities $q_{ii}, q_{ij}q_{ji}$ for the Nichols algebra do not depend on the deformation: The Ω has on the Nichols algebra just the effect of a Doi twist by the deforming group 2-cocycle).

Proof. The essential idea is clear: Quantum monodromy numbers F_- can be written by Theorem 5.11 as quantum symmetrizer of some $\tilde{\mathrm{F}}_-$ for the same $q_{ij} = e^{\pi \mathrm{i} m_{ij}}$, so they vanish for every formal linear combination in the kernel of the quantum symmetrizer. But the Nichols algebra is by definition the quotient by the kernel of this quantum symmetrizer. Then for commutative, cocommutative $\mathcal{V}$ this implies the $\mathfrak{Z}_{\alpha_i}$ obey Nichols algebra relations. More generally (e.g. the deformation $\mathcal{V}_\Lambda^{(\Omega,1)}$ below) if $\mathcal{V}$ is non-commutative up to scalars, then these additional scalars enter as well and the $\mathfrak{Z}_{\alpha_i}$ generate the Nichols algebra for a modified braiding matrix. The technical condition of smallness implies by 5.9 the property smallness-F on the m_{ij}; the quantum symmetrizer formula holds only under the condition smallness-F on m_i, m_{ij}, but the vanishing in the Nichols algebra is

[11]the choice of the basis is not so important as in later application, where a Virasoro action is present; any set of elements satisfying smallness generates a Nichols algebra.

completely independent on m_i, so the vanishing result holds independent of m_i (but the poles do cause crucial nontrivial effects Section 6.2).

We now make this precise: Let $\mathbb{C}^I = \mathbb{C}\otimes_\mathbb{Z}\Lambda$ be the braided vector space spanned by the $\alpha_1, \ldots, \alpha_{\mathrm{rank}}$ with two alternative braidings $q_{ij} = e^{\pi \mathrm{i}\,(\alpha_i,\alpha_j)}$ and $q_{ij}^\Omega = e^{\pi \mathrm{i}\,(\alpha_i,\alpha_j)}\cdot\Omega(\alpha_i,\alpha_j)$. We consider the tensor algebra

$$A, A^\Omega := \bigoplus_{n\geq 0}(\mathbb{C}^I)^{\otimes n}$$

and two alternative actions of the braid group via q_{ij}, q_{ij}^Ω. (note that due to $\Omega(x,y)\Omega(y,x) = 1$ the deformation term factorizes over the symmetric group).

We wish to proof that the endomorphisms $\mathfrak{Z}_{\alpha_i}$ fulfill the relations of the Nichols algebra in each degree n. This means, let formally $X = \{1, \ldots, n\}$ and consider colorings $f : X \to I$. Define the braiding matrices and -factors by $(q_f)_{xy} := q_{f(x)f(y)}$ and $q_f(\sigma)$ and respectively for $q_f^\Omega(\sigma) = \Omega_f(\sigma)q_f(\sigma)$ where $\Omega_f(\sigma)$ gives an action of the symmetric group.

We wish to prove (using our generalized associativity and quantum symmetrizer formulae for F_-) that if a formal finite linear combination in the free algebra

$$e_c := \sum_{f:X\to C} c_f\ e_{f(1)} \otimes \cdots e_{f(n)}$$

vanishes in the Nichols algebra A, A^Ω, then the linear combination

$$\mathfrak{Z}_c = \sum_{f:X\to C} c_f \mathfrak{Z}_{\alpha_{f(1)}} \cdots \mathfrak{Z}_{\alpha_{f(n)}}$$

should vanish. The former means by the defining property of a Nichols algebra

$$\begin{aligned} 0 \overset{!}{=} \text{Ш}_q(e_c) &:= \sum_{\sigma\in\mathbb{S}_n}\sum_{f:X\to I} q_f(\sigma)\cdot c_f\ e_{f(\sigma(1))}\otimes\cdots e_{f(\sigma(n))} \\ &= \sum_{f:X\to I}\left(\sum_\sigma q_{f\sigma}(\sigma)c_{f\sigma}\right)\ e_{f(1)}\otimes\cdots e_{f(n)} \end{aligned}$$

respectively for $0 = \text{Ш}_{q^\Omega}(e_c)$ in A^Ω that

$$= \sum_{f:X\to I}\left(\sum_\sigma \Omega_{f\sigma}(\sigma)q_{f\sigma}(\sigma)c_{f\sigma}\right)\ e_{f(1)}\otimes\cdots e_{f(n)}$$

The key idea is that we have written in Theorem 5.11 the quantum monodromy numbers F_- as a quantum symmetrizer of $\tilde{\mathrm{F}}_-$ for the same q_{ij} (regardless of Ω).

$$\mathrm{F}_-((m_x, m_{x,y})_{1\leq x,y\leq n}) = \sum_{\sigma\in\mathbb{S}_n} q_f(\sigma)\,\tilde{\mathrm{F}}_-((m_{\sigma^{-1}(x)}, m_{\sigma^{-1}(x)\sigma^{-1}(y)})_{x,y})$$

For the application we assume that the arguments factorize $m_{x,y} = m_{f(x),f(y)}$ and $m_x = m_{f(x)} + k_x$ with $k_x \in \mathbb{N}_0$ in a formal basis $[k_1, \ldots, k_n]$:

$$\sum_{(k_x)_x\in\mathbb{N}_0^n} [k_1, \ldots, k_n] \cdot \sum_f c_f\,\mathrm{F}_-((m_{f(x)} + k_x, m_{f(x),f(y)})_{1\leq x,y\leq n})$$
$$= \sum_{(k_x)_x\in\mathbb{N}_0^n} [k_1, \ldots, k_n] \cdot \sum_f c_f \sum_{\sigma\in\mathbb{S}_n} q_f(\sigma)\,\tilde{\mathrm{F}}_-((m_{f(\sigma^{-1}(x))} + k_{\sigma^{-1}(x)}, m_{f(\sigma^{-1}(x))f(\sigma^{-1}(y))})_{x,y})$$
$$= \sum_f \sum_{\sigma\in\mathbb{S}_n} q_{f\sigma}(\sigma) c_{f\sigma} \sum_{(k_x)_x\in\mathbb{N}_0^n} \tilde{\mathrm{F}}_-((m_{f(x)} + k_x, m_{f(x)f(y)})_{x,y})\,[k_{\sigma(1)}, \ldots, k_{\sigma(n)}]$$

Assume additionally an invariant property for our formal symbols $[k_{\sigma(1)}, \ldots, k_{\sigma(n)}] = [k_1, \ldots, k_n]$ resp. $= \Omega(\sigma)[k_1, \ldots, k_n]$, then our assumption $0 = \text{Ш}_{q^\Omega}(e_c)$ implies:

$$= \sum_f \left(\sum_{\sigma\in\mathbb{S}_n} \Omega_{f\sigma}(\sigma) q_{f\sigma}(\sigma) c_{f\sigma}\right) \left(\sum_{(k_x)_x\in\mathbb{N}_0^n} \tilde{\mathrm{F}}_-((m_{f(x)} + k_x, m_{f(x)f(y)})_{x,y})\,[k_1, \ldots, k_n]\right) = 0$$

So it remains to study how the quantum monodromy numbers enter in the associativity and prove in the particular case $\mathcal{V}_\Lambda$ resp. $\mathcal{V}_\Lambda^{(\Omega,1)}$ that the additional invariance enters (due to commutativity, cocommutativity): Our Associativity Theorem 4.3 implies for $a_x = \mathrm{e}^{\phi_{\alpha_{f(x)}}}$ grouplike with $\langle a_x, a_y\rangle = 1 \cdot z^{m_{f(x)f(y)}},\ m_{ij} = (\alpha_{f(x)}, \alpha_{f(x)})$:

$$\left(\prod_{x=1}^n \mathrm{ResY}(a_x)\right) v = \sum_{(k_x)_x\in\mathbb{N}_0^n} \sum_{(m_x, m_{xy})_{x,y}} \prod_{1\leq x\leq n} \langle a_x^{(-n)}, v^{(-x)}\rangle_{m_x} \prod_{1\leq x<y\leq n} \langle a_x^{-(y-1)}, a_y^{(-x)}\rangle_{m_{xy}}$$
$$\cdot\, v^{(0)} \prod_{x=n}^{1} \frac{\partial^{k_x}}{k_x!} a_x^{(0)} \cdot \mathrm{F}_-((m_x + k_x, m_{xy})_{x,y})$$

$$\sum_f c_f \left(\prod_{i=1}^n \mathfrak{Z}_{\alpha_{f(x)}}\right) v = \sum_{(k_x)_x\in\mathbb{N}_0^n} \left(\sum_{(m_x)_x} \prod_{1\leq x\leq n} \langle \mathrm{e}^{\phi_{\alpha_{f(x)}}}, v^{(-x)}\rangle_{m_{f(x)}} \cdot v^{(0)} \prod_{x=n}^{1} \frac{\partial^{k_x}}{k_x!} \mathrm{e}^{\phi_{\alpha_{f(x)}}}\right)$$
$$\cdot \sum_f c_f\,\mathrm{F}_-((m_{f(x)} + k_x, m_{f(x)f(y)})_{x,y})$$

The large bracket is a function in the $[k_1, \ldots k_n] \in \mathbb{N}_0^n$, and we now discuss their behaviour under permutation of $\{1, \ldots, n\}$ if simultaneously the k_x are permuted: Because $\mathcal{V}_\Lambda, \mathcal{V}_\Lambda^{(\Omega,1)}$ are cocommutative, the Hopf pairings with v are permutation invariant. The subsequent product is over polynomials $P_{\alpha_{f(x)},k_x}$, which are central in $\mathcal{V}_\Lambda, \mathcal{V}_\Lambda^{(\Omega,1)}$, times exponentials $\mathrm{e}^{\phi_{\alpha_{f(x)}}}$. The latter are also permutation invariant in $\mathcal{V}_\Lambda$ (which is commutative altogether), while in $\mathcal{V}_\Lambda^{(\Omega,1)}$ they form by definition the twisted groupring $\mathbb{C}_\kappa[\Lambda]$ with 2-cocycle κ and associate alternating bicharacter Ω determining the non-cocommutativity. Hence altogether the large bracket is invariant under permutations for $\mathcal{V}_\Lambda$ and picks up a factor $\Omega(\sigma)$ for $\mathcal{V}_\Lambda^{(\Omega,1)}$.

This shows by the previous argument that the assumption $\text{Ш}_q \sum_f c_f \; e_{f(1)} \otimes \cdots e_{f(n)} = 0$ (resp. Ш_{q^Ω}) implies $\sum_f c_f \left(\prod_{i=1}^n \mathfrak{Z}_{\alpha_{f(x)}}\right) \; v = 0$; which is the assertion. $\square$

Remark 6.2. *The author would conjecture that usually the endomorphisms $\mathfrak{Z}_{\alpha_i}$ even generates precisely the Nichols algebra i.e. the quantum monodromy numbers fulfill precisely the Nichols algebra relations. This seems to be obvious from the fact that the Nichols algebra is the smallest Hopf algebra quotient; indeed we have an action of $\mathfrak{Z}_{\alpha_i}$ by derivations, but these derivations are only derivations with respect to the nonassociative algebra $\mathrm{Y}(a)_{-1}b$.*

Example 6.3 (Quantum groups)**.** *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with root lattice $\Lambda_\mathfrak{g}$ and Killing form $(-,-)_\mathfrak{g}$. Choose an even integer $\ell = 2p \in \mathbb{N}$. Consider the rescaled root lattice $\Lambda := \frac{1}{\sqrt{p}}\Lambda_\mathfrak{g}$ with basis $\alpha_i/\sqrt{p}$ and braiding matrix $q_{ij} = e^{\pi \mathrm{i}(\alpha_i/\sqrt{p},\alpha_j/\sqrt{p})} = e^{\frac{2\pi \mathrm{i}}{\ell}(\alpha_i,\alpha_j)_\mathfrak{g}}$. This is precisely the braiding matrix underlying the Nichols algebra $u_q(\mathfrak{g})^+$ with q a primitive ℓ-th root of unity.*

Thus Theorem 6.1 shows that the operators $E_i \mapsto \mathfrak{Z}_{\alpha_i/\sqrt{p}}$ constitute a representation of the Nichols algebra $u_q(\mathfrak{g})^+$ on $\mathcal{V}_{\frac{1}{\sqrt{p}}\Lambda_\mathfrak{g}}$. Morover the relations in Lemma 3.14 easily show that $K_i \mapsto e^{\pi \mathrm{i}\, \partial_{\alpha_i/\sqrt{p}}}$ extends this to a representation of $u_q(\mathfrak{g})^{\geq}$.

6.2. **Weyl reflection operators.** Besides the relations of $\mathfrak{Z}_{\alpha_i}$ determined by the Nichols algebra we have further relations that only hold on specific subspaces $\mathcal{V}_\lambda$, corresponding to certain combinations of $F(m_{ij}, m_i)$ that only vanish for certain m_i. For example the following vanishing result relates directly to vanishing $\mathrm{ad}_{x_i}^k \, x_j$ in the adjoint representation (quantum Serre Relation) or with different k on other representations:

Lemma 6.4. *Take the case were all parameters m_{ij} are equal, and all parameters m_i are equal modulo $\mathbb{Z}$. If there is some number $n \in \mathbb{N}$ such that $2m_i + (n-1)m_{ij} \in 2\mathbb{Z}$, then the coefficient in the definition of $\tilde{\mathrm{F}}_-$ vanishes*

$$\frac{1}{(2\pi \mathrm{i})^n} \sum_{k=0}^n (-1)^k \left(\prod_{i=k+1}^n e^{2\pi \mathrm{i}\, m_i}\right) \sum_{\eta \in \mathbb{S}_{k,\overline{n-k}}} \left(\prod_{i<j,\; \eta(i)>\eta(j)} e^{\pi \mathrm{i}\, m_{ij}}\right) = 0$$

In consequence, under the assumption of the Quantum Symmetrizer Formula 5.11 (smallness-F and none of the poles*) we have:*

$$\boxed{2(\alpha,\lambda)+(n-1)(\alpha,\alpha)\in 2\mathbb{Z}\quad\Longrightarrow\quad \mathfrak{Z}_\alpha^n v_\lambda=0}$$

Reformulated as braidings, the condition is very familiar from Nichols algebras:

$$q_{ij}q_{ji}q_{ii}^{n-1}=e^{\pi\mathrm{i}(\alpha,\lambda)}e^{\pi\mathrm{i}(\lambda,\alpha)}(e^{\pi\mathrm{i}(\alpha,\alpha)})^{n-1}=1$$

Proof. Denote $q:=e^{\mathrm{i}\pi m_{ij}}$ then by assumption $q^{-(n-1)}=e^{2\pi\mathrm{i}\,m_i}$. We calculate

$$\sum_{k=0}^n(-1)^k\left(\prod_{i=k+1}^n e^{2\pi\mathrm{i}\,m_i}\right)\sum_{\eta\in\mathbb{S}_{k,\overline{n-k}}}\left(\prod_{i<j,\;\eta(i)>\eta(j)}e^{\pi\mathrm{i}\,m_{ij}}\right)$$
$$=q^{-(n-1)n}\sum_{k=0}^n(-1)^kq^{(n-1)k}\sum_{\eta\in\mathbb{S}_{k,\overline{n-k}}}q^{\ell(\eta)}$$

Now we prove the vanishing of this sum by constructing a perfect matching on

$$\bigcup_{k\ odd}\mathbb{S}_{k,\overline{n-k}}\quad\leftrightarrow\quad\bigcup_{k\ even}\mathbb{S}_{k,\overline{n-k}}$$

compatible with the q-weights: For every k-subset A of $\{1...n\}$ with $1\notin A$ we consider the $k+1$-subset $\{1\}\cup A$. Interpreting the subsets as the image $\eta(\{1,\ldots,k\})$ this subset fixes η_A uniquely: $\{1,\ldots,k\}$ maps in order to A and the rest maps to the complement in reversed order. Then compared to $\eta_{A'}\in\mathbb{S}_{k+1,\overline{n-(k+1)}}$ the permutation $\eta_A\in\mathbb{S}_{k,\overline{n-k}}$ maps additionally $\{n\}$ to $\{1\}$ and thus $\ell(\eta_{A'})=\ell(\eta_A)+(n-1)$. Altogether we compare

$$q^{(n-1)(k+1)}q^{\ell(\eta_{A'})}=q^{(n-1)k}q^{\ell(\eta_{A'})+(n-1)}=q^{(n-1)k}q^{\ell(\eta_A}$$

Thus the terms (k,η_A) and $(k+1,\eta_{A'}$ always cancel and the overall sum vanishes as asserted. □

Similar vanishing results hold for more complicated expressions. *However* the assertion that $\mathrm{Sel}(m_i,m_{ij})$ has no poles is crucial, since the vanishing result holds only a specific λ i.e. m_i (for the Nichols relation we have extended the vanishing result for generic m_i to the discrete pole set).

Example 6.5. *Let $p,n\in\mathbb{N}$ and assume all $m_{ij}=2/p$ and all $m_i=-(n-1)/p-1$. Then the assertion above holds:*

$$2m_i+(n-1)m_{ij}=-2\ \in 2\mathbb{Z}$$

But $\mathrm{Sel}(m_i+k_i,m_{ij}),k_i\in\mathbb{N}_0$ *has a pole precisely for the smallest $k_1,\ldots,k_n=0$, consequently*

$$\mathfrak{Z}_\alpha^n\mathrm{e}^{\phi_\lambda}=\mathrm{F}_-(-(n-1)/p-1,2/p)\cdot\mathrm{e}^{\phi_{\lambda-n\alpha}}\neq 0$$

In this case we could easily determine the residue from the Selberg integral formula in terms of Γ-functions, and the quantum symmetrizer formula returns the precise value.

To summarize, the entire “Nichols algebra part” vanishes (by Nichols algebra representation relations) but there is a non-zero summand living in the “Lie algebra part”. These singular maps are remarkable, the previous example appears already in [TW13]. More concretely, with our techniques we can prove in general e.g. that they are Virasoro algebra homomorphisms, in contrast to other expressions in $\mathfrak{Z}_\alpha$'s. As we shall discuss in the next section, they are precisely the Weyl reflections around the Steinberg point inside the fundamental alcove (the exmaple above is the case $\mathfrak{sl}_2$).

6.3. **Outlook on Logarithmic Kazhdan-Lusztig Correspondence.** In this article we have constructed a graded infinite-dimensional representation of a diagonal Nichols algebra $\mathcal{B}((q_{ij})_{i,j})$, and in particular of a quantum group Borel part $u_q(\mathfrak{g})^+$, by letting it act via screening operators on any fractional lattice VOA $\mathcal{V}_\Lambda$ where $q_{ij} = e^{\pi \mathrm{i}(\alpha_i,\alpha_j)}$ (and smallness) hold. We now want to sketch a conjectural path to Kazhdan-Lusztig correspondence (also in a more general setting of Nichols algebras). Essentially the quantum field theory faciliates a invertible bimodule with an action of the affine Lie algebra and of the quantum group, giving the category equivalence. We want to be more precise:

Fact 6.6. *For a given basis of the root lattice (resp. generalized Weyl chamber for Nichols algebra) there is a* conformal structure *on the lattice VOA $\mathcal{V}_\Lambda$ i.e. an additional representation of the Virasoro algebra. This has been shown in* [FT10] *for quantum groups (simply-laced) Lie algebras and in* [ST12] *in the Nichols algebra setting.*

Conjecture 6.7. *More precisely: There is a so-called* short *basis α_i^{short} of the fractional lattice Λ and an associated* long *basis α_i^{long} of some integral sublattice Λ^{long}, such that the conformal weights of all these e^{ϕ_α} are 1. This implies the associated* long screening operators *$\mathfrak{Z}_{\alpha_i^{long}}$ are Virasoro algebra homomrphisms (and* morally*, so are the short screening operators, but this is not precisely true!).*

Again for simply-laced Lie algebras this is already proven in [FT10] *and they identify $\Lambda^{long} = \sqrt{p}\Lambda_\mathfrak{g}$ as again the rescaled root lattice. The author would conjecture that in the general Lie algebra case $\Lambda^{long} = \sqrt{p}\Lambda_\mathfrak{g}^\vee$ is the dual root system and in the general Nichols algebra case the subset of Cartan-like roots* [Ang15] *generates the respective root lattice of the Lie algebra.*

The previous conjecture seems to be rather easy, requiring only calculations with conformal dimensions and central charges. Now, in particular our article shows:

Theorem (see 6.1)**.** *The short screenings $\mathfrak{Z}_{\alpha_i^{short}}$ consitute a representation of the Nichols algebra in question.*

The next conjecture was already proven for simply-laced Lie algebras in [FT10] and it the analogy of our results for long screenings (requiring no fractional calculations):

Conjecture 6.8. *The long screenings* $\mathfrak{Z}_{\alpha_i^{long}}$ *constitute a representation of the resp. Lie algebra (dual root system for multiply-laced or Cartan-type roots for Nichols algebras), by construction commuting with the action of the Virasoro algebra.*

Note that this conjecture requires technically to change to a deformation $\mathcal{V}_\Lambda^{(\Omega,1)}$*, because naturally the Lie algebra will incorprate commutators and anticommutators, see the example in Lemma 3.17.*

Now these operators can be used to define subspaces of lattice VOA, which are by themselves much more intersting VOA's. This method of presenting a VOA is in general called *free-field realization* [FF88] resp. [FGST06a][FT10]:

Conjecture 6.9. *The subspace of the (non-fractional) lattice VOA* $\mathcal{V}_{\Lambda^{long}}$

$$\mathrm{W} := \mathcal{V}_{\Lambda^{long}} \cap \bigcap_i \ker \mathfrak{Z}_{\alpha_i^{short}} \cap \bigcap_i \ker \mathfrak{Z}_{\alpha_i^{long}}$$

is in the Lie algebra case isomorphic to the respective W*-algebra, i.e. the so-called Hamilton- or Quantum-Drinfel'd-Sokolov-reduction of the affine Lie algebra* $\hat{g}$ *at level* ℓ*.*

This alternative construction for W*-algebras is true for generic* q [FF88] *and otherwise widely believed, but technically a hard conjecture of great independent interest!*

Definition 6.10. *The larger subspace of the (non-fractional) lattice VOA* $\mathcal{V}_{\Lambda^{long}}$

$$\mathcal{W} := \mathcal{V}_{\Lambda^{long}} \cap \bigcap_i \ker \mathfrak{Z}_{\alpha_i^{short}}$$

should be the interesting Logarithmic conformal field theory (LCFT)*.*

Conjecture 6.11. *The subspace* $\mathcal{W}^{log} \supset \mathrm{W}$ *is a vertex subalgebra with an action of the Lie algebra* $\mathfrak{g}$ *from the long screenings. It is generated as such by a pure exponential* $\mathrm{e}^{\phi_{-\alpha}}$ *for highest roots* α*, spanning the adjoint representation of* $\mathfrak{g}$*.*

Conjecture 6.12. *The representation theory of* $\mathcal{W}^{log}$ *should be interesting:*

- *It is a non-semisimple modular tensor category.*
- *It is as abelian category equivalent to the representation category of the respective quantum group (or Drinfel'd double of the Nichols algebra).*
- *Is is as modular tensor category equivalent to the represention category of a quasi-Hopf algebra generalizing the quantum group* [GR15]
- *The action of the mapping class group* $\mathrm{SL}_2(\mathbb{Z})$ *on the VOA representation and on the quantum group center coincide* [FGST06b][RT14].

Moreover the representation theory of the smaller kernel of the long screenings should be equivalent to the representation theory of Lusztig's infinite quantum group of divided powers (or more generally the post-Nichols algebra [AAB15]*).*

Once these conjectures are proven, they explain Kazhdan-Lusztig corrensponდence as a category equivalence between these modular tensor categories (which is a finite extension of the infinite representation theory of the W-algebra). More precisely the correspondence would arise from the existence of a quantum field theory with commuting actions of an affine Lie algebra and the respective quantum group.

Conjecture 6.13. *In the authors opinion one should try to prove this via intermediate steps, which also make more clear why the correspondence should hold:*

- *Prove that all irreducible modules of the LCFT arise as unique irreducible quotients of restrictions of the (known) irreducible modules of the lattice VOA (which Feigin accordingly calls "Verma modules"). This gives the category equivalence on the level of the Grothendieck ring of the categories. The modules seem accessable (see below), in particular their groundstates, but showing these are all irreducible should be an independent calculation in term of Zhu's algebra*
- *More precisely, introduce for each element of the Weyl group (resp. Weyl groupoid of the Nichols algbebra) an action of the Weyl group on the representations of the lattice VOA, using respective combinations of short screenings. While usually short screening are surprisingly bad behaved with respect to Virasoro action, these specific combinations precisely intertwine the two different Virasoro actions from the first conjecture. This proof succeeds with the techniques in this article, see Section 6.2.*
- *The key idea is that short screenings (at least these combinations) become VOA-module homomorphisms if we restrict our lattice VOA to the kernel of those screenings, which is precisely the VOA. This shows many of the submodules in the restrictions appearing and the conjecture is that they completely determine the decomposition behaviour of the restricted module.*
- *In particular, the formerly irreducibles should over the LCFT decompose into new irreducibles in a way, that prove for these new irreducibles non-vanishing* Ext*-groups precisely for orbits of the Weyl group, and hence precisely shows the same nontrivial* Ext*-groups as the other side of the correspondence demand by the* linkage principle.
- *As a last step, we should need to construct a bimodule, that consists of all reflections of the lattice VOA glued together along those reflections. Such has (without this interpretation) done by hand in* [FGST06a] *for* $\mathfrak{sl}_2$, $\ell = 4$. *In other words*

this means constructing a projective cover of the trivial (vacuum-) representation on the VOA side and showing that it has a $u_q(\mathfrak{g})$-action (rather than just a $u_q(\mathfrak{g})^+$-action). One approach to construct such a projective cover is through regularized screening in a current collaborative work of the author with Feigin and Semikhatov for $\mathfrak{sl}_2$.

The author proposes to extend the question to more and more general Hopf algebras. In essence, the ideal picture would be for each modular tensor category a realizing LCFT. The impression is that this can only work though existing classification on the Hopf algebra side: For example [AS10] shows that every finite-dimensional *pointed* Hopf algebra over an abelian groupring looks like a quantum group, but one has additionally the possibilties of other diagonal Nichols algebras. One step further, the author has in [Len12][Len14] constructed non-diagonal Nichols algebras by a *folding procedure* and [HV14] show that these are essentially all such Nichols algebras (probably the only ones leading to nontrivial modular tensor categories). The authors conjectures

Conjecture 6.14. *Assume $\mathfrak{g}$ a Lie algebra, $\ell = 4$ and θ a diagram automorphism. Then we can consider*

- *An orbifold model of the LCFT above for $\mathfrak{g}, \ell$ by the automorphism θ.*
- *A factorizable Hopf algebra associated to the authors non-diagonal Nichols algebra associated to $\mathfrak{g}, \theta$.*

Are their representation categories equivalent?

Even one more step to non-pointed examples becomes more difficult, because the groupring of the lattice VOA needs to be replaced by a semisimple Hopf algebra. Are there such free-field theories ?

6.4. **Example $\mathfrak{sl}_2$ at $\ell = 4$.** We will make the vague outlook in the previous section precise on the example $\mathfrak{g} = \mathfrak{sl}_2$ where the main conjectures have been proven in [FGST06a] for $\ell = 4$ and subsequently [NT11] for arbitrary ℓ.

Let $(\alpha, \alpha) = 2$ and $\ell = 2p \in \mathbb{N}$ and consider the lattice VOA $\mathcal{V}_{\frac{1}{\sqrt{p}}\alpha\mathbb{Z}} = \mathcal{V}_{\frac{2}{\sqrt{p}}\mathbb{Z}}$. As one can check by elementary calculations there is a conformal structure[12] (i.e. a Virasoro algebra action) on this VOA with an L_0-grading (made explicitly below), such that precisely the two pure exponentials $\mathrm{e}^{-\alpha/\sqrt{p}}$ and $\mathrm{e}^{+\alpha\sqrt{p}}$ are L_0-eigenvectors to eigenvalue (=conformal weight) 1.

[12]Energy-stress tensor $\frac{1}{4}\partial\phi_\alpha\partial\phi_\alpha - (\sqrt{p} - 1/\sqrt{p})\partial^2\phi_{\alpha/2}$ with central charge $13 - 6(p + \frac{1}{p})$

This means the following choice for short resp. long *screening operators* should be studied as they have nice compatibility with the Virasoro action:

$$\mathfrak{Z}_{-\alpha/\sqrt{p}} \qquad \mathfrak{Z}_{+\alpha\sqrt{p}}$$

Our results in this article show from $q_{11} = e^{\pi \mathrm{i}(-\alpha/\sqrt{p},-\alpha/\sqrt{p})} = e^{\frac{2\pi \mathrm{i}}{p}}$ that the Nichols algebra relation $(\mathfrak{Z}_{-\alpha/\sqrt{p}})^p = 0$ holds and thus the short screening generates $u_q(\mathfrak{sl}_2)^+$. On the other hand $e^{\pi \mathrm{i}(+\alpha\sqrt{p},+\alpha\sqrt{p})} = e^{2\pi \mathrm{i} p} = 1$ and the long screening $\mathfrak{Z}_{+\alpha\sqrt{p}}$ generates the Borel part $U(\mathfrak{sl}_2)^+$. Moreover since $e^{\pi \mathrm{i}(-\alpha/\sqrt{p},+\alpha\sqrt{p})} = e^{-2\pi \mathrm{i}} = 1$ these two actions commute.

The kernel of the short screening is known as the *triplet algebra*

$$\mathcal{W}(p) \cong \mathcal{V}_{\sqrt{p}\alpha\mathbb{Z}} \cap \ker \mathfrak{Z}_{+\alpha/\sqrt{p}}$$

This sub-VOA is known to be generated by a pure exponential and the $\mathfrak{sl}_2$-action spanning an adjoint representation:

$$\begin{aligned}
\mathcal{W}^- &:= e^{-\phi_{\sqrt{p}\alpha}} \\
\mathcal{W}^0 &:= \mathfrak{Z}_{+\alpha\sqrt{p}} e^{-\phi_{\sqrt{p}\alpha}} \\
\mathcal{W}^+ &:= \mathfrak{Z}_{+\alpha\sqrt{p}} \mathfrak{Z}_{+\alpha\sqrt{p}} e^{-\phi_{\sqrt{p}\alpha}}
\end{aligned}$$

Note that $\mathcal{W}^0$ is always a pure differential polynomial.

Example 6.15. *For the trivial case $p = 1$ get the affine Lie algebra at trivial level:*

$$\begin{aligned}
\mathcal{W}^- &= \mathrm{e}^{\phi_{-\alpha}} \\
\mathcal{W}^0 &= \partial\phi_\alpha \\
\mathcal{W}^+ &= -2\mathrm{e}^{+\phi_\alpha}
\end{aligned}$$

For the first nontrivial case $p = 2$, and was studied in terms of symplectic fermions

$$\begin{aligned}
\mathcal{W}^- &:= \mathrm{e}^{\phi_{-\alpha\sqrt{2}}} \\
\mathcal{W}^0 &:= \frac{1}{3!}\left(\partial\phi_{\alpha\sqrt{2}}\partial\phi_{\alpha\sqrt{2}}\partial\phi_{\alpha\sqrt{2}} + 3\partial\phi_{\alpha\sqrt{2}}\partial^2\phi_{\alpha\sqrt{2}} + \partial^3\phi_{\alpha\sqrt{2}}\right) \\
\mathcal{W}^+ &:= 8\left(\partial\phi_{\alpha\sqrt{2}}\partial\phi_{\alpha\sqrt{2}} + \partial^2\phi_{\alpha\sqrt{2}}\right)\mathrm{e}^{\phi_{+\alpha\sqrt{2}}}
\end{aligned}$$

The representation theory $\mathcal{W}(p)$ is nowadays known:

For the trivial case $p = 1$ the representation theory is semisimple, for $p > 1$ the VOA will be logarithmic modular in the following sense:

Theorem 6.16 ([AM08]). *The representation theory of $\mathcal{W}(p)$, $p \geq 2$ is as follows*

(1) *It has $2p$ irreducible representations denoted $\Lambda(i), \Pi(i), i = 1 \ldots p$.*
(2) *It is non-semisimple (i.e L_0 acts non-diagonalizable on some modules i.e. physical correlators have logarithmic singularities in addition to poles).*
(3) *It is C_2-cofinite* [CF06].

We now discuss the restriction of $\mathcal{V}_\Lambda$-representations to $\mathcal{W}(p)$ in our approach:

The representations of any $\mathcal{V}_\Lambda$ are enumerated by cosets $[\lambda] \in \Lambda/\Lambda^*$, here

$$\Lambda/\Lambda^* = \frac{1}{\sqrt{p}}\alpha\mathbb{Z}/\sqrt{p}\frac{\alpha}{2}\mathbb{Z} \cong \mathbb{Z}_{2p}$$

We enumerate those elements in each coset by

$$\lambda_k := k\frac{\alpha}{2}/\sqrt{p} \qquad \text{for a coset } [k \bmod 2p]$$

The L_0-eigenvalues of the pure exponentials $\mathrm{e}^{\phi_{\lambda_k}}$ give a parabola

$$\frac{(k-(p-1))^2}{4p} + \frac{c-1}{24} = \frac{k(k-2p+2)}{4p} = h_{k+1,1}$$

The choice for c is precisely the $(1,p)$ *minimal model* and the L_0-eigenvalues h are precisely the discrete series of $\mathcal{V}ir_c$-modules.

The groundstates (in the sense of Zhu's algebra) of the module $\mathcal{V}_{[\lambda]}$ are $\mathrm{e}^{\phi_\lambda}$ for those representatives λ with minimal distance to the *Steinberg point* $Q = \lambda_{p-1} = (p-1)\frac{\alpha}{2}/\sqrt{p}$. So the groundstate is 2-dimensional with $k = -1, 2p-1$ for the class $[-1]$ and 1-dimensional with $k \in \{0, \ldots 2p-2\}$ for $[k]$. They begin with $k = 0, h_{1,1} = 0$ for the adjoint representation $\mathcal{V}_{[0]} = \mathcal{V}$ (vaccum representation) and ends with again $k = p-2, h_{2p,1} = 0$ and then the unique representation with 2-dimensional groundstates $k = 2p-1, -1$.

Example 6.17 ($p = 2$).

$k =$	(-1)	0	$1 = p-1$	2	$3 = 2p-1$
$h_{k+1,1} =$	$(\frac{3}{8})$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$

We now study the restriction of these modules to $\mathcal{W}(p)$-modules, denoted $\Lambda(1), \ldots \Lambda(p)$ with 1-dimensional groundstate of L_0 eigenvalue $h_{1,1} \cdots h_{p,1}$ and $\Pi(1), \ldots \Pi(p)$ with 2-dimensional groundstate of common L_0 eigenvalue $h_{3p-1,1} \cdots h_{3p-p,1}$.

From the short screenings $\mathfrak{Z}_{-\alpha/\sqrt{p}}$ we build Weyl reflections around Steinberg point:

$$\mathfrak{Z}_{-\alpha/\sqrt{p}}^k : \mathcal{V}_{[\lambda_{(p-1)+k}]} \longrightarrow \mathcal{V}_{[\lambda_{(p-1)-k}]}$$

For $k = p-1$ (Steinberg point) and $k = 2p-1$ (mapping to $[-1] = [2p-1]$) this operator maps the module to itself and they stay irreducible over $\mathcal{W}(p)$, more precisely the irreducibles[13] $\Lambda(p), \Pi(p)$.

All other modules arise in pairs $(p-1)+k, (p-1)-k$ and decompose accordingly into extensions of kernel and image of $(\mathfrak{Z}_{-\alpha/\sqrt{p}})^k$

$$0 \to \Lambda(i) \to \mathcal{V}_{[\lambda_{i-1}]} \to \Pi(p-i) \to 0 \qquad i = 1, \ldots, p-1$$

$$0 \to \Pi(p-i) \to \mathcal{V}_{[\lambda_{2p-2-(i-1)}]} \to \Lambda(i) \to 0 \qquad i = 1, \ldots, p-1$$

One can show this abelian category is equivalent to $u_q(\mathfrak{sl}_2)$ at $\ell = 2p$ [NT11].

We want to stress that usually short screenings are *not* $\mathcal{V}ir$-homomorphisms and usually map to infinite sums. The reflections around the Steinberg point are extremely rare and show a different behaviour. This assumption is implicit in [Fel89] and for $\mathfrak{sl}_2$ this was proven by [TW13]. Our article now explains this behaviour in generality in Lemma 6.4, because the term should vanish by Nichols algbebra reasons, but a pole in the Selberg integral causes single nonzero summands (we can show this yields in general a Virasoro-homomorphism, but this is beyond the scope of this article).

The case $\mathfrak{sl}_2$ corresponds to example 6.5:

$$(\mathfrak{Z}_{-\alpha/\sqrt{p}})^k : \ \mathcal{V}_{[p-1+k]} \mapsto \mathcal{V}_{[p-1-k]}$$

with coefficients $m_{ij} = 2/p, \ i,j \leq k$ and $m_i = -(p-1+k)/p$.

Remark 6.18 (see [AM08]). *We can easily convince ourselves on the level of graded characters and modular forms that this decompostition corresponds to a decomposition of theta function into related $3p-1$-vector-valued modular functions with componentes given by characters*

$$\mathrm{ch}_{1,\mathcal{V}_{[\lambda_{p-1\pm k}]}}(q) = \frac{\Theta_{p,k}(q)}{\eta(q)}$$

$$\mathrm{ch}_{1,\Lambda(i)}(q) = \frac{i\ \Theta_{p,p-i}(q) + 2p\ \partial\Theta_{p,p-i}}{p\ \eta(q)}$$

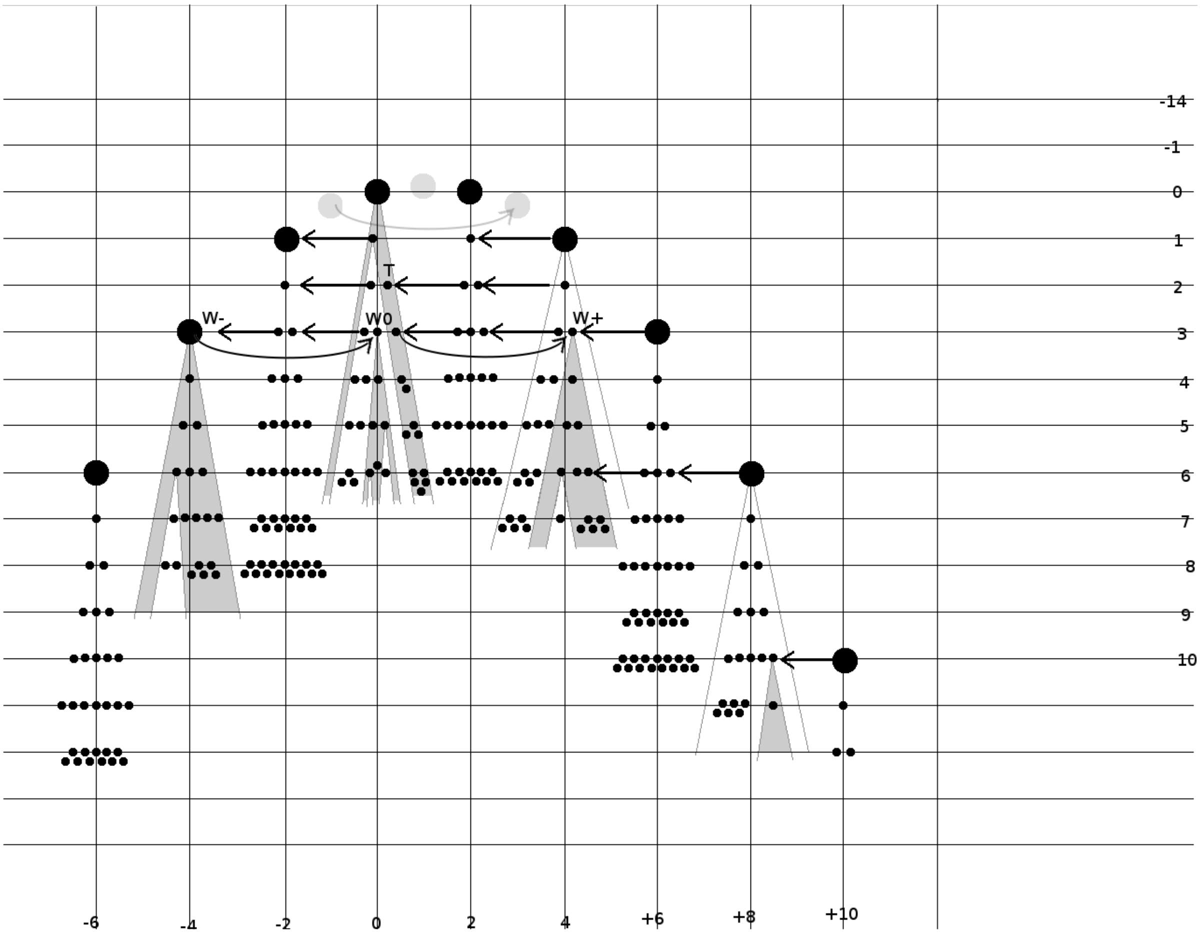
$$\mathrm{ch}_{1,\Pi(i)}(q) = \frac{i\ \Theta_{p,i}(q) - 2p\ \partial\Theta_{p,i}}{p\ \eta(q)}$$

and $p-1$ pseudocharacters [CG16]*, where one defines*

$$\Theta_{p,k}(q) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{4p}(k+2pn)^2} \qquad \partial\Theta_{p,k}(q) := \sum_{n\in\mathbb{Z}} (n + \frac{k}{2p}) q^{\frac{1}{4p}(k+2pn)^2}$$

[13] This notation is due to Semikhatov and the letters Λ, Π should visualize the 1- resp. 2-dim groundstates.

We will summarize this on a picture in the case $p=2$: Dots denote basis vectors, the Y-axis denotes L_0 eigenvalues and the X-axis denotes k; recall that all $k\mathrm{mod}4$ belong to the same representation (the grey dots should remind on the odd-k representation which are not part of $\frac{1}{\sqrt{p}}\Lambda$ and hence not in the principal block). The straight short arrows denote short screenings $\mathfrak{Z}^2_{-\alpha/\sqrt{p}}=0$ and the bent long arrows denote long screenings giving the $\mathfrak{sl}_2$ action. Grey areas are the kernel of the short screening in the adjoint (vacuum-) representation $\mathcal{V}_{[0]}=\mathcal{V}_{\sqrt{p}\Lambda}$ i.e. the LCFT $\mathcal{W}(p)$.



Acknowledgement. *I am indebted and greatful to A. Semikhatov, for hospitality in Moscow and for countless hours of exposing me to this topic and his ideas, to C. Schweigert, whose continuing support and inspiration made all of this possible, and to B. Feigin, who created such beauty, continuously answered my questions and explained the big picture. Also many thanks to Christian Reiher for enduring help on the analysis side and regarding the smallness-condition.*

My visit in Moscow with the DAAD PRIME program was funded by the BMBF Germany and the Marie Curie Action of the EU. I am also greatful for partial support from the DFG Graduiertenkolleg 1670 at the University Hamburg.

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