INHABITANTS OF INTERESTING SUBSETS OF THE BOUSFIELD LATTICE

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Abstract. The set of Bousfield classes has some important subsets such as the distributive lattice $\mathbf{DL}$ of all classes $\langle E\rangle$ which are smash idempotent and the complete Boolean algebra $\mathbf{cBA}$ of closed classes. We provide examples of spectra that are in $\mathbf{DL}$, but not in $\mathbf{cBA}$; in particular, for every prime p, the Bousfield class of the Eilenberg-MacLane spectrum $\langle H\mathbb{F}_p\rangle$ is in $\mathbf{DL}\backslash\mathbf{cBA}$.

1. Introduction & Definitions

In the original paper [1] introducing the Bousfield lattice $\mathbf{B}$, Bousfield also introduces its subsets $\mathbf{BA}$ and $\mathbf{DL}$ and identifies the location of many explicit Bousfield classes. In [4, Definition 6.3], Hovey and Palmieri add a third interesting subset, denoted by $\mathbf{cBA}$. (We shall give definitions below.) It is easy to see that

$$\mathbf{BA} \subseteq \mathbf{cBA} \subseteq \mathbf{DL} \subseteq \mathbf{B}.$$

In this paper, we deal with the question of which and how many spectra live in the various parts of $\mathbf{B}$ defined by this chain of inclusions. The main cardinality results of this paper (lower bounds) are graphically represented as in Figure 1 and concern the dark grey parts.

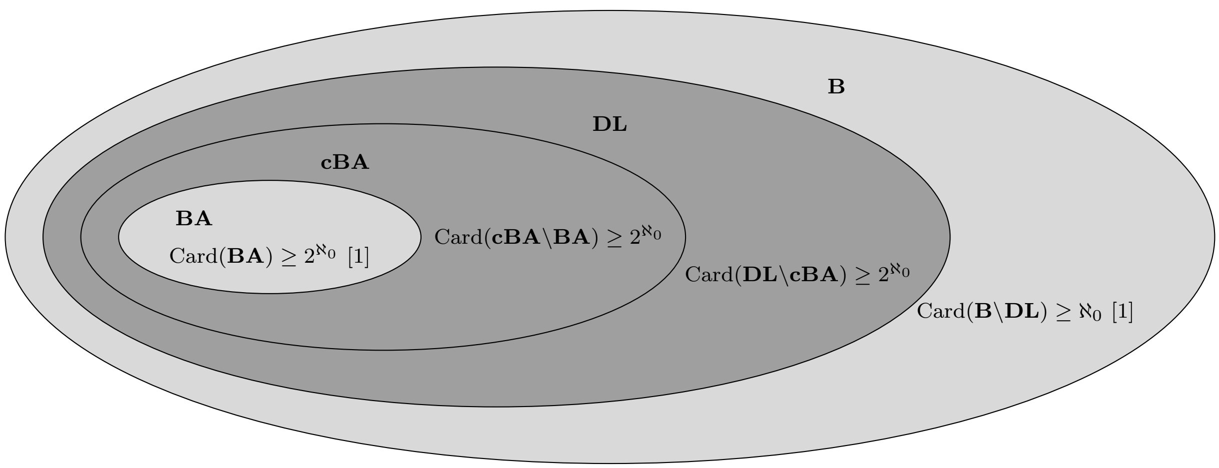

Figure 1. Lower bounds for the sizes of the four differences of subsets of $\mathbf{B}$.

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2. Definitions

In order to fix notation, we give the relevant definitions, following closely the exposition in [4]. We consider the Bousfield equivalence of spectra [1]: two spectra X and Y are equivalent if for all spectra E, $X_*(E) = 0$ if and only $Y_*(E) = 0$ (alternatively put: $X \wedge E \simeq *$ if and only if $Y \wedge E \simeq *$). For a spectrum X, we write $\langle X \rangle$ for the class of all spectra E with $X_*(E) = 0$. The class of all Bousfield classes is denoted by $\mathbf{B}$. By a theorem of Ohkawa [5, 2], it is known that $\mathbf{B}$ is a set and

$$2^{\aleph_0} \leq \mathrm{Card}(\mathbf{B}) \leq 2^{2^{\aleph_0}}.$$

This set is a poset with respect to reverse inclusion: $\langle X \rangle \leq \langle Y \rangle$ if and only if for all spectra Z, $Y_* Z = 0$ implies $X_* Z = 0$. The poset $(\mathbf{B}, \leq)$ has a largest element $\mathbf{1} := \langle S \rangle$ where S is the sphere spectrum and we denote by $\mathbf{0}$ the minimal element which is the Bousfield class of the trivial spectrum. We work at a fixed but arbitrary prime p, *i.e.*, we consider p-local spectra.

For every prime p, $K(n)$ denotes the nth Morava K-theory spectrum with coefficients $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$ where the degree of v_n is $2p^n - 2$. We use the convention that $K(\infty)$ is the mod p Eilenberg-MacLane spectrum, $H\mathbb{F}_p$. For any subset $S \subseteq \mathbb{N} \cup \{\infty\}$, we denote by $K(S)$ the spectrum $\bigvee_{n \in S} K(n)$.

The topological operations $\wedge$ and $\vee$ of taking smash products and wedges, respectively, are well-defined on $\mathbf{B}$; the class $\langle \bigvee_{i \in I} X_i \rangle$ is the least upper bound ("join") in the structure $(\mathbf{B}, \leq)$ of the classes $\langle X_i \rangle$ [1, (2.2)], but in general, $\wedge$ does not produce the greatest lower bound. We can define the greatest lower bound ("meet") by

$$\bigcurlywedge \mathcal{X} := \bigvee \{Z \,;\, \forall X \in \mathcal{X} (Z \leq X)\},$$

and observe that $\wedge$ and $\curlywedge$ can differ quite a bit: the Brown-Comenetz dual I of the p-local sphere spectrum satisfies $\langle I \rangle \wedge \langle I \rangle = \mathbf{0} \neq \langle I \rangle = \langle I \rangle \curlywedge \langle I \rangle$ [1, Lemma 2.5].

The complete lattice $(\mathbf{B}, \curlywedge, \vee)$ is endowed with a pseudo-complementation function

$$aX := \bigvee \{Z \,;\, Z \wedge X = 0\}$$

which is well-defined on Bousfield classes, *i.e.*, $a\langle X \rangle := \langle aX \rangle$ is independent of the choice of representative X of $\langle X \rangle$. The function a is not in general a complement. While $a^2 = \mathrm{id}$ and $a\langle X \rangle \wedge \langle X \rangle = \mathbf{0}$, we may not have $a\langle X \rangle \vee \langle X \rangle = \mathbf{1}$ [1, Lemma 2.7]. Bousfield defined two subclasses of $\mathbf{B}$ as follows:

$$\mathbf{BA} := \{\langle X \rangle \,;\, \langle X \rangle \vee a\langle X \rangle = \mathbf{1}\}, \text{ and}$$
$$\mathbf{DL} := \{\langle X \rangle \,;\, \langle X \rangle \wedge \langle X \rangle = \langle X \rangle\}.$$

Many examples for classes in $\mathbf{BA}$ or $\mathbf{DL}$ are known. Bousfield showed in [1] that every Moore spectrum of an abelian group is in $\mathbf{BA}$ and so are the periodic topological K-theory spectra $\langle KO \rangle = \langle KU \rangle$; furthermore, he shows that (arbitrary joins of) finite CW spectra also give classes in $\mathbf{BA}$. Every class of a ring spectrum is in $\mathbf{DL}$ but not necessarily in $\mathbf{BA}$ [1, § 2.6]; in particular, all Eilenberg-MacLane spectra of rings are in $\mathbf{DL}$, but, *e.g.*, the class of the Eilenberg-MacLane spectrum of the integers, $\langle H\mathbb{Z} \rangle$, is in $\mathbf{DL} \backslash \mathbf{BA}$ [1, Lemma 2.7]. However, the Brown-Comenetz duals of (p-local) spheres are not in $\mathbf{DL}$ [1, Lemma 2.5].

We have that $\mathbf{BA} \subseteq \mathbf{DL}$; on $\mathbf{DL}$, $\wedge$ and $\curlywedge$ coincide, and $(\mathbf{DL}, \wedge, \vee)$ is a distributive lattice. Furthermore, on $\mathbf{BA}$, a is a true complement, so $(\mathbf{BA}, \wedge, \vee, \mathbf{0}, \mathbf{1}, a)$ is a Boolean algebra, but not complete.

There is a retraction from $\mathbf{B}$ to $\mathbf{DL}$ defined by

$$r\langle X \rangle := \bigvee \{\langle Z \rangle \,;\, \langle Z \rangle \in \mathbf{DL} \text{ and } \langle Z \rangle \leq \langle X \rangle\}.$$

The pseudo-complementation function a may not respect $\mathbf{DL}$, *i.e.*, it could be that $\langle X \rangle \in \mathbf{DL}$, but $a\langle X \rangle \notin \mathbf{DL}$. On $\mathbf{DL}$, we therefore define a new pseudo-complement by

$$A\langle X \rangle := ra\langle X \rangle.$$

While $A^3 = A$ and $\langle X \rangle \leq A^2 \langle X \rangle$, it is not in general the case that $A^2 = \mathrm{id}$. It is known [4, Lemma 6.2(d)] that A converts joins to meets, *i.e.*,

$$A(\bigvee \mathcal{X}) = \bigcurlywedge \{A\langle X \rangle \,;\, X \in \mathcal{X}\}.$$

Following [4, Definition 6.3], we define

$$\mathbf{cBA} := \{\langle X \rangle \in \mathbf{DL} \,;\, A^2 \langle X \rangle = \langle X \rangle\}.$$

The set **cBA** carries a complete Boolean algebra structure [4, Theorem 6.4]; however, it is not $(\mathbf{cBA}, \wedge, \vee, \mathbf{0}, \mathbf{1}, A)$, but instead $(\mathbf{cBA}, \wedge, \curlyvee, \mathbf{0}, \mathbf{1}, A)$ with $\curlyvee$ defined by

$$\bigcurlyvee \mathcal{X} := A^2 \bigvee \mathcal{X}.$$

3. Results

We start with an observation on joins of elements in **BA** and use this to derive lower bounds for the size of **DL****cBA** and **cBA****BA**.

Lemma 1. *If $\mathcal{X} \subseteq \mathbf{BA}$, then $\bigcurlyvee \mathcal{X} = \bigvee \mathcal{X}$. In particular, $\bigvee \mathcal{X} \in \mathbf{cBA}$.*

Proof. We have that

$$\bigcurlyvee \mathcal{X} = A^2 \bigvee \mathcal{X} = rara \bigvee \mathcal{X},$$

and as a converts joins to meets, the latter is equal to

$$rar \bigcurlywedge \{a\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}.$$

Since every $a\langle X\rangle$ is in **BA**, it is also in **DL**, and as **DL** is complete,

$$\Xi := \bigcurlywedge \{a\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\} \in \mathbf{DL}$$

and hence $r\Xi = \Xi$. Therefore, as a sends meets to joins,

$$\begin{aligned} rar\Xi &= ra\Xi \\ &= r \bigvee \{a^2\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\} \\ &= r \bigvee \{\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\} \\ &= \bigvee \mathcal{X}. \end{aligned}$$

$\square$

Proposition 2. *If $S \subseteq \mathbb{N}$ is infinite, then $\langle K(S)\rangle = \bigvee_{i\in S} \langle K(i)\rangle \in \mathbf{cBA}\backslash\mathbf{BA}$ and $\langle K(S)\rangle \geq \langle I\rangle$.*

Proof. By Lemma 1, $\langle K(S)\rangle$ is in **cBA**. Hovey showed [3, Proof of Theorem 3.6] that the mod-p Moore spectrum, $M(p)$ is $K(S)$-local, so in particular $K(S)$ has a finite local and [4, Proposition 7.2] gives that $\langle K(S)\rangle \geq \langle I\rangle$. If $K(S)$ were in **BA**, having a finite local implies [4, Lemma 7.9] that $\langle K(S) \wedge I\rangle \neq \mathbf{0}$. But we know that $\langle K(n) \wedge I\rangle = \mathbf{0}$ and hence using distributivity we get that $\langle K(S) \wedge I\rangle = \mathbf{0}$. $\square$

Corollary 3. *We have a proper inclusion $\mathbf{BA} \subsetneqq \mathbf{cBA}$; in fact, the set **cBA****BA** has size continuum.*

Proof. Because **BA** is a Boolean algebra, $a\langle X\rangle \in \mathbf{BA}$ for elements $\langle X\rangle \in \mathbf{BA} \subseteq \mathbf{DL}$. Therefore, $A\langle X\rangle = ra\langle X\rangle = a\langle X\rangle$. But $a^2 = \mathrm{id}$, so "$\subseteq$" holds. For the non-equality, if $S \neq S'$ are infinite subsets of $\mathbb{N}$, then Dwyer and Palmieri showed that $\langle K(S)\rangle \neq \langle K(S')\rangle$ [2, Lemma 3.4], so there are continuum many elements in the complement. $\square$

To sum up, we have

$$\mathbf{BA} \subsetneqq \mathbf{cBA} \subseteq \mathbf{DL} \subsetneqq \mathbf{B}.$$

Hovey and Palmieri argue that the middle inclusion is also proper:

> This argument also implies that A^2 is not the identity—indeed, if A^2 were the identity, one can check that A would have to convert meets to joins. However, we do not know a specific spectrum X in **DL** for which $A^2\langle X\rangle \neq \langle X\rangle$. [4, p. 185]

We analyse the argument sketched in the above quote:

Lemma 4. *Let $\mathcal{X} \subseteq \mathbf{DL}$ be any set such that A^2 is the identity for each $\langle X\rangle \in \mathcal{X}$ and for $\bigvee\{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}$. Then*

$$A(\bigcurlywedge \mathcal{X}) = \bigvee \{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}.$$

Proof. Since A converts joins to meets, under the assumption of the lemma, we have

$$\begin{aligned} A(\curlywedge \mathcal{X}) &= A \curlywedge \{A^2\langle X\rangle\, ;\, \langle X\rangle \in \mathcal{X}\} \\ &= A^2 \bigvee \{A\langle X\rangle\, ;\, \langle X\rangle \in \mathcal{X}\} \\ &= \bigvee \{A\langle X\rangle\, ;\, \langle X\rangle \in \mathcal{X}\}. \end{aligned}$$

□

Corollary 5 (Hovey-Palmieri)**.** *The operation A^2 is not the identity on* **DL***;* i.e., $\mathbf{cBA} \subsetneq \mathbf{DL}$.

Proof. Let $X := K(\mathbb{N})$, $Y := H\mathbb{F}_p = K(\infty)$, and $\mathcal{X} := \{X, Y\} \subseteq \mathbf{DL}$. We assume towards a contradiction that A^2 is the identity on $\mathbf{DL}$, so in particular, the assumptions of Lemma 4 are satisfied for $\mathcal{X}$. But $\langle X\rangle \curlywedge \langle Y\rangle = \langle X\rangle \wedge \langle Y\rangle = \mathbf{0}$, hence $A(\langle X\rangle \curlywedge \langle Y\rangle) = \mathbf{1}$. On the other hand, $A\langle X\rangle \vee A\langle Y\rangle \leq a\langle I\rangle < \mathbf{1}$, in contradiction to Lemma 4. □

The proof of Corollary 5 due to Hovey and Palmieri yields a trichotomy result: at least one of $\langle K(\mathbb{N})\rangle$, $\langle H\mathbb{F}_p\rangle$, and $A\langle K(\mathbb{N})\rangle \vee A\langle H\mathbb{F}_p\rangle$ is not in $\mathbf{cBA}$. We improve this in our Dichotomy Lemma 7 to a dichotomy which will allow us to identify concrete elements in $\mathbf{DL}\backslash\mathbf{cBA}$.

Lemma 6. *For any spectrum, the condition $A\langle E\rangle < \mathbf{1}$ is equivalent to $\langle E\rangle \neq \mathbf{0}$.*

Proof. If $\langle E\rangle = \mathbf{0}$, then clearly $A\langle E\rangle = \mathbf{1}$. Conversely, if $A\langle E\rangle = \mathbf{1}$, then $a\langle E\rangle \geq A\langle E\rangle = \mathbf{1}$, and so

$$\langle E\rangle = \mathbf{1} \wedge \langle E\rangle = a\langle E\rangle \wedge \langle E\rangle = \mathbf{0}.$$

□

Lemma 7 (Dichotomy Lemma)**.** *Let X and Y be spectra, and let E be a spectrum such that $\langle E\rangle \neq \mathbf{0}$. Suppose that the following conditions hold:*

(1) $\langle X\rangle \in \mathbf{DL}$,
(2) $\langle Y\rangle \in \mathbf{DL}$,
(3) $\langle X\rangle \wedge \langle Y\rangle = \mathbf{0}$,
(4) $\langle E\rangle \leq \langle X\rangle$, *and*
(5) $\langle E\rangle \leq \langle Y\rangle$.

Then $\langle X\rangle$ or $\langle Y\rangle$ is not in **cBA**.

Note that conditions (4) and (5) are equivalent to saying that $\langle X\rangle \curlywedge \langle Y\rangle \neq \mathbf{0}$, and thus the Dichotomy Lemma extracts the failure of $A^2 = \mathrm{id}$ from the discrepancy between $\curlywedge$ and $\wedge$ in $\mathbf{B}$.

Proof. Assume that $A^2\langle X\rangle = \langle X\rangle$ and $A^2\langle Y\rangle = \langle Y\rangle$. Since A converts joins to meets, we get by our assumption on X and Y

$$\mathbf{1} = A\mathbf{0} = A(\langle X\rangle \wedge \langle Y\rangle) = A(A^2\langle X\rangle \wedge A^2\langle Y\rangle) = A^2(A\langle X\rangle \vee A\langle Y\rangle)$$

and the latter is $A\langle X\rangle \curlyvee A\langle Y\rangle$ by definition of $\curlyvee$. As A is order-reversing we get $A\langle X\rangle \leq A\langle E\rangle$ and $A\langle Y\rangle \leq A\langle E\rangle$ and hence (using Lemma 6)

$$\mathbf{1} = A^2(A\langle X\rangle \vee A\langle Y\rangle) = A\langle X\rangle \curlyvee A\langle Y\rangle \leq A\langle E\rangle \curlyvee A\langle E\rangle = A\langle E\rangle < \mathbf{1},$$

a contradiction, showing that our assumption that both $\langle X\rangle$ and $\langle Y\rangle$ are in **cBA** cannot hold. □

As usual, we call a set $S \subset \mathbb{N} \cup \{\infty\}$ *coinfinite*, if its complement $(\mathbb{N} \cup \{\infty\})\backslash S$ is infinite.

Theorem 8. *For any coinfinite set $S \subseteq \mathbb{N} \cup \{\infty\}$ with $\infty \in S$, we have that $\langle K(S)\rangle$ is not in* **cBA**.

Proof. In Lemma 7, choose E to be the Brown-Comenetz dual of the p-local sphere spectrum, I. We know by [4, Lemma 7.1(c)] that $\langle H\mathbb{F}_p\rangle \geq \langle I\rangle$, and hence $\langle K(S)\rangle \geq \langle I\rangle$. As the complement $\overline{S} := (\mathbb{N} \cup \{\infty\})\backslash S$ is infinite, we get by Proposition 2 that $\langle K(\overline{S})\rangle \geq \langle I\rangle$. Both, $\langle K(S)\rangle$ and $\langle K(\overline{S})\rangle$ are in $\mathbf{DL}$ and $\langle K(S)\rangle \wedge \langle K(\overline{S})\rangle = \mathbf{0}$. Thus all conditions of the Dichotomy Lemma are satisfied, and we get that one of $\langle K(S)\rangle$ and $\langle K(\overline{S})\rangle$ is not in **cBA**. However, by Corollary 3, $\langle K(\overline{S})\rangle \in \mathbf{cBA}$, so $\langle K(S)\rangle \in \mathbf{DL}\backslash\mathbf{cBA}$. □

Corollary 9. *There are at least $2^{\aleph_0}$ Bousfield classes in* $\mathbf{DL}\backslash\mathbf{cBA}$.

Proof. This follows directly from Theorem 8 and [2, Lemma 3.4], as there are $2^{\aleph_0}$ many coinfinite subsets of $\mathbb{N} \cup \{\infty\}$. $\square$

4. Applications

Several conjectures made by Hovey and Palmieri in [4] suggest that $\langle H\mathbb{F}_p \rangle$ is not in **cBA** [4, Proposition 6.14]. This follows directly from our Theorem 8:

Corollary 10. *For every prime p, we have that $\langle H\mathbb{F}_p \rangle \in \mathbf{DL} \backslash \mathbf{cBA}$.*

Proof. This is clear from Theorem 8, as $\langle H\mathbb{F}_p \rangle = \langle K(\infty) \rangle = \langle K(\{\infty\}) \rangle$ where $\{\infty\}$ is coinfinite in $\mathbb{N} \cup \{\infty\}$. $\square$

Our method also identifies several other explicit Bousfield classes in $\mathbf{DL} \backslash \mathbf{cBA}$. The following examples exploit the fact that for any self-map of a spectrum X, $f \colon \Sigma^{|f|} X \to X$ one gets by [6, Lemma 1.34] that

$$\langle X \rangle = \langle C_f \rangle \vee \langle X[f^{-1}] \rangle.$$

Here, C_f denotes the cofiber of f and $X[f^{-1}]$ is the telescope. Then the Bousfield class of the Eilenberg-MacLane spectrum of the p-local integers, $H\mathbb{Z}_{(p)}$, is $\langle K(\{0, \infty\}) \rangle$. This is a special case of a truncated Brown-Peterson spectrum $BP\langle n \rangle$ with $\pi_*(BP\langle n \rangle) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ ($|v_i| = 2p^i - 2$). Multiplication by v_n is a self-map on $BP\langle n \rangle$ with cofiber $BP\langle n-1 \rangle$ and $BP\langle n \rangle[v_n^{-1}] = E(n)$. An iteration then gives (cf. [6, Theorem 2.1]) $\langle BP\langle n \rangle \rangle = \langle E(n) \rangle \vee \langle H\mathbb{F}_p \rangle$. As the Bousfield class of $E(n)$ is $\langle K(0) \rangle \vee \ldots \vee \langle K(n) \rangle$ we obtain $\langle BP\langle n \rangle \rangle = \langle K(\{0, \ldots, n, \infty\}) \rangle$.

Corollary 11. *For every prime p and every natural number n, we have that $\langle H\mathbb{Z}_{(p)} \rangle$ and $\langle BP\langle n \rangle \rangle$ are in $\mathbf{DL} \backslash \mathbf{cBA}$.*

Proof. The subsets $\{0, \infty\}$ and $\{0, \ldots, n, \infty\}$ are coinfinite in $\mathbb{N} \cup \{\infty\}$. $\square$

For the connective Morava K-theory $k(n)$ (with $\pi_* k(n) = \mathbb{F}_p[v_n]$) we get $\langle k(n) \rangle = \langle K(n) \rangle \vee \langle H\mathbb{F}_p \rangle = \langle K(\{n, \infty\}) \rangle$.

Corollary 12. *For every natural number n, $\langle k(n) \rangle \in \mathbf{DL} \backslash \mathbf{cBA}$.*

Proof. This follows from Theorem 8, as $\{n, \infty\}$ is coinfinite in $\mathbb{N} \cup \{\infty\}$. $\square$

Similar to the Morava K-theory spectra $K(n)$ we can consider the telescopes $T(n)$ of v_n-maps. (Cf. [4, §5] for details.) It is known that

$$\langle T(n) \rangle = \langle K(n) \rangle \vee \langle A(n) \rangle$$

where $A(n)$ is the spectrum describing the failure of the telescope conjecture. We set $\langle T(\infty) \rangle = \langle H\mathbb{F}_p \rangle$. The classes $\langle T(n) \rangle$ and $\langle A(n) \rangle$ are in **BA** but $\bigvee_{\mathbb{N}} \langle T(n) \rangle \notin \mathbf{BA}$ by [4, Corollary 7.10]. By Lemma 1, we know that for any $S \subseteq \mathbb{N}$, we have that $\bigvee_{n \in S} \langle T(n) \rangle \in \mathbf{cBA}$. An argument similar to the proof of Proposition 2 yields Proposition 13.

Proposition 13. *If $S \subseteq \mathbb{N}$ is infinite, then $\langle T(S) \rangle = \bigvee_{i \in S} \langle T(i) \rangle \in \mathbf{cBA} \backslash \mathbf{BA}$ and $\langle T(S) \rangle \geq \langle I \rangle$.*

Theorem 14. *Let $S \subseteq \mathbb{N} \cup \{\infty\}$ be a coinfinite subset with $\infty \in S$. Then $\langle T(S) \rangle$ is not in **cBA**.*

Proof. Again, we use the Brown-Comenetz dual of the p-local sphere as E in the Dichotomy Lemma. Let $\overline{S}$ be the complement of S. As $\langle T(n) \rangle \geq \langle K(n) \rangle$ and as $\infty \in S$ we have that

$$\bigvee_{n \in S} \langle T(n) \rangle \geq \bigvee_{n \in S} \langle K(n) \rangle \geq \langle I \rangle$$

and $\bigvee_{n \in \overline{S}} \langle T(n) \rangle \geq \langle I \rangle$. The telescopes satisfy $\langle T(n) \rangle \wedge \langle T(m) \rangle = \mathbf{0}$ for $m \neq n$: cf. [4, §5] for the cases $n \neq \infty \neq m$ and cf. the proof of [4, Proposition 6.14] for $\langle H\mathbb{F}_p \rangle) \wedge \bigvee_{\mathbb{N}} \langle T(n) \rangle = \mathbf{0}$. Therefore we obtain that one of $\bigvee_{n \in S} \langle T(n) \rangle$ or $\bigvee_{n \in \overline{S}} \langle T(n) \rangle$ cannot be an element of **cBA**, but $\bigvee_{n \in \overline{S}} \langle T(n) \rangle$ is in **cBA** by Proposition 13. $\square$

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