Pseudo-Riemannian almost quaternionic homogeneous spaces with irreducible isotropy

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Abstract

We show that pseudo-Riemannian almost quaternionic homogeneous spaces with index 4 and an $\mathbb{H}$-irreducible isotropy group are locally isometric to a pseudo-Riemannian quaternionic Kähler symmetric space if the dimension is at least 16. In dimension 12 we give a non-symmetric example.

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1 Introduction

In [AZ] Ahmed and Zeghib studied pseudo-Riemannian almost complex homogeneous spaces of index 2 with a $\mathbb{C}$-irreducible isotropy group. They showed that these spaces are already pseudo-Kähler if the dimension is at least 8. If furthermore the Lie algebra of the isotropy group is $\mathbb{C}$-irreducible then the space is locally isometric to one of five symmetric spaces.

There are two different quaternionic analogues of Kähler manifolds, namely hyper-Kähler and quaternionic Kähler manifolds. In the first case, the complex structure is replaced by three complex structures assembling into a hyper-complex structure (I, J, K), in the second by the more general notion of a quaternionic structure $Q \subset \operatorname{End} TM$ on the underlying manifold M. Riemannian as well as pseudo-Riemannian quaternionic Kähler manifolds are Einstein and therefore of particular interest in pseudo-Riemannian geometry.

In [CM] the authors investigated the hyper-complex analogue of the topic studied by Ahmed and Zeghib, namely pseudo-Riemannian almost hyper-complex homogeneous spaces

of index 4 with an $\mathbb{H}$-irreducible isotropy group. It turned out that these spaces of dimension greater or equal than 8 are already locally isometric to the flat space $\mathbb{H}^{1,n}$ except in dimension 12, where non-symmetric examples exist.
In this article we study the quaternionic analogue, that is we consider pseudo-Riemannian almost quaternionic homogeneous spaces of index 4 with an $\mathbb{H}$-irreducible isotropy group. The main result of our analysis is the following theorem.

Theorem 1.1. *Let (M, g, Q) be a connected almost quaternionic pseudo-Hermitian manifold of index 4 and $\dim M = 4n+4 \geq 16$, such that there exists a connected Lie subgroup $G \subset \mathrm{Iso}(M, g, Q)$ acting transitively on M. If the isotropy group $H := G_p$, $p \in M$, acts $\mathbb{H}$-irreducibly, then (M, g, Q) is locally isometric to a quaternionic Kähler symmetric space.*

Here $\mathrm{Iso}(M, g, Q)$ denotes the subgroup of the isometry group $\mathrm{Iso}(M, g)$ which preserves the almost quaternionic structure Q of M. A consequence of the theorem is that the homogeneous space M itself is quaternionic Kähler and locally symmetric. Notice that pseudo-Riemannian quaternionic Kähler symmetric spaces have been classified in [AC]. In Section 3.2 we show, by construction of a non-symmetric example in dimension 12, that the hypothesis $\dim M \geq 16$ in Theorem 1.1 cannot be omitted. Moreover, we classify in Proposition 3.1 all examples with the same isotropy algebra $\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(1,2) \oplus \mathfrak{so}(4) \subset \mathfrak{gl}(\mathbb{R}^{1,2} \otimes \mathbb{R}^4) \cong \mathfrak{gl}(12, \mathbb{R})$ in terms of the solutions of a system of four quadratic equations for six real variables.

The strategy of the proof of Theorem 1.1 is as follows. We consider the $\mathbb{H}$-irreducible isotropy group H as a subgroup of $\mathrm{Sp}(1,n)\mathrm{Sp}(1)$ and classify the possible Lie algebras. Then we consider the covering G/H^0 of $M = G/H$ and show by taking into account the possible Lie algebras that it is a reductive homogeneous space. Finally, we show that the universal covering $\tilde{M}$ is a symmetric space. The invariance of the fundamental 4-form under G then implies that the symmetric space is quaternionic Kähler.
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2 About subgroups of Sp(1,*n*)Sp(1)

Lemma 2.1 (Goursat's theorem)**.** *Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras. There is a one-to-one correspondence between Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and quintuples $\mathcal{Q}(\mathfrak{h}) = (A, A_0, B, B_0, \theta)$, with $A \subset \mathfrak{g}_1$ $B \subset \mathfrak{g}_2$ Lie subalgebras, $A_0 \subset A$, $B_0 \subset B$ ideals and $\theta : A/A_0 \to B/B_0$ is a Lie algebra isomorphism.*

Proof: Let $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie subalgebra and denote by $\pi_i : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_i$, $i = 1, 2$, the natural projections. Set $A := \pi_1(\mathfrak{h}) \subset \mathfrak{g}_1$, $B := \pi_2(\mathfrak{h}) \subset \mathfrak{g}_2$, $A_0 := \ker(\pi_{2|\mathfrak{h}})$ and $B_0 := \ker(\pi_{1|\mathfrak{h}})$. It is not hard to see that A_0 and B_0 can be identified with ideals in A

and B respectively. Now we can define a map $\tilde{\theta} : A \to B/B_0$ as follows. For $X \in A$ take any $Y \in B$ such that $X + Y \in \mathfrak{h}$ and define $\tilde{\theta}(X) := Y + B_0$. It is easy to check that this map is well defined. Its kernel is A_0 so $\tilde{\theta}$ induces a Lie algebra isomorphism $\theta : A/A_0 \to B/B_0$. This defines a map $\mathfrak{h} \mapsto \mathcal{Q}(\mathfrak{h})$.
Conversely, a quintuple $Q = (A, A_0, B, B_0, \theta)$ as above defines a Lie subalgebra $\mathfrak{h} = \mathcal{G}(Q) \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ by setting

$$\mathfrak{h} := \{X + Y \in A \oplus B \mid \theta(X + A_0) = Y + B_0\}.$$

It is not hard to see that the maps $\mathcal{G}$ and $\mathcal{Q}$ are inverse to each other. $\square$

We will use the following two classification results for $\mathbb{H}$-irreducible subgroups of $\mathrm{Sp}(1, n)$.

Theorem 2.1 ([CM, Corollary 2.1])**.** *Let $H \subset \mathrm{Sp}(1, n)$ be a connected and $\mathbb{H}$-irreducible Lie subgroup. Then H is conjugate to one of the following groups:*

(i) $\mathrm{SO}^0(1, n)$, $\mathrm{SO}^0(1, n) \cdot \mathrm{U}(1)$, $\mathrm{SO}^0(1, n) \cdot \mathrm{Sp}(1)$ *if* $n \geq 2$,

(ii) $\mathrm{SU}(1, n)$, $\mathrm{U}(1, n)$,

(iii) $\mathrm{Sp}(1, n)$,

(iv) $U^0 = \{A \in \mathrm{Sp}(1, 1) \mid A\Phi = \Phi A\} \cong \mathrm{Spin}^0(1, 3)$ *with* $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *if* $n = 1$.

Proposition 2.1 ([CM, Proposition 2.4])**.** *Let $H \subset \mathrm{Sp}(1, n)$ be an $\mathbb{H}$-irreducible subgroup. Then one of the following is true.*

(i) *H is discrete.*

(ii) $H^0 = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$ *or* $H^0 = \mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$.

(iii) *H^0 is $\mathbb{H}$-irreducible.*

(iv) $n = 1$ *and H^0 is one of the groups* $\mathrm{SO}^0(1, 1)$, $\mathrm{SO}^0(1, 1) \cdot \mathrm{U}(1)$, $\mathrm{SO}^0(1, 1) \cdot \mathrm{Sp}(1)$ *or*

$$S = \left\{ e^{ibt} \begin{pmatrix} \cosh(at) & \sinh(at) \\ \sinh(at) & \cosh(at) \end{pmatrix} \middle| \; t \in \mathbb{R} \right\},$$

for some non-zero real numbers a, b.

We denote by $\pi_1 : \mathfrak{sp}(1, n) \oplus \mathfrak{sp}(1) \to \mathfrak{sp}(1, n)$ and $\pi_2 : \mathfrak{sp}(1, n) \oplus \mathfrak{sp}(1) \to \mathfrak{sp}(1)$ the canonical projections.

Proposition 2.2. *Let $n \geq 2$ and $H \subset \mathrm{Sp}(1, n)\mathrm{Sp}(1)$ be an $\mathbb{H}$-irreducible closed subgroup. Then the Lie algebra $\mathfrak{h}$ is one of the following:*

(i) $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ *with* $\mathfrak{h}_0 \in \{\{0\}, \mathfrak{so}(1, n)\}$, $\mathfrak{c} \subset \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1} \oplus \mathfrak{sp}(1)$ *and* $\pi_1(\mathfrak{c}) = \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}$, $\pi_2(\mathfrak{c}) = \mathfrak{sp}(1)$, $\mathfrak{c} \cap \mathfrak{sp}(1, n) = \{0\}$, $\mathfrak{c} \cap \mathfrak{sp}(1) = \{0\}$,

(ii) $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ *with* $\mathfrak{h}_0 \in \{\{0\}, \mathfrak{so}(1, n), \mathfrak{su}(1, n)\}$, $\mathfrak{c} \subset \mathfrak{u}(1) \cdot \mathbb{1}_{n+1} \oplus \mathfrak{u}(1)$ *and* $\pi_1(\mathfrak{c}) = \mathfrak{u}(1) \cdot \mathbb{1}_{n+1}$, $\pi_2(\mathfrak{c}) = \mathfrak{u}(1)$, $\mathfrak{c} \cap \mathfrak{sp}(1, n) = \{0\}$, $\mathfrak{c} \cap \mathfrak{sp}(1) = \{0\}$,

(iii) $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ *where* $\mathfrak{h}_0 \subset \mathfrak{sp}(1,n)$ *is one of the following Lie algebras*

$$\mathfrak{sp}(1,n), \quad \mathfrak{u}(1,n), \quad \mathfrak{su}(1,n), \quad \mathfrak{so}(1,n) \oplus \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \quad \mathfrak{so}(1,n) \oplus \mathfrak{u}(1) \cdot \mathbb{1}_{n+1},$$

$$\mathfrak{so}(1,n), \quad \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \quad \mathfrak{u}(1) \cdot \mathbb{1}_{n+1}, \quad \{0\},$$

and $\mathfrak{c} \subset \mathfrak{sp}(1)$ *is* $\{0\}, \mathfrak{u}(1)$ *or* $\mathfrak{sp}(1)$.

Proof: The idea is to apply Goursat's theorem (Lemma 2.1) to $\mathfrak{h} \subset \mathfrak{sp}(1,n) \oplus \mathfrak{sp}(1)$. The Lie subalgebras A, A_0, B and B_0 are given by $\pi_1(\mathfrak{h})$, $\mathfrak{h} \cap \mathfrak{sp}(1)$, $\pi_2(\mathfrak{h})$ and $\mathfrak{h} \cap \mathfrak{sp}(1)$. Let $p : \mathrm{Sp}(1,n) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1,n)$ be the natural projection. Notice that $H \subset \mathrm{Sp}(1,n)\mathrm{Sp}(1)$ is $\mathbb{H}$-irreducible if and only if $p(\hat{H}) \subset \mathrm{Sp}(1,n)$ is $\mathbb{H}$-irreducible, where $\hat{H}$ is the preimage of H under the two-fold covering $\mathrm{Sp}(1,n) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1,n)\mathrm{Sp}(1)$. By Proposition 2.1 and Theorem 2.1 we know that $p(\hat{H})$ is either discrete or $(p(\hat{H}))^0$ is one of the following subgroups of $\mathrm{Sp}(1,n)$:

$$\mathrm{Sp}(1,n), \quad \mathrm{U}(1,n), \quad \mathrm{SU}(1,n), \quad \mathrm{SO}^0(1,n)\,(\mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}), \quad \mathrm{SO}^0(1,n)\,(\mathrm{U}(1) \cdot \mathbb{1}_{n+1}),$$

$$\mathrm{SO}^0(1,n), \quad \mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}, \quad \mathrm{U}(1) \cdot \mathbb{1}_{n+1}.$$

Since $dp = \pi_1$ we immediately obtain all possibilities for $\pi_1(\mathfrak{h})$. Furthermore $\mathfrak{h} \cap \mathfrak{sp}(1,n)$ is an ideal of the Lie algebra $\pi_1(\mathfrak{h})$. We can read off from the above list a decomposition of $\pi_1(\mathfrak{h})$ into ideals, which gives us all possibilities for $\mathfrak{h} \cap \mathfrak{sp}(1,n)$. The resulting list of pairs (A, A_0) is displayed in a table below.
On the other side there are only three Lie subalgebras of $\mathfrak{sp}(1)$, namely $\mathfrak{sp}(1)$ itself, $\mathfrak{u}(1)$ and $\{0\}$. It follows that $\pi_2(\mathfrak{h})$ is one of these three. Again, $\mathfrak{h} \cap \mathfrak{sp}(1)$ is an ideal of $\pi_2(\mathfrak{h})$. It follows that the only possibilites for $\mathfrak{h} \cap \mathfrak{sp}(1)$ are the same as for $\pi_2(\mathfrak{h})$.
By Goursat's theorem we have a Lie algebra isomorphism $\theta : A/A_0 \to B/B_0$. Since we know all possibilities for B and B_0, it follows that A/A_0 is isomorphic to $\mathfrak{sp}(1)$, $\mathfrak{u}(1)$ or $\{0\}$. Therefore we need to consider all possibilities for A and A_0, as listed in the following table, and keep only those for which A/A_0 is isomorphic to $\mathfrak{sp}(1)$, $\mathfrak{u}(1)$ or $\{0\}$.

A	A_0
$\mathfrak{sp}(1,n)$	$\mathfrak{sp}(1,n)$ $\{0\}$
$\mathfrak{su}(1,n) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(1,n) \oplus \mathfrak{u}(1)$ $\mathfrak{su}(1,n)$ $\mathfrak{u}(1)$ $\{0\}$
$\mathfrak{su}(1,n)$	$\mathfrak{su}(1,n)$ $\{0\}$
$\mathfrak{so}(1,n) \oplus \mathfrak{sp}(1)$	$\mathfrak{so}(1,n) \oplus \mathfrak{sp}(1)$ $\mathfrak{so}(1,n)$ $\mathfrak{sp}(1)$ $\{0\}$
$\mathfrak{so}(1,n) \oplus \mathfrak{u}(1)$	$\mathfrak{so}(1,n) \oplus \mathfrak{u}(1)$ $\mathfrak{so}(1,n)$ $\mathfrak{u}(1)$ $\{0\}$

$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n)$ $\{0\}$
$\mathfrak{sp}(1)$	$\mathfrak{sp}(1)$ $\{0\}$
$\mathfrak{u}(1)$	$\mathfrak{u}(1)$ $\{0\}$
$\{0\}$	$\{0\}$

If $B/B_0 \cong \mathfrak{sp}(1)$ then $B = \mathfrak{sp}(1)$ and $B_0 = \{0\}$. The possibilities for (A, A_0) are

$$(\mathfrak{so}(1,n) \oplus \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \mathfrak{so}(1,n)) \quad \text{and} \quad (\mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \{0\}).$$

This gives us case (i). Analogously we get the remaining Lie algebras in (ii) and (iii). $\square$

3 Main results

3.1 Proof of the main theorem

Lemma 3.1 ([CM, Lemma 3.1]). *Let $n \geq 3$ and $\alpha \in \otimes^3 V^*$, where $V = \mathbb{H}^{1,n}$ is considered as real vector space. If α is $\mathrm{SO}^0(1,n)$-invariant, then $\alpha = 0$.*

Remark 3.1. *The $\mathrm{SO}^0(1,n)$-invariant elements of $\otimes^3 V^*$ are in one-to-one correspondence to the $\mathrm{SO}^0(1,n)$-equivariant bilinear maps from $V \times V$ to V. It follows from Lemma 3.1 that the corresponding bilinear maps also vanish.*

Proof of Theorem 1.1: Let $\rho : H \to \mathrm{GL}(T_pM)$ be the isotropy representation. We identify H with its image $\rho(H)$. Since H preserves the metric g and the almost quaternionic structure Q, we can consider H as a subgroup of $\mathrm{Sp}(1,n)\mathrm{Sp}(1)$.

In our first step we consider the covering G/H^0 of $M = G/H$ and show that it is a reductive homogeneous space, i.e. there exists an H^0-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

We apply Proposition 2.2 to H^0. The existence of a subspace $\mathfrak{m}$ is clear if $\mathfrak{h}$ is one of the semi-simple Lie algebras in the list. If $\mathfrak{h}$ is one of the abelian Lie algebras contained in $\mathfrak{u}(1) \cdot \mathbb{1}_{n+1} \oplus \mathfrak{u}(1)$, then the closure of $\mathrm{Ad}(H^0) \subset \mathrm{GL}(\mathfrak{g})$ is compact and hence there exists an $\mathrm{Ad}(H^0)$-invariant subspace $\mathfrak{m}$. The remaining Lie algebras in the list have the form $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{z}$ where $\mathfrak{s}$ is semi-simple containing $\mathfrak{so}(1,n)$ and $\mathfrak{z}$ is the non-trivial centre. Then $\mathfrak{g}$ decomposes into $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus \mathfrak{m}$ with respect to the action of $\mathfrak{s}$. If we consider the action of $\mathfrak{s}$ on $\mathfrak{m} \cong \mathbb{H}^{1,n}$ as a complex representation, then $\mathfrak{m}$ is either $\mathbb{C}$-irreducible or decomposes into two $\mathbb{C}$-irreducible subrepresentations. Since the elements of $\mathfrak{z}$ commute with $\mathfrak{s}$, they preserve the sum of all non-trivial $\mathfrak{s}$-submodules, which is precisely $\mathfrak{m}$. Thus we have shown that G/H^0 is a reductive homogeneous space.

Next we show that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a symmetric Lie algebra. It is sufficient to show that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. We restrict the Lie bracket $[\cdot,\cdot]$ to $\mathfrak{m} \times \mathfrak{m}$ and denote its projection to $\mathfrak{m}$ by β. It is an antisymmetric bilinear map which is $\mathrm{Ad}(H)$-equivariant. Since $\mathfrak{m} \cong \mathbb{H}^{1,n}$, we

can consider β as an element of $\otimes^3(\mathbb{H}^{1,n})^*$. It is also H^{Zar}-invariant, where H^{Zar} denotes the Zariski closure. Since H^{Zar} is an algebraic group, it has only finitely many connected components, see [Mi]. Now we show that $(H^{Zar})^0$ is non-compact.
Assume that $(H^{Zar})^0$ is compact. Since H^{Zar} has only finitely many connected components it follows that H^{Zar} is compact and therefore contained in a maximal compact subgroup of $\mathrm{Sp}(1,n)\mathrm{Sp}(1)$. Hence, H^{Zar} is conjugate to a subgroup of $(\mathrm{Sp}(1)\times\mathrm{Sp}(n))\mathrm{Sp}(1)$ but this contradicts the $\mathbb{H}$-irreducibility of H^{Zar}. So we have shown that $(H^{Zar})^0$ is non-compact.
Now we apply Proposition 2.2 to H^{Zar}. Since H^{Zar} is non-compact we see from the list there that $(H^{Zar})^0$ contains $\mathrm{SO}^0(1,n)$. Hence, β is $\mathrm{SO}^0(1,n)$-equivariant. Since $n \geq 3$ it follows from Remark 3.1 that β vanishes. This shows that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a symmetric Lie algebra and that the universal covering $\tilde{M} = \tilde{G}/\tilde{G}_p$ of M is a symmetric space. The fundamental 4-form Ω of $\tilde{M}$ is $\tilde{G}$-invariant and since $\tilde{M}$ is a symmetric space Ω is parallel. In particular Ω is closed. It is known that for dimension ≥ 12 an almost quaternionic Hermitian manifold is quaternionic Kähler if $d\Omega = 0$, see [S]. This shows that $\tilde{M}$ is furthermore a quaternionic Kähler manifold. Summarizing, we have shown that M is locally isometric to a quaternionic Kähler symmetric space. $\square$

3.2 A class of non-symmetric examples in dimension 12

In Theorem 1.1 we did not consider the dimension 12. This is because the arguments used in the proof to show that M is a reductive homogeneous space do not apply in this dimension, although still $\mathrm{SO}^0(1,n) \subset H^{Zar}$ holds. In fact, the proof relies on Lemma 3.1 which holds for dimension $4n+4 \geq 16$. If $\dim M = 12$ then $n = 2$ and then there exist non-trivial anti-symmetric bilinear forms $\mathbb{H}^{1,2} \times \mathbb{H}^{1,2} \to \mathbb{H}^{1,2}$ which are invariant under $\mathrm{SO}^0(1,2)$. Therefore in dimension 12 we cannot be sure if the manifolds are symmetric.
In the following we will give a non-symmetric example by specifying a Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is a Lie algebra of the list in Proposition 2.2. The pair $(\mathfrak{g}, \mathfrak{h})$ defines a simply connected homogeneous space $M = G/H$ where G is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and H is the closed connected Lie subgroup of G with Lie algebra $\mathfrak{h}$.

Let $\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{c}$ with $\mathfrak{c} = \{(X \cdot \mathbb{1}_3, X) \in \mathfrak{sp}(1) \cdot \mathbb{1}_3 \oplus \mathfrak{sp}(1) \mid X \in \mathfrak{sp}(1)\}$, see Proposition 2.2 (i). Then we consider the vector space direct sum $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} = \mathbb{H}^{1,2}$ and define a Lie bracket on $\mathfrak{g}$ in the following way. For elements $A, B \in \mathfrak{h}$ we take the standard Lie bracket of $\mathfrak{h}$, i.e. $[A,B] = AB - BA$. Then we define $[A,x] = -[x,A] = Ax$ for $A \in \mathfrak{h}$ and $x \in \mathfrak{m}$. Note that, as an $\mathfrak{h}$-module, we can decompose $\mathfrak{m} = \mathbb{H}^{1,2} = \mathbb{R}^{1,2} \otimes \mathbb{H} = \mathbb{R}^{1,2} \otimes \mathbb{R}^4$, where the action of $\mathfrak{so}(1,2)$ is by the defining representation on the first factor and trivial on the second and the action of $\mathfrak{c} \cong \mathfrak{so}(3) \subset \mathfrak{so}(4)$ is trivial on the first factor and by the standard four-dimensional representation $\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ on the second. Finally we have to define the Lie bracket for elements in $\mathfrak{m} = \mathbb{R}^{1,2} \otimes \mathbb{R}^4$.
Let $K : \mathbb{R}^{1,2} \to \mathfrak{so}(1,2)$ be an isomorphism of Lie algebras where $\mathbb{R}^{1,2}$ is endowed with

the Lorentzian cross product, $\iota : \mathfrak{sp}(1) \to \mathfrak{c}$, $X \to X \cdot \mathbb{1}_3 + X$, and let η be the standard Lorentz metric on $\mathbb{R}^{1,2}$. Furthermore denote $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^4$. Let $x = u \otimes p$, $y = v \otimes q \in \mathbb{R}^{1,2} \otimes \mathbb{R}^4$ and write $p = p_0 + \vec{p}$, $q = q_0 + \vec{q}$, where $p_0, q_0 \in \mathbb{R}$ and $\vec{p}, \vec{q} \in \operatorname{Im}\mathbb{H} = \mathbb{R}^3$. We set

$$[x, y] = \underbrace{\langle \vec{p}, \vec{q} \rangle \cdot K(u \times v) - \frac{1}{2}\eta(u, v)\iota(\vec{p} \times \vec{q})}_{\in \mathfrak{h}} + \underbrace{u \times v(p_0 q_0 - \langle \vec{p}, \vec{q} \rangle)}_{\in \mathbb{R}^{1,2} \subset \mathbb{H}^{1,2} = \mathfrak{m}},$$

where $\vec{p} \times \vec{q}$ is the Euclidian cross product in $\operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$ and $u \times v$ the Lorentzian cross product in $\mathbb{R}^{1,2}$. This extends the partially defined bracket to an anti-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies the Jacobi-identity. Hence $\mathfrak{g}$ becomes a Lie algebra. We claim that $(\mathfrak{g}, \mathfrak{h})$ is not a symmetric pair. In fact, every $\mathfrak{h}$-invariant complement $\mathfrak{m}'$ of $\mathfrak{h}$ in $\mathfrak{g}$ contains $\mathbb{R}^{1,2} \otimes \mathbb{R}^3$ (there is no other equivalent $\mathfrak{h}$-submodule in $\mathfrak{g}$) and thus we see from the formula for the bracket that $[\mathfrak{m}', \mathfrak{m}'] \not\subseteq \mathfrak{h}$.

For a general classification of the homogeneous spaces with $\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{c}$ we need to classify all the Lie algebra structures on the vector $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^{1,2} \otimes \mathbb{R}^4$ such that the Lie bracket restricts to the Lie bracket of $\mathfrak{h}$ and to the given representation of $\mathfrak{h}$ on $\mathbb{R}^{1,2} \otimes \mathbb{R}^4$. For this one has to describe all the $\mathfrak{h}$-invariant tensors of $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{g} \cong \Lambda^2\mathfrak{m}^* \otimes \mathfrak{h} \oplus \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}$ which satisfy the Jacobi-identity. With the above notation, these bilinear maps have the following form

$$\begin{aligned} [x, y] &= (a \cdot p_0 q_0 + b\,\langle \vec{p}, \vec{q} \rangle) \cdot K(u \times v) + \eta(u, v)\,(c \cdot \iota(\vec{p} \times \vec{q}) + d\,(p_0 \vec{q} - q_0 \vec{p})) \\ &\quad + u \times v \cdot \left(a_1 \cdot p_0 q_0 + a_2 \cdot \langle \vec{p}, \vec{q} \rangle + \frac{a_3}{2}\,(p_0 \vec{q} + q_0 \vec{p}) \right), \end{aligned}$$

where $a, b, c, d, a_1, a_2, a_3 \in \mathbb{R}$. The bracket satisfies the Jacobi-identity if and only if the following equations hold

$$\begin{aligned} 0 &= d, \\ 0 &= a + \frac{a_1 a_3}{2} - \frac{a_3^2}{4}, && (1) \\ 0 &= b + 2c + \frac{a_2 a_3}{2}, && (2) \\ 0 &= b + a_1 a_2 - \frac{a_2 a_3}{2}, && (3) \\ 0 &= -\frac{b a_3}{2} + a a_2. && (4) \end{aligned}$$

Summarizing we obtain the following proposition.

Proposition 3.1. *Every solution (a, b, c, a_1, a_2, a_3) of the quadratic system (1)-(4) defines a connected and simply connected homogeneous almost quaternionic pseudo-Hermitian manifold G/H with isotropy algebra $\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(1,2) \oplus \mathfrak{so}(4) \subset \mathfrak{gl}(\mathbb{R}^{1,2} \otimes \mathbb{R}^4) \cong \mathfrak{gl}(12, \mathbb{R})$. Conversely, every such homogeneous space arises by this construction.*

The above example corresponds to $a = 0$, $b = 1$, $c = -\frac{1}{2}$, $d = 0$, $a_1 = 1$, $a_2 = -1$ and $a_3 = 0$.

References

[AC] D.V. Alekseevsky and V. Cortés, *Classification of pseudo-Riemannian symmetric spaces of quaternionic Kähler type*, Amer. Math. Soc. Transl. (2) 213 (2005), 33-62.

[AZ] A. ben Ahmed and A. Zeghib, *On homogeneous Hermite-Lorentz spaces*, Preprint, `arXiv:math.DG/1106.4145v1` (2011).

[CM] V. Cortés and B. Meinke, *Pseudo-Riemannian almost hypercomplex homogeneous spaces with irreducible isotropy*, Preprint, `arXiv:1606.06486` (2016).

[Mi] J. Milnor, *On the Betti numbers of real varieties*, Proc. Amer. Math. Soc. **15** (1964) 275-280. Press, 1974, 49-87.

[S] A. F. Swann, *HyperKähler and Quaternionic Kähler Geometry*, Math. Ann. **289** (1991), no. 1, 421-450.