

Tangle-tree duality: in graphs, matroids and beyond*

Reinhard Diestel
Mathematisches Seminar, Universität Hamburg

Sang-il Oum
KAIST, Daejeon, 34141 South Korea

January 10, 2017

Abstract

We apply a recent duality theorem for tangles in abstract separation systems to derive tangle-type duality theorems for width-parameters in graphs and matroids. We further derive a duality theorem for the existence of clusters in large data sets.

Our applications to graphs include new, tangle-type, duality theorems for tree-width, path-width, and tree-decompositions of small adhesion. Conversely, we show that carving width is dual to edge-tangles. For matroids we obtain a duality theorem for tree-width.

Our results can be used to derive short proofs of all the classical duality theorems for width parameters in graph minor theory, such as path-width, tree-width, branch-width and rank-width, as well as of a general bramble-type duality theorem of Amini, Mazoit, Nisse, and Thomassé which unifies these classical theorems.

1 Introduction

There are a number of theorems in the structure theory of sparse graphs that assert a duality between high connectivity present somewhere in the graph and an overall tree structure. For example, a graph has small tree-width if and only if it contains no large-order bramble. Amini, Mazoit, Nisse, and Thomassé [1] generalized the notion of a bramble to give similar duality theorems for other width parameters, including branch-width, rank-width and matroid tree-width. The highly cohesive substructures, or *HCSs*, dual to low width in all these cases are what we call *concrete* HCSs: like brambles, they are sets of edges that hang together in a certain specified way.

In [9] we introduced another type of HCSs for graphs and matroids, which we call *abstract* HCS. These are modelled on the notion of a *tangle* introduced by Robertson and Seymour [18] for the proof of the graph minor theorem. They are orientations of all the separations of a graph or matroid, up to some given order, that are ‘consistent’ in a way specified by a set \mathcal{F} . This \mathcal{F} can be varied

*This is an extended version available only in preprint form.

to give different notions of consistency, leading to different notions of \mathcal{F} -tangles. We then proved a general duality theorem for \mathcal{F} -tangles [9, Theorem 4.3], which says that graphs or matroids not containing an \mathcal{F} -tangle have a certain type of tree structure, the type depending on the choice of \mathcal{F} . In each case, this tree structure clearly precludes the existence of an \mathcal{F} -tangle, and thus provides an easily checked certificate for their possible nonexistence.

Our first aim in this paper is to show that our duality theorem for abstract HCSs from [9] implies the known duality theorems for tangles: the classical Robertson-Seymour one for tangles and branch-width in graphs [18], and its analogue for matroids derived from [18] by Geelen, Gerards and Whittle [11].

As a tool for proving an Erdős-Pósa-type theorem for edge-disjoint immersions of graphs, Liu [14] recently introduced the notion of *edge-tangles*. We obtain a duality theorem for these, too. Interestingly, the tree structures dual to edge-tangles have been studied before [20], but their duality was never observed.

Our next aim will be to derive new duality theorems, in terms of \mathcal{F} -tangles, for the classical graph width parameters of tree-width and path-width, as well as for some lesser known ones such as rank-width [16] and carving-width [20]. By tweaking \mathcal{F} , we can obtain tailor-made duality theorems also for particular kinds of tree-decompositions as desired, such as those of some specified adhesion.

Matroid tree-width was introduced only more recently, by Hliněný and Whittle [12], and we shall obtain a tangle-type duality theorem for this too.

Our results will imply, and hence offer new proofs for, the bramble-type duality theorem for tree-width of Seymour and Thomas [19] and their duality theorem for path-width with Bienstock and Robertson [2]. We also find an \mathcal{F} that expresses the generalized brambles of Amini, Mazoit, Nisse, and Thomassé [1] as \mathcal{F} -tangles, and are thus able to deduce their unified theorem too.

We first proved the duality theorem of [9] that we keep applying here for ‘separations of sets’: a common generalization of graph and matroid separations. This version already implied all the results mentioned so far. However we then noticed that we needed much less to express, and to prove, this duality theorem. This had some interesting consequences, so let us briefly explain what happened.

The oriented separations in a graph or matroid are partially ordered in a natural way, as $(A, B) \leq (C, D)$ whenever $A \subseteq C$ and $B \supseteq D$. This partial ordering is inverted by the involution $(A, B) \mapsto (B, A)$. Let us call any poset (\vec{S}, \leq) with an order-reversing involution $\vec{s} \mapsto \bar{s}$ an *abstract separation system*. If this poset is a lattice, we call it a *separation universe*. It is *submodular* if there is a real *order* function $\vec{s} \mapsto |\vec{s}|$ satisfying $|\vec{s}| = |\bar{s}|$ and $|\vec{r} \vee \vec{s}| + |\vec{r} \wedge \vec{s}| \leq |\vec{r}| + |\vec{s}|$ for all $\vec{r}, \vec{s} \in \vec{S}$.

All the necessary ingredients of \mathcal{F} -tangles in graphs, and of their dual tree structures, can be expressed in terms of (\vec{S}, \leq) . Indeed, two separations are nested if and only if they have orientations that are comparable under \leq . And the consistency requirement for classical tangles is, essentially, that if \vec{r} and \vec{s} ‘lie in’ the tangle (i.e., if the tangle orients r as \vec{r} and s as \vec{s}) then so does $\vec{r} \vee \vec{s}$ if it is in \vec{S} . It turned out that this was not a special case: we could express the entire duality theorem in its proof in this abstract setting.¹ Put more pointedly, we never need that our separations actually ‘separate’ anything: all we ever use is how they relate to each other in terms of (\vec{S}, \leq) .

¹Despite the increase in generality, this reduction to the essential also made the proof more readable.

The fact that our duality theorem works in abstract separation systems makes it applicable to contexts that are quite far from graphs and matroids. For example, the bipartitions of the set D of pixels of an image form a separation system: they are partially ordered by inclusion of their sides, and the involution of flipping the sides of the bipartition inverts this ordering. Depending on the application, some ways of cutting the image in two will be more natural than others, which gives rise to a cost function on these separations of D . Taking this cost of a separation as its ‘order’ then gives rise to tangles: abstract HCSs signifying regions of the image. Unlike regions defined by simply specifying a subset of D , regions defined by tangles are allowed to be fuzzy in terms of which pixels they ‘contain’ – much like regions in real-world images.

If the cost function on the separations of our image is submodular – which in practice is not a severe restriction – the abstract duality theorem from [9] can be applied to these tangles. For every integer k , our application of this theorem will either find a region of order at least k or produce a nested ‘tree’ set of bipartitions, all of order $< k$, which together witness that no such region exists [10]. This information could be used, for example, to assess the quality of an image, eg. after sending it through a noisy channel.

Our paper is organized as follows. We begin in Section 2 with a brief description of abstract separation systems, just enough to state in Section 3, as Theorem 3.2, the main duality theorem of [9] that we shall be applying throughout.

In Section 4 we prove a duality theorem for classical tangles as introduced by Robertson and Seymour [18]. Our result differs from their tangle-branchwidth duality theorem in the way we describe the tree-like structure that graphs without large tangles must have. But the theorems are similar and interderivable, and so our result may be seen as just a new proof of tangle-branchwidth duality.

In Section 5 we apply Theorem 3.2 to set partitions: of the vertex set of a graph, the ground set of a matroid, or any set with a natural submodular order function on its bipartitions. By specifying this order function we obtain duality theorems for rank-width, edge-tangles, and carving-width in graphs, for tangles in matroids and, as an example of an application beyond graphs and matroids, for coherent features in pixellated images.

In Sections 6 and 7 we obtain our new duality theorems for tree-width and path-width, and show how to derive from these the existing but different duality theorems for these parameters. In Section 8 we prove our duality theorem for matroid tree-width. In Section 9 we derive duality theorems for tree-decompositions of bounded adhesion.

In Section 10, finally, we show how our duality theorem for abstract tangles, Theorem 3.2, implies the duality theorem for abstract brambles of Amini, Mazoit, Nisse, and Thomassé [1].

2 Abstract separation systems

In this section we introduce abstract separation systems – just enough to state the main duality theorem from [9] in Section 3, and thus make this paper self-contained.

A *separation of a set V* is a set $\{A, B\}$ such that $A \cup B = V$. The ordered pairs (A, B) and (B, A) are its *orientations*. The *oriented separations* of V are the orientations of its separations. Mapping every oriented separation (A, B) to

its *inverse* (B, A) is an involution that reverses the partial ordering

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Note that this is equivalent to $(D, C) \leq (B, A)$. Informally, we think of (A, B) as *pointing towards* B and *away from* A . Similarly, if $(A, B) \leq (C, D)$, then (A, B) *points towards* $\{C, D\}$, while (C, D) *points away from* $\{A, B\}$.

Generalizing these properties of separations of sets, we now give an axiomatic definition of ‘abstract’ separations. A *separation system* $(\vec{S}, \leq, *)$ is a partially ordered set \vec{S} with an order-reversing involution $*$. Its elements are called *oriented separations*. When a given element of \vec{S} is denoted as \vec{s} , its *inverse* \vec{s}^* will be denoted as \bar{s} , and vice versa. The assumption that $*$ be *order-reversing* means that, for all $\vec{r}, \vec{s} \in \vec{S}$,

$$\vec{r} \leq \vec{s} \Leftrightarrow \bar{r} \geq \bar{s}. \quad (1)$$

A *separation* is a set of the form $\{\vec{s}, \bar{s}\}$, and then denoted by s . We call \vec{s} and \bar{s} the *orientations* of s . The set of all such sets $\{\vec{s}, \bar{s}\} \subseteq \vec{S}$ will be denoted by S . If $\vec{s} = \bar{s}$, we call both \vec{s} and s *degenerate*.

When a separation is introduced ahead of its elements and denoted by a single letter s , its elements will then be denoted as \vec{s} and \bar{s} .² Given a set $S' \subseteq S$ of separations, we write $\vec{S}' := \bigcup S' \subseteq \vec{S}$ for the set of all the orientations of its elements. With the ordering and involution induced from \vec{S} , this is again a separation system.³

Separations of sets, and their orientations, are clearly an instance of this if we identify $\{A, B\}$ with $\{(A, B), (B, A)\}$.

If there are binary operations \vee and \wedge on our separation system \vec{S} such that $\vec{r} \vee \vec{s}$ is the supremum and $\vec{r} \wedge \vec{s}$ the infimum of \vec{r} and \vec{s} in \vec{S} , we call $(\vec{S}, \leq, *, \vee, \wedge)$ a *universe* of (oriented) separations. By (1), it satisfies De Morgan’s law:

$$(\vec{r} \vee \vec{s})^* = \bar{r} \wedge \bar{s}. \quad (2)$$

The oriented separations of a set V form such a universe: if $\vec{r} = (A, B)$ and $\vec{s} = (C, D)$, say, then $\vec{r} \vee \vec{s} := (A \cup C, B \cap D)$ and $\vec{r} \wedge \vec{s} := (A \cap C, B \cup D)$ are again oriented separations of V , and are the supremum and infimum of \vec{r} and \vec{s} . Similarly, the oriented separations of a graph form a universe. Its oriented separations of order $< k$ for some fixed k , however, form a separation system inside this universe that may not itself be a universe with respect to \vee and \wedge as defined above.

A separation $\vec{r} \in \vec{S}$ is *trivial* in \vec{S} , and \bar{r} is *co-trivial*, if there exists $s \in S$ such that $\vec{r} < \vec{s}$ as well as $\bar{r} < \bar{s}$. Note that if \vec{r} is trivial in \vec{S} then so is every $\vec{r}' \leq \vec{r}$. If \vec{r} is trivial, witnessed by \vec{s} , then $\vec{r} < \vec{s} < \bar{r}$ by (1). Separations \vec{s} such that $\vec{s} \leq \bar{s}$, trivial or not, will be called *small*.

The trivial oriented separations of a set V , for example, are those of the form $\vec{r} = (A, B)$ with $A \subseteq C \cap D$ and $B \supseteq C \cup D = V$ for some $s = \{C, D\} \neq r$. The small separations (A, B) of V are all those with $B = V$.

²It is meaningless here to ask which is which: neither \vec{s} nor \bar{s} is a well-defined object just given s . But given one of them, both the other and s will be well defined. They may be degenerate, in which case $s = \{\vec{s}\} = \{\bar{s}\}$.

³For $S' = S$, our definition of \vec{S}' is consistent with the existing meaning of \vec{S} . When we refer to oriented separations using explicit notation that indicates orientation, such as \vec{s} or (A, B) , we sometimes leave out the word ‘oriented’ to improve the flow of words. Thus, when we speak of a ‘separation (A, B) ’, this will in fact be an oriented separation.

Two separations r, s are *nested* if they have comparable orientations; otherwise they *cross*. Two oriented separations \vec{r}, \vec{s} are *nested* if r and s are nested.⁴ We say that \vec{r} *points towards* s , and \vec{r} *points away from* s , if $\vec{r} \leq \vec{s}$ or $\vec{r} \leq \vec{s}$. Then two nested oriented separations are either comparable, or point towards each other, or point away from each other. A set of separations is *nested* if every two of its elements are nested.

A set $O \subseteq \vec{S}$ of oriented separations is *antisymmetric* if it does not contain the inverse of any of its nondegenerate elements. It is *consistent* if there are no distinct $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\vec{r}, \vec{s} \in O$. (Informally: if it does not contain orientations of distinct separations that point away from each other.) An *orientation* of S is a maximal antisymmetric subset of \vec{S} : a subset that contains for every $s \in S$ exactly one of its orientations $\vec{s}, \bar{\vec{s}}$.

Every consistent orientation of S contains all separations \vec{r} that are trivial in \vec{S} , because it cannot contain their inverse $\bar{\vec{r}}$: if the triviality of \vec{r} is witnessed by $s \in S$, say, then $\bar{\vec{r}}$ would be inconsistent with both \vec{s} and $\bar{\vec{s}}$.

Given a set \mathcal{F} , a consistent orientation of S is an \mathcal{F} -*tangle* if it avoids \mathcal{F} , i.e., has no subset $F \in \mathcal{F}$. We think of \mathcal{F} as a collection of ‘forbidden’ subsets of \vec{S} . Avoiding \mathcal{F} adds another degree of consistency to an already formally consistent orientation of S , one that can be tailored to specific applications by designing \mathcal{F} in different ways. The idea is always that the oriented separations in a set $F \in \mathcal{F}$ collectively point to an area (of the ground set or structure which the separations in S are thought to ‘separate’) that is too small to accommodate some particular type of highly cohesive substructure.

A set σ of nondegenerate oriented separations, possibly empty, is a *star of separations* if they point towards each other: if $\vec{r} \leq \vec{s}$ for all distinct $\vec{r}, \vec{s} \in \sigma$ (Fig. 1). Stars of separations are clearly nested. They are also consistent: if \vec{r}, \vec{s} lie in the same star we cannot have $\vec{r} < \vec{s}$, since also $\vec{s} \leq \vec{r}$ by the star property. A star σ need not be antisymmetric; but if $\{\vec{s}, \bar{\vec{s}}\} \subseteq \sigma$, then any other $\vec{r} \in \sigma$ will be trivial.

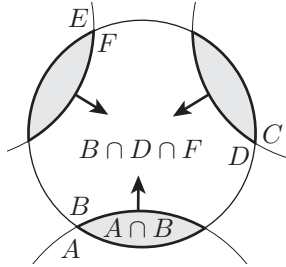


Figure 1: The separations $(A, B), (C, D), (E, F)$ form a 3-star

Let S be a set of separations. An S -*tree* is a pair (T, α) of a tree⁵ T and a function $\alpha: \vec{E}(T) \rightarrow \vec{S}$ from the set

$$\vec{E}(T) := \{ (x, y) : \{x, y\} \in E(T) \}$$

of the *orientations* of its *edges* to \vec{S} such that, for every edge xy of T , if $\alpha(x, y) = \vec{s}$ then $\alpha(y, x) = \bar{\vec{s}}$. It is an S -tree is *over* $\mathcal{F} \subseteq 2^{\vec{S}}$ if, in addition, for every

⁴Terms introduced for unoriented separations may be used informally for oriented separations too if the meaning is obvious, and vice versa.

⁵Trees have at least one node [6].

node t of T we have $\alpha(\vec{F}_t) \in \mathcal{F}$, where

$$\vec{F}_t := \{(x, t) : xt \in E(T)\}.$$

We shall call the set $\vec{F}_t \subseteq \vec{E}(T)$ the *oriented star at t in T* (even if it is empty). Its image $\alpha(\vec{F}_t) \in \mathcal{F}$ is said to be *associated with t in (T, α)* .

An important example of S -trees are *irredundant S -trees over stars*: those over some \mathcal{F} all of whose elements are stars of separations. In such an S -tree (T, α) the map α preserves the *natural partial ordering* on $\vec{E}(T)$ defined by letting $(x, y) < (u, v)$ if $\{x, y\} \neq \{u, v\}$ and the unique $\{x, y\}$ – $\{u, v\}$ path in T joins y to u (see Figure 2):

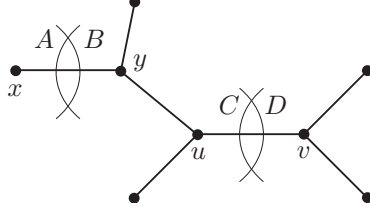


Figure 2: Edges $(x, y) < (u, v)$ and separations $(A, B) = \alpha(x, y) \leq \alpha(u, v) = (C, D)$

3 Tangle duality in abstract separation systems

The tangle duality theorem for abstract separation systems, the result from [9] which we seek to apply in this paper to various different contexts, says the following. Let (\vec{S}, \leq) be a separation system and \mathcal{F} a collection of ‘forbidden’ sets of separations. Then, under certain conditions, either S has an \mathcal{F} -tangle or there exists an S -tree over \mathcal{F} . We now define these conditions and state the theorem formally. We then prove a couple of lemmas that will help us apply it.

Let \vec{r} be a nontrivial and nondegenerate element of a separation system $(\vec{S}, \leq, *)$ contained in some universe $(\vec{U}, \leq, *, \vee, \wedge)$ of separations, the ordering and involution on \vec{S} being induced by those of \vec{U} . Consider any $\vec{s}_0 \in \vec{S}$ such that $\vec{r} \leq \vec{s}_0$. As \vec{r} is nontrivial and nondegenerate, so is \vec{s}_0 .

Let $S_{\geq \vec{r}}$ be the set of all separations $s \in S$ that have an orientation $\vec{s} \geq \vec{r}$. Since \vec{r} is nontrivial, only one of the two orientations \vec{s} of every $s \in S_{\geq \vec{r}} \setminus \{\vec{r}\}$ satisfies $\vec{s} \geq \vec{r}$. Letting

$$f \downarrow_{\vec{s}_0}^{\vec{r}}(\vec{s}) := \vec{s} \vee \vec{s}_0 \quad \text{and} \quad f \downarrow_{\vec{s}_0}^{\vec{r}}(\vec{s}) := (\vec{s} \vee \vec{s}_0)^*$$

for all $\vec{s} \geq \vec{r}$ in $\vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$ thus defines a map $\vec{S}_{\geq \vec{r}} \rightarrow \vec{U}$, the *shifting map* $f \downarrow_{\vec{s}_0}^{\vec{r}}$ (Fig. 3, right). Note that $f \downarrow_{\vec{s}_0}^{\vec{r}}(\vec{r}) = \vec{s}_0$, since $\vec{r} \leq \vec{s}_0$. Shifting maps preserve the partial ordering on a separation system, and in particular map stars to stars:

Lemma 3.1. [9] *The map $f = f \downarrow_{\vec{s}_0}^{\vec{r}}$ preserves the ordering \leq on $\vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$. In particular, f maps stars to stars.*

Let us say that \vec{s}_0 *emulates* \vec{r} in \vec{S} if $\vec{s}_0 \geq \vec{r}$ and every $\vec{s} \in \vec{S} \setminus \{\vec{r}\}$ with $\vec{s} \geq \vec{r}$ satisfies $\vec{s} \vee \vec{s}_0 \in \vec{S}$. We call \vec{S} *separable* if for every two nontrivial and nondegenerate $\vec{r}, \vec{r}' \in \vec{S}$ such that $\vec{r} \leq \vec{r}'$ there exists an $s_0 \in S$ with an orientation \vec{s}_0 that emulates \vec{r} and its inverse \vec{s}_0 emulating \vec{r}' .

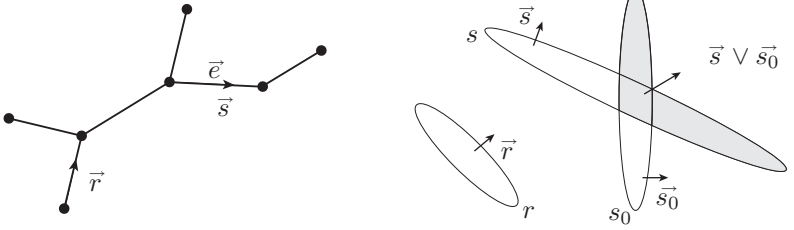


Figure 3: Shifting \vec{s} to $\vec{s} \vee \vec{s}_0$

Given a set $\mathcal{F} \in 2^{\vec{U}}$ of stars of separations, we say that $\vec{s}_0 \in \vec{S}$ *emulates* $\vec{r} \in \vec{S}$ in \vec{S} for \mathcal{F} if \vec{s}_0 emulates \vec{r} in \vec{S} and for any star $\sigma \subseteq \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$ in \mathcal{F} that has an element $\vec{s} \geq \vec{r}$ we also have $f \downarrow_{\vec{s}_0}^{\vec{r}}(\sigma) \in \mathcal{F}$.

Let us say that a set \mathcal{F} *forces* the separations $\vec{s} \in \vec{S}$ for which $\{\vec{s}\} \in \mathcal{F}$. And that \vec{S} is \mathcal{F} -*separable* if for all nontrivial and nondegenerate $\vec{r}, \vec{r}' \in \vec{S}$ that are not forced by \mathcal{F} and satisfy $\vec{r} \leq \vec{r}'$ there exists an $s_0 \in S$ with an orientation \vec{s}_0 that emulates \vec{r} in \vec{S} for \mathcal{F} and such that \vec{s}_0 emulates \vec{r}' in \vec{S} for \mathcal{F} . (As earlier, any such \vec{s}_0 will also be nontrivial and nondegenerate.)

Recall that an orientation O of S is an \mathcal{F} -*tangle* if it is consistent and avoids \mathcal{F} . We call \mathcal{F} *standard* for \vec{S} if it forces all $\vec{s} \in \vec{S}$ that are trivial in \vec{S} . The ‘strong duality theorem’ from [9] now reads as follows.

Theorem 3.2 (Tangle duality theorem for abstract separation systems).

Let $(\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations containing a separation system $(\vec{S}, \leq, *)$. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for \vec{S} . If \vec{S} is \mathcal{F} -separable, exactly one of the following assertions holds:

- (i) There exists an \mathcal{F} -tangle of S .
- (ii) There exists an S -tree over \mathcal{F} .

Often, the proof that \vec{S} is \mathcal{F} -separable can be split into two easier parts, a proof that \vec{S} is separable and one that \mathcal{F} is *closed under shifting* in \vec{S} : that whenever $\vec{s}_0 \in \vec{S}$ emulates (in \vec{S}) some nontrivial and nondegenerate $\vec{r} \leq \vec{s}_0$ not forced by \mathcal{F} , then it does so for \mathcal{F} . Indeed, the following lemma is immediate from the definitions:

Lemma 3.3. If \vec{S} is separable and \mathcal{F} is closed under shifting in \vec{S} , then \vec{S} is \mathcal{F} -separable. \square

The separability of \vec{S} will often be established as follows. Let us call a real function $\vec{s} \mapsto |\vec{s}|$ on a universe $(\vec{U}, \leq, *, \vee, \wedge)$ of oriented separations an *order function* if it is non-negative, symmetric and submodular, that is, if $0 \leq |\vec{s}| = |\vec{s}|$ and

$$|\vec{r} \vee \vec{s}| + |\vec{r} \wedge \vec{s}| \leq |\vec{r}| + |\vec{s}|$$

for all $\vec{r}, \vec{s} \in \vec{U}$. We then call $|s| := |\vec{s}|$ the *order* of s and of \vec{s} . For every positive integer k ,

$$\vec{S}_k := \{\vec{s} \in \vec{U} : |\vec{s}| < k\}$$

is a separation system (though not necessarily a universe). A universe with an order function will be called *submodular*.

Lemma 3.4. *Every such \vec{S}_k is separable.*

Proof. Given nontrivial and nondegenerate $\vec{r}, \vec{r}' \in \vec{S}_k$ such that $\vec{r} \leq \vec{r}'$, we have to find an $\vec{s}_0 \in \vec{S}_k$ such that \vec{s}_0 emulates \vec{r} in \vec{S}_k and \vec{s}_0 emulates \vec{r}' in \vec{S}_k . We choose $\vec{s}_0 \in \vec{U}$ of minimum order with $\vec{r} \leq \vec{s}_0 \leq \vec{r}'$. Since \vec{r} is a candidate for \vec{s}_0 , we have $|\vec{s}_0| \leq |\vec{r}|$ and hence $\vec{s}_0 \in \vec{S}_k$. We show that \vec{s}_0 emulates \vec{r} ; by symmetry, this will imply also that \vec{s}_0 emulates \vec{r}' .

Let us show that every $\vec{s} \geq \vec{r}$ in \vec{S}_k satisfies $\vec{s} \vee \vec{s}_0 \in \vec{S}_k$. We prove this by showing that $|\vec{s} \vee \vec{s}_0| \leq |\vec{s}|$, which will follow from submodularity once we have shown that $|\vec{s} \wedge \vec{s}_0| \geq |\vec{s}_0|$. This, however, holds since $\vec{s} \wedge \vec{s}_0$ was a candidate for the choice of \vec{s}_0 : we have $\vec{r} \leq \vec{s} \wedge \vec{s}_0$ since $\vec{r} \leq \vec{s}$ and $\vec{r} \leq \vec{s}_0$, while $\vec{s} \wedge \vec{s}_0 \leq \vec{s}_0 \leq \vec{r}'$. \square

For the rest of this paper except in Sections 5 and 10, whenever we consider a graph $G = (V, E)$ it will have at least one vertex, and we consider the submodular universe \vec{U} of its (oriented) *vertex separations*, the separations (A, B) of V such that G has no edge between $A \setminus B$ and $B \setminus A$, with the order function

$$|A, B| := |A \cap B|.$$

Note that A and B are allowed to be empty. For each positive integer k , the set $\vec{S}_k = \{\vec{s} \in \vec{U} : |\vec{s}| < k\}$ will be a separable separation system, by Lemma 3.4.

4 Tangle duality in graphs

A *tangle of order k* in a finite graph $G = (V, E)$, as introduced by Robertson and Seymour [18], is (easily seen to be equivalent to) an orientation of S_k that avoids

$$\mathcal{T} := \{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq \vec{U} : G[A_1] \cup G[A_2] \cup G[A_3] = G\}.$$

(The three separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct.) Clearly, \mathcal{T} forces all the small separations in \vec{U} , those of the form (A, V) . Hence $\mathcal{T} \cap \vec{S}_k$ is a standard subset of \vec{S}_k , for every integer $k > 0$.

Notice that any \mathcal{T} -avoiding orientation O of S_k is consistent, and therefore an \mathcal{T} -tangle in our sense, since for any pair of separations $(C, D) \leq (A, B)$ we have $G[D] \cup G[A] \supseteq G[B] \cup G[A] = G$ and hence $\{(D, C), (A, B)\} \in \mathcal{T}$. Similarly, O must contain all (A, B) with $|A| < k$: it cannot contain (B, A) , as $(A, V) \in O$ by $\{(V, A)\} \in \mathcal{T}$ but $\{(B, A), (A, V)\} \in \mathcal{T}$.

Since our duality theorems, so far, only work with sets \mathcal{F} consisting of stars of separations, let us consider the set \mathcal{T}^* of those sets in \mathcal{T} that are stars.

Theorem 4.1 (Tangle duality theorem for graphs).

For every $k > 0$, every graph G satisfies exactly one of the following assertions:

- (i) *G has a \mathcal{T}^* -tangle of S_k .*
- (ii) *G has an S_k -tree over \mathcal{T}^* .*

Proof. By Theorem 3.2 and Lemmas 3.3–3.4, all we need to show is that \mathcal{T}^* is closed under shifting in $\vec{S} = \vec{S}_k$. This is easy from the definitions. Informally, if $(X, Y) \in \vec{S}$ emulates some $\vec{r} \leq (X, Y)$ not forced by \mathcal{T} and we shift a star

$$\sigma = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$$

with $\vec{r} \leq (A_1, B_1)$, say, then we replace (A_1, B_1) with $(A_1 \cup X, B_1 \cap Y)$, and (A_i, B_i) with $(A_i \cap Y, B_i \cup X)$ for $i \geq 2$. As any vertex or edge that is not in $G[Y]$ lies in $G[X]$, this means that $\bigcup_i G[A_i] = G$ remains unchanged. \square

Our tangle duality theorem can easily be extended to include the classical duality theorem of Robertson and Seymour [18] for tangles and branch-width. In order to do so, we first show that all \mathcal{T}^* -tangles are in fact \mathcal{T} -tangles, so these two notions coincide. Secondly, we will check that our tree structure witnesses for the non-existence of a tangle coincide with those used by Robertson and Seymour: that a graph has an S_k -tree over \mathcal{T}^* if and only it has branch-width $< k$.

Using the submodularity of our order function $\{A, B\} \mapsto |A, B|$, we can easily show that \mathcal{T}^* -tangles of S_k are in fact \mathcal{T} -tangles:

Lemma 4.2. *Every consistent \mathcal{T}^* -avoiding orientation O of S_k avoids \mathcal{T} , as long as $|G| \geq k$.*

Proof. Suppose O has a subset $\sigma \in \mathcal{T}$. We show that as long as this set is not an inclusion-minimal nested set in \mathcal{T} , we can either delete one of its elements, or replace it by a smaller separation in O , so that the resulting set $\sigma' \subseteq O$ is still in \mathcal{T} but is smaller or contains fewer pairs of crossing separations. Iterating this process, we eventually arrive at a minimal nested set in \mathcal{T} that is still a subset of O . By its minimality, this set is an antichain (compare the definition of \mathcal{T}), and all consistent nested antichains are stars.⁶ Our subset of O will thus lie in \mathcal{T}^* , contradicting our assumption that O avoids \mathcal{T}^* .

If σ has two comparable elements, we delete the smaller one and retain a subset of O in \mathcal{T} . We now assume that σ is an antichain, but that it contains two crossing separations, $\vec{r} = (A, B)$ and $\vec{s} = (C, D)$ say. As these and their inverses lie in \vec{S}_k , submodularity implies that one of the separations $(A \cap D, B \cup C) \leq (A, B)$ and $(B \cap C, A \cup D) \leq (C, D)$ also lies in \vec{S}_k . Let us assume the former; the other case is analogous.

Let σ' be obtained from σ by replacing \vec{r} with $\vec{r}' := (A \cap D, B \cup C) \in \vec{S}_k$. Then σ' is still in \mathcal{T} , since any vertex or edge of $G[A]$ that is not in $G[A \cap D]$ lies in $G[C]$, and (C, D) is still in σ' . Moreover, while \vec{r} crosses \vec{s} , clearly \vec{r}' does not. To complete the proof, we just have to show that \vec{r}' cannot cross any separation $\vec{t} \in \sigma'$ that was nested with \vec{r} .

If $\vec{r} \leq \vec{t}$ or $\vec{r} \leq \vec{t}$, then $\vec{r}' \leq \vec{r}$ is nested with \vec{t} , as desired. If not then $\vec{t} \leq \vec{r}$, since $\{\vec{r}, \vec{t}\} \subseteq O$ is consistent. This contradicts our assumption that σ is an antichain. \square

The following elementary lemma provides the link between our S -trees and branch-decompositions as defined by Robertson and Seymour [18]:

Lemma 4.3. *For every integer $k \geq 3$,⁷ our graph G has branch-width $< k$ if and only if G has an S_k -tree over \mathcal{T}^* .*

Proof. If $|E(G)| \leq 1$, then the branch-width is 0 by definition, and G has an S_k -tree over \mathcal{T}^* with two nodes and a single edge mapped to (V, V) both ways. We now assume that G has at least 2 edges.

⁶Here we use that $|G| \geq k$: otherwise $\{(V, V)\} \in \mathcal{T} \setminus \mathcal{T}^*$ lies in O .

⁷See the remark after Theorem 4.4.

Let us prove the forward implication first. We may assume that G has no isolated vertices, because we can easily add a leaf in an S_k -tree corresponding to an isolated vertex.

Suppose (T, L) is a branch-decomposition of width $< k$. For each edge $e = st$ of T , let T_s and T_t be the components of $T - e$ containing s and t , respectively. Let $A_{s,t}$ and $B_{s,t}$ be the sets of vertices incident with an edge in $L^{-1}(V(T_s))$ and $L^{-1}(V(T_t))$, respectively, except that if $A_{s,t}$ contains only two vertices we always put both these also in $B_{s,t}$ (and similarly vice versa). Note that $B_{s,t} = A_{t,s}$.

For all adjacent nodes $s, t \in T$ let $\alpha(s, t) := (A_{s,t}, B_{s,t})$. Since G has no isolated vertices these $\alpha(s, t)$ are separations, and since (T, L) has width $< k$ they lie in \tilde{S}_k . For each internal node t of T and its three neighbours s_1, s_2, s_3 , every edge of G has both ends in one of the $G[A_{s_i, t}]$, so

$$\alpha(\vec{F}_t) = \{\alpha(s_1, t), \alpha(s_2, t), \alpha(s_3, t)\} \in \mathcal{T}.$$

As $A_{s_i, t} \subseteq A_{t, s_j} = B_{s_j, t}$ for all $i \neq j$, the set $\alpha(\vec{F}_t)$ is a star. For leaves t of T , the associated star $\alpha(\vec{F}_t)$ has the form $\{(V, A)\}$ with $|A| = 2$, which is in \mathcal{T}^* since $k \geq 3$. This proves that (T, α) is an S_k -tree over \mathcal{T}^* .

Now let us prove the converse. We may again assume that G has no isolated vertex. Let (T, α) be an S_k -tree over \mathcal{T}^* . For each edge e of G , let us orient the edges st of T towards t whenever $\alpha(s, t) = (A, B)$ is such that B contains both ends of e . If e has its ends in $A \cap B$, we choose an arbitrary orientation of st . As T has fewer edges than nodes, there exists a node $t =: L(e)$ such that every edge at t is oriented towards t .

Let us choose an S_k -tree (T, α) and $L: E \rightarrow V(T)$ so that the number of leaves in $L(E)$ is maximized, and subject to this with $|V(T)|$ minimum. We claim that, for every edge e of G , the node $t = L(e)$ is a leaf of T . Indeed, if not, let us extend T to make $L(e)$ a leaf. If t has degree 2, we attach a new leaf t' to t and put $\alpha(t', t) = (V(e), V)$ and $L(e) = t'$, where $V(e)$ denotes the set of ends of e . If t has degree 3 then, by definition of \mathcal{T} , there is a neighbour t' of t such that $e \in G[A]$ for $(A, B) = \alpha(t', t)$. As $t = L(e)$, this means that e has both ends in $A \cap B$. Subdivide the edge tt' , attach a leaf t^* to the subdividing vertex t'' , put $\alpha(t', t'') = \alpha(t'', t) = (A, B)$ and $\alpha(t^*, t'') = (V(e), V)$, and let $L(e) = t^*$. In both cases, (T, α) is still an S_k -tree over \mathcal{T}^* .

In the same way one can show that L is injective. Indeed, if $L(e') = t = L(e'')$ for distinct $e', e'' \in E$, we could increase the number of leaves in $L(E)$ by joining two new leaves t', t'' to the current leaf t , letting $\alpha(t', t) = (V(e'), V)$ and $\alpha(t'', t) = (V(e''), V)$, and redefine $L(e')$ as t' and $L(e'')$ as t'' .

By the minimality of $|V(T)|$, every leaf of T is in $L(E)$, since we could otherwise delete it. Finally, no node t of T has degree 2, since contracting an edge at t while keeping α unchanged on the remaining edges would leave an S_k -tree over \mathcal{T}^* . (Here we use that G has no isolated vertices, and that $L(e) \neq t$ for every edge e of G .)

Hence L is a bijection from E to the set of leaves of T , and T is a ternary tree. Thus, (T, L) is a branch-decomposition of G , clearly of width less than k . \square

We can now derive, and extend, the Robertson-Seymour [18] duality theorem for tangles and branch-width:

Theorem 4.4 (Tangle duality theorem for graphs, extended).

The following assertions are equivalent for all finite graphs $G \neq \emptyset$ and $k \geq |G|$:

- (i) G has a tangle of order k .
- (ii) G has a \mathcal{T} -tangle of S_k .
- (iii) G has a \mathcal{T}^* -tangle of S_k .
- (iv) G has no S_k -tree over \mathcal{T}^* .
- (v) G has branch-width at least k , or $k \leq 2$ and G is a disjoint union of stars and isolated vertices and has at least one edge.

Proof. If $k = 1$, then all statements are true. If $k = 2$, they are all true if G has an edge, and all false if not. Assume now that $k \geq 3$.

(i) \leftrightarrow (ii) follows from the definition of a tangle at the start of this section, and our observation that they are consistent.

(ii) \rightarrow (iii) is trivial; the converse is Lemma 4.2.

(iii) \leftrightarrow (iv) is an application of Theorem 3.2.

(iv) \leftrightarrow (v) is Lemma 4.3. □

The exception in (v) for $k \leq 2$ is due to a quirk in the notion of branch-width, which results from its emphasis on separating individual edges. The branch-width of all nontrivial trees other than stars is 2, but it is 1 for stars $K_{1,n}$. For a clean duality theorem (even one just in the context of [18]) it should be 2 also for stars: every graph with at least one edge has a tangle of order 2, because we can orient all separations in S_2 towards a fixed edge. Similarly, the branch-width of a disjoint union of edges is 0, but its tangle number is 2.

5 Tangle duality for set partitions: rank-width, carving-width and edge-tangles in graphs, matroid tangles, and image segmentation

The concepts of branch-width and tangles were introduced by Robertson and Seymour [18] not only for graphs but more generally for hypergraphs. They proved all their relevant lemmas more generally for arbitrary order functions $(A, B) \mapsto |A, B|$ rather than just $|A, B| = |A \cap B|$. Geelen, Gerards, Robertson, and Whittle [11] applied this explicitly to the submodular connectivity function in matroids.

Our first aim in this section is to derive from Theorem 3.2 a duality theorem for tangles in arbitrary submodular universes of set separations.⁸ This will imply the above branch-width duality theorems for hypergraphs and matroids, as well as their cousins for carving width [20] and rank-width of graphs [16]. It will also yield a duality theorem for *edge-tangles*, tangles of bipartitions of the vertex set of a graph whose order is the number of edges across. We shall then recast the theorem in the language of cluster analysis to derive a duality theorem for the existence of clusters in data sets.

Recall that an oriented *separation* of a set V is a pair (A, B) such that $A \cup B = V$. Often, the separations considered will be bipartitions of V , but

⁸Recall that these are more general than set partitions: the two sides may overlap.

in general we allow $A \cap B \neq \emptyset$. We also allow A and B to be empty. Recall that *order functions* are non-negative, symmetric and submodular functions on a separation system.

Let \vec{U} be any submodular universe of separations of a set V of at least two elements, with an order function $(A, B) \mapsto |A, B|$. Given $k > 0$, call an orientation of

$$S_k = \{ \{A, B\} \in U : |A, B| < k \}$$

a *tangle of order k* if it avoids

$$\mathcal{F} = \{ \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq \vec{S}_k : A_1 \cup A_2 \cup A_3 = V \} \\ \cup \{ \{(A, B)\} \subseteq \vec{S}_k : |B| = 1 \}.$$

Here, $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct. In particular, \mathcal{F} is standard and tangles are consistent, so the tangles of \vec{U} are precisely its \mathcal{F} -tangles.

Let $\mathcal{F}^* \subseteq \mathcal{F}$ be the set of stars in \mathcal{F} . As in the proof of Theorem 4.1, it is easy to prove that \mathcal{F}^* is closed under shifting in every \vec{S}_k . We also have the following analogue of Lemma 4.2, with the same proof:

Lemma 5.1. *Every consistent \mathcal{F}^* -avoiding orientation of S_k avoids \mathcal{F} , as long as $|V| \geq k$. \square*

By Lemmas 3.4 and 5.1, Theorem 3.2 now specializes as follows:

Theorem 5.2 (Tangle duality theorem for set separations).

Given a submodular universe \vec{U} of separations of a set V and $k \geq V$, the following assertions are equivalent:

- (i) \vec{U} has a tangle of order k .
- (ii) \vec{U} has an \mathcal{F}^* -tangle of S_k .
- (iii) \vec{U} has no \vec{S}_k -tree over \mathcal{F}^* . \square

Applying Theorem 5.2 with the appropriate order functions yields duality theorems for all known width parameters based on set separations. For example, let V be the vertex set of a graph G , with bipartitions as separations. Counting the edges across a bipartition defines an order function whose \mathcal{F} -tangles are known as the *edge-tangles* of G , so Theorem 5.2 yields a duality theorem for these. See Liu [14] for an application of edge-tangles to an Erdős-Pósa-type problem for immersion.

The duals to edge-tangles of order k are S_k -trees over \mathcal{F}^* . These were introduced by Seymour and Thomas [20] as *carvings*. The least k such that G has a carving is its *carving-width*. We thus have a duality theorem between edge-tangles and carving-width.

Taking as the order of a vertex bipartition the rank of the adjacency matrix of the bipartite graph that this partition induces gives rise to a width parameter called rank-width. In our terminology, G has *rank-width* $< k$ if and only if it admits an S_k -tree over \mathcal{F}^* . The corresponding \mathcal{F} -tangles of S_k , then, are necessary and sufficient witnesses for having rank-width $\geq k$, and we have a duality theorem for rank-width.

If V is the vertex set of a hypergraph or the ground set of a matroid, the \mathcal{F} -tangles coincide, just as for graphs, with the hypergraph tangles of [18] or the matroid tangles of [11]. As in the proof of Lemma 4.3, a hypergraph or matroid has branch-width $< k$ if and only if it has an S_k -tree over \mathcal{F}^* . Theorem 5.2 thus yields the original duality theorems of [18] and [11] in this case.

Our tangle duality theorem for set separations can also be applied in contexts quite different from graphs and matroids. In [10] we considered the separations of a pixellated image obtained by cutting it in two. Assigning small integers to natural cuts and larger integers to less natural ones defines an order function which makes these cuts into a submodular universe of separations. Its tangles then correspond to natural regions of this image: parts that cannot be cut into halves of roughly equal size by natural cuts.

Theorem 5.2 then applies to these regions: if there is no region of some given order, or ‘coherence’, then this is witnessed by a nested set of separations which cut the image up, recursively, into single pixels. This nested set of cuts, therefore, will be an easily checkable witness for the non-existence of a highly coherent region in the image.

This approach to image segmentation has an important advantage over more traditional ways of identifying the regions of an image. Real-world pixellated images tend to be fuzzy, and tangles can capture their regions despite their fuzziness. Indeed, consider a large grid in a graph. For every low-order separation, most of the grid will lie on the same side, so the grid ‘orients’ that separation towards this side. But every single vertex will lie on the ‘wrong’ side for *some* low-order separation, the side not containing most of the grid; for example, it may be separated off by its four neighbours. The grid, therefore, defines a unique k -tangle for some large k , but the ‘location’ of this tangle is not represented correctly by any one of its vertices – just as for a fuzzy region of an image it may be impossible to say which pixels exactly belong to that region and which do not.

The application in [10] of Theorem 5.2 to image analysis reads as follows:

Corollary 5.3. [10] *For every picture π on a canvas and every integer $k > 0$, either π has a non-trivial region of coherence at least k , or there exists a laminar set of lines of order $< k$ all whose splitting stars are void 3-stars or single pixels. For no picture do both these happen at once.*

6 Tangle duality for tree-width in graphs

We now apply our abstract duality theorem to obtain a new duality theorem for tree-width in graphs. Its witnesses for large tree-width will be orientations of S_k , like tangles, and thus different from brambles (or ‘screens’), the dual objects in the classical tree-width duality theorem of Seymour and Thomas [19].

This latter theorem, which ours easily implies, says that a finite graph either has a tree-decomposition of width less than $k - 1$ or a bramble of order at least k , but not both. The original proof of this theorem is as mysterious as the result is beautiful. The shortest known proof is given in [5] (where we refer the reader also for definitions), but it is hardly less mysterious. A more natural, if slightly longer, proof due to Mazoit is presented in [6]. The proof via our abstract duality theorem, as outlined below, is perhaps not shorter all told, but it seems to be the simplest available so far.

Given a finite graph $G = (V, E)$, we consider its submodular separation universe \vec{U} and the separation systems $\vec{S}_k \subseteq \vec{U}$ as defined at the end of Section 3. For every integer $k > 0$ let

$$\mathcal{F}_k := \left\{ \sigma \subseteq \vec{U} \mid \sigma = \{(A_i, B_i) : i = 0, \dots, n\} \text{ is a star with } \left| \bigcap_{i=0}^n B_i \right| < k \right\}.$$

(We take $\bigcap_{i=0}^n B_i := V$ if $\sigma = \emptyset$, so K_1 is an S_k -tree over \mathcal{F}_k if $|G| < k$.)

Since \mathcal{F}_k forces all the small nondegenerate separations in \vec{S}_k , the separations $(A, V) \in \vec{S}_k$ with $A \neq V$, it is standard for every S_k . We have also seen that \vec{S}_k is separable (Lemma 3.4). To apply Theorem 3.2 we thus only need the following lemma (cf. Lemma 3.3) – whose proof contains the only bit of magic now left in tree-width duality:

Lemma 6.1. *For every integer $k > 0$, the set \mathcal{F}_k is closed under shifting in S_k .*

Proof. Consider a separation $\vec{s}_0 = (X, Y) \in \vec{S}_k =: \vec{S}$ that emulates, in \vec{S} , some nontrivial and nondegenerate $\vec{r} \in \vec{S}$ not forced by \mathcal{F}_k . Let

$$\sigma = \{(A_i, B_i) : i = 0, \dots, n\} \subseteq \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$$

be a star in \mathcal{F}_k with $\vec{r} \leq (A_0, B_0)$. Then

$$\vec{r} \leq (A_0, B_0) \leq (B_i, A_i) \text{ for all } i \geq 1. \quad (3)$$

We have to show that

$$\sigma' = \{(A'_i, B'_i) : i = 0, \dots, n\} \in \mathcal{F}_k$$

for $(A'_i, B'_i) := f \downarrow_{\vec{s}_0}^{\vec{r}} (A_i, B_i)$.

From Lemma 3.1 we know that σ' is a star. Since (X, Y) emulates \vec{r} in \vec{S} , we have $\sigma' \subseteq \vec{S}$ by (3). It remains to show that $|\bigcap_{i=0}^n B'_i| < k$. The trick will be to rewrite this intersection as the intersection of the two sides of a suitable separation that we know to lie in $S = S_k$.

By (3) we have $(A'_0, B'_0) = (A_0 \cup X, B_0 \cap Y)$, while $(A'_i, B'_i) = (A_i \cap Y, B_i \cup X)$ for $i \geq 1$. Since the (A_i, B_i) are separations, i.e. in \vec{U} , so is $(\bigcap_{i=1}^n B_i, \bigcup_{i=1}^n A_i)$. As trivially $(V, B_0) \in \vec{U}$, this implies that, for $B^* := \bigcap_{i=1}^n B_i$, also

$$\left(\bigcap_{i=1}^n B_i \cap V, \bigcup_{i=1}^n A_i \cup B_0 \right) \stackrel{(3)}{=} (B^*, B_0) \in \vec{U}.$$

Since $\sigma \in \mathcal{F}_k$ we have $|B^*, B_0| = |\bigcap_{i=0}^n B_i| < k$, so $(B^*, B_0) \in \vec{S}_k = \vec{S}$ (Fig. 4).⁹ As also $\vec{r} \leq (B^*, B_0)$ by (3), the fact that (X, Y) emulates \vec{r} in \vec{S} therefore implies that $(B^* \cup X, B_0 \cap Y) \in \vec{S} = \vec{S}_k$. But then

$$\left| \bigcap_{i=0}^n B'_i \right| = \left| (B_0 \cap Y) \cap \bigcap_{i=1}^n (B_i \cup X) \right| = |B_0 \cap Y, B^* \cup X| < k,$$

which means that $\sigma' \in \mathcal{F}_k$. □

Theorem 6.2 (Tangle-treewidth duality theorem for graphs).

For every $k > 0$, every graph G satisfies exactly one of the following assertions:

- (i) G has an \mathcal{F}_k -tangle of S_k .
- (ii) G has an S_k -tree over \mathcal{F}_k .

Proof. Apply Theorem 3.2 and Lemmas 3.3, 3.4 and 6.1. □

⁹The A_i , of course, are ‘more disjoint’ than they appear in the figure.

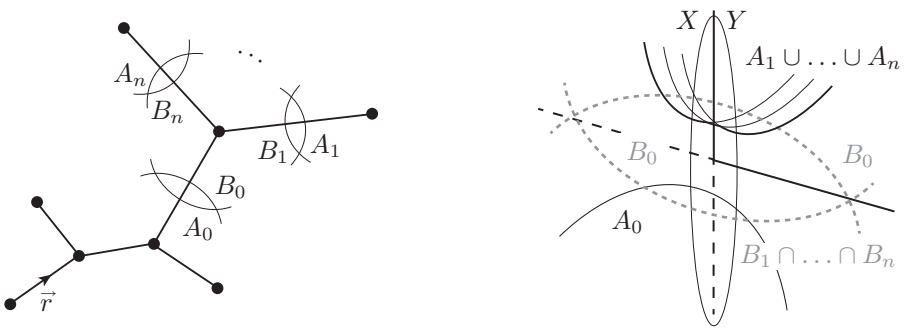


Figure 4: Shifting the separation (B^*, B_0)

Condition (ii) above can be expressed in terms of the tree-width $\text{tw}(G)$ of G :

Lemma 6.3. *A graph G has an S_k -tree over \mathcal{F}_k if and only if $\text{tw}(G) < k - 1$. More precisely, G has an S_k -tree (T, α) over \mathcal{F}_k if and only if it admits a tree-decomposition (T, \mathcal{V}) of width $< k - 1$.*

Proof. Given any S -tree (T, α) of G over a set \mathcal{F} of stars, let $\mathcal{V} = (V_t)_{t \in T}$ be defined by letting

$$V_t = \bigcap \{ B : (A, B) = \alpha(s, t), \text{ } st \in E(T) \}. \quad (4)$$

It is easy to check [4] that (T, \mathcal{V}) is a tree-decomposition of G with adhesion sets $V_t \cap V_{t'} = A \cap B$ whenever $(A, B) = \alpha(t, t')$. If $S = S_k$ and $\mathcal{F} = \mathcal{F}_k$, we have $|V_t| < k$ at all $t \in T$, so (T, \mathcal{V}) has width less than $k - 1$.

Conversely, given a tree-decomposition (T, \mathcal{V}) with $\mathcal{V} = (V_t)_{t \in T}$, say, define $\alpha: \vec{E}(T) \rightarrow \vec{S}_k$ as follows. Given $t_1 t_2 \in E(T)$, let T_i be the component of $T - t_1 t_2$ containing t_i , and put $U_i := \bigcup_{t \in V(T_i)} V_t$ ($i = 1, 2$). Then let $\alpha(t_1, t_2) := (U_1, U_2)$. One easily checks [5] that $U_1 \cap U_2 = V_{t_1} \cap V_{t_2}$, so α takes its values in \vec{S}_k if (T, \mathcal{V}) has width $< k - 1$. Moreover, every part V_t satisfies (4), so if (T, \mathcal{V}) has width $< k - 1$ then (T, α) is over \mathcal{F}_k . \square

If desired, we can derive from Theorem 6.2 the tree-width duality theorem of Seymour and Thomas [19]. This is cast in terms of brambles, or ‘screens’, as they originally called them. (See [6] for a definition and some background.)

Brambles have an interesting history. After Robertson and Seymour had invented tangles, they looked for a tangle-like type of highly cohesive substructure, or HCS, dual to low tree-width. Their plan was that this should be a map β assigning to every set X of fewer than k vertices one component of $G - X$. The question, in our language, was how to make these choices consistent: so that they would define an abstract HCS.

The obvious consistency requirement, that $\beta(Y) \subseteq \beta(X)$ whenever $X \subseteq Y$, is easily seen to be too weak. Yet asking that $\beta(X) \cap \beta(Y) \neq \emptyset$ for all X, Y turned out to be too strong. In [19], Seymour and Thomas then found a requirement that worked: that any two such sets, $\beta(X)$ and $\beta(Y)$, should *touch*: that either they share a vertex or G has an edge between them. Such maps β are now called *havens*, and it is easy to show that G admits a haven of order k (one defined on all sets X of fewer than k vertices) if and only if G has a bramble of order at least k .

Lemma 6.4. *G has a bramble of order at least k if and only if G has an \mathcal{F}_k -tangle of S_k .*

Proof. Let \mathcal{B} be a bramble of order at least k . For every $\{A, B\} \in S_k$, since $A \cap B$ is too small to cover \mathcal{B} but every two sets in \mathcal{B} touch and are connected, exactly one of the sets $A \setminus B$ and $B \setminus A$ contains an element of \mathcal{B} . Thus,

$$O = \{ (A, B) \in \vec{S}_k : B \setminus A \text{ contains an element of } \mathcal{B} \}$$

is an orientation of S_k , which for the same reason is clearly consistent.

To show that O avoids \mathcal{F}_k , let $\sigma = \{(A_1, B_1), \dots, (A_n, B_n)\} \in \mathcal{F}_k$ be given. Then $|\bigcap_{i=1}^n B_i| < k$, so some $C \in \mathcal{B}$ avoids this set and hence lies in the union of the sets $A_i \setminus B_i$. But these sets are disjoint, since σ is a star. Hence C lies in one of them, $A_1 \setminus B_1$ say, putting (B_1, A_1) in O . But then $(A_1, B_1) \notin O$, so $\sigma \not\subseteq O$ as claimed.

Conversely, let O be an \mathcal{F}_k -tangle of S_k . We shall define a bramble \mathcal{B} containing for every set X of fewer than k vertices exactly one component of $G - X$, and no other sets. Such a bramble will have order at least k , since no such set X covers it.

Given such a set X , note first that $X \neq V$. For if $|V| < k$ then $\emptyset \in \mathcal{F}_k$, contradicting our assumption that O has no subset in \mathcal{F}_k . Let C_1, \dots, C_n be the vertex sets of the components of $G - X$. Consider the separations (A_i, B_i) with $A_i = C_i \cup N(C_i)$ and $B_i = V \setminus C_i$. Since

$$\sigma_X := \{ (A_i, B_i) \mid i = 1, \dots, n \}$$

is a star in \mathcal{F}_k , not all the (A_i, B_i) lie in O . So $(B_i, A_i) \in O$ for some i , and since O is consistent this i is unique. Let us make C_i an element of \mathcal{B} .

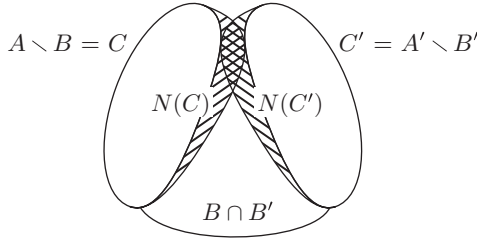


Figure 5: If C, C' do not touch, then (B, A) and (B', A') are inconsistent.

It remains to show that every two sets in \mathcal{B} touch. Given $C, C' \in \mathcal{B}$, there are sets X and X' such that σ_X contains a separation (A, B) with $A = C \cup N(C)$ and $(B, A) \in O$, and likewise for C' . If C and C' do not touch, then $C' \subseteq B \setminus A$ and hence $A' \subseteq B$ (Fig. 5), and similarly $A \subseteq B'$. Hence $(A, B) \leq (B', A') \in O$ but also $(B, A) \in O$, contradicting the consistency of O . \square

Theorem 6.2 can thus be extended to incorporate tree-width and brambles:

Theorem 6.5 (Tangle-bramble-treewidth duality theorem for graphs).

The following assertions are equivalent for all finite graphs G and $k > 0$:

- (i) G has a bramble of order at least k .
- (ii) G has an \mathcal{F}_k -tangle of S_k .

- (iii) G has no S_k -tree over \mathcal{F}_k .
- (iv) G has tree-width at least $k - 1$.

Proof. (i) \leftrightarrow (ii) is Lemma 6.4. (ii) \leftrightarrow (iii) is Theorem 6.2. (iii) \leftrightarrow (iv) is Lemma 6.3. \square

7 Tangle duality for path-width in graphs

A *path-decomposition* of a graph G is a tree-decomposition of G whose decomposition tree is a path. The *path-width* of G is the least width of such a tree-decomposition. By Lemma 6.3, G has path-width $< k - 1$ if and only if it has an S_k -tree over

$$\mathcal{F}_k^{(2)} := \{ \sigma \subseteq \vec{S}_k \mid \sigma = \{(A_1, B_1), (A_2, B_2)\} \text{ is a star with } |B_1 \cap B_2| < k \},$$

the stars in \mathcal{F}_k of order at most 2. Theorem 6.2 has the following analogue:

Theorem 7.1 (Tangle-pathwidth duality theorem for graphs).

For every $k > 0$, every graph G satisfies exactly one of the following assertions:

- (i) G has an $\mathcal{F}_k^{(2)}$ -tangle of S_k .
- (ii) G has an S_k -tree over $\mathcal{F}_k^{(2)}$.

Proof. Apply Theorem 3.2 and Lemmas 3.3, 3.4 and 6.1. \square

Bienstock, Robertson, Seymour and Thomas [2] also found tangle-like HCSs dual to path-width, which they call ‘blockages’.¹⁰ Let us define these, and then incorporate their result into our duality theorem with a unified proof.

Given a set X of vertices in $G = (V, E)$, let us write $\partial(X)$ for the set of vertices in X that have a neighbour outside X . A *blockage of order $k - 1$* , according to [2], is a collection \mathcal{B} of sets $X \subseteq V$ such that

- (B1) $|\partial(X)| < k$ for all $X \in \mathcal{B}$;
- (B2) $X' \in \mathcal{B}$ whenever $X' \subseteq X \in \mathcal{B}$ and $|\partial(X')| < k$;
- (B3) for every $\{X_1, X_2\} \in S_k$, exactly one of X_1 and X_2 lies in \mathcal{B} .

To deduce the duality theorem of [2] from Theorem 3.2, we just need to translate blockages into orientations of S_k :

Theorem 7.2 (Tangle-blockage-pathwidth duality theorem for graphs).

The following assertions are equivalent for finite graphs $G \neq \emptyset$ and $k > 0$:

- (i) G has a blockage of order $k - 1$.
- (ii) G has an $\mathcal{F}_k^{(2)}$ -tangle of S_k .
- (iii) G has no S_k -tree over $\mathcal{F}_k^{(2)}$.
- (iv) G has path-width at least $k - 1$.

¹⁰They go on to show that all graphs with a blockage of order $k - 1$ – which are precisely the graphs of path-width at least $k - 1$ – contain every forest of order k as a minor. This corollary is perhaps better known than the path-width duality theorem itself.

Proof. Theorem 7.1 asserts the equivalence of (ii) and (iii), while (iii) is equivalent to (iv) by Lemma 6.3.

(i)→(ii): Suppose that G has a blockage \mathcal{B} of order $k-1$. By (B2) and (B3),

$$O = \{ (X, Y) \in \vec{S}_k : X \in \mathcal{B} \}$$

is a consistent orientation of S_k .

For a proof that O avoids every singleton star $\{(A, X)\} \in \mathcal{F}_k^{(2)}$ it suffices to show that \mathcal{B} contains every set X of order $< k$: then $(X, A) \in O$ and hence $(A, X) \notin O$. To show that $X \in \mathcal{B}$, consider the separation $\{X, V\} \in S_k$. If $V \in \mathcal{B}$, then also $X \in \mathcal{B}$ by (B2), contradicting (B3). Hence $V \notin \mathcal{B}$, and thus $X \in \mathcal{B}$ by (B3).

To complete the proof that O avoids $\mathcal{F}_k^{(2)}$ consider $\{(A_1, B_1), (A_2, B_2)\} \in \mathcal{F}_k^{(2)}$ with $(A_1, B_1) \neq (A_2, B_2)$, and suppose that $(A_1, B_1) \in O$. Since $\{(A_1, B_1), (A_2, B_2)\}$ is a star, $\{B_1, B_2\}$ is a separation. As $|B_1 \cap B_2| < k$ by definition of $\mathcal{F}_k^{(2)}$, it lies in S_k . Applying (B3) three times, we deduce from our assumption of $A_1 \in \mathcal{B}$ that $B_1 \notin \mathcal{B}$, and hence $B_2 \in \mathcal{B}$, and hence $A_2 \notin \mathcal{B}$. Thus, $(A_2, B_2) \notin O$.

(ii)→(i): Let O be an $\mathcal{F}_k^{(2)}$ -tangle of S_k . We claim that

$$\mathcal{B} := \{ X : (X, Y) \in O \}$$

is a blockage of order $k-1$. Clearly, \mathcal{B} satisfies (B1).

Given $\{X_1, X_2\} \in S_k$ as in (B3), assume that $(X_1, X_2) \in O$. Then $X_1 \in \mathcal{B}$. If also $X_2 \in \mathcal{B}$, there exists Y_2 such that $(X_2, Y_2) \in O$. Then $\{X_1 \cap Y_2, X_2\}$ is still a separation of V , and clearly in S_k . As $(X_1 \cap Y_2, X_2) \leq (X_1, X_2) \in O$ and O is consistent, we have $(X_1 \cap Y_2, X_2) \in O$. Then $\{(X_1 \cap Y_2, X_2), (X_2, Y_2)\}$ is a star in $\mathcal{F}_k^{(2)}$, contradicting our assumption.

Given $X' \subseteq X \in \mathcal{B}$ as in (B2), with $(X, Y) \in O$ say, let $Y' := \partial(X') \cup (V \setminus X')$ and $Z := \partial(X) \cup (V \setminus X)$. Then $Z \subseteq Y$ and hence $|X \cap Z| \leq |X \cap Y| < k$, so $\{X, Z\} \in S_k$. By (B3) we have $Z \notin \mathcal{B}$ and hence $(Z, X) \notin O$, so $(X, Z) \in O$. Since O is consistent and $\vec{S}_k \ni (X', Y') \leq (X, Z)$, we thus obtain $(X', Y') \in O$ and hence $X' \in \mathcal{B}$, as desired. \square

8 Tangle duality for tree-width in matroids

Hliněný and Whittle [12, 13] generalized the notion of tree-width from graphs to matroids.¹¹ Let us show how Theorem 3.2 implies a duality theorem for tree-width in matroids.

Let $M = (E, I)$ be a matroid with rank function r . Its *connectivity function* is defined as

$$\lambda(X) := r(X) + r(E \setminus X) - r(M).$$

We consider the universe \vec{U} of all bipartitions (X, Y) of E . Since

$$|X, Y| := \lambda(X) = \lambda(Y)$$

is non-negative, submodular and symmetric, it is an order function on \vec{U} , so our universe \vec{U} is submodular.

¹¹In our matroid terminology we follow Oxley [17].

A *tree-decomposition* of M is a pair (T, τ) , where T is a tree and $\tau: E \rightarrow V(T)$ is any map. Let t be a node of T , and let T_1, \dots, T_d be the components of $T - t$. Then the *width* of t is the number

$$\sum_{i=1}^d r(E \setminus F_i) - (d-1)r(M),$$

where $F_i = f^{-1}(V(T_i))$. (If t is the only node of T , we let its width be $r(M)$.) The *width* of (T, τ) is the maximum width of the nodes of T . The *tree-width* of M is the minimum width over all tree-decompositions of M .

Matroid tree-width was designed so as to generalize the tree-width of graphs:

Theorem 8.1 (Hliněný and Whittle [12, 13]). *The tree-width of a finite graph containing at least one edge equals the tree-width of its cycle matroid.*

In order to specialize Theorem 3.2 to a duality theorem for tree-width in matroids, we consider for $k > 0$ the set

$$S_k = \{ \{A, B\} \in U : |A, B| < k \};$$

then \vec{S}_k is separable by Lemma 3.4. For stars $\sigma = \{(A_i, B_i) : i = 0, \dots, n\} \subseteq \vec{U}$ we write

$$\langle \sigma \rangle := r(M) + \sum_{i=0}^n (r(B_i) - r(M)).$$

We consider

$$\mathcal{F}_k := \{ \sigma \subseteq \vec{U} : \sigma \text{ is a star with } \langle \sigma \rangle < k \}.$$

Clearly, the singleton stars $\{(A, B)\}$ in \mathcal{F}_k are precisely those with $r(B) < k$, and the empty star lies in \mathcal{F}_k if and only if $r(M) < k$. We remark that requiring $\sigma \subseteq \vec{S}_k$ in the definition of \mathcal{F}_k would not spare us a proof of the following lemma, which we shall need in the proof of Lemma 8.4.

Lemma 8.2. *Every $\sigma \in \mathcal{F}_k$ is a subset of \vec{S}_k .*

Proof. We show that every A_i in a star $\sigma = \{(A_i, B_i) : i = 0, \dots, n\} \subseteq \vec{U}$ satisfies $\lambda(A_i) \leq \langle \sigma \rangle$; if $\sigma \in \mathcal{F}_k$, this implies that $|A_i, B_i| < k$ as desired. Our proof will be for $i = 0$; the other cases then follow by symmetry.

Since σ is a star we have $A_i \subseteq B_j$ whenever $i \neq j$, and in particular $A_{i+1} \subseteq B_i^* := B_1 \cap \dots \cap B_i$ for $i = 1, \dots, n-1$. Hence $B_i^* \cup B_{i+1} \supseteq E$. Submodularity of the rank function now gives

$$r(B_i^*) + r(B_{i+1}) \geq r(B_i^* \cap B_{i+1}) + r(B_i^* \cup B_{i+1}) = r(B_{i+1}^*) + r(M)$$

for each $i = 1, \dots, n-1$. Summing these inequalities over $i = 1, \dots, n-1$, and noting that $B_1^* = B_1$, yields

$$r(B_1) + \dots + r(B_n) \geq r(B_1 \cap \dots \cap B_n) + (n-1)r(M).$$

Using that σ is a star and hence $A_0 \subseteq B_1 \cap \dots \cap B_n$, we deduce

$$\begin{aligned} \langle \sigma \rangle &= \sum_{i=0}^n r(B_i) - n r(M) \geq r(B_0) + r(B_1 \cap \dots \cap B_n) - r(M) \\ &\geq r(B_0) + r(A_0) - r(M) \\ &= \lambda(A_0). \end{aligned}$$

as desired. □

In order to apply Theorem 3.2, we have to prove that \vec{S}_k is \mathcal{F}_k -separable:

Lemma 8.3. \vec{S}_k is \mathcal{F}_k -separable.

Proof. Let $\vec{r}, \vec{r}' \in \vec{S}_k$ be given: nondegenerate, nontrivial, not forced by \mathcal{F}_k , and satisfying $\vec{r} \leq \vec{r}'$. Pick $(X, Y) \in \vec{S}_k$ with $\vec{r} \leq (X, Y) \leq \vec{r}'$ and $|X, Y|$ minimum. We claim that (X, Y) emulates \vec{r} in \vec{S}_k for \mathcal{F}_k , and that (Y, X) emulates \vec{r}' in \vec{S}_k for \mathcal{F}_k . By symmetry, it is enough to prove that (X, Y) emulates \vec{r} for \mathcal{F}_k .

The proof of Lemma 3.4 shows that (X, Y) emulates \vec{r} .¹² To show that it does so for \mathcal{F}_k , consider a nonempty star

$$\sigma = \{(A_i, B_i) : i = 0, \dots, n\} \subseteq \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$$

in \mathcal{F}_k (where $\vec{S} := \vec{S}_k$) with $\vec{r} \leq (A_0, B_0)$. Then

$$\vec{r} \leq (A_0, B_0) \leq (B_i, A_i) \text{ for all } i \geq 1. \quad (5)$$

We have to show that

$$\sigma' = \{(A'_i, B'_i) : i = 0, \dots, n\} \in \mathcal{F}_k$$

for $(A'_i, B'_i) := f \downarrow_{\vec{s}_0}^{\vec{r}} (A_i, B_i)$.

From Lemma 3.1 we know that σ' is a star. Since (X, Y) emulates \vec{r} , we have $\sigma' \subseteq \vec{S}_k$ by (5). It remains to show that $\langle \sigma' \rangle < k$. We show that, in fact,

$$(\langle \sigma' \rangle) = r(Y \cap B_0) + \sum_{i=1}^n r(X \cup B_i) - nr(M) \leq \langle \sigma \rangle; \quad (6)$$

as $\langle \sigma \rangle < k$ by our assumption that $\sigma \in \mathcal{F}_k$, this will complete the proof.

By submodularity of the rank function, we have

$$\begin{aligned} r(Y \cap B_0) + r(Y \cup B_0) &\leq r(Y) + r(B_0) \\ \text{and } r(X \cup B_i) + r(X \cap B_i) &\leq r(X) + r(B_i) \quad \text{for } i = 1, \dots, n. \end{aligned}$$

For our proof of (6) we need to show that the sum of the first terms in these $n + 1$ inequalities is at most the sum of the last terms. This will follow from these inequalities once we know that the sum of the second terms is at least the sum of the third terms. So let us prove this, i.e., that

$$r(Y \cup B_0) + \sum_{i=1}^n r(X \cap B_i) \geq r(Y) + nr(X). \quad (7)$$

For $i = 1, \dots, n$ let us abbreviate $A_i^* := A_1 \cup \dots \cup A_i$ and $B_i^* := B_1 \cap \dots \cap B_i$.

Since σ is a star we have $A_i \subseteq B_j$ whenever $i \neq j$. Hence $A_n^* \subseteq B_0$, giving

$$r(Y \cup B_0) \geq r(Y \cup A_n^*), \quad (8)$$

and $A_{i+1} \subseteq B_i^*$ for $i \geq 1$. Hence $B_i^* \cup B_{i+1} \supseteq E$. By submodularity, this implies

$$\begin{aligned} r(X \cap B_i^*) + r(X \cap B_{i+1}) &\geq r(X \cap (B_i^* \cap B_{i+1})) + r(X \cap (B_i^* \cup B_{i+1})) \\ &= r(X \cap B_{i+1}^*) + r(X) \end{aligned}$$

for each $i = 1, \dots, n - 1$. Summing this for $i = 1, \dots, n - 1$, and recalling that

¹²Technically, we do not need this fact at this point and could use Lemma 8.2 to deduce it from the fact that all σ' as below lie in \mathcal{F}_k . But that seems heavy-handed.

$B_1^* = B_1$, we obtain

$$\sum_{i=1}^n r(X \cap B_i) \geq r(X \cap B_n^*) + (n-1)r(X). \quad (9)$$

Since $\{X, Y\}$ and $\{B_n^*, A_n^*\}$ are bipartitions of E , so is $\{X \cap B_n^*, Y \cup A_n^*\}$. Moreover, we have $\vec{r} \leq (X \cap B_n^*, Y \cup A_n^*)$ since $\vec{r} \leq (X, Y)$ and $\vec{r} \leq (B_n^*, A_n^*)$ by (5), and we also have $(X \cap B_n^*, Y \cup A_n^*) \leq (X, Y) \leq \vec{r}$. It would therefore contradict our choice of (X, Y) if we had $|X \cap B_n^*, Y \cup A_n^*| < |X, Y|$. Hence $|X \cap B_n^*, Y \cup A_n^*| \geq |X, Y|$, and therefore

$$r(X \cap B_n^*) + r(Y \cup A_n^*) \geq r(X) + r(Y). \quad (10)$$

Adding up inequalities (8), (9), (10) we obtain (7), proving (6). \square

Lemma 8.4. *M has an S_k -tree over \mathcal{F}_k if and only if it has tree-width $< k$. More precisely, M has an S_k -tree (T, α) over \mathcal{F}_k if and only if it admits a tree-decomposition (T, τ) of width $< k$.*

Proof. For the forward implication, consider any S_k -tree (T, α) of M . Given $e \in E$, orient every edge st of T , with $\alpha(s, t) = (A, B)$ say, towards t if $e \in B$, and let τ map e to the unique sink of T in this orientation. Then (T, τ) is a tree-decomposition of M . If (T, α) is over \mathcal{F}_k , the decomposition is easily seen to have width less than k .

Conversely, let (T, τ) be a tree-decomposition of M of width $< k$. For every edge $e = st$ of T , let T_s and T_t be the components of $T - e$ containing s and t , respectively. Let

$$\alpha(s, t) := (\tau^{-1}(T_s), \tau^{-1}(T_t)) \in \vec{U}.$$

Since every node t has width less than k , its associated star $\{\alpha(s, t) : st \in E(T)\}$ of separations is in \mathcal{F}_k . (This includes the case of $|T| = 1$.) By Lemma 8.2 this implies that $\alpha(\vec{E}(T)) \subseteq \vec{S}_k$, so (T, α) is an S_k -tree over \mathcal{F}_k . \square

Theorem 3.2 now yields the following duality theorem for matroid tree-width.

Theorem 8.5 (Tangle-treewidth duality theorem for matroids).

Let M be a matroid, and let $k > 0$ be an integer. Then the following statements are equivalent:

- (i) M has tree-width at least k .
- (ii) M has no S_k -tree over \mathcal{F}_k .
- (iii) M has an \mathcal{F}_k -tangle of S_k . \square

9 Tree-decompositions of small adhesion

As a case in point, let us illustrate the versatility of Theorem 3.2 by deducing a duality theorem for a new width parameter: one that bounds the width and the adhesion of a tree-decomposition independently, that is, allows the first bound to be greater.

Recall that the *adhesion* of a tree-decomposition (T, \mathcal{V}) of a graph $G = (V, E)$ is the largest size of an attachment set, the number $\max_{st \in E(T)} |V_s \cap V_t|$. (If T has only one node t , we set the adhesion to 0.) Trivially if a tree-decomposition

has width $< k - 1$ it has an adhesion $< k$, and it is easy to convert it to a tree-decomposition of the same width and adhesion $< k - 1$.

The idea now is to have a duality theorem whose tree structures are the tree-decompositions of adhesion $< k$ and width less than $w - 1 \geq k - 1$. For $w = k$ this should default to the duality for tree-width discussed in Section 6.

Let \vec{U} and \vec{S}_k be as defined at the end of Section 3. Recall that \vec{S}_k is separable, by Lemma 3.4. Let

$$\mathcal{F}_k^w = \{ \sigma \subseteq \vec{S}_k \mid \sigma = \{(A_i, B_i) : i = 0, \dots, n\} \text{ is a star with } |\bigcap_{i=0}^n B_i| < w \},$$

(As before, we let $\bigcap_{i=0}^n B_i := V$ if $\sigma = \emptyset$, so K_1 is an S_k -tree over \mathcal{F}_k if $|G| < w$.) Note that, for $w = k$, we have $\mathcal{F}_k^w = \mathcal{F}_k$ as defined in Section 6.

Lemma 9.1. *S_k is \mathcal{F}_k^w -separable.*

Proof. Let $\vec{r}, \vec{r}' \in \vec{S}_k$ be given: nondegenerate, nontrivial, not forced by \mathcal{F}_k^w , and satisfying $\vec{r} \leq \vec{r}'$. Pick $\vec{s}_0 = (X, Y) \in \vec{S}_k$ with $\vec{r} \leq (X, Y) \leq \vec{r}'$ and $|X, Y|$ minimum. We claim that (X, Y) emulates \vec{r} in \vec{S}_k for \mathcal{F}_k^w , and that (Y, X) emulates \vec{r}' in \vec{S}_k for \mathcal{F}_k^w . By symmetry, it is enough to prove that (X, Y) emulates \vec{r} for \mathcal{F}_k^w .

The proof of Lemma 3.4 shows that (X, Y) emulates \vec{r} . To show that it does so in \mathcal{F}_k^w , consider a nonempty star

$$\sigma = \{(A_i, B_i) : i = 0, \dots, n\} \subseteq \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$$

in \mathcal{F}_k^w (where $\vec{S} := \vec{S}_k$) with $\vec{r} \leq (A_0, B_0)$. Then

$$\vec{r} \leq (A_0, B_0) \leq (B_i, A_i) \text{ for all } i \geq 1. \quad (11)$$

We have to show that

$$\sigma' = \{(A'_i, B'_i) : i = 0, \dots, n\} \in \mathcal{F}_k^w$$

for $(A'_i, B'_i) := f \downarrow_{\vec{s}_0}^{\vec{r}} (A_i, B_i)$.

From Lemma 3.1 we know that σ' is a star. Since (X, Y) emulates \vec{r} , we have $\sigma' \subseteq \vec{S}_k$ by (11). It remains to show that $|\bigcap_{i=0}^n B'_i| < w$. As in Lemma 6.1, we shall prove this by rewriting the intersection of all the B'_i as an intersection of the two sides of a suitable separation, and use submodularity and the choice of (X, Y) to show that this separation has order $< w$.

By (11) and the definition of $f \downarrow_{\vec{s}_0}^{\vec{r}}$, we have $(A'_0, B'_0) = (A_0 \cup X, B_0 \cap Y)$, while $(A'_i, B'_i) = (A_i \cap Y, B_i \cup X)$ for $i \geq 1$. Since the (A_i, B_i) are separations, i.e. in \vec{U} , so is $(\bigcap_{i=1}^n B_i, \bigcup_{i=1}^n A_i)$. As trivially $(V, B_0) \in \vec{U}$, this implies that, for $B^* := \bigcap_{i=1}^n B_i$, also

$$\left(\bigcap_{i=1}^n B_i \cap V, \bigcup_{i=1}^n A_i \cup B_0 \right) \stackrel{(11)}{=} (B^*, B_0) \in \vec{U}.$$

Note that

$$|B^*, B_0| = |B^* \cap B_0| = \left| \bigcap_{i=0}^n B_i \right| < w \quad (12)$$

since $\sigma \in \mathcal{F}_k^w$.

As $\vec{r} \leq (X, Y)$, and also $\vec{r} \leq (B^*, B_0)$ by (11), we further have

$$\vec{r} \leq (X \cap B^*, Y \cup B_0) \leq (X, Y) \leq \vec{r}.$$

Hence if $|X \cap B^*, Y \cup B_0| < |X, Y|$ then this would contradict our choice of (X, Y) . Therefore $|X \cap B^*, Y \cup B_0| \geq |X, Y|$. As

$$|X \cap B^*, Y \cup B_0| + |X \cup B^*, Y \cap B_0| \leq |X, Y| + |B^*, B_0|$$

by submodularity, we deduce that

$$|X \cup B^*, Y \cap B_0| \leq |B^*, B_0| < w$$

by (12). Hence

$$\left| \bigcap_{i=0}^n B'_i \right| = \left| (B_0 \cap Y) \cap \bigcap_{i=1}^n (B_i \cup X) \right| = |B_0 \cap Y, B^* \cup X| < w,$$

which means that $\sigma' \in \mathcal{F}_k^w$ as desired. \square

The following translation lemma is proved like Lemma 6.3:

Lemma 9.2. *G has an S_k -tree over \mathcal{F}_k^w if and only if it has a tree-decomposition of width $< w - 1$ and adhesion $< k$.*

Theorem 3.2 and our two lemmas imply the following duality theorem:

Theorem 9.3 (Tangle-treewidth duality in graphs with given adhesion).

The following assertions are equivalent for all finite graphs $G \neq \emptyset$ and integers $w \geq k > 0$:

- (i) G has an \mathcal{F}_k^w -tangle of S_k .
- (ii) G has no S_k -tree over \mathcal{F}_k^w .
- (iii) G has no tree-decomposition of width $< w - 1$ and adhesion $< k$. \square

10 Weakly Submodular Partition Functions

Amini, Mazoit, Nisse and Thomassé [1], and Lyaudet, Mazoit and Thomassé [15], proposed a framework to unify duality theorems in graph minor theory which, unlike ours, is based exclusively on partitions. Their work, presented to us by Mazoit in the summer of 2013, inspired us to look for possible simplifications, for generalizations to separations that are not partitions, and for applications to tangle-like dense objects not covered by their framework. Our findings are presented in this paper and its sequel [8]. Although our approach differs from theirs, we remain indebted to Mazoit and his coauthors for this inspiration.

Since the applications of our abstract duality theorem include the applications of [1], it may seem unnecessary to ask whether our result also implies theirs directly. However, for completeness we address this question now.

A *partition* of a finite set E is a set of disjoint subsets of E , possibly empty, whose union is E . We write $\mathcal{P}(E)$ for the set of all partitions of E . In [1], any function $\mathcal{P}(E) \rightarrow \mathbb{R} \cup \{\infty\}$ is called a *partition function* of E . We abbreviate $\psi(\{A_1, \dots, A_n\})$ to $\psi(A_1, \dots, A_n)$, but note that the partition remains unordered. A partition function ψ is called *weakly submodular* in [1] if, for every pair $(\mathcal{A}, \mathcal{B})$ of partitions of E and every choice of $A_0 \in \mathcal{A}$ and $B_0 \in \mathcal{B}$, one of the following holds with $\mathcal{A} =: \{A_0, \dots, A_n\}$ and $\mathcal{B} =: \{B_0, \dots, B_m\}$:

(i) there exists a set F such that $A_0 \subseteq F \subseteq A_0 \cup (E \setminus B_0)$ and $\psi(A_0, \dots, A_n) > \psi(F, A_1 \setminus F, \dots, A_n \setminus F)$;

(ii) $\psi(B_0, \dots, B_m) \geq \psi(B_0 \cup (E \setminus A_0), B_1 \cap A_0, \dots, B_m \cap A_0)$.

Let us translate this to our framework. Given $A \subseteq E$, let $\bar{A} := E \setminus A$. Then $\vec{U} := \{(A, \bar{A}) : A \subseteq E\}$ is a universe. Given a partition function ψ of E , let

$$\vec{S}_k = \{(A, \bar{A}) \in \vec{U} : \psi(A, \bar{A}) < k\}.$$

Every partition $\mathcal{A} = \{A_0, \dots, A_n\}$ defines a star $\{(A_0, \bar{A}_0), \dots, (A_n, \bar{A}_n)\} \subseteq \vec{U}$, which we denote by $\sigma(\mathcal{A})$. Let

$$\mathcal{F}_k := \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{P}(E) \text{ and } \psi(\mathcal{A}) < k\} \cup \{(\bar{X}, X) : |X| \leq 1, \psi(\bar{X}, X) < k\}.$$

If all the stars in \mathcal{F}_k are subsets of \vec{S}_k , we call ψ *monotone*. All the weakly submodular partition functions used in [1] for applications are monotone, and we do not know whether any exist that are not.

Lemma 10.1. *If ψ is weakly submodular, then \vec{S}_k is \mathcal{F}_k -separable.*

Proof. Let $\vec{S} = \vec{S}_k$. Let $\vec{r}, \vec{r}' \in \vec{S}$ that are not forced by \mathcal{F}_k and satisfy $\vec{r} \leq \vec{r}'$ be given. Pick $\vec{s}_0 = (X, Y) \in \vec{S}$ with $\vec{r} \leq \vec{s}_0 \leq \vec{r}'$ and $\psi(X, Y)$ minimum. We claim that \vec{s}_0 emulates \vec{r} in \vec{S} for \mathcal{F}_k , and that \vec{s}_0 emulates \vec{r}' in \vec{S} for \mathcal{F}_k . By symmetry, it is enough to prove that \vec{s}_0 emulates \vec{r} for \mathcal{F}_k .

We first prove that \vec{s}_0 emulates \vec{r} . Let $\vec{s} = (A, B) \in \vec{S}_{\geq \vec{r}} \setminus \{\vec{r}\}$ be given. Since ψ is weakly submodular, one of the following assertions holds:

- (i) there exists F such that $Y \subseteq F \subseteq Y \cup B$ and $\psi(X, Y) > \psi(\bar{F}, F)$;
- (ii) $\psi(A, B \geq \psi(A \cup X, B \cap Y)$.

Since $A \cap X \subseteq \bar{F} \subseteq X$, we have $\vec{r} \leq (\bar{F}, F) \leq \vec{r}'$. So (i) does not hold, by the choice of (X, Y) . So by (ii), $\vec{s}_0 \vee \vec{s} = (A \cup X, B \cap Y) \in \vec{S}$. This proves that \vec{s}_0 emulates \vec{r} .

Now let us show that stars can be shifted. Let

$$\sigma = \{(A_i, B_i) : i = 0, \dots, n\}$$

be a star in $\mathcal{F}_k \cap \vec{S}_{\geq \vec{r}}$, with $(A_0, B_0) \geq \vec{r}$. We have to show that

$$\sigma' = \{(A'_i, B'_i) : i = 0, \dots, n\} \in \mathcal{F}_k$$

for $(A'_i, B'_i) := f \downarrow_{\vec{s}_0}^{\vec{r}} (A_i, B_i)$. Since $\vec{r} \leq (A_0, B_0) \leq (B_i, A_i)$ for $i \geq 1$, we have $(A'_0, B'_0) = (A_0 \cup X, B_0 \cap Y)$, while $(A'_i, B'_i) = (A_i \cap Y, B_i \cup X)$ for $i \geq 1$.

If $n = 0$, then $|B'_0| \leq |B_0| \leq 1$ and so $\sigma' = \{(A'_0, B'_0)\} \in \mathcal{F}_k$. If $n \neq 0$, then $\psi(A_0, \dots, A_n) < k$. By the minimal choice of $\vec{s}_0 = (X, Y)$, there exists no F such that $X \subseteq F \subseteq X \cup A_0$ and $\psi(X, Y) > \psi(F, \bar{F})$ (as earlier). Applying the weak submodularity of ψ with (X, Y) and (A_0, \dots, A_n) , we deduce that

$$\psi(A'_0, \dots, A'_n) = \psi(A_0 \cup X, A_1 \cap Y, \dots, A_n \cap Y) \leq \psi(A_0, \dots, A_n) < k.$$

Thus, $\sigma' \in \mathcal{F}_k$. □

In [1], a k -*bramble* for a weakly submodular partition function ψ of E is a non-empty set \mathcal{B} of pairwise intersecting subsets of E that contains an element from every partition \mathcal{A} of E with $\psi(\mathcal{A}) < k$. It is *non-principal* if it contains no singleton set $\{e\}$. In our terminology, Amini et al. [1] prove that there exists a non-principal k -bramble for ψ if and only if there is no S_k -tree over \mathcal{F}_k ; they call this a ‘partitioning k -search tree’.

Now any k -bramble \mathcal{B} defines an orientation O of S_k : given $\{A, B\} \in S_k$ exactly one of A, B must lie in \mathcal{B} , and if B does we put (A, B) in O . Clearly O is consistent and avoids non-singleton stars in \mathcal{F}_k , and if \mathcal{B} is non-principal it avoids all singleton stars in \mathcal{F}_k . Conversely, given an orientation O of S_k , let $\mathcal{B} := \{B : (A, B) \in O\}$. If O is consistent, no two elements of \mathcal{B} are disjoint. If O avoids singleton stars in \mathcal{F}_k , then \mathcal{B} is non-principal. And finally, if ψ is monotone and O avoids \mathcal{F}_k , then \mathcal{B} contains an element from every partition $\mathcal{A} = \{A_1, \dots, A_n\}$ of E with $\psi(\mathcal{A}) < k$: since $\sigma(\mathcal{A}) \in \mathcal{F}_k$ there is $(A_i, \bar{A}_i) \in \sigma(\mathcal{A}) \setminus O$, which means that $(\bar{A}_i, A_i) \in O$ and thus $A_i \in \mathcal{B}$.

Lemma 10.1 and Theorem 3.2 thus imply the duality theorem of Amini et al. [1] for monotone weakly submodular partition functions:

Theorem 10.2. *The following assertions are equivalent for all monotone weakly submodular partition functions ψ of a finite set E and $k > 0$:*

- (i) *There exists a non-principal k -bramble for ψ .*
- (ii) *S_k has an \mathcal{F}_k -tangle.*
- (iii) *There exists no S_k -tree over \mathcal{F}_k .*
- (iv) *There exists no partitioning k -search tree.*

11 Further applications

There are some obvious ways in which we can modify the sets \mathcal{F} considered so far in this section to create new kinds of highly cohesive substructures and obtain associated duality theorems as corollaries of Theorem 3.2. For example, we might strengthen the notion of a tangle by forbidding not just all the 3-sets of separations whose small sides together cover the entire graph or matroid, but forbid all such m -sets with m up to some fixed value $n > 3$. The resulting set \mathcal{F} can then be replaced by its subset \mathcal{F}^* of stars without affecting the set of consistent orientations avoiding \mathcal{F} , just as in Lemma 4.2.

Similarly, we might like tree-decompositions whose decomposition trees have degrees of at least n at all internal nodes: graphs with such a tree-decomposition, of width and adhesion $< k$ say, would ‘decay fast’ along $(< k)$ -separations. Such tree-decompositions can be described as S_k -trees over the subset of all $(\geq n)$ -sets and singletons in the \mathcal{F}_k defined in Section 6.

Another ingredient we might wish to change are the singleton stars in \mathcal{F} associated with leaves. For example, we might be interested in tree-decompositions whose leaf parts are planar, while its internal parts need not be planar but might have to be small. Theorem 3.2 would offer dual objects also for such decompositions.

Conversely, it would be interesting to see whether other concrete highly cohesive substructures than those discussed in the preceding sections can be

described as \mathcal{F} -tangles for some \mathcal{F} of a suitable set S of separations – of a graph or something else.

Bowler [3] answered this in the negative for complete minors in graphs, a natural candidate. Using the terminology of [5] for minors H of G , let us say that a separation (A, B) of G *points to* an $IH \subseteq G$ if this IH has a branch set in $B \setminus A$ but none in $A \setminus B$. A set of oriented separations *points to* a given IH if each of its elements does. Clearly, for every $IK_k \subseteq G$ exactly one of (A, B) and (B, A) in \tilde{S}_k points to this IK_k .

Theorem 11.1. [3] *For every $k \geq 5$ there exists a graph G such that for no set $\mathcal{F} \subseteq 2^{\tilde{S}_k}$ of stars are the \mathcal{F} -tangles of S_k precisely the orientations of S_k that point to some $IK_k \subseteq G$.*

To prove this, Bowler considered as G a subdivision of K_k obtained by subdividing every edge of K_k exactly once. He constructed an orientation O such that every star $\sigma \subseteq O$ points to an IK_k but the entire O does not. This O , then, avoids every \mathcal{F} consisting only of stars not pointing to any IK_k . But any $\mathcal{F} \subseteq 2^{\tilde{S}_k}$ such that the orientations of S_k pointing to an IK_k are precisely the \mathcal{F} -tangles must consist of stars not pointing to an IK_k , since any star that does is contained in the unique orientation of S_k pointing to the IK_k to which this star points.

However, K_k minors can be captured by \mathcal{F} -avoiding orientations of S_k if we do not insist that \mathcal{F} contain only stars but allow it to contain *weak stars*: sets of oriented separations that pairwise either cross or point towards each other (formally: consistent antichains in \tilde{S}_k). In [8] we prove a duality theorem for orientations of separation systems avoiding such collections \mathcal{F} of weak stars.

In [7], we show that Theorem 3.2 implies duality theorems for k -blocks and for any given subset of k -tangles.

References

- [1] O. Amini, F. Mazoit, N. Nisse, and S. Thomassé. Submodular partition functions. *Discrete Appl. Math.*, 309(20):6000–6008, 2009.
- [2] D. Bienstock, N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a forest. *J. Combin. Theory Ser. B*, 52(2):274–283, 1991.
- [3] N. Bowler. Presentation at *Hamburg workshop on graphs and matroids*, Spiekeroog 2014.
- [4] J. Carmesin, R. Diestel, F. Hundertmark, and M. Stein. Connectivity and tree structure in finite graphs. *Combinatorica*, 34(1):1–35, 2014.
- [5] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.
- [6] R. Diestel. *Graph Theory* (5th edition). Springer-Verlag, 2017. Electronic edition available at <http://diestel-graph-theory.com/>.
- [7] R. Diestel, J. Erde, and P. Eberenz. Duality theorem for tangles in graphs. arXiv:1605.09139, 2016.
- [8] R. Diestel and S. Oum. Unifying duality theorems for width parameters, II. General duality. arXiv:1406.3798, 2014.

- [9] R. Diestel and S. Oum. Tangle-tree duality in abstract separation systems. ArXiv, 2017.
- [10] R. Diestel and G. Whittle. Tangles and the Mona Lisa. arXiv:1603.06652, 2016.
- [11] J. Geelen, B. Gerards, N. Robertson, and G. Whittle. Obstructions to branch-decomposition of matroids. *J. Combin. Theory (Series B)*, 96:560–570, 2006.
- [12] P. Hliněný and G. Whittle. Matroid tree-width. *European J. Combin.*, 27(7):1117–1128, 2006.
- [13] P. Hliněný and G. Whittle. Addendum to matroid tree-width. *European J. Combin.*, 30:1036–1044, 2009.
- [14] C.-H. Liu. Packing and covering immersions in 4-edge-connected graphs. arXiv:1505.00867, 2015.
- [15] L. Lyaudet, F. Mazoit, and S. Thomassé. Partitions versus sets: a case of duality. *European J. Combin.*, 31(3):681–687, 2010.
- [16] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.
- [17] J. Oxley. *Matroid Theory*. Oxford University Press, 1992.
- [18] N. Robertson and P. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory (Series B)*, 52:153–190, 1991.
- [19] P. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. *J. Combin. Theory (Series B)*, 58(1):22–33, 1993.
- [20] P. Seymour and R. Thomas. Call routing and the ratcatcher. *Combinatorica*, 14(2):217–241, 1994.