

# Eilenberg-Watts calculus for finite categories and a bimodule Radford $S^4$ theorem

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## ABSTRACT

We obtain Morita invariant versions of Eilenberg-Watts type theorems, relating Deligne products of finite linear categories to categories of left exact as well as of right exact functors. This makes it possible to switch between different functor categories as well as Deligne products, which is often very convenient. For instance, we can show that applying the equivalence from left exact to right exact functors to the identity functor, regarded as a left exact functor, gives a Nakayama functor. The equivalences of categories we exhibit are compatible with the structure of module categories over finite tensor categories. This leads to a generalization of Radford's  $S^4$ -theorem to bimodule categories. We also explain the relation of our construction to relative Serre functors on module categories that are constructed via inner Hom functors.

# 1 Introduction

A classical result in algebra, the Eilenberg-Watts theorem [Ei, Wa], states that, given two unital rings  $R$  and  $S$ , any right exact functor  $F: R\text{-mod} \rightarrow S\text{-mod}$  that preserves small coproducts is naturally isomorphic to tensoring with an  $S$ - $R$ -bimodule. Further, this bimodule can be expressed explicitly through the functor  $F$ , namely as the left  $S$ -module  $F({}_R R)$  endowed with a natural right  $R$ -action.

While this formulation of the statement is extremely useful, it hides the categorical nature of the situation: For a category that can be realized as the category of modules over a ring, that ring is determined only up to Morita equivalence. But nevertheless the bimodule in the theorem appears to make explicit use of the choice of ring. This can pose a problem in situations in which the category is given in more abstract terms and a Morita invariant formulation is desired.

Moreover, it is known that similar results exist for left exact functors, expressing them, under certain finiteness conditions, in terms of a Hom functor (see e.g. [Iv]) applied to a bimodule. When taken together, for sufficiently nice categories one thus deals with two categories – of left exact functors and of right exact functors. respectively – that should both be related to suitable categories of bimodules.

In the present paper, we place ourselves under the following finiteness conditions which in particular allows us to rely on results from [ENO, Sh1] and to relate them to the Eilenberg-Watts equivalences: We fix an algebraically closed field  $\mathbb{k}$  and work with finite  $\mathbb{k}$ -linear categories, i.e. categories that are equivalent to categories of finite-dimensional modules over a finite-dimensional  $\mathbb{k}$ -algebra. This ensures the existence of various categorical constructions, in particular of the Deligne product and of certain ends and coends.

Besides giving a Morita invariant formulation of the Eilenberg-Watts equivalences, this paper connects these equivalences with other representation theoretic concepts, in particular with Nakayama functors and, for the case of rigid monoidal categories, with Radford's theorem on the fourth power of the antipode of a Hopf algebra.

To give a more detailed account of our findings, let us first recall that, given two finite linear categories  $\mathcal{A} = A\text{-mod}$  and  $\mathcal{B} = B\text{-mod}$ , the opposite category  $\mathcal{A}^{\text{opp}}$  can be identified with the category of right  $A$ -modules and the Deligne product with  $B$ - $A$ -bimodules,

$$\mathcal{B} \boxtimes \mathcal{A}^{\text{opp}} \cong B\text{-bimod-}A. \quad (1.1)$$

A first result towards a categorical formulation is a categorical variant of the Peter-Weyl theorem: We can express the regular  $A$ -bimodule  $A = {}_A A_A$  and the co-regular  $A$ -bimodule  $A^* = {}_A A_A^*$  as an end and as a coend, respectively, of a functor with values in  $A$ -bimod; specifically, we have

$$A = \int_{m \in A\text{-mod}} m \boxtimes m^* \quad \text{and} \quad A^* = \int^{m \in A\text{-mod}} m \boxtimes m^* \quad (1.2)$$

as  $A$ -bimodules. More generally, we show in Proposition 2.8 that for any  $\mathbb{k}$ -linear functor  $G: A\text{-mod} \rightarrow B\text{-mod}$  the functor from  $A\text{-mod} \times A\text{-mod}^{\text{opp}}$  to  $B\text{-bimod-}A$  that is defined by  $m \times \bar{n} \mapsto G(m) \otimes_{\mathbb{k}} n^*$  has as an end the  $B$ - $A$ -bimodule

$$\int_{m \in A\text{-mod}} G(m) \otimes_{\mathbb{k}} m^* = G(A), \quad (1.3)$$

and as a coend the  $B$ - $A$ -bimodule

$$\int^{m \in A\text{-mod}} G(m) \otimes_{\mathbb{k}} m^* = G(A^*) \quad (1.4)$$

(with structure morphisms as given in (2.20) and (2.22), respectively).

With the help of this result we can set up a triangle

$$\begin{array}{ccc} & \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} & \\ \Psi^l \nearrow & & \nwarrow \Psi^r \\ \mathcal{L}ex(\mathcal{A}, \mathcal{B}) & \xleftarrow[\Gamma^r]{\Gamma^l} & \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \end{array} \quad (1.5)$$

of Eilenberg-Watts type equivalences of finite linear categories. Here  $\mathcal{L}ex(-, -)$  is the category of left exact functors and  $\mathcal{R}ex(-, -)$  the one of right exact functors. The Deligne product  $\mathcal{B} \boxtimes \mathcal{A}^{\text{opp}} \cong \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$  plays the role of the category of bimodules, as in the relation (1.1) above. The Eilenberg-Watts equivalences for categories of left exact functors have appeared in [ENO, Rem. 2.2(i)] and [Sh1, Section 3.4]; they are

$$\begin{aligned} \Phi^l : \quad \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} &\longrightarrow \mathcal{L}ex(\mathcal{A}, \mathcal{B}) \\ \bar{a} \boxtimes b &\longmapsto \text{Hom}_{\mathcal{A}}(a, -) \otimes b \end{aligned}$$

and

$$\begin{aligned} \Psi^l : \quad \mathcal{L}ex(\mathcal{A}, \mathcal{B}) &\longrightarrow \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \\ F &\longmapsto \int^{a \in \mathcal{A}} \bar{a} \boxtimes F(a). \end{aligned} \quad (1.6)$$

That these functors are quasi-inverses is essentially a consequence of the Yoneda lemma. The case of right exact functors uses instead of the Hom-pairing  $\text{Hom}_{\mathcal{A}}$  (which constitutes a left exact functor  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{A} \rightarrow \text{vect}$ ) a pairing that is familiar from the study of Serre functors: the right exact pairing  $\text{Hom}_{\mathcal{A}}(-, -)^*$  obtained by using the vector space dual. (Recall that the morphism spaces of the categories under consideration are finite-dimensional.) Explicitly, we show that the following pair of functors are quasi-inverse equivalences:

$$\begin{aligned} \Phi^r : \quad \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} &\longrightarrow \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \\ \bar{a} \boxtimes b &\longmapsto \text{Hom}_{\mathcal{A}}(-, a)^* \otimes b \end{aligned}$$

and

$$\begin{aligned} \Psi^r : \quad \mathcal{R}ex(\mathcal{A}, \mathcal{B}) &\longrightarrow \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \\ G &\longmapsto \int_{a \in \mathcal{A}} \bar{a} \boxtimes G(a). \end{aligned} \quad (1.7)$$

We note that these results provide a convenient way to think about Deligne products in terms of left or right exact functors, respectively. Regarding  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$  as a categorification of matrix elements, we describe functors in terms of matrix elements and obtain a categorified matrix calculus. This should prove useful in various applications, e.g. in situations in which relative tensor products and relative (twisted) centers play a role, like in topological field theories. Specifically, one can generalize constructions established for semisimple categories that involve sums over simple objects to the non-semisimple case by using instead ends or coends. This can e.g. be a key step in the passage to non-semisimple variants of modular functors.

Composing the equivalences (1.6) and (1.7) leads to natural equivalences  $\Gamma^{\text{lr}}$  and  $\Gamma^{\text{rl}}$  between the categories of left exact and right exact functors. We can then in particular consider the right exact endofunctor

$$N_{\mathcal{X}}^{\text{r}} := \Gamma^{\text{rl}}(\text{id}_{\mathcal{X}}) = \int^{x \in \mathcal{X}} \text{Hom}_{\mathcal{X}}(-, x)^* \otimes x. \quad (1.8)$$

For  $\mathcal{X} = A\text{-mod}$  a category of modules, by the Peter-Weyl formula (1.2) this turns out to be the Nakayama functor

$$N_{A\text{-mod}}^{\text{r}} = A^* \otimes_A - \cong \text{Hom}_{A\text{-mod}}(-, A)^*. \quad (1.9)$$

In other words, we have arrived at a Morita invariant description of the Nakayama functor. It follows immediately from the triangle (1.5) of equivalences that there is also a left exact analogue of the Nakayama functor, namely

$$N_{\mathcal{X}}^{\text{l}} := \Gamma^{\text{lr}}(\text{id}_{\mathcal{X}}) = \int_{x \in \mathcal{X}} \text{Hom}_{\mathcal{X}}(x, -) \otimes x. \quad (1.10)$$

In the case  $\mathcal{X} = A\text{-mod}$  of modules over an algebra the so defined left exact functor is given by  $N_{A\text{-mod}}^{\text{l}} = \text{Hom}_{\text{mod-}A}(-^*, A) \cong \text{Hom}_{A\text{-mod}}(A^*, -)$ .

In the Radford theorem iterated duals enter crucially. Similarly, the following property of the Nakayama functors (1.8) and (1.10), proven in Theorem 3.18, turns out to be essential: For  $F: \mathcal{A} \rightarrow \mathcal{B}$  a left exact functor for which the left adjoint  $F^{\text{l.l.a.}}$  of its left adjoint  $F^{\text{l.a.}}$  exists and is again left exact, there is a natural isomorphism

$$N_{\mathcal{B}}^{\text{l}} \circ F \cong F^{\text{l.l.a.}} \circ N_{\mathcal{A}}^{\text{l}} \quad (1.11)$$

of functors. An analogous result holds for right exact functors.

In the second part of this paper we consider the particular case that the finite  $\mathbb{k}$ -linear category  $\mathcal{M}$  has the additional structure of a bimodule category over finite tensor categories  $\mathcal{A}$  and  $\mathcal{B}$ . We show, in Theorem 4.4, that in this case the left exact Nakayama functor  $N_{\mathcal{M}}^{\text{l}}$  of the category  $\mathcal{M}$  has a canonical structure of a bimodule functor

$$N_{\mathcal{M}}^{\text{l}}: \mathcal{M} \rightarrow {}^{\text{rr}}\mathcal{M}^{\text{ll}}, \quad (1.12)$$

that is, there are coherent isomorphisms

$$N_{\mathcal{M}}^{\text{l}}(a.m.b) \cong a^{\vee\vee}. N_{\mathcal{M}}^{\text{l}}(m).{}^{\vee\vee}b \quad (1.13)$$

for all  $m \in \mathcal{M}$ ,  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . Here  $a^{\vee}$  is the right dual and  ${}^{\vee}a$  the left dual of  $a \in \mathcal{A}$ . Similarly (Theorem 4.5) for the Nakayama functor there are coherent isomorphisms

$$N_{\mathcal{M}}^{\text{r}}(a.m.b) \cong {}^{\vee\vee}a. N_{\mathcal{M}}^{\text{r}}(m).b^{\vee\vee} \quad (1.14)$$

These statements actually imply and generalize Radford's classical  $S^4$ -theorem for Hopf algebras; indeed, as shown in Lemma 4.10, for  $\mathcal{A}$  regarded as a bimodule category over itself the Nakayama functors are, up to dualities, given by tensoring with the distinguished invertible object  $D \in \mathcal{A}$ . It follows that the quadruple dual of  $\mathcal{A}$  satisfies

$$-{}^{\vee\vee\vee\vee} \cong D \otimes - \otimes D^{-1}, \quad (1.15)$$

which is just the categorical formulation of Radford's theorem as obtained in [ENO, Thm. 3.3].

Another direct application of our results, pointed out to us by Shimizu [Sh3], is Theorem 4.14: A finite multitensor category is equivalent as a linear category to the category of modules over a symmetric Frobenius algebra if and only if it is unimodular and its double dual functor is isomorphic to the identity functor.

Nakayama functors are closely related with Serre functors. Recall that a right Serre functor on a linear Hom-finite additive category  $\mathcal{X}$  is an additive endofunctor  $G$  together with a natural family of isomorphisms between the morphism spaces  $\text{Hom}_{\mathcal{X}}(x, y)$  and  $\text{Hom}_{\mathcal{X}}(y, G(c))^*$ ; left Serre functors are defined analogously. In our setting, a Serre functor exists if and only if  $\mathcal{X}$  is semisimple. On the other hand, if  $\mathcal{X} = \mathcal{M}$  is a module category, the *inner Hom functor*  $\underline{\text{Hom}}(-, -)$  allows one to define the notion of *relative Serre functors*  $S_{\mathcal{M}}^r$  and  $S_{\mathcal{M}}^l$  (see also [Sc, Sect. 4.4]) as functors that instead come with families

$$\underline{\text{Hom}}(m, n)^{\vee} \xrightarrow{\cong} \underline{\text{Hom}}(n, S_{\mathcal{M}}^r(m)) \quad \text{and} \quad {}^{\vee}\underline{\text{Hom}}(m, n) \xrightarrow{\cong} \underline{\text{Hom}}(S_{\mathcal{M}}^l(n), m) \quad (1.16)$$

of isomorphisms, natural in  $m, n \in \mathcal{M}$ , for a right and a left relative Serre functor, respectively. A necessary and sufficient condition for the existence of right and left relative Serre functors for a module category  $\mathcal{M}$  is that  $\mathcal{M}$  is an exact module category (Proposition 4.23).

Finally we show in Theorem 4.25 that for an exact module category  $\mathcal{M}$  over a finite tensor category  $\mathcal{A}$  with distinguished invertible object  $D_{\mathcal{A}}$  there is an equivalence

$$N_{\mathcal{M}}^l \cong D_{\mathcal{A}} \cdot S_{\mathcal{M}}^l \quad (1.17)$$

between the Nakayama functor and the left relative Serre functor composed with the action by the distinguished element. This result suggests in particular that an exact module category  $\mathcal{M}$  should be called unimodular iff there exists a module natural isomorphism between the Nakayama functor and the right relative Serre functor of  $\mathcal{M}$ . Then in particular the finite tensor category  $\mathcal{A}$  is unimodular iff the regular  $\mathcal{A}$ -module category  ${}_{\mathcal{A}}\mathcal{A}$  is unimodular.

To conclude this survey of results, let us stress a major virtue of our categorical formulation of the Eilenberg-Watts theorem: it allows us, for finite linear categories, to switch back and forth between Deligne products of categories and categories of right exact as well as of left exact functors and thereby also between those two types of functor categories. As a consequence we can understand features of Deligne products in terms of functor categories and, conversely, aspects of functors in terms of a categorified matrix calculus; for instance, certain ends and coends can be naturally interpreted as providing a categorified matrix multiplication (see Corollary 3.7). We take the fact that the structures which arise when setting up an Eilenberg-Watts calculus for a (finite linear) category  $\mathcal{C}$  fit well with the additional structure of a module category on  $\mathcal{C}$  as a further indication that the structures investigated in this paper are natural and can be of much avail.

## 2 Preliminary results

### 2.1 Notation and background

Throughout the paper we make the following assumptions. We work over a fixed field  $\mathbb{k}$ ; all categories are  $\mathbb{k}$ -linear and all functors  $\mathbb{k}$ -linear or bilinear. All  $\mathbb{k}$ -algebras and all modules and bimodules over them are finite-dimensional. For such (unital associative) algebras  $A$  and  $B$  we denote by  $A\text{-mod}$ ,  $\text{mod-}A$  and  $B\text{-bimod-}A$  the categories of finite-dimensional left and right

$A$  modules and of finite-dimensional  $B$ - $A$ -bimodules, respectively, and we write  $\mathrm{Hom}_A(-, -)$  for morphisms of left  $A$  modules and  $\mathrm{Hom}_{B|A}(-, -)$  for morphisms of  $B$ - $A$ -bimodules. Two distinguished  $A$ -bimodules are the regular bimodule  ${}_A A_A$  with the actions given by the product of  $A$  and the co-regular bimodule  ${}_A A_A^*$  for which the actions are obtained by dualizing those for  ${}_A A_A$ .

Furthermore, finiteness properties of the categories involved are essential for our work: we require, unless specified otherwise, that categories are in addition *finite*. As in [EO] we assume that  $\mathbb{k}$  is algebraically closed; a  $\mathbb{k}$ -linear category is finite iff [EO, Sect. 2.1] it is equivalent as a linear category to the category of finite-dimensional (left or right) modules over some finite-dimensional  $\mathbb{k}$ -algebra  $A$ . Our main interest lies in abstract finite linear categories and Morita invariant statements rather than in statements involving a concrete choice of algebra  $A$ . But we do have to make use of pertinent information about finite-dimensional algebras.

We start by recalling variants (see e.g. [Iv, Thms. 2.4, 2.6, 2.7] and [Sh1, Lemma 2.4]) of the Eilenberg-Watts theorem. Denote the monoidal category of finite-dimensional vector spaces over  $\mathbb{k}$  by  $\mathrm{vect}$ , and for  $v \in \mathrm{vect}$  the dual vector space  $\mathrm{Hom}_{\mathrm{vect}}(v, \mathbb{k})$  by  $v^*$ . Then we have the following standard results:

**Lemma 2.1.** Let  $A$  and  $B$  be finite-dimensional  $\mathbb{k}$ -algebras and  $F, G: A\text{-mod} \rightarrow B\text{-mod}$   $\mathbb{k}$ -linear functors. Then the following statements are equivalent:

- (L1)  $F$  is left exact.
- (L2)  $F$  admits a left adjoint.
- (L3)  $F \cong \mathrm{Hom}_A(M, -)$  with  $M = (F({}_A A_A^*))^*$  (thus in particular  $F$  is representable).

Likewise, the following are equivalent:

- (R1)  $G$  is right exact.
- (R2)  $G$  admits a right adjoint.
- (R3)  $G \cong G({}_A A_A) \otimes_A -$ .
- (R4)  $G \cong \mathrm{Hom}_A(-, G({}_A A_A)^*)^*$ .

Note that a functor that is left or right adjoint to a linear functor is linear as well. Also, the vector spaces  $(F({}_A A_A^*))^*$  in (L3) (and analogously  $G({}_A A_A)$  in (R3) and  $(G({}_A A_A^*))^*$  in (R4)) is endowed with the appropriate natural structure of a  $B$ - $A$ -bimodule, as e.g. described explicitly in the proof of Proposition 2.8 below. Further, the isomorphism in (R4) is obtained from the one in (R3) with the help of the description

$$\mathrm{Hom}_A({}_A X, (M_A)^*)^* \cong \mathrm{Hom}_{\mathbb{k}}(M_A \otimes_A X, \mathbb{k})^* = (M_A \otimes_A X)^{**} \cong M_A \otimes_A X \quad (2.1)$$

of the tensor product  $M_A \otimes_A X$  of finite-dimensional modules over an algebra. Note that  $(-)^*: A\text{-mod} \rightarrow \mathrm{mod}\text{-}A$  is an exact functor because it is an equivalence, and while  $\mathrm{Hom}_A: A\text{-mod}^{\mathrm{opp}} \times A\text{-mod} \rightarrow \mathrm{vect}$  is left exact,  $\mathrm{Hom}_A(-, -)^*: A\text{-mod} \times A\text{-mod}^{\mathrm{opp}} \rightarrow \mathrm{vect}$  is right exact.

We write  $F^{\mathrm{l.a.}}$  and  $F^{\mathrm{r.a.}}$  for the left and right adjoint of a functor  $F$ , respectively (provided they exist). For abelian categories  $\mathcal{C}$  and  $\mathcal{D}$  we denote by  $\mathcal{L}ex(\mathcal{C}, \mathcal{D})$  and  $\mathcal{R}ex(\mathcal{C}, \mathcal{D})$  the categories of left exact and right exact linear functors from  $\mathcal{C}$  to  $\mathcal{D}$ , respectively.

The calculus of Eilenberg-Watts also leads directly to the following result, which involves the Nakayama functor that we will discuss in more detail in Section 3.5.

**Lemma 2.2.** (i) For  $A$  and  $B$  finite-dimensional algebras there is an equivalence between the categories  $\mathcal{L}ex(A\text{-mod}, B\text{-mod})$  and  $(A\text{-bimod}\text{-}B)^{\mathrm{opp}}$  which maps a functor  $F$  to  $(F({}_A A_A^*))^*$ ,

as well as an equivalence  $\mathcal{R}ex(A\text{-mod}, B\text{-mod}) \simeq (A\text{-bimod-}B)^{\text{opp}}$ , mapping a functor  $G$  to the  $B$ - $A$ -bimodule  $G({}_A A_A)^*$ .

(ii) For any finite-dimensional algebra  $A$  there is an equivalence  $\mathcal{L}ex(A\text{-mod}, A\text{-mod}) \xrightarrow{\cong} \mathcal{R}ex(A\text{-mod}, A\text{-mod})$ . This equivalence maps the identity functor  $\text{id}_{A\text{-mod}}$ , regarded as left exact functor, to the right exact functor  $({}_A A_A)^* \otimes_A -$ .

*Proof.* (i) is an easy consequence of the parts (L3) and (R3), respectively, of Lemma 2.1. (ii) follows by composing the two equivalences from (i) in the case  $B = A$ .  $\square$

By the equivalence of finite linear categories to those of modules over finite-dimensional  $\mathbb{k}$ -algebras, Lemma 2.1 immediately gives (compare [DSS, Prop. 1.7 & Cor. 1.10])

**Corollary 2.3.** Let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be linear functors between finite linear categories. The following statements are equivalent:

(L1)  $F \in \mathcal{L}ex(\mathcal{C}, \mathcal{D})$ .

(L2)  $F$  admits a left adjoint.

Likewise, the following are equivalent:

(R1)  $G \in \mathcal{R}ex(\mathcal{C}, \mathcal{D})$ .

(R2)  $G$  admits a right adjoint.

Moreover for  $\mathcal{D} = \text{vect}$  in addition the following statements are equivalent to (L1) and (L2) and to (R1) and (R2), respectively:

(L3)  $F$  is representable, i.e.  $G \cong \text{Hom}_{\mathcal{C}}(c, -)$  for some  $c \in \mathcal{C}$ .

(R3)  $G$  is ‘dually representable’, i.e.  $G \cong \text{Hom}_{\mathcal{C}}(-, d)^*$  for some  $d \in \mathcal{C}$ .

In the sequel we will use the abbreviated notation

$$\text{Hom}_{\mathcal{C}}(c, d) =: {}_c \langle c, d \rangle \quad (2.2)$$

for morphism spaces. Apart from its brevity the motivation for this notation is that we like to think informally of the Hom functor as a categorified inner product. Moreover, if  $\mathcal{C}$  is  $\text{vect}$  or a category of (bi)modules we further abbreviate  ${}_{\text{vect}} \langle -, - \rangle =: {}_{\mathbb{k}} \langle -, - \rangle$  and

$${}_{A\text{-mod}} \langle -, - \rangle =: {}_A \langle -, - \rangle, \quad \text{and} \quad {}_{B\text{-bimod-}A} \langle -, - \rangle =: {}_{B|A} \langle -, - \rangle, \quad (2.3)$$

respectively.

Whenever it does not make expressions too clumsy, we will write  $\bar{c}$  and  $\bar{f}$  for the objects and morphisms in the opposite category  $\mathcal{C}^{\text{opp}}$  that correspond to an object  $c$  and a morphism  $f$  in  $\mathcal{C}$ . Thus with the notation (2.2) we have  ${}_{\mathcal{C}^{\text{opp}}} \langle d, c \rangle = {}_c \langle c, d \rangle$ . If  $\mathcal{C}$  is monoidal then so is  $\mathcal{C}^{\text{opp}}$ , and we take its tensor product to be given by  $\bar{c} \otimes_{\mathcal{C}^{\text{opp}}} \bar{c}' = \overline{c \otimes_{\mathcal{C}} c'}$ . Then the structure of a right duality on  $\mathcal{C}$  induces a left duality on  $\mathcal{C}^{\text{opp}}$  and vice versa. For a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  the *opposite functor* is the functor  $F^{\text{opp}}: \mathcal{C}^{\text{opp}} \rightarrow \mathcal{D}^{\text{opp}}$  that consists of the same maps on the class of objects and on the morphism sets as  $F$ , i.e.

$$F^{\text{opp}}(\bar{c}) = \overline{F(c)} \quad \text{and} \quad F^{\text{opp}}(\overline{m' \xrightarrow{f} m}) = \overline{F(m \xrightarrow{f} m')}. \quad (2.4)$$

If  $F$  is right exact, then  $F^{\text{opp}}$  is left exact, and vice versa.

We are accustomed with the distinguished role that the Hom functor, i.e. left exact (bi)functor  ${}_c \langle -, - \rangle: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \text{vect}$ , plays in many contexts. As it turns out, for us the right

exact (bi)functor obtained from  $\text{Hom}$  by taking duals in  $\text{vect}$ , which is familiar from the definition of Nakayama and Serre functors (and which was already used in formula (2.1) and part (R3) of Corollary 2.3), is of similar importance. We therefore introduce it formally:

**Definition 2.4.** For  $\mathcal{C}$  a finite linear category, the *dual Hom* functor is the functor

$$\begin{aligned} {}_c\langle -, - \rangle^*: \mathcal{C}^{\text{opp}} \times \mathcal{C} &\longrightarrow \text{vect} \\ (\bar{c}, c') &\longmapsto ({}_c\langle c', c \rangle)^* \end{aligned} \quad (2.5)$$

Just like in the case of  $\text{Hom}$ , we will use the term dual Hom functor also for the right exact functor  ${}_c\langle -, c \rangle^*: \mathcal{C} \rightarrow \text{vect}$  with fixed  $c \in \mathcal{C}$  and for the contravariant functor  ${}_c\langle c, - \rangle^*$ . Finally recall that a linear category is a module category over  $\text{vect}$ ; by definition, for  $c \in \mathcal{C}$  and  $v \in \text{vect}$  the object  $c \otimes v \cong v \otimes c \in \mathcal{C}$  is the one that corepresents the functor  ${}_c\langle -, c \rangle \otimes_{\mathbb{k}} v$ , so that

$${}_c\langle c \otimes v, c' \otimes v' \rangle \cong v^* \otimes_{\mathbb{k}} {}_c\langle c, c' \rangle \otimes_{\mathbb{k}} v' \cong {}_{\text{vect}}\langle v, {}_c\langle c, c' \rangle \otimes_{\mathbb{k}} v' \rangle \quad (2.6)$$

for  $c, c' \in \mathcal{C}$  and  $v, v' \in \text{vect}$ . Note that in  $\mathcal{C}^{\text{opp}}$  we have  $\overline{v \otimes c} = v^* \otimes \bar{c}$ .

## 2.2 Ends and coends

Recall [Mac, Ch. IX.4] that a dinatural transformation  $F \Rightarrow x$  from a functor  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{D}$  to an object  $x \in \mathcal{D}$  is a family  $\varphi = \{\varphi_c: F(\bar{c}, c) \rightarrow x\}_{c \in \mathcal{C}}$  of morphisms satisfying  $\varphi_{c'} \circ F(\bar{c}', f) = \varphi_c \circ F(\bar{f}, c)$  for all  $f \in {}_c\langle c, c' \rangle$ . A coend  $(z, \iota)$  for a functor  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{D}$  is an object  $z \in \mathcal{D}$  together with a dinatural transformation  $\iota$  from  $F$  to  $z$  having the universal property that for any dinatural transformation  $\varphi: F \Rightarrow x$  to some  $x \in \mathcal{D}$  there is a unique morphism  $\kappa = \kappa(\varphi) \in {}_{\mathcal{D}}\langle z, x \rangle$  such that  $\varphi_c = \kappa \circ \iota_c$  for all  $c \in \mathcal{C}$ . The notion of an end for a functor  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{D}$  is defined dually. We often suppress the universal dinatural transformation and denote the coend and end of a functor  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{D}$ , as well as the underlying objects, by  $\int^{c \in \mathcal{C}} F(\bar{c}, c)$  and by  $\int_{c \in \mathcal{C}} F(\bar{c}, c)$ , respectively.

Properties of ends and coends will play a crucial role in the formulation of our results. In the present and the next subsection we collect the most basic of those properties. We start by recalling the following fundamental preservation results in which, unlike in the rest of the paper,  $\mathcal{A}$  is not assumed to be linear or finite.

**Lemma 2.5.**

(i) Let  $H: \mathcal{A}^{\text{opp}} \times \mathcal{A} \rightarrow \mathcal{C}$  be a functor whose coend exists. Then there is a natural isomorphism

$${}_c\langle \int^{a \in \mathcal{A}} H(\bar{a}, a), - \rangle \cong \int_{a \in \mathcal{A}} {}_c\langle H(\bar{a}, a), - \rangle \quad (2.7)$$

of functors from  $\mathcal{C}$  to  $\text{vect}$ .

(ii) Let  $H: \mathcal{A}^{\text{opp}} \times \mathcal{A} \rightarrow \mathcal{C}$  be a functor whose end exists. Then there is a natural isomorphism

$${}_c\langle -, \int_{a \in \mathcal{A}} H(\bar{a}, a) \rangle \cong \int_{a \in \mathcal{A}} {}_c\langle -, H(\bar{a}, a) \rangle. \quad (2.8)$$

Another result that will often be used and holds in generality is that for any pair of functors  $F, G: \mathcal{A} \rightarrow \mathcal{B}$  there is a bijection [Mac, Eq. IX.5(2)]

$$\text{Nat}(F, G) \cong \int_{a \in \mathcal{A}} {}_{\mathcal{B}}\langle F(a), G(a) \rangle \quad (2.9)$$



between the set of natural transformations and the end on the right hand side. The dinatural transformation for this end just consists of writing a natural transformation as a tuple in its components.

Restricting our attention from now on again to finite linear categories, we know from Corollary 2.3 that left exact functors  $F: \mathcal{B} \rightarrow \mathcal{C}$  are representable and thus continuous, while right exact functors  $G: \mathcal{B} \rightarrow \mathcal{C}$  are cocontinuous. Now the end can be written as a (small) limit and thus commutes with continuous functors, while the coend commutes with cocontinuous functors. Thus in particular we have

**Lemma 2.6.** Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be finite linear categories and  $H: \mathcal{A}^{\text{opp}} \times \mathcal{A} \rightarrow \mathcal{B}$  be a functor whose (co)end exists. There are canonical isomorphisms

$$F\left(\int_{a \in \mathcal{A}} H(\bar{a}, a)\right) \cong \int_{a \in \mathcal{A}} F(H(\bar{a}, a)) \quad \text{for any } F \in \mathcal{L}ex(\mathcal{B}, \mathcal{C}) \quad (2.10)$$

and

$$G\left(\int^{a \in \mathcal{A}} H(\bar{a}, a)\right) \cong \int^{a \in \mathcal{A}} G(H(\bar{a}, a)) \quad \text{for any } G \in \mathcal{R}ex(\mathcal{B}, \mathcal{C}). \quad (2.11)$$

As another simple relationship we mention that for any functor  $H: \mathcal{A}^{\text{opp}} \times \mathcal{A} \rightarrow \mathcal{B}$  between finite linear categories there is a canonical isomorphism

$$\overline{\int^{a \in \mathcal{A}} H(\bar{a}, a)} \cong \int_{a \in \mathcal{A}} H^{\text{opp}}(a, \bar{a}) \quad (2.12)$$

with  $H^{\text{opp}}: \mathcal{A} \times \mathcal{A}^{\text{opp}} \rightarrow \mathcal{B}^{\text{opp}}$  and the obvious relation between the dinatural families of the coend and end.

## 2.3 Generalized Yoneda lemmas

For any functor  $F: \mathcal{A} \rightarrow \text{vect}$  we have

$$\int_{b \in \mathcal{A}} \text{vect} \langle \mathcal{A} \langle a, b \rangle, F(b) \rangle \stackrel{(2.9)}{\cong} \text{Nat}(\mathcal{A} \langle a, - \rangle, F) \cong F(a) \quad (2.13)$$

by the Yoneda lemma. This can be used to show that the Hom functor and the dual Hom functor (as introduced in Definition 2.4) constitute units for a kind of convolution product of linear functors. This property of the Hom and dual Hom functors plays a key role in various calculations below (as well as e.g. in topological [Ly] and conformal [FS] quantum field theory). Therefore we give an explicit proof, even though the result is known (see [Ri, Cor. 1.4.5 & Ex. 1.4.6]):

**Proposition 2.7.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be finite linear categories and  $F: \mathcal{A} \rightarrow \mathcal{C}$  a linear functor.

(i) There is a natural isomorphism

$$\int^{a \in \mathcal{A}} \mathcal{A} \langle a, - \rangle \otimes F(a) \cong F \quad (2.14)$$

of linear functors. In particular, the coend on the left hand side exists as an object of the category  $\mathcal{F}un(\mathcal{A}, \mathcal{C})$  of linear functors from  $\mathcal{A}$  to  $\mathcal{C}$ . More specifically, functorially in  $x \in \mathcal{A}$  the object  $F(x)$  has the properties of a coend  $\int^{a \in \mathcal{A}} \mathcal{A} \langle a, x \rangle \otimes F(a)$ .

(ii) There is a natural isomorphism

$$\int_{a \in \mathcal{A}} \mathcal{A}\langle -, a \rangle^* \otimes F(a) \cong F \quad (2.15)$$

of linear functors.

*Proof.*

(i) For any  $x \in \mathcal{A}$  and  $c \in \mathcal{C}$  we have

$$\begin{aligned} c \langle \int_{a \in \mathcal{A}} \mathcal{A}\langle a, x \rangle \otimes F(a), c \rangle &\stackrel{(2.7)}{\cong} \int_{a \in \mathcal{A}} c \langle \mathcal{A}\langle a, x \rangle \otimes F(a), c \rangle \\ &\stackrel{(2.6)}{\cong} \int_{a \in \mathcal{A}} \mathcal{A}\langle a, x \rangle^* \otimes_{\mathbb{k}} c \langle F(a), c \rangle \cong \int_{a \in \mathcal{A}} \text{vect} \langle \mathcal{A}\langle a, x \rangle, c \langle F(a), c \rangle \rangle \\ &\stackrel{(2.9)}{\cong} \text{Nat}(\mathcal{A}\langle -, x \rangle, c \langle F(-), c \rangle) \stackrel{(2.13)}{\cong} c \langle F(x), c \rangle \end{aligned} \quad (2.16)$$

natural in  $x \in \mathcal{A}$ . Here in the last line we use the (contravariant) Yoneda lemma. The claim now follows by invoking the Yoneda lemma once again. The universal dinatural transformation consists of morphisms from  $\mathcal{A}\langle a, x \rangle \otimes F(a)$  to  $F(x)$ , for  $a \in \mathcal{A}$ . Using the isomorphism  $c \langle \mathcal{A}\langle a, x \rangle \otimes F(a), F(x) \rangle \cong_{\mathbb{k}} \langle \mathcal{A}\langle a, x \rangle, c \langle F(a), F(x) \rangle \rangle$ , this morphism comes from the map  $f \mapsto F(f)$  for all  $f \in \mathcal{A}\langle a, x \rangle$ .

(ii) Analogously as in the case of (i) the proof follows from

$$\begin{aligned} c \langle c, \int_{a \in \mathcal{A}} \mathcal{A}\langle x, a \rangle^* \otimes F(a) \rangle &\stackrel{(2.8)}{\cong} \int_{a \in \mathcal{A}} c \langle c, \mathcal{A}\langle x, a \rangle^* \otimes F(a) \rangle \\ &\stackrel{(2.6)}{\cong} \int_{a \in \mathcal{A}} \text{vect} \langle \mathcal{A}\langle x, a \rangle, c \langle c, F(a) \rangle \rangle \stackrel{(2.13)}{\cong} c \langle c, F(x) \rangle. \end{aligned} \quad (2.17)$$

This time the covariant Yoneda lemma has been used, while for completing the proof one implements the contravariant Yoneda lemma. Finally, the universal dinatural transformation for the end comes from the isomorphism  $c \langle F(x), \mathcal{A}\langle x, a \rangle^* \otimes F(a) \rangle \cong_{\mathbb{k}} \langle \mathcal{A}\langle a, x \rangle, c \langle F(a), F(x) \rangle \rangle$  for  $a \in \mathcal{A}$ ; as in (i), this is induced by the map  $f \mapsto F(f)$  for all  $f \in \mathcal{A}\langle a, x \rangle$ .  $\square$

In the case  $\mathcal{B} = \text{vect}$  the property (2.15) reduces to the statement of the Yoneda lemma as given in (2.13). Actually, as the proof indicates, it is merely a repackaging of that statement and could therefore just be referred to as a Yoneda lemma itself, and analogously (2.14) as a co-Yoneda lemma [Ri]. For concreteness, interpreting loosely ends and coends as analogues of integrals, we will instead in the sequel refer to these results as the *convolution properties* of the Hom functor and of the dual Hom functor, respectively.

## 2.4 A Peter-Weyl expression for (co)regular bimodules

For  $A$  and  $B$  finite-dimensional  $\mathbb{k}$ -algebras and  $G$  a  $\mathbb{k}$ -linear functor from  $A\text{-mod}$  to  $B\text{-mod}$ , the object  $G(A)$  has a natural structure of a  $B$ - $A$ -bimodule. To see this, note that the right multiplication  $r_\alpha: A \rightarrow A$  by  $\alpha \in A$  on  $A$  is a morphism of left  $A$ -modules. As a consequence,  $G(r_\alpha): G(A) \rightarrow G(A)$  is a morphism of left  $B$ -modules, and one can take the ring homomorphism  $\alpha \mapsto G(r_\alpha) \in \text{End}(G(A))$  as the definition of the right  $A$ -module structure on  $G(A)$ . By construction, this commutes with the left  $B$ -module structure on  $G(A)$ .

This bimodule structure on  $G(A)$  is used in the following statement in which (unlike in the Eilenberg-Watts results) no exactness property needs to be assumed.

**Proposition 2.8.** Let  $A$  and  $B$  be finite-dimensional algebras and  $G: A\text{-mod} \rightarrow B\text{-mod}$  a linear functor. The end of the functor

$$\begin{aligned} \tilde{G} : A\text{-mod} \times A\text{-mod}^{\text{opp}} &\longrightarrow B\text{-bimod-}A \\ m \times \bar{n} &\longmapsto G(m) \otimes_{\mathbb{k}} n^* \end{aligned} \quad (2.18)$$

is given as a  $B$ - $A$ -bimodule by

$$\int_{m \in A\text{-mod}} G(m) \otimes_{\mathbb{k}} m^* = G(A) \quad (2.19)$$

with the natural  $B$ - $A$ -bimodule structure on  $G(A)$ . Its dinatural family is

$$i_m^A := G(\hat{\rho}_m) : G(A) \rightarrow G(m) \otimes_{\mathbb{k}} m^* \quad \text{with} \quad \hat{\rho}_m := (\rho_m \otimes_{\mathbb{k}} \text{id}_{m^*}) \circ (\text{id}_A \otimes \beta_m) \quad (2.20)$$

for  $(m, \rho_m : A \otimes m \rightarrow m) \in A\text{-mod}$ , with  $\beta_m \in {}_{\mathbb{k}}\langle \mathbb{k}, m \otimes_{\mathbb{k}} m^* \rangle$  the coevaluation in  $\text{vect}$  and with  $m \otimes_{\mathbb{k}} m^*$  considered as an  $A$ -module via the action of  $A$  on the left tensorand.

Similarly, the coend of the functor  $\tilde{G}$  is

$$\int^{m \in A\text{-mod}} G(m) \otimes_{\mathbb{k}} m^* = G(A^*) \quad (2.21)$$

with dinatural family

$$i_m^{A^*} := G(\tilde{\rho}_m) \quad \text{with} \quad \tilde{\rho}_m := (\text{id}_{A^*} \otimes_{\mathbb{k}} [\delta_m \circ (\rho_m \otimes_{\mathbb{k}} \text{id}_{m^*})]) \circ (\beta_{A^*} \otimes_{\mathbb{k}} \text{id}_m \otimes_{\mathbb{k}} \text{id}_{m^*}), \quad (2.22)$$

where  $\delta_m$  is the evaluation in  $\text{vect}$ .

*Proof.* We show the statement for the end; the proof for the coend is analogous.

(i) By linearity of  $G$  there is a natural isomorphism  $G(m \otimes_{\mathbb{k}} m^*) \cong G(m) \otimes_{\mathbb{k}} m^*$  of left  $B$ -modules. Since  $\hat{\rho}_m$  is a morphism of left  $A$ -modules,  $G(\hat{\rho}_m)$  is a morphism of left  $B$ -modules. From the functoriality of  $G$  and the module property of  $m$  it follows that

$$i_m^A \circ G(r_\alpha) = (\text{id}_{G(m)} \otimes_{\mathbb{k}} r_\alpha^*) \circ i_m^A \quad (2.23)$$

for all  $\alpha \in A$ , which tells us that  $i_m^A$  is a morphism of right  $A$ -modules. Thus in total we have

$$i_m^A \in {}_{B|A}\langle G(A), \tilde{G}(m, \bar{m}) \rangle, \quad (2.24)$$

i.e.  $i_m^A$  is well defined. Further, invoking the fact that  $\rho_m$  is a bimodule morphism and the naturality of the coevaluation one now sees that the family  $i^A = (i_m^A)_{m \in A\text{-mod}}$  is dinatural.

(ii) Assume that  $j_m^z \in {}_{B|A}\langle z, \tilde{G}(m, \bar{m}) \rangle$  is any dinatural family from an object  $z \in B\text{-bimod-}A$  to  $\tilde{G}$ . Applying dinaturalness of  $j^z$  to the morphisms  $(\text{id}_{G(A)} \otimes_{\mathbb{k}} \mu^*) \circ i_m^A$  for  $m \in A\text{-mod}$  and  $\mu^* \in m^*$  gives

$$(G(r_\mu) \otimes_{\mathbb{k}} \text{id}_{A^*}) \circ j_A^z = (\text{id}_{G(m)} \otimes_{\mathbb{k}} r_\mu^*) \circ j_m^z. \quad (2.25)$$

We denote, for  $w \in \text{vect}$ , by  $\delta_w \in {}_{\mathbb{k}}\langle w^* \otimes_{\mathbb{k}} w, \mathbb{k} \rangle$  the evaluation map and by  $f^* = (\delta_w \otimes_{\mathbb{k}} \text{id}_{v^*}) \circ (\text{id}_{w^*} \otimes_{\mathbb{k}} f \otimes_{\mathbb{k}} \text{id}_{v^*}) \circ (\text{id}_{w^*} \otimes_{\mathbb{k}} \beta_v)$  the  $\text{vect}$ -dual of a linear map  $f \in {}_{\mathbb{k}}\langle v, w \rangle$ . Post-composing the

equality (2.25) with  $\text{id}_{G(m)} \otimes_{\mathbb{k}} \eta_A^*$ , where  $\eta_A^* = \delta_A \circ (\text{id}_{A^*} \otimes \eta_A)$  is the dual of the unit  $\eta_A \in {}_{\mathbb{k}}\langle \mathbb{k}, A \rangle$  of  $A$ , results in  $(G(r_\mu) \otimes_{\mathbb{k}} \eta_A^*) \circ j_A^z = (\text{id}_{G(m)} \otimes_{\mathbb{k}} \mu^*) \circ j_m^z$  for all  $\mu \in m$ , implying that

$$(i_m^A \otimes_{\mathbb{k}} \eta_A^*) \circ j_A^z = (\text{id}_{G(m)} \otimes_{\mathbb{k}} \text{id}_{m^*}) \circ j_m^z = j_m^z. \quad (2.26)$$

Thus the dinatural family for  $z$  is related to the one for  $A$  by

$$j_m^z = i_m^A \circ \kappa_z \quad (2.27)$$

for all  $m \in A\text{-mod}$ , with  $\kappa_z := (\text{id}_{G(A)} \otimes_{\mathbb{k}} \eta_A^*) \circ j_A^z$ .

(iii) It remains to show that  $\kappa_z$  is the unique morphism satisfying (2.27) for all  $m$ . This is seen by noting that  $i_A^A \circ \kappa = j_A^z$  as a special case of (2.27), and that, by the calculation

$$\begin{aligned} (\text{id}_{G(A)} \otimes_{\mathbb{k}} \eta_A^*) \circ i_A^A &= (\text{id}_{G(A)} \otimes_{\mathbb{k}} \eta_A^*) \circ (\psi_A \otimes_{\mathbb{k}} \text{id}_{A^*}) \circ (\text{id}_{G(A)} \otimes_{\mathbb{k}} \beta_A) \\ &= \psi_A \circ (\text{id}_{G(A)} \otimes_{\mathbb{k}} \eta_A) = G(r_{\eta_A}) = G(\text{id}_A) = \text{id}_{G(A)}, \end{aligned} \quad (2.28)$$

$i_A^A$  has a left inverse. □

Taking  $G$  to be the identity functor, as a special case of Proposition 2.8 we have the following Peter-Weyl type result which provides a Morita invariant characterization of regular and co-regular bimodules:

**Corollary 2.9.** For any finite-dimensional  $\mathbb{k}$ -algebra  $A$  there are isomorphisms

$$\int_{m \in A\text{-mod}} m \otimes_{\mathbb{k}} m^* \cong {}_A A_A \quad (2.29)$$

and

$$\int^{m \in A\text{-mod}} m \otimes_{\mathbb{k}} m^* \cong {}_A A_A^* \quad (2.30)$$

of  $A$ -bimodules. The dinatural transformations of the end and coend are given by the families  $i_m^A := (\rho_m \otimes_{\mathbb{k}} \text{id}_{m^*}) \circ (\text{id}_A \otimes \beta_m)$  and

$$i_m^{A^*} := (\text{id}_{A^*} \otimes \delta_m) \circ (\text{id}_{A^*} \otimes \rho_m \otimes \text{id}_{m^*}) \circ (\beta_A \otimes \text{id}_m \otimes \text{id}_{m^*}) \quad (2.31)$$

of linear maps for  $(m, \rho_m) \in A\text{-mod}$ , respectively.

As already pointed out, no exactness of the functors is assumed in these results. Since for any finite linear category  $\mathcal{M}$  we can choose an algebra  $A$  and an equivalence  $\mathcal{M} \simeq A\text{-mod}$ , we can transport them to the setting of finite linear categories. For doing so we also need an analogue of the tensor product over  $\mathbb{k}$ ; this is provided by the Deligne product.

Recall [De, Sect.5] that the Deligne product  $\mathcal{A} \boxtimes \mathcal{B}$  of two finite linear categories is the linear category that is universal for right exact functors with domain  $\mathcal{A} \times \mathcal{B}$ . That is, there is a functor  $\boxtimes: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \boxtimes \mathcal{B}$ , exact in each variable, such that for any linear category  $\mathcal{X}$  every right exact (in both variables) functor  $F$  from  $\mathcal{A} \times \mathcal{B}$  factorizes uniquely (up to unique natural isomorphism) as  $F = \tilde{F} \circ \boxtimes$  with  $\tilde{F}: \mathcal{A} \boxtimes \mathcal{B} \rightarrow \mathcal{X}$  being right exact. Thus in particular we have an equivalence  $\mathcal{R}ex(\mathcal{A} \times \mathcal{B}, \mathcal{X}) \simeq \mathcal{R}ex(\mathcal{A} \boxtimes \mathcal{B}, \mathcal{X})$ , functorial in  $\mathcal{X}$ , of categories of right exact functors. By considering opposite functors and using that the opposite of a right exact functor is left exact, the Deligne product is universal with respect to *left* exact functors as well. Also note that for  $A$  and  $B$  finite-dimensional  $\mathbb{k}$ -algebras we have linear isomorphisms

$$A\text{-mod} \boxtimes \text{mod-}B \cong A\text{-mod} \boxtimes B^{\text{opp}}\text{-mod} \cong (A \otimes_{\mathbb{k}} B^{\text{opp}})\text{-mod} \cong A\text{-bimod-}B. \quad (2.32)$$

The Peter-Weyl results above thus imply

**Corollary 2.10.** For every linear functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  between finite linear categories, the end  $\int_{m \in \mathcal{M}} F(m) \boxtimes \bar{m}$  and the coend  $\int^{m \in \mathcal{M}} F(m) \boxtimes \bar{m}$  in  $\mathcal{N} \boxtimes \mathcal{M}^{\text{opp}}$  exist.

**Remark 2.11.** For the case that  $A$  is a Hopf algebra, the assertion of Corollary 2.9 has already been shown in [FSS, Prop. A.3]. The proof in [FSS] is obtained from the one above by considering the right  $A$ -action on  $A$  that is obtained from the left actions by means of the antipode of  $A$ .

### 3 Eilenberg-Watts calculus for finite linear categories

If  $\mathcal{A}$  and  $\mathcal{B}$  are finite linear categories, then there are finite-dimensional algebras  $A$  and  $B$  such that  $\mathcal{A} \simeq A\text{-mod}$  and  $\mathcal{B} \simeq B\text{-mod}$ . The Eilenberg-Watts results of Lemma 2.1 then imply that the functor category  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  is equivalent to  $B\text{-}A\text{-bimod}$ , which in turn is equivalent to the Deligne product  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$ . In principle the resulting equivalence

$$\mathcal{R}ex(\mathcal{A}, \mathcal{B}) \simeq \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \quad (3.1)$$

of finite linear categories might depend on the choice of algebras  $A$  and  $B$ . In the sequel we will analyze this equivalence, as well as a corresponding equivalence for left exact functors, without having to refer to any choice of algebras. Thus in particular these equivalences do not depend on those individual algebras, but only on their Morita classes. Our discussion will strongly rely on the use of ends and coends, the relevance of which may be traced back to Proposition 2.8.

#### 3.1 A triangle of adjoint equivalences

In this subsection we establish, for any two finite  $\mathbb{k}$ -linear categories  $\mathcal{A}$  and  $\mathcal{B}$ , adjoint equivalences between the Deligne product  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$  and the categories  $\mathcal{L}ex(\mathcal{A}, \mathcal{B})$  and  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  of left exact and right exact functors, respectively, from  $\mathcal{A}$  to  $\mathcal{B}$ . These equivalences imply in particular that these functor categories are finite linear categories. For the case of  $\mathcal{L}ex(\mathcal{A}, \mathcal{B})$  this equivalence has been established in [ENO, Rem. 2,2(i)] and [Sh1, Lemma 3.2]; the result for  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  follows by considering in addition to the Hom functor also the dual Hom functor.

We start by

**Definition 3.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be finite linear categories. The (abstract) *Eilenberg-Watts functors* for  $\mathcal{A}$  and  $\mathcal{B}$  are the following four linear functors:

$$\begin{aligned} \Phi^l \equiv \Phi_{\mathcal{A}, \mathcal{B}}^l : \quad & \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \longrightarrow \mathcal{L}ex(\mathcal{A}, \mathcal{B}) \\ & \bar{a} \boxtimes b \longmapsto {}_{\mathcal{A}}\langle a, - \rangle \otimes b, \\ \Psi^l \equiv \Psi_{\mathcal{A}, \mathcal{B}}^l : \quad & \mathcal{L}ex(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \\ & F \longmapsto \int^{a \in \mathcal{A}} \bar{a} \boxtimes F(a), \\ \Phi^r \equiv \Phi_{\mathcal{A}, \mathcal{B}}^r : \quad & \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \longrightarrow \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \\ & \bar{a} \boxtimes b \longmapsto {}_{\mathcal{A}}\langle -, a \rangle^* \otimes b, \\ \Psi^r \equiv \Psi_{\mathcal{A}, \mathcal{B}}^r : \quad & \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \\ & G \longmapsto \int_{a \in \mathcal{A}} \bar{a} \boxtimes G(a). \end{aligned} \quad (3.2)$$

Note that the functors  $\Phi^l$  and  $\Phi^r$  take a simple form only when applied to  $\boxtimes$ -factorized objects. We will have to apply them also on objects that are ends or coends, in which case we write for example

$$\Phi^l\left(\int^a \bar{a} \boxtimes F(a)\right) = \left[\mathrm{Hom}_{\mathcal{A}}(\mathcal{I}, -) \otimes ?\right]\left(\int^a \bar{a} \boxtimes F(a)\right). \quad (3.3)$$

We also abbreviate

$$\begin{aligned} \Gamma^{rl} &:= \Phi^r \circ \Psi^l : \mathcal{L}ex(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \\ \text{and } \Gamma^{lr} &:= \Phi^l \circ \Psi^r : \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{L}ex(\mathcal{A}, \mathcal{B}). \end{aligned} \quad (3.4)$$

Using that the Hom functor, being left exact, preserves ends and that the dual Hom functor preserves coends, this implies

$$\Gamma^{rl}(F) = \int^{a \in \mathcal{A}} \mathcal{A}\langle -, a \rangle^* \otimes F(a) \quad \text{and} \quad \Gamma^{lr}(G) = \int_{a \in \mathcal{A}} \mathcal{A}\langle a, - \rangle \otimes G(a) \quad (3.5)$$

for  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$  and for  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{B})$ , respectively. In the definition of  $\Phi^l$  we use that the Deligne product is exact and universal with respect to left exact functors. In the definition of  $\Psi^l$  and  $\Psi^r$  we use the fact, seen in Corollary 2.10, that the coend and end, respectively, exist, and that they are functorial with respect to natural transformations.

For ease of reference we collect these functors in the following diagram:

$$\begin{array}{ccc} & \mathcal{A}^{\mathrm{opp}} \boxtimes \mathcal{B} & \\ \Psi^l \nearrow & & \nwarrow \Psi^r \\ \Phi^l & & \Phi^r \\ \mathcal{L}ex(\mathcal{A}, \mathcal{B}) & \xleftrightarrow[\Gamma^{lr}]{\Gamma^{rl}} & \mathcal{R}ex(\mathcal{A}, \mathcal{B}) \end{array} \quad (3.6)$$

**Theorem 3.2.** [Categorical Eilenberg-Watts theorem]

For any pair of finite  $\mathbb{k}$ -linear categories  $\mathcal{A}$  and  $\mathcal{B}$  the functors in the triangle (3.6) constitute quasi-inverse pairs of adjoint equivalences

$$\mathcal{L}ex(\mathcal{A}, \mathcal{B}) \simeq \mathcal{A}^{\mathrm{opp}} \boxtimes \mathcal{B} \simeq \mathcal{R}ex(\mathcal{A}, \mathcal{B}). \quad (3.7)$$

*Proof.* (i) That the functor  $\Phi^l$  is an equivalence is shown in [Sh1, Lemma 3.2], and that  $\Psi^l$  is a quasi-inverse of  $\Phi^l$  in [Sh1, Lemma. 3.3].

(ii) To obtain the second equivalence in (3.7) we apply the first one after taking opposites. Note that for  $\bar{a} \in \mathcal{A}^{\mathrm{opp}}$  and  $v \in \mathrm{vect}$  we have

$$\bar{a} \otimes v = \overline{\bar{a} \otimes v^*} \in \mathcal{A}^{\mathrm{opp}}, \quad (3.8)$$

and that by Lemma 2.2  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  is equivalent to  $(\mathcal{L}ex(\mathcal{A}^{\mathrm{opp}}, \mathcal{B}^{\mathrm{opp}}))^{\mathrm{opp}}$ . The equivalences  $\Psi^l$  and  $\Psi^r$  give a pair of quasi-inverse equivalences

$$\begin{aligned} (\Phi^l)^{\mathrm{opp}} &: \mathcal{A}^{\mathrm{opp}} \boxtimes \mathcal{B} \simeq (\mathcal{A} \boxtimes \mathcal{B}^{\mathrm{opp}})^{\mathrm{opp}} \xrightarrow{\simeq} (\mathcal{L}ex(\mathcal{A}^{\mathrm{opp}}, \mathcal{B}^{\mathrm{opp}}))^{\mathrm{opp}} \quad \text{and} \\ (\Psi^l)^{\mathrm{opp}} &: (\mathcal{L}ex(\mathcal{A}^{\mathrm{opp}}, \mathcal{B}^{\mathrm{opp}}))^{\mathrm{opp}} \xrightarrow{\simeq} (\mathcal{A} \boxtimes \mathcal{B}^{\mathrm{opp}})^{\mathrm{opp}} \simeq \mathcal{A}^{\mathrm{opp}} \boxtimes \mathcal{B}. \end{aligned} \quad (3.9)$$

The so defined functor  $(\Phi^1)^{\text{opp}}$  maps the object  $\bar{a} \boxtimes b \in \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$  to the opposite of the functor  $\text{Hom}_{\mathcal{A}^{\text{opp}}}(a, -) \otimes \bar{b}$ , i.e.  $(\Phi^1)^{\text{opp}}(\bar{a} \boxtimes b) \stackrel{(3.8)}{=} (\mathcal{A}\langle -, a \rangle)^* \otimes b = \Phi^r(\bar{a} \boxtimes b)$ , while the functor  $(\Psi^1)^{\text{opp}}$  equals  $\Psi^r$ : it maps  $\bar{F}$  with  $F \in \mathcal{L}ex(\mathcal{A}^{\text{opp}}, \mathcal{B}^{\text{opp}})$  to

$$(\Psi^1)^{\text{opp}}(\bar{F}) = \overline{\int^{a \in \mathcal{A}^{\text{opp}}} a \boxtimes \bar{F}(a)} \cong \int_{a \in \mathcal{A}} \bar{a} \boxtimes F(a) = \Psi^r(\bar{F}). \quad (3.10)$$

(iii) Using formula (2.9) we have  $\int_{a' \in \mathcal{A}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \bar{a}' \boxtimes F(a'), \bar{a} \boxtimes b \rangle \cong \text{Nat}(F, \Phi^1(\bar{a} \boxtimes b))$  and hence

$$\begin{aligned} \mathcal{F}un(\mathcal{A}, \mathcal{B}) \langle F, \Phi^1(x) \rangle &\equiv \text{Nat}(F, \Phi^1(x)) \cong \int_{a \in \mathcal{A}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \bar{a} \boxtimes F(a), x \rangle^* \\ &\cong \int_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}} \langle \int^{a \in \mathcal{A}} \bar{a} \boxtimes F(a), x \rangle = \int_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}} \langle \Psi^1(F), x \rangle \end{aligned} \quad (3.11)$$

for  $x \in \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}$ . Thus  $\Psi^1$  is left adjoint to  $\Phi^1$ . Similarly, the calculation

$$\begin{aligned} \mathcal{F}un(\mathcal{A}, \mathcal{B}) \langle \Phi^r(\bar{a} \boxtimes b), G \rangle &\cong \int_{a' \in \mathcal{A}} \mathcal{B} \langle \mathcal{A}\langle a', a \rangle^* \otimes b, G(a') \rangle \cong \int_{a' \in \mathcal{A}} \mathcal{A}\langle a', a \rangle \otimes_{\mathbb{k}} \mathcal{B} \langle b, G(a') \rangle \\ &\cong \int_{a' \in \mathcal{A}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \bar{a} \boxtimes b, \bar{a}' \boxtimes G(a') \rangle \\ &\cong \int_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}} \langle \bar{a} \boxtimes b, \int_{a' \in \mathcal{A}} \bar{a}' \boxtimes G(a') \rangle = \int_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}} \langle \bar{a} \boxtimes b, \Psi^r(G) \rangle \end{aligned} \quad (3.12)$$

for  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  shows that (also invoking universality of the Deligne product)  $\Psi^r$  is right adjoint to  $\Phi^r$ . Since the respective pairs of functors are equivalences, the adjunctions are two-sided.

(iv) That the functors  $\Gamma^{\text{rl}}$  and  $\Gamma^{\text{lr}}$  form an adjoint equivalence follows directly by composing the relevant statements for the functors  $\Phi^1$ ,  $\Psi^1$  and  $\Phi^r$ ,  $\Psi^r$ .  $\square$

The following rewritings will be convenient:

**Lemma 3.3.** For any functor  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$  we have

$$\Psi^1(F) \cong \int_{a \in \mathcal{A}} \bar{a} \boxtimes \Gamma^{\text{rl}}(F)(a), \quad (3.13)$$

and for any functor  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{B})$  we have

$$\Psi^r(G) \cong \int^{a \in \mathcal{A}} \bar{a} \boxtimes \Gamma^{\text{lr}}(G)(a). \quad (3.14)$$

*Proof.* The right hand side of (3.13) is nothing but  $\Psi^r(\Gamma^{\text{rl}}(F)) \cong \Psi^r \circ \Phi^r \circ \Psi^1(F)$  and thus,  $\Psi^r$  and  $\Phi^r$  being quasi-inverse, equals the left hand side. The isomorphism (3.14) follows analogously by recognizing the right hand side as  $\Psi^1(\Gamma^{\text{lr}}(G))$ .  $\square$

### 3.2 Hom pairings, compositions and (co)ends for finite categories

In the proof of Theorem 3.2 we encountered the relations (3.11) and (3.12) for morphism spaces which involve objects of the form  $\Psi^1(F)$  or  $\Psi^r(G)$ . As we deal with an adjoint equivalence,  $\Psi^1$  is not only left, but also right adjoint to  $\Phi^1$ . As a consequence there also exist relations for morphism spaces in which with respect to those in (3.11) and (3.12) the domain and codomain are interchanged. This provides the following ‘opposed’ variant of the basic (co)end preservation statement of Lemma 2.5, which is valid in the finite linear case considered here; this result will be a rich source of identities relevant to us.

**Proposition 3.4.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be finite linear categories. Then for  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{C})$  we have

$${}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{a} \boxtimes c, \int^{b \in \mathcal{A}} \bar{b} \boxtimes F(b) \rangle \cong \int^{b \in \mathcal{A}} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{a} \boxtimes c, \bar{b} \boxtimes F(b) \rangle, \quad (3.15)$$

while for  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{C})$  we have

$${}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \int_{b \in \mathcal{A}} \bar{b} \boxtimes G(b), \bar{a} \boxtimes c \rangle \cong \int_{b \in \mathcal{A}} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{b} \boxtimes G(b), \bar{a} \boxtimes c \rangle \quad (3.16)$$

respectively, natural in  $a \in \mathcal{A}$  and  $c \in \mathcal{C}$ .

*Proof.* (i) To obtain the isomorphism (3.15) we use that  $\Psi^1$  has  $\Phi^1$  as a quasi-inverse and invoke (2.9) and the (Yoneda) convolution property of the dual Hom functor to compute

$$\begin{aligned} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{a} \boxtimes c, \Psi^1(F) \rangle &\cong {}_{\mathcal{L}ex(\mathcal{A}, \mathcal{C})} \langle \Phi^1(\bar{a} \boxtimes c), F \rangle = \text{Nat}({}_{\mathcal{A}} \langle a, - \rangle \otimes c, F) \\ &\cong \int_{b \in \mathcal{A}} c \langle {}_{\mathcal{A}} \langle a, b \rangle \otimes c, F(b) \rangle \cong \int_{b \in \mathcal{A}} {}_{\mathcal{A}} \langle a, b \rangle^* \otimes_{\mathbb{k}} c \langle c, F(b) \rangle \cong c \langle c, F(a) \rangle. \end{aligned} \quad (3.17)$$

On the other hand, the (co-Yoneda) convolution property of the Hom functor implies that the same result can be obtained as follows:

$$\int^{b \in \mathcal{A}} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{a} \boxtimes c, \bar{b} \boxtimes F(b) \rangle \cong \int^{b \in \mathcal{A}} {}_{\mathcal{A}} \langle b, a \rangle \otimes_{\mathbb{k}} {}_{\mathcal{A}} \langle F^{1.a.}(c), b \rangle \cong c \langle c, F(a) \rangle. \quad (3.18)$$

The claim (3.15) now follows by inserting the definition of  $\Psi^1$ .

(ii) Analogously we calculate

$$\begin{aligned} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \Psi^r(G), \bar{a} \boxtimes c \rangle &\cong {}_{\mathcal{R}ex(\mathcal{A}, \mathcal{C})} \langle G, \Phi^r(\bar{a} \boxtimes c) \rangle = \text{Nat}(G, {}_{\mathcal{A}} \langle -, a \rangle^* \otimes c) \\ &\cong \int_{b \in \mathcal{A}} c \langle G(b), {}_{\mathcal{A}} \langle b, a \rangle^* \otimes c \rangle \cong \int_{b \in \mathcal{A}} c \langle G(b), c \rangle \otimes_{\mathbb{k}} {}_{\mathcal{A}} \langle b, a \rangle^* \\ &\cong \int_{\bar{b} \in \mathcal{A}^{\text{opp}}} c \langle G(b), c \rangle \otimes_{\mathbb{k}} {}_{\mathcal{A}^{\text{opp}}} \langle \bar{a}, \bar{b} \rangle^* \cong c \langle G(a), c \rangle. \end{aligned} \quad (3.19)$$

as well as (using now in addition that, being right exact, by Corollary 2.3  $G$  has a right adjoint)

$$\begin{aligned} \int^{b \in \mathcal{A}} {}_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{b} \boxtimes G(b), \bar{a} \boxtimes c \rangle &\cong \int^{b \in \mathcal{A}} {}_{\mathcal{A}} \langle a, b \rangle \otimes_{\mathbb{k}} {}_{\mathcal{A}} \langle b, G^{r.a.}(c) \rangle \\ &\cong {}_{\mathcal{A}} \langle a, G^{r.a.}(c) \rangle \cong c \langle G(a), c \rangle. \end{aligned} \quad (3.20)$$

Again the claim now follows by comparison, after inserting the definition of  $\Psi^r$ .  $\square$



The left hand side of each of the isomorphisms (3.15) and (3.16) defines a left exact (contravariant, respectively covariant) functor from  $\mathcal{A}^{\text{opp}} \times \mathcal{C}$  to  $\text{vect}$ , and thus, by the universal property of the Deligne product, a corresponding functor from  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}$  to  $\text{vect}$ . It may, however, happen that the so defined functor cannot be expressed as a coend in the same way as on the right hand side of (3.15) and (3.16). But if we take the coend on the right hand side in the category of left exact functors, then we do get such an expression. Using, as in [Ly], the symbol  $\oint$  for such coends in categories of left exact functors, we have

**Corollary 3.5.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be finite linear categories. Then for any  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{C})$  there is an isomorphism

$$\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle -, \int^{b \in \mathcal{A}} \bar{b} \boxtimes F(b) \rangle \cong \oint^{b \in \mathcal{A}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle -, \bar{b} \boxtimes F(b) \rangle, \quad (3.21)$$

in  $\mathcal{L}ex(\mathcal{A} \boxtimes \mathcal{C}^{\text{opp}}, \text{vect})$ , while for any  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{C})$  there is an isomorphism

$$\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle \int_{b \in \mathcal{A}} \bar{b} \boxtimes G(b), - \rangle \cong \oint^{b \in \mathcal{A}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle \bar{b} \boxtimes G(b), - \rangle \quad (3.22)$$

in  $\mathcal{R}ex(\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}, \text{vect})$ .

*Proof.* The coends on the right hand side of these expressions exist as objects of the relevant functor categories. Moreover, when restricted to  $\boxtimes$ -factorized objects, by Proposition 3.4 they coincide with the functors on the left hand side. The statement thus follows by the universal property (for left exact functors) of the Deligne product.  $\square$

We will frequently use Proposition 3.4 in the form that is more directly seen in the proof, i.e. as

**Corollary 3.6.** For  $\mathcal{A}$  and  $\mathcal{C}$  be finite linear categories, we have

$$\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle \bar{a} \boxtimes c, \Psi^l(F) \rangle \cong {}_c \langle c, F(a) \rangle \quad \text{for } F \in \mathcal{L}ex(\mathcal{A}, \mathcal{C}) \quad (3.23)$$

and

$$\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C} \langle \Psi^r(G), \bar{a} \boxtimes c \rangle \cong {}_c \langle G(a), c \rangle \quad \text{for } G \in \mathcal{R}ex(\mathcal{A}, \mathcal{C}). \quad (3.24)$$

Invoking the Yoneda lemma, this determines  $\Psi^l(F)$  and  $\Psi^r(G)$ . The same type of arguments gives the following result for the composition of functors, by which coends and ends can be understood as a kind of categorified matrix multiplication:

**Corollary 3.7.** Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  and  $F': \mathcal{B} \rightarrow \mathcal{C}$  be left exact functors, and  $G: \mathcal{A} \rightarrow \mathcal{B}$  and  $G': \mathcal{B} \rightarrow \mathcal{C}$  right exact functors. Then there are isomorphisms

$$\int^{b \in \mathcal{B}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \bar{a} \boxtimes b, \Psi_{\mathcal{A}, \mathcal{B}}^l(F) \rangle \otimes_{\mathbb{k}} \mathcal{B}^{\text{opp}} \boxtimes \mathcal{C} \langle \bar{b} \boxtimes c, \Psi_{\mathcal{B}, \mathcal{C}}^l(F') \rangle \cong {}_c \langle c, F' \circ F(a) \rangle \quad (3.25)$$

and

$$\int^{b \in \mathcal{B}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \Psi_{\mathcal{A}, \mathcal{B}}^r(G), \bar{a} \boxtimes b \rangle \otimes_{\mathbb{k}} \mathcal{B}^{\text{opp}} \boxtimes \mathcal{C} \langle \Psi_{\mathcal{B}, \mathcal{C}}^r(G'), \bar{b} \boxtimes c \rangle \cong {}_c \langle G' \circ G(a), c \rangle \quad (3.26)$$

natural in  $a \in \mathcal{A}$  and in  $c \in \mathcal{C}$ . Analogously, there are natural isomorphisms

$$\int_{b \in \mathcal{B}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \bar{a} \boxtimes b, \Psi_{\mathcal{A}, \mathcal{B}}^1(F) \rangle^* \otimes_{\mathbb{k}} \mathcal{B}^{\text{opp}} \boxtimes \mathcal{C} \langle \bar{b} \boxtimes c, \Psi_{\mathcal{B}, \mathcal{C}}^1(F') \rangle^* \cong {}_c \langle c, F' \circ F(a) \rangle^* \quad (3.27)$$

and

$$\int_{b \in \mathcal{B}} \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \langle \Psi_{\mathcal{A}, \mathcal{B}}^r(G), \bar{a} \boxtimes b \rangle^* \otimes_{\mathbb{k}} \mathcal{B}^{\text{opp}} \boxtimes \mathcal{C} \langle \Psi_{\mathcal{B}, \mathcal{C}}^r(G'), \bar{b} \boxtimes c \rangle^* \cong {}_c \langle G' \circ G(a), c \rangle^*. \quad (3.28)$$

*Proof.* Invoking the isomorphism (3.15), the coend on the left hand side of (3.25) becomes  $\int_{b \in \mathcal{B}} \mathcal{B} \langle b, F(a) \rangle \otimes_{\mathbb{k}} {}_c \langle c, F'(b) \rangle$ . After using that the functor  $F'$  has a left adjoint  $F'^{\text{l.a.}}$  we can apply the convolution property of the Hom functor to reduce this expression to  $\mathcal{B} \langle F'^{\text{l.a.}}(c), F(a) \rangle \cong {}_c \langle c, F' \circ F(a) \rangle$ . This proves (3.25). To show the isomorphism (3.26) we use analogously the result (3.16) together with the fact that  $G'$  has a right adjoint. The other two isomorphisms are shown by the same type of argument.  $\square$

### 3.3 Adjunctions and (co)ends

Next we establish a compatibility between taking (co)ends and adjunctions.

**Lemma 3.8.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be finite linear categories. For  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{C})$  and  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{C})$  there are isomorphisms

$$\Psi^1(F) \equiv \int_{a \in \mathcal{A}} \bar{a} \boxtimes F(a) \cong \int_{c \in \mathcal{C}} \overline{F^{\text{l.a.}}(c)} \boxtimes c \quad (3.29)$$

and

$$\Psi^r(G) \equiv \int_{a \in \mathcal{A}} \bar{a} \boxtimes G(a) \cong \int_{c \in \mathcal{C}} \overline{G^{\text{r.a.}}(c)} \boxtimes c \quad (3.30)$$

of coends and of ends in  $\mathcal{A}^{\text{opp}} \boxtimes \mathcal{C}$ , respectively.

Moreover, for any two left exact functors  $F_1: \mathcal{A} \rightarrow \mathcal{B}$  and  $F_2: \mathcal{B} \rightarrow \mathcal{C}$  the two isomorphisms

$$\int_{a \in \mathcal{A}} \bar{a} \boxtimes F_2 \circ F_1(a) \xrightarrow{\cong} \int_{c \in \mathcal{C}} \overline{F_1^{\text{l.a.}} \circ F_2^{\text{l.a.}}(c)} \boxtimes c \quad (3.31)$$

resulting from the isomorphism (3.29) are equal. Analogously also the isomorphism in (3.30) is coherent in this sense.

*Proof.* Applying the functor  $\Phi^1$  to the left hand side of (3.29) gives  $\Phi^1 \circ \Psi^1(F) \cong F$ . The same functor is obtained on the right hand side: using that  $\Phi^1$ , being an equivalence, is right exact and thus preserves the coend, we have

$$\begin{aligned} \Phi^1 \left( \int_{c \in \mathcal{C}} \overline{F^{\text{l.a.}}(c)} \boxtimes c \right) &\cong \int_{c \in \mathcal{C}} \Phi^1(\overline{F^{\text{l.a.}}(c)} \boxtimes c) = \int_{c \in \mathcal{C}} \mathcal{M} \langle F^{\text{l.a.}}(c), - \rangle \otimes c \\ &\cong \int_{c \in \mathcal{C}} {}_c \langle c, F(-) \rangle \otimes c \stackrel{(2.14)}{\cong} F. \end{aligned} \quad (3.32)$$

The claim thus follows directly from the fact that  $\Phi^1$  is an equivalence. The isomorphism (3.30) is analogously obtained from the isomorphism

$$\begin{aligned} \Phi^r(\int_{c \in \mathcal{C}} \overline{G^{r.a.}(c)} \boxtimes c) &\cong \int_{c \in \mathcal{C}} \Phi^r(\overline{G^{r.a.}(c)} \boxtimes c) = \int_{c \in \mathcal{C}} c \langle -, G^{r.a.}(c) \rangle^* \otimes c \\ &\cong \int_{c \in \mathcal{C}} c \langle G(-), c \rangle^* \otimes c \stackrel{(2.15)}{\cong} G \end{aligned} \quad (3.33)$$

of functors. The coherence of these isomorphisms with respect to the composition of functors follows directly from the definition of the isomorphisms above.  $\square$

For a right exact functor  $G$ , the functor  $\Gamma^{lr}(G)$  is left exact and thus has a left adjoint, and analogously for a left exact functor  $F$ . Using adjoints and the triangle (3.6) of equivalences one obtains two potentially different equivalences from  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  to  $\mathcal{R}ex(\mathcal{B}, \mathcal{A})$ , namely mapping a right exact functor  $G$  either to  $\Gamma^{rl}(G^{r.a.})$  or to  $(\Gamma^{lr}(G))^{l.a.}$ . The next result, which uses Lemma 3.8, shows that these two functors in fact coincide, and likewise do the analogous equivalences of left exact functors.

**Corollary 3.9.** For any functor  $G \in \mathcal{R}ex(\mathcal{A}, \mathcal{B})$  the functor  $\Gamma^{rl}(G^{r.a.}) \in \mathcal{R}ex(\mathcal{B}, \mathcal{A})$  is left adjoint to the functor  $\Gamma^{lr}(G) \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$ , and for any  $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$  the functor  $\Gamma^{lr}(F^{l.a.}) \in \mathcal{L}ex(\mathcal{B}, \mathcal{A})$  is right adjoint to  $\Gamma^{rl}(F) \in \mathcal{R}ex(\mathcal{A}, \mathcal{B})$ .

*Proof.* The statements follow by direct computation. We have

$$\begin{aligned} \mathcal{B} \langle b, \Gamma^{lr}(G)(a) \rangle &= \mathcal{B} \langle b, \int_{c \in \mathcal{A}} \mathcal{A} \langle c, a \rangle \otimes G(c) \rangle \cong \int_{c \in \mathcal{A}} \mathcal{A} \langle c, a \rangle \otimes_{\mathbb{k}} \mathcal{B} \langle b, G(c) \rangle \\ &\cong \mathcal{A} \langle \int^{c \in \mathcal{A}} \mathcal{B} \langle b, G(c) \rangle^* \otimes c, a \rangle \\ &\stackrel{(3.29)}{\cong} \mathcal{A} \langle \int^{d \in \mathcal{B}} \mathcal{B} \langle b, d \rangle^* \otimes G^{r.a.}(d), a \rangle \cong \mathcal{A} \langle \Gamma^{rl}(G^{r.a.})(b), a \rangle \end{aligned} \quad (3.34)$$

for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ ; this shows the first of the two claimed adjunctions. The second adjunction can be seen analogously, or alternatively by re-interpreting (3.34), with  $G^{r.a.} \in \mathcal{R}ex(\mathcal{B}, \mathcal{A})$  playing the role of  $F$ .  $\square$

### 3.4 Coends over Deligne products

In the sequel we consider coends taken over a Deligne product of finite linear categories. Recall that for  $\mathcal{A}$  and  $\mathcal{B}$  finite linear categories, the functor categories  $\mathcal{L}ex(\mathcal{A}, \mathcal{B})$  and  $\mathcal{R}ex(\mathcal{A}, \mathcal{B})$  are finite linear as well. Also recall the notation  $\oint^\bullet$  for coends in categories of left exact functors which was introduced in Corollary 3.5; analogously we write  $\oint_\bullet$  for an end taken in a category of right exact functors. Our result is based on the following statement on the coend of the Hom functor of a Deligne product.

**Lemma 3.10.** For  $\mathcal{C} = \mathcal{A} \boxtimes \mathcal{B}$  a Deligne product of finite linear categories there are isomorphisms

$$\int^{c \in \mathcal{C}} c_{\text{opp}} \boxtimes c \langle \overline{y} \boxtimes x, \overline{c} \boxtimes c \rangle \cong c \langle x, y \rangle \cong \oint^{a \in \mathcal{A}} \int^{b \in \mathcal{B}} c_{\text{opp}} \boxtimes c \langle \overline{y} \boxtimes x, \overline{a} \boxtimes \overline{b} \boxtimes a \boxtimes b \rangle \quad (3.35)$$

and

$$\int_{c \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{y} \boxtimes x, \bar{c} \boxtimes c \rangle^* \cong {}_{\mathcal{C}} \langle x, y \rangle^* \cong \oint_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{y} \boxtimes x, \overline{a \boxtimes b} \boxtimes a \boxtimes b \rangle^* \quad (3.36)$$

natural in  $x \in \mathcal{C}$  and  $\bar{y} \in \mathcal{C}^{\text{opp}}$ .

*Proof.* The first of the two isomorphisms (3.35) follows from the convolution property (2.14) of the Hom functor after noting that  ${}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{x} \boxtimes y, \bar{c} \boxtimes c \rangle \cong {}_{\mathcal{C}} \langle y, c \rangle \otimes_{\mathbb{k}} {}_{\mathcal{C}} \langle c, x \rangle$ . To establish the second isomorphism, we choose finite-dimensional algebras  $A$  and  $B$  with equivalences  $\mathcal{A} \simeq A\text{-mod}$  and  $\mathcal{B} \simeq \text{mod-}B$ . Then we also have an equivalence  $\mathcal{A} \boxtimes \mathcal{B} \simeq A\text{-bimod-}B$ , and the right hand side of formula (3.35) gives the expression  $\oint^{m \in A\text{-mod}} \int^{n \in \text{mod-}B} \text{Hom}_{A|B}(x, m \otimes_{\mathbb{k}} n) \otimes_{\mathbb{k}} \text{Hom}_{A|B}(m \otimes_{\mathbb{k}} n, y)$  for  $x = {}_A x_B, y = {}_A y_B \in A\text{-bimod-}B$ ,  $m = {}_A m \in A\text{-mod}$  and  $n = n_B \in \text{mod-}B$ . The existence of these coends as well as the equivalence to the Hom functor is shown by the following chain of isomorphisms:

$$\begin{aligned} & \oint^{m \in A\text{-mod}} \int^{n \in \text{mod-}B} {}_{A|B} \langle x, m \otimes_{\mathbb{k}} n \rangle \otimes_{\mathbb{k}} {}_{A|B} \langle m \otimes_{\mathbb{k}} n, y \rangle \\ & \cong \oint^{m \in A\text{-mod}} \int^{n \in \text{mod-}B} {}_B \langle m^* \otimes_A x, n \rangle \otimes_{\mathbb{k}} {}_B \langle n, (y^* \otimes_A m)^* \rangle \\ & \cong \oint^{m \in A\text{-mod}} {}_B \langle m^* \otimes_A x, (y^* \otimes_A m)^* \rangle \cong \oint^{m \in A\text{-mod}} {}_{A|B} \langle (m \otimes_{\mathbb{k}} m^*) \otimes_A x, y \rangle \\ & \cong \oint^{m \in A\text{-mod}} {}_{A|A} \langle m \otimes_{\mathbb{k}} m^* \otimes_A (x \otimes_B y^*), A \rangle \\ & \stackrel{(3.22)}{\cong} {}_{A|A} \langle \int_{m \in A\text{-mod}} m \otimes_{\mathbb{k}} (m^* \otimes_A (x \otimes_B y^*)), A \rangle \cong {}_{A|A} \langle A \otimes_A x \otimes_B y^*, A \rangle \cong {}_{A|B} \langle x, y \rangle. \end{aligned} \quad (3.37)$$

Here in the first, third and fourth step we apply explicit formulas for the adjunctions inside the respective Hom spaces, the second step is the convolution property of the Hom functor in  $\text{mod-}B$ , and the fifth step uses the commutation of the Hom functor with the coend according to Corollary 3.5 for the right exact functor  $-\otimes_A x \otimes_B y^*: \text{mod-}A \rightarrow \text{mod-}A$ .

This proves (3.35). The first isomorphism in (3.36) is a direct consequence of the convolution property (2.15) of the dual Hom functor, while the second isomorphism in (3.36) follows from (3.35) by applying the duality functor and noting that dualizing turns a left exact coend into a right exact end.  $\square$

Using these results we obtain

**Lemma 3.11.** Let  $\mathcal{C} = \mathcal{A} \boxtimes \mathcal{B}$  and  $\mathcal{X}$  be finite linear categories. For  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{X}$  a left exact bilinear functor whose coend exists one has

$$\int_{c \in \mathcal{C}} F(\bar{c}, c) \cong \int_{a \in \mathcal{A}} \oint^{b \in \mathcal{B}} F(\bar{a} \boxtimes \bar{b}, a \boxtimes b). \quad (3.38)$$

Analogously, if the end of a right exact functor bilinear  $G: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{X}$  exists, then one has

$$\int_{c \in \mathcal{C}} G(\bar{c}, c) \cong \int_{a \in \mathcal{A}} \oint_{b \in \mathcal{B}} G(\bar{a} \boxtimes \bar{b}, a \boxtimes b). \quad (3.39)$$

*Proof.* By the convolution property of the Hom functor we have

$$\begin{aligned}
F(\bar{x}, y) &\cong \int^{\bar{c} \in \mathcal{C}^{\text{opp}}} {}_{\mathcal{C}^{\text{opp}}} \langle \bar{c}, \bar{x} \rangle \otimes_{\mathbb{k}} F(\bar{c}, y) \cong \int^{\bar{c} \in \mathcal{C}^{\text{opp}}} \int^{d \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}}} \langle \bar{c}, \bar{x} \rangle \otimes_{\mathbb{k}} {}_{\mathcal{C}} \langle d, y \rangle \otimes F(\bar{c}, d) \\
&\cong \int^{\bar{c} \in \mathcal{C}^{\text{opp}}} \int^{d \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \bar{x} \boxtimes y \rangle \otimes F(\bar{c}, d) \\
&\cong \oint^{\bar{c} \in \mathcal{C}^{\text{opp}}} \oint^{d \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \bar{x} \boxtimes y \rangle \otimes F(\bar{c}, d).
\end{aligned} \tag{3.40}$$

Here the last isomorphism holds because the respective functors are already left exact. The isomorphism (3.38) then follows by the calculation

$$\begin{aligned}
\int^{x \in \mathcal{C}} F(\bar{x}, x) &\cong \oint^{x \in \mathcal{C}} \oint^{\bar{c} \in \mathcal{C}^{\text{opp}}} \oint^{d \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \bar{x} \boxtimes x \rangle \otimes F(\bar{c}, d) \\
&\cong \oint^{\bar{c} \in \mathcal{C}^{\text{opp}}} \oint^{d \in \mathcal{C}} \oint^{x \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \bar{x} \boxtimes x \rangle \otimes F(\bar{c}, d) \\
&\stackrel{(3.35)}{\cong} \oint^{\bar{c} \in \mathcal{C}^{\text{opp}}} \oint^{d \in \mathcal{C}} \oint^{a \in \mathcal{A}} \oint^{b \in \mathcal{B}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \overline{a \boxtimes b} \boxtimes a \boxtimes b \rangle \otimes F(\bar{c}, d) \\
&\cong \oint^{a \in \mathcal{A}} \oint^{b \in \mathcal{B}} \oint^{\bar{c} \in \mathcal{C}^{\text{opp}}} \oint^{d \in \mathcal{C}} {}_{\mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}} \langle \bar{c} \boxtimes d, \overline{a \boxtimes b} \boxtimes a \boxtimes b \rangle \otimes F(\bar{c}, d) \\
&\cong \oint^{a \in \mathcal{A}} \oint^{b \in \mathcal{B}} F(\bar{a} \boxtimes \bar{b}, a \boxtimes b) \cong \int^{a \in \mathcal{A}} \oint^{b \in \mathcal{B}} F(\bar{a} \boxtimes \bar{b}, a \boxtimes b).
\end{aligned} \tag{3.41}$$

Here in the second and fourth line we invoke the Fubini theorem for left exact coends (see [Ly, Thm. B.2]). The isomorphism (3.39) is seen analogously.  $\square$

**Corollary 3.12.** For  $\mathcal{C} = \mathcal{A} \boxtimes \mathcal{B}$  the Deligne product of finite linear categories  $\mathcal{A}$  and  $\mathcal{B}$  the isomorphisms

$$\int^{c \in \mathcal{C}} \bar{c} \boxtimes c \cong \int^{a \in \mathcal{A}} \int^{b \in \mathcal{B}} \bar{a} \boxtimes \bar{b} \boxtimes a \boxtimes b \tag{3.42}$$

and

$$\int_{c \in \mathcal{C}} \bar{c} \boxtimes c \cong \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} \bar{a} \boxtimes \bar{b} \boxtimes a \boxtimes b \tag{3.43}$$

hold.

*Proof.* The functor  $\boxtimes: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{C}^{\text{opp}} \boxtimes \mathcal{C}$  is exact, hence both of the isomorphisms (3.38) and (3.39) are applicable. Moreover, for all finite linear categories  $\mathcal{X}$  and all objects  $x \in \mathcal{X}$  we have  $\oint^{b \in \mathcal{B}} x \boxtimes \bar{b} \boxtimes b \cong x \boxtimes \int^{b \in \mathcal{B}} \bar{b} \boxtimes b$  because in this case the left exact coend is also an ordinary coend. An analogous statement applies to the right exact end.  $\square$

With the help of this corollary we arrive at the following result, in which with respect to Lemma 3.11 the role of ends and coends is interchanged:

**Lemma 3.13.** For  $\mathcal{C} = \mathcal{A} \boxtimes \mathcal{B}$  and  $\mathcal{X}$  finite linear categories and  $F: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{X}$  a left exact bilinear functor whose end exists the isomorphism

$$\int_{c \in \mathcal{C}} F(\bar{c}, c) \cong \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} F(\bar{a} \boxtimes \bar{b}, a \boxtimes b) \tag{3.44}$$

of ends holds. Analogously, if the coend of a right exact bilinear functor  $G: \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{X}$  exists, then there is an isomorphism

$$\int^{c \in \mathcal{C}} G(\bar{c}, c) \cong \int^{a \in \mathcal{A}} \int^{b \in \mathcal{B}} G(\bar{a} \boxtimes \bar{b}, a \boxtimes b) \quad (3.45)$$

of coends.

*Proof.* Since  $F$  is left exact, it induces a left exact linear functor  $\widehat{F}: \mathcal{C}^{\text{opp}} \boxtimes \mathcal{C} \rightarrow \mathcal{X}$ . We then have

$$\begin{aligned} \int_{c \in \mathcal{C}} F(\bar{c}, c) &\cong \int_{c \in \mathcal{C}} \widehat{F}(\bar{c} \boxtimes c) \stackrel{(2.10)}{\cong} \widehat{F}\left(\int_{c \in \mathcal{C}} \bar{c} \boxtimes c\right) \stackrel{(3.43)}{\cong} \widehat{F}\left(\int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} \bar{a} \boxtimes \bar{b} \boxtimes a \boxtimes b\right) \\ &\stackrel{(2.10)}{\cong} \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} \widehat{F}(\bar{a} \boxtimes \bar{b} \boxtimes a \boxtimes b) \cong \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} F(\bar{a} \boxtimes \bar{b}, a \boxtimes b). \end{aligned} \quad (3.46)$$

The isomorphism (3.45) follows analogously with the help of (3.42).  $\square$

### 3.5 Nakayama functors for finite linear categories

Let  $A = (A, \mu, \eta)$  be a finite-dimensional  $\mathbb{k}$ -algebra; recall that we assume all modules and bimodules to be finite-dimensional. As in Section 2.1 we denote the finite  $\mathbb{k}$ -linear category of finite-dimensional left  $A$ -modules by  $A\text{-mod}$  and write the Hom functor on  $A\text{-mod}^{\text{opp}} \times A\text{-mod}$  as  ${}_A\langle -, - \rangle$ . We denote the algebra opposite to  $A$  by  $A^{\text{opp}}$ . In the sequel, whenever it is required in order for an expression to make sense that  $A$  or the dual vector space  $A^* = {}_{\mathbb{k}}\langle A, \mathbb{k} \rangle$  has the structure of a left or right  $A$ -module or of an  $A$ -bimodule, then  $A$  is to be regarded as being endowed with the regular left or right  $A$ -action(s), while  $A^*$  is regarded as being endowed with the duals of these actions (i.e., for short, as the *co-regular* left, right or bi-module).

The *Nakayama functor* of  $A\text{-mod}$  is the right exact endofunctor

$$N_A^r := {}_A\langle -, A \rangle^* \quad (3.47)$$

of  $A\text{-mod}$ . We also consider its left exact cousin, the endofunctor

$$N_A^l := {}_{A^{\text{opp}}}\langle -^*, A \rangle \quad (3.48)$$

of  $A\text{-mod}$ . These functors can also be described (see [AsSS, Lemma III.2.9]: and [Iv, Prop. 3.1]) by the isomorphic functors

$$N_A^r \cong A^* \otimes_A - \quad \text{and} \quad N_A^l \cong {}_A\langle A^*, - \rangle, \quad (3.49)$$

respectively, with  $A^*$  the co-regular  $A$ -bimodule. The restrictions of these two endofunctors to the subcategories of projective and of injective  $A$ -modules, respectively, are quasi-inverse equivalences  $A\text{-Projmod} \xrightarrow{\cong} A\text{-Injmod}$  [Iv, Sect. 3].

In this paper we are interested in finite linear categories  $\mathcal{X}$  as abstract categories, not in their particular realization as  $A\text{-mod}$  for some specific algebra  $A$ . Accordingly we wish to work with a purely categorical variant of the Nakayama functor and its left exact analogue. Note that the identity functor  $\text{id}_{\mathcal{X}}$  is both left and right exact, so for  $\mathcal{X}$  a finite linear category we can apply both of the functors (3.4) to it. We can thus give

**Definition 3.14.** The *Nakayama functor* of a finite linear category  $\mathcal{X}$  is the endofunctor

$$N_{\mathcal{X}}^r := \Gamma^{\text{rl}}(\text{id}_{\mathcal{X}}) = \int^{x \in \mathcal{X}} \mathcal{X}\langle -, x \rangle^* \otimes x \in \mathcal{R}ex(\mathcal{X}, \mathcal{X}), \quad (3.50)$$

i.e. the image of the identity functor  $\text{id}_{\mathcal{X}}$ , seen as a left exact functor, in  $\mathcal{R}ex(\mathcal{X}, \mathcal{X})$ .

The *left exact analogue of the Nakayama functor* of  $\mathcal{X}$  is the image

$$N_{\mathcal{X}}^l := \Gamma^{\text{lr}}(\text{id}_{\mathcal{X}}) = \int_{x \in \mathcal{X}} \mathcal{X}\langle x, - \rangle \otimes x \in \mathcal{L}ex(\mathcal{X}, \mathcal{X}) \quad (3.51)$$

of  $\text{id}_{\mathcal{X}}$ , seen as a right exact functor, in  $\mathcal{L}ex(\mathcal{X}, \mathcal{X})$ .

The following considerations will show that this terminology is appropriate. To connect the functors  $N_{\mathcal{X}}^l$  and  $N_{\mathcal{X}}^r$  to the representation theoretic structures occurring in (3.47)–(3.49), let us choose a finite-dimensional algebra  $A$  with an equivalence  $A\text{-mod} \simeq \mathcal{X}$ . From now on we tacitly identify  $\mathcal{X}$  with  $A\text{-mod}$ . Doing so we can state

**Lemma 3.15.** The Nakayama functor of a finite linear category  $\mathcal{X} \simeq A\text{-mod}$  satisfies

$$N_{\mathcal{X}}^r \cong ({}_A A_A)^* \otimes_A - . \quad (3.52)$$

*Proof.* The Nakayama functor  $N^r \equiv N_{\mathcal{X}}^r$  is right exact, hence upon the identification  $\mathcal{X} = A\text{-mod}$ , according to (R3) of Lemma 2.1 there is a natural isomorphism

$$N^r \cong N^r({}_A A_A) \otimes_A - . \quad (3.53)$$

Now for any  $y \in \mathcal{X}$  we have an isomorphism  ${}_A \langle {}_A A_A, y \rangle \cong y$  as left  $A$ -modules and hence an isomorphism  ${}_A \langle {}_A A_A, y \rangle^* \cong y^*$  of right  $A$ -modules. This implies that

$$N^r({}_A A_A) = \int^{y \in A\text{-mod}} {}_A \langle {}_A A_A, y \rangle^* \otimes y \cong \int^{y \in A\text{-mod}} y^* \otimes y \stackrel{(2.30)}{\cong} ({}_A A_A)^* \quad (3.54)$$

as  $A$ -bimodules. Combining this isomorphism with the expression (3.53) for  $N^r$  gives (3.52).  $\square$

Next we note that, by construction,  $N_{\mathcal{X}}^l$  is left exact and thus has a left adjoint, while  $N_{\mathcal{X}}^r$  is right exact and thus has a right adjoint. Taking  $G = \text{id}_{\mathcal{X}}$  in Corollary 3.9 we have in fact

**Lemma 3.16.** For any finite linear category  $\mathcal{X}$  the functor  $N_{\mathcal{X}}^r$  is left adjoint to  $N_{\mathcal{X}}^l$ .

We finally note that the expressions for the Nakayama functors have natural generalizations obtained by replacing the identity functor by an arbitrary linear functor. Indeed, given a right exact linear functor  $G: A\text{-mod} \rightarrow B\text{-mod}$ , the linear functor  $\Gamma^{\text{lr}}(G): A\text{-mod} \rightarrow B\text{-mod}$  can, with the help of the Peter-Weyl isomorphism (2.19), be written as

$$\begin{aligned} \Gamma^{\text{lr}}(G) &= \int_{m \in A\text{-mod}} {}_A \langle m, - \rangle \otimes G(m) \cong {}_A \langle \int^{m \in A\text{-mod}} G(m)^* \otimes m, - \rangle \\ &\cong {}_A \langle (\int_{m \in A\text{-mod}} G(m) \otimes m^*)^*, - \rangle \cong {}_A \langle G(A)^*, - \rangle. \end{aligned} \quad (3.55)$$

This provides the explicit form of the Eilenberg-Watts equivalence between right and left exact functors that exists according to Lemma 2.2(i). Note that the calculation in (3.55) still makes sense if  $G$  is just required to be a linear functor. Thus the prescription for  $\Gamma^{\text{lr}}$  in

(3.5) in fact defines a functor from the category  $\mathcal{F}un(A\text{-mod}, B\text{-mod})$  of *all* linear functors to  $\mathcal{L}ex(A\text{-mod}, B\text{-mod})$ . Similarly it follows from (2.21), using the description (2.1) of the tensor product over  $A$ , that

$$\begin{aligned}
\int^{m \in A\text{-mod}} {}_A\langle -, m \rangle^* \otimes F(m) &\cong \int^{m \in A\text{-mod}} {}_A\langle -, m \otimes_{\mathbb{k}} F(m)^* \rangle^* \\
&\stackrel{(2.8)}{\cong} \left( \int_{m \in A\text{-mod}} {}_A\langle -, m \otimes_{\mathbb{k}} F(m)^* \rangle \right)^* \cong {}_A\langle -, \int_{m \in A\text{-mod}} m \otimes_{\mathbb{k}} F(m)^* \rangle^* \\
&\cong {}_A\langle -, (\int^{m \in A\text{-mod}} F(m) \otimes_{\mathbb{k}} m^*)^* \rangle^* \stackrel{(2.21)}{\cong} {}_A\langle -, F(A^*)^* \rangle^* \stackrel{(2.1)}{\cong} F(A^*) \otimes_A -.
\end{aligned} \tag{3.56}$$

This shows that

$$\Gamma^{\text{rl}}(F) \cong F(A^*) \otimes_A - \tag{3.57}$$

and that the prescription for  $\Gamma^{\text{rl}}$  in (3.5) extends to a functor from  $\mathcal{F}un(A\text{-mod}, B\text{-mod})$  to  $\mathcal{R}ex(A\text{-mod}, B\text{-mod})$ .

**Remark 3.17.** For  $\mathcal{X}$  a finite linear category, the Nakayama functors and the equivalences appearing in the triangle (3.6) can be used to endow the category  $\mathcal{M} = \mathcal{X}^{\text{opp}} \boxtimes \mathcal{X}$  with the structure of a Grothendieck-Verdier (GV) category. Recall [BD] that a GV-structure on a monoidal category  $\mathcal{M}$  amounts to an object  $N \in \mathcal{M}$  and an equivalence  $D_N: \mathcal{M} \rightarrow \mathcal{M}^{\text{opp}}$  with a natural family of isomorphisms  ${}_{\mathcal{M}}\langle x \otimes y, N \rangle \cong {}_{\mathcal{M}}\langle y, D_N(x) \rangle$ .

In our case we use the equivalence of  $\mathcal{M}$  with the monoidal category  $\mathcal{R}ex(\mathcal{X}, \mathcal{X})$  to describe the GV-structure. The category  $\mathcal{R}ex(\mathcal{X}, \mathcal{X})$  has an obvious monoidal structure by composition of functors. For the object  $N$  we take the Nakayama functor  $N_{\mathcal{X}}^{\text{r}} \in \mathcal{R}ex(\mathcal{X}, \mathcal{X})$ . As the equivalence  $D_N$  we take the functor that maps  $G \in \mathcal{R}ex(\mathcal{X}, \mathcal{X})$  to the functor  $\Gamma^{\text{rl}}(G^{\text{r.a.}}) \in \mathcal{R}ex(\mathcal{X}, \mathcal{X})^{\text{opp}}$ . Using that  $N_{\mathcal{X}}^{\text{r}} = \Gamma^{\text{rl}}(\text{id}_{\mathcal{X}})$  with  $\Gamma^{\text{rl}} = \Phi^{\text{r}} \circ \Psi^{\text{l}}$  and invoking Corollary 3.5, for  $G, H \in \mathcal{R}ex(\mathcal{X}, \mathcal{X})$  we obtain

$$\begin{aligned}
\text{Nat}_{\mathcal{R}ex}(H \circ G, N_{\mathcal{X}}^{\text{r}}) &= \text{Nat}_{\mathcal{R}ex}(H \circ G, \Phi^{\text{r}} \circ \Psi^{\text{l}}(\text{id}_{\mathcal{X}})) \cong {}_{\mathcal{M}}\langle \Psi^{\text{r}}(H \circ G), \Psi^{\text{l}}(\text{id}_{\mathcal{X}}) \rangle \\
&\stackrel{(3.2)}{\cong} {}_{\mathcal{M}}\langle \int_{x \in \mathcal{X}} \bar{x} \boxtimes H \circ G(x), \int_{y \in \mathcal{X}} \bar{y} \boxtimes y \rangle \\
&\stackrel{(3.21), (3.22)}{\cong} \oint^{x \in \mathcal{X}} \oint^{y \in \mathcal{X}} {}_{\mathcal{M}}\langle \bar{x} \boxtimes H \circ G(x), \bar{y} \boxtimes y \rangle \\
&\cong \oint^{x \in \mathcal{X}} \oint^{y \in \mathcal{X}} {}_{\mathcal{X}}\langle y, x \rangle \otimes_{\mathbb{k}} {}_{\mathcal{X}}\langle H \circ G(x), y \rangle
\end{aligned} \tag{3.58}$$

and analogously (using that  $H$  has a right adjoint  $H^{\text{r.a.}}$ )

$$\begin{aligned}
\text{Nat}_{\mathcal{R}ex}(G, \Gamma^{\text{rl}}(H^{\text{r.a.}})) &= \text{Nat}_{\mathcal{R}ex}(G, \Phi^{\text{r}} \circ \Psi^{\text{l}}(H^{\text{r.a.}})) \cong {}_{\mathcal{M}}\langle \Psi^{\text{r}}(G), \Psi^{\text{l}}(H^{\text{r.a.}}) \rangle \\
&\cong \oint^{x \in \mathcal{X}} \oint^{y \in \mathcal{X}} {}_{\mathcal{X}}\langle y, x \rangle \otimes_{\mathbb{k}} {}_{\mathcal{X}}\langle G(x), H^{\text{r.a.}}(y) \rangle.
\end{aligned} \tag{3.59}$$

When combined with the adjunction we thus arrive at a natural isomorphism

$$\text{Nat}_{\mathcal{R}ex}(H \circ G, N_{\mathcal{X}}^{\text{r}}) \cong \text{Nat}_{\mathcal{R}ex}(G, \Gamma^{\text{rl}}(H^{\text{r.a.}})) \tag{3.60}$$

for all  $G, H \in \mathcal{R}ex(\mathcal{X}, \mathcal{X})$ .

General results on GV-categories [Man, Prop. 1.2] imply that the equivalence  $D_N$  induces another monoidal structure on  $\mathcal{R}ex(\mathcal{X}, \mathcal{X})$  for which the Nakayama functor  $N_{\mathcal{X}}^{\text{r}}$  is the monoidal unit. This monoidal structure can be recognized as the one that is obtained by identifying  $\mathcal{R}ex(\mathcal{X}, \mathcal{X})$ , via the functors  $\Gamma^{\text{lr}}$  and  $\Gamma^{\text{rl}}$ , with the monoidal category  $\mathcal{L}ex(\mathcal{X}, \mathcal{X})$ .



### 3.6 Properties of Nakayama functors

In this subsection we use the categorical formulation of the Nakayama functor to describe its behavior with respect to the composition with functors, to the Deligne product, and to taking opposites, and we characterize the finite linear categories whose Nakayama functor is an equivalence. We also determine the Nakayama functors of the categories of left (right) exact endofunctors between linear categories. The corresponding results in the case of modules over algebras are rather evident. Still, the categorical formulation has advantages, for instance when discussing the coherence natural isomorphisms in the following result.

When the double left adjoint of a functor  $F$  exists, we denote it by  $F^{\text{l.l.a.}}$ ; analogously we denote the double right adjoint by  $G^{\text{r.r.a.}}$ . We have

**Theorem 3.18.** (i) Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor between finite linear categories having a left exact left adjoint  $F^{\text{l.a.}}$  whose left adjoint is again left exact. Then there is a natural isomorphism

$$\varphi_F^{\text{l}}: \quad N_{\mathcal{B}}^{\text{l}} \circ F \cong F^{\text{l.l.a.}} \circ N_{\mathcal{A}}^{\text{l}} \quad (3.61)$$

of functors. Analogously, for a right exact functor  $G: \mathcal{A} \rightarrow \mathcal{B}$  having a right exact right adjoint  $G^{\text{r.a.}}$  whose right adjoint is again right exact, there is a natural isomorphism

$$\varphi_G^{\text{r}}: \quad N_{\mathcal{B}}^{\text{r}} \circ G \cong G^{\text{r.r.a.}} \circ N_{\mathcal{A}}^{\text{r}}. \quad (3.62)$$

(ii) The isomorphisms  $\varphi_F^{\text{l}}$  and  $\varphi_G^{\text{r}}$  are coherent in the following sense: If  $F_1: \mathcal{A} \rightarrow \mathcal{B}$  and  $F_2: \mathcal{B} \rightarrow \mathcal{C}$  are left exact functors, then the diagram

$$\begin{array}{ccc} N_{\mathcal{C}}^{\text{l}} \circ F_2 \circ F_1 & \xrightarrow{\varphi_{F_2}^{\text{l}} \circ \text{id}} & F_2^{\text{l.l.a.}} \circ N_{\mathcal{B}}^{\text{l}} \circ F_1 \\ \downarrow \varphi_{F_2 \circ F_1}^{\text{l}} & & \downarrow \text{id} \circ \varphi_{F_1}^{\text{l}} \\ (F_2 \circ F_1)^{\text{l.l.a.}} \circ N_{\mathcal{A}}^{\text{l}} & \xrightarrow{\cong} & F_2^{\text{l.l.a.}} \circ F_1^{\text{l.l.a.}} \circ N_{\mathcal{A}}^{\text{l}} \end{array} \quad (3.63)$$

commutes, with the unnamed isomorphism in the lower row obtained from the canonical isomorphism  $(F_2 \circ F_1)^{\text{l.l.a.}} \cong F_2^{\text{l.l.a.}} \circ F_1^{\text{l.l.a.}}$ . The analogous diagram for  $\varphi^{\text{r}}$  commutes as well.

*Proof.* For  $x \in \mathcal{A}$  we have

$$\begin{aligned} N_{\mathcal{B}}^{\text{l}}(F(x)) &= \int_{b \in \mathcal{B}} {}_{\mathcal{B}}\langle b, F(x) \rangle \otimes b \cong \int_{b \in \mathcal{B}} {}_{\mathcal{A}}\langle F^{\text{l.a.}}(b), x \rangle \otimes b \\ &\cong [{}_{\mathcal{A}}\langle \text{?}, x \rangle \otimes ?] \left( \int_{b \in \mathcal{B}} \overline{F^{\text{l.a.}}(b)} \boxtimes b \right) \\ &\stackrel{(3.30)}{\cong} [{}_{\mathcal{A}}\langle \text{?}, x \rangle \otimes ?] \left( \int_{a \in \mathcal{A}} \bar{a} \boxtimes F^{\text{l.l.a.}}(a) \right) \cong \int_{a \in \mathcal{A}} {}_{\mathcal{A}}\langle a, x \rangle \otimes F^{\text{l.l.a.}}(a) \\ &\cong F^{\text{l.l.a.}} \left( \int_{a \in \mathcal{A}} {}_{\mathcal{A}}\langle a, x \rangle \otimes a \right) = F^{\text{l.l.a.}}(N_{\mathcal{A}}^{\text{l}}(x)) \end{aligned} \quad (3.64)$$

naturally in  $x$ . Here in the second and third line we use that  ${}_{\mathcal{A}}\langle \text{?}, x \rangle \otimes ?$  is left exact and thus commutes with the end, and the first isomorphism in the third line is (3.30) with  $G = F^{\text{l.l.a.}}$ ; the last line holds because  $F^{\text{l.l.a.}}$  is required to be left exact.

The coherence statement with respect to the composition of the functors follows easily from the one in Lemma 3.8.  $\square$

Since the double left adjoint of the right adjoint of a functor is just a left adjoint, we have directly

**Corollary 3.19.** (i) Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor whose left adjoint is again left exact. Then there is a natural isomorphism  $N_{\mathcal{A}}^1 \circ F^{r.a.} \cong F^{l.a.} \circ N_{\mathcal{B}}^1$ . Similarly, if  $G: \mathcal{A} \rightarrow \mathcal{B}$  is an exact functor whose right adjoint is again right exact, then there is a natural isomorphism  $N_{\mathcal{A}}^r \circ G^{l.a.} \cong G^{r.a.} \circ N_{\mathcal{B}}^r$ .

(ii) If in addition  $N_{\mathcal{A}}^1$  is an equivalence, then one can express the right adjoint of  $F$  in terms of the left adjoint as

$$F^{r.a.} \cong N_{\mathcal{A}}^r \circ F^{l.a.} \circ N_{\mathcal{B}}^1. \quad (3.65)$$

And similarly, if  $N_{\mathcal{B}}^1$  is an equivalence, then  $F^{l.a.} \cong N_{\mathcal{A}}^1 \circ F^{r.a.} \circ N_{\mathcal{B}}^r$ .

We continue with further categorical properties of the Nakayama functors.

**Proposition 3.20.** (i) The Nakayama functors for the Deligne product  $\mathcal{A} \boxtimes \mathcal{B}$  of two finite linear categories can be expressed in terms of the Nakayama functors for its factors as

$$N_{\mathcal{A} \boxtimes \mathcal{B}}^1 \cong N_{\mathcal{A}}^1 \boxtimes N_{\mathcal{B}}^1 \quad \text{and} \quad N_{\mathcal{A} \boxtimes \mathcal{B}}^r \cong N_{\mathcal{A}}^r \boxtimes N_{\mathcal{B}}^r. \quad (3.66)$$

(ii) The Nakayama functors of the opposite category of a finite linear category  $\mathcal{A}$  are

$$N_{\mathcal{A}^{\text{opp}}}^1 \cong (N_{\mathcal{A}}^r)^{\text{opp}} \quad \text{and} \quad N_{\mathcal{A}^{\text{opp}}}^r \cong (N_{\mathcal{A}}^1)^{\text{opp}}. \quad (3.67)$$

*Proof.* The claim (i) is seen by the computation

$$\begin{aligned} N_{\mathcal{A} \boxtimes \mathcal{B}}^1 &\equiv \int_{x \in \mathcal{A} \boxtimes \mathcal{B}} \mathcal{A} \boxtimes \mathcal{B} \langle x, - \rangle \otimes x \stackrel{(3.44)}{\cong} \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} \mathcal{A} \boxtimes \mathcal{B} \langle a \boxtimes b, - \rangle \otimes (a \boxtimes b) \\ &\cong \int_{a \in \mathcal{A}} \mathcal{A} \langle a, - \rangle \otimes a \boxtimes \int_{b \in \mathcal{B}} \mathcal{B} \langle b, - \rangle \otimes b = N_{\mathcal{A}}^1 \boxtimes N_{\mathcal{B}}^1. \end{aligned} \quad (3.68)$$

The statement for  $N_{\mathcal{A} \boxtimes \mathcal{B}}^r$  follows by taking left adjoints in formula (3.68) which, when combined with Lemma 3.16 implies that

$$N_{\mathcal{A} \boxtimes \mathcal{B}}^r \cong (N_{\mathcal{A} \boxtimes \mathcal{B}}^1)^{l.a.} \cong (N_{\mathcal{A}}^1 \boxtimes N_{\mathcal{B}}^1)^{l.a.} \cong (N_{\mathcal{A}}^1)^{l.a.} \boxtimes (N_{\mathcal{B}}^1)^{l.a.} \cong N_{\mathcal{A}}^r \boxtimes N_{\mathcal{B}}^r. \quad (3.69)$$

To show the first isomorphism in (ii), for  $\bar{a} \in \mathcal{A}^{\text{opp}}$  we compute

$$\begin{aligned} N_{\mathcal{A}^{\text{opp}}}^1(\bar{a}) &= \int_{\bar{c} \in \mathcal{A}^{\text{opp}}} \mathcal{A}^{\text{opp}} \langle \bar{c}, \bar{a} \rangle \otimes \bar{c} \cong \int_{c \in \mathcal{A}} \overline{\mathcal{A} \langle \bar{a}, c \rangle^* \otimes c} \\ &\cong \overline{\int_{c \in \mathcal{A}} \mathcal{A} \langle \bar{a}, c \rangle^* \otimes c} = (N_{\mathcal{A}}^r)^{\text{opp}}(\bar{a}). \end{aligned} \quad (3.70)$$

The second isomorphism in (ii) follows from the first by interchanging  $\mathcal{A}$  with  $\mathcal{A}^{\text{opp}}$ .  $\square$

Under the equivalences (3.2) of the Deligne product of linear categories to the categories of left and right exact functors, the Nakayama functor has the following explicit description.

**Lemma 3.21.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be finite linear categories. There are canonical isomorphisms

$$N_{\mathcal{L}ex(\mathcal{A}, \mathcal{B})}^1(F) \cong N_{\mathcal{B}}^1 \circ F \circ N_{\mathcal{A}}^1 \quad \text{for } F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B}) \quad (3.71)$$

and

$$N_{\mathcal{R}ex(\mathcal{A}, \mathcal{B})}^r(G) \cong N_{\mathcal{B}}^r \circ G \circ N_{\mathcal{A}}^r \quad \text{for } G \in \mathcal{R}ex(\mathcal{A}, \mathcal{B}). \quad (3.72)$$

*Proof.* The equivalence  $\Phi^1: \mathcal{A}^{\text{opp}} \boxtimes \mathcal{B} \rightarrow \mathcal{L}ex(\mathcal{A}, \mathcal{B})$  meets the requirements of Theorem 3.18. It follows that there is a natural isomorphism  $N_{\mathcal{L}ex(\mathcal{A}, \mathcal{B})}^1 \circ \Phi^1 \cong \Phi^1 \circ N_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}}^1$ . By Proposition 3.20 we have  $N_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}}^1 \cong (N_{\mathcal{A}}^r)^{\text{opp}} \boxtimes N_{\mathcal{B}}^l$ .

For linear endofunctors  $F$  of  $\mathcal{A}$  and  $G$  of  $\mathcal{B}$  denote by  $L_G$  and  $R_F$  the linear endofunctors of  $\mathcal{L}ex(\mathcal{A}, \mathcal{B})$  given by pre- and post-composition with  $G$  and  $F$ , respectively. We compute

$$\begin{aligned} \Phi^1 \circ N_{\mathcal{A}^{\text{opp}} \boxtimes \mathcal{B}}^1(\bar{a} \boxtimes b) &\cong {}_{\mathcal{A}}\langle N_{\mathcal{A}}^r(a), - \rangle \otimes N_{\mathcal{B}}^l(b) \\ &\cong {}_{\mathcal{A}}\langle a, N_{\mathcal{A}}^l(-) \rangle \otimes N_{\mathcal{B}}^l(b) = L_{N_{\mathcal{B}}^l} \circ R_{N_{\mathcal{A}}^l} \circ \Phi^1(\bar{a} \boxtimes b), \end{aligned} \quad (3.73)$$

where the second isomorphism holds by Lemma 3.16. This is isomorphic to  $N_{\mathcal{L}ex(\mathcal{A}, \mathcal{B})}^1 \circ \Phi^1(\bar{a} \boxtimes b)$ .

Since  $\Phi^1$  is an equivalence, we conclude that there are isomorphisms (3.71).

The isomorphisms (3.72) follow analogously by considering the equivalence  $\Phi^r$ .  $\square$

A natural question is under which conditions the Nakayama functors  $N_{\mathcal{X}}^l$  and  $N_{\mathcal{X}}^r$  are equivalences. Consider first the case that  $\mathcal{X}$  is the category  $A\text{-mod}$  of finite-dimensional modules over some finite-dimensional algebra  $A$ . We have (see e.g. [AuRS, Prop. IV.3.1])

**Lemma 3.22.** The following properties of a finite-dimensional  $\mathbb{k}$ -algebra  $A$  are equivalent:

- (i) The Nakayama functor (3.47) and the functor (3.48) are quasi-inverse.
- (ii)  $A$  is injective as a left  $A$ -module,
- (iii)  $A$  is injective as a right  $A$ -module,
- (iv) The class of (left or right) projective modules coincides with the class of (left or right) injective modules.

An algebra with any (and thus all) of these properties is called *self-injective*. For self-injective  $A$  the functors (3.47) and (3.48) are in particular exact and there are natural isomorphisms [Iv, Thm. 3.3]

$$N_A^r \cong {}_A\langle {}_A\langle A^*, A \rangle, - \rangle \quad \text{and} \quad N_A^l \cong {}_A\langle A^*, A \rangle \otimes_A -, \quad (3.74)$$

where  ${}_A\langle A^*, A \rangle$  is seen as a bimodule and hence  ${}_A\langle {}_A\langle A^*, A \rangle, {}_A m \rangle$  as a left module.

If  $A$  has a structure of Frobenius algebra, then it is in particular self-injective. The Nakayama automorphism of a Frobenius algebra  $A$  with Frobenius form  $\kappa$  is the automorphism  $\nu$  of  $A$  defined by the equalities  $\kappa(\alpha, \nu(\beta)) = \kappa(\beta, \alpha)$  for all  $\alpha, \beta \in A$ . The functors  $N_A^r$  and  $N_A^l$  amount to twisting the action of  $A$  on a module by the Nakayama automorphism  $\nu$  of  $A$  and by  $\nu^{-1}$ , respectively. It follows that if  $A$  is Frobenius, then  $A^*$  and  ${}_A\langle A^*, A \rangle$  are isomorphic as bimodules to the bimodules obtained from the regular bimodule  ${}_A A_A$  by twisting the right and left  $A$ -action, respectively, by the Nakayama automorphism [Iv, Cor. 3.4]. A Frobenius algebra is symmetric iff the Nakayama automorphism is inner, and thus iff  $A^*$  and  $A$  are isomorphic as bimodules, and iff the Nakayama functor is isomorphic to the identity functor.

The properties of being self-injective and of having the structure of a symmetric Frobenius algebra are Morita invariant. Accordingly, following a suggestion by Shimizu [Sh3] we give

**Definition 3.23.** (i) A finite linear category is called *self-injective* iff it is equivalent as a linear category to the category of modules over a self-injective  $\mathbb{k}$ -algebra.

(ii) A finite linear category is called *symmetric Frobenius* iff it is equivalent as a linear category to the category of modules over a symmetric Frobenius  $\mathbb{k}$ -algebra.

Combining the previous statements with Lemma 3.15 we conclude:

**Proposition 3.24.** Let  $\mathcal{X}$  be a finite linear category.

- (i) The Nakayama functors  $N_{\mathcal{X}}^l$  and  $N_{\mathcal{X}}^r$  are equivalences iff  $\mathcal{X}$  is self-injective. In this case they are quasi-inverse to each other.
- (ii)  $N_{\mathcal{X}}^l$  and  $N_{\mathcal{X}}^r$  are isomorphic to the identity functor iff  $\mathcal{X}$  is symmetric Frobenius.

In fact, structures of a symmetric Frobenius algebra on an algebra  $A$  over a field are in bijection with bimodule isomorphisms from  $A$  to  $A^*$ . Thus for  $\mathcal{X} \simeq A\text{-mod}$  there is a bijection between isomorphisms from  $N_{\mathcal{X}}^l$  (or  $N_{\mathcal{X}}^r$ ) to  $\text{id}_{\mathcal{X}}$  and structures of a symmetric Frobenius algebra on  $A$ . It is thus tempting to call, for  $\mathcal{X}$  a finite linear category, an isomorphism between (say)  $N_{\mathcal{X}}^r$  and the identity functor a *symmetric Frobenius structure* on  $\mathcal{X}$ .

### 3.7 Nakayama functors and dualities

Taking linear categories as objects, either right or left exact linear functors between them as 1-morphisms, and the Deligne product as a monoidal structure, one obtains two monoidal 2-categories  $\mathcal{R}ex$  ( $\mathcal{L}ex$ ) of linear categories, right (left) exact linear functors and natural transformations. (The restriction to either left or right exact functors is required because the Deligne product of a left and a right exact functor is not defined, in general.) In both cases the symmetric monoidal structure is the Deligne product. By taking isomorphism classes of functors these 2-categories give two monoidal categories  $h\mathcal{R}ex$  and  $h\mathcal{L}ex$ . In the sequel we show that a linear category and its opposite are dual both in  $h\mathcal{R}ex$  and in  $h\mathcal{L}ex$  and express the Nakayama functors as combinations of the corresponding duality structures in  $\mathcal{R}ex$  and  $\mathcal{L}ex$ . Thus for any finite linear category  $\mathcal{M}$  we consider the pair

$$\begin{aligned} b_{\mathcal{M}} : \text{vect} &\longrightarrow \mathcal{M} \boxtimes \mathcal{M}^{\text{opp}} & \text{and} & & d_{\mathcal{M}} : \mathcal{M}^{\text{opp}} \boxtimes \mathcal{M} &\longrightarrow \text{vect} \\ v &\longmapsto v \otimes \int_{m \in \mathcal{M}} m \boxtimes \bar{m} & & & \bar{m} \boxtimes m' &\longmapsto {}_{\mathcal{M}}\langle m', m \rangle^* \end{aligned} \quad (3.75)$$

of functors, and similarly the pair

$$\begin{aligned} \tilde{b}_{\mathcal{M}} : \text{vect} &\longrightarrow \mathcal{M}^{\text{opp}} \boxtimes \mathcal{M} & \text{and} & & \tilde{d}_{\mathcal{M}} : \mathcal{M} \boxtimes \mathcal{M}^{\text{opp}} &\longrightarrow \text{vect} \\ v &\longmapsto v \otimes \int_{m \in \mathcal{M}} \bar{m} \boxtimes m & & & m \boxtimes \bar{m}' &\longmapsto {}_{\mathcal{M}}\langle m', m \rangle. \end{aligned} \quad (3.76)$$

Being functors from  $\text{vect}$  to a linear category,  $b_{\mathcal{M}}$  and  $\tilde{b}_{\mathcal{M}}$  are exact, while by definition  $d_{\mathcal{M}}$  is right exact and  $\tilde{d}_{\mathcal{M}}$  is left exact.

We can express the functors  $N_{\mathcal{M}}^l$  and  $N_{\mathcal{M}}^r$  in terms of these functors as follows. Denote by  $\sigma_{\mathcal{M}, \mathcal{N}} : m \boxtimes n \mapsto n \boxtimes m$  the symmetry of the Deligne product  $\mathcal{M} \boxtimes \mathcal{N} \cong \mathcal{N} \boxtimes \mathcal{M}$  and abbreviate  $\sigma_{\mathcal{M}} := \sigma_{\mathcal{M}, \mathcal{M}^{\text{opp}}}$ . Then

$$N_{\mathcal{M}}^l = (\tilde{d}_{\mathcal{M}} \boxtimes \text{id}_{\mathcal{M}}) \circ [\text{id}_{\mathcal{M}} \boxtimes (\sigma_{\mathcal{M}} \circ b_{\mathcal{M}})] \quad (3.77)$$

and

$$N_{\mathcal{M}}^r = (\text{id}_{\mathcal{M}} \boxtimes d_{\mathcal{M}}) \circ [(\sigma_{\mathcal{M}^{\text{opp}}} \circ \tilde{b}_{\mathcal{M}}) \boxtimes \text{id}_{\mathcal{M}}]. \quad (3.78)$$

**Lemma 3.25.** The functor  $\tilde{d}_{\mathcal{M}}$  is right adjoint to  $b_{\mathcal{M}}$ , and  $d_{\mathcal{M}}$  is left adjoint to  $\tilde{b}_{\mathcal{M}}$ .

*Proof.* By definition we have

$$\tilde{b}_{\mathcal{M}} = - \otimes \Psi^l(\text{id}_{\mathcal{M}}) \quad \text{and} \quad \sigma_{\mathcal{M}} \circ b_{\mathcal{M}} \cong - \otimes \Psi^r(\text{id}_{\mathcal{M}}). \quad (3.79)$$

For any  $v \in \text{vect}$ ,  $x \in \mathcal{M}$  and  $\bar{y} \in \mathcal{M}^{\text{opp}}$  the right adjoint  $b_{\mathcal{M}}^{\text{r.a.}}$  satisfies

$$\begin{aligned} v^* \otimes_{\mathbb{k}} b_{\mathcal{M}}^{\text{r.a.}}(x \boxtimes \bar{y}) &\cong_{\mathbb{k}} \langle v, (b_{\mathcal{M}}^{\text{r.a.}} \circ \sigma_{\mathcal{M}^{\text{opp}}})(\bar{y} \boxtimes x) \rangle \cong_{\mathcal{M}^{\text{opp}} \boxtimes \mathcal{M}} \langle \sigma_{\mathcal{M}} \circ b_{\mathcal{M}}(v), \bar{y} \boxtimes x \rangle \\ &\cong v^* \otimes_{\mathbb{k}} {}_{\mathcal{M}^{\text{opp}} \boxtimes \mathcal{M}} \langle \Psi^{\text{r}}(\text{id}_{\mathcal{M}}), \bar{y} \boxtimes x \rangle \stackrel{(3.16)}{\cong} v^* \otimes_{\mathbb{k}} {}_{\mathcal{M}} \langle y, x \rangle. \end{aligned} \quad (3.80)$$

Thus  $b_{\mathcal{M}}^{\text{r.a.}}(x \boxtimes \bar{y}) \cong_{\mathcal{M}} \langle y, x \rangle = \tilde{d}_{\mathcal{M}}(x \boxtimes \bar{y})$ . Similarly, for  $\tilde{b}_{\mathcal{M}}^{\text{l.a.}}$  we have

$$\begin{aligned} v \otimes_{\mathbb{k}} (\tilde{b}_{\mathcal{M}}^{\text{l.a.}}(\bar{y} \boxtimes x))^* &\cong_{\mathbb{k}} \langle \tilde{b}_{\mathcal{M}}^{\text{l.a.}}(\bar{y} \boxtimes x), v \rangle \\ &\cong v \otimes_{\mathbb{k}} {}_{\mathcal{M}^{\text{opp}} \boxtimes \mathcal{M}} \langle \bar{y} \boxtimes x, \Psi^{\text{l}}(\text{id}_{\mathcal{M}}) \rangle \stackrel{(3.15)}{\cong} v \otimes_{\mathbb{k}} {}_{\mathcal{M}} \langle x, y \rangle \end{aligned} \quad (3.81)$$

and thus  $\tilde{b}_{\mathcal{M}}^{\text{l.a.}}(\bar{y} \boxtimes x) \cong_{\mathcal{M}} \langle x, y \rangle = d_{\mathcal{M}}(\bar{y} \boxtimes x)$ .  $\square$

Similar calculations show that the other adjoints of  $b$  and  $\tilde{b}$  are given by

$$\tilde{b}_{\mathcal{M}}^{\text{r.a.}}(\bar{y} \boxtimes x) \cong \int_{m \in \mathcal{M}} {}_{\mathcal{M}} \langle y, m \rangle \otimes_{\mathbb{k}} {}_{\mathcal{M}} \langle m, x \rangle \quad (3.82)$$

and

$$b_{\mathcal{M}}^{\text{l.a.}}(x \boxtimes \bar{y}) \cong \left( \int_{m \in \mathcal{M}} {}_{\mathcal{M}} \langle x, m \rangle \otimes_{\mathbb{k}} {}_{\mathcal{M}} \langle m, y \rangle \right)^*. \quad (3.83)$$

**Remark 3.26.** The functors (3.75) and (3.76) satisfy zigzag identities up to natural isomorphisms. In the case of (3.75) we have natural isomorphisms

$$\begin{aligned} (\text{id}_{\mathcal{M}} \boxtimes d_{\mathcal{M}}) \circ (b_{\mathcal{M}} \boxtimes \text{id}_{\mathcal{M}}) : m' &\longmapsto \int_{m \in \mathcal{M}} m \otimes {}_{\mathcal{M}} \langle m', m \rangle^* \cong m' \quad \text{and} \\ (d_{\mathcal{M}} \boxtimes \text{id}_{\mathcal{M}^{\text{opp}}}) \circ (\text{id}_{\mathcal{M}^{\text{opp}}} \boxtimes b_{\mathcal{M}}) : \bar{m}' &\longmapsto \int_{m \in \mathcal{M}} {}_{\mathcal{M}} \langle m, m' \rangle^* \otimes \bar{m} \\ &= \int_{\bar{m} \in \mathcal{M}^{\text{opp}}} {}_{\mathcal{M}^{\text{opp}}} \langle \bar{m}', \bar{m} \rangle^* \otimes \bar{m} \cong \bar{m}', \end{aligned} \quad (3.84)$$

where in both cases we use the convolution property (as well as (2.10) applied to the Deligne product, which is an exact functor). The calculation for the functors (3.76) is similar. In view of these identities one may wish to think about the functors (3.75) as furnishing a notion of duality for the functor category  $\mathcal{R}ex(\mathcal{M}, \mathcal{N})$  (and analogously (3.76) for  $\mathcal{L}ex(\mathcal{M}, \mathcal{N})$ ). Accordingly one may put

$$G^* := (d_{\mathcal{N}} \boxtimes \text{id}_{\mathcal{M}^{\text{opp}}}) \circ (\text{id}_{\mathcal{N}^{\text{opp}}} \boxtimes G \boxtimes \text{id}_{\mathcal{M}^{\text{opp}}}) \circ (\text{id}_{\mathcal{N}^{\text{opp}}} \boxtimes b_{\mathcal{M}}) : \mathcal{N}^{\text{opp}} \rightarrow \mathcal{M}^{\text{opp}} \quad (3.85)$$

for any  $G \in \mathcal{F}un(\mathcal{M}, \mathcal{N})$ ; this gives  $G^*(\bar{n}) = \int_{m \in \mathcal{M}} {}_{\mathcal{N}} \langle G(m), n \rangle^* \otimes \bar{m}$ . If  $G$  is right exact, then it has a right adjoint  $G^{\text{r.a.}}$ , and by direct calculation one concludes that  $G^* \cong (G^{\text{r.a.}})^{\text{opp}}$ . Note that  $G^{\text{r.a.}}$  is left exact, so its opposite  $G^{\text{r.a.opp}}$  is right exact, hence the mapping (3.85) sends right exact functors to right exact ones. Further, noting that  $(G^{\text{r.a.}})^{\text{opp}} \cong (G^{\text{opp}})^{\text{l.a.}}$ , it follows in particular that

$$G^{**} \cong (((G^{\text{r.a.}})^{\text{opp}})^{\text{r.a.}})^{\text{opp}} \cong (((G^{\text{opp}})^{\text{l.a.}})^{\text{r.a.}})^{\text{opp}} \cong (G^{\text{opp}})^{\text{opp}} \cong G. \quad (3.86)$$

## 4 Eilenberg-Watts calculus for module categories

### 4.1 Module categories

We now study the compatibility between the constructions considered above and the structure of a module category over a finite tensor category on the finite linear category in question. Recall that a (left) module category over a finite tensor category  $\mathcal{A}$  (or, for short, an  $\mathcal{A}$ -module), is a finite linear category  $\mathcal{M} = {}_{\mathcal{A}}\mathcal{M}$  together with a bilinear functor, exact in the first variable, from  $\mathcal{A} \times \mathcal{M}$  to  $\mathcal{M}$ , which we call the action of  $\mathcal{A}$  and simply denote by a dot, as well as natural isomorphisms  $\mu$  and  $\lambda$  with components  $\mu_{a,b,m} \in {}_{\mathcal{M}}\langle (a \otimes b).m, a.(b.m) \rangle$  and  $\lambda_m \in {}_{\mathcal{M}}\langle \mathbf{1}_{\mathcal{A}}.m, m \rangle$  that satisfy pentagon and triangle relations analogous to the associator and unit constraint of a monoidal category. Right  $\mathcal{A}$ -modules and  $\mathcal{A}$ - $\mathcal{B}$ -bimodules are defined analogously.

Our conventions concerning dualities of a rigid category  $\mathcal{C}$  are as follows. The right dual of an object  $c$  is denoted as  $c^\vee$ , and the right evaluation and coevaluation are morphisms

$$\mathrm{ev}_c^r \in {}_{\mathcal{C}}\langle c^\vee \otimes c, \mathbf{1} \rangle \quad \text{and} \quad \mathrm{coev}_c^r \in {}_{\mathcal{C}}\langle \mathbf{1}, c \otimes c^\vee \rangle, \quad (4.1)$$

while the left evaluation and coevaluation are

$$\mathrm{ev}_c^l \in {}_{\mathcal{C}}\langle c \otimes {}^\vee c, \mathbf{1} \rangle \quad \text{and} \quad \mathrm{coev}_c^l \in {}_{\mathcal{C}}\langle \mathbf{1}, {}^\vee c \otimes c \rangle \quad (4.2)$$

with  ${}^\vee c$  the left dual of  $c$ .

In view of their importance in some of the calculations below, we state the behavior of the action with respect to dualities explicitly:

**Lemma 4.1.** Let  $\mathcal{A}$  be a rigid monoidal category, and let  $\mathcal{M} = {}_{\mathcal{A}}\mathcal{M}$  and  $\mathcal{N} = \mathcal{N}_{\mathcal{A}}$  be left and right  $\mathcal{A}$ -modules, respectively. Then we have natural isomorphisms

$${}_{\mathcal{M}}\langle a.m, m' \rangle \cong {}_{\mathcal{M}}\langle m, {}^\vee a.m' \rangle \quad (4.3)$$

for all  $a \in \mathcal{A}$  and all  $m, m' \in \mathcal{M}$ , and

$${}_{\mathcal{N}}\langle n.a, n' \rangle \cong {}_{\mathcal{N}}\langle n, n'.a^\vee \rangle \quad (4.4)$$

for all  $a \in \mathcal{A}$  and all  $n, n' \in \mathcal{N}$ .

*Proof.* This is seen in the same way as in the case of a (not necessarily strict) tensor product: The defining properties of the left duality and the naturality of the mixed associator  $\mu$  directly imply that the linear maps

$${}_{\mathcal{M}}\langle a.m, m' \rangle \ni f \longmapsto (\mathrm{id}_{{}^\vee a} \cdot f) \circ \mu_{{}^\vee a, a, m} \circ (\mathrm{coev}_a^l \cdot \mathrm{id}_m) \in {}_{\mathcal{M}}\langle m, {}^\vee a.m' \rangle \quad (4.5)$$

and

$${}_{\mathcal{M}}\langle m, {}^\vee a.m' \rangle \ni g \longmapsto (\mathrm{ev}_a^l \cdot \mathrm{id}_{m'}) \circ \mu_{a, {}^\vee a, m}^{-1} \circ (\mathrm{id}_a \cdot g) \in {}_{\mathcal{M}}\langle a.m, m' \rangle \quad (4.6)$$

are mutually inverse, thus establishing (4.3). The validity of (4.4) follows analogously, now using the properties of the right duality.  $\square$

**Remark 4.2.** The left action with a fixed object  $a \in \mathcal{A}$  on an  $\mathcal{A}$ -module  ${}_{\mathcal{A}}\mathcal{M}$  defines a linear endofunctor

$$F_a : m \longmapsto a.m \quad (4.7)$$

of  $\mathcal{M}$ . According to Lemma 4.1 the functor  $F_{{}^\vee a}$  is right adjoint to  $F_a$ . Moreover, reading the formula (4.4) backwards shows that  $F_a$  also has a left adjoint, namely  $F_{{}^\vee a}$ , and thus the endofunctor  $F_a$  is exact. Similarly, the right action with  $a$  on  $\mathcal{N}_{\mathcal{A}}$  defines an exact endofunctor  $G_a$  which has  $G_{{}^\vee a}$  and  $G_{{}^\vee a}$  as a right and left adjoint, respectively.

Combining the observation in this remark with Lemma 3.8 yields the following result for (co)ends taken over module categories:

**Corollary 4.3.** Let  $\mathcal{M}$  be a  $\mathcal{A}$ - $\mathcal{B}$ -bimodule category over finite tensor categories. Then we have

$$\int_{m \in \mathcal{M}} \overline{m} \boxtimes a.m.b \cong \int_{m \in \mathcal{M}} \overline{\nabla a.m.b^\vee} \boxtimes m \quad (4.8)$$

as well as

$$\int_{m \in \mathcal{M}} \overline{m} \boxtimes a.m.b \cong \int_{m \in \mathcal{M}} \overline{a^\vee.m.\nabla b} \boxtimes m \quad (4.9)$$

naturally in  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  and coherently with respect to the monoidal structures of  $\mathcal{A}$  and  $\mathcal{B}$ .

## 4.2 Radford theorems for bimodule categories

Just like for any finite linear category, the Nakayama functor can be defined for a finite module or bimodule category  $\mathcal{M}$  over a finite tensor category. It is then natural to study how the functors  $N_{\mathcal{M}}^l$  and  $N_{\mathcal{M}}^r$  relate to the module category structure.

For  $\mathcal{M}$  an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule category over finite tensor categories  $\mathcal{A}$  and  $\mathcal{B}$ , denote by  ${}^{\text{rr}}\mathcal{M}^{\text{ll}}$  the  $\mathcal{A}$ - $\mathcal{B}$ -bimodule that coincides with  $\mathcal{M}$  as a finite linear category but for which the left  $\mathcal{A}$ -action is twisted by the double right dual functor and the right  $\mathcal{B}$ -action is twisted by the double left dual functor, and by  ${}^{\text{ll}}\mathcal{M}^{\text{rr}}$  the one for which the left  $\mathcal{A}$ -action is twisted by the double left dual functor and the right  $\mathcal{B}$ -action by the double right dual functor. We have

**Theorem 4.4.** [Radford theorem for finite bimodule categories]

Let  $\mathcal{A}$  and  $\mathcal{B}$  be finite tensor categories and  $\mathcal{M}$  an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule category. Then the endofunctor  $N_{\mathcal{M}}^l$  of the category  $\mathcal{M}$  has a natural structure of a bimodule functor

$$N_{\mathcal{M}}^l: \mathcal{M} \rightarrow {}^{\text{rr}}\mathcal{M}^{\text{ll}}, \quad (4.10)$$

i.e. there are coherent isomorphisms

$$N_{\mathcal{M}}^l(a.m.b) \cong a^{\vee\vee}.N_{\mathcal{M}}^l(m).\vee\vee b \quad (4.11)$$

for all  $m \in \mathcal{M}$ ,  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .

The terminology *Radford theorem* is appropriate because, as we will learn in the next subsection (see Remark 4.13), for the case of a finite tensor category seen as bimodule category over itself, the statement reduces to a variant of the Radford theorem for finite tensor category as obtained in [ENO].

*Proof.* The isomorphisms (4.11) are obtained by applying Theorem 3.18 to the functor (4.7) given by left action.  $\square$

Analogously, considering the functor  $N_{\mathcal{M}}^r$  leads to the following alternative version of the Radford theorem:

**Theorem 4.5.** For  $\mathcal{A}$  and  $\mathcal{B}$  finite tensor categories, the Nakayama functor  $N_{\mathcal{M}}^r$  of an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule category  $\mathcal{M}$  extends to a bimodule functor  $N_{\mathcal{M}}^r: \mathcal{M} \rightarrow {}^{\text{ll}}\mathcal{M}^{\text{rr}}$ : we have

$$N_{\mathcal{M}}^r(a.m.b) \cong \vee\vee a.N_{\mathcal{M}}^r(m).b^{\vee\vee} \quad (4.12)$$

coherently for all  $m \in \mathcal{M}$ ,  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .

Next we recall [EO]

**Definition 4.6.** An  $\mathcal{A}$ -module  $\mathcal{M}$  is called an *exact* module category iff for every projective  $p \in \mathcal{A}$  and every  $m \in \mathcal{M}$  the object  $p.m \in \mathcal{M}$  is projective.

Any additive module functor  $\mathcal{M} \rightarrow \mathcal{N}$  from an exact module category  $\mathcal{M}$  is exact [EGNO, Prop. 7.6.9]; as a consequence the two previous results imply immediately that for  $\mathcal{M}$  an exact module category the Nakayama functor  $N_{\mathcal{M}}^l$  as well as  $N_{\mathcal{M}}^r$  are exact. Moreover, an exact module category satisfies the criterion (i) of Lemma 3.22 [EGNO, Cor. 7.6.4]. The lemma thus gives

**Corollary 4.7.** For  $\mathcal{M}$  an exact module category over a finite tensor category the Nakayama functor  $N_{\mathcal{M}}^r$  and the functor  $N_{\mathcal{M}}^l$  are quasi-inverse equivalences and are thus in particular exact.

**Remark 4.8.** In view of Proposition 3.24 we can conclude that, for  $\mathcal{M}$  an exact module category over any finite tensor category, any algebra  $B \in \text{vect}$  such that  $B\text{-mod}$  is equivalent to  $\mathcal{M}$  as a  $\mathbb{k}$ -linear category is self-injective as a  $\mathbb{k}$ -algebra. This is equivalent to the fact [EGNO, Cor. 7.6.4] that every projective object of an exact module category over a finite (multi-)tensor category is injective and vice versa.

By considering the twisted bimodule structure from Theorem 4.4 for the special case of an exact module category one obtains

**Corollary 4.9.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be module categories over a finite tensor category, and let the  $\mathcal{A}$ -module  $\mathcal{M}$  be exact. Then for any module functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  the natural isomorphism (3.61) is a module natural transformation between module functors from  $\mathcal{M}$  to  ${}^{\text{rr}}\mathcal{N}^{\text{ll}}$ .

*Proof.* That we deal with a module natural isomorphism is verified by direct calculation, using that the action of  $\mathcal{A}$  is exact and hence commutes with the end and that  $F$  is a module functor.  $\square$

### 4.3 Distinguished objects

The previous considerations remain valid if we work with (bi)module categories over finite *multi* tensor categories, i.e. [EGNO, Def. 4.1.1] we do not need to assume that the monoidal unit  $\mathbf{1}$  of the tensor category  $\mathcal{A}$  is absolutely simple, i.e. satisfies  ${}_{\mathcal{A}}\langle \mathbf{1}, \mathbf{1} \rangle \cong \mathbb{k}$ . We now examine the case that the bimodule category under consideration is a finite multitensor category  $\mathcal{M} = \mathcal{A}$  regarded as a bimodule category over itself, which is automatically an exact bimodule category [EGNO, Ex. 7.5.5]. We can then make use of trivial identities like  $N_{\mathcal{A}}^l(a) \cong N_{\mathcal{A}}^l(\mathbf{1} \otimes a) = N_{\mathcal{A}}^l(\mathbf{1}.a)$ . By setting  $m = \mathbf{1}$  and either  $a = \mathbf{1}$  or  $b = \mathbf{1}$  in Theorems 4.4 and 4.5 we learn that

$$\begin{aligned} N_{\mathcal{A}}^l(a) &\cong N_{\mathcal{A}}^l(\mathbf{1}) \otimes {}^{\vee\vee}a \cong a^{\vee\vee} \otimes N_{\mathcal{A}}^l(\mathbf{1}) && \text{and} \\ N_{\mathcal{A}}^r(a) &\cong \tilde{N}_{\mathcal{A}}^l(\mathbf{1}) \otimes a^{\vee\vee} \cong {}^{\vee\vee}a \otimes \tilde{N}_{\mathcal{A}}^l(\mathbf{1}). \end{aligned} \tag{4.13}$$

Thus we obtain

**Lemma 4.10.** Let  $\mathcal{A}$  be a finite multitensor category. Then there are isomorphisms

$$N_{\mathcal{A}}^l \cong D_{\mathcal{A}} \otimes {}^{\vee\vee}- \cong -{}^{\vee\vee} \otimes D_{\mathcal{A}} \tag{4.14}$$

and

$$N_{\mathcal{A}}^r \cong \tilde{D}_{\mathcal{A}} \otimes -{}^{\vee\vee} \cong {}^{\vee\vee}- \otimes \tilde{D}_{\mathcal{A}} \tag{4.15}$$



of functors, with

$$D_{\mathcal{A}} := N_{\mathcal{A}}^1(\mathbf{1}) = \int_{a \in \mathcal{A}} {}_{\mathcal{A}}\langle a, \mathbf{1} \rangle \otimes a \quad \text{and} \quad \tilde{D}_{\mathcal{A}} := N_{\mathcal{A}}^r(\mathbf{1}) = \int^{a \in \mathcal{A}} {}_{\mathcal{A}}\langle \mathbf{1}, a \rangle^* \otimes a. \quad (4.16)$$

Note that  $N_{\mathcal{A}}^1$  is by construction left exact; from the explicit expressions given in (4.14) and (4.15) we see directly that both  $N_{\mathcal{A}}^r$  and  $N_{\mathcal{A}}^1$  are in fact exact, in agreement with Corollary 4.7 as applied to the exact bimodule category  ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ . If the category  $\mathcal{A}$  is clear from the context, we write just  $D$  in place of  $D_{\mathcal{A}}$ .

We can combine Lemmas 3.16 and 4.10 and the fact that the double dual is an equivalence to see that  $\tilde{D} \cong {}^{\vee\vee}D$ . Also, for the composition of the two functors  $N_{\mathcal{A}}^1$  and  $N_{\mathcal{A}}^r$  we get

$$N_{\mathcal{A}}^1 \circ N_{\mathcal{A}}^r \cong D \otimes {}^{\vee\vee}\tilde{D} \otimes - \cong \tilde{D}^{\vee\vee} \otimes -{}^{\vee\vee\vee\vee} \otimes D \cong {}^{\vee}D \otimes -{}^{\vee\vee\vee\vee} \otimes D, \quad (4.17)$$

and similar formulas for  $N_{\mathcal{A}}^r \circ N_{\mathcal{A}}^1$ . As a consequence we obtain

**Lemma 4.11.** Let  $\mathcal{A}$  be a finite multitensor category. Then the object  $D \in \mathcal{A}$  is invertible, with inverse  $\tilde{D}$ .

*Proof.* Since  $\mathcal{A}$  is exact as bimodule category over itself, by Corollary 4.7  $N_{\mathcal{A}}^1$  is an equivalence with quasi-inverse  $N_{\mathcal{A}}^r$ . Formula (4.17) then implies that the endofunctor  $D \otimes {}^{\vee\vee}\tilde{D} \otimes -$  of  $\mathcal{A}$  is equivalent to  $\text{id}_{\mathcal{A}}$ , hence the object  $D \otimes {}^{\vee\vee}\tilde{D}$  is isomorphic to  $\mathbf{1}$ .  $\square$

Next note that Lemma 4.10 implies in particular that the quadruple dual satisfies  $-{}^{\vee\vee\vee\vee} \otimes D \cong D \otimes -$ . Thus we have

**Corollary 4.12.** The quadruple dual endofunctor of a finite tensor category  $\mathcal{A}$  is naturally isomorphic to conjugation by the object  $D_{\mathcal{A}}$ :

$$-{}^{\vee\vee\vee\vee} \cong D \otimes - \otimes D^{-1}. \quad (4.18)$$

**Remark 4.13.** Comparison with [Sh1, Lemma 5.1] shows that the object  $D_{\mathcal{A}}$  defined by (4.16) coincides with the *distinguished invertible object* of  $\mathcal{A}$  as defined in [ENO, Def. 3.1]. Accordingly, Corollary 4.12 is a version of the Radford  $S^4$  theorem for finite tensor categories as obtained in [ENO, Thm. 3.3]. Also note that in case  $\mathcal{A} = H\text{-mod}$  is the category of finite-dimensional modules over a finite-dimensional Hopf algebra  $H$ , the object  $D_{\mathcal{A}}$  is the one-dimensional  $H$ -module that is furnished by the distinguished group-like element of the dual Hopf algebra  $H^*$ .

Recall from Proposition 3.24 that a finite linear category  $\mathcal{C}$  is symmetric Frobenius iff its Nakayama functor is naturally isomorphic to the identity. If  $\mathcal{C}$  is even a finite multitensor category, then by evaluating Lemma 4.10 at the monoidal unit  $\mathbf{1} \in \mathcal{C}$  we see that a necessary condition for  $\mathcal{C}$  to be symmetric Frobenius is that it is unimodular. Again from Lemma 4.10 we thus have the following theorem which generalizes the result [OS, Folgerung 3.3.2], see also [Hu] for a related result, that a Hopf algebra has the structure of a symmetric Frobenius algebra if and only if it is unimodular and the square of the antipode is an inner automorphism:

**Theorem 4.14.** [Sh3] A finite multitensor category  $\mathcal{C}$  is symmetric Frobenius if and only if it is unimodular and the double dual functor on  $\mathcal{C}$  is naturally isomorphic to the identity as linear functor.

In particular we have

**Corollary 4.15.** A pivotal or spherical finite multitensor category is symmetric Frobenius iff it is unimodular.

If  $\mathcal{A}$  is just a finite monoidal category, with the monoidal structure not necessarily a biexact functor, we can still define the objects  $D_{\mathcal{A}}$  and  $\widetilde{D}_{\mathcal{A}}$  by the formulas (4.16). They are then no longer necessarily invertible, but in case either left or right duals exist in  $\mathcal{A}$ , the corresponding isomorphisms in Lemma 4.10 are still valid.

**Example 4.16.** An interesting specific case is obtained when  $\mathcal{A}$  is the finite monoidal category  $\mathcal{L}ex(\mathcal{X}, \mathcal{X})$  of left exact endofunctors of a finite linear category  $\mathcal{X}$ . From Lemma 3.21 it follows that

$$D_{\mathcal{L}ex(\mathcal{X}, \mathcal{X})} = N_{\mathcal{L}ex(\mathcal{X}, \mathcal{X})}^1(\text{id}) \cong N_{\mathcal{X}}^1 \circ N_{\mathcal{X}}^1. \quad (4.19)$$

Albeit  $\mathcal{L}ex(\mathcal{X}, \mathcal{X})$  is, in general, not a finite multitensor category owing to the lack of right duals, it follows that  $D_{\mathcal{L}ex(\mathcal{X}, \mathcal{X})}$  is invertible if  $N_{\mathcal{X}}^1$  is an equivalence.

**Remark 4.17.** From Theorem 3.18 we learn that for  $F: \mathcal{A} \rightarrow \mathcal{B}$  an exact functor between finite monoidal categories having a left exact left adjoint and satisfying  $F(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$  we have

$$D_{\mathcal{B}} \cong F^{l.l.a.}(D_{\mathcal{A}}). \quad (4.20)$$

If  $F$  is an equivalence, so that we can take  $F^{l.a.} = F^{r.a.}$ , this reduces to  $D_{\mathcal{B}} \cong F(D_{\mathcal{A}})$ , thereby reproducing Corollary 3.14 of [Sh2].

Inspired by [Sh3] we apply our results to monoidal functors between finite multitensor categories to relate their left and right adjoints and to simplify and unify proofs of some known results. First we recall a general

**Lemma 4.18.** Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an exact monoidal functor between finite multitensor categories.

(i) There exist canonical isomorphisms

$$F^{r.a.}(b^{\vee}) \cong (F^{l.a.}(b))^{\vee} \quad (4.21)$$

natural in  $b \in \mathcal{B}$ . Analogously there are isomorphisms  $F^{r.a.}({}^{\vee}b) \cong {}^{\vee}(F^{l.a.}(b))$ , and similar isomorphisms with the role of left and right adjoints interchanged.

(ii)  $F^{l.a.}$  and  $F^{r.a.}$  commute with the double dual functors of  $\mathcal{B}$ .

*Proof.* (i) follows directly from a Yoneda type argument using that  $F$ , being monoidal, commutes with the duality functors. (ii) is obtained by applying (i) twice.  $\square$

We can thus state

**Proposition 4.19.** Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an exact monoidal functor between finite multitensor categories whose left adjoint is again left exact. Then the right adjoint of  $F$  can be expressed in terms of the left adjoint as

$$F^{r.a.}(b) \cong D_{\mathcal{A}}^{-1} \otimes F^{l.a.}(D_{\mathcal{B}} \otimes b) \quad (4.22)$$

for  $b \in \mathcal{B}$ , while the double left adjoint of  $F$  obeys

$$F^{l.l.a.}(a) \cong D_{\mathcal{B}} \otimes F(D_{\mathcal{A}}^{-1} \otimes a) \quad (4.23)$$

for  $a \in \mathcal{A}$ .

*Proof.* For the first statement we invoke with Corollary 3.19 to compute

$$\begin{aligned}
F^{\text{r.a.}}(b) &\stackrel{(3.65)}{\cong} N_{\mathcal{A}}^{\text{r}}(F^{\text{l.a.}}(N_{\mathcal{B}}^{\text{l}}(b))) \stackrel{(4.14)}{\cong} N_{\mathcal{A}}^{\text{r}}(F^{\text{l.a.}}(D_{\mathcal{B}} \otimes {}^{\vee\vee}b)) \cong N_{\mathcal{A}}^{\text{r}}(F^{\text{l.a.}}({}^{\vee\vee}(D_{\mathcal{B}} \otimes b))) \\
&\cong N_{\mathcal{A}}^{\text{r}}({}^{\vee\vee}(F^{\text{l.a.}}(D_{\mathcal{B}} \otimes b))) \cong D_{\mathcal{A}}^{-1} \otimes F^{\text{l.a.}}(D_{\mathcal{B}} \otimes b).
\end{aligned} \tag{4.24}$$

Here step three holds because  ${}^{\vee\vee}D_{\mathcal{B}} \cong D_{\mathcal{B}}$  by invertibility of  $D_{\mathcal{B}}$ , and Lemma 4.18 is used in the fourth step. The isomorphism (4.23) follows analogously.  $\square$

When evaluated at the monoidal unit of a finite tensor category, the equality (4.23) reproduces [Sh2, Thm. 4.8]. Also, if we apply (4.22) to the forgetful functor  $U: \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$  from the Drinfeld center of a finite tensor category, then using that  $\mathcal{Z}(\mathcal{A})$  is factorizable [ENO, Prop. 4.4] and thus unimodular, we obtain Lemma 4.7 of [Sh1].

## 4.4 Inner Hom and relative Serre functors

Recall (see e.g. [RV]) that a right Serre functor on a linear Hom-finite additive category  $\mathcal{C}$  is an additive endofunctor  $G$  together with a natural family of isomorphisms  ${}_c\langle c, d \rangle \xrightarrow{\cong} {}_c\langle d, G(c) \rangle^*$ ; left Serre functors are defined analogously. Right Serre functors are fully faithful; if a right Serre functor is even an equivalence, then it is called a *Serre functor*. The category  $\mathcal{C}$  has a Serre functor iff the dual Hom functors  ${}_c\langle c, - \rangle^*$  and  ${}_c\langle -, c \rangle^*$  are representable for every  $c \in \mathcal{C}$ . For a finite linear category  $\mathcal{C}$  it follows immediately from Corollary 2.3 that the existence of a Serre functor is equivalent to semisimplicity.

**Remark 4.20.** Assume for a moment that a finite linear category  $\mathcal{M}$  admits left and right Serre functors  $S_{\mathcal{M}}^{\text{l}}$  and  $S_{\mathcal{M}}^{\text{r}}$ . Then they are equivalences and may be taken to form an adjoint pair. The calculation

$$\begin{aligned}
N_{\mathcal{M}}^{\text{r}}(x) &= \int^{y \in \mathcal{M}} {}_{\mathcal{M}}\langle x, y \rangle^* \otimes y \cong \int^{y \in \mathcal{M}} {}_{\mathcal{M}}\langle S_{\mathcal{M}}^{\text{l}}(y), x \rangle \otimes y \\
&\cong \int^{y \in \mathcal{M}} {}_{\mathcal{M}}\langle y, (S_{\mathcal{M}}^{\text{l}})^{\text{r.a.}}(x) \rangle \otimes y \stackrel{(2.14)}{\cong} (S_{\mathcal{M}}^{\text{l}})^{\text{r.a.}}(x),
\end{aligned} \tag{4.25}$$

where the first equality is the definition of the Nakayama functor  $N_{\mathcal{M}}^{\text{r}}$  and the second the definition of the Serre functor  $S_{\mathcal{M}}^{\text{l}}$ , then shows that the Nakayama functor  $N_{\mathcal{M}}^{\text{r}}$  of  $\mathcal{M}$  is isomorphic to  $(S_{\mathcal{M}}^{\text{l}})^{\text{r.a.}}$ , and analogously one sees that that  $N_{\mathcal{M}}^{\text{l}}$  is isomorphic to  $(S_{\mathcal{M}}^{\text{r}})^{\text{l.a.}}$ . Now as noted above  $\mathcal{M}$  in fact does *not* possess left and right Serre functors unless it is semisimple. Still, by this observation it is tempting to think of the Nakayama functors  $N_{\mathcal{M}}^{\text{l}}$  and  $N_{\mathcal{M}}^{\text{r}}$  as substitutes for left and right Serre functors of  $\mathcal{M}$ . This fits with the description of the Nakayama functors in Equations (3.77) and (3.78), which look familiar from the relation between Serre functors and dualizability structure. One might be tempted to conclude that the Nakayama functors always provide Serre functors. Recall, however, that the Deligne product of a left exact and a right exact functor is not, in general, defined; as a consequence, the right hand sides of (3.77) and (3.78) do not, in general, constitute Serre functors.

In the rest of this subsection  $\mathcal{A}$  is a finite tensor category. Recall that for a left  $\mathcal{A}$ -module  $\mathcal{M} = {}_{\mathcal{A}}\mathcal{M}$  the inner Hom  $\underline{\text{Hom}}(-, -)$  is defined via a family of natural isomorphisms  $\text{Hom}_{\mathcal{A}}(a, \underline{\text{Hom}}(m, n)) \cong {}_{\mathcal{M}}\langle a.m, n \rangle$  for  $m, n \in \mathcal{M}$  and  $a \in \mathcal{A}$ . This naturally defines a left exact

functor  $\underline{\text{Hom}}(-, -): \mathcal{M}^{\text{opp}} \times \mathcal{M} \rightarrow \mathcal{A}$ . The module category  $\mathcal{M}$  is an exact  $\mathcal{A}$ -module if and only if the functor  $\underline{\text{Hom}}(-, -)$  is also right exact [EO, Cor. 3.15 & Prop. 3.16].

The notion of inner Hom allows us to introduce a relative version of (left or right) Serre functor, as an adaptation of [Sc, Def. 4.29]:

**Definition 4.21.** Let  $\mathcal{M}$  be a left  $\mathcal{A}$ -module. A *right relative Serre functor* on  $\mathcal{M}$  is an endofunctor  $S_{\mathcal{M}}^r$  of  $\mathcal{M}$  together with a family

$$\underline{\text{Hom}}(m, n)^\vee \xrightarrow{\cong} \underline{\text{Hom}}(n, S_{\mathcal{M}}^r(m)) \quad (4.26)$$

of isomorphisms natural in  $m, n \in \mathcal{M}$ . Analogously, a *left relative Serre functor*  $S_{\mathcal{M}}^l$  comes with a family

$${}^\vee \underline{\text{Hom}}(m, n) \xrightarrow{\cong} \underline{\text{Hom}}(S_{\mathcal{M}}^l(n), m) \quad (4.27)$$

of natural isomorphisms.

**Lemma 4.22.** A right relative Serre functor on  $\mathcal{M}$  is a twisted module functor in the sense that there are coherent natural isomorphisms

$$S_{\mathcal{M}}^r(a.m) \cong a^{\vee\vee}.S_{\mathcal{M}}^r(m). \quad (4.28)$$

Similarly there are coherent natural isomorphisms  $S_{\mathcal{M}}^l(a.m) \cong {}^{\vee\vee}a.S_{\mathcal{M}}^l(m)$ .

*Proof.* We have a chain

$$\begin{aligned} \underline{\text{Hom}}(n, S_{\mathcal{M}}^r(a.m)) &\xrightarrow{\cong} \underline{\text{Hom}}(a.m, n)^\vee \xrightarrow{\cong} a^{\vee\vee} \otimes \underline{\text{Hom}}(m, n) \\ &\xrightarrow{\cong} a^{\vee\vee} \otimes \underline{\text{Hom}}(n, S_{\mathcal{M}}^r(m)) \xrightarrow{\cong} \underline{\text{Hom}}(n, a^{\vee\vee}.S_{\mathcal{M}}^r(m)) \end{aligned} \quad (4.29)$$

of isomorphisms is natural in  $m, n \in \mathcal{M}$  and in  $a \in \mathcal{A}$ . By the enriched Yoneda lemma (see e.g. [Sc, Lemma 4.11] for an adaption to module categories) this induces the natural isomorphisms 4.28, and these are by construction coherent with respect to the monoidal structure. The statement for  $S_{\mathcal{M}}^l$  is derived analogously.  $\square$

It follows directly from the definitions that if  $\mathcal{M}_{\mathcal{A}}$  has both a left and a right relative Serre functor, then these are quasi-inverse to each other. Since the inner Hom functor is left exact in both arguments, the functors  $\underline{\text{Hom}}(-, -)^\vee$  and  ${}^\vee \underline{\text{Hom}}(-, -)$  from  $\mathcal{M} \times \mathcal{M}^{\text{opp}}$  to  $\mathcal{A}$  are right exact in both arguments. It follows that if a module category has a left and a right Serre functor, then the inner Hom functor is exact so that, by the proof of Proposition 3.13 of [EO],  $\mathcal{M}_{\mathcal{A}}$  is exact. Conversely, if  $\mathcal{M}_{\mathcal{A}}$  is an exact module category, then the functors  $\underline{\text{Hom}}(-, -)^\vee$  and  ${}^\vee \underline{\text{Hom}}(-, -)$  are exact and hence, by the enriched Yoneda lemma, representable. Thus an exact module category has both relative Serre functors. We summarize these findings in

**Proposition 4.23.** An  $\mathcal{A}$ -module  $\mathcal{M}_{\mathcal{A}}$  has a left and a right relative Serre functor if and only if it is exact as a module category over  $\mathcal{A}$ .

Similarly as the Hom functor, the inner Hom functor  $\underline{\text{Hom}}$  is left exact in each argument, while  ${}^\vee \underline{\text{Hom}}$  is right exact. The respective adjoint functors can be given explicitly. To see this, note that every object  $m \in \mathcal{M}$  furnishes a module functor

$$\begin{aligned} F_m: \quad {}_{\mathcal{A}}\mathcal{A} &\longrightarrow {}_{\mathcal{A}}\mathcal{M} \\ a &\longmapsto a.m. \end{aligned} \quad (4.30)$$

We have

**Lemma 4.24.** For  $\mathcal{M}$  an exact module category over a finite tensor category  $\mathcal{A}$  and  $m \in \mathcal{M}$ , the functor  $\underline{\text{Hom}}(m, -)$  is right adjoint to the functor  $F_m$ , and  ${}^\vee \underline{\text{Hom}}(-, m)$  is left adjoint to  $F_m$ .

*Proof.* The first of the claims is shown by the calculation

$${}_{\mathcal{A}}\langle a, F_m^{\text{r.a.}}(n) \rangle \cong {}_{\mathcal{M}}\langle F_m(a), n \rangle \cong {}_{\mathcal{A}}\langle a, \underline{\text{Hom}}(m, n) \rangle, \quad (4.31)$$

combined with the Yoneda lemma. Similarly, after noticing that

$${}_{\mathcal{A}}\langle {}^\vee \underline{\text{Hom}}(n, m), a \rangle \cong {}_{\mathcal{A}}\langle a^\vee, \underline{\text{Hom}}(n, m) \rangle \cong {}_{\mathcal{M}}\langle a^\vee \cdot n, m \rangle \stackrel{(4.3)}{\cong} {}_{\mathcal{M}}\langle n, a \cdot m \rangle, \quad (4.32)$$

the second claim follows by  ${}_{\mathcal{A}}\langle F_m^{\text{l.a.}}(n), a \rangle \cong {}_{\mathcal{M}}\langle n, F_m(a) \rangle \cong {}_{\mathcal{A}}\langle {}^\vee \underline{\text{Hom}}(n, m), a \rangle$ .  $\square$

If we regard the finite tensor category  $\mathcal{A}$  as an (exact) module category over itself, then the relative Serre functors are given by the right and left double dual functor, respectively. Accordingly we can regard the relative Serre functors as generalizations of the double dual functors. Then, recalling from Corollary 4.7 that for an exact module category the Nakayama functors are equivalences, we obtain the following result which generalizes Lemma 4.10 from the regular left module category  ${}_{\mathcal{A}}\mathcal{A}$  to arbitrary exact module categories.

**Theorem 4.25.** Let  $\mathcal{M}$  be an exact module category over a finite tensor category  $\mathcal{A}$ . Then there are isomorphisms

$$N_{\mathcal{M}}^{\text{l}} \cong D \cdot S_{\mathcal{M}}^{\text{l}} \quad (4.33)$$

and

$$N_{\mathcal{M}}^{\text{r}} \cong D^{-1} \cdot S_{\mathcal{M}}^{\text{r}} \quad (4.34)$$

of module endofunctors, with  $D$  the distinguished object of  $\mathcal{A}$ .

*Proof.* According to Theorem 3.18, for any exact  $\mathcal{A}$ -module  $\mathcal{N}$  and any module functor  $F: \mathcal{N} \rightarrow \mathcal{M}$  there is an equivalence  $F^{\text{l.l.a.}} \circ N_{\mathcal{M}}^{\text{l}} \cong N_{\mathcal{N}}^{\text{l}} \circ F$ . Further, recalling from Lemma 4.24 that the functor the functor  $F_m$  introduced in (4.30) satisfies  $F_m^{\text{l.a.}} \cong {}^\vee \underline{\text{Hom}}(-, m)$ , we can calculate

$${}_{\mathcal{A}}\langle a, F_m^{\text{l.a.}}(n) \rangle \cong {}_{\mathcal{A}}\langle a, {}^\vee \underline{\text{Hom}}(n, m) \rangle \stackrel{(4.27)}{\cong} {}_{\mathcal{A}}\langle a, \underline{\text{Hom}}(S_{\mathcal{M}}^{\text{l}}(m), n) \rangle \cong {}_{\mathcal{A}}\langle a \cdot S_{\mathcal{M}}^{\text{l}}(m), n \rangle. \quad (4.35)$$

By the module version of the Yoneda lemma [Sc, Lemma 4.11 & Prop. 4.12], this shows that we have  $F_m^{\text{l.l.a.}}(-) \cong - \cdot S_{\mathcal{M}}^{\text{l}}(m)$  as module functors. Using the isomorphism  $F_m^{\text{l.l.a.}} \circ N_{\mathcal{A}}^{\text{l}} \cong N_{\mathcal{M}}^{\text{l}} \circ F_m$ , evaluated at the monoidal unit  $\mathbf{1} \in \mathcal{A}$ , together with the expression (4.14) for  $N_{\mathcal{A}}^{\text{l}}$  we obtain the stated expression for  $N_{\mathcal{M}}^{\text{l}}$ . Analogously, after invoking  $F_m^{\text{r.a.}} \cong \underline{\text{Hom}}(m, -)$  from Lemma 4.24 we can show that  $F_m^{\text{r.a.}}(a) = a \cdot S_{\mathcal{M}}^{\text{r}}(m)$  which, in turn, together with Theorem 3.18 and the expression (4.15) for  $N_{\mathcal{A}}^{\text{r}}$  implies (4.34).  $\square$

**Remark 4.26.** A finite tensor category  $\mathcal{A}$  is called unimodular iff its distinguished invertible object is isomorphic to the monoidal unit [ENO, Def. 4.1]. This generalizes the Hopf algebraic notion of unimodularity: if  $\mathcal{A} = H\text{-mod}$  is the category of finite-dimensional modules over a finite-dimensional Hopf algebra  $H$ , then  $\mathcal{A}$  is unimodular iff  $H$  is unimodular, i.e. has a two-sided integral. Now note that for a unimodular finite tensor category  $\mathcal{A}$  the Nakayama functors are isomorphic to the (left or right) double dual, and the quadruple dual of  $\mathcal{A}$  is trivial. In view of Theorem 4.25 it is thus tempting to extend the notion in the following way further to exact

module categories: We call an exact module category  ${}_{\mathcal{A}}\mathcal{M}$  over a (not necessarily unimodular) finite tensor category  $\mathcal{A}$  *unimodular* iff there exists a module natural isomorphism  $N_{\mathcal{M}}^l \cong S_{\mathcal{M}}^r$  between the Nakayama functor and the right relative Serre functor of  ${}_{\mathcal{A}}\mathcal{M}$ .

This indeed generalizes the notion of unimodular tensor category: For a finite tensor category  $\mathcal{A}$ , the right relative Serre functor of the regular  $\mathcal{A}$ -module  ${}_{\mathcal{A}}\mathcal{A}$  is the double right dual functor. Thus by the characterization of  $N_{\mathcal{A}}^l$  in Lemma 4.10,  ${}_{\mathcal{A}}\mathcal{A}$  is unimodular iff there exists an isomorphism  $D_{\mathcal{A}} \cong \mathbf{1}$ .

Note that any two module natural isomorphisms  $N_{\mathcal{M}}^l \cong S_{\mathcal{M}}^r$  differ by a module natural automorphism of  $S_{\mathcal{M}}^r$ . Now every exact module category can be written as the direct sum of indecomposable ones [EO]. If  $\mathcal{M}$  is an indecomposable module category, then it follows from the invertibility of  $S_{\mathcal{M}}^r$  that  $\mathcal{F}un_{\mathcal{A}}(S_{\mathcal{M}}^r, S_{\mathcal{M}}^r) \cong \mathbb{k}$  is one-dimensional, so that in this case the module natural isomorphism  $N_{\mathcal{M}}^l \cong S_{\mathcal{M}}^r$  is unique up to a scalar.

If the finite tensor category  $\mathcal{A}$  itself is unimodular, it follows that an exact module category  ${}_{\mathcal{A}}\mathcal{M}$  is unimodular iff there exists a module natural isomorphism  $S^l \cong S^r$  between its relative Serre functors.

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## References

- [AsSS] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Vol. 1: Techniques of Representation Theory* (Cambridge University Press, Cambridge 2006)
- [AuRS] M. Auslander, I. Reiten, and S.O. Smalø, *Representation Theory of Artin Algebras* (Cambridge University Press, Cambridge 1995)
- [BD] M. Boyarchenko and V. Drinfeld, *A duality formalism in the spirit of Grothendieck and Verdier*, Quantum Topology 4 (2013) 447–489 [[math.QA/1108.6020](#)]
- [De] P. Deligne, *Catégories tannakiennes*, in: *The Grothendieck Festschrift*, vol. II, P. Cartier et al., eds. (Birkhäuser, Boston 1990), p. 111–195
- [DSS] C.L. Douglas, C. Schommer-Pries, and N. Snyder, *The balanced tensor product of module categories*, preprint [math.QA/1406.4204](#)
- [Ei] S. Eilenberg, *Abstract description of some basic functors*, J.Indian Math. Soc. 24 (1960) 231–234
- [EGNO] P.I. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor Categories* (American Mathematical Society, Providence 2015)
- [ENO] P.I. Etingof, D. Nikshych, and V. Ostrik, *An analogue of Radford’s  $S^4$  formula for finite tensor categories*, Int. Math. Res. Notices (2004) 2915–2933 [[math.QA/0404504](#)]
- [EO] P.I. Etingof and V. Ostrik, *Finite tensor categories*, Moscow Math. J. 4 (2004) 627–654 [[math.QA/0301027](#)]
- [FS] J. Fuchs and C. Schweigert, *Consistent systems of correlators in non-semisimple conformal field theory*, Adv. Math. 307 (2017) 598–639 [[math.QA/1604.01143](#)]
- [FSS] J. Fuchs, C. Schweigert, and C. Stigner, *Modular invariant Frobenius algebras from ribbon Hopf algebra automorphisms*, J. Algebra 363 (2012) 29–72 [[math.QA/1106.0210](#)]
- [Hu] J.E. Humphreys, *Symmetry for finite dimensional Hopf algebras*, Proc. Amer. Math. Soc. 68 (1978) 143–146
- [Iv] S.O. Ivanov, *Nakayama functors and Eilenberg-Watts theorems*, J. Math. Sci. 183 (2012) 675–680
- [Ly] V.V. Lyubashenko, *Ribbon abelian categories as modular categories*, J. Knot Theory and its Ramif. 5 (1996) 311–403
- [Mac] S. Mac Lane, *Categories for the Working Mathematician* (Springer Verlag, New York 1971)
- [Man] Yu.I. Manin, *Grothendieck-Verdier duality patterns in quantum algebra*, preprint [math.QA/1701.01261](#)
- [OS] U. Oberst and H.-J. Schneider, *Über Untergruppen endlicher algebraischer Gruppen*, manuscripta mathematica 3 (1973) 217–241
- [RV] I. Reiten and M. Van den Bergh, *Noetherian hereditary categories satisfying Serre duality*, J. Amer. Math. Soc. 15 (2002) 295–366 [[math.RT/9911242](#)]
- [Ri] E. Riehl, *Categorical Homotopy Theory* (Cambridge University Press, Cambridge 2014)

- [Sc] G. Schaumann, *Pivotal tricategories and a categorification of inner-product modules*, *Algebr. Represent. Theory* 18 (2015) 1407–1479 [[math.QA/1405.5667](#)]
- [Sh1] K. Shimizu, *On unimodular finite tensor categories*, *Int. Math. Res. Notices* (2016) – [[math.QA/1402.3482v4](#)]
- [Sh2] K. Shimizu, *The relative modular object and Frobenius extensions of finite Hopf algebras*, *J. Algebra* 471 (2017) 75–112 [[math.CT/1412.0211](#)]
- [Sh3] K. Shimizu, private communication.
- [Wa] C.E. Watts, *Intrinsic characterizations of some additive functors*, *Proc. Amer. Math. Soc.* 11 (1960) 5–8