MINIMUM VERTEX DEGREE CONDITION FOR TIGHT HAMILTONIAN CYCLES IN 3-UNIFORM HYPERGRAPHS

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Dedicated to the memory of András Hajnal

ABSTRACT. We show that every 3-uniform hypergraph with n vertices and minimum vertex degree at least $(5/9 + o(1))\binom{n}{2}$ contains a tight Hamiltonian cycle. Known lower bound constructions show that this degree condition is asymptotically optimal.

§1. INTRODUCTION

G. A. Dirac [7] proved that every graph $G = (V, E)$ on at least 3 vertices and with minimum vertex degree $\delta(G) \geqslant |V|/2$ contains a Hamiltonian cycle. This result is best possible, as there are graphs G with minimum degree $\delta(G) = \lceil |V|/2 \rceil - 1$ not containing a Hamiltonian cycle.

We continue the study to which extent Dirac's theorem can be generalised to hypergraphs. Here we shall restrict to 3-uniform hypergraphs and if not mentioned otherwise by a hypergraph we will mean a 3-uniform hypergraph. Note that in this case there are at least two natural concepts of a minimum degree condition and several notions of cycle, and we briefly introduce some of them below.

For a hypergraph $H = (V, E)$ and a vertex $v \in V$ we denote by $d_H(v)$ *the degree of v* defined as

$$d_H(v) = \big|\{e \in E \colon v \in e\}\big|,$$

and by $\delta(H) = \min d_H(v)$ the *minimum vertex degree* of H taken over all $v \in V$. Similarly, for any two vertices $u, v \in V$ we denote by $d_H(u, v)$ their *pair degree*, which is the number of edges containing u and v, i.e.,

$$d_H(u, v) = \big|N_H(u, v)\big| = \big|\{e \in E \colon u, v \in e\}\big|,$$

and let $\delta_2(H) = \min d_H(u, v)$ be the *minimum pair degree* over all pairs of vertices of H.

An early notion of cycles in hypergraphs appeared in the work of Berge [1] (see, also [2]) more than 40 years ago. More recently, Katona and Kierstead [14] considered the following types of paths and cycles.

A hypergraph P is a *tight path of length ℓ*, if $|V(P)| = \ell + 2$ and there is an ordering of the vertices $V(P) = \{x_1, \ldots, x_{\ell+2}\}$ such that a triple e forms a hyperedge of P if and only if $e = \{x_i, x_{i+1}, x_{i+2}\}$ for some $i \in [\ell]$. The ordered pairs (x_1, x_2) and $(x_{\ell+1}, x_{\ell+2})$ are the *end-pairs* of P and we say that P is a tight (x_1, x_2)-$(x_{\ell+1}, x_{\ell+2})$ path. This definition of end-pairs is not symmetric and implicitly fixes a direction on P and the order of the end-pairs. Hence, we may refer to (x_1, x_2) as the *starting pair* and to $(x_{\ell+1}, x_{\ell+2})$ as the *ending pair*. All other vertices of P are called *internal*. We sometimes identify such a path P with the sequence of its vertices $x_1 \ldots x_{\ell+2}$. Moreover, a *tight cycle C of length $\ell \geqslant 4$* consists of a path $x_1 \ldots x_\ell$ of length $\ell - 2$ and the two additional hyperedges $\{x_{\ell-1}, x_\ell, x_1\}$ and $\{x_\ell, x_1, x_2\}$. In both cases the *length* of a tight cycle and of a tight path is measured by the number of hyperedges and we will use the same convention for the length of cycles, paths, and walks in graphs. For simplicity we denote edges and hyperedges by xy and xyz instead of $\{x, y\}$ and $\{x, y, z\}$.

Roughly speaking, one may think of tight paths and cycles as ordered hypergraphs such that "consecutive" edges overlap in exactly two vertices. Similarly, one may consider so-called *loose* paths and cycles, where the overlap is restricted to one vertex only. The optimal approximate minimum pair and vertex degree conditions for the existence of loose Hamiltonian cycles were obtained in [3, 15] and precise versions for large hypergraphs appeared in [6, 12].

Results on pair degree conditions implying tight Hamiltonian cycles were obtained in [19, 20]. For minimum vertex degrees, $(5/9 - o(1))n^2/2$ provides a lower bound (see Examples 1.2 (*i*) – (*iii*) below), which was conjectured to be optimal. So far only suboptimal upper bounds were obtained in [11, 17, 18]. We close this gap here, as the following result provides an asymptotically optimal minimum vertex degree condition for tight Hamiltonian cycles.

Theorem 1.1. *For every $\alpha > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H with $n \geqslant n_0$ vertices and with minimum vertex degree $\delta(H) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ contains a tight Hamiltonian cycle.*

A recent result of Cooley and Mycroft [5] establishes the existence of an almost spanning tight cycle under the same degree condition as in Theorem 1.1. Moreover, both these results are asymptotically best possible, as the following well known examples show.

Example 1.2. (i) Consider a partition $X \cup Y = V$ of a vertex set V of size n with $|X| = \lceil (n+1)/3 \rceil$ and let H be the hypergraph containing all triples $e \in V^{(3)}$ such that $|e \cap X| \neq 2$. It is not hard to show that H contains no tight Hamiltonian cycle, since two consecutive vertices in X cannot be connected to Y (see, e.g., [17]). Moreover, we have $\delta(H) \geqslant (5/9 - o(1))n^2/2$.

(ii) Similarly, one may consider a partition $X \cup Y = V$ with $|X| = \lceil 2n/3 \rceil$ and let H be the hypergraph consisting of all triples $e \in V^{(3)}$ such that $|e \cap X| \neq 2$. Again H has $\delta(H) \geqslant (5/9 - o(1))n^2/2$ and it contains no tight Hamiltonian cycle.

(iii) The last example utilises the fact that every tight Hamiltonian cycle contains a matching of size $\lfloor n/3 \rfloor$. Again we consider a partition $X \cup Y = V$ this time with $|X| = \lfloor n/3 \rfloor - 1$ and let H consist of all triples having at least one vertex in X. Consequently, H contains no matching of size $\lfloor n/3 \rfloor$ and, hence, no tight Hamiltonian cycle. On the other hand, $\delta(H) \geqslant (5/9 - o(1))n^2/2$.

We also would like to mention that in addition to the results on Hamiltonian cycles in 3-uniform hypergraphs discussed here, quite a few extensions and related results for k-uniform hypergraphs already appeared in the literature and we refer to the surveys [16, 22] (and the references therein) for a more detailed discussion.

Organisation. The proof of Theorem 1.1 is based on the *absorption method* developed in [20] and we discuss this approach in Section 2.1. In Section 2.2 we introduce the main concepts and lemmas for the proof of Theorem 1.1 and deduce the theorem based on the lemmas. Each of the subsequent Sections $3-7$ is devoted to the proof of one of the main lemmas from Section 2.2.

§2. Building Hamiltonian cycles in hypergraphs

2.1. **Absorption method.** In [20] the *absorption method* was introduced, which turned out to be a very well suited approach for extremal degree-type problems forcing the existence of spanning subhypergraphs. Our proof is also guided by this strategy, which in the context of Hamiltonian cycles can be summarised as follows: Construct an almost spanning cycle C that contains a special, so-called *absorbing path* P_A. The absorbing path has the special property that it can *absorb* the vertices outside C in such a way that a Hamiltonian cycle is created.

For example, in the context of graphs a vertex v outside C could be easily added to C if it formed a triangle with some edge xy of C, i.e., we would replace the edge xy of C by the path x-v-y of length 2. Obviously, this would have no effect on the remainder of C, since xy and the path x-v-y have the same end vertices. However, in order to repeat such

a procedure for m vertices outside C it would be convenient if each such vertex would form a triangle with at least m mutually disjoint edges in $P_A \subseteq C$. Then we could absorb one vertex after another in a greedy manner into P_A and its extensions. However, in the proof we may not have much control on the set of vertices left out by the almost spanning cycle C.

In order to prepare for such a scenario we ensure that P_A can absorb any set of vertices, which is not too large. For this it would be desirable to know that for every vertex v there exist many edges that form a triangle with v, i.e., there are many *v-absorbers*. Let us remark that if one would like to prove an approximate version of Dirac's theorem for n-vertex graphs G with $\delta(G) \geqslant (1/2 + \alpha)n$, then these edges would exist. Indeed, one can observe that the degree assumption forces for every vertex v at least $\alpha n^2/2$ triangles containing it. Based on this fact one can show that εn edges selected independently at random will contain, with high probability, at least δn v-absorbers for any vertex v, for some suitably chosen constants satisfying $\alpha > \varepsilon > \delta > 0$. Moreover, the degree condition allows us to put all these edges onto one path, an absorbing path P_A with the desired property. Consequently, the problem of finding a Hamiltonian cycle reduces to finding an almost spanning cycle C containing P_A and covering all but at most δn vertices of G.

In the context of Dirac's theorem for graphs such a "reduction" seems to be somewhat going overboard, as much simpler proofs even of the exact result are known. However, for hypergraphs no such simple proof surfaced yet and the absorption method seems to provide an appropriate approach.

For tight cycles in 3-uniform hypergraphs, the following absorbers were considered in [20]: two hyperedges xyz and yzw (which themselves form a tight (x, y)-(z, w)-path of length 2) are a v-absorber if v forms a hyperedge with each of the three consecutive pairs xy, yz, and zw. These three hyperedges allow us to insert v between y and z, leading to a tight path of length three with the same end-pairs (x, y) and (z, w). It is not hard to show that the minimum pair degree condition $\delta_2(H) \geqslant (1/2 + \alpha)n$ for an n-vertex hypergraph H guarantees the existence of $\Omega(n^4)$ v-absorbers for any vertex v, which is a good starting point for building an absorbing path in this context. However, for building such a path (and for creating an almost spanning tight cycle C) we would also need to connect the end-pairs of absorbers (and eventually the end-pairs of paths to build up C). Again, the minimum pair degree assumption was utilised for these connections in [20] and it could be shown that any pair of pairs can be connected by a relatively short tight path.

For the proof of Theorem 1.1, however, we only have a minimum vertex degree condition at hand and this calls for more complex v-absorbers and a more complicated connecting mechanism. In [17, 18] this problem was addressed by removing hyperedges containing

pairs with too small degree, which led to suboptimal minimum degree conditions. For the asymptotically optimal condition of $\left(\frac{5}{9} + o(1)\right)\frac{n^2}{2}$ new ideas for the absorbers and the connectability were required.

Roughly speaking, the absorbers we shall use here consist of two parts. First, we show that there are $\Omega(n)$ vertices z for which there exist $\Omega(n^4)$ tight paths $P_z = xyy'x'$, which can absorb z in the way described above, and we call such vertices z *absorbable* (see Figure 6.1). Moreover, for every vertex v and every absorbable vertex z there are at least $\Omega(n^4)$ quadruples (a, b, c, d) such that both vertices v and z form a hyperedge with all three pairs ab, bc, and cd. In particular, $abvcd$ and $abzcd$ form tight paths of length three in H. Consequently, the two-edge path $P_z = xyy'x'$ together with the three-edge path $abzcd$ can absorb v without changing the end pairs of P_z and of $abzcd$. Indeed, we may replace z in $abzcd$ by v and then include z between y and y' in P_z (see Definition 6.1 and Figure 6.1). Most importantly, for every vertex v such an argument would give rise to $\Omega(n^9)$ absorbers consisting of a tight path $abzcd$ of length three and a tight two-edge path P_z, which, in principle, would allow us to apply the absorption method in a similar manner as in [20].

However, connecting the end-pairs of paths arising in the proof requires more involved changes. In [20], the minimum pair degree assumption allows a *Connecting Lemma* which asserts that for every pair of disjoint pairs of vertices there exists a relatively short tight path connecting them.

A similar statement in the context of Theorem 1.1 fails to be true. In fact, there might be pairs of vertices that are not contained in any hyperedge at all. More interestingly, even when restricting to pairs of degree $\Omega(n)$, a corresponding connecting lemma might fail, as the following example shows.

Similarly as in Examples 1.2 (i) and (ii) consider a hypergraph $H = (V, E)$ with partition $X \cup Y = V$, where $|X| = \xi n$ for some $\xi < 1/3$, and with an edge set defined by $E = \{e \in V^{(3)} \colon |X \cap e| \neq 2\}$. For sufficiently large n such a hypergraph H satisfies the degree condition in Theorem 1.1, but every tight path P starting with a pair of vertices in X is bound to stay in X, i.e., $V(P) \subseteq X$. Owing to such examples we will define a suitable notion of *connectable pairs*, i.e., pairs of vertices for which a restricted Connecting Lemma can be proved (see Definition 2.5 and Proposition 2.6 in the next subsection). On the other hand, this notion must be flexible and general enough, so that we can show that all paths considered in the proof have such connectable pairs as ends. In fact, this adjustment led to a few, somewhat technical, problems that we had to address here. In the next section we present the notion of connectable pairs and the main lemmas which lead to the proof of Theorem 1.1.

2.2. **Outline of the proof.** In this section we present the proof of Theorem 1.1 based on Propositions 2.3, 2.6, 2.7, 2.9, and 2.10. These propositions will be stated here and we defer their proofs to separate later sections. The interplay of these propositions makes use of some auxiliary constants. For a simpler presentation we will note their dependencies along the way by writing $a \gg b$ to indicate that b will be chosen sufficiently small depending on a (and other constants appearing to the left of b).

More precisely, we are first given $\alpha > 0$ by Theorem 1.1 and without loss of generality we may assume that $1 \gg \alpha$. Then we fix the following auxiliary constants β, ζ_*, ζ_{**}, ϑ_*, $\vartheta_{**} > 0$ and integers ℓ, $n \in \mathbb{N}$ obeying the following hierarchy

$$1 \gg \alpha \gg \beta, \frac{1}{\ell}, \zeta_* \gg \vartheta_* \gg \zeta_{**} \gg \vartheta_{**} \gg \frac{1}{n}. \qquad (2.1)$$

These constants will be introduced together with the propositions and the quantification of the propositions will allow us to fix them under the hierarchy given in (2.1).

Theorem 1.1 concerns n-vertex hypergraphs $H = (V, E)$ with minimum vertex degree $\delta(H) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$. This degree condition implies a corresponding edge density of the *link graphs* defined below.

Definition 2.1. *For a 3-uniform hypergraph $H = (V, E)$ and a vertex $v \in V$ we define the* link graph L_v *of v as the graph with vertex set $V(L_v) = V$ and edge set*

$$E(L_v) = \{yz \colon vyz \in E(H)\}.$$

Observe that v is an isolated vertex in the link graph L_v and $e(L_v) = d_H(v) \geqslant \delta(H)$. The minimum degree assumption of Theorem 1.1 implies that every link graph has density at least $5/9 + \alpha$ and in Section 3 we investigate structural properties of such graphs. In particular, we shall show that these link graphs contain a "well connected" large subgraph, which will allow us to build and connect tight paths in the hypergraph (see Proposition 2.6 below). More precisely, we consider subgraphs satisfying the following property.

Definition 2.2. *A graph R is said to be (β, ℓ)-robust if for any two distinct vertices x and y of R the number of x-y-paths in R of length ℓ is at least $\beta|V(R)|^{\ell-1}$.*

The following proposition, which will be proved in Section 3, asserts that all link graphs contain a robust subgraph with many vertices and edges. For a graph G and $A, B \subseteq V(G)$, let $e_G(A, B)$ be the number of edges of G with one vertex in A and one in B.

Proposition 2.3 (Robust subgraphs). *For every $\alpha > 0$ there are $\beta > 0$ and an odd integer $\ell \geqslant 3$ such that for sufficiently large n every n-vertex graph $L = (V, E)$ with $|E| \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ contains an induced subgraph $R \subseteq L$ satisfying*

  (i) $|V(R)| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n$,

(ii) $e_L\big(V(R), V \smallsetminus V(R)\big) \leqslant \alpha n^2/4$ and $e(R) \geqslant \big(\tfrac{5}{9} + \tfrac{\alpha}{2}\big) \tfrac{n^2}{2} - \tfrac{(n-|V(R)|)^2}{2}$,

(iii) and R is (β, ℓ)-robust.

For the proof of Theorem 1.1 we fix for every vertex $v \in V$ a (β, ℓ)-robust subgraph $R_v \subseteq L_v$ as guaranteed by Proposition 2.3. In other words, after $\alpha > 0$ was revealed in Theorem 1.1, we use Proposition 2.3 to define constants $\beta > 0$ and $\ell \in \mathbb{N}$. We indicate this dependency by

$$\alpha \gg \beta, \frac{1}{\ell}\,.$$

Moreover, we may assume that n is sufficiently large, as it will be the last constant to be chosen in the proof of Theorem 1.1. Consequently, for any given hypergraph $H = (V, E)$ concerned in Theorem 1.1 we can appeal to Proposition 2.3 and this way we fix a (β, ℓ)-robust subgraph $R_v \subseteq L_v$ for every vertex $v \in V$. We summarise this in the following setup.

Setup 2.4. *Suppose α, $\beta > 0$, suppose $\ell \geqslant 3$ is an odd integer, and suppose $H = (V, E)$ is a 3-uniform hypergraph with $|V| = n$ sufficiently large, with $\delta(H) \geqslant \big(\tfrac{5}{9} + \alpha\big) \tfrac{n^2}{2}$, and with (β, ℓ)-robust subgraphs $R_v \subseteq L_v$ for every $v \in V$ given by Proposition 2.3.*

As discussed in Section 2.1, under the degree assumption of Theorem 1.1 it is not necessarily true that any two pairs of vertices can be connected at all by a tight path, even if we only consider pairs of high degree. Still there is a reasonably large collection of pairs admitting such mutual connections. In fact, pairs that are contained in sufficiently many robust subgraphs can be connected by tight paths in H. This will be made precise in the definition of *connectable pairs* and in the *Connecting Lemma* stated below.

Definition 2.5. *Given Setup 2.4 and $\zeta > 0$, an unordered pair xy of vertices in V is said to be ζ-connectable if the set*

$$U_{xy} = \{v \in V \colon xy \in E(R_v)\}$$

of all vertices v having xy as an edge of their robust subgraph has size $|U_{xy}| \geqslant \zeta|V|$. The ordered pair (x, y) is called ζ-connectable if xy is.

The Connecting Lemma below asserts that pairs of connectable pairs can be connected by many tight paths. Section 4 is devoted to the proof of Proposition 2.6.

Proposition 2.6 (Connecting Lemma)**.** *Given Setup 2.4 and $\zeta > 0$, there exists $\vartheta > 0$ such that every two disjoint ζ-connectable ordered pairs (x, y) and (z, w) are connected by at least $\vartheta n^{3\ell+1}$ tight (x, y)-(z, w)-paths of length $3(\ell + 1)$ in H.*

The Connecting Lemma plays a crucial role in building an absorbing path P_A (guaranteed by Proposition 2.9), as well as in building an almost spanning cycle C (see Proposition 2.10

below). For the former application we shall fix $\zeta_* \ll \alpha$ and the Connecting Lemma will yield some constant $\vartheta_* \ll \zeta_*$. Given ϑ_* we will then choose ζ_{**} for the latter application, obtaining $\vartheta_{**} \ll \zeta_{**}$. This gives rise to the hierarchy

$$\alpha \gg \zeta_* \gg \vartheta_* \gg \zeta_{**} \gg \vartheta_{**} \,,$$

as declared in (2.1).

The Connecting Lemma will allow us to connect tight paths that start and end with a connectable pair. However, in the process of building longer paths, we must not interfere with already constructed subpaths. For that we put a small *reservoir* of vertices aside and in the proof of Proposition 2.10 connections will only be created by using new vertices from this reservoir. The existence of such a reservoir set is given by the following proposition and its probabilistic proof is given in Section 5.

Proposition 2.7 (Reservoir Lemma). *Given Setup 2.4 and, in addition, let ϑ_*, $\zeta_{**} > 0$ and suppose that $\vartheta_{**} = \vartheta_{**}(\alpha, \beta, \ell, \zeta_{**}) > 0$ is given by Proposition 2.6.*

Then there exists a reservoir set $\mathcal{R} \subseteq V$ with $\frac{\vartheta_*^2}{2}n \leqslant |\mathcal{R}| \leqslant \vartheta_*^2 n$ such that for all disjoint pairs of ζ_{**}-connectable pairs (x, y) and (z, w) there are at least $\vartheta_{**}|\mathcal{R}|^{3\ell+1}/2$ tight (x, y)-(z, w)-paths of length $3(\ell + 1)$ in H whose internal vertices belong to $\mathcal{R}$.

We summarise the situation by the following setup extending Setup 2.4.

Setup 2.8. *Given Setup 2.4 and constants as stated in (2.1), where $\vartheta_* = \vartheta_*(\alpha, \beta, \ell, \zeta_*)$ and $\vartheta_{**} = \vartheta_{**}(\alpha, \beta, \ell, \zeta_{**})$ are given by Proposition 2.6. In addition, let $\mathcal{R} \subseteq V$ be a reservoir set given by Proposition 2.7.*

After these preparatory propositions we are ready to build a Hamiltonian cycle. As outlined above, we first create and put aside an *absorbing path* P_A, which at the end of the proof will allow us to 'absorb' an arbitrary (but not too large) set X of leftover vertices into an almost spanning tight cycle, and thus creating a tight Hamiltonian cycle.

Proposition 2.9 (Absorbing path). *Given Setup 2.8, then there exists a tight (absorbing) path $P_A \subseteq H - \mathcal{R}$ such that*

    (i) $|V(P_A)| \leqslant \vartheta_* n$,

    (ii) the end-pairs of P_A are ζ_*-connectable, and

    (iii) for every set $X \subseteq V \smallsetminus V(P_A)$ with $|X| \leqslant 2\vartheta_*^2 n$ there is a tight path in H whose set of vertices is $V(P_A) \cup X$ and whose end-pairs are the same as those of P_A.

The proof of Proposition 2.9 is the content of Section 6. The last proposition (see Section 7 for its proof) establishes the existence of an almost spanning tight cycle containing P_A and covering all but at most $2\vartheta_*^2 n$ vertices of H.

Proposition 2.10 (Almost spanning cycle). *Given Setup 2.8 and a tight absorbing path $P_A \subseteq H - \mathcal{R}$ from Proposition 2.9, then there exists a tight cycle $C \subseteq H$ containing P_A and passing through at least $(1 - 2\vartheta_*^2)n$ vertices.*

Finally, we observe that combining Propositions 2.9 and 2.10 implies the existence of a Hamiltonian tight cycle in H.

Proof of Theorem 1.1. Given $\alpha > 0$ we choose all auxiliary constants as described above and assume Setup 2.8. Then Proposition 2.10 guarantees the existence of an almost spanning cycle C which contains the absorbing path P_A and covers all but at most $2\vartheta_*^2 n$ vertices. Property (iii) of the absorbing path P_A guaranteed by Proposition 2.9 allows us to absorb the remaining vertices into the cycle. This concludes the proof of Theorem 1.1. $\square$

It is left to prove Propositions 2.3, 2.6, 2.7, 2.9, and 2.10, which is the content of Sections 3–7.

§3. Robust subgraphs

In this section we establish the existence of robust subgraphs within the link graphs of the given hypergraph H. The proof of Proposition 2.3 splits into two parts. In the first part (rendered in Lemma 3.2 below) we establish the existence of a subgraph R satisfying properties (i) and (ii) of Proposition 2.3, and the following strong connectivity property.

Definition 3.1. *A graph R is said to be μ-inseparable if $\delta(R) \geqslant \mu|V(R)|$ and for every partition $X \uplus Y = V(R)$ into parts of size at least $\mu|V(R)|$ we have $e(X,Y) \geqslant \mu^2|V(R)|^2$.*

Lemma 3.2. *For every $\alpha > 0$ and sufficiently large n every n-vertex graph $L = (V,E)$ with $|E| \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ contains an induced subgraph $R \subseteq L$ satisfying*

(i) *$|V(R)| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n$,*
(ii) *$e_L\big(V(R), V \smallsetminus V(R)\big) \leqslant \alpha n^2/4$ and $e(R) \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |V(R)|)^2}{2}$,*
(iii) *and R is $(\alpha/72)$-inseparable.*

In the second part of the proof we deduce Proposition 2.3 from Lemma 3.2 and for that we utilise the inseparability of R to deduce the robustness. We first give the proof of the lemma.

Proof of Lemma 3.2. We may assume $\alpha \in (0, 4/9]$, since otherwise no graph L satisfying the assumption exists. For convenience set

$$\mu = \frac{\alpha}{72} \tag{3.1}$$

and for sufficiently large n let $L = (V,E)$ be an n-vertex graph with $e(L) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$.

Defining the subgraph R. We fix the maximum $t \in \mathbb{N}$ for which there exists a partition $V_1 \cup \ldots \cup V_t = V$ with

    (a) $|V_1| \geqslant \cdots \geqslant |V_t| \geqslant \mu n/2$ and

    (b) $\sum_{1 \leqslant i < j \leqslant t} e_L(V_i, V_j) \leqslant 2(t-1)\mu^2 n^2$.

Since the trivial partition $V_1 = V$ satisfies properties (a) and (b) we know $t \geqslant 1$ and from (a) we infer that $t \leqslant 2/\mu$. Moreover, the upper bound on t combined with (b) implies

$$\sum_{1 \leqslant i < j \leqslant t} e_L(V_i, V_j) < 4\mu n^2. \tag{3.2}$$

Let $\eta \in (0, 1]$ be given by

$$|V_1| = \eta n.$$

It is easy to check that $\eta > 1/3$, as otherwise

$$e(L) = \sum_{i=1}^t e_L(V_i) + \sum_{1 \leqslant i < j \leqslant t} e_L(V_i, V_j) < \sum_{i=1}^t \frac{|V_i|^2}{2} + 4\mu n^2$$

$$\leqslant \frac{n}{3} \sum_{i=1}^t \frac{|V_i|}{2} + 4\mu n^2 = \left(\frac{1}{3} + 8\mu\right) \frac{n^2}{2} \overset{(3.1)}{\leqslant} \frac{5}{9} \frac{n^2}{2}$$

contradicts our assumption on $e(L)$. However, below we even show $\eta > 2/3$ and in the proof of that we will consider a quadratic inequality where the weak bound $\eta > 1/3$ from above rules out one interval of possible solutions. In fact, we have

$$\frac{\eta^2 n^2}{2} \geqslant e_L(V_1) > e(L) - \frac{(n - |V_1|)^2}{2} - 4\mu n^2 \geqslant \left(\frac{5}{9} + \alpha - (1-\eta)^2 - 8\mu\right) \frac{n^2}{2}.$$

This leads to the quadratic inequality

$$\eta^2 \geqslant \frac{5}{9} + \alpha - (1-\eta)^2 - 8\mu \quad \Longleftrightarrow \quad \left(\eta - \frac{1}{3}\right)\left(\eta - \frac{2}{3}\right) \geqslant \frac{\alpha}{2} - 4\mu.$$

Since assuming that $\eta \in (\frac{1}{3}, \frac{2}{3} + \frac{2}{3}\alpha)$ would yield

$$\left(\eta - \frac{1}{3}\right)\left(\eta - \frac{2}{3}\right) < \left(\eta - \frac{1}{3}\right) \cdot \frac{2}{3}\alpha < \frac{2}{3} \cdot \frac{2}{3}\alpha = \frac{\alpha}{2} - \frac{\alpha}{18} = \frac{\alpha}{2} - 4\mu,$$

we have

$$|V_1| = \eta n \geqslant \left(\frac{2}{3} + \frac{2}{3}\alpha\right) n \qquad \text{and} \qquad e_L(V_1) > \frac{2}{9}n^2 \geqslant \mu n^2. \tag{3.3}$$

Let $W = \{w_1, \ldots, w_m\} \subseteq V_1$ be a maximal (ordered) subset such that

$$\left| N_L(w_i) \cap (V_1 \setminus \{w_1, \ldots, w_{i-1}\}) \right| < \mu n$$

for every $i \in [m]$. Owing to the second part of (3.3) we have $V_1 \setminus W \neq \varnothing$. Moreover, by definition $V_1 \setminus W$ induces a subgraph of minimum degree at least μn in L and we set

$$U = V_1 \setminus W \qquad \text{and} \qquad R = L[U],$$

and below we verify that R has the desired properties.

Verifying the properties of R. We first observe that $|W| < \mu n/2$. Suppose for a contradiction that there exists a subset $W' = \{w_1, \ldots, w_{\lceil \mu n/2 \rceil}\} \subseteq W$. Then we can replace the set V_1 in the partition $V_1 \cup \ldots \cup V_t = V$ by $W' \cup (V_1 \setminus W')$ and obtain a partition into $t + 1$ parts, which satisfies (a), as

$$|V_1 \setminus W'| \geqslant |V_1 \setminus W| = |U| > \delta(R) \geqslant \mu n. \tag{3.4}$$

Moreover, the ordering of the vertices in W yields

$$e_L(W', V_1 \setminus W') \leqslant \sum_{w_i \in W'} \left| N_L(w_i) \cap (V_1 \setminus \{w_1, \ldots, w_{i-1}\}) \right| < \mu n \cdot |W'| \leqslant \mu^2 n^2, \tag{3.5}$$

which shows that the partition $W' \cup (V_1 \setminus W') \cup V_2 \cup \ldots \cup V_t = V$ also satisfies (b). Consequently, this partition would contradict the maximal choice of t and, hence, we have indeed $|W| < \mu n/2$.

Property (i) of Lemma 3.2 then follows from

$$|V(R)| = |U| = |V_1 \setminus W| = |V_1| - |W|$$

$$> |V_1| - \frac{\mu}{2}n = (\eta - \frac{\mu}{2})n \overset{(3.3)}{\geqslant} \left(\frac{2}{3} + \frac{2\alpha}{3} - \frac{\mu}{2}\right) n \overset{(3.1)}{\geqslant} \left(\frac{2}{3} + \frac{\alpha}{2}\right) n.$$

For property (ii), note that

$$e_L(U, V \setminus U) = \sum_{i=2}^{t} e_L(U, V_i) + e_L(U, W)$$

$$\leqslant \sum_{i=2}^{t} e_L(V_1, V_i) + \mu n |W| \overset{(b)}{\leqslant} 2(t-1)\mu^2 n^2 + \mu^2 n^2 < 4\mu n^2,$$

where we used $t \leqslant 2/\mu$ in the last inequality. Consequently, the first inequality of property (ii) follows from the choice of μ in (3.1). The second inequality is a direct consequence of the first and the lower bound on $e(L)$ given by the assumption of the lemma

$$e(R) = e(L) - e_L(U, V \setminus U) - e_L(V \setminus U) \geqslant \left(\frac{5}{9} + \alpha\right) \frac{n^2}{2} - \frac{\alpha}{2} \frac{n^2}{2} - \frac{(n - |U|)^2}{2}.$$

For property (iii) we first note that we already observed the required minimum degree condition $\delta(R) \geqslant \mu |U|$ in (3.4). For the second property in Definition 3.1 consider an arbitrary partition $X \cup Y = U$ with parts of size at least $\mu |U| > 2\mu n/3$. We appeal to the maximality of t and infer from (b) that

$$e_L(X, V_1 \setminus X) > 2\mu^2 n^2.$$

Consequently, since $V_1 \setminus X = Y \cup W$, we have

$$e_R(X, Y) = e_L(X, Y \cup W) - e_L(X, W) \geqslant e_L(X, V_1 \setminus X) - e_L(U, W) \geqslant 2\mu^2 n^2 - \mu^2 n^2 = \mu^2 n^2,$$

which implies that R is μ-inseparable and this concludes the proof of Lemma 3.2.   □

Next we deduce Proposition 2.3 from Lemma 3.2.

Proof of Proposition 2.3. For $\alpha \in (0, 4/9]$ set $\mu = \alpha/72$. We set ℓ to the smallest odd integer such that

$$\ell > \frac{8}{\mu^2} + 1 \qquad \text{and set} \qquad \beta = \frac{1}{72}\left(\frac{\mu}{2}\right)^{5\ell}. \tag{3.6}$$

For sufficiently large n let $L = (V, E)$ be an n-vertex graph with $e(L) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$. Moreover, let $U \subseteq V$ and $R = L[U]$ be the induced subgraph guaranteed by Lemma 3.2. In particular, $V(R) = U$,

$$|U| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n, \quad e(R) \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |V(R)|)^2}{2}, \quad \text{and} \quad \delta(R) \geqslant \mu|U|. \tag{3.7}$$

It remains to show that R is (β, ℓ)-robust for the choice of β and ℓ in (3.6). This proof will be carried out in three steps. First we show that for every pair of distinct vertices x, $z \in V(R)$ there exist at least $\Omega(n^{s-1})$ x-z-walks in R of length $s = s(x, z) \leqslant \ell$ (see (3.9) below). In the second step we ensure that $s(x, z)$ can be chosen to be odd (see (3.12)) and in the last step we show that we can insist that the walks have length ℓ independent of the pair x and z. Noting that most of these walks will indeed be paths, then concludes the proof. Below we give the details of each of the three steps.

First step. For an arbitrary vertex $x \in U$ and for every integer $i \geqslant 1$ we define

$$Y_x^i = \{y \in U \colon \text{there are at least } (\mu^4/4)^s|U|^{s-1} \ x\text{-}y\text{-walks of length } s \text{ in } R \text{ for some } s \leqslant i\}.$$

For every $i \geqslant 2$ we have $Y_x^i \supseteq Y_x^{i-1}$ and, consequently,

$$|Y_x^i| \geqslant |Y_x^1| \geqslant |N_R(x)| \geqslant \delta(R) \geqslant \mu|U|.$$

Next we show that for every integer i with $1 \leqslant i \leqslant 2/\mu^2$ at least one of the following holds:

$$\left|U \smallsetminus Y_x^i\right| < \mu|U| \qquad \text{or} \qquad \left|Y_x^{i+1} \smallsetminus Y_x^i\right| \geqslant \frac{\mu^2}{2}|U|. \tag{3.8}$$

If $|U \smallsetminus Y_x^i| \geqslant \mu|U|$, then the μ-inseparability of R implies

$$e_L\left(Y_x^i, U \smallsetminus Y_x^i\right) \geqslant \mu^2|U|^2.$$

This means however that at least $\mu^2|U|/2$ vertices $U \smallsetminus Y_x^i$ have at least $\mu^2|U|/2$ neighbours in Y_x^i. For every such vertex in $U \smallsetminus Y_x^i$ at least $1/i \geqslant \mu^2/2$ proportion of its neighbours in Y_x^i are connected to x by walks of the same length, which implies $\left|Y_x^{i+1} \smallsetminus Y_x^i\right| \geqslant \mu^2|U|/2$ and this establishes (3.8).

From (3.8) we infer that for $j = \lfloor 2/\mu^2 \rfloor$ we have $|U \smallsetminus Y_x^j| < \mu|U|$. Since $x \in U$ was arbitrary, the same conclusion holds for every vertex $z \in U$, i.e., we also have $|U \smallsetminus Y_z^j| < \mu|U|$.

Therefore, at least $|U| - 2\mu|U| > |U|/2$ vertices y are contained in the intersection $Y_x^j \cap Y_z^j$. Each of these vertices gives rise to constants s_1, $s_2 \leqslant j \leqslant 2/\mu^2$ such that there are at least $(\mu^4/4)^{s_1}|U|^{s_1-1}$ x-y-walks of length s_1 and there are at least $(\mu^4/4)^{s_2}|U|^{s_2-1}$ z-y-walks of length s_2. Consequently, for $s_y = s_1 + s_2 \geqslant 2$ there are at least $(\mu^4/4)^{s_y}|U|^{s_y-2}$ x-z-walks of length s_y in R passing through y. Repeating this argument for all vertices $y \in Y_x^j \cap Y_z^j$ shows that there is a subset of at least $\frac{|U|}{2}/\frac{4}{\mu^2}$ vertices with the same value s_y. Consequently, for some $s(x, z)$ with $2 \leqslant s(x, z) \leqslant 4/\mu^2$ there are at least

$$\frac{\mu^2}{8}|U| \cdot \left(\frac{\mu^4}{4}\right)^{s(x,z)} |U|^{s(x,z)-2} \geqslant \left(\frac{\mu}{2}\right)^{5s(x,z)} |U|^{s(x,z)-1} \tag{3.9}$$

x-z-walks of length $s(x, z)$ in R. It will be convenient to define for every pair of vertices x, $z \in U$ the set

$$\mathcal{S}_{x,z} = \left\{s \geqslant 2 \colon \text{there are at least } (\mu/2)^{5s}|U|^{s-1} \ x\text{-}z\text{-walks in } R\right\} \tag{3.10}$$

and (3.9) asserts $\mathcal{S}_{x,z} \cap [2, 4/\mu^2] \neq \varnothing$. This concludes the discussion of the first step.

Second step. We elaborate on (3.9) and show that we can obtain a similar formula with the additional restriction that $s(x, z)$ is odd. For that let $x \in U$ be arbitrary and consider the disjoint sets

$$Y_x^{\mathrm{odd}} \mathbin{\dot\cup} Y_x^{\mathrm{even}} \subseteq U$$

defined through the parity of the integers $s(x, y)$ for which the lower bound in (3.9) holds for the number of x-y-walk in R, i.e.,

$$Y_x^{\mathrm{odd}} = \left\{y \in U \colon \mathcal{S}_{x,y} \cap [2, 4/\mu^2] \text{ contains only odd integers}\right\}$$

and

$$Y_x^{\mathrm{even}} = \left\{y \in U \colon \mathcal{S}_{x,y} \cap [2, 4/\mu^2] \text{ contains only even integers}\right\}.$$

Moreover, we consider the set Y_x^{flex} of "parity-wise flexible" vertices covering the remainder of U, i.e.,

$$Y_x^{\mathrm{flex}} = \left\{y \in U \colon \mathcal{S}_{x,y} \cap [2, 4/\mu^2 + 1] \text{ contains both odd and even integers}\right\}.$$

Owing to the additional "+1" in the definition, the set Y_x^{flex} may not be disjoint from $Y_x^{\mathrm{odd}} \cup Y_x^{\mathrm{even}}$. However, all three sets together cover U. More importantly, the vertices $y \in Y_x^{\mathrm{flex}}$ connect to x by many odd and many even walks of short length, which will allow us to "fix" the parity for every vertex $z \in U$ by first connecting z with some $y \in Y_x^{\mathrm{flex}}$ and then, depending on the parity of the z-y-walk, continuing by a walk of different parity to x.

Obviously, for such an approach it will be useful that Y_x^{flex} indeed contains many vertices and, therefore, below we show

$$\left|Y_x^{\text{flex}}\right| \geqslant \frac{n}{36} \geqslant \frac{|U|}{36}. \tag{3.11}$$

For that we note that $Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}$ induces at most $\mu|U|^2$ edges, as otherwise some vertex in $y \in Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}$ would have at least $2\mu|U|$ neighbours in Y_x^{odd}. Any such a neighbour y' and its odd x-y'-walks can be used to build even x-y-walks of length at most $4/\mu^2 + 1$ and at least a $(2/\mu^2 + 1)^{-1}$ proportion of these walks would have the same length. Consequently, there would be some even integer contained in $\mathcal{S}_{x,y} \cap [2, 4/\mu^2 + 1]$, which contradicts $y \in Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}$. Applying the same argument to $Y_x^{\text{even}} \smallsetminus Y_x^{\text{flex}}$ tells us

$$e_R\left(Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}\right) + e_R\left(Y_x^{\text{even}} \smallsetminus Y_x^{\text{flex}}\right) \leqslant 2\mu|U|^2.$$

Since, trivially, $e_R\left(Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}, Y_x^{\text{even}} \smallsetminus Y_x^{\text{flex}}\right) \leqslant |U|^2/4$ and all edges of R not counted by $e_R\left(Y_x^{\text{odd}} \smallsetminus Y_x^{\text{flex}}, Y_x^{\text{even}} \smallsetminus Y_x^{\text{flex}}\right)$ are incident with a vertex in Y_x^{flex}, we have

$$e(R) \leqslant \left(\frac{1}{2} + 4\mu\right)\frac{|U|^2}{2} + \sum_{v \in Y_x^{\text{flex}}} d_R(v).$$

On the other hand, we have

$$e(R) \overset{(3.7)}{\geqslant} \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |U|)^2}{2}.$$

For ϱ defined by $|U| = \varrho n$ these two estimates on $e(R)$ lead to

$$\frac{2}{n}|Y_x^{\text{flex}}| \geqslant \frac{2}{n}\sum_{v \in Y_x^{\text{flex}}}\frac{d_R(v)}{n} \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right) - (1 - \varrho)^2 - \left(\frac{1}{2} + 4\mu\right)\varrho^2 \geqslant \varrho\left(2 - \frac{3}{2}\varrho\right) - \frac{4}{9},$$

where we used the choice $\mu = \alpha/72 < \alpha/8$ for the last inequality. Since $\varrho \in (2/3, 1]$, the right-hand side is minimised for $\varrho = 1$ and (3.11) follows.

Having established (3.11) below we shall show that for every vertex $z \in U$ there exists some odd integer $s'(x, z) \leqslant 8/\mu^2 + 1$ such that there are at least

$$\frac{1}{36}\left(\frac{\mu}{2}\right)^{5s'(x,z)+2}|U|^{s'(x,z)-1} \tag{3.12}$$

x-z-walks of length $s'(x, z)$ in R. In fact, for every vertex z and every $y \in Y_x^{\text{flex}}$ we appeal to (3.10) and obtain many z-y-walks of length $s(z, y)$. Since $y \in Y_x^{\text{flex}}$, there is some

$$s(y, x) \in \mathcal{S}_{y,x} \cap [2, 4/\mu^2 + 1]$$

of different parity than $s(z, y)$ and connecting the corresponding walks gives us

$$\left(\frac{\mu}{2}\right)^{5s(z,y)}|U|^{s(z,y)-1} \times \left(\frac{\mu}{2}\right)^{5s(y,x)}|U|^{s(y,x)-1} = \left(\frac{\mu}{2}\right)^{5s(z,y)+5s(y,x)}|U|^{s(z,y)+s(y,x)-2}$$

x-z-walks of odd length $s(z,y) + s(y,x) \leqslant 8/\mu^2 + 1$ passing through y. Similarly as in the first step we repeat this argument for all vertices $y \in Y_x^{\mathrm{flex}}$ and conclude that there must be a subset of $\frac{|U|}{36} / \frac{4}{\mu^2}$ vertices leading to odd walks of the same length $s'(x,z)$ and, hence, there are at least

$$\frac{|U|}{36} \cdot \frac{\mu^2}{4} \cdot \left(\frac{\mu}{2}\right)^{5s'(x,z)} |U|^{s'(x,z)-2}$$

x-z-walks of length $s'(x,z)$ in R and (3.12) follows.

Third step. In the last step we finally show that R is (β, ℓ)-robust. So far we achieved in the second step that for every pair of vertices there are many short walks of odd length connecting them. However, so far the length may depend on the pair that is connected and below we extend many walks so that they all have the same length ℓ independent of the pair. In fact, we shall show that for every pair of distinct vertices x and z in R there are at least $2\beta|U|^{\ell-1}$ x-z-walks of length ℓ in R.

For an arbitrary vertex $x \in U$ we consider its neighbourhood $N_R(x)$ and let $S_R(x)$ be its second neighbourhood, i.e., the set of vertices connected by a walk of length two with x in R. In particular, $N_R(x)$ and $S_R(x)$ might not be disjoint. Since $\delta(R) \geqslant \mu|U|$, we have

$$\left|N_R(x)\right| \geqslant \mu|U| \qquad \text{and} \qquad e_R\big(N_R(x), S_R(x)\big) \geqslant \frac{1}{2}\mu|U| \cdot \left|N_R(x)\right| \geqslant \frac{\mu^2}{2}|U|^2\,, \qquad (3.13)$$

where the factor $1/2$ takes into account that $N_R(x)$ and $S_R(x)$ may not be disjoint. Consequently one can show that there are subsets $N_x \subseteq N_R(x)$ and $S_x \subseteq S_R(x)$ of size at least $\mu^2|U|/4$ such that for every vertex $y \in N_x$ we have

$$\left|N_R(y) \cap S_x\right| \geqslant \frac{\mu^2}{4}|U|$$

and, similarly, $\left|N_R(y') \cap N_x\right| \geqslant \mu^2|U|/4$ for every $y' \in S_x$. Indeed the sets N_x and S_x exist, as otherwise we could keep deleting edges incident to vertices of small degree in $N_R(x)$ (resp. $S_R(x)$). More precisely, we consider vertices one by one and if $v \in N_x$ (resp. S_x) has at most $\mu^2|U|/4$ neighbours in S_x (resp. N_x), then we remove the edges between v and its neighbourhood in S_x (resp. N_x). However, this way less than

$$\big(\left|N_R(x)\right| + \left|S_R(x)\right|\big) \cdot \frac{\mu^2}{4}|U| \leqslant \frac{\mu^2}{2}|U|^2$$

edges would be deleted altogether, which by (3.13) implies that the procedure ends with a non-empty subgraph with the required degree condition. Therefore, for every vertex $y' \in S_x$ and every odd integer s'' there exist at least $(\mu^2/4)^{s''}|U|^{s''}$ walks of length s'' that start in y' and end in $N_x \subseteq N_R(x)$.

Let $z \in U$ be distinct from x. For every $y' \in S_x$ there is an odd integer $s'(z,y') \leqslant 8/\mu^2 + 1$ such that (3.12) holds for the vertex pair (z, y'). Since ℓ and $s'(z,y')$ are odd and since

$s'(z, y') \leqslant 8/\mu^2 + 1 < \ell$, for the odd integer

$$s'' = \ell - s'(z, y') - 1 \geqslant 1$$

there are $(\mu^2/4)^{s''}|U|^{s''}$ walks of length s'' from y' to some vertex $y \in N_x \subseteq N_R(x)$, which then extends to a z-x-walk of length ℓ. In other words for every $y' \in S_x$ there are at least

$$\frac{1}{36}\left(\frac{\mu}{2}\right)^{5s'(z,y')+2}|U|^{s'(z,y')-1} \times \left(\frac{\mu^2}{4}\right)^{s''}|U|^{s''} \geqslant \frac{1}{36}\left(\frac{\mu}{2}\right)^{5\ell-2}|U|^{\ell-2}$$

x-z-walks of length ℓ in R passing through y'. Repeating this argument for every vertex $y' \in S_x$ leads by our choice of β in (3.6) to $2\beta|U|^{\ell-1}$ x-z-walks of length ℓ and for sufficiently large n at least half of these walks are indeed paths of length ℓ. Since x and z were arbitrary this shows that R is (β, ℓ)-robust and concludes the proof of Proposition 2.3. $\square$

We close this section with the observation that two graphs R and R' on the same vertex set, obtained by applications of Proposition 2.3, must share quite a few edges. This will be essential in the proof of Theorem 1.1 as it asserts that any pair of robust subgraphs from two link graphs share some edges.

Proposition 3.3. *Let V be a set of n vertices and let $R = (U, E)$ and $R' = (U', E')$ be graphs on vertex sets U, $U' \subseteq V$. If for some $\alpha > 0$ we have*

$$|U| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n \quad and \quad |E| \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |U|)^2}{2}$$

and

$$|U'| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n \quad and \quad |E'| \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |U'|)^2}{2}$$

then $|E \cap E'| \geqslant \alpha n^2/2$.

Proof. Define the real numbers ϱ, ϱ', and η by

$$|U| = \varrho n, \quad |U'| = \varrho' n, \quad and \quad |E \cap E'| = \eta\frac{n^2}{2}.$$

The assumptions on the sizes of U and U' assert

$$\varrho, \varrho' \in \left[\tfrac{2}{3} + \tfrac{\alpha}{2}, 1\right]. \tag{3.14}$$

Similarly, the assumptions on $|E|$ and $|E'|$ and the sieve formula yield

$$|E \cup E'| \geqslant \left(\frac{10}{9} + \alpha - (1 - \varrho)^2 - (1 - \varrho')^2 - \eta\right)\frac{n^2}{2}. \tag{3.15}$$

On the other hand, we have

$$|E \cup E'| \leqslant \left|\binom{U}{2} \cup \binom{U'}{2}\right| = \binom{|U|}{2} + \binom{|U'|}{2} - \binom{|U \cap U'|}{2}.$$

Now $|U \cap U'| \geq (\varrho + \varrho' - 1)n$ and by (3.14) the expression $\varrho + \varrho' - 1$ is positive, so

$$\left|E \cup E'\right| \leq \left(\varrho^2 + (\varrho')^2 - (\varrho + \varrho' - 1)^2\right) \frac{n^2}{2} \, .$$

Together with (3.15) this gives

$$\varrho^2 + (\varrho')^2 - (\varrho + \varrho' - 1)^2 \geq \frac{10}{9} + \alpha - (1 - \varrho)^2 - (1 - \varrho')^2 - \eta \, ,$$

i.e.,

$$(\varrho - \varrho')^2 + \eta \geq \frac{1}{9} + \alpha \, .$$

But (3.14) implies $(\varrho - \varrho')^2 < 1/9$, and thus we have indeed $\eta \geq \alpha$. $\square$

§4. Connectable pairs

In this section we establish the Connecting lemma (Proposition 2.6) and, therefore, justify the notion of connectable pairs from Definition 2.5 by showing that such pairs indeed can be connected by tight paths in H.

Proof of Proposition 2.6. Let $\zeta > 0$ be given and set

$$\vartheta = \frac{1}{2} \left(\frac{2}{3}\right)^{\ell^2 - 1} \left(\frac{\alpha\beta\zeta}{2}\right)^{\ell+1} \, . \tag{4.1}$$

We recall Definition 2.5, set $t = \lceil \zeta n \rceil$, and let

$$\{u(1), \dots, u(t)\} \subseteq U_{xy} \quad \text{as well as} \quad \{v(1), \dots, v(t)\} \subseteq U_{zw}$$

be arbitrary t-subsets of U_{xy} and U_{zw}, respectively.

Let us define

$$I_{ab} = \left\{i \in [t] \colon ab \in E(R_{u(i)}) \cap E(R_{v(i)})\right\}$$

for any ordered pair (a, b) of vertices from V. Then double counting shows that

$$\sum_{(a,b) \in V^2} |I_{ab}| = \sum_{i=1}^{t} \left|E(R_{u(i)}) \cap E(R_{v(i)})\right| \geq \frac{\alpha}{2} n^2 t \, , \tag{4.2}$$

where the last inequality follows by Proposition 3.3. We intend to estimate the number T of all tight (x, y)-(z, w)-walks of the form

$$xyu(i_1)r_1 r_2 u(i_2) \dots r_{\ell-2} r_{\ell-1} u(i_{(\ell+1)/2}) abv(j_1) s_1 s_2 v(j_2) \dots s_{\ell-2} s_{\ell-1} v(j_{(\ell+1)/2}) zw \, , \tag{4.3}$$

where tight walks are defined similarly like tight paths, but vertices are allowed to repeat. Such walks can be represented by sextuples

$$\left(\vec{i}, \vec{j}, \vec{r}, \vec{s}, a, b\right) \in [t]^{(\ell+1)/2} \times [t]^{(\ell+1)/2} \times V^{\ell-1} \times V^{\ell-1} \times V \times V.$$

Intuitively, these walks connect (x, y) to (z, w) via an arbitrary "middle pair" (a, b) (see Figure 4.1). The construction of such walks can be reduced to a 2-uniform problem in link graphs by demanding that for every $k \in [(\ell + 1)/2]$ we have:

(a) $i_k, j_k \in I_{ab}$,

(b) $yr_1 \ldots r_{\ell-1}a$ is a path in $R_{u(i_k)}$,

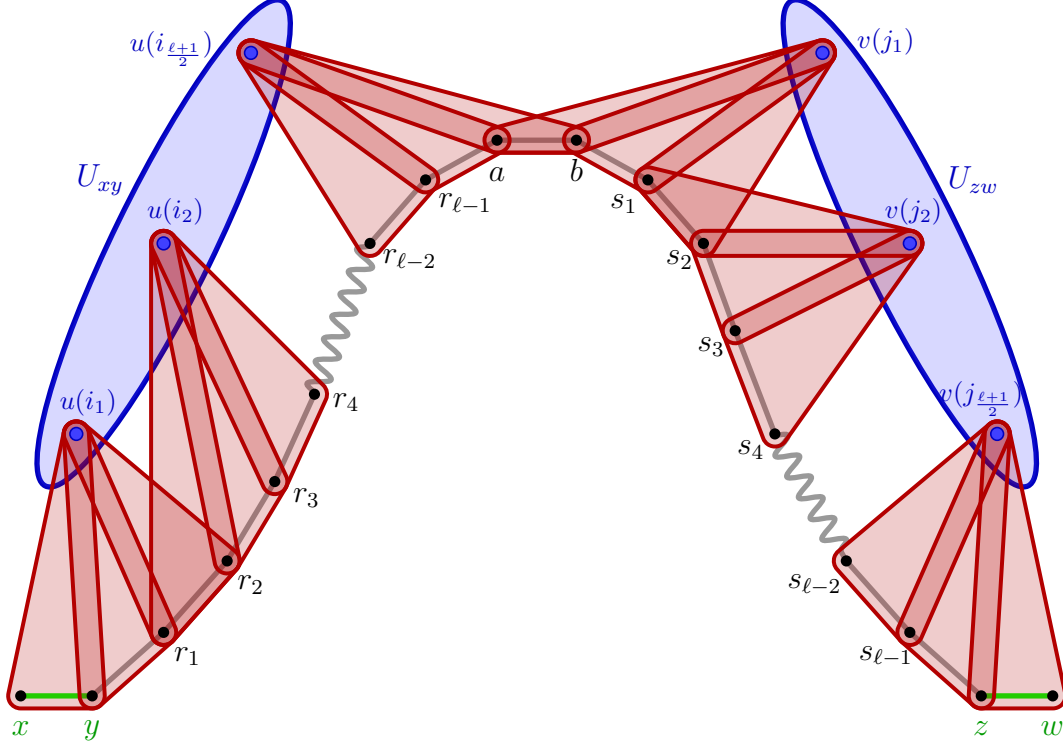(c) and $bs_1 \ldots s_{\ell-1}z$ is a path in $R_{v(j_k)}$.

FIGURE 4.1. Connecting the ζ-connectable pairs (x, y) and (z, w) through middle pair (a, b) using vertices from the sets U_{xy} and U_{zw}.

In other words, if T^* denotes the number of sextuples $(\vec{i}, \vec{j}, \vec{r}, \vec{s}, a, b)$ satisfying the conditions (a), (b), and (c), then $T \geqslant T^*$. Note that the hyperedges $xyu(i_1)$ and $v(j_{(\ell+1)/2})zw$ are not forced by (a) and (b), but are a direct consequence of $u(i_1) \in U_{xy}$ and $v(j_{(\ell+1)/2}) \in U_{zw}$. Similarly, the required hyperedges $u(i_{(\ell+1)/2})ab$ and $abv(j_1)$ are a consequence of (a). On the other hand, conditions (a)–(c) imply several additional hyperedges, which are not required for the (x, y)-(z, w)-walk. Hence, indeed we have $T \geqslant T^*$. Below we shall show

$$T^* \geqslant 2\vartheta n^{3\ell+1}. \tag{4.4}$$

Since at most $O(n^{3\ell})$ of the corresponding walks (4.3) can fail to be a path (due to the presence of repeated vertices), this trivially implies Proposition 2.6.

As a first step towards the proof of (4.4) we will fix for a while the middle vertices a and b and study the number T_{ab} of possibilities to complete a walk of the desired kind by an appropriate choice of the $3\ell - 1$ remaining vertices. Evidently

$$T_{ab} = R_{ab}S_{ab}\,, \tag{4.5}$$

where R_{ab} denotes the number of possibilities to choose $i_1, \ldots, i_{(\ell+1)/2} \in I_{ab}$ and vertices $r_1, \ldots, r_{\ell-1} \in V$ such that (b) holds and S_{ab} has a similar meaning with respect to the numbers j_k, the vertices s_k, and property (c). Given any sequence $\vec{r} = (r_1, \ldots, r_{\ell-1}) \in V^{\ell-1}$ of vertices, we set

$$D(\vec{r}) = \left\{i \in I_{ab} \colon y\vec{r}a \text{ is a path in } R_{u(i)}\right\}.$$

Then

$$R_{ab} = \sum_{\vec{r} \in V^{\ell-1}} |D(\vec{r})|^{(\ell+1)/2}$$

and from the (β, ℓ)-robustness of $R_{u(i)}$ combined with property (i) of Proposition 2.3 applied for every $i \in I_{ab}$, we infer by means of double counting that

$$\sum_{\vec{r} \in V^{\ell-1}} |D(\vec{r})| \geqslant |I_{ab}| \cdot \beta \left(\frac{2}{3}n\right)^{\ell-1}.$$

Thus a standard convexity argument shows

$$R_{ab} \geqslant n^{\ell-1} |I_{ab}|^{(\ell+1)/2} \beta^{(\ell+1)/2} \left(\frac{2}{3}\right)^{(\ell^2-1)/2}.$$

Applying this argument also to S_{ab}, using (4.5), and summing over all $(a,b) \in V^2$ we deduce

$$T^* = \sum_{(a,b) \in V^2} T_{ab} \geqslant \left(\frac{2}{3}\right)^{\ell^2-1} \beta^{\ell+1} n^{2(\ell-1)} \times \sum_{(a,b) \in V^2} |I_{ab}|^{\ell+1}$$

$$\overset{(4.2)}{\geqslant} \left(\frac{2}{3}\right)^{\ell^2-1} \beta^{\ell+1} n^{2(\ell-1)} \times \left(\frac{\alpha}{2}t\right)^{\ell+1} n^2\,,$$

where we used Jensen's inequaility in the last step. Recalling the choice of ϑ in (4.1) and that $t = \lceil \zeta n \rceil$ entails (4.4) and this concludes the proof. $\square$

We close this section with the following immediate consequence of Definition 2.5, which we shall use at several occasions in the subsequent sections.

Fact 4.1. *Given Setup 2.4 and $\zeta > 0$, there are at most ζn^3 triples $(x, y, z) \in V^3$ with $xy \in E(R_z)$ such that the pair xy fails to be ζ-connectable.*

Proof. If an (unordered) pair xy fails to be ζ-connectable, then it follows from Definition 2.5 that $|U_{xy}| \leqslant \zeta n$ and, hence, xy is an edge in R_z for at most $\zeta|V| = \zeta n$ vertices $z \in V$. Since there are at most n^2 ordered pairs $(x, y) \in V^2$, the fact follows. $\square$

§5. Reservoir

In this section we focus on the Reservoir lemma (Proposition 2.7). The existence of such a reservoir set is established by a standard probabilistic argument.

Proof of Proposition 2.7. Consider a random subset $\mathcal{R} \subseteq V$ with elements included independently with probability

$$p = \left(1 - \frac{1}{10\ell}\right)\vartheta_*^2 \,.$$

Consequently, $|\mathcal{R}|$ is binomially distributed and we infer from Chernoff's inequality that

$$\mathbb{P}\big(|\mathcal{R}| < \vartheta_*^2 n/2\big) = o(1) \,. \tag{5.1}$$

Moreover, since $\vartheta^2 n \geqslant (4/3)^{\frac{1}{3\ell+1}} pn \geqslant (1+c)\mathbb{E}[|\mathcal{R}|]$ for some sufficiently small $c = c(\ell) > 0$, we also have

$$\mathbb{P}\big(|\mathcal{R}| > \vartheta_*^2 n\big) \leqslant \mathbb{P}\Big(|\mathcal{R}| > (4/3)^{\frac{1}{3\ell+1}} pn\Big) = o(1) \,. \tag{5.2}$$

Recall that for every disjoint pair (x,y) and (z,w) of ζ_{**}-connectable pairs Proposition 2.6 ensures the existence of at least $\vartheta_{**} n^{3\ell+1}$ tight (x,y)-(z,w)-paths of length $3(\ell+1)$ (having $3\ell+5$ vertices in total). Let $X = X((x,y),(z,w))$ be a random variable counting the number of (x,y)-(z,w)-paths with all $3\ell+1$ internal vertices in $\mathcal{R}$. Consequently

$$\mathbb{E}[X] \geqslant p^{3\ell+1} \cdot \vartheta_{**} n^{3\ell+1} \,. \tag{5.3}$$

Since including or not including a particular vertex into $\mathcal{R}$ affects the random variable X by at most $(3\ell+1)n^{3\ell}$, the Azuma–Hoeffding inequality (see, e.g., [13, Corollary 2.27]) asserts

$$\mathbb{P}\big(X \leqslant \tfrac{2}{3}\vartheta_{**}(pn)^{3\ell+1}\big) \overset{(5.3)}{\leqslant} \mathbb{P}\big(X \leqslant \tfrac{2}{3}\mathbb{E}[X]\big)$$
$$\leqslant \exp\left(-\frac{\mathbb{E}[X]^2}{18 \cdot n \cdot ((3\ell+1)n^{3\ell})^2}\right) = \exp\big(-\Omega(n)\big) \,. \tag{5.4}$$

Since there are at most n^4 pairs of ζ_{**}-connectable pairs that we have to consider, in view of (5.2), the union bound combined with (5.4), implies that a.a.s. the set $\mathcal{R}$ has the property that for every pair of connectable pairs at least $\vartheta_{**}|\mathcal{R}|^{3\ell+1}/2$ tight connecting paths have all internal vertices in $\mathcal{R}$. In addition, due to (5.1) and (5.2) a.a.s. the set $\mathcal{R}$ also satisfies $\vartheta_*^2 n/2 \leqslant |\mathcal{R}| \leqslant \vartheta_*^2 n$. Consequently, a reservoir set $\mathcal{R}$ with all required properties indeed exists. □

In Section 7 we will frequently need to connect ζ_{**}-connectable pairs through the reservoir. Whenever such a connection is made, the part of the reservoir that may still be used for further connections shrinks by $3\ell+1$ vertices. Although $\Omega(n)$ such connections are needed,

we shall be able to keep the reservoir almost intact throughout this process, which in turn guarantees that there will always be some permissible connections left.

Lemma 5.1. *Given Setup 2.8 with a reservoir set $\mathcal{R} \subseteq V$, let $\mathcal{R}' \subseteq \mathcal{R}$ be an arbitrary subset of size at most $2\vartheta_{**}^2 n$. Then for all disjoint pairs of ζ_{**}-connectable pairs (x, y) and (z, w) there is a tight (x,y)-(z,w)-path of length $3(\ell + 1)$ in H whose internal vertices belong to $\mathcal{R} \smallsetminus \mathcal{R}'$.*

Proof. Recalling $|\mathcal{R}| \geqslant \vartheta_*^2 n/2$ and the hierarchy (2.1) yields $|\mathcal{R}'| \leqslant 2\vartheta_{**}^2 n \leqslant \frac{\vartheta_{**}}{8\ell}|\mathcal{R}|$. Moreover, every given vertex in $\mathcal{R}'$ is a member of at most $(3\ell + 1)|\mathcal{R}|^{3\ell}$ tight (x,y)-(z,w)-paths of length $3(\ell + 1)$ in H whose internal vertices belong to $\mathcal{R}$. Consequently, there are still at least

$$\frac{\vartheta_{**}}{2}|\mathcal{R}|^{3\ell+1} - |\mathcal{R}'| \cdot (3\ell + 1)|\mathcal{R}|^{3\ell} \geqslant \frac{\vartheta_{**}}{2}|\mathcal{R}|^{3\ell+1} - \frac{\vartheta_{**}}{8\ell} \cdot (3\ell + 1)|\mathcal{R}|^{3\ell+1} > 0$$

such paths with all internal vertices in $\mathcal{R} \smallsetminus \mathcal{R}'$. $\qquad\square$

§6. ABSORBING PATH

In this section we we prove Proposition 2.9, that is, we establish the existence of an absorbing path. The following special hypergraph (the so-called v-absorber, see Figure 6.1a) will allow us to absorb a given vertex v into a path containing a v-absorber (see Figure 6.1b).

Definition 6.1. *Given Setup 2.8 and a vertex $v \in V$, a 9-tuple $(a, b, c, d, z, x, y, y', x') \in (V \smallsetminus \{v\})^9$ of distinct vertices such that*

(i) $zab, zbc, zcd, zxy, zyy', zy'x', xyy', yy'x' \in E$, and

(ii) the pairs ab, cd, xy, and $y'x'$ are ζ_*-connectable

is called a v-absorber if, in addition, $vab, vbc, vcd \in E$.

An important property of these configurations proved in Lemma 6.7 below asserts that for every vertex $v \in V$ there exist $\Omega(n^9)$ such v-absorbers. For standard probabilistic reasons this will lead us to a family $\mathcal{F}$ of $\Omega(n)$ set-wise mutually disjoint 9-tuples containing for each $v \in V$ at least $\Omega(n)$ absorbers (see Lemma 6.8). Owing to condition (*ii*) of Definition 6.1, we may then use the Connecting Lemma (Proposition 2.6) for connecting them, i.e., for producing an *absorbing path* P_A of length $\Omega(n)$, which contains for every v-absorber $(a, b, c, d, z, x, y, y', x') \in \mathcal{F}$ the subpaths $abzcd$ and $xyy'x'$. If at the end of the proof of Theorem 1.1 the need to absorb v arises, we shall simply replace in P_A, for one such v-absorber, the subpaths $abzcd$ and $xyy'x'$ by $abvcd$ and $xyzy'x'$ (see Figure 6.1b).

Towards the goal of estimating the number of v-absorbers from below, we shall at first only deal with configurations consisting of the five vertices z, x, y, y', and x'. In the lemma that follows we do not pay attention to connectability demands yet. For potential
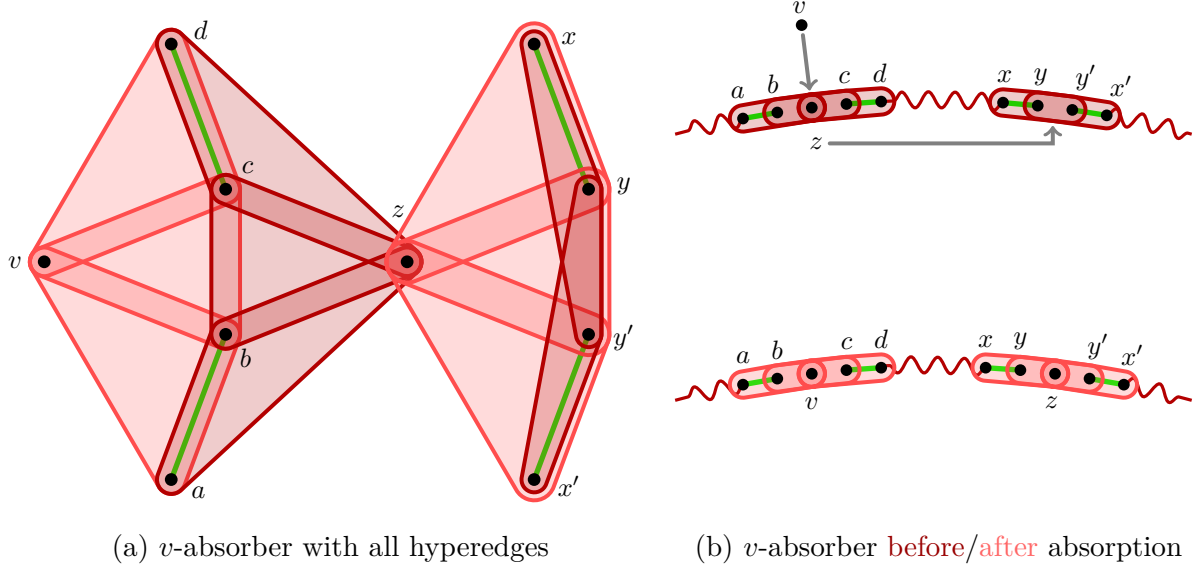
(a) v-absorber with all hyperedges   (b) v-absorber before/after absorption

FIGURE 6.1. A v-absorber, where the ζ_*-connectable pairs are indicated in green, hyperedges used before absorption of v are dark red and hyperedges used after absorption of v are light red.

future references we point out that its proof requires only a less restrictive minimum degree condition than the one provided by Theorem 1.1.

Lemma 6.2. *For every hypergraph $H = (V, E)$ with n vertices and $\delta(H) \geqslant \frac{6}{11} \cdot \frac{n^2}{2}$ there exist at least $n^5/28^4$ quintuples $(x, y, y', x', z) \in V^5$ with the following properties:*

(i) $xyz, yy'z, x'y'z, xyy', yy'x' \in E$;

(ii) $d(y, z) > \frac{5}{12}n$.

Proof. We consider the function $f \colon E \to \mathbb{R}$ defined by

$$f(x, y, z) = \frac{n}{d(x, y)} + \frac{n}{d(x, z)} + \frac{n}{d(y, z)}$$

and note that by double counting we have

$$\sum_{xyz \in E} f(x, y, z) = n \cdot |\partial H| \leqslant \frac{n^3}{2}, \tag{6.1}$$

where ∂H denotes the set of those pairs in $V^{(2)}$ that are contained in at least one edge of H. An edge $e \in E(H)$ is said to be *central* if $f(e) \leqslant \frac{28}{5}$. In view of (6.1) the set C of central edges satisfies $\frac{28}{5}|E \smallsetminus C| \leqslant \frac{n^3}{2}$, i.e., $|E \smallsetminus C| \leqslant \frac{5}{56}n^3$. On the other hand, the minimum degree condition imposed on H yields $|E| \geqslant \frac{1}{11}n^3$ and thus we have

$$|C| = |E| - |E \smallsetminus C| \geqslant \frac{n^3}{11} - \frac{5n^3}{56} = \frac{n^3}{11 \cdot 56} > \frac{n^3}{28^2}. \tag{6.2}$$

Next we will show the following statement.

Claim 6.3. *If $yy'z$ is a central edge with*

$$d(y, y') \geqslant d(y, z) \geqslant d(y', z), \tag{6.3}$$

then

$$|N(y, z) \cap N(y, y')| \geqslant \frac{n}{28}, \quad |N(y', z) \cap N(y, y')| \geqslant \frac{n}{28}, \tag{6.4}$$

and (ii) *of Lemma 6.2 holds.*

Proof. Let $yy'z$ be a central edge satisfying (6.3). Due to $f(y, y', z) \leqslant \frac{28}{5}$ we have

$$\frac{2n}{d(y, y')} + \frac{n}{d(y', z)} \leqslant \frac{28}{5}.$$

Moreover, the Cauchy–Schwarz inequality yields

$$\left(\frac{2}{d(y, y')} + \frac{1}{d(y', z)} \right) \left(d(y, y') + d(y', z) \right) \geqslant (\sqrt{2} + 1)^2 > \frac{29}{5}.$$

Hence

$$d(y, y') + d(y, z) \geqslant d(y, y') + d(y', z) > \tfrac{29}{28}n,$$

which implies (6.4).

Finally, $f(y, y', z) \leqslant \frac{28}{5}$, $d(y, y') \leqslant n$, and (6.3) lead to

$$\frac{28}{5} \geqslant \frac{n}{d(y, y')} + \frac{n}{d(y, z)} + \frac{n}{d(y', z)} \geqslant 1 + \frac{2n}{d(y, z)},$$

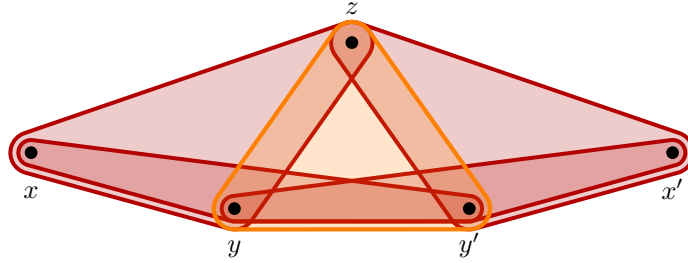which proves (ii) of Lemma 6.2.   □

FIGURE 6.2. Quintuple (x, y, y', x', z) from Lemma 6.2 with central edge zyy'.

Having thus established the above claim, we continue with the proof of Lemma 6.2. To this end we remark that for every central edge $yy'z$ satisfying (6.3) the estimates (6.4) imply that there are at least $\frac{n}{28}$ choices of x and at least $\frac{n}{28}$ choices of x' such that (i) of Lemma 6.2 holds. Applying this argument to all central edges and taking (6.2) into account we deduce the existence of at least $\frac{n^5}{28^4}$ quintuples $(x, y, y', x', z) \in V^5$ with the desired properties.   □

Next we prove that there are $\Omega(n)$ vertices which are capable of playing the rôle of z in many absorbers.

Definition 6.4. *Given Setup 2.8, a vertex $z \in V$ is said to be* absorbable *if there exist at least $\frac{n^4}{2^{21}}$ quadruples $(x, y, y', x') \in V^4$ such that*

(a) *the five triples xyz, yzy', $zy'x'$, xyy', and $yy'x'$ belong to E,*

(b) *and the pairs xy, $y'x'$ are ζ_*-connectable.*

Lemma 6.5. *Given Setup 2.8, there exist at least $\frac{n}{2^{21}}$ absorbable vertices.*

Proof. Let $A \subseteq V^5$ denote the set of all quintuples (x, y, y', x', z) satisfying the conclusion of Lemma 6.2, which then states that

$$|A| \geqslant \frac{n^5}{28^4}.$$

We intend to show that for "most" of these quintuples the pairs xy and $x'y'$ are ζ_*-connectable. This will then imply Lemma 6.5 in view of an easy counting argument.

As we shall verify below, the following three sets of "exceptional" quintuples are small:

$$Q_1 = \{(x, y, y', x', z) \in A \colon y \notin V(R_z)\},$$

$$Q_2 = \{(x, y, y', x', z) \in A \colon y \in V(R_z) \text{ but } \{x, x', y'\} \nsubseteq V(R_z)\},$$

and

$$Q_3 = \{(x, y, y', x', z) \in A \colon xy, x'y' \in E(R_z) \text{ but one of these pairs is not } \zeta_*\text{-connectable}\}.$$

If $(x, y, y', x', z) \in Q_1$, then by clause (ii) of Lemma 6.2 we have $d_{L_z}(y) \geqslant \frac{5n}{12} = \left(\frac{1}{3} + \frac{1}{12}\right)n$ and due to $|V(R_z)| \geqslant \frac{2}{3}n$ it follows that y is incident to at least $\frac{n}{12}$ edges in the link graph L_z running from $V \smallsetminus V(R_z)$ to $V(R_z)$. Owing to condition (ii) from Proposition 2.3 there are for each $z \in V$ at most $3\alpha n$ vertices y with this property and, consequently, we have

$$|Q_1| \leqslant 3\alpha n^5.$$

Similarly if $(x, y, y', x', z) \in Q_2$, then at least one of the pairs xy, yy', or $x'y'$ connects $V(R_z)$ to its complement in the link graph L_z, which shows

$$|Q_2| \leqslant \tfrac{6}{4}\alpha n^5.$$

Moreover, the case $\zeta = \zeta_*$ of Fact 4.1 leads to

$$|Q_3| \leqslant 2\zeta_* n^5.$$

Finally, taking $\zeta_*, \alpha \ll 1$ into account we get

$$|A \smallsetminus (Q_1 \cup Q_2 \cup Q_3)| \geqslant \left(\frac{1}{28^4} - \frac{9}{2}\alpha - 2\zeta_* \right) n^5 > \frac{n^5}{2^{20}} \,.$$

Definition 6.4 guarantees that for at most $\frac{n^5}{2^{21}}$ of the quintuples

$$(x, y, y', x', z) \in A \smallsetminus (Q_1 \cup Q_2 \cup Q_3)$$

the vertex z can fail to be absorbable. Conversely every absorbable vertex can account for at most n^4 such quintuples. Thus there are indeed at least $\frac{n}{2^{21}}$ absorbable vertices. □

It remains to consider the other part of our absorbers, i.e., the six hyperedges spanned by a, b, c, d together with v and z (see Figure 6.1).

Lemma 6.6. *Given Setup 2.8, for every vertex $v \in V$ there are at least $\alpha^4 n^5$ quintuples $(a, b, c, d, z) \in V^5$ such that*

    (i) $vab, vbc, vcd, zab, zbc, zcd \in E$,

    (ii) ab and cd are ζ_*-connectable,

    (iii) and z is absorbable.

Proof. For every vertex $v \in V$ and every fixed absorbable vertex $z \in V$ Proposition 3.3 tells us $|E(R_v) \cap E(R_z)| \geqslant \alpha \frac{n^2}{2}$ and a result due to Blakley and Roy [4] (asserting the validity of Sidorenko's conjecture [9,21] for paths) entails that there are at least $\alpha^3 n^4$ quadruples $(a, b, c, d) \in V^4$ forming a three-edge path in both graphs R_v and R_z. Together with Lemma 6.5 this shows that there are at least $\frac{\alpha^3}{2^{21}} n^5$ quintuples $(a, b, c, d, z) \in V^5$ satisfying properties (i) and (iii) of Lemma 6.6, and $ab, cd \in E(R_v) \cap E(R_z)$ instead of property (ii). As a consequence of Fact 4.1, however, there are at most $2\zeta_* n^5$ such quintuples for which one of these two pairs fails to be ζ_*-connectable. As $\zeta_* \ll \alpha \ll 1$ implies $\frac{\alpha^3}{2^{21}} - 2\zeta_* > \alpha^4$, the lemma follows. □

Lemma 6.6 easily implies that there are $\Omega(n^9)$ v-absorbers for every vertex $v \in V$. In addition we can also ensure that these absorbers are outside the reservoir $\mathcal{R}$.

Lemma 6.7. *Given Setup 2.8, for every $v \in V$ the number of v-absorbers in $(V \smallsetminus \mathcal{R})^9$ is at least $\alpha^5 n^9$.*

Proof. Combining Lemma 6.6 with Definition 6.1, we learn that there are at least $\frac{\alpha^4}{2^{21}} n^9$ 9-tuples meeting all requirements from that definition except that some of the 10 vertices $v, a, \ldots, x'$ might coincide. However, there can be at most $45 n^8$ such bad 9-tuples. Moreover, at most $9\vartheta_*^2 n^9$ members of V^9 can use a vertex from the reservoir and, consequently, the number of desired v-absorbers is at least $\left(\frac{\alpha^4}{2^{21}} - \frac{45}{n} - 9\vartheta_*^2 \right) n \geqslant \alpha^5 n$. □

Having established that there are at least $\Omega(n^9)$ v-absorbers with connectable pairs for every $v \in V$ we can build the absorbing path by a standard probabilistic argument. First we find a suitable selection of $\Omega(n)$ disjoint 9-tuples that contain many v-absorbers for every v, which is rendered by the following lemma. In a second step we utilise the ζ_*-connectable pairs and connect these 9-tuples to the absorbing path avoiding the reservoir set $\mathcal{R}$.

Lemma 6.8. *Given Setup 2.8, there is a set $\mathcal{F} \subseteq (V \smallsetminus \mathcal{R})^9$ with the following properties:*

(i) $|\mathcal{F}| \leqslant 8\alpha^{-5}\vartheta_*^2 n$,

(ii) *all vertices of every 9-tuple in $\mathcal{F}$ are distinct and the 9-tuples in $\mathcal{F}$ are pairwise disjoint,*

(iii) *if $(a, b, c, d, z, x, y, y', x') \in \mathcal{F}$, then $abz, bzc, zcd, xyy', yy'x' \in E$ and the pairs $ab, cd, xy, x'y'$ are ζ_*-connectable,*

(iv) *and for every $v \in V$ there are at least $2\vartheta_*^2 n$ many v-absorbers in $\mathcal{F}$.*

Proof. Set
$$\gamma = \frac{4\vartheta_*^2}{\alpha^5}$$
and consider a random selection $\mathcal{X} \subseteq (V \smallsetminus R)^9$ containing each such 9-tuple independently with probability $p = \gamma n^{-8}$. Since $\mathbb{E}[|\mathcal{X}|] \leqslant pn^9 = \gamma n$, Markov's inequality yields

$$\mathbb{P}\big(|\mathcal{X}| > 2\gamma n\big) \leqslant \frac{1}{2}. \tag{6.5}$$

Let us call two distinct 9-tuples from V^9 *overlapping* if there is a vertex occurring in both. Evidently, there are at most $81n^{17}$ ordered pairs of overlapping 9-tuples. Hence the random variable P giving the number of such pairs both of whose components are in $\mathcal{X}$ satisfies

$$\mathbb{E}[P] \leqslant 81n^{17}p^2 = 81\gamma^2 n\,.$$

By $\vartheta_* \ll \alpha$ we have $18\gamma \leqslant \vartheta^*$ and thus a further application of Markov's inequality discloses

$$\mathbb{P}\big(P > \vartheta_*^2 n\big) \leqslant \mathbb{P}\big(P > 324\gamma^2 n\big) \leqslant \frac{1}{4}\,. \tag{6.6}$$

In view of Lemma 6.7 for each vertex $v \in V$ the set $\mathcal{A}_v$ containing all v-absorbers from $(V \smallsetminus R)^9$ has the property $\mathbb{E}[|\mathcal{A}_v \cap \mathcal{X}|] \geqslant \alpha^5 n^9 p = \alpha^5 \gamma n = 4\vartheta_*^2 n$. Since $|\mathcal{A}_v \cap \mathcal{X}|$ is binomially distributed, Chernoff's inequality gives for every $v \in V$

$$\mathbb{P}\big(|\mathcal{A}_v \cap \mathcal{X}| \leqslant 3\vartheta_*^2 n\big) \leqslant \exp\big(-\Omega(n)\big) < \frac{1}{5n}\,. \tag{6.7}$$

Owing to (6.5), (6.6), and (6.7) there is an "instance" $\mathcal{F}_*$ of $\mathcal{X}$ satisfying the following:

- $|\mathcal{F}_*| \leqslant 2\gamma n$,
- $\mathcal{F}_*$ contains at most $\vartheta_*^2 n$ overlapping pairs,
- and for every $v \in V$ the number of v-absorbers in $\mathcal{F}_*$ is at least $3\vartheta_*^2 n$.

To obtain the desired set $\mathcal{F}$ we delete from $\mathcal{F}_*$ all 9-tuples containing some vertex more than once, all 9-tuples belonging to an overlapping pair, and all 9-tuples violating (iii). Then (i) is immediate from $|\mathcal{F}| \leqslant |\mathcal{F}^*|$, (ii) and (iii) hold by construction, and for (iv) we recall that v-absorbers satisfy (iii) by definition. $\square$

Finally, we are ready to build an absorbing path and thus establish Proposition 2.9.

Proof of Proposition 2.9. Let $\mathcal{F} \subseteq (V \smallsetminus \mathcal{R})^9$ be as obtained in Lemma 6.8. By condition (iii) from this lemma for every $(a, b, c, d, z, x, y, y', x') \in \mathcal{F}$ we may consider the tight paths $abzcd$ and $xyy'x'$. By (ii) these paths are mutually vertex-disjoint and by (i) the set $\mathcal{G}$ of all these paths satisfies $|\mathcal{G}| = 2|\mathcal{F}| \leqslant 16\alpha^{-5}\vartheta_*^2 n$.

Using the connecting lemma we will now prove that there is a path $P_A \subseteq H - \mathcal{R}$

(a) containing all members of $\mathcal{G}$ as subpaths,

(b) whose end-pairs are ζ_*-connectable,

(c) and whose length is at most $(3\ell + 6)|\mathcal{G}|$.

Essentially, the reason why such a path exists is that starting with any member of $\mathcal{G}$ we can construct P_A by $|\mathcal{G}| - 1$ successive applications of the connecting lemma attaching one further path from $|\mathcal{G}|$ in each step. When carrying this plan out, we need to avoid entering the reservoir and we need to be careful not to use the same vertex multiple times.

To show that this is possible we consider a maximal subset $\mathcal{G}^* \subseteq \mathcal{G}$ such that some path $P_A^* \subseteq H - \mathcal{R}$ has the properties (a), (b), and (c) enumerated above with $\mathcal{G}$ replaced by $\mathcal{G}^*$. As the end-pairs of members of $\mathcal{G}$ are by definition ζ_*-connectable we have $P_A^* \neq \varnothing$. From (c) and $1 \gg \alpha, \ell^{-1} \gg \vartheta_*$ we infer

$$|V(P_A^*)| \leqslant 2 + (3\ell + 6)|\mathcal{G}^*| \leqslant 4\ell|\mathcal{G}| \leqslant 64\ell\alpha^{-5}\vartheta_*^2 n \leqslant \vartheta_*^{3/2} n \qquad (6.8)$$

and thus our upper bound on the size of the reservoir leads to

$$|V(P_A^*)| + |\mathcal{R}| \leqslant 2\vartheta_*^{3/2} n \leqslant \frac{\vartheta_* n}{2(3\ell + 1)} \,. \qquad (6.9)$$

Assume for the sake of contradiction that $\mathcal{G}^* \neq \mathcal{G}$. Let (z, w) be the ending pair of P_A^* and let P be an arbitrary path from $\mathcal{G} \smallsetminus \mathcal{G}^*$ with starting pair (x, y). Since both (z, w) and (x, y) are ζ_*-connectable, Proposition 2.6 tells us that there are at least $\vartheta_* n^{3\ell+1}$ tight (z, w)-(x, y)-path of length $3(\ell + 1)$. By (6.9) at least half of these are disjoint from $V(P_A^*) \cup \mathcal{R}$. In particular, there is at least one such connection giving rise to a path $P_A^{**} \subseteq H - \mathcal{R}$ starting with P_A^*, ending with P and satisfying

$$|V(P_A^{**})| = |V(P_A^*)| + (3\ell + 1) + |V(P)| \leqslant |V(P_A^*)| + (3\ell + 6) \leqslant 2 + (3\ell + 6)(|\mathcal{G}^*| + 1) \,.$$

So P_A^{**} exemplifies that $\mathcal{G}^* \cup \{P\}$ contradicts the maximality of $\mathcal{G}^*$ and this contradiction proves that we have indeed $\mathcal{G}^* = \mathcal{G}$, i.e., that a path P_A with the properties (a), (b), and (c) promised above does really exist.

As proved in (6.8) this path satisfies in particular the above condition (i) of Proposition 2.9. Moreover, (ii) is the same as (b). To finally establish (iii) of Proposition 2.9 one absorbs the up to at most $2\vartheta_*^2 n$ vertices from X one by one into P_A. This is possible due to (a) combined with clause (iv) from Lemma 6.8. $\qquad\square$

§7. Almost spanning cycle

This section is dedicated to the proof of Proposition 2.10. Most of the work we need to perform concerns the construction of a long path Q in the induced subhypergraph $\widehat{H} = H - V(P_A)$ that covers "almost all" vertices, but leaves the reservoir set $\mathcal{R}$ "almost intact." Besides, the end-pairs of this path should be sufficiently connectable so that it can easily be included into C. These properties of Q are made precise by the following statement.

Lemma 7.1. *Given Setup 2.8, there is a path $Q \subseteq \widehat{H}$ such that*

(i) $|V(\widehat{H}) \smallsetminus (\mathcal{R} \cup V(Q))| \leqslant \vartheta_*^2 n,$

(ii) $|V(Q) \cap \mathcal{R}| \leqslant \vartheta_{**}^2 n,$

(iii) *and the end-pairs of Q are ζ_{**}-connectable.*

Before we prove Lemma 7.1, we deduce Proposition 2.10 from the lemma.

Proof of Proposition 2.10. Given the path $Q \subseteq H - P_A$ by Lemma 7.1, one simply connects the end-pairs of P_A with the end-pairs of Q through "free vertices" from the reservoir using Lemma 5.1. The connectability assumption of that lemma is satisfied by condition (ii) from Proposition 2.9 and by condition (iii) from Lemma 7.1. Each of these connections uses exactly $3\ell + 1$ vertices of $\mathcal{R}$. Consequently, it follows from Lemma 7.1 (ii) that at most $\vartheta_{**}^2 n + (3\ell + 1) < 2\vartheta_{**}^2 n$ vertices from $\mathcal{R}$ need to be avoided and Lemma 5.1 applies. The resulting tight cycle C contains all but at most $\vartheta_*^2 n$ vertices from $V \smallsetminus \mathcal{R}$ (see Lemma 7.1 (i)). Furthermore, since $|\mathcal{R}| \leqslant \vartheta_*^2 n$ (see Setup 2.8 and Proposition 2.7) it follows that C covers all but at most $2\vartheta_*^2 n$ as required by Proposition 2.10. $\qquad\square$

It remains to establish Lemma 7.1. This proof will occupy the remainder of this section and, as explained in Section 2, it completes the proof of our main result. In the proof we make use of the following extension of the Erdős–Gallai theorem [8] concerning the extremal problem for *long* paths. We state the result of Faudree and Schelp [10, page 151] in a form tailored for our application.

Theorem 7.2 (Faudree and Schelp). *If $G = (V, E)$ is a graph not containing a path of length $\lambda|V|$ for $\lambda > 1/2$, then $|E| \leqslant \left(\lambda^2 + (1 - \lambda)^2\right)|V|^2/2$.* $\square$

Proof of Lemma 7.1. We fix an integer M satisfying the conditions

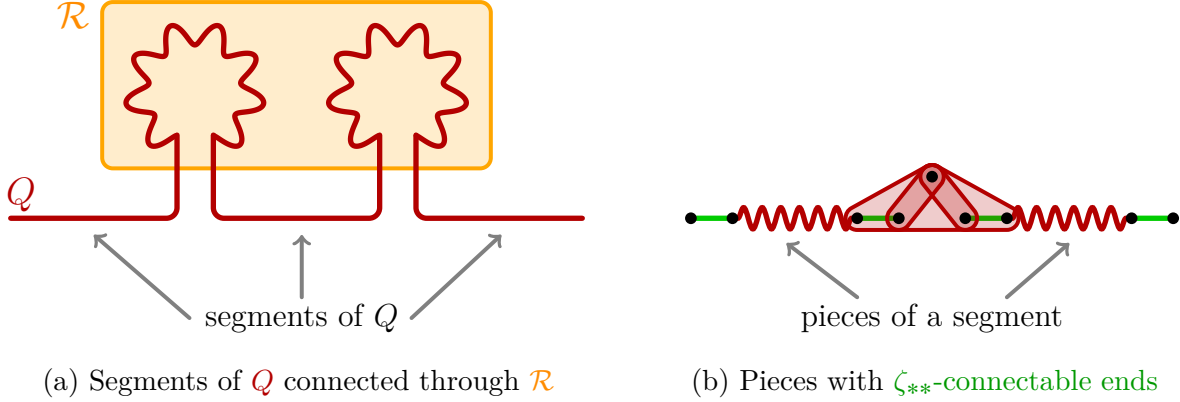$$\vartheta_{**} \gg \frac{1}{M} \gg \frac{1}{n} \quad \text{and} \quad M \equiv 2 \pmod 3. \tag{7.1}$$

(a) Segments of Q connected through $\mathcal{R}$

(b) Pieces with ζ_{**}-connectable ends

FIGURE 7.1. Segments and pieces of the tight path Q.

The desired path Q will consist of many "segments" from $\widehat{H} - \mathcal{R}$ that are connected with each other through the reservoir $\mathcal{R}$ (see Figure 7.1a). For technical reasons it will be helpful to assume that every segment F satisfies

$$|V(F)| \equiv -1 \pmod{M + 1}$$

and that it has ζ_{**}-connectable end-pairs. The former property of these segments allows us to think of them as being composed of several "pieces" consisting of M vertices each, such that any two consecutive pieces are connected with each other through one further vertex (see Figure 7.1b). These pieces will be taken from the set

$$\mathscr{P} = \left\{P \subseteq \widehat{H} - \mathcal{R} \colon P \text{ is an } M\text{-vertex tight path whose end-pairs are } \zeta_{**}\text{-connectable}\right\}.$$

Roughly speaking, the strategy of the proof below is to show that a path Q of the kind just described will satisfy the conclusion of Lemma 7.1 as soon as it is "maximal" in the sense we will make precise next. To formulate this maximality condition, it will be convenient to talk not only about the path Q itself but also about the set $\mathscr{C} \subseteq \mathscr{P}$ of pieces used in its construction. We collect all properties that we require from the pair $(\mathscr{C}, Q)$ into the definition that follows.

Candidates. A pair $(\mathscr{C}, Q)$ consisting of a subset $\mathscr{C} \subseteq \mathscr{P}$ whose members are mutually vertex-disjoint and a tight path $Q \subseteq \widehat{H}$ is said to be a *candidate* if

  (*a*) every $P \in \mathscr{C}$ is a subpath of Q,
  (*b*) if $P', P'' \in \mathscr{C}$ with $P' \neq P''$ lie on Q in such a way that no $P \in \mathscr{C}$ lies between them, then between P' and P'' there is
      (*i*) either a single vertex
      (*ii*) or there are only vertices from $\mathcal{R}$,
  (*c*) provided $\mathscr{C} \neq \varnothing$, the path Q starts and ends with a path from $\mathscr{C}$,
  (*d*) and $|V(Q) \cap \mathcal{R}| \leqslant 19\alpha^{-1}\ell|\mathscr{C}|$.

For instance, the pair consisting of the empty set and the empty path is a candidate. Now let $(\mathscr{C}, Q)$ be a candidate with $|\mathscr{C}|$ as large as possible. Suppose we would know that the set

$$U = V(\widehat{H}) \smallsetminus \big(\mathcal{R} \cup V(Q)\big) = V\big(H - P_A - \mathcal{R} - Q\big)$$

of unused vertices outside the reservoir satisfies

$$|U| \leqslant \vartheta_*^2 n. \tag{7.2}$$

We claim that then Q would have all the desired properties. Indeed by (7.2) it would satisfy (*i*) of Lemma 7.1. Since the members of $\mathscr{C}$ are mutually disjoint, we have $|\mathscr{C}| \leqslant \frac{n}{M}$ so from (*d*) and $M \gg \vartheta_{**}^{-1}, \ell, \alpha^{-1}$ we get (*ii*). Moreover (*c*) implies part (*iii*) of Lemma 7.1.

Hence, for the rest of the argument we assume that (7.2) is false and intend to derive a contradiction by constructing a "better" candidate $(\mathscr{C}', Q')$ with a larger family $\mathscr{C}'$. Obviously, the path of such a candidate will need to contain some vertices from U and to prepare ourself for a later stage of the argument we will now deal with the connectability properties of the robust subgraphs of these vertices. More precisely, for each $u \in U$ we define a subgraph $\overline{R}_u \subseteq R_u$ with the same set of vertices by deleting all edges that are not ζ_{**}-connectable. Owing to Fact 4.1 we have, in particular,

$$\sum_{u \in U} \big(e(R_u) - e(\overline{R}_u)\big) \leqslant \frac{\zeta_{**}}{2}n^3.$$

Consequently, the set

$$U_{\mathrm{bad}} = \big\{u \in U : e(\overline{R}_u) \leqslant e(R_u) - \tfrac{1}{8}\alpha n^2\big\}$$

satisfies $|U_{\mathrm{bad}}| \leqslant 8\zeta_{**}\alpha^{-1}n$ and, by $\zeta_{**} \ll \vartheta_*, \alpha$, this leads to

$$|U_{\mathrm{bad}}| \leqslant \frac{1}{2}\vartheta_*^2 n. \tag{7.3}$$

For each $u \in U \smallsetminus U_{\mathrm{bad}}$ we introduce the real number $\eta_u \in \big[\tfrac{2}{3} + \tfrac{\alpha}{2}, 1\big]$ by

$$\big|V\big(\overline{R}_u\big)\big| = |V(R_u)| = \eta_u n \tag{7.4}$$

and observe that part (ii) from Proposition 2.3 implies

$$e\left(\overline{R}_u\right) \geqslant e\left(R_u\right) - \frac{1}{8}\alpha n^2 \geqslant \left(\frac{5}{9} + \frac{\alpha}{4} - (1 - \eta_u)^2\right)\frac{n^2}{2}. \tag{7.5}$$

Useful societies. Let $B_1, \ldots, B_{|\mathscr{C}|}$ be the vertex sets of the paths belonging to $\mathscr{C}$ and fix an arbitrary partition

$$U = B_{|\mathscr{C}|+1} \uplus \ldots \uplus B_\nu \uplus B',$$

with

$$|B_{|\mathscr{C}|+1}| = \cdots = |B_\nu| = M \qquad \text{and} \qquad |B'| < M.$$

The sets belonging to

$$\mathscr{B} = \{B_1, \ldots, B_\nu\}$$

will be referred to as *blocks*. The size of their union

$$B = B_1 \uplus B_2 \uplus \ldots \uplus B_\nu, \tag{7.6}$$

in view of candidacy property (b), can be bounded from below by

$$|B| = M\nu \geqslant n - |V(P_A)| - |\mathcal{R}| - |\mathscr{C}| - |B'| \geqslant (1 - \vartheta_* - \vartheta_*^2)n - \nu - M,$$

where we used bounds on $|V(P_A)|$ and $|\mathcal{R}|$ from Propositions 2.9 and 2.7. Thus, observing that $\nu \leqslant n/M$ and recalling that $M \gg 1/\vartheta_*$, we obtain

$$|B| \geqslant (1 - 2\vartheta_*)n.$$

Consequently, by (7.4) and (7.5), recalling that $\vartheta_* \ll \alpha$, for every $u \in U \smallsetminus U_{\text{bad}}$

$$\frac{|V\left(\overline{R}_u\right) \cap B|}{M\nu} \leqslant \frac{\eta_u}{1 - 2\vartheta_*} \leqslant \eta_u + \frac{\alpha}{36} \qquad \text{and} \qquad \frac{e_{\overline{R}_u}(B)}{M^2\binom{\nu}{2}} \geqslant \frac{5}{9} + \frac{2\alpha}{9} - (1 - \eta_u)^2, \tag{7.7}$$

where $e_G(A)$ stands for the number of edges in $G[A]$, the subgraph of G induced by a subset of vertices $A \subseteq V(G)$.

A *society* is a set of m blocks, where

$$m = 1 + \left\lceil \frac{36}{\alpha} \right\rceil. \tag{7.8}$$

The collection of all $\binom{\nu}{m}$ societies will be denoted by $\mathfrak{S}$.

Definition 7.3. *A society $\mathcal{S} \in \mathfrak{S}$ is said to be useful for a vertex $u \in U \smallsetminus U_{\text{bad}}$ if for its union $S = \bigcup \mathcal{S}$ and the real number τ defined by $|S \cap V(\overline{R}_u)| = \tau|S|$,*

$$e_{\overline{R}_u}\left(S \cap V(\overline{R}_u)\right) \geqslant \left(\frac{5}{9} + \frac{\alpha}{9} - (1 - \eta_u)(1 + \eta_u - 2\tau)\right)\frac{|S|^2}{2}.$$

The following claim may explain the terminology used in Definition 7.3.

Claim 7.4. *If a society $\mathcal{S} \in \mathfrak{S}$ is useful for a vertex $u \in U \smallsetminus U_{\text{bad}}$, then the graph $\overline{R}_u$ contains a graph path on $\frac{2}{3}(M+1)(m+6)$ vertices all of which belong to $S = \bigcup \mathcal{S}$.*

Proof. Notice that by (7.1) the number $\frac{2}{3}(m+6)(M+1)$ is indeed an integer. Since

$$1 + \frac{36}{\alpha} \overset{(7.8)}{\leqslant} m \overset{(7.8)}{\leqslant} 2 + \frac{36}{\alpha} \overset{(2.1)}{\leqslant} \frac{37}{\alpha}$$

it follows from the position of M in our hierarchy (7.1) that

$$\frac{36m}{\alpha m - 36} \leqslant \frac{36m}{\alpha} \leqslant \frac{36 \cdot 37}{\alpha^2} \leqslant M.$$

This implies that $\frac{2}{3}(M+1)(m+6) \leqslant \left(\frac{2}{3} + \frac{\alpha}{9}\right)Mm$. Thus it suffices to find a path in the graph $\overline{R}_u$ traversing $\left(\frac{2}{3} + \frac{\alpha}{9}\right)Mm$ vertices all of which belong to $S = \bigcup \mathcal{S}$. Let us define a real number ϱ by

$$e_{\overline{R}_u}\big(S \cap V(\overline{R}_u)\big) = \varrho \frac{(Mm)^2}{2} = \varrho \frac{|S|^2}{2}.$$

Clearly $\tau^2 \geqslant \varrho$ and the definitions of τ and ϱ yield

$$e_{\overline{R}_u}\big(S \cap V(\overline{R}_u)\big) = \frac{\varrho}{\tau^2} \cdot \frac{|S \cap V(\overline{R}_u)|^2}{2}. \tag{7.9}$$

Moreover, the usefulness of $\mathcal{S}$ implies

$$\tau^2 \geqslant \varrho \geqslant \frac{5}{9} + \frac{\alpha}{9} - (1 - \eta_u)(1 + \eta_u - 2\tau), \tag{7.10}$$

and thus we have

$$\tau\left(\tau - \frac{2}{3}\right) \geqslant \frac{\alpha}{9} + \left(\eta_u - \frac{2}{3}\right)\left(\eta_u + \frac{2}{3} - 2\tau\right). \tag{7.11}$$

We have defined η_u in (7.4) so that $\eta_u \geqslant \frac{2}{3} + \frac{\alpha}{2}$. Consequently, if $\tau < \frac{2}{3} + \frac{\alpha}{9}$, then

$$\frac{\alpha}{9} + \left(\eta_u - \frac{2}{3}\right)\left(\eta_u + \frac{2}{3} - 2\tau\right) > \frac{\alpha}{9} > \tau\left(\tau - \frac{2}{3}\right),$$

a contradiction with (7.11). This proves that

$$\tau \geqslant \frac{2}{3} + \frac{\alpha}{9}. \tag{7.12}$$

The right-hand side of (7.10) rewrites as $\frac{5}{9} + \frac{\alpha}{9} - (1-\tau)^2 + (\eta_u - \tau)^2$ and for this reason we have

$$\varrho \geqslant \frac{5}{9} + \frac{\alpha}{9} - (1-\tau)^2. \tag{7.13}$$

Owing to (7.12), we deduce from Theorem 7.2 (applied with $\lambda = (2/3 + \alpha/9)/\tau$ to the induced subgraph of $\overline{R}_u$ on the set $S \cap V(\overline{R}_u)$ of size $\tau|S| = \tau m M$) that the failure of our claim would imply

$$\frac{\varrho}{\tau^2} \cdot \frac{|S \cap V(\overline{R}_u)|^2}{2} \overset{(7.9)}{=} e_{\overline{R}_u}\big(S \cap V(\overline{R}_u)\big) \leqslant \left(\left(\frac{\frac{2}{3} + \frac{\alpha}{9}}{\tau}\right)^2 + \left(1 - \frac{\frac{2}{3} + \frac{\alpha}{9}}{\tau}\right)^2\right) \frac{|S \cap V(\overline{R}_u)|^2}{2}.$$

Consequently, we arrive at

$$\left(\frac{2}{3}+\frac{\alpha}{9}\right)^2 + \left(\tau - \frac{2}{3} - \frac{\alpha}{9}\right)^2 \geqslant \varrho \overset{(7.13)}{\geqslant} \frac{5}{9} + \frac{\alpha}{9} - (1-\tau)^2\,,$$

whence

$$\left(\frac{2}{3}+\frac{\alpha}{9}\right)^2 + \left(\frac{1}{3} - \frac{\alpha}{9}\right)^2 \geqslant \frac{5}{9} + \frac{\alpha}{9}\,,$$

i.e.,

$$\frac{2}{27}\alpha + \frac{2}{81}\alpha^2 \geqslant \frac{1}{9}\alpha\,,$$

contrary to $\alpha \ll 1$. This completes the proof of Claim 7.4. $\square$

A counting argument shows that there exists a society that is useful for many vertices.

Claim 7.5. *There is a society $\mathcal{S}' \in \mathfrak{S}$ useful for at least $\frac{\alpha}{18}|U \smallsetminus U_{\mathrm{bad}}|$ vertices $u \in U \smallsetminus U_{\mathrm{bad}}$.*

Proof. The claim follows by double counting from the assertion that for every $u \in U \smallsetminus U_{\mathrm{bad}}$ the number of useful societies is at least $\frac{\alpha}{18}|\mathfrak{S}|$, which we verify below. For that consider a vertex $u \in U \smallsetminus U_{\mathrm{bad}}$. Suppose that $\gamma\binom{\nu}{m}$ of all $|\mathfrak{S}| = \binom{\nu}{m}$ societies are useful for u. For $i \in [\nu]$ set

$$|B_i \cap V(\overline{R}_u)| = \tau_i M$$

and for all i and j with $1 \leqslant i < j \leqslant \nu$ set

$$e_{\overline{R}_u}\big(B_i \cap V(\overline{R}_u), B_j \cap V(\overline{R}_u)\big) = \varrho_{ij} M^2\,.$$

By Definition 7.3, if the society $\mathcal{S} = \{B_1, \dots, B_m\}$ is not useful for u, then

$$\sum_{1 \leqslant i < j \leqslant m} \varrho_{ij} \leqslant \frac{e_{\overline{R}_u}(S \cap V(\overline{R}_u))}{M^2} < \left(\frac{5}{9} + \frac{\alpha}{9} - (1-\eta_u^2)\right)\frac{m^2}{2} + (1-\eta_u)m\sum_{1 \leqslant i \leqslant m} \tau_i\,.$$

If $\mathcal{S}$ is useful we still have the trivial bound

$$\sum_{1 \leqslant i < j \leqslant m} \varrho_{ij} \leqslant \binom{m}{2}\,.$$

Summing over all societies we infer

$$\binom{\nu - 2}{m - 2}\sum_{1 \leqslant i < j \leqslant \nu} \varrho_{ij} \leqslant \gamma\binom{\nu}{m}\binom{m}{2} + \binom{\nu}{m}\left(\frac{5}{9} + \frac{\alpha}{9} - (1-\eta_u^2)\right)\frac{m^2}{2}$$

$$+ (1-\eta_u)m\binom{\nu - 1}{m - 1}\sum_{1 \leqslant i \leqslant \nu} \tau_i\,.$$

Dividing by $\binom{\nu-2}{m-2}\binom{\nu}{2} = \binom{\nu}{m}\binom{m}{2}$ one learns that the set B introduced in (7.6) satisfies

$$\frac{e_{\overline{R}_u}(B)}{M^2\binom{\nu}{2}} \leqslant \gamma + \frac{m}{m-1}\left(\frac{5}{9} + \frac{\alpha}{9} - (1-\eta_u^2)\right) + 2(1-\eta_u)\frac{m}{m-1}\frac{|V(\overline{R}_u) \cap B|}{M\nu}\,.$$

Owing to (7.7) this yields

$$\frac{5}{9} + \frac{2\alpha}{9} - (1 - \eta_u)^2 \leqslant \gamma + \left(1 + \frac{1}{m-1}\right)\left(\frac{5}{9} + \frac{\alpha}{9} - (1 - \eta_u^2) + 2(1 - \eta_u)\left(\eta_u + \frac{\alpha}{36}\right)\right)$$

whence

$$\frac{\alpha}{9} \leqslant \gamma + \frac{\alpha}{18}(1 - \eta_u) + \frac{1}{m-1}\,.$$

Hence, the choice of m in (7.8) and the bound $\eta_u > \frac{2}{3} > \frac{1}{2}$ yield indeed that $\gamma \geqslant \frac{\alpha}{18}$.   □

For the rest of the proof let $\mathcal{S}'$ from Claim 7.5 be fixed. By (7.3) and the purported falsity of (7.2) this means that the set

$$U' = \{u \in U \smallsetminus U_{\text{bad}} : \mathcal{S}' \text{ is useful for } u\}$$

satisfies $|U'| \geqslant \frac{\alpha\vartheta_*^2}{36}n$. Now we apply Claim 7.4 to each $u \in U'$. Each time the outcome may be regarded as a sequence of $\frac{2}{3}(M+1)(m+6)$ distinct vertices from the set $S' = \bigcup \mathcal{S}'$. Due to $|S'| = Mm$ there are no more than $(Mm)!$ such sequences and thus there is a set $U'' \subseteq U'$ with

$$|U''| \geqslant \frac{\alpha\vartheta_*^2 n}{36(Mm)!} \geqslant \frac{1}{3}(M+1)(m+6) - 1$$

such that all graphs $\overline{R}_u$ with $u \in U''$ contain a common path W on $\frac{2}{3}(M+1)(m+6)$ vertices.

Augmenting Q. Using the vertices of W and $\frac{1}{3}(M+1)(m+6) - 1$ arbitrary vertices from U'' we obtain a tight path $T \subseteq (\widehat{H} - \mathcal{R})$ with $|V(T)| = (M+1)(m+6) - 1$ and every vertex of the T with a position divisible by 3 is a vertex from U'' (see Figure 7.2).
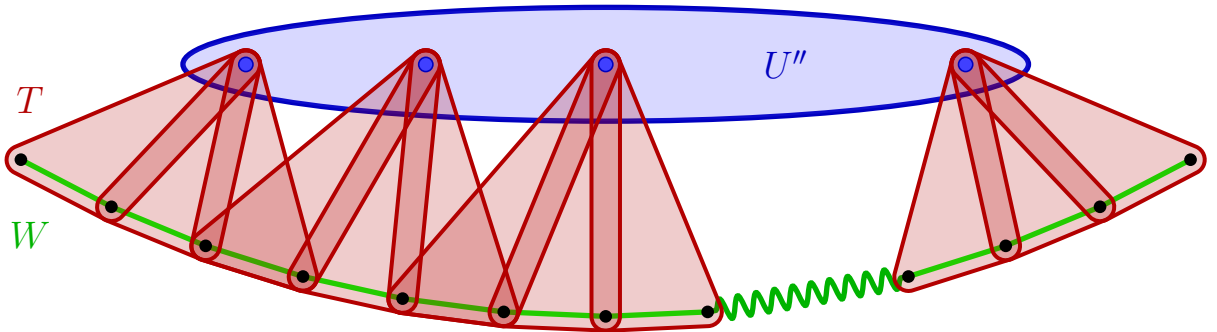

FIGURE 7.2. Tight path T on the graph path W of ζ_{**}-connectable pairs.

Next we split the path T into $(m+6)$ tight paths $P_1, \ldots, P_{m+6}$ on M vertices each such that $T = P_1 x_1 P_2 x_2 \ldots x_{m+5} P_{m+6}$ for some $x_1, \ldots, x_{m+5} \in V$. In fact, owing to (7.1) the vertices $x_1, \ldots, x_{m+5}$ have a position divisible by 3 on the path T and, therefore, they

belong to U''. Consequently, the end-pairs of the paths $P_1, \ldots, P_{m+6}$ consist of consecutive vertices from W and, hence, they are ζ_{**}-connectable. In other words,

$$P_1, \ldots, P_{m+6} \in \mathscr{P}.$$

Now let $\mathscr{C}^-$ be the collection of those paths from $\mathscr{C}$ whose vertex sets belong to the society $\mathcal{S}'$, i.e., the paths from $\mathscr{C}^- \subseteq \mathscr{C}$ are blocks from the society $\mathcal{S}'$, and put

$$\mathscr{C}^0 = (\mathscr{C} \smallsetminus \mathscr{C}^-) \cup \{P_1, \ldots, P_{m+6}\}.$$

Since $|\mathscr{C}^-| \leqslant |\mathcal{S}'| = m$, we have $|\mathscr{C}^0| \geqslant |\mathscr{C}| + 6$ and thus to derive the desired contradiction it is enough to construct a path Q^0 such that $(\mathscr{C}^0, Q^0)$ is a candidate. The idea for doing so is to take the subpaths into which the removal of $\mathscr{C}^-$ splits Q as well as the path T and to connect all of them by means of Lemma 5.1. Of course we may also need to remove several vertices of the type mentioned in condition $(b)\,(i)$ and in case $\mathscr{C}^-$ should contain the initial or terminal part of Q we might also need to disregard some $\mathcal{R}$-vertices in order to achieve that Q^0 satisfies (c). The things that remain to be checked are

   (1) that we still have enough space in the reservoir to create the desired connections by applications of Lemma 5.1

   (2) and that the new pair $(\mathscr{C}^0, Q^0)$ will again obey condition (d).

Since $|\mathscr{C}^-| \leqslant m$ at most $m + 1$ successive applications of Lemma 5.1 are required to connect all pieces for building Q^0. Since $(\mathscr{C}, Q)$ satisfies (d), we know

$$|V(Q) \cap \mathcal{R}| \leqslant \frac{19\ell|\mathscr{C}|}{\alpha} \leqslant \frac{19\ell n}{\alpha M} \leqslant \vartheta_{**}^2 n - (3\ell + 1)m$$

and, hence, there arises no problem with (1).

Utilising the same condition (d) more carefully we obtain

$$|V(Q^0) \cap \mathcal{R}| \leqslant |V(Q) \cap \mathcal{R}| + (3\ell + 1)(m + 1) \leqslant \frac{19\ell|\mathscr{C}|}{\alpha} + (3\ell + 1)(m + 1).$$

So our choice of m in (7.8) and $1 \gg \alpha, \ell^{-1}$ lead to

$$|V(Q^0) \cap \mathcal{R}| \leqslant \frac{19\ell|\mathscr{C}|}{\alpha} + (3\ell + 1)\left(\frac{36}{\alpha} + 3\right) \leqslant \frac{19\ell}{\alpha}(|\mathscr{C}| + 6).$$

In the light of $|\mathscr{C}^0| \geqslant |\mathscr{C}| + 6$ this shows that $(\mathscr{C}^0, Q^0)$ obeys condition (d) and, hence, it is indeed a candidate. As it contradicts the maximality of $(\mathscr{C}^0, Q^0)$ we have thereby proved the validity of (7.2) and as said above Lemma 7.1 is thereby proved as well. $\square$

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