Completeness of projective special Kähler and quaternionic Kähler manifolds

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Dedicated to Simon Salamon on the occasion of his 60th birthday

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Abstract

We prove that every projective special Kähler manifold with *regular boundary behaviour* is complete and defines a family of complete quaternionic Kähler manifolds depending on a parameter $c \geq 0$. We also show that, irrespective of its boundary behaviour, every complete projective special Kähler manifold with *cubic prepotential* gives rise to such a family. Examples include non-trivial deformations of non-compact symmetric quaternionic Kähler manifolds.

Keywords: Special Kähler manifolds, quaternionic Kähler manifolds, c-map, Ferrara-Sabharwal metric, one-loop deformation, completeness

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Contents

Introduction

Quaternionic Kähler manifolds constitute a much studied class of Einstein manifolds of special holonomy [B]. All known complete examples of positive scalar curvature are symmetric of compact type (Wolf spaces) and it has been conjectured that there are no more complete quaternionic Kähler manifolds of positive scalar curvature [LS]. Besides the noncompact duals of the Wolf spaces, there exist also nonsymmetric complete examples of negative scalar curvature including locally symmetric spaces, nonsymmetric homogeneous spaces (Alekseevsky spaces) and deformations of quaternionic hyperbolic space [L]. Our work is motivated by the desire to obtain further complete examples of quaternionic Kähler manifolds using ideas from supergravity and string theory.

Based on general supersymmetry arguments [BW] and dimensional reduction in field theory it has been known for a long time in the physics community that projective special Kähler manifolds (see Definition 3) are related to quaternionic Kähler manifolds of negative scalar curvature. This correspondence, known as the supergravity c-map, was established by Ferrara and Sabharwal [FS] who explicitly associated a quaternionic Kähler metric with every projective special Kähler domain (see Definition 5), cf. [Hi] for another proof. It was shown in [CHM] that the supergravity c-map maps every complete projective special Kähler manifold to a complete quaternionic Kähler manifold.

Motivated by the fact that in the low energy limit string theory is described by supergravity, Robles Llana, Saueressig and Vandoren [RSV] proposed a deformation of the Ferrara-Sabharwal metric (or supergravity c-map metric) depending on a real parameter. This deformation, know as the one-loop deformation, is interpreted as the full perturbative quantum correction (with no higher loop corrections) of supergravity when embedded into string theory. It was proven in [ACDM] using an indefinite version of the HK/QK correspondence [ACM] that the one-loop deformation of the Ferrara-Sabharwal metric is indeed quaternionic Kähler on its domain of positivity. As a corollary, one obtains a new proof of the quaternionic Kähler property for the (undeformed) Ferrara-Sabharwal metric. It was also found that the completeness of the metric depends on the sign of the deformation parameter. In particular, it was shown that the one-loop deformation of the complex hyperbolic plane is complete for positive deformation parameter and incomplete for negative deformation parameter.

The purpose of this paper is to give general completeness results for projective special Kähler manifolds and one-loop deformations of Ferrara-Sabharwal metrics. These results make it possible to construct many new explicit complete quaternionic Kähler manifolds of negative scalar curvature by the supergravity c-map and its one-loop quantum correction.

After reviewing some basic definitions and facts concerning special Kähler manifolds in the first section, we introduce the notion of regular boundary behaviour for special Kähler manifolds in the second section. The main result of that section is that every projective special Kähler manifold with regular boundary behaviour is complete, see Theorem 7 and its Corollary 8 for projective special Kähler domains.

In the third section we study the one-loop deformation of Ferrara-Sabharwal metrics for nonnegative deformation parameter. We show that the one-loop deformation is not only defined in the case of projective special Kähler domains but is a globally defined one-parameter family of quaternionic Kähler metrics for every projective special Kähler manifold, see Theorem 12. Moreover, we show that the resulting quaternionic Kähler manifolds carry a globally defined integrable complex structure subordinate to the quaternionic structure.

In the fourth section we prove the completeness of the one-loop deformation for nonnegative deformation parameter under the assumption that the initial projective special Kähler manifold has either regular boundary behaviour (see Theorem 13) or is complete with cubic prepotential (see Theorem 27). The latter projective special Kähler manifolds are precisely those which can be obtained by dimensional reduction from five-dimensional supergravity [DV] with complete scalar geometry [CHM]. The corresponding construction is known as the supergravity r-map, which maps projective special real manifolds to projective special Kähler domains.

As the simplest[1] application of Theorem 27 (see Example 28) we discuss a one-parameter deformation of the metric of the noncompact symmetric space $G_2^*/SO(4)$ by locally inhomogeneous complete quaternionic Kähler metrics, where G_2^* denotes the non-compact real form of the complex Lie group of type G_2. In fact, Theorem 27 implies the completeness of the one-loop deformation for all the symmetric quaternionic Kähler manifolds of noncompact type with exception of the quaternionic hyperbolic spaces (which are not in the image of the supergravity c-map) and the spaces $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$.

Similarly, applying Theorem 13 to the complex hyperbolic space (which is a projective special Kähler domain with regular boundary behaviour) we obtain the completeness of the one-parameter deformation of the remaining symmetric spaces $\tilde{X}(n+1)$, see Example 14.

Based on the effective necessary and sufficient completeness criterion for projective special real manifolds provided in [CNS, Thm. 2.6], it is easy to construct many more examples of complete projective special Kähler domains with cubic prepotential (see for example [CDL] and work in progress by Jüngling, Lindemann and the first two authors) and corresponding one-loop deformed quaternionic Kähler manifolds by Theorem 27.

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1 Preliminaries

1.1 Conical and projective special Kähler manifolds

First we recall some basic facts and definitions of special Kähler geometry [ACD, CM].

Definition 1. *A* conical affine special Kähler manifold (M, J, g, ∇, ξ) *is a pseudo-Kähler manifold* (M, J, g) *endowed with a flat torsion-free connection* ∇ *and a vector field* ξ *such that*

[1]The corresponding projective special real manifold is a point.

1. $\nabla\omega = 0$, *where* $\omega = g(J.,.)$ *is the Kähler form,*

2. $d^{\nabla} J = 0$, *where* J *is considered as a 1-form with values in* TM,

3. $\nabla\xi = D\xi = \mathrm{Id}$, *where* D *is the Levi-Civita connection and*

4. g *is positive definite on the distribution* $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ *and negative definite on* $\mathcal{D}^{\perp}$.

Note that the affine special Kähler metric g has the global Kähler potential $f = g(\xi, \xi)$ in the sense that

$$\frac{i}{2}\partial\bar{\partial} f = \omega.$$

Furthermore the vector fields ξ and $J\xi$ generate a holomorphic homothetic action of a 2-dimensional Abelian[2] Lie algebra and $J\xi$ is a Killing vector field.

Proposition 2. *Let* (M, J, g, ∇, ξ) *be a conical affine special Kähler manifold such that the vector fields* ξ *and* $J\xi$ *generate a principal* $\mathbb{C}^*$*-action. Then the degenerate symmetric tensor field*

$$g' := -\frac{g}{f} + \frac{\alpha^2 + (J^*\alpha)^2}{f^2}, \tag{1.1}$$

where $\alpha := g(\xi, \cdot) = \frac{1}{2}df$, *induces a Kähler metric* $\bar{g}$ *on the quotient (complex) manifold* $\bar{M}$.

Proof: It suffices to check that the kernel of g' is exactly $\mathcal{D}$, the distribution tangent to the $\mathbb{C}^*$-orbits, and that g' is invariant under the $\mathbb{C}^*$-action. □

Definition 3. *A* projective special Kähler manifold $(\bar{M}, \bar{g})$ *is a quotient as in the previous proposition with canonical projection* $\pi\colon M \to \bar{M}$.

Notice that the projective special Kähler metric is related to the tensor field (1.1) by $g' = \pi^*\bar{g}$.

1.2 Conical and projective special Kähler domains

In this section we describe an important class of special Kähler manifolds, the so-called special Kähler domains. It is known that every special Kähler manifold is locally isomorphic to a special Kähler domain [ACD].

Let $F : M \to \mathbb{C}$ be a holomorphic function on a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that

[2] Note that a (real) holomorphic vector field X always commutes with JX: $\mathcal{L}_X(JX) = (\mathcal{L}_X J)X = 0$

(i) F is homogeneous of degree 2, that is $F(az) = a^2F(z)$ for all $z \in M$, $a \in \mathbb{C}^*$,

(ii) the real matrix $(N_{IJ}(z))_{I,J=0,\dots n}$, defined by

$$N_{IJ}(z) := 2\mathrm{Im}\, F_{IJ}(z) = -i(F_{IJ}(z) - \overline{F_{IJ}(z)}),$$

is of signature $(1, n)$ for all $z \in M$, where $F_I := \frac{\partial F}{\partial z^I}$, $F_{IJ} := \frac{\partial^2 F}{\partial z^I \partial z^J}$ etc.,

(iii) $f(z) := \sum N_{IJ}(z) z^I \bar{z}^J > 0$ for all $z \in M$.

Definition 4. *A* conical special Kähler domain (M, g, F) *is a* $\mathbb{C}^*$*-invariant domain* $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ *endowed with a holomorphic function* F *(called* holomorphic prepotential*) as above and with the pseudo-Riemannian metric*

$$g = \sum N_{IJ} dz^I d\bar{z}^J.$$

Notice that g has signature $(2, 2n)$ and is pseudo-Kähler with the Kähler potential f. A conical special Kähler domain becomes a conical special Kähler manifold if we endow it with the complex structure J and the position vector field ξ induced from the ambient space $\mathbb{C}^{n+1}$. The flat connection ∇ is induced by the standard flat connection on $\mathbb{R}^{2n+2}$ via the immersion $M \ni (z^0, \dots, z^n) \mapsto \mathrm{Re}(z^0, \dots, z^n, F_0, \dots, F_n)$.

Next we consider the domain $\bar{M} = \pi(M) \subset \mathbb{C}P^n$ which is the image of M under the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n.$$

The quotient manifold $\bar{M}$ inherits a (positive definite) Kähler metric $\bar{g}$ uniquely determined by

$$\pi^* \bar{g} = -\frac{g}{f} + \frac{\alpha^2 + (J^*\alpha)^2}{f^2}, \tag{1.2}$$

where $\alpha := g(\xi, \cdot) = \frac{1}{2} df$.

Definition 5. *A* projective special Kähler domain $(\bar{M}, \bar{g})$ *is the quotient* $\bar{M}$ *of a conical special Kähler domain* M *by the natural* $\mathbb{C}^*$*-action, endowed with its canonical Kähler metric* $\bar{g}$.

Now we describe a local Kähler potential for the projective special Kähler metric $\bar{g}$ in a neighborhood of a point $p \in \bar{M}$. This yields a local Kähler potential $\mathcal{K}$ for projective special Kähler manifolds. Let λ be any linear function on $\mathbb{C}^{n+1}$ such that p lies in the affine chart $\{\lambda \neq 0\} \subset \mathbb{C}P^n$. The function $\frac{f}{\lambda\bar{\lambda}}$ is homogeneous of degree 0 on $M \cap \{\lambda \neq 0\}$ and therefore well defined on $\pi(M \cap \{\lambda \neq 0\}) = \bar{M} \cap \{\lambda \neq 0\}$. Then

$$\mathcal{K} := -\log\left(\frac{f}{\lambda\bar{\lambda}}\right)$$

is a Kähler potential for the metric $\bar{g}$ on the open subset $\bar{M} \cap \{\lambda \neq 0\}$. By an appropriate choice of linear coordinates $(z^0, \dots, z^n)$ on $\mathbb{C}^{n+1}$ we can assume that $\lambda = z^0$.

2 Special Kähler manifolds with regular boundary behaviour

Now we consider certain compactifications of projective special Kähler manifolds by adding a boundary. As a first step we consider conical affine special Kähler manifolds with boundary.

Definition 6. *A* conical affine special Kähler manifold with regular boundary behaviour *is a conical affine special Kähler manifold* (M, J, g, ∇, ξ) *which admits an embedding* $i\colon M \to \mathcal{M}$ *into a manifold with boundary* $\mathcal{M}$ *such that* $i(M) = \mathrm{int}\, \mathcal{M} := \mathcal{M} \setminus \partial\mathcal{M}$ *and the tensor fields* (J, g, ξ) *smoothly extend to* $\mathcal{M}$ *such that, for all boundary points* $p \in \partial\mathcal{M}$, $f(p) = 0$, $df_p \neq 0$ *and* g_p *is negative semi-definite on* $\mathcal{H}_p := T_p\partial\mathcal{M} \cap J(T_p\partial\mathcal{M})$ *with kernel* $\mathrm{span}\{\xi_p, J\xi_p\}$, *where* $f = g(\xi, \xi)$.

Note that for the smooth extendability of the metric g it is sufficient to assume that J and f smoothly extend to the boundary. Indeed this follows from the fact that f is a Kähler potential for g.

As in the case of empty boundary, we will assume that ξ and $J\xi$ generate a principal $\mathbb{C}^*$-action on the manifold $\mathcal{M}$. Then $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^*$ is a manifold with boundary and its interior $\bar{M} = M/\mathbb{C}^*$ is a projective special Kähler manifold with projective special Kähler metric $\bar{g}$. If the manifold $\bar{\mathcal{M}}$ with boundary is compact, then we will call $(\bar{M}, \bar{g})$ a projective special Kähler manifold with regular boundary behaviour.

The projective special Kähler domains considered in Remark 1 below, are examples of projective special Kähler manifolds with regular boundary behaviour.

Theorem 7. *Every projective special Kähler manifold with regular boundary behaviour is complete.*

Proof: Consider the underlying conical affine special Kähler manifold (M, J, g, ∇, ξ) with regular boundary behaviour. We first show that g_p is nondegenerate for every point $p \in \partial\mathcal{M}$. By definition of regular boundary behavior we have $g|_{\mathcal{H}_p \times \mathcal{H}_p} \leq 0$ with kernel $\mathrm{span}\{\xi_p, J\xi_p\}$. Let $\mathcal{H}'_p \subset \mathcal{H}_p$ be a complex hyperplane not containing ξ_p. Then g_p is negative definite on $\mathcal{H}'_p$. For dimensional reasons $\mathcal{H}_p$ is a real codimension one subspace of $T_p\partial\mathcal{M}$. Let w be a vector in the complement of $\mathcal{H}_p$ in $T_p\partial\mathcal{M}$. By applying the Gram-Schmidt procedure we can assume that w is g_p-orthogonal to $\mathcal{H}'_p$ in $T_p\partial\mathcal{M}$. Then $\mathrm{span}\{w, Jw\}$ is g_p-orthogonal to $\mathcal{H}'_p$ by the J-invariance of g_p. Since the real 4-dimensional vector space $\mathrm{span}\{\xi_p, J\xi_p, w, Jw\}$ is g_p-orthogonal to $\mathcal{H}'_p$ in $T_p\mathcal{M}$ it suffices to show that

g_p is nondegenerate on span$\{\xi_p, J\xi_p, w, Jw\}$. By continuity of df and ξ we know that

$$2g_p(\xi_p, .) = df_p.$$

Since $Jw \notin T_p\partial\mathcal{M}$ and $w \in T_p\partial\mathcal{M}$ we have

$$0 \neq df_p(Jw) = 2g_p(\xi_p, Jw) = -2g_p(J\xi_p, w) \text{ and } 0 = df_p(w) = 2g_p(\xi_p, w) = 2g_p(J\xi_p, Jw).$$

Now by considering the representing matrix of g_p on span$\{\xi_p, J\xi_p, w, Jw\}$ and using that g_p vanishes on span$\{\xi_p, J\xi_p\}$ we see that g_p is nondegenerate. This proves that g_p is nondegenerate and, therefore, of signature $(2, 2n)$ by continuity.

Let $\gamma : I \to \bar{M}$, $I = [0, b)$, $0 < b \leq \infty$, be a curve which is not contained in any compact subset of $\bar{M}$. We will show that γ has infinite length under the assumption of regular boundary behaviour. Call a point $p \in \bar{\mathcal{M}}$ an accumulation point of γ if there exists a sequence $t_i \in I$ such that $\lim t_i = b$ and $\lim \gamma(t_i) = p$. By our assumption, γ has at least one accumulation point $\bar{p}_0$ on the boundary. We distinguish two cases:

1^{st} case: γ has exactly one accumulation point $\overline{p}_0$ which necessarily lies on the boundary. Under this hypothesis, for every neighborhood of $\bar{p}_0$ we can find $a \in I$ such that $\gamma([a, b))$ is fully contained in that neighborhood.

Choose a point $p_0 \in \pi^{-1}(\bar{p}_0) \subseteq \partial\mathcal{M}$. Since the signature of g_{p_0} is $(2, 2n)$, there exists a complex hyperplane $E \subset T_{p_0}\mathcal{M}$ on which $-g$ is positive definite. Let M' denote a complex hypersurface through p_0 tangent to E such that $-g|_{TM' \times TM'}$ is positive definite.

The pullback of the projective special Kähler metric can be estimated on $N = \mathrm{int}(M')$ as follows

$$(\pi^*\bar{g})|_N = -\left.\frac{g}{f}\right|_N + \left.\frac{\alpha^2 + (J^*\alpha)^2}{f^2}\right|_N \geq \left.\frac{\alpha^2}{f^2}\right|_N = \frac{df^2}{4f^2}. \tag{2.1}$$

Now we show how this implies that γ has infinite length. We can assume by shifting the initial point of the interval I that γ is fully contained in $\pi(N) \subset \bar{M}$. Let $\gamma_N : I \to N$ be the curve which projects to γ under $\pi|_N$. Then there exists a sequence $t_i \in [0, b)$ such that $f(\gamma_N(t_i)) \to 0$ and $\gamma_N([0, t_i]) \subset \gamma_N(I) \subset N$. In view of (2.1), we have

$$\begin{aligned} L(\gamma) &\geq L\big(\gamma|_{[0,t_i]}\big) = L^{\pi^*\bar{g}}(\gamma_N|_{[0,t_i]}) \geq \frac{1}{2}\int_0^{t_i} \left|\frac{d}{dt}\log f \circ \gamma_N\right| dt \\ &\geq -\frac{1}{2}\int_0^{t_i} \frac{d}{dt}\log f \circ \gamma_N \, dt = \frac{1}{2}\big(\log f(\gamma_N(0)) - \log f(\gamma_N(t_i))\big) \to \infty. \end{aligned}$$

This shows that γ has infinite length.

2^{nd} case: γ has at least two accumulation points. Let $\overline{p}_0 \neq \overline{p}_1$ be such accumulation points. We know that at least one accumulation point, e.g. $\overline{p}_0$, lies in the boundary. Under the assumption that there exists a second accumulation point, we now show that the second accumulation point can be taken arbitrarily near to $\overline{p}_0$. In other words, we claim that for every given neighborhood U of $\bar{p}_0$ there exists an accumulation point $\bar{p}_2 \in U\backslash\{\bar{p}_0\}$. Indeed let us denote by $B_r^{aux}(\bar{p}_0)$ the ball of radius $r > 0$ centered at $\bar{p}_0$ with respect to an auxiliary Riemannian metric on $\bar{\mathcal{M}}$. Choose $r > 0$ such that $\overline{B_r^{aux}(\bar{p}_0)} \subset U$. If $\bar{p}_1 \in U$ there is nothing to prove. If $\bar{p}_1 \notin U$ choose sequences $s_i < t_i < s_{i+1}$ such that $\lim_{i\to\infty} \gamma(s_i) = \bar{p}_0$ and $\lim_{i\to\infty} \gamma(t_i) = \bar{p}_1$. We can assume that $\gamma(s_i) \in B_{r/2}^{aux}(\bar{p}_0)$ and $\gamma(t_i) \notin \overline{B_r^{aux}(\bar{p}_0)}$ for all i. Then there exists a sequence $u_i \in (s_i, t_i)$ with $\gamma(u_i) \in B_r^{aux}(\bar{p}_0) \setminus B_{r/2}^{aux}(\bar{p}_0)$. The sequence $\gamma(u_i)$ has an accumulation point $\bar{p}_2 \in \overline{B_r^{aux}(\bar{p}_0)} \subset U$. We will continue to denote this accumulation point arbitrarily close to $\overline{p}_0$ by $\bar{p}_1$.

If $\overline{p}_1 \in \overline{M}$ it is easy to see that γ has infinite length. In fact consider a geodesically convex ball $B_\delta(\bar{p}_1)$ of radius $\delta > 0$ centered at $\bar{p}_1$ with respect to $\bar{g}$. We take δ sufficiently small such that $B_\delta(\bar{p}_1)$ is relatively compact in $\bar{M}$. Since the curve γ intersects the ball $B_{\delta/2}(\bar{p}_1)$ an arbitrarily large number of times k, the length of γ is larger or equal than $k\delta \to \infty$.

Thus we can assume that $\bar{p}_1$ lies in the boundary as well. By restricting U we can assume that U is in the image of a complex hypersurface $M' \subset \overline{\mathcal{M}}$ as above. We can further assume that $f \leq \epsilon$ on M'. Since $g' = \pi^*\bar{g}$ is given by (1.1) the Riemannian metric $\pi^*\bar{g}|_N$ on $N = \mathrm{int}(M')$ is bounded from below by the Riemannian metric

$$- \left.\frac{g}{f}\right|_N \geq -\frac{1}{\epsilon} g|_N. \tag{2.2}$$

Let us denote by $B'_r(p)$ the ball centered at $p \in M'$ of radius $r > 0$ with respect to the Riemannian metric $-g|_{M'}$ on M'. We choose $\delta > 0$ such that $B'_\delta(p_0)$ is relatively compact in M'. Then every curve in $B'_\delta(p_0)$ from $B'_{\delta/2}(p_0) \subset M'$ ($p_0 := (\pi|_{M'})^{-1}(\bar{p}_0)$) which leaves $B'_\delta(p_0)$ has length with respect to $-g|_{M'}$ bounded from below by some positive constant c (in fact $c = \delta/2$). Since we can assume that $p_1 := (\pi|_{M'})^{-1}(\bar{p}_1)$ is arbitrarily close to p_0 we can assume that $p_1 \in B'_\delta(p_0)$ and there exist disjoint balls $B'_{\delta'}(p_0), B'_{\delta'}(p_1) \subset B'_\delta(p_0)$ which have distance with respect to $-g|_{M'}$ bounded from below by some positive constant. By reducing the above constant c, if necessary, we can assume that this constant is again c. Then we can conclude that every curve which connects a point in $B'_{\delta'}(p_0)$ with a point in $B'_{\delta'}(p_1)$ has length with respect to $-g|_{M'}$ bounded from below by c. Since p_0 and p_1 are accumulation points of γ either γ leaves the set $\pi(N)$ infinitely often, in which case γ has infinite length, or γ stays eventually inside $\pi(N)$, in which case it can be eventually identified with a curve γ_N in N by the projection $\pi|_N$. Since $\bar{p}_0$ and $\bar{p}_1$ are accumulation points of γ_N there exists an infinite number of arcs of γ_N in N connecting $B'_{\delta'}(p_0)$ with $B'_{\delta'}(p_1)$. Again the length is infinite. In both cases we used the estimate (2.2) together with the lower bound c on the length of arcs with respect to $-g|_N$. □

Remark 1. *In the case of conical affine special Kähler domains the description of regular boundary behaviour simplifies as follows. Let* $(\bar{M}, \bar{g})$ *be a projective special Kähler domain with underlying conical special Kähler domain* (M, g, F)*. Suppose that the affine Kähler potential* f *extends to a smooth function (denoted again by* f*) on some neighborhood of* $\mathrm{cl}(M) \setminus \{0\}$*, where* $\mathrm{cl}(M)$ *denotes the closure of* M*, such that* $f(p) = 0$*,* $df_p \neq 0$*, and that* g_p *is negative semi-definite on* $T_p\partial M \cap J(T_p\partial M)$ *with kernel* $\mathbb{C}\xi_p = \mathbb{C}p$ *for all boundary points* $p \in \partial M \setminus \{0\}$*. Then* (M, g, F) *is an example of a conical affine special Kähler manifold with regular boundary behaviour and* $(\bar{M}, \bar{g})$ *an example of a projective special Kähler manifold with regular boundary behaviour.*

The following result is an immediate consequence of Theorem 7.

Corollary 8. *Under the above assumptions on the boundary behaviour of the affine Kähler potential* f *in Remark 1, the Riemannian manifold* $(\bar{M}, \bar{g})$ *is complete.*

3 One-loop deformed Ferrara-Sabharwal metric

In this section we will recall the definition of the one-loop (quantum) deformation of the Ferrara-Sabharwal metric which is a one-parameter family of quaternionic Kähler metrics associated with a projective special Kähler domain [RSV, ACDM]. The fact that the metric is quaternionic Kähler was proven in [ACDM] with the help of an indefinite version of Haydys' HK/QK correspondence [Ha] developed in [ACM]. This implies that the reduced scalar curvature $\nu = \frac{\mathrm{scal}}{4m(m+2)}$ is negative and more precisely given by $\nu = -2$ with the present normalizations. Here m is the quaternionic dimension of the quaternionic Kähler manifold. In the special case of the (undeformed) Ferrara-Sabharwal metric the quaternionic Kähler property was obtained by different methods in [FS, Hi].

Every projective special Kähler manifold admits a covering by projective special Kähler domains and we will show that the one-loop deformed Ferrara-Sabharwal metrics associated with the domains can be consistently glued to a globally defined (quaternionic Kähler; to be shown) metric. This generalizes the result that the Ferrara-Sabharwal metric, which was originally defined for special Kähler domains [FS], is globally defined for every projective special Kähler manifold [CHM]. We will also show that the above quantum deformed quaternionic Kähler manifolds admit a globally defined integrable complex structure J_1 subordinate to the quaternionic structure, generalizing results of [CLST] for the Ferrara-Sabharwal metric.

3.1 The supergravity c-map

Let $(\bar{M}, \bar{g})$ be a projective special Kähler domain of complex dimension n. The **supergravity c-map** [FS] associates with $(\bar{M}, \bar{g})$ a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}})$ of dimension $4n+4$. Following the conventions of [CHM], we have $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ and

$$\begin{aligned} g_{\bar{N}} &= \bar{g} + g_G, \\ g_G &= \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(m) d\zeta^I d\zeta^J \\ &\quad + \frac{1}{2\rho}\sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(m)d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m)d\zeta^L), \end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on $m \in \bar{M}$ and $\mathcal{I}(m)$ is invertible with the inverse $\mathcal{I}^{-1}(m) =: (\mathcal{I}^{IJ}(m))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK}z^K \sum_L N_{JL}z^L}{\sum_{IJ} N_{IJ}z^I z^J}, \quad N_{IJ} := 2\mathrm{Im}F_{IJ}, \tag{3.1}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates (z^I) on the underlying conical special Kähler domain $M \to \bar{M}$. Notice that the expressions are homogeneous of degree zero and, hence, well-defined functions on $\bar{M}$. It is shown in [CHM, Cor. 5] that the matrix $\mathcal{I}(m)$ is positive definite and hence invertible and that the metric $g_{\bar{N}}$ does not depend on the choice of special coordinates [CHM, Thm. 9]. It is also shown that $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, \bar{g})$ is complete [CHM, Thm. 5]. Using $(p_a)_{a=1,\ldots,2n+2} := (\tilde{\zeta}_I, \zeta^J)_{IJ=0,\ldots,n}$ and the positive definite matrix [CHM]

$$(\hat{H}^{ab}) := \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix},$$

we can combine the last two terms of g_G into $\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b$, i.e. the quaternionic Kähler metric is given by

$$g_{FS} := g_{\bar{N}} = \bar{g} + \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)^2 + \frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b. \tag{3.2}$$

This metric is known as the **Ferrara-Sabharwal** metric.

3.2 The one-loop deformation

Now we consider a family of metrics g^c_{FS} depending on a real parameter c such that $g^0_{FS} = g_{FS}$. To define this family we assume for the moment that $z^0 \neq 0$ on the conical affine special Kähler domain $M \subset \mathbb{C}^{n+1}$. Under this assumption we can consider the projective special Kähler domain as a subset $\bar{M} \subset \mathbb{C}^n \subset \mathbb{C}P^n$.

Definition 9. *For any $c \in \mathbb{R}$, the metric*

$$\begin{aligned} g_{FS}^{c} &= \frac{\rho+c}{\rho}\,\bar{g} + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2 + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c \mathcal{K})^2 \\ &\quad + \frac{1}{2\rho}\sum_{a,\,b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2} e^{\mathcal{K}} \left| \sum_{I=0}^{n} (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I) \right|^2 \end{aligned} \tag{3.3}$$

is defined on the domains

$$\begin{aligned} N'_{(4n+4,\,0)} &:= \{\rho > -2c,\ \rho > 0\} \subset \bar{N}, \\ N'_{(4n,\,4)} &:= \{-c < \rho < -2c\} \subset \bar{N}, \\ N'_{(4,\,4n)} &:= \bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3} \end{aligned} \tag{3.4}$$

*for any projective special Kähler domain $\bar{M}$ defined by a holomorphic function F on the underlying conical affine special Kähler domain M, where $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$, $(X^\mu)_{\mu=1,\dots,n}$ are standard inhomogeneous holomorphic coordinates on $\bar{M} \subset \mathbb{C}^n$, $X^0 := 1$, the real coordinate ρ corresponds to the second factor, $(\tilde{\phi},\, \tilde{\zeta}_I,\, \zeta^I)_{I=0,\dots,n}$ are standard real coordinates on $\mathbb{R}^{2n+3}$, and $\mathcal{K} := -\log \sum_{I,\,J=0}^{n} X^I N_{IJ}(X) \bar{X}^J$ is the Kähler potential for $\bar{g}$. The metric g_{FS}^c is called the **one-loop deformed Ferrara-Sabharwal metric**.*

Proposition 10. *Let $\bar{M} \subset \mathbb{C}^n \subset \mathbb{C}P^n$ be a projective special Kähler domain and g_{FS}^c, $g_{FS}^{c'}$ one-loop deformed Ferrara-Sabharwal metrics for positive deformation parameters $c,\, c' \in \mathbb{R}^{>0}$ defined on $\bar{N} = N'_{(4n+4,\,0)}$. Then $(\bar{N},\, g_{FS}^c)$ and $(\bar{N},\, g_{FS}^{c'})$ are isometric.*

Proof: Any $e^\lambda \in \mathbb{R}^{>0}$ acts diffeomorphically on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ as follows:

$$\bar{N} \to \bar{N}, \quad (m,\, \rho,\, \tilde{\phi},\, \tilde{\zeta}_I,\, \zeta^I)_{I=0,\dots,n} \mapsto (m,\, e^\lambda \rho,\, e^\lambda \tilde{\phi},\, e^{\lambda/2}\tilde{\zeta}_I,\, e^{\lambda/2}\zeta^I)_{I=0,\dots,n}.$$

Under this action, $g_{FS}^c \mapsto g_{FS}^{e^{-\lambda}c}$. Choosing $e^\lambda = c/c'$, this shows that $(\bar{N},\, g_{FS}^c)$ and $(\bar{N},\, g_{FS}^{c'})$ are isometric. □

3.3 Globalization of the one-loop deformed metric

Let $(\bar{M}, \bar{g})$ be a projective special Kähler manifold with underlying conical affine special Kähler manifold (M, J, g, ∇, ξ). Consider a covering of $\bar{M}$ by open subsets $\bar{M}_\alpha$ isomorphic to projective special Kähler domains. Over the preimage $M_\alpha := \pi^{-1}(\bar{M}_\alpha)$ we have a system of so-called conical affine special coordinates $(z^I)_{0 \le I \le n}$ which correspond to the natural coordinates in the underlying conical affine special Kähler domain equipped with the holomorphic prepotential F. Notice that the map $\phi_\alpha \colon M_\alpha \to \mathbb{C}^{2n+2}$, $p \mapsto (z^I, F_I)|_p$, where F_I denotes the I-th partial derivative at the point $z = (z^0, \dots, z^n)$, is a conical nondegenerate Lagrangian immersion in the sense of [CM]. Further note that the coordinates as well as the prepotential depend on α. To indicate this dependence we will write z^I_α, F_α etc. Since any pair of conical nondegenerate Lagrangian immersions is related by a real linear symplectic transformation [ACD, CM] there exists an element

$$\mathcal{O} = \mathcal{O}_{\beta,\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(\mathbb{R}^{2n+2})$$

such that $\phi_\beta = \mathcal{O} \circ \phi_\alpha$ on $M_\alpha \cap M_\beta$.

Define $\bar{N}_\alpha := \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ and $N_\alpha := M_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$, where $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ is endowed with the standard coordinate system $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^J) = (\rho_\alpha, \tilde{\phi}_\alpha, \tilde{\zeta}_{I,\alpha}, \zeta^J_\alpha) =: (\rho_\alpha, \tilde{\phi}_\alpha, v_\alpha)$ and $S^1_c := \mathbb{R}/2\pi c\mathbb{Z}$. Notice that S^1_c can be canonically identified with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ by $[x] \mapsto [cx]$ if $c \neq 0$ and that $S^1_0 = \mathbb{R}$.

Next we define an equivalence relation on the disjoint union of the $\bar{N}_\alpha$ (and similarly on the disjoint union of the N_α)

$$\begin{aligned} (m_\alpha, \rho_\alpha, \tilde{\phi}_\alpha, v_\alpha) &\sim (m_\beta, \rho_\beta, \tilde{\phi}_\beta, v_\beta) \\ :\Leftrightarrow m_\alpha = m_\beta,\ \rho_\alpha = \rho_\beta,\ \tilde{\phi}_\beta &= \tilde{\phi}_\alpha - ic\log\left(\frac{z^0_\alpha \bar{z}^0_\beta}{z^0_\beta \bar{z}^0_\alpha}\right),\ v_\beta = (\mathcal{O}^t_{\beta,\alpha})^{-1} v_\alpha. \end{aligned}$$

Proposition 11. *The quotient $\bar{N} := \cup_\alpha \bar{N}_\alpha / \sim$ is a smooth manifold of real dimension $4n+4$ fibering over the projective special Kähler manifold $\bar{M}$ as a bundle of flat symplectic manifolds modeled on the quotient of a symplectic vector space $\mathbb{R}^{2n+2}$ by a cyclic group of translations (the cyclic group is trivial for $c = 0$). By π we denote the induced natural projection $\bar{N} \to \bar{M}$. Similarly, the quotient $N := \cup_\alpha N_\alpha / \sim$ is a bundle over the conical affine special Kähler manifold M with flat symplectic fibers.*

Proof: It is clear that $\bar{N}$ is a fibre bundle with standard fibre $\mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$. By taking the logarithm of ρ one can identify the standard fibre with the quotient $\mathbb{R} \times S^1_c \times \mathbb{R}^{2n+2}$ of $\mathbb{R}^{2n+4}$ by the group of translations $2\pi c\mathbb{Z}$ acting on the second coordinate. Since the transition functions take values in the group of affine symplectic transformations of $\mathbb{R} \times S^1_c \times \mathbb{R}^{2n+2}$, the fibers of the resulting bundle naturally carry a flat symplectic structure. In fact, the linear part of the transition functions takes values in the subgroup $\{\mathrm{Id}_{\mathbb{R}^2}\} \times \mathrm{Sp}(\mathbb{R}^{2n+2}) \subset \mathrm{Sp}(\mathbb{R}^{2n+4})$. □

To avoid a parameter-dependence of the domain of definition of the metric we will assume from now on for simplicity that the one-loop parameter $c > 0$.

Theorem 12. *The quaternionic Kähler metrics $g^c_{FS,\alpha}$, $c > 0$, given by (3.3) on each coordinate domain $\bar{N}_\alpha$ of $\bar{N}$ using the coordinates $(X^\mu, \rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^J) = (X^\mu_\alpha, \rho_\alpha, \tilde{\phi}_\alpha, \tilde{\zeta}_{I,\alpha}, \zeta^J_\alpha)$ induce a well-defined quaternionic Kähler metric g^c_{FS} on $\bar{N}$. Furthermore there exists a* *globally defined integrable complex structure J_1 subordinate to the parallel skew-symmetric quaternionic structure Q of $(\bar{N}, g^c_{FS})$.*

Proof: First we show that the quaternionic Kähler metrics defined on the domains $\bar{N}_\alpha$ are consistent. The terms $\frac{\rho+c}{\rho}\bar{g}$ and $\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2$ in (3.3) are manifestly coordinate independent, since the transition functions do not act on ρ. The one-form $\eta_{can} := \sum_{I=0}^n (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)$ is obviously invariant under linear symplectic transformations and therefore also coordinate independent. The invariance of the term $\sum_{a,b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b$ was shown in [CHM, Lemma 4]. Next we show the invariance of $d\tilde{\phi} + cd^c\mathcal{K}$. Since

$$\sum_{I,J} X^I N_{IJ} \bar{X}^J = \frac{f}{z^0 \bar{z}^0},$$

where $f = g(\xi, \xi) = \sum_{I,J} z^I N_{IJ} \bar{z}^J$ is coordinate independent (but defined on N, not on $\bar{N}$), we see that

$$cd^c\mathcal{K}_\beta - cd^c\mathcal{K}_\alpha = cd^c \log\left(\frac{z^0_\beta \bar{z}^0_\beta}{z^0_\alpha \bar{z}^0_\alpha}\right) = icd\log\left(\frac{z^0_\alpha \bar{z}^0_\beta}{z^0_\beta \bar{z}^0_\alpha}\right),$$

where we have used that $d^c = -J^* d$ on functions. By the transition rule for $\tilde{\phi}$ we have

$$d\tilde{\phi}_\beta = d\tilde{\phi}_\alpha - icd\log\left(\frac{z^0_\alpha \bar{z}^0_\beta}{z^0_\beta \bar{z}^0_\alpha}\right).$$

This shows the invariance of $d\tilde{\phi} + cd^c\mathcal{K}$.

Finally we show the invariance of $e^{\mathcal{K}}\left|\sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2$. By rewriting this as

$$\frac{1}{\sum X^I N_{IJ}(X)\bar{X}^J}\left|\sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 = \frac{z^0\bar{z}^0}{f}\left|\sum_{I=0}^n \frac{z^I}{z^0}d\tilde{\zeta}_I + F_I(\tfrac{z}{z^0})d\zeta^I\right|^2 = \frac{1}{f}\left|\sum_{I=0}^n z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I\right|^2$$

we see that the term is coordinate independent. In fact, the sum $\sum_{I=0}^n z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I$ is obtained from the natural pairing between $\mathbb{C}^{2n+2}$ and $(\mathbb{C}^{2n+2})^* \supset (\mathbb{R}^{2n+2})^*$ which is, in particular, invariant under linear symplectic transformations. Summarizing we have shown that the metric g^c_{FS} is well defined on $\bar{N}$.

Since g^c_{FS} is quaternionic Kähler (of negative scalar curvature) on each of the domains $\bar{N}_\alpha$ it follows that g^c_{FS} is a quaternionic Kähler metric. In fact, the locally defined parallel skew-symmetric quaternionic structures on the domains $\bar{N}_\alpha$ are uniquely determined by the Lie algebra of the holonomy group of $g^c_{FS}|_{\bar{N}_\alpha}$ and therefore extend to a globally defined quaternionic structure Q. It can be also checked by direct calculations (see below) that the locally defined quaternionic structures Q_α on $\bar{N}_\alpha$ are consistent. In fact, the description of the quaternionic Kähler structure on $\bar{N}_\alpha$ in terms of the HK/QK-correspondence [ACDM] yields an almost hypercomplex structure (J_1, J_2, J_3) on $\bar{N}_\alpha$ which defines the quaternionic structure Q_α. The structure is defined by the three Kähler forms $\omega_i = g^c_{FS}(J_i\cdot, \cdot)$, $i = 1, 2, 3$. These are given by

$$\omega_i = -d\theta_i + 2\theta_j \wedge \theta_k,$$

where (i, j, k) is a cyclic permutation of $\{1, 2, 3\}$ and the one-forms θ_i on $\bar{N}_\alpha$ are defined by

$$\begin{aligned}
\theta_1 &= -\frac{1}{4\rho}\big(d\tilde{\phi} + (\rho + c)d^c\mathcal{K} - \eta_{can}\big) \\
\theta_2 + i\theta_3 &= i\frac{\sqrt{\rho + c}}{\rho}e^{\mathcal{K}/2}\sum_{I=0}^{n} X^I A_I, \quad A_I := d\tilde{\zeta}_I + \sum_J F_{IJ}d\zeta^J.
\end{aligned}$$

Next we prove that Q admits a global section J_1 by showing that the Kähler form ω_1 is invariantly defined, i.e. coordinate independent. First we remark that θ_1 can be decomposed as

$$\theta_1 = -\frac{1}{4\rho}\big(d\tilde{\phi} + cd^c\mathcal{K} - \eta_{can}\big) - \frac{1}{4}d^c\mathcal{K},$$

where the first was already shown to be invariant. Using that $\mathcal{K} = -\log\frac{f}{(r^0)^2}$, where $z^0 = r^0e^{i\varphi^0}$, the second term can be decomposed as

$$-\frac{1}{4}d^c\mathcal{K} = \frac{1}{4}d^c\log f + \frac{1}{2}d^c\log r^0 = \frac{1}{4}d^c\log f - \frac{1}{2}d\varphi^0.$$

Since the first term on the right-hand side is invariant we see that

$$\theta_1 = \theta_1^{inv} - \frac{1}{2}d\varphi^0,$$

where θ_1^{inv} is coordinate independent. This implies that $d\theta_1$ is invariant. Now we observe that

$$\sum z^I A_I$$

is invariant (defined on N). This follows from

$$\sum z^I A_I = \sum z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I,$$

where the right-hand side was already observed to be invariant. As a consequence, the two-form

$$\theta_2 \wedge \theta_3 = -\frac{1}{2i}(\theta_2 + i\theta_3) \wedge (\theta_2 - i\theta_3)$$

is also invariant, since

$$\theta_2 + i\theta_3 = \frac{i}{\rho}\left(\frac{\rho+c}{f}\right)^{\frac{1}{2}} e^{-i\varphi^0} \sum z^I A_I,$$

which implies that $e^{i\varphi^0}(\theta_2 + i\theta_3)$ is a well defined one-form on N.

Combining these results we have shown that $\omega_1 = -d\theta_1 + 2\theta_2 \wedge \theta_3$ is invariant. By similar calculations it is easy to show that a conformal multiple $e^{i\varphi^0}\omega$ of the $(2,0)$-form

$$\omega = \omega_2 + i\omega_3$$

with respect to J_1 is invariantly defined on N (and horizontal with respect to the projection $N \to \bar{N}$ induced by $M \to \bar{M}$). This implies that the complex plane spanned by ω and $\bar{\omega}$ is invariantly defined on $\bar{N}$ and therefore the real plane spanned by ω_2 and ω_3. This reproves the fact that the quaternionic structure is well-defined.

Now we prove the integrability of J_1. It is sufficent to check this on $\bar{N}_\alpha$. In the case $c = 0$ this was previously shown in [CLST]. With the definition of ω_1 above we compute

$$\begin{aligned}
\omega_1 &= \frac{1}{4\rho}\left(d\rho \wedge d^c\mathcal{K} + (\rho+c)\, dd^c\mathcal{K} - 2\sum_{I=0}^{n} d\tilde{\zeta}_I \wedge d\zeta^I\right) + \frac{1}{\rho} d\rho \wedge \theta_1 \\
&\quad + \frac{\rho+c}{\rho^2} e^{\mathcal{K}} i (\sum_I X^I A_I) \wedge (\sum_J \bar{X}^J \bar{A}_J) \\
&= \frac{\rho+c}{\rho}\frac{1}{4} dd^c\mathcal{K} + \frac{i}{2}\frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\tau \wedge \bar{\tau} - \frac{i}{2}\frac{1}{\rho}\sum_{I,\,J=0}^{n} N^{IJ} A_I \wedge \bar{A}_J \\
&\quad + \frac{i}{2}\frac{2\rho+2c}{\rho^2} e^{\mathcal{K}} (\sum_I X^I A_I) \wedge (\sum_J \bar{X}^J \bar{A}_J),
\end{aligned} \tag{3.5}$$

where

$$\tau := d\tilde{\phi} + \eta_{can} + c d^c\mathcal{K} + i\frac{\rho+2c}{\rho+c} d\rho$$

and we used that

$$\sum_{I,\,J=0}^{n} iN^{IJ} A_I \wedge \bar{A}_J = \sum_{I,\,J,\ K=0}^{n} iN^{IJ}(F_{IK} - \bar{F}_{IK}) d\zeta^K \wedge \tilde{\zeta}_J = \sum_{I=0}^{n} d\tilde{\zeta}_I \wedge d\zeta^I.$$

Together with the expression

$$g^c_{FS} = \frac{\rho+c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}|\tau|^2 - \frac{1}{\rho}\sum_{I,\,J=0}^{n} N^{IJ} A_I \bar{A}_J + \frac{2\rho+2c}{\rho^2} e^{\mathcal{K}} |\sum_{I=0}^{n} X^I A_I|^2$$

for the deformed Ferrara-Sabharwal metric, which can be proven using [ACDM, Lemma 3], (3.5) shows that

$$(\tau,\, dX^\mu,\, A_I)^{\mu=1,\,\dots,\,n}_{I=0,\,\dots,\,n}$$

is a coframe of holomorphic one-forms with respect to J_1. This can be linearly combined into the coframe

$$\big(\tau + 2ic\partial\mathcal{K} - 2\sum_{I=0}^{n} \zeta^I A_I - \sum_{I,\,J,\,K=0}^{n} \zeta^I F_{IJK}(X)\zeta^J dX^K,$$
$$dX^\mu,\ \frac{1}{2}(A_I - \sum_{J,\,K=0}^{n} F_{IJK}(X)\zeta^J dX^K)\big)$$

of closed holomorphic one-forms which corresponds to the J_1-holomorphic coordinate system

$$(\chi,\ X^\mu,\ w_I = \frac{1}{2}(\tilde{\zeta}_I + \sum_{J=0}^{n} F_{IJ}(X)\zeta^J))^{\mu=1,\,\dots,\,n}_{I=0,\,\dots,\,n},$$

where

$$\chi := \tilde{\phi} + i(\rho + c(\mathcal{K} + \log(\rho + c))) - \sum_{I=0}^{n} \zeta^I \tilde{\zeta}_I - \sum_{I,\,J=0}^{n} \zeta^I F_{IJ}(X)\zeta^J.$$

This proves the integrability of J_1. □

4 Completeness of the one-loop deformation

4.1 Completeness of the one-loop deformation for projective special Kähler manifolds with regular boundary behaviour

In this and the next section, we prove under two different types of natural assumptions the completeness of the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} (see Definition 9 and Theorem 12) on $\bar{N}$ for $c \geq 0$. For $c < 0$ and the case of projective special Kähler domains, $(N'_{(4n+4,\,0)},\, g^c_{FS})$ is known to be incomplete [ACDM, Rem. 9].

Theorem 13. *Let $(\bar{M}, \bar{g})$ be a projective special Kähler manifold with regular boundary behaviour and $(\bar{N}, g^c_{FS})$ the one-loop deformed Ferrara-Sabharwal (quaternionic Kähler) manifold associated to $(\bar{M}, \bar{g})$. Then $(\bar{N}, g^c_{FS})$ is complete for all $c \geq 0$.*

Example 14. *The projective special Kähler manifold $\mathbb{C}H^n$ with quadratic holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^{n}(z^\mu)^2)$ on the conical affine special Kähler domain $M := \{|z^0|^2 > \sum_{\mu=1}^{n}|z^\mu|^2\}$ has regular boundary behaviour in the sense of Definition 6. Thus Corollary 8 implies the completeness of the projective special Kähler domain $\mathbb{C}H^n$.*

We know that $(\bar{N},\, g_{FS})$ is isometric to the series of Wolf spaces

$$\tilde{X}(n+1) = \frac{SU(n+1,\,2)}{S[U(n+1) \times U(2)]} \tag{4.1}$$

of non-compact type, see e.g. [DV].

Corollary 15. *For any $n \in \mathbb{N}_0$ and $c \in \mathbb{R}^{\geq 0}$, the deformed Ferrara-Sabharwal metric*

$$
\begin{aligned}
g_{FS}^{c} = & \frac{\rho + c}{\rho} \frac{1}{1 - \|X\|^2} \Big(\sum_{\mu=1}^{n} dX^\mu d\bar{X}^\mu + \frac{1}{1 - \|X\|^2} \big| \sum_{\mu=1}^{n} \bar{X}^\mu dX^\mu \big|^2 \Big) \\
& + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 - \frac{2}{\rho} (dw_0 d\bar{w}_0 - \sum_{\mu=1}^{n} dw_\mu d\bar{w}_\mu) \\
& + \frac{\rho + c}{\rho^2} \frac{4}{1 - \|X\|^2} \big| dw_0 + \sum_{\mu=1}^{n} X^\mu dw_\mu \big|^2 \\
& + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \Big(d\tilde{\phi} - 4Im\big(\bar{w}_0 dw_0 - \sum_{\mu=1}^{n} \bar{w}_\mu dw_\mu\big) + \frac{2c}{1 - \|X\|^2} Im \sum_{\mu=1}^{n} \bar{X}^\mu dX^\mu \Big)^2
\end{aligned}
$$

with $w_0 := \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$, $w_\mu := \frac{1}{2}(\tilde{\zeta}_\mu - i\zeta^\mu)$, $\mu = 1, \ldots, n$, on[3]

$$\bar{N} = \{(X, \rho, \tilde{\phi}, w) \in \mathbb{C}^n \times \mathbb{R}^{>0} \times \mathbb{R} \times \mathbb{C}^{n+1} \mid \|X\|^2 < 1\}$$

defined by the holomorphic function

$$F = \frac{i}{2} \left((z^0)^2 - \sum_{\mu=1}^{n} (z^\mu)^2 \right) \text{ on } M := \left\{ |z^0|^2 > \sum_{\mu=1}^{n} |z^\mu|^2 \right\}$$

is a complete quaternionic Kähler metric. Furthermore $(\bar{N}, g_{FS})$ is isometric to the symmetric space $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$.

Proof of Theorem 13. Let $\gamma \colon [0, b) \to \bar{N}$ be a smooth curve which leaves every compact subset of $\bar{N}$, $b \in (0, \infty]$. We have to show that γ has infinite length. By Theorem 7 we know that $(\bar{M}, \bar{g})$ is complete.

Lemma 16. *For every complete Riemannian manifold (M, g) and $c \geq 0$ the Riemannian manifold*

$$\left(M \times \mathbb{R}^{>0}, \frac{\rho + c}{\rho} g + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 \right)$$

is complete. Here ρ denotes the $\mathbb{R}^{>0}$-coordinate.

Proof: This follows from the estimate

$$\frac{\rho + c}{\rho} g + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 \geq g + \frac{1}{4} (d \log \rho)^2.$$

□

[3] In the case of a projective special Kähler domain $\bar{M}$ we consider $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ as in Definition 9, rather than its cyclic quotient $\bar{M} \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ on which the metric is also defined.

We consider the projection

$$\bar{N} \to \bar{M} \times \mathbb{R}^{>0},\ p \mapsto (\pi(p), \rho(p)),$$

where $\pi\colon \bar{N} \to \bar{M}$ is the fibre bundle projection introduced in Proposition 11. Since the metric

$$\frac{\rho + c}{\rho} g + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c} d\rho^2$$

on the base $\bar{M} \times \mathbb{R}^{>0}$ is complete by the previous lemma, the projection $\bar{\gamma}$ of γ to $\bar{M}\times\mathbb{R}^{>0}$ either stays in a compact set or has infinite length. In the latter case γ has infinite length. So we can assume that $\bar{\gamma}$ stays in a compact set.

Using similar arguments as in the proof of Theorem 7 we can assume that $\bar{\gamma}$ has a unique accumulation point $(\bar{p}_0, \rho_0)$. In fact, the existence of two different accumulation points implies that $\bar{\gamma}$ and, hence, γ have infinite length. There exists a sequence $t_i \to b$ with $\bar{\gamma}(t_i) \to (\bar{p}_0, \rho_0) \in \bar{M} \times \mathbb{R}^{>0}$ and $\gamma(t_i)$ leaves every compact subset of $\bar{N}_\alpha \cong \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$, where $\bar{M}_\alpha$ is a projective special Kähler domain containing $(\bar{p}_0, \rho_0)$ and $\bar{N}_\alpha$ is the corresponding trivial fibre bundle endowed with the one-loop deformed Ferrara-Sabharwal metric associated to the projective special Kähler domain $\bar{M}_\alpha$. Note that $\pi_{\mathbb{R}^{2n+2}}(\gamma(t_i)) \in \mathbb{R}^{2n+2}$ is unbounded.

Lemma 17. *For $\varepsilon > 0$ and sufficiently small relatively compact $\bar{M}_\alpha \subset \bar{M}$ we have*[4] *$g^c_{FS} \geq \delta \cdot g_{FS}$ on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$ for some $\delta = \delta(\alpha, \varepsilon) > 0$.*

Proof: Choose linear coordinates $(z^0, \ldots, z^n)$ for the underlying conical affine special Kähler domain M_α such that g_α restricted to the $(z^1, \ldots, z^n)$-plane is positive definite. This can always be achieved by restricting the coordinate domain. Then it follows from (1.1) that $\bar{g}_\alpha \geq \frac{k}{4}(d^c\mathcal{K})^2$ for some $k > 0$. Let $\varepsilon > 0$ be given. We claim that

$$g^c_{FS} \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c} g_{FS}$$

on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$. Note first that

$$\bar{g} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2 \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c}\left(\bar{g} + \frac{1}{4\rho^2}d\rho^2\right).$$

Next the last two expressions in the definition of g^c_{FS} can be estimated from below

$$\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I)\right|^2 \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon + c}\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b,$$

[4]Here g_{FS} denotes the metric on $\bar{N}_\alpha = \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ induced by the metric g_{FS} on $\bar{M}_\alpha \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. Alternatively one can compare the metrics by pulling back g^c_{FS} to the cyclic covering $\bar{M}_\alpha \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3} \to \bar{N}_\alpha$.

since $\frac{k\epsilon}{k\epsilon+c} \leq 1$ and $(\hat{H}^{ab})$ is positive definite. Last setting

$$\theta_0 := d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I),$$

we conclude

$$\begin{aligned}
&\frac{c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}(\theta_0 + cd^c\mathcal{K})^2 \\
\geq\ & \frac{kc}{4\rho}(d^c\mathcal{K})^2 + \frac{1}{4\rho^2}\underbrace{\frac{\rho+c}{\rho+2c}}_{\frac{1}{2}\leq\ldots\leq 1}\left(\underbrace{\frac{c}{k\varepsilon+c}(\theta_0+(k\varepsilon+c)d^c\mathcal{K})^2}_{\geq 0} + \frac{k\varepsilon}{k\varepsilon+c}\theta_0^2 - kc\varepsilon(d^c\mathcal{K})^2\right) \\
\geq\ & \frac{1}{2}\frac{k\varepsilon}{k\varepsilon+c}\frac{1}{4\rho^2}\theta_0^2 + \frac{ck}{4\rho^2}(\rho-\varepsilon)(d^c\mathcal{K})^2 \\
\geq\ & \frac{1}{2}\frac{k\varepsilon}{k\varepsilon+c}\frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)\right)^2,
\end{aligned}$$

where the last inequality follows from $\rho > \varepsilon$. Combining these three inequalities, we have shown that

$$g^c_{FS} \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon+c}g_{FS}$$

on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$. □

Choose $\varepsilon > 0$ such that $\rho_0 \geq 2\varepsilon$. For the undeformed metric g_{FS} on $\bar{N}_\alpha$ we have $g_{FS} = \bar{g}|_{\bar{M}_\alpha} + g_G$, where g_G is a family of left invariant metrics on $G = \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ endowed with the Lie group structure defined in [CHM].

Since $\bar{M}_\alpha \subset \bar{M}$ is relatively compact, we can estimate $g_G \geq \mathrm{const}\, g^0_G$ for some left invariant metric g^0_G on the group fibre G. This implies that the curve γ has infinite length, since every homogenous Riemannian metric is complete and the length of γ can be estimated by the length of its projection to G. □

4.2 Completeness of the one-loop deformation for complete projective special Kähler manifolds with cubic prepotential

In this section, we prove completeness of the one-loop deformation g^c_{FS} in the case of complete projective special Kähler manifolds in the image of the supergravity r-map. We will recall the definition of the latter manifolds below. They are also know as *projective special Kähler manifolds with cubic prepotential* or *projective very special Kähler manifolds*.

In Section 4.2.1, we introduce projective special real geometry and the supergravity r-map. The latter assigns a complete projective special Kähler manifold to each complete projective special real manifold. In Section 4.2.2, we derive a sufficient condition for the completeness of $(N'_{(4n+4,\,0)},\, g^c_{FS})$ for $c \in \mathbb{R}^{\geq 0}$. Recall that we construct $(N'_{(4n+4,\,0)},\, g^c_{FS})$ from a projective special Kähler manifold. We prove the completeness of $(N'_{(4n+4,\,0)},\, g^c_{FS})$ in the case that the projective special Kähler manifold is obtained from a complete projective special real manifold via the supergravity r-map and in the case of $\mathbb{C}H^n$.

As a corollary, we obtain deformations by complete quaternionic Kähler metrics of all known homogeneous quaternionic Kähler manifolds of negative scalar curvature (including symmetric spaces), except for quaternionic hyperbolic space. In the case of the series $\tilde{X}(n+1) = \frac{SU(n+1,\,2)}{S[U(n+1)\times U(2)]}$, which corresponds to the projective special Kähler domains $\mathbb{C}H^n$ with quadratic prepotential, we already gave a simple and explicit expression for the deformed metric in Corollary 15.

In this chapter, we only discuss positive definite quaternionic Kähler metrics.

4.2.1 Projective special real geometry and the supergravity r-map

Definition 18. *Let h be a homogeneous cubic polynomial in n variables with real coefficients and let $U \subset \mathbb{R}^n\backslash\{0\}$ be an $\mathbb{R}^{>0}$-invariant domain such that $h|_U > 0$ and such that $g_{\mathcal{H}} := -\partial^2 h\big|_{\mathcal{H}}$ is a Riemannian metric on the hypersurface $\mathcal{H} := \{x \in U \mid h(x) = 1\} \subset U$. Then $(\mathcal{H},\, g_{\mathcal{H}})$ is called a* ***projective special real*** *(PSR) manifold.*

Define $\bar{M} := \mathbb{R}^n + iU \subset \mathbb{C}^n$. We endow $\bar{M}$ with the standard complex structure and use holomorphic coordinates $(X^\mu = y^\mu + ix^\mu)_{\mu=1,\,\dots,\,n} \in \mathbb{R}^n + iU$. We define a Kähler metric

$$\begin{aligned}\bar{g} = 2\sum_{\mu,\,\nu=1}^{n} g_{\mu\bar{\nu}} dX^\mu d\bar{X}^\nu &:= \sum_{\mu,\,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu} dX^\mu d\bar{X}^\nu \\ &= \frac{1}{2}\sum_{\mu,\,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu}(dX^\mu \otimes d\bar{X}^\nu + d\bar{X}^\nu \otimes dX^\mu)\end{aligned}$$

on $\bar{M}$ with Kähler potential

$$\mathcal{K}(X,\,\bar{X}) := -\log 8h(x) = -\log h\left(i(\bar{X} - X)\right). \tag{4.2}$$

Definition 19. *The correspondence $(\mathcal{H},\, g_{\mathcal{H}}) \mapsto (\bar{M},\, \bar{g})$ is called the* ***supergravity r-map****.*

Remark 2. With $\frac{\partial}{\partial X^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial y^\mu} - i\frac{\partial}{\partial x^\mu}\right)$, we have

$$\begin{aligned}2\bar{g}\left(\frac{\partial}{\partial X^\mu},\, \frac{\partial}{\partial \bar{X}^\nu}\right) &= 2g_{\mu\bar{\nu}} = \frac{\partial^2 \mathcal{K}(X,\,\bar{X})}{\partial X^\mu \partial \bar{X}^\nu} =: \mathcal{K}_{\mu\bar{\nu}} \\ &= -\frac{1}{4}\frac{\partial^2 \log h(x)}{\partial x^\mu \partial x^\nu} = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)},\end{aligned} \tag{4.3}$$

where $h_\mu(x) := \frac{\partial h(x)}{\partial x^\mu}$, $h_{\mu\nu}(x) := \frac{\partial^2 h(x)}{\partial x^\mu \partial x^\nu}$, etc., for $\mu,\, \nu = 1,\, \ldots,\, n$.

The inverse $(\mathcal{K}^{\bar{\nu}\lambda})_{\nu,\, \lambda=1,\, \ldots,\, n}$ of $(\mathcal{K}_{\mu\bar{\nu}})_{\mu,\, \nu=1,\, \ldots,\, n}$ is given by

$$\mathcal{K}^{\bar{\nu}\lambda} = -4h(x)h^{\nu\lambda}(x) + 2x^\nu x^\lambda. \tag{4.4}$$

This can be shown using the fact that h is a homogeneous polynomial of degree three:

$$\sum_{\mu=1}^{n} h_\mu(x) x^\mu = 3h(x), \quad \sum_{\nu=1}^{n} h_{\mu\nu}(x) x^\nu = 2h_\mu(x),$$
$$\sum_{\rho=1}^{n} h_{\mu\nu\rho}(x) x^\rho = h_{\mu\nu}, \quad h_{\mu\nu\rho\sigma} = 0. \tag{4.5}$$

Remark 3. Note that any manifold $(\bar{M},\, \bar{g})$ in the image of the supergravity r-map is a projective special Kähler domain. The corresponding conical affine special Kähler domain is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1,\, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*,\ X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}$$

endowed with the standard complex structure J and the metric g_M defined by the holomorphic function

$$F : M \to \mathbb{C}, \quad F(z^0,\, \ldots,\, z^n) = \frac{h(z^1,\, \cdots,\, z^n)}{z^0}.$$

Note that in general, the flat connection[5] ∇ on M is not the standard one induced from $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. The homothetic vector field ξ is given by $\xi = \sum_{I=0}^{n}(z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I})$. To check that $\bar{g}$ is the corresponding projective special Kähler metric, one uses the fact that

$$8|z^0|^2 h(x) = \sum_{I,\, J=0}^{n} z^I N_{IJ}(z,\, \bar{z})\bar{z}^J, \tag{4.6}$$

where as above, $x = (\operatorname{Im} X^1,\, \ldots,\, \operatorname{Im} X^n) = (\operatorname{Im} \frac{z^1}{z^0},\, \ldots,\, \operatorname{Im} \frac{z^n}{z^0}) \in U$ (see [CHM]).

Definition 20. *A Kähler manifold $(\bar{M},\, \bar{g})$ in the image of the supergravity r-map is called a* ***projective very special Kähler manifold****.*

Due to the following two results, projective special real geometry constitutes a powerful tool for the construction of complete projective special Kähler manifolds.

Theorem 21. [CHM]
The supergravity r-map preserves completeness, i.e. it assigns a complete projective special Kähler manifold to each complete projective special real manifold.

[5] ∇ is defined by $x^I = \operatorname{Re} z^I$ and $y_I = \operatorname{Re} F_I(z)$ being flat, $I = 0,\, \ldots,\, n$ (see [ACD]).

The question of completeness for a projective special real manifold $(\mathcal{H},\, g_{\mathcal{H}})$ reduces to a simple topological question for the hypersurface $\mathcal{H} \subset \mathbb{R}^n$:

Theorem 22. [CNS, Thm. 2.6.]
Let $(\mathcal{H},\, g_{\mathcal{H}})$ be a projective special real manifold of dimension $n-1$. If $\mathcal{H} \subset \mathbb{R}^n$ is closed, then $(\mathcal{H},\, g_{\mathcal{H}})$ is complete.

Remark 4. In low dimensions, it is possible to classify all complete projective special real manifolds up to linear isomorphisms of the ambient space. In the case of curves, there are exactly two examples [CHM]. In the case of surfaces, there exist precisely five discrete examples and a one-parameter family [CDL].

4.2.2 The completeness theorem

Definition 23. *The **q-map** is the composition of the supergravity r- and c-map. It assigns a $(4n+4)$-dimensional quaternionic Kähler manifold to each $(n-1)$-dimensional projective special real manifold.*

Remark 5. Except for quaternionic hyperbolic space $\mathbb{H}H^{n+1}$, all Wolf spaces of non-compact type and all known homogeneous, non-symmetric quaternionic Kähler manifolds (called normal quaternionic Kähler manifolds or Alekseevsky spaces) are in the image of the supergravity c-map. While the series $\tilde{X}(n+1) = Gr_{0,2}(\mathbb{C}^{n+1,2})$ of non-compact Wolf spaces can be obtained via the supergravity c-map from the projective special Kähler manifold $\mathbb{C}H^n$ (with holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^{n}(z^\mu)^2)$), which is not in the image of the supergravity r-map, all the other manifolds mentioned above are in the image of the q-map.

Below, we prove the completeness of the one-loop deformation of the Ferrara-Sabharwal metric with positive deformation parameter $c \in \mathbb{R}^{\geq 0}$ for all manifolds in the image of the q-map.

Due to the following result, both the supergravity c-map and the q-map preserve completeness:

Theorem 24. [CHM]
The supergravity c-map assigns a complete quaternionic Kähler manifold of dimension $4n+4$ to each complete projective special Kähler manifold of dimension $2n$.

Let $(\bar{M}, \bar{g})$ be a projective special Kähler domain with underlying conical special Kähler domain (M, g, F). As in Section 3.2, we assume that $M \subset \{z^0 \neq 0\} \subset \mathbb{C}^{n+1}$

and identity $\bar{M}$ with $M \cap \{z^0 = 1\}$. Then, by restricting the tensor field $\frac{g}{f}$ to $\bar{M} \subset M$, we can write

$$\bar{g} = -\frac{g}{f} + (\partial \mathcal{K})(\bar{\partial} \mathcal{K}) = -\frac{g}{f} + \frac{1}{4}(d\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2. \tag{4.7}$$

We consider the one-loop deformed Ferrara-Sabharwal metric (see Eq. (3.3))

$$\begin{aligned} g^c_{FS} = \frac{\rho + c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c}d\rho^2 + \frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c\mathcal{K})^2 \\ + \frac{1}{2\rho}\sum_{a,\, b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum_{I=0}^{n}(X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 \end{aligned} \tag{4.8}$$

for $c \in \mathbb{R}^{\geq 0}$ defined on $N'_{(4n+4,\,0)} = \bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ endowed with global coordinates

$$(X^\mu,\, \rho,\, \tilde{\phi},\, \tilde{\zeta}_I,\, \zeta^I)^{\mu=1,\dots,n}_{I=0,\dots,n}.$$

Proposition 25. *If $(\bar{M}, \bar{g})$ is complete and $\bar{g} \geq \frac{k}{4}(d^c\mathcal{K})^2$, for some $k \in \mathbb{R}^{>0}$, then $(\bar{N},\, g^c_{FS})$ is complete for every $c \in \mathbb{R}^{\geq 0}$.*

Proof: $(\bar{N},\, g^0_{FS})$ is complete by Theorem 24. Since every curve on $(\bar{N},\, g^c_{FS})$ approaching $\rho = 0$ has infinite length, we can restrict to $\{\rho > \epsilon\} \subset \bar{N}$ for some $\epsilon > 0$. With the same argument as in Lemma 17 one shows

$$g^c_{FS} \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon + c}g^0_{FS}$$

using that $\bar{g} \geq \frac{k}{4}(d^c\mathcal{K})^2$. Since $(\bar{N},\, g^0_{FS})$ is complete, this shows that $(\bar{N},\, g^c_{FS})$ is complete as well for $c \in \mathbb{R}^{\geq 0}$. □

For quaternionic Kähler manifolds in the image of the q-map, the prepotential is $F(z) = \frac{h(z^1,\dots,z^n)}{z^0}$.

Lemma 26. *For projective special Kähler manifolds in the image of the supergravity r-map we have*

$$\bar{g} \geq \frac{1}{12}(d^c\mathcal{K})^2.$$

Proof: First, we show that

$$\tilde{g} := -\sum_{\mu,\nu=1}^{n}\frac{h_{\mu\nu}(x)}{h(x)}dy^\mu dy^\nu \geq -\frac{2}{3}(d^c\mathcal{K})^2. \tag{4.9}$$

Considering $\tilde{g}$ as a family of pseudo-Riemannian metrics on $\mathbb{R}^n$ depending on a parameter $x \in U$, the left hand side is positive definite on the orthogonal complement $Y^{\perp_{\tilde{g}}}$ of

$Y := \sum_{\mu=1}^n x^\mu \partial_{y^\mu}$, while the right hand side is zero, since $\tilde{g}(Y, \cdot) = 2d^c\mathcal{K}$. In the direction of Y, we have $\tilde{g}(Y, Y) = -6 = -\frac{2}{3}(d^c\mathcal{K})^2(Y, Y)$.

Equation (4.9) implies

$$\bar{g} \geq \frac{1}{4h(x)} \sum_{\mu,\nu=1}^{n} \left(-h_{\mu\nu}(x) + \frac{h_\mu(x)h_\nu(x)}{h(x)} \right) dy^\mu dy^n \geq -\frac{1}{6}(d^c\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2 = \frac{1}{12}(d^c\mathcal{K})^2.$$

□

This shows that the assumption of Proposition 25 is fulfilled with $k = 1/3$ for projective special Kähler manifolds in the image of the supergravity r-map and proves the following theorem.

Theorem 27. *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a complete projective special real manifold of dimension $n-1$ and g_{FS}^c, $c \in \mathbb{R}^{\geq 0}$, the one-loop deformed Ferrara-Sabharwal metric on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ defined by the projective special Kähler domain $(\bar{M}, \bar{g})$ obtained from $(\mathcal{H}, g_{\mathcal{H}})$ via the supergravity r-map. Then $(\bar{N}, g_{FS}^c)$ is a complete quaternionic Kähler manifold. $(\bar{N}, g_{FS}^0)$ is the complete quaternionic Kähler manifold obtained from $(\mathcal{H}, g_{\mathcal{H}})$ via the q-map.*

Example 28. For the case $n = 1$ ($h = x^3$), $(\bar{N}, g_{FS}^0)$ is isometric to the symmetric space $G_2^*/SO(4)$. In this case we checked using computer algebra software that the squared pointwise norm of the Riemann tensor with respect to the metric is

$$\sum_{i,j,k,l,\tilde{i},\tilde{j},\tilde{k},\tilde{l}=1}^{8} R_{ijkl} g^{i\tilde{i}} g^{j\tilde{j}} g^{k\tilde{k}} g^{l\tilde{l}} R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = \frac{128 \begin{pmatrix} 528c^7 + 2112c^6\rho + 3664c^5\rho^2 + 3568c^4\rho^3 \\ + 2110c^3\rho^4 + 764c^2\rho^5 + 161c\rho^6 + 17\rho^7 \end{pmatrix}}{3(c+\rho)(2c+\rho)^6}.$$

For $c > 0$, this function is non-constant, which shows that $(\bar{N}, g_{FS}^c)$ is not locally homogeneous for $c > 0$.

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