Modular characteristic classes for representations over finite fields

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Abstract

The cohomology of the degree-n general linear group over a finite field of characteristic p, with coefficients also in characteristic p, remains poorly understood. For example, the lowest degree previously known to contain nontrivial elements is exponential in n. In this paper, we introduce a new system of characteristic classes for representations over finite fields, and use it to construct a wealth of explicit nontrivial elements in these cohomology groups. In particular we obtain nontrivial elements in degrees linear in n. We also construct nontrivial elements in the mod p homology and cohomology of the automorphism groups of free groups, and the general linear groups over the integers. These elements reside in the unstable range where the homology and cohomology remain poorly understood.

1 Introduction

We introduce a new system of modular characteristic classes for representations of groups over finite fields, and use it to construct explicit nontrivial elements in the modular cohomology of the general linear groups over finite fields. The cohomology groups $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F})$ were computed by Quillen [Qui72] in the case where $\mathbb{F}$ is a field of characteristic different from p, but he remarked that determining them in the modular case where the characteristic of $\mathbb{F}$ is p "seems to be a difficult problem once $N \geq 3$" [Qui72, p. 578]. Indeed, the modular cohomology has since resisted computation for four decades. Complete calculations exist only for $N \leq 4$ [Agu80, TY83b, TY83a, AMM90]. Much attention has focused on the case where N is small compared to p, e.g. [Bar04, BNP12a, BNP12b, Spr15a].

To our knowledge, when $N > \max\{p, 4\}$, the only previously constructed nonzero elements of $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$ are those due to Milgram and Priddy

[MP87], in the case $r = 1$. These reside in exponentially high degree: at least p^{N-2}. On the other hand, the cohomology is known to vanish in degrees less than $N/2$, by the stability theorem of Maazen [Maa79] together with Quillen's observation [Qui72] that the stable limit is zero. This leaves a large degree gap where it was not known whether the cohomology groups are nontrivial. We narrow this gap considerably by providing nonzero classes in degrees linear in N. We obtain:

Theorem 1 (see Theorem 39)**.** *Let $N \geq 2$, and let n be the natural number satisfying*

$$p^{n-1} < N \leq p^n.$$

Then

$$H^*(GL_N\mathbb{F}_{p^r}; \mathbb{F}_p)$$

has a nonzero element in degree $r(2p^n - 2p^{n-1} - 1)$. Moreover, it has is a non-nilpotent element in degree $2r(p^n - 1)$ if p is odd and in degree $r(2^n - 1)$ if $p = 2$. □

Notice that the degrees in the theorem grow linearly with N: if d is any of the degrees mentioned in the theorem, then

$$r(N-1) \leq d < 2r(pN-1).$$

In the case $r = 1$, we obtain stronger results, for instance:

Theorem 2 (see Theorem 41)**.** *For all $N \geq 2$,*

$$H^*(GL_N\mathbb{F}_2; \mathbb{F}_2)$$

has a non-nilpotent element of degree d for every d with at least $\lceil \log_2 N \rceil$ ones in its binary expansion. □

Our characteristic classes are defined for representations of dimension $N \geq 2$ over the finite field $\mathbb{F}_{p^r}$, and they are modular in the sense that they take values in group cohomology with coefficients in a field $\mathbb{F}$ of characteristic p. Thus they are interesting even for p-groups. The family of characteristic classes is parametrized by the cohomology of $GL_2\mathbb{F}_{p^r}$. We will show that many classes in this family are nonzero by finding representations ρ on which they are nontrivial. This will produce a family of nonzero cohomology classes on the general linear groups, namely the "universal classes" obtained by applying the characteristic classes to the defining representation of $GL_N\mathbb{F}_{p^r}$ where N is the dimension of ρ.

The characteristic classes are defined in terms of a push–pull construction featuring a transfer map. This construction was previously studied by the second author in [Spr15a], where he proved that it yields an injective map

$$H^*(GL_2\mathbb{F}_{p^r};\mathbb{F}_p) \longrightarrow H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$$

for $2 \leq N \leq p$. The present work was inspired by computations of the first author in string topology of classifying spaces [Lah16] featuring similar push–pull constructions.

In addition to the groups $GL_N\mathbb{F}_{p^r}$, our characteristic classes can be used to study other groups with interesting representations over finite fields. For example:

Theorem 3 (see Theorem 45 and Proposition 33)**.** *For all $n \geq 1$,*

$$H^*(\mathrm{Aut}(F_{p^n});\mathbb{F}_p) \qquad \textit{and} \qquad H^*(GL_{p^n}\mathbb{Z};\mathbb{F}_p)$$

have a non-nilpotent element of degree $2d$ for every d with the following property: the sum of the p-ary digits of d is equal to $k(p-1)$ for some $k \geq n$. In particular, there is a non-nilpotent element of degree $2p^n - 2$. (For $p = 2$, divide degrees by 2.) □

These classes live in the unstable range where the cohomology groups remain poorly understood.

The paper proceeds as follows. In section 2, we summarize the behavior of our characteristic classes and give two equivalent constructions. In section 3, we show that the universal classes on $GL_N\mathbb{F}_{p^r}$ for various N are related by parabolic induction maps. In section 4, we prove that our characteristic classes vanish on representations decomposable as a direct sum. In sections 5 and 6, we recall the cohomology of $GL_2\mathbb{F}_{p^r}$ and describe a coalgebra structure on it. In section 7, we develop a formula for the characteristic classes of a representation decomposable as a "wedge sum." In section 8, we introduce a family of "basic representations" which decompose as such wedge sums, and in section 9, we show that many of the characteristic classes of these representations are nonzero. In section 10, we study representations of elementary abelian groups, proving that their characteristic classes agree with those of a certain subrepresentation. In section 11, we exploit this property to construct high-dimensional representations with the same characteristic classes as the basic representations, and deduce our main results on the cohomology of general linear groups (Theorems 39 and 41). In section 12, we restrict to the prime fields $\mathbb{F}_p$, and give an alternative description of some of the characteristic classes of the basic representations in terms of Dickson invariants. This allows us to deduce that a certain subset of the universal classes is algebraically independent (Theorem 44). Finally, in section 13, we present applications to the homology and cohomology of automorphism groups of free groups and general linear groups over integers.

Conventions

Throughout the paper, p is a prime and $q = p^r$ is a power of p. A "representation" means a finite-dimensional representation over the field $\mathbb{F}_q$. Unless noted otherwise, homology and cohomology will be with coefficients in a field $\mathbb{F}$ of characteristic p, which will be omitted from notation. The main interest is in the cases $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F} = \mathbb{F}_q$, the latter being interesting since it allows for an explicit description of $H^*(GL_2\mathbb{F}_q)$.

As our characteristic classes χ_α are only defined for representations of dimension at least 2, whenever the notation $\chi_\alpha(\rho)$ appears, it is implicitly assumed that the representation ρ has dimension at least 2.

2 Characteristic classes

We will now define our characteristic classes. Given $\alpha \in H^*(GL_2\mathbb{F}_q)$, to each N-dimensional representation ρ over $\mathbb{F}_q$ of a group G (with $N \geq 2$), we will associate a class

$$\chi_\alpha(\rho) \in H^*(G).$$

only depending on the isomorphism class of ρ. As required of a characteristic class, the assignment $\rho \mapsto \chi_\alpha(\rho)$ will be natural with respect to group homomorphisms in the sense that

$$\chi_\alpha(f^*\rho) = f^*\chi_\alpha(\rho)$$

for any group homomorphism f to the domain of ρ. In fact we will give two alternative constructions of the characteristic classes, Definitions 5 and 6 below, and prove their equivalence. The following theorem summarizes the basic properties of the classes.

Theorem 4. *The characteristic classes χ_α have the following properties.*

(i) (Nontriviality, Remark 8) For the identity representation of $GL_2\mathbb{F}_q$, we have

$$\chi_\alpha(\mathrm{id}_{GL_2\mathbb{F}_q}) = \alpha$$

for all $\alpha \in H^{>0}(GL_2\mathbb{F}_q)$.

(ii) *(Vanishing on decomposables, Corollary 16) If ρ and η are nonzero representations of G, then*

$$\chi_\alpha(\rho \oplus \eta) = 0.$$

(iii) *(Wedge sum formula, Theorem 27) Suppose ρ and η are representations of G, and $v_0 \in \rho$ and $w_0 \in \eta$ are vectors fixed by G. Then*

$$\chi_\alpha(\rho \vee \eta) = -\sum \chi_{\alpha_{(1)}}(\rho) \cdot \chi_{\alpha_{(2)}}(\eta),$$

where $\rho \vee \eta = \rho \oplus \eta / \langle v_0 - w_0 \rangle$ and

$$\Delta(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)}$$

is a coproduct on the cohomology of $GL_2\mathbb{F}_q$ which will be described in section 6.

(iv) *(Reduction to J_1, Theorem 34) If ρ is a representation of an elementary abelian p-group G, then*

$$\chi_\alpha(\rho) = \chi_\alpha(J_1(\rho)),$$

where $J_1(\rho) \subset \rho$ is the subrepresentation consisting of the vectors annihilated by the second power of the augmentation ideal in the group ring $\mathbb{F}_qG$.

(v) *(Dependence on α, Remark 8) For a fixed representation ρ of a group G, the map*

$$H^*(GL_2\mathbb{F}_q; \mathbb{F}) \longrightarrow H^*(G; \mathbb{F}), \quad \alpha \longmapsto \chi_\alpha(\rho)$$

is $\mathbb{F}$-linear and commutes with the action of the mod p Steenrod algebra and the operation Fr_* induced by the Frobenius map of the coefficient field $\mathbb{F}$. Moreover, it commutes with coefficient field extension

$$H^*(-; \mathbb{F}) \longrightarrow H^*(-; \mathbb{E}) = H^*(-; \mathbb{F}) \otimes_{\mathbb{F}} \mathbb{E},$$

i.e. $\chi_{\alpha \otimes 1}(\rho) = \chi_\alpha(\rho) \otimes 1$.

In part (v), the mod p Steenrod algebra acts on cohomology with $\mathbb{F}$-coefficients by extension of scalars from cohomology with $\mathbb{F}_p$-coefficients.

We now give our two definitions of the χ_α-classes.

Definition 5. For $\alpha \in H^{>0}(GL_2\mathbb{F}_q)$ and $N \geq 2$, let

$$\chi_\alpha^{(N)} = (i_! \circ \pi^*)(\alpha)$$

in $H^d(GL_N\mathbb{F}_q)$, where i denotes the inclusion of the parabolic subgroup

$$P = \begin{bmatrix} GL_2 & * \\ & GL_{N-2} \end{bmatrix} \leq GL_N\mathbb{F}_q \tag{1}$$

into $GL_N\mathbb{F}_q$, $\pi\colon P \to GL_2\mathbb{F}_q$ is the projection onto the diagonal copy of $GL_2\mathbb{F}_q$, and $i_!$ denotes the transfer map induced by i. By convention, set $\chi_\alpha^{(N)} = 0$ when $\dim \alpha = 0$. (This convention simplifies the statement of Theorem 27).

Now let G be a group and let ρ be a representation of G of dimension $N \geq 2$. We define

$$\chi_\alpha(\rho) = \rho^*(\chi_\alpha^{(N)}) \in H^*(G).$$

On the right side, the notation ρ denotes the homomorphism $G \to GL_N\mathbb{F}_q$ associated to the representation, well defined up to conjugacy. It is evident from the definition that $\chi_\alpha(\rho)$ only depends on the isomorphism class of ρ and satisfies the required naturality. We call the underlying class $\chi_\alpha^{(N)} = \chi_\alpha(\mathrm{id}_{GL_N\mathbb{F}_q}) \in H^d(GL_N\mathbb{F}_q)$ the *universal χ_α-class for N-dimensional representations.*

Now we give the second definition. If V and W are vector spaces, write $\mathrm{Emb}(V, W)$ for the set of linear embeddings of V into W. If X is a G-space or G-set, we write $X/\!\!/G$ for the topological bar construction $B(\mathrm{pt}, G, X)$ [May75, Section 7], a model for the homotopy orbit space $EG \times_G X$. In particular, $\mathrm{pt}/\!\!/G$ is a model for the classifying space BG.

Definition 6. Let G be a group and let ρ be a representation of G of dimension $N \geq 2$. Using the diagram

$$\begin{array}{ccc} & \mathrm{Emb}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q & \\ {}^{!}\swarrow_{\pi} & & \searrow^{\tau} \\ \mathrm{pt}/\!\!/G & & \mathrm{pt}/\!\!/GL_2\mathbb{F}_q \end{array} \tag{2}$$

define

$$\chi_\alpha(\rho) = (\pi_! \circ \tau^*)(\alpha) \in H^*(G)$$

for $\alpha \in H^{>0}GL_2\mathbb{F}_q$. Here $G \times GL_2\mathbb{F}_q$ acts on $\mathrm{Emb}(\mathbb{F}_q^2, \rho)$ by

$$(g, A) \cdot f = g \circ f \circ A^{-1}, \tag{3}$$

and π and τ are the evident projection maps. Observe that the action of the $GL_2\mathbb{F}_q$-factor on $\mathrm{Emb}(\mathbb{F}_q^2, \rho)$ is free. Therefore, the map π factors as the composite

$$\mathrm{Emb}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q \xrightarrow{\simeq} (\mathrm{Emb}(\mathbb{F}_q^2, \rho)/GL_2\mathbb{F}_q)/\!\!/G \longrightarrow \mathrm{pt}/\!\!/G,$$

of a homotopy equivalence and a covering space with fibres modeled on the Grassmannian $\mathrm{Gr}_2(\rho)$, a finite set. Thus π indeed admits a transfer map $\pi_!$ on cohomology. As before, set $\chi_\alpha(\rho) = 0$ if $\dim \alpha = 0$. Clearly $\chi_\alpha(\rho)$ only depends on the isomorphism class of ρ, and compatibility of transfer maps with pullbacks implies that χ_α satisfies the required naturality.

Proposition 7. *The classes $\chi_\alpha(\rho)$ of Definitions 5 and 6 agree.*

Proof. Write temporarily $\tilde{\chi}_\alpha$ for the the classes given by Definition 6. By naturality, it suffices to show that $\tilde{\chi}_\alpha(\mathrm{id}_{GL_N\mathbb{F}_q}) = \chi_\alpha^{(N)}$ for all $N \geq 2$. Observe that for $\rho = \mathrm{id}_{GL_N\mathbb{F}_q}$, the $GL_N\mathbb{F}_q \times GL_2\mathbb{F}_q$-action on $\mathrm{Emb}(\mathbb{F}_q^2, \mathbb{F}_q^N)$ given by (3) is transitive, with the stabilizer of the inclusion $\mathbb{F}_q^2 \hookrightarrow \mathbb{F}_q^N$, $v \mapsto (v, 0)$ given by the subgroup

$$\left\{ \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, A_{11} \right) \in GL_N\mathbb{F}_q \times GL_2\mathbb{F}_q \;\middle|\; \begin{matrix} A_{11} \in GL_2\mathbb{F}_q \\ A_{22} \in GL_{N-2}\mathbb{F}_q \\ A_{12} \in \mathrm{Mat}_{N,N-2}\mathbb{F}_q \end{matrix} \right\}$$

of $GL_N\mathbb{F}_q \times GL_2\mathbb{F}_q$. Thus diagram (2) for computing $\tilde{\chi}_\alpha(\mathrm{id}_{GL_N\mathbb{F}_q})$ reduces to the diagram

$$\mathrm{pt}/\!\!/GL_N\mathbb{F}_q \xleftarrow{\;!\;} \mathrm{pt}/\!\!/P \longrightarrow \mathrm{pt}/\!\!/GL_2\mathbb{F}_q$$

where P is the parabolic subgroup (1) of Definition 5 and the two maps are induced by the inclusion $i\colon P \hookrightarrow GL_N\mathbb{F}_q$ and the projection $\pi\colon P \to GL_2\mathbb{F}_q$ of Definition 5. The claim follows. □

Remark 8. In the case $N = 2$, the maps i and π of Definition 5 both reduce to the identity map of $GL_2\mathbb{F}_q$. Thus part (i) of Theorem 4 follows. Part (v) of Theorem 4 is immediate from the fact that induced maps and transfer maps on cohomology have the properties in question.

3 Parabolic induction

The aim of this section is to show that the universal χ_α-classes $\chi_\alpha^{(n)}$ for various n are related by what we call *parabolic induction maps.*

Definition 9. For $m \le n$, we define the *parabolic induction map* $\Phi_{m,n}$ to be the composite

$$\Phi_{m,n} \colon H^*(GL_m\mathbb{F}_q) \xrightarrow{\pi^*} H^*(P) \xrightarrow{i_!} H^*(GL_n\mathbb{F}_q)$$

where P is the parabolic subgroup

$$P = \begin{bmatrix} GL_m\mathbb{F}_q & * \\ 0 & GL_{n-m}\mathbb{F}_q \end{bmatrix} \le GL_n\mathbb{F}_q,$$

$i \colon P \hookrightarrow GL_n\mathbb{F}_q$ is the inclusion, and $\pi \colon P \to GL_m\mathbb{F}_q$ is the projection onto the diagonal copy of $GL_m\mathbb{F}_q$.

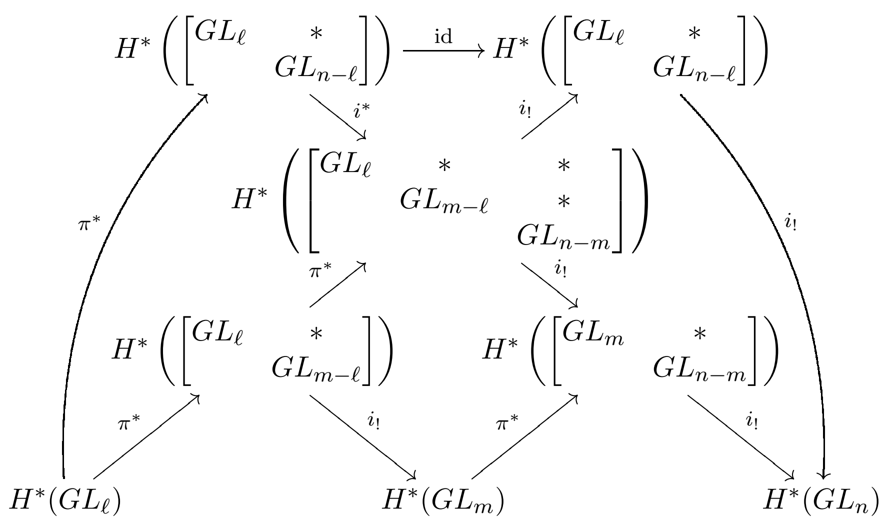
Remark 10. Comparing Definitions 5 and 9, we see that $\chi_\alpha^{(n)} = \Phi_{2,n}(\alpha)$ for all $n \ge 2$ and $\alpha \in H^{>0}(GL_2\mathbb{F}_q)$.

The parabolic induction maps are compatible with one another under composition:

Proposition 11. *For all $\ell \le m \le n$,*

$$\Phi_{m,n} \circ \Phi_{\ell,m} = \Phi_{\ell,n}.$$

Proof. Consider the diagram



where for brevity we have written GL_a for $GL_a\mathbb{F}_q$ and where the various i's and π's stand for the evident inclusion and projection maps, respectively. Observe that the diagram commutes: For the two cells with curved arrows, commutativity is immediate; for the diamond in the middle, commutativity follows from the fact that the two projection maps π involved have the same kernel; and for the triangle on top, commutativity follows from the observation that the index of the subgroup

$$\begin{bmatrix} GL_\ell & * & * \\ & GL_{m-\ell} & * \\ & & GL_{n-m} \end{bmatrix} \le \begin{bmatrix} GL_\ell & * \\ & GL_{n-\ell} \end{bmatrix}$$

equals the number of points in the Grassmannian $\mathrm{Gr}_{m-\ell}(\mathbb{F}_q^{n-\ell})$, which is

$$\binom{n-\ell}{m-\ell}_q \equiv 1 \mod q.$$

The claim now follows by observing that the composite from $H^*(GL_\ell)$ to $H^*(GL_n)$ along the bottom of the diagram equals $\Phi_{m,n} \circ \Phi_{\ell,m}$, while the composite along the top of the diagram equals $\Phi_{\ell,n}$. □

Combining Proposition 11 and Remark 10, we obtain the desired connection between the classes $\chi_\alpha^{(n)}$ for different values of n.

Corollary 12. $\chi_\alpha^{(n)} = \Phi_{m,n}(\chi_\alpha^{(m)})$ *for all* $2 \le m \le n$ *and* $\alpha \in H^*(GL_2\mathbb{F}_q)$. □

This in turn implies

Corollary 13. *If* $\chi_\alpha^{(n)}$ *is nonzero or non-nilpotent, so is* $\chi_\alpha^{(m)}$ *for every* $2 \le m \le n$.

Proof. Only the claim regarding non-nilpotence requires comment. Since non-nilpotence can be verified using Steenrod powers, the claim follows by observing that the parabolic induction maps commute with the Steenrod operations, because induced maps and transfer maps [Eve68] do. □

In view of Corollary 13, to establish the nonvanishing of the universal classes $\chi_\alpha^{(m)}$ for a large range of m's, one should try to find as high-dimensional representations ρ as possible with $\chi_\alpha(\rho) \neq 0$. This is our aim in section 11.

4 Vanishing on decomposables

Our goal in this section is to prove the following result.

Theorem 14. *If ρ is a representation which has a trivial subrepresentation of dimension 2, then*

$$\chi_\alpha(\rho) = 0$$

for all α.

Remark 15. Representations over $\mathbb{F}_q$ (or any field of characteristic p) have an "upper-triangularization principle" with respect to mod-p cohomology. Indeed, if $P \leq G$ is a Sylow p-subgroup, then restriction from G to P in cohomology is injective, while the restriction of any representation of G to P is unipotent upper-triangular with respect to some basis: in particular, it has a fixed line.

Hence, Theorem 14 implies:

Corollary 16. *If ρ and η are nonzero representations, then*

$$\chi_\alpha(\rho \oplus \eta) = 0$$

for all α. □

Remark 17. For a representation of a p-group P over a field of characteristic p, the condition of having just one-dimensional fixed-point space is quite restrictive. Indeed, such representations are precisely the subrepresentations of the regular representation of P.

Remark 18. Taking η to be the trivial 1-dimensional representation shows that the universal classes $\chi_\alpha^{(n)} \in H^*(GL_n\mathbb{F}_q)$ vanish upon restriction to $H^*(GL_{n-1}\mathbb{F}_q)$. That is, they are annihilated by the stabilization maps.

Remark 19. Corollary 16 implies that the universal classes $\chi_\alpha^{(n)}$ are primitives in the bialgebra

$$\bigoplus_{n=0}^{\infty} H^*(GL_n\mathbb{F}_q)$$

where the coproduct is induced by the block-sum homomorphisms

$$GL_a\mathbb{F}_q \times GL_b\mathbb{F}_q \to GL_{a+b}\mathbb{F}_q.$$

We now turn to the proof of Theorem 14. Let $V \leq \rho^G$ be a subspace of the set of fixed-points in ρ. Define

$$\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho) = \left\{ f \in \mathrm{Emb}(\mathbb{F}_q^2, \rho) \;\middle|\; \mathrm{Im}(f) \cap \rho^G = V \right\},$$

the set of embeddings whose image contains precisely the fixed-points in V (nonempty only for $\dim V \leq 2$). This decomposition

$$\mathrm{Emb}(\mathbb{F}_q^2, \rho) = \coprod_{V \leq \rho^G} \mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)$$

as $(G \times GL_2\mathbb{F}_q)$-sets yields a disjoint union decomposition of the space $\mathrm{Emb}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q$. Consequently, Definition 6 splits up as a sum

$$\chi_\alpha(\rho) = \sum_{V \leq \rho^G} ((\pi_V)_! \circ \tau_V^*)(\alpha),$$

where π_V and τ_V are the evident projections fitting into the diagram

$$\mathrm{pt}/\!\!/G \xleftarrow[\pi_V]{!} \mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q \xrightarrow{\tau_V} \mathrm{pt}/\!\!/GL_2\mathbb{F}_q$$

The following lemma says that, in calculating $\chi_\alpha(\rho)$, one needs to consider only those embeddings whose image contains all of the fixed points, and also contains nonfixed points.

Lemma 20. *Let ρ be a representation of G, and $V \leq \rho^G$ a subspace. Then*

$$(\pi_V)_! \circ \tau_V^* = 0$$

in positive degrees if either

1. $\dim V = 2$, *or*
2. $V < \rho^G$ *is a proper subspace.*

Proof. First suppose $\dim V = 2$. Then the G-action on $\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)$ is trivial, so τ_V factors through

$$\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)/\!\!/GL_2\mathbb{F}_q = \mathrm{Iso}(\mathbb{F}_q^2, V)/\!\!/GL_2\mathbb{F}_q,$$

which is contractible.

Now we turn to the case in which $V \leq \rho^G$ is proper. If $\dim(V) > 2$, we are done since in this case $\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)$ is empty. In light of the previous case, we may therefore assume that $\dim V < 2$. Pick some fixed line $L \leq \rho^G$ which is not contained in V. Let X be the image of $\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)$ under the map

$$\mathrm{Hom}(\mathbb{F}_q^2, \rho) \longrightarrow \mathrm{Hom}(\mathbb{F}_q^2, \rho/L)$$

induced by the quotient homomorphism $r\colon \rho \to \rho/L$.

Observe that the fibre over $f \in X$ of the resulting surjection

$$\mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho) \longrightarrow X$$

has a free $\mathrm{Hom}(\mathbb{F}_q^2/f^{-1}rV, L)$-action. Consequently, the cardinalities of the fibres are divisible by $q^{2-\dim(V)}$, and hence by p. The same is true after passing to homotopy orbits, so that in the commutative diagram

$$\begin{array}{ccccc} & & X/\!\!/G \times GL_2\mathbb{F}_q & & \\ & \swarrow^{\pi'_V} & \uparrow \phi & {}^{\tau'_V}\searrow & \\ & & \mathrm{Emb}^{(V)}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q & & \\ & \swarrow_{\pi_V} & & {}_{\tau_V}\searrow & \\ \mathrm{pt}/\!\!/G & & & & \mathrm{pt}/\!\!/GL_2\mathbb{F}_q, \end{array}$$

(where π'_V and τ'_V are the evident projections), the map ϕ is a union of covering spaces whose multiplicities are divisible by p. It follows that

$$(\pi_V)_! \circ \tau_V^* = (\pi'_V)_! \circ (\phi_! \circ \phi^*) \circ (\tau'_V)^* = (\pi'_V)_! \circ 0 \circ (\tau'_V)^* = 0. \qquad \square$$

Given a subspace $V \leq \rho^G$, let us write $\mathrm{Emb}^{(\geq V)}(\mathbb{F}_q^2, \rho)$ for the set of embeddings whose image contains V. For later reference, we record the following corollary of Lemma 20.

Corollary 21. *Let ρ be a representation of G and let $V \leq \rho^G$ be a subspace. For all $\alpha \in H^{>0}(GL_2\mathbb{F}_q)$ we have*

$$\chi_\alpha(\rho) = (\pi_! \circ \tau^*)(\alpha)$$

where π and τ are the evident projections fitting into the diagram

$$\begin{array}{ccccc} & & \mathrm{Emb}^{(\geq V)}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q & & \\ & \overset{!}{\swarrow}_{\pi} & & {}_{\tau}\searrow & \\ \mathrm{pt}/\!\!/G & & & & \mathrm{pt}/\!\!/GL_2\mathbb{F}_q \end{array} \qquad \square$$

Proof of Theorem 14. Since we are assuming $\dim \rho^G \geq 2$, any $V \leq \rho^G$ satisfying $\dim V < 2$ is proper. So by the lemma, $(\pi_V)_! \circ \tau_V^* = 0$ for every V, showing $\chi_\alpha(\rho) = 0$. □

5 The cohomology of $GL_2\mathbb{F}_q$

Because our family of characteristic classes is indexed on the modular cohomology of $GL_2\mathbb{F}_q$, we now describe the cohomology for the reader's convenience. The result described in this section is well-known [Agu80, Qui72, FP83] [Spr15b, ch. 1]. To make the description more explicit, we assume in this section that the coefficient field $\mathbb{F}$ for cohomology is an extension of $\mathbb{F}_q$.

The p-Sylow subgroup of $GL_2\mathbb{F}_q$ is

$$\mathbb{F}_q = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix},$$

the additive group of the finite field. Restriction gives an isomorphism

$$H^*(GL_2\mathbb{F}_q) \approx H^*(\mathbb{F}_q)^{\mathbb{F}_q^\times}$$

with the invariants of the multiplicative group

$$\mathbb{F}_q^\times \approx C_{q-1}$$

acting on the cohomology ring of the additive group.

Remark 22. The same remarks apply with $GL_2\mathbb{F}_q$ replaced by its subgroup

$$\mathrm{Aff}_1\mathbb{F}_q = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

Hence the restriction in mod p cohomology from $GL_2\mathbb{F}_q$ to $\mathrm{Aff}_1\mathbb{F}_q$ is an isomorphism. Since the index is a unit modulo p, the transfer $H^*(\mathrm{Aff}_1\mathbb{F}_q) \to H^*(GL_2\mathbb{F}_q)$ is an isomorphism as well.

By the assumption that the coefficient field $\mathbb{F}$ is an extension of $\mathbb{F}_q$, the action of $\mathbb{F}_q^\times$ on $H^*(\mathbb{F}_q)$ diagonalizes, as all $(q-1)$-th roots are present and distinct. That is, there is a weight-space decomposition

$$H^*(\mathbb{F}_q) = \bigoplus_{k \in \mathbb{Z}/(q-1)} E_k, \tag{4}$$

with E_k consisting of those elements on which $c \in \mathbb{F}_q^\times$ acts as multiplication by c^k. In particular, restriction gives an isomorphism

$$H^*(GL_2\mathbb{F}_q) \approx H^*(\mathbb{F}_q)^{\mathbb{F}_q^\times} = E_0.$$

Now, one can find canonical (up to scalar multiple) generators for the cohomology:

$$H^*(\mathbb{F}_{p^r}) = \begin{cases} \mathbb{F}[y_0, \ldots, y_{r-1}] & \text{if } p = 2, \\ \mathbb{F}\langle x_0, \ldots, x_{r-1} \rangle & \\ \qquad \otimes \mathbb{F}[y_0, \ldots, y_{r-1}] & \text{if } p \text{ odd}, \end{cases}$$

where

$$x_k, y_k \in E_{p^k}$$

and $\deg x_i = 1$, $\deg y_i = 2$ (for $p = 2$, $\deg y_i = 1$). The angle braces indicate an exterior algebra. The monomials

$$x^A y^B = \prod_{k=0}^{r-1} x_k^{a_k} y_k^{b_k}$$

for

$$A = (a_0, \ldots, a_{r-1}) \in \{0,1\}^r \quad \text{and} \quad B = (b_0, \ldots, b_{r-1}) \in \mathbb{N}^r$$

now form an additive basis for the cohomology, consisting of eigenvectors for the action of the multiplicative group.

From this, we obtain the following description of the invariants:

Lemma 23. *Under the assumption that $\mathbb{F}$ is an extension of $\mathbb{F}_q$, the restriction*

$$H^*(GL_2\mathbb{F}_q; \mathbb{F}) \longrightarrow H^*(\mathbb{F}_q; \mathbb{F})$$

is an isomorphism onto the subspace (and subring) generated by the monomials

$$x^A y^B$$

with the property that

$$(p^r - 1) \mid \sum_{k=0}^{r-1} p^k (a_k + b_k).$$

(In the case of odd p; for $p = 2$ of course there are no exterior terms.) □

6 The coproduct on the cohomology of $GL_2\mathbb{F}_q$

Because $\mathbb{F}_q$ is an abelian group, its addition map $\mu\colon \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ gives its cohomology the structure of a (coassociative cocommutative counital) coalgebra. We will show that, although $GL_2\mathbb{F}_q$ is not abelian, its cohomology inherits a coproduct from the Sylow p-subgroup $\mathbb{F}_q$ of $GL_2\mathbb{F}_q$. Observe that the transfer map

$$\mathrm{tr}\colon H^*(\mathbb{F}_q) \longrightarrow H^*(GL_2\mathbb{F}_q)$$

is a retraction onto the image of $\mathrm{res} : H^*(GL_2\mathbb{F}_q) \hookrightarrow H^*(\mathbb{F}_q)$.

Proposition 24. *For any field $\mathbb{F}$ of characteristic p, the group cohomology $H^*(GL_2\mathbb{F}_q; \mathbb{F})$ admits a unique coassociative cocommutative counital coproduct Δ making*

$$\mathrm{tr}\colon H^*(\mathbb{F}_q; \mathbb{F}) \longrightarrow H^*(GL_2\mathbb{F}_q; \mathbb{F})$$

into a homomorphism of coalgebras.

Proof. It suffices to verify that the kernel $J = \ker(\mathrm{tr})$ is a coideal, that is, that J is annihilated by the counit $H^*(\mathbb{F}_q; \mathbb{F}) \to \mathbb{F}$ and that

$$\mu^* J \subset H^*(\mathbb{F}_q; \mathbb{F}) \otimes_{\mathbb{F}} J + J \otimes_{\mathbb{F}} H^*(\mathbb{F}_q; \mathbb{F}).$$

The first of these conditions is immediate form the observation that tr is an isomorphism in degree 0. To verify the second condition, observe that if it holds for the field $\mathbb{F}$, it also holds for any subfield of $\mathbb{F}$, by the compatibility of transfers and induced maps with extending the coefficient field for cohomology. Thus it is enough to check the condition in the case where $\mathbb{F}$ is an extension of $\mathbb{F}_q$. In that case, we have the weight-space decomposition (4), with respect to which the inclusion $H^*(GL_2\mathbb{F}_q) \hookrightarrow H^*(\mathbb{F}_q)$ corresponds to the inclusion $E_0 \hookrightarrow \bigoplus_k E_k$ and the transfer tr to the projection $\bigoplus_k E_k \to E_0$. In particular, we have

$$J = \bigoplus_{k \neq 0} E_k.$$

Because the addition map $\mu : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ is equivariant with respect to the action of the multiplicative group, the induced map $\mu^*\colon H^*(\mathbb{F}_q) \to H^*(\mathbb{F}_q) \otimes H^*(\mathbb{F}_q)$ satisfies

$$\mu^*(E_k) \subset \bigoplus_{i+j=k} E_i \otimes E_j.$$

Thus

$$\begin{aligned}\mu^*(J) &\subset \bigoplus_{i+j\neq 0} E_i \otimes E_j \\ &\subset \bigoplus_{\substack{i\neq 0 \text{ or} \\ j\neq 0}} E_i \otimes E_j \\ &= J \otimes H^*(\mathbb{F}_q;\mathbb{F}) + H^*(\mathbb{F}_q;\mathbb{F}) \otimes J. \qquad \square\end{aligned}$$

We warn the reader that the coproduct does not make $H^*(GL_2\mathbb{F}_q)$ into a Hopf algebra, as it is not compatible with the cup product.

Explicitly, we can describe Δ in terms of the monomial basis of Lemma 23 under the assumption that $\mathbb{F}$ is an extension of $\mathbb{F}_q$:

$$\Delta\left(x^A y^B\right) = \sum_{\substack{A'+A''=A \\ B'+B''=B \\ P(A'+B')}} \binom{B}{B'} x^{A'} y^{B'} \otimes x^{A''} y^{B''}.$$

Here we have used the shorthand

$$\binom{B}{B'} = \binom{b_0}{b'_0} \cdots \binom{b_{r-1}}{b'_{r-1}},$$

and the sum runs over only those pairs with the divisibility property

$$P(C) = \left[(p^r - 1) \mid \sum_{k=0}^{r-1} p^k c_k\right]. \tag{5}$$

(As usual, for $p = 2$ one must remove the exterior terms.) Informally, one simply performs the usual coproduct on $H^*(\mathbb{F}_q)$ (which is multiplicative, with the generators primitive), and then throws out all terms which do not satisfy the divisibility condition (5).

7 Wedge sum formula

While our characteristic classes vanish on direct sums by Corollary 16, they exhibit interesting behavior with respect to the following wedge sum construction, which we can use to combine two representations without proliferating their fixed-points.

Definition 25. A *pointed representation* of a group G is a representation ρ of G equipped with the choice of a *basepoint*, a non-zero vector $v_0 \in \rho$ fixed by the G-action. If (ρ, v_0) and (η, w_0) are pointed representations of groups G and H, respectively, we define their *wedge sum* to be the representation

$$\rho \vee \eta = \rho \oplus \eta / \langle v_0 - w_0 \rangle,$$

of $G \times H$, a pointed representation with basepoint $v_0 = w_0$. The isomorphism type of $\rho \vee \eta$ as a pointed representation remains unchanged if the basepoints v_0 and w_0 are replaced by non-zero multiples. Hence for representations with unique fixed lines, we will take the liberty to form wedge sums without explicitly specifying the basepoints.

Remark 26. Working with pointed representations may appear overly restrictive, since not all representations admit a non-zero fixed vector. However, from the point of view of mod-p cohomology it is no loss, due to the upper-triangularization principle of Remark 15.

We describe how to calculate the classes of a wedge sum in terms of the classes of the two factors:

Theorem 27. *If ρ and η are pointed representations of G and H, respectively, then for all $\alpha \in H^*(GL_2\mathbb{F}_q)$*

$$\chi_\alpha(\rho \vee \eta) = -\sum \chi_{\alpha_{(1)}}(\rho) \times \chi_{\alpha_{(2)}}(\eta) \tag{6}$$

as an element of $H^*(G \times H)$, where

$$\Delta(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)}$$

is the coproduct constructed in section 6.

We begin the proof by establishing an "affine version" of the push-pull construction in Definition 6, better suited to working with pointed representations. Let $i : \mathbb{F}_q \to GL_2\mathbb{F}_q$ be the inclusion.

Proposition 28. *Let (ρ, v_0) be a pointed representation of a group G with $\dim(\rho) \geq 2$. The characteristic classes of ρ are given by following the diagram*

$$\begin{array}{ccccc} & & \rho /\!\!/ G \times \mathbb{F}_q & & \\ & {}^{!}\swarrow_{\pi} & & \searrow^{\tau} & \\ \mathrm{pt}/\!\!/G & & & & \mathrm{pt}/\!\!/\mathbb{F}_q \xrightarrow{\ i\ } \mathrm{pt}/\!\!/GL_2\mathbb{F}_q \end{array}$$

where the action of $G \times \mathbb{F}_q$ on ρ is given by

$$(g, c) \cdot v = gv - cv_0.$$

More precisely,

$$\chi_\alpha(\rho) = -(\pi_! \circ \tau^* \circ i^*)(\alpha)$$

for all $\alpha \in H^*(GL_2\mathbb{F}_q)$.

Proof. We first observe that the formula is correct when $\deg \alpha = 0$. In that case, $\chi_\alpha(\rho)$ is zero by convention, while the right hand side is multiplication by the multiplicity of the covering π, which is

$$|\rho/\langle v_0\rangle| = q^{\dim \rho - 1},$$

a multiple of q (so zero on cohomology).

Hence we assume $\alpha \in H^{>0}(GL_2\mathbb{F}_q)$. Now, the expression on the right hand side splits as a sum of two terms, corresponding to the equivariant decomposition

$$\rho = \langle v_0\rangle \coprod (\rho - \langle v_0\rangle).$$

The former term vanishes, because $\langle v_0\rangle /\!\!/ G \times \mathbb{F}_q \to \mathrm{pt}/\!\!/\mathbb{F}_q$ factors through the contractible space $\langle v_0\rangle /\!\!/ \mathbb{F}_q$. So we must verify that the latter term in the sum agrees with $\chi_\alpha(\rho)$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{pt}/\!\!/G & \xleftarrow[\;!\;]{\pi'} & \mathrm{Emb}^{(\geq \langle v_0\rangle)}(\mathbb{F}_q^2, \rho)/\!\!/G \times GL_2\mathbb{F}_q & \xrightarrow{\tau'} & \mathrm{pt}/\!\!/GL_2 \\
 & \nwarrow^{\pi''}_{!} & \uparrow \phi & & \uparrow i \\
 & & \rho - \langle v_0\rangle /\!\!/ G \times \mathbb{F}_q & \xrightarrow{\tau''} & \mathrm{pt}/\!\!/\mathbb{F}_q
\end{array}$$

where $\mathrm{Emb}^{(\geq \langle v_0\rangle)}(\mathbb{F}_q^2, \rho)$ is as in Corollary 21 and where ϕ is induced by the map sending $v \in \rho - \langle v_0\rangle$ to the embedding $e_1 \mapsto v_0$, $e_2 \mapsto v$.

Following the diagram along the bottom, i.e. $\pi''_! \circ (\tau'')^* \circ i^*$, yields the right hand side of the proposition, without the minus sign. Meanwhile, following the top of the diagram, i.e. $\pi'_! \circ (\tau')^*$, yields $\alpha \mapsto \chi_\alpha(\rho)$, by Corollary 21. Now, up to homotopy equivalence, ϕ agrees with the covering map

$$\left(\frac{\rho}{\langle v_0\rangle} - 0\right) /\!\!/ G \longrightarrow \mathbb{P}\left(\frac{\rho}{\langle v_0\rangle}\right) /\!\!/ G$$

which has multiplicity $q - 1$. (We used that the actions of $GL_2\mathbb{F}_q$ and $\mathbb{F}_q$ are free, to replace their homotopy quotients with strict quotients.) So

$\phi_! \circ \phi^* = -1$, and we get

$$\begin{aligned}\chi_\alpha(\rho) &= (\pi'_! \circ \tau'^*)(\alpha) \\ &= -(\pi'_! \circ (\phi_! \circ \phi^*) \circ \tau'^*)(\alpha) \\ &= -(\pi''_! \circ \tau''^* \circ i^*)(\alpha).\end{aligned}$$

□

We will also need the following fact:

Lemma 29. *In the notation of Proposition 28,*

$$\pi_! \circ \tau^* \circ (i^* \circ i_!) = \pi_! \circ \tau^*.$$

Proof. Because $i_! \circ i^* = 1$, it suffices to show that $\pi_! \circ \tau^* \colon H^*(\mathbb{F}_q) \to H^*(G)$ factors through $i_! \colon H^*(\mathbb{F}_q) \to H^*(GL_2\mathbb{F}_q)$. To do so, it suffices to show that it factors through the transfer $H^*(\mathbb{F}_q) \to H^*(\mathrm{Aff}_1\mathbb{F}_q)$, because the further transfer $H^*(\mathrm{Aff}_1\mathbb{F}_q) \to H^*(GL_2\mathbb{F}_q)$ is an isomorphism by Remark 22. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{pt}/\!\!/G & \xleftarrow{\;!\;} & \rho/\!\!/G \times \mathrm{Aff}_1\mathbb{F}_q & \xrightarrow{\;\tau'\;} & \mathrm{pt}/\!\!/\mathrm{Aff}_1\mathbb{F}_q \\ & \overset{!}{\underset{\pi}{\nwarrow}} & \phi \big\uparrow ! & & j \big\uparrow ! \\ & & \rho/\!\!/G \times \mathbb{F}_q & \xrightarrow{\;\tau\;} & \mathrm{pt}/\!\!/\mathbb{F}_q \end{array}$$

where $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \in \mathrm{Aff}_1\mathbb{F}_q$ acts on ρ by $v \mapsto b^{-1}v - b^{-1}av_0$. The square is a pullback and τ' is a fibration, so $\phi_! \circ \tau^* = \tau'^* \circ j_!$, yielding the desired factorization. □

Proof of Theorem 27. Let $\mu \colon \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ be the addition map, and $i \colon \mathbb{F}_q \to GL_2\mathbb{F}_q$ the inclusion. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{pt}/\!\!/G \times H & \xleftarrow[\pi']{\;!\;} & \rho \vee \eta/\!\!/G \times H \times \mathbb{F}_q & \xrightarrow[\tau']{} & \mathrm{pt}/\!\!/\mathbb{F}_q & \xrightarrow[i]{} & \mathrm{pt}/\!\!/GL_2\mathbb{F}_q \\ & \overset{!}{\underset{\pi}{\nwarrow}} & \varphi \big\uparrow \simeq & & \mu \big\uparrow & & \\ & & \rho \times \eta/\!\!/G \times \mathbb{F}_q \times H \times \mathbb{F}_q & \xrightarrow[\tau]{} & \mathrm{pt}/\!\!/\mathbb{F}_q \times \mathbb{F}_q & & \end{array}$$

Here φ is induced by the quotient map $\rho \times \eta \to \rho \vee \eta$ and the addition map μ. The action of $(G \times \mathbb{F}_q) \times (H \times \mathbb{F}_q)$ on $\rho \times \eta$ is the product of those in Proposition 28. Notice that π and τ are each a product of two projection maps. Observe also that the kernel of μ acts freely on $\rho \times \eta$, with quotient $\rho \vee \eta$. Thus the map φ is a homotopy equivalence.

Applying Proposition 28 to both ρ and η shows that the right hand side of equation (6) agrees with $-(\pi_! \circ \tau^* \circ (i \times i)^*)(\Delta\alpha)$. Inserting

$$\Delta = (i \times i)_! \circ \mu^* \circ i^*$$

and applying Lemma 29 to remove $(i \times i)^*(i \times i)_!$ yields $-(\pi_! \circ \tau^* \circ \mu^* \circ i^*)(\alpha)$. The diagram shows that this is equal to $-(\pi'_! \circ \tau'^* \circ i^*)(\alpha)$, which by Proposition 28 agrees with the left hand side of equation (6). □

8 The basic representations

We now define a family of representations whose characteristic classes we shall be able to describe. For any $\mathbb{F}_q$-vector space V, regard V as an elementary abelian group under addition, and define its "basic representation" ρ_V to be

$$\rho_V = \mathbb{F}_p \oplus V^*$$

as a vector space, with V acting as

$$v \cdot (c, u) = (c + u(v), u).$$

In coordinates, it looks like

$$\rho_{\mathbb{F}_q^n} : (a_1, \ldots, a_n) \longmapsto \begin{bmatrix} 1 & a_1 & \cdots & a_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

We equip ρ_V with the basepoint $(1, 0) \in \mathbb{F}_q \oplus V^*$. If V and W are two vector spaces, we then have

$$\rho_V \vee \rho_W = \rho_{V \oplus W}.$$

Therefore, in terms of a basis we have a decomposition

$$\rho_{\mathbb{F}_q^n} = \rho_{\mathbb{F}_q} \vee \cdots \vee \rho_{\mathbb{F}_q}$$

as representations of $\mathbb{F}_q^n$. Hence by Theorem 27,

$$\chi_\alpha(\rho_{\mathbb{F}_q^n}) = (-1)^{n-1} \sum \chi_{\alpha_{(1)}}(\rho_{\mathbb{F}_q}) \times \cdots \times \chi_{\alpha_{(n)}}(\rho_{\mathbb{F}_q}).$$

But $\rho_{\mathbb{F}_q}$ is the (2-dimensional) representation given by the inclusion of $\mathbb{F}_q$ as the Sylow p-subgroup of $GL_2\mathbb{F}_q$, by which we identified $H^*(GL_2\mathbb{F}_q)$ as a subspace of $H^*(\mathbb{F}_q)$. Therefore by Theorem 4.(i),

$$\chi_\alpha(\rho_{\mathbb{F}_q^n}) = \begin{cases} \alpha & \text{if } \deg \alpha > 0, \\ 0 & \text{if } \deg \alpha = 0. \end{cases}$$

Consequently, $\chi_\alpha(\rho_{\mathbb{F}_q^n})$ is obtained from the iterated coproduct $\Delta^{n-1}(\alpha)$ by throwing away all those terms which have degree 0 in some factor. Let us write the result explicitly, assuming that $\mathbb{F}$ extends $\mathbb{F}_q$. To begin with, write the iterated coproduct as

$$\Delta^{n-1}\left(x^A y^B\right) = \sum_{\substack{A_1+\cdots+A_n=A \\ B_1+\cdots+B_n=B \\ P(A_i+B_i)\ \forall i}} \binom{B}{B_1;\cdots;B_n} x^{A_1}y^{B_1}\otimes\cdots\otimes x^{A_n}y^{B_n}.$$

Here we have used the shorthand

$$\binom{B}{B_1;\cdots;B_n} = \binom{b_0}{b_{0,1},\ldots,b_{0,n}}\cdots\binom{b_{r-1}}{b_{r-1,1},\ldots,b_{r-1,n}}, \tag{7}$$

where the terms on the right hand side are multinomial coefficients, and $B_i = (b_{0,i},\ldots,b_{r-1,i})$. Now,

$$\chi_{x^Ay^B}(\rho_{\mathbb{F}_q^n}) = (-1)^{n-1} \sum_{\substack{A_1+\cdots+A_n=A \\ B_1+\cdots+B_n=B \\ P(A_i+B_i)\ \forall i \\ A_i+B_i\neq 0\ \forall i}} \binom{B}{B_1;\cdots;B_n} x^{A_1}y^{B_1}\times\cdots\times x^{A_n}y^{B_n}. \tag{8}$$

9 Nontrivial classes

We now consider the question of which characteristic classes of the basic representation are nonzero. In equation (8), the monomials $x^{A_1}y^{B_1}\times\cdots\times x^{A_n}y^{B_n}$ in the sum are all linearly independent. So

$$\chi_{x^Ay^B}(\rho_{\mathbb{F}_q^n}) \neq 0$$

if and only if there exists a nonzero term in the sum. By iterating Lucas' theorem on the value of binomial coefficients mod p, we see that the coefficient (7) appearing in (8) is nonzero mod p precisely when there is not a carry when adding together the numbers $b_{j,1},\ldots,b_{j,n}$ in base p for any $0\leq j\leq r-1$. It is now straightforward to check that by choosing

$$A_i = (0,\ldots,0) \quad\text{and}\quad B_i = p^{i-1}(p-1)\cdot(1,\ldots,1),$$

we obtain a nonzero term appearing in a sum of type (8), as we do by choosing

$$A_1 = (1,\ldots,1) \quad\text{and}\quad B_1 = (p-2)\cdot(1,\ldots,1),$$

and for $i > 1$

$$A_i = (0, \ldots, 0) \quad \text{and} \quad B_i = p^{i-2}((p-2)p+1) \cdot (1, \ldots, 1).$$

Summing over i to obtain A and B, we have the following result.

Proposition 30. *For p odd:*

(i) The class

$$\chi_\alpha(\rho_{\mathbb{F}_q^n}) \in H^{2r(p^n-1)}(\mathbb{F}_q^n; \mathbb{F}_q)$$

is non-nilpotent for

$$\alpha = (y_0 \cdots y_{r-1})^{p^n-1}.$$

(ii) The class

$$\chi_\alpha(\rho_{\mathbb{F}_q^n}) \in H^{r(2p^n-2p^{n-1}-1)}(\mathbb{F}_q^n; \mathbb{F}_q)$$

is nonzero for

$$\alpha = x_0 \cdots x_{r-1}(y_0 \cdots y_{r-1})^{p^n-p^{n-1}-1}.$$

Part (i) is also valid for $p = 2$, with the degree halved. □

Now restrict to the case $r = 1$, where we can give a complete description of which classes are nonzero.

Definition 31. For $m \in \mathbb{N}$, define $s_p(m)$ to be the sum of the digits of m in p-ary notation. That is,

$$s_p(m) = \sum_k c_k$$

where

$$m = \sum_k c_k p^k \quad \text{with} \quad 0 \le c_k < p \quad \forall k.$$

First we give a bound for m in terms of $s_p(m)$.

Lemma 32. *Let $s \in \mathbb{N}$. The lowest m such that $s_p(m) = s$ is given by*

$$m = (d+1)p^c - 1,$$

where c and d are determined by

$$s = c(p-1) + d, \qquad 0 \le d < p-1.$$

Proof. The lowest m must have a p-ary representation of the form

$$d'.\underbrace{(p-1).\cdots.(p-1)}_{c'} \quad \text{with} \quad 0 \leq d' < p-1$$

since, if not, we may decrease m while preserving $s_p(m)$ by "moving digits to the right." For the digits to sum to s, we must have $d' = d$ and $c' = c$, which yields the claimed value for m. □

By Lemma 23 and the congruence

$$m \equiv s_p(m) \mod p-1,$$

the characteristic class χ_{y^m} is defined if and only if $s_p(m)$ is a multiple of $p-1$. Similarly, the characteristic class χ_{xy^m} is defined precisely when

$$s_p(m) \equiv -1 \mod p-1.$$

The following proposition determines in terms of $s_p(m)$ which of the classes $\chi_{y^m}(\rho_{\mathbb{F}_p^n})$ and $\chi_{xy^m}(\rho_{\mathbb{F}_p^n})$ are nonzero.

Proposition 33. *Fix $n \geq 1$.*

(i) Let $p = 2$. For all $m \geq 1$, the class

$$\chi_{y^m}(\rho_{\mathbb{F}_2^n}) \in H^m(\mathbb{F}_2^n; \mathbb{F}_2)$$

is nonzero (and non-nilpotent) if and only if

$$s_2(m) \geq n.$$

The lowest-dimensional such class occurs in degree $2^n - 1$.

(ii) Let p be odd. The class

$$\chi_{y^m}(\rho_{\mathbb{F}_p^n}) \in H^{2m}(\mathbb{F}_p^n; \mathbb{F}_p)$$

is non-nilpotent if

$$s_p(m) = k(p-1) \quad \textit{with} \quad k \geq n.$$

The lowest-dimensional such class occurs in degree $2p^n - 2$. All other $\chi_{y^m}(\rho_{\mathbb{F}_p^n})$ are zero.

(iii) Let p be odd. The class

$$\chi_{xy^m}(\rho_{\mathbb{F}_p^n}) \in H^{2m+1}(\mathbb{F}_p^n; \mathbb{F}_p)$$

is nonzero if

$$s_p(m) = k(p-1) - 1 \quad \text{with} \quad k \geq n.$$

The lowest-dimensional such class occurs in degree $2p^n - 2p^{n-1} - 1$. All other $\chi_{xy^m}(\rho_{\mathbb{F}_p^n})$ are zero.

Proof. We will prove part (iii); the other parts are quite similar except that x does not appear.

By writing m in p-ary notation, we can form a multiset

$$\{x, y, \ldots, y, y^p, \ldots, y^p, \ldots, y^{p^k}, \ldots, y^{p^k}\},$$

where each power of p appears less than p times, and the product of the elements is our class xy^m. The number of elements is $1 + s_p(m)$.

Now, assuming $1 + s_p(m) = k(p-1)$ with $k \geq n$, the multiset can be partitioned into a disjoint union of n nonempty submultisets, each of which has a multiple of $p-1$ elements. Choosing such a partition yields a factorization $xy^m = (x^{a_1}y^{b_1}) \cdots (x^{a_n}y^{b_n})$ which corresponds to a nonzero summand in equation (8).

Conversely, suppose $\chi_{xy^m}(\rho_{\mathbb{F}_p^n}) \neq 0$. Let A_i, B_i, $i = 1, \ldots, n$, be the exponents appearing in some nonzero summand in equation (8) (with $A = 1$, $B = m$). The condition

$$\binom{m}{B_1, \ldots, B_n} \neq 0 \mod p$$

implies that

$$s_p(m) = s_p(B_1) + \cdots + s_p(B_n).$$

But each $A_i + B_i > 0$, and

$$A_i + s_p(B_i) \equiv A_i + B_i \equiv 0 \mod (p-1)$$

by the condition $P(A_i + B_i)$. Thus $A_i + s_p(B_i)$ is a positive multiple of $p-1$ for all i. Summing up,

$$1 + s_p(m) = k(p-1)$$

for some $k \geq n$, as desired.

The statement about degrees follows by applying Lemma 32. □

10 Representations of elementary abelian groups

Let G be a p-group, and let M be a representation of G. We define a filtration

$$0 = J_{-1}(M) \leq J_0(M) \leq J_1(M) \leq \cdots \leq M$$

of M by setting $J_{-1}(M) = 0$ and inductively defining $J_\ell(M)$ to be the preimage of the fixed-point subspace $(M/J_{\ell-1}(M))^G$ under the quotient map $M \to M/J_{\ell-1}(M)$. In particular, $J_0(M) = M^G$. More generally,

$$J_i(M) = \{ v \in M \mid I^{i+1} \cdot v = 0 \}$$

where $I \subset \mathbb{F}_q G$ is the augmentation ideal of the group ring. By Remark 15, every inclusion in the filtration is strict until $J_\ell(M)$ becomes equal to M, so $J_{\dim(M)-1}(M) = M$. As the filtration is preserved by maps between representations, it follows in particular that every d-dimensional submodule of M must be contained in $J_{d-1}(M)$.

Now we consider the case where G is elementary abelian.

Theorem 34. *Suppose ξ is a representation of an elementary abelian p-group. Then*

$$\chi_\alpha(\xi) = \chi_\alpha(J_1(\xi)).$$

That is, only the subrepresentation $J_1(\xi)$ matters when calculating $\chi_\alpha(\xi)$.

Proof. Definition 6 involves the transfer associated to the covering

$$\mathrm{Emb}(\mathbb{F}_q^2, \rho) /\!\!/ G \times GL_2\mathbb{F}_q \simeq \mathrm{Gr}_2(\rho) /\!\!/ G \to \mathrm{pt} /\!\!/ G.$$

This decomposes as a sum of the transfer maps associated to each of the components of the cover. Since $G = \pi_1(\mathrm{pt} /\!\!/ G)$ is elementary abelian, all such transfers vanish except for those associated to trivial (1-sheeted) components. These correspond to the 2-planes $V \in \mathrm{Gr}_2(\rho)$ such that $g \cdot V = V$ for all g, i.e., the 2-dimensional subrepresentations of ρ. Since all such subrepresentations are contained in $J_1(\rho)$, the result is unchanged by replacing the cover $\mathrm{Gr}_2(\rho)$ with its subspace $\mathrm{Gr}_2(J_1(\rho))$. □

The $J_\bullet$ filtration is compatible with tensor products, in the sense that

$$J_i(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{a_1 + \cdots + a_n = i} J_{a_1}(\xi_1) \otimes \cdots \otimes J_{a_n}(\xi_n),$$

where ξ_i is a representation of the p-group G_i, and the tensor product is external (i.e., considered as a representation of $\prod G_i$). In other words, the $J_\bullet$ filtration of a tensor product is just the tensor product of the filtrations of the factors. So:

Lemma 35. *Suppose ρ and η are representations of G, G' whose fixed-point subspaces are one-dimensional. Then*

$$J_1(\rho \otimes \eta) = J_1(\rho) \vee J_1(\eta)$$

as representations of $G \times G'$. □

Corollary 36. *If ρ and η are representations of elementary abelian groups G, G', then*

$$\chi_\alpha(\rho \otimes \eta) = -\sum \chi_{\alpha_{(1)}}(\rho) \times \chi_{\alpha_{(2)}}(\eta),$$

where

$$\Delta(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)}.$$

Proof. Both ρ and η must of course have at least a one-dimensional fixed-point space. And if either of them has more than one fixed line, then so does the tensor product. Consequently, by Lemma 14, both sides of the equation vanish. So we assume that $\dim J_0(\rho) = \dim J_0(\eta) = 1$. Now the result is immediate from Lemma 35, Theorem 34 and Theorem 27. □

Lastly, let us restrict our attention for a moment to the case $r = 1$. In that case, the calculation of $\chi_\alpha(\rho_A)$ in section 8 actually suffices to compute $\chi_\alpha(\bullet)$ for all representations of elementary abelian p-groups. Indeed:

Theorem 37. *Let $\xi : A \to GL_N\mathbb{F}_p$ be a representation of an elementary abelian p-group A. Then*

$$\chi_\alpha(\xi) = \pi^*\chi_\alpha(\rho_{A/B})$$

if $\dim J_0(\xi) = 1$, and zero otherwise. Here $B \leq A$ is the kernel of the action of A on $J_1(\xi)$ and $\pi : A \to A/B$ is the projection.

Proof. Assume $J_0(\xi)$ is one-dimensional; otherwise $\chi_\alpha(\xi) = 0$ by Theorem 14. By Theorem 34, the left hand side depends only on $J_1(\xi)$. Since the same is true for the right hand side, we may assume that $\xi = J_1(\xi)$. Then B is the kernel of ξ, so

$$\chi_\alpha(\xi) = \pi^*\chi_\alpha(\tilde{\xi}),$$

where $\tilde{\xi}$ is the corresponding representation of the quotient A/B, a faithful representation. Hence we have reduced to the case of a representation ξ of A which is faithful and satisfies $J_1(\xi) = \xi$ and $\dim J_0(\xi) = 1$. In this case, the claim reads

$$\chi_\alpha(\xi) = \chi_\alpha(\rho_A),$$

so the proof is completed by verifying that under these conditions

$$\xi \approx \rho_A.$$ □

11 Cohomology of $GL_N\mathbb{F}_q$

In this section, we aim to prove nontriviality of $\chi_\alpha^{(N)} \in H^*(GL_N\mathbb{F}_q)$ for N as large as possible. We do so by constructing a representation whose characteristic classes will coincide with those studied in section 9 but whose dimension is much larger:

Lemma 38. *For each $n \geq 1$, there is a representation ξ_A of the elementary abelian group $A = \mathbb{F}_q^n$ such that*

$$\dim(\xi_A) = p^n \quad \textit{and} \quad J_1(\xi_A) \approx \rho_A.$$

In the case $r = 1$, ξ_A may be taken to be the regular representation of A.

Proof. In light of Lemma 35 and the fact that $\rho_V \vee \rho_W = \rho_{V\oplus W}$, we may immediately reduce to the case $n = 1$ by defining

$$\xi_{\mathbb{F}_q^n} = \underbrace{\xi_{\mathbb{F}_q} \otimes \cdots \otimes \xi_{\mathbb{F}_q}}_{n}.$$

In the case $n = 1$, we may take

$$\xi_{\mathbb{F}_q} = \mathrm{Sym}^{p-1}(\mathbb{F}_q^2),$$

a p-dimensional representation of $\mathbb{F}_q \leq GL_2\mathbb{F}_q$. In terms of the basis $\mathbb{F}_q^2 = \langle e_1, e_2\rangle$,

$$\xi_{\mathbb{F}_q} = \langle e_1^{p-1}, e_1^{p-2}e_2, \ldots, e_2^{p-1}\rangle,$$

and we have

$$J_i(\xi_{\mathbb{F}_q}) = \langle e_1^{p-1-j}e_2^j \mid j \leq i\rangle.$$

In particular, the action on $J_1(\xi_{\mathbb{F}_q})$ is the standard one

$$\rho_{\mathbb{F}_q} : a \longmapsto \begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}.$$

Now assume $r = 1$. We will check that $\xi_{\mathbb{F}_p^n}$ as defined above is isomorphic to the regular representation of $\mathbb{F}_p^n$. Since both representations split up as an n-fold tensor product, it suffices to prove this in the case $n = 1$. In that case, we need to check that the two endomorphism of order p have the same Jordan structure. Now observe that they both have a single fixed line, hence a single block. □

Using the characteristic classes of the representation in Lemma 38, we can now produce many nonzero cohomology classes on the general linear groups:

Theorem 39. *Fix $n \geq 1$. For all*

$$2 \leq N \leq p^n,$$

(i) The class

$$\chi_\alpha^{(N)} \in H^*(GL_N\mathbb{F}_q; \mathbb{F}_q) \quad \text{with} \quad \alpha = (y_0 \cdots y_{r-1})^{p^n - 1}$$

is a non-nilpotent element of degree $2r(p^n - 1)$ for p odd, or degree $r(2^n - 1)$ for $p = 2$.

(ii) For p odd, the class

$$\chi_\alpha^{(N)} \in H^*(GL_N\mathbb{F}_q; \mathbb{F}_q) \quad \text{with} \quad \alpha = x_0 \cdots x_{r-1}(y_0 \cdots y_{r-1})^{p^n - p^{n-1} - 1}$$

is a nonzero element of degree $r(2p^n - 2p^{n-1} - 1)$.

Proof. In the special case $N = p^n$, we need only combine Lemma 38 with Theorem 34 and Proposition 30. The general case now follows from Corollary 13. □

Remark 40. The universal classes $\chi_\alpha^{(n)}$ are not the only nonzero cohomology classes on $GL_N\mathbb{F}_q$ which can be detected using our characteristic classes: there are also "dual classes"

$$\chi_\alpha((\mathbb{F}_q^n)^*) = \chi_\alpha(T : GL_n \to GL_n) = T^*(\chi_\alpha^{(n)}),$$

where

$$T : GL_n\mathbb{F}_q \to GL_n\mathbb{F}_q$$

is the inverse-transpose automorphism $A \mapsto (A^{-1})^t$. These dual classes are in general distinct from the $\chi_\alpha^{(n)}$ classes. Indeed, for $n > 2$, let $V = \mathbb{F}_q^{n-1}$ and consider the dual $\nu = \rho_V^*$ of the basic representation. Then

$$\nu^*\chi_\alpha^{(n)} = \chi_\alpha(\nu) = 0$$

for all α by Theorem 14, while

$$\nu^*T^*\chi_\alpha^{(n)} = \chi_\alpha(\nu^*) = \chi_\alpha(\rho_V)$$

which is often nonzero, as we saw in section 9.

In the case $r = 1$, we have from Proposition 33 (along with Theorem 34, Lemma 38, and Corollary 13):

Theorem 41. *Fix $n \geq 1$. For all*

$$2 \leq N \leq p^n,$$

(i) *If $p = 2$, the class*

$$\chi_{y^d}^{(N)} \in H^d(GL_N\mathbb{F}_2; \mathbb{F}_2)$$

is a non-nilpotent element whenever the sum of the binary digits of d is at least n.

(ii) *If p is odd, the class*

$$\chi_{y^d}^{(N)} \in H^{2d}(GL_N\mathbb{F}_p; \mathbb{F}_p)$$

is a non-nilpotent element whenever the sum of the p-ary digits of d is $k(p-1)$ for some $k \geq n$.

(iii) *If p is odd, the class*

$$\chi_{xy^d}^{(N)} \in H^{2d+1}(GL_N\mathbb{F}_p; \mathbb{F}_p)$$

is a nonzero element whenever the sum of the p-ary digits of d is $k(p-1) - 1$ for some $k \geq n$. □

12 Dickson invariants

For this section, we restrict to the case $r = 1$ and assume $\mathbb{F} = \mathbb{F}_p$. We give an alternative description of the classes

$$\chi_{y^k}(\rho_{\mathbb{F}_p^n}) \in H^*(\mathbb{F}_p^n).$$

By equation (8), these classes belong to the polynomial subalgebra

$$\mathbb{F}_p[\mathbb{F}_p^{n*}] \subset H^*(\mathbb{F}_p^n),$$

where we interpret $\mathbb{F}_p^{n*}$ as $H^1(\mathbb{F}_p)$ if $p = 2$ or as the image of the Bockstein map $\beta : H^1(\mathbb{F}_p^n) \to H^2(\mathbb{F}_p^n)$ if p is odd. Because the image of $\rho_{\mathbb{F}_p^n}$ in $GL_{n+1}\mathbb{F}_p$ is normalized by a copy of $GL_n\mathbb{F}_p$, these classes live in the invariant subring

$$\mathbb{F}_p[\mathbb{F}_p^{n*}]^{GL_n\mathbb{F}_p} = \mathbb{F}_p[D_{p^n-p^{n-1}}, \ldots, D_{p^n-1}]$$

where the elements $D_{p^n - p^i}$ are the Dickson invariants of $\mathbb{F}_p^{n*}$. Our aim is to express the classes $\chi_{y^k}(\rho_{\mathbb{F}_p^n})$ as polynomials in the Dickson invariants.

We begin with an alternative, coordinate-independent description of the classes $\chi_{y^k}(\rho_{\mathbb{F}_p^n})$. So far, we have defined these classes when k is a multiple of $p-1$. It is convenient to extend the definition by setting $\chi_{y^k} = 0$ when $p-1$ does not divide k. Then:

Lemma 42. *For every $k > 0$, we have*

$$\chi_{y^k}(\rho_{\mathbb{F}_p^n}) = -\sum_{z \in \mathbb{F}_p^{n*}} z^k.$$

Proof. By expanding $z = c_1 z_1 + \cdots + c_n z_n$ according to the dual $\{z_1, \ldots, z_n\}$ of the standard basis, we obtain

$$\begin{aligned}
\sum_{z \in \mathbb{F}_p^{n*}} z^k &= \sum_{c_1, \cdots, c_n \in \mathbb{F}_p} (c_1 z_1 + \cdots c_n z_n)^k \\
&= \sum_{i_1 + \cdots + i_n = k} \binom{k}{i_1, \ldots, i_n} \left(\sum_{c \in \mathbb{F}_p} c^{i_1} z^{i_1} \right) \times \cdots \times \left(\sum_{c \in \mathbb{F}_p} c^{i_n} z^{i_n} \right) \\
&= (-1)^n \sum_{\substack{i_1 + \cdots + i_n = k \\ (p-1) | i_j > 0 \ \forall j}} \binom{k}{i_1, \ldots, i_n} z^{i_1} \times \cdots \times z^{i_n},
\end{aligned}$$

which agrees with the right hand side of equation (8) (in the case $q = p, A = 0, B = k$) except for a factor of -1. The last equality holds because $\sum_{c \in \mathbb{F}_p} c^i$ is equal to -1 if $(p-1)$ divides i and $i \neq 0$, and to 0 otherwise. □

In other words, the classes $-\chi_{y^k}(\rho_{\mathbb{F}_p^n})$ can be viewed as the power sum symmetric polynomials evaluated on the elements of $\mathbb{F}_p^{n*}$. Since the Dickson polynomials are the elementary symmetric polynomials evaluated on the same elements, Newton's identity gives a relationship between the two. To express it conveniently, write total classes (as elements of the degree-wise completed cohomology)

$$D = \sum_{i=0}^{n} D_{p^n - p^i} = \prod_{z \in \mathbb{F}_p^{n*}} (1 + z) = \sum_{k=0}^{p^n} \sigma_k(\mathbb{F}_p^{n*}),$$

$$A = \sum_{k>0} (-1)^k \chi_{y^k}(\rho_{\mathbb{F}_p^n}) = \sum_{k>0} (-1)^{k-1} p_k(\mathbb{F}_p^{n*}).$$

Here σ_k is the k-th elementary symmetric polynomial, and $p_k(x_1, \dots, x_\ell) = \sum_i x_i^k$ is the k-th power sum polynomial.

From Newton's identity

$$\left(\sum_{i=0}^{\infty} \sigma_i\right)\left(\sum_{i=1}^{\infty} (-1)^{i-1} p_i\right) = \sum_{k=1}^{\infty} k\sigma_k,$$

we get

$$DA = \sum_{k>0} kD_k = -D_{p^n-1}.$$

The last equality holds because, for each k, either $p|k$ or $D_k = 0$ or $k = p^n - 1$. Consequently,

Theorem 43. *We have*

$$A = -D_{p^n-1} \cdot D^{-1}. \qquad \square$$

We can write the lowest terms more explicitly:

$$A = -D_{p^n-1}(1 - D_{p^n-p^{n-1}} - \cdots - D_{p^n-1} + \text{higher terms})$$

using the calculation

$$D^{-1} = (1 + \tilde{D})^{-1} = 1 - \tilde{D} + \tilde{D}^2 - \cdots = 1 - \tilde{D} + \text{higher terms}$$

where $\tilde{D} = D_{p^n-p^{n-1}} - \cdots - D_{p^n-1}$. Consequently, we have

$$\chi_{y^{2p^n-p^i-1}}(\rho_{\mathbb{F}_p^n}) = \pm D_{p^n-1} D_{p^n-p^i}, \quad 0 \le i \le n,$$

and these are the only $\chi_{y^k}(\rho_{\mathbb{F}_p^n})$'s which are nonzero for $k \le 2(p^n - 1)$. Because the Dickson polynomials are algebraically independent, it follows that the n classes

$$D_{p^n-1} \quad \text{and} \quad D_{p^n-1} D_{p^n-p^i}, \quad 1 \le i \le n-1,$$

are algebraically independent. We can deduce:

Theorem 44. *Suppose $n + 1 \le N \le p^n$. Then the universal classes*

$$\chi^{(N)}_{y^{2p^n-p^i-1}} \in H^*(GL_N\mathbb{F}_p; \mathbb{F}_p), \quad 1 \le i \le n$$

are algebraically independent. In particular, our characteristic classes $\chi^{(N)}_\bullet$ generate a subring of Krull dimension at least n.

Proof. In light of the above discussion and Theorem 34, it suffices to produce a representation ξ of $\mathbb{F}_p^n$ with $\dim(\xi) = N$ and $J_1(\xi) = \rho_{\mathbb{F}_p^n}$. In the case $N = p^n$, we have the representation $\xi = \xi_{\mathbb{F}_p^n}$ of Lemma 38. In the general case, any N-dimensional subrepresentation ξ with

$$\rho_{\mathbb{F}_p^n} = J_1(\xi_{\mathbb{F}_p^n}) \leq \xi \leq \xi_{\mathbb{F}_p^n}$$

will suffice. Such subrepresentations exist with every dimension between $n+1 = \dim \rho_{\mathbb{F}_p^n}$ and $p^n = \dim \xi_{\mathbb{F}_p^n}$; they can be constructed by upper-triangularizing the quotient $\xi_{\mathbb{F}_p^n}/\rho_{\mathbb{F}_p^n}$. □

13 Applications to $\mathrm{Aut}(F_n)$ and $GL_n\mathbb{Z}$

We assume $r = 1$ and $\mathbb{F} = \mathbb{F}_p$ throughout the section. In this section, our aim is to demonstrate the usefulness of our characteristic classes for studying groups other than general linear groups over finite fields by presenting applications of our computations to the homology and cohomology of automorphism groups of free groups and general linear groups over the integers. The applications are of a broadly similar type to those we have presented for the cohomology of general linear groups over finite fields: we construct large families of explicit nontrivial elements in the homology and cohomology of $\mathrm{Aut}(F_{p^n})$ and $GL_{p^n}\mathbb{Z}$ living in the unstable range where the homology and cohomology groups remain poorly understood. In the case of homology, the classes are not only nontrivial, but in fact indecomposable in the rings $H_*(\coprod_{n\geq 0} B\mathrm{Aut}(F_n))$ and $H_*(\coprod_{n\geq 0} BGL_n\mathbb{Z})$. (We will indicate the ring structures below.)

Our starting point is the observation that the regular representation ρ_{reg} of $\mathbb{F}_p^n$ factors as the composite

$$\mathbb{F}_p^n \xrightarrow{c} \Sigma_{p^n} \xrightarrow{i} \mathrm{Aut}(F_{p^n}) \xrightarrow{\pi_{\mathrm{ab}}} GL_{p^n}\mathbb{Z} \xrightarrow{\rho_{\mathrm{can}}} GL_{p^n}\mathbb{F}_p \tag{9}$$

where c is the Cayley embedding, i is the embedding sending a permutation to the corresponding automorphism of the free group F_{p^n} given by permuting generators, π_{ab} is given by abelianization, and ρ_{can} is the canonical representation of $GL_{p^n}\mathbb{Z}$ on $\mathbb{F}_p^{p^n}$ given by reduction mod p. By Lemma 38 and Theorem 34, in the case $r = 1$, the characteristic classes of the regular representation ρ_{reg} of $\mathbb{F}_p^n$ agree with those of the basic representation $\rho_{\mathbb{F}_p^n}$:

$$\chi_\alpha(\rho_{\mathrm{reg}}) = \chi_\alpha(\rho_{\mathbb{F}_p^n}) \tag{10}$$

for all α. Thus we obtain

Theorem 45. *Suppose $\alpha \in H^*(GL_2\mathbb{F}_p; \mathbb{F}_p)$ is any class for which $\chi_\alpha(\rho_{\mathbb{F}_p^n}) \neq 0$. Then*

$$\chi_\alpha(\rho_{\text{can}}) \neq 0 \in H^*(GL_{p^n}\mathbb{Z}; \mathbb{F}_p)$$

and

$$\chi_\alpha(\pi_{\text{ab}}^*(\rho_{\text{can}})) \neq 0 \in H^*(\text{Aut}(F_{p^n}); \mathbb{F}_p).$$

Moreover, if $\chi_\alpha(\rho_{\mathbb{F}_p^n})$ is non-nilpotent, so are $\chi_\alpha(\rho_{\text{can}})$ and $\chi_\alpha(\pi_{\text{ab}}^*(\rho_{\text{can}}))$. □

We remind the reader that the question of when $\chi_\alpha(\rho_{\mathbb{F}_p^n})$ is nonzero was given a complete answer in Proposition 33.

We now turn to the application to homology groups. The homology groups of each of the spaces

$$\coprod_{n\geq 0} B\Sigma_n, \quad \coprod_{n\geq 0} B\text{Aut}(F_n), \quad \coprod_{n\geq 0} BGL_n\mathbb{Z} \quad \text{and} \quad \coprod_{n\geq 0} BGL_n\mathbb{F}_p$$

are highly structured. First, in each case the homology carries a ring structure. These structures are induced by disjoint unions of sets (yielding homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$); by free products of free groups (yielding homomorphisms $\text{Aut}(F_n) \times \text{Aut}(F_m) \to \text{Aut}(F_{n+m})$); and by direct sums of free modules (yielding the block-sum homomorphisms on general linear groups). Second, with the apparent exception of automorphism groups of free groups, in each case the homology groups carry an additional product, which we denote by $\circ$. These products are induced by direct products of sets (yielding homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{nm}$) and tensor products of free modules (yielding homomorphisms $GL_nR \times GL_mR \to GL_{nm}R$). The maps induced by the analogues

$$\Sigma_n \xrightarrow{i} \text{Aut}(F_n) \xrightarrow{\pi_{\text{ab}}} GL_n\mathbb{Z} \xrightarrow{\rho_{\text{can}}} GL_n\mathbb{F}_p, \quad n \geq 0$$

of the maps appearing in (9) are compatible with all this structure in the sense that all give ring homomorphisms, and both $(\pi_{\text{ab}}i)_*$ and $(\rho_{\text{can}})_*$ preserve the $\circ$-product. For each $k \geq 0$, let $z_k \in H_*(\mathbb{F}_p)$ be the dual of $y^k \in H^*(\mathbb{F}_p)$, and write

$$\begin{aligned} E_k &= c_*(z_k) \in H_*(\Sigma_p) \\ E_k^{\mathbb{Z}} &= (\pi_{\text{ab}}i)_*(E_k) \in H_*(GL_p\mathbb{Z}) \\ E_k^{\mathbb{F}_p} &= (\rho_{\text{can}})_*(E_k^{\mathbb{Z}}) \in H_*(GL_p\mathbb{F}_p), \end{aligned}$$

where $c\colon \mathbb{F}_p \to \Sigma_p$ is the Cayley embedding. With this notation, we have:

Theorem 46. *Suppose $B_1, \ldots, B_n \in \mathbb{Z}$ are positive multiples of $p-1$ such that there is no carry when $B_1, \ldots, B_n$ are added together in base p. Let $B = B_1 + \cdots + B_n$. Then the following elements are indecomposable in their respective rings:*

(i) $i_*(E_{B_1} \circ \cdots \circ E_{B_n}) \in H_{2B}(\mathrm{Aut}(F_{p^n}); \mathbb{F}_p)$ *in* $H_*(\bigsqcup_{k \geq 0} B\mathrm{Aut}(F_k); \mathbb{F}_p)$

(ii) $E^{\mathbb{Z}}_{B_1} \circ \cdots \circ E^{\mathbb{Z}}_{B_n} \in H_{2B}(GL_{p^n}\mathbb{Z}; \mathbb{F}_p)$ *in* $H_*(\bigsqcup_{k \geq 0} BGL_k\mathbb{Z}; \mathbb{F}_p)$

(iii) $E^{\mathbb{F}_p}_{B_1} \circ \cdots \circ E^{\mathbb{F}_p}_{B_n} \in H_{2B}(GL_{p^n}\mathbb{F}_p; \mathbb{F}_p)$ *in* $H_*(\bigsqcup_{k \geq 0} BGL_k\mathbb{F}_p; \mathbb{F}_p)$.

(For $p = 2$, replace $2B$ by B.)

Proof. The first two elements map to the third under the ring homomorphisms $(\pi_{\mathrm{ab}})_*$ and $(\rho_{\mathrm{can}})_*$, so it suffices to prove the third claim. Observe that the Cayley embedding $\mathbb{F}_p^n \to \Sigma_{p^n}$ factors up to conjugacy as

$$\mathbb{F}_p^n \xrightarrow{c^{\times n}} \Sigma_p^n \longrightarrow \Sigma_{p^n}$$

where the latter map induces the iterated $\circ$-product. Thus, by the factorization (9) of the regular representation, we have

$$E^{\mathbb{F}_p}_{B_1} \circ \cdots \circ E^{\mathbb{F}_p}_{B_n} = (\rho_{\mathrm{reg}})_*(z_{B_1} \times \cdots \times z_{B_n}).$$

We obtain

$$\begin{aligned} \langle \chi^{(p^B)}_{y^B}, E^{\mathbb{F}_p}_{B_1} \circ \cdots \circ E^{\mathbb{F}_p}_{B_n} \rangle &= \langle \chi_{y^B}(\rho_{\mathrm{reg}}), z_{B_1} \times \cdots \times z_{B_n} \rangle \\ &= \langle \chi_{y^B}(\rho_{\mathbb{F}_p^n}), z_{B_1} \times \cdots \times z_{B_n} \rangle \\ &= (-1)^{n-1} \binom{B}{B_1, \ldots, B_n} \\ &\neq 0 \in \mathbb{F}_p. \end{aligned}$$

Here the second equality follows from equation (10), the third from equation (8), and the final inequality from the choice of $B_1, \ldots, B_n$. The claim now follows from Remark 19. ☐

Remark 47. In the case $p \neq 2$, we can obtain further indecomposables in the aforementioned rings by a similar argument by taking into account the exterior class in $H^*(\mathbb{F}_p)$. We leave the formulation of the result to the reader.

Remark 48. Analogues of Theorems 45 and 46 hold, by the same arguments, for many other interesting groups through which the regular representation factors. An example is $GL_{p^n}R$ for R any ring surjecting onto $\mathbb{F}_p$.

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