Edge-disjoint double rays in infinite graphs: a Halin type result

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Abstract

We show that any graph that contains k edge-disjoint double rays for any $k \in \mathbb{N}$ contains also infinitely many edge-disjoint double rays. This was conjectured by Andreae in 1981.

1 Introduction

We say a graph G has *arbitrarily many vertex-disjoint* H if for every $k \in \mathbb{N}$ there is a family of k vertex-disjoint subgraphs of G each of which is isomorphic to H. Halin's Theorem says that every graph that has arbitrarily many vertex-disjoint rays, also has infinitely many vertex-disjoint rays [5]. In 1970 he extended this result to vertex-disjoint double rays [6]. Jung proved a strengthening of Halin's Theorem where the initial vertices of the rays are constrained to a certain vertex set [7].

We look at the same questions with 'edge-disjoint' replacing 'vertex-disjoint'. Consider first the statement corresponding to Halin's Theorem. It suffices to prove this statement in locally finite graphs, as each graph with arbitrarily many edge-disjoint rays contains a locally finite union of tails of these rays. But the statement for locally finite graphs follows from Halin's original Theorem applied to the line-graph.

This reduction to locally finite graphs does not work for Jung's Theorem or for Halin's statement about double rays. Andreae proved an analog of Jung's Theorem for edge-disjoint rays in 1981, and conjectured that a Halin-type Theorem would be true for edge-disjoint double rays [1]. Our aim in the current paper is to prove this conjecture.

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More precisely, we say a graph G has *arbitrarily many edge-disjoint H* if for every $k \in \mathbb{N}$ there is a family of k edge-disjoint subgraphs of G each of which is isomorphic to H, and our main result is the following.

Theorem 1. *Any graph that has arbitrarily many edge-disjoint double rays has infinitely many edge-disjoint double rays.*

Even for locally finite graphs this theorem does not follow from Halin's analogous result for vertex-disjoint double rays applied to the line graph. For example a double ray in the line graph may correspond, in the original graph, to a configuration as in Figure 1.

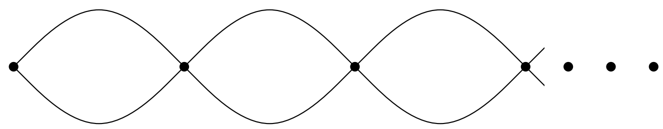

Figure 1: A graph that does not include a double ray but whose line graph does.

A related notion is that of ubiquity. A graph H is *ubiquitous* with respect to a graph relation $\leq$ if $nH \leq G$ for all $n \in \mathbb{N}$ implies $\aleph_0 H \leq G$, where nH denotes the disjoint union of n copies of H. For example, Halin's Theorem says that rays are ubiquitous with respect to the subgraph relation. It is known that not every graph is ubiquitous with respect to the minor relation [2], nor is every locally finite graph ubiquitous with respect to the subgraph relation [8, 9], or even the topological minor relation [2, 3]. However, Andreae has conjectured that every locally finite graph is ubiquitous with respect to the minor relation [2]. For more details see [3]. In Section 6 (the outlook) we introduce a notion closely related to ubiquity.

The proof is organised as follows. In Section 3 we explain how to deal with the cases that the graph has infinitely many ends, or an end with infinite vertex-degree. In Section 4 we consider the 'two ended' case: That in which there are two ends ω and ω' both of finite vertex-degree, and arbitrarily many edge-disjoint double rays from ω to ω'.

The only remaining case is the 'one ended' case: That in which there is a single end ω of finite vertex-degree and arbitrarily many edge-disjoint double rays from ω to ω. One central idea in the proof of this case is to consider 2-rays instead of double rays. Here a 2-ray is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path. The remainder of the proof is subdivided into two parts: In Subsection 5.3 we show that if there are arbitrarily many edge-disjoint 2-rays into ω, then there are infinitely many such 2-rays. In Subsection 5.2 we show that if there are infinitely many edge-disjoint 2-rays into ω, then there are infinitely many edge-disjoint double rays from ω to ω.

We finish by discussing the outlook and mentioning some open problems.

2 Preliminaries

All our basic notation for graphs is taken from [4]. In particular, two rays in a graph are equivalent if no finite set separates them. The equivalence classes of this relation are called the *ends* of G. We say that a ray in an end ω *converges* to ω. A double ray *converges* to all the ends of which it includes a ray.

2.1 The structure of a thin end

It follows from Halin's Theorem that if there are arbitrarily many vertex-disjoint rays in an end of G, then there are infinitely many such rays. This fact motivated the central definition of the *vertex-degree* of an end ω: the maximal cardinality of a set of vertex-disjoint rays in ω.

An end is *thin* if its vertex-degree is finite, and otherwise it is *thick*. A pair (A, B) of edge-disjoint subgraphs of G is a *separation* of G if $A \cup B = G$. The number of vertices of $A \cap B$ is called the *order* of the separation.

Definition 2. Let G be a locally finite graph and ω a thin end of G. A countable infinite sequence $((A_i, B_i))_{i\in\mathbb{N}}$ of separations of G *captures* ω if for all $i \in \mathbb{N}$

- $A_i \cap B_{i+1} = \emptyset$,
- $A_{i+1} \cap B_i$ is connected,
- $\bigcup_{i\in\mathbb{N}} A_i = G$,
- the order of (A_i, B_i) is the vertex-degree of ω, and
- each B_i contains a ray from ω.

Lemma 3. *Let G be a locally finite graph with a thin end ω. Then there is a sequence that captures ω.*

Proof. Without loss of generality G is connected, and so is countable. Let $v_1, v_2, \ldots$ be an enumeration of the vertices of G. Let k be the vertex-degree of ω. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a set of vertex-disjoint rays in ω and let S be the set of their start vertices. We pick a sequence $((A_i, B_i))_{i\in\mathbb{N}}$ of separations and a sequence (T_i) of connected subgraphs recursively as follows. We pick (A_i, B_i) such that S is included in A_i, such that there is a ray from ω included in B_i, and such that B_i does not meet $\bigcup_{j<i} T_j$ or $\{v_j \mid j \leq i\}$: subject to this we minimise the size of the set X_i of vertices in $A_i \cap B_i$. Because of this minimization B_i is connected and X_i is finite. We take T_i to be a finite connected subgraph of B_i including X_i. Note that any ray that meets all of the B_i must be in ω.

By Menger's Theorem [4] we get for each $i \in \mathbb{N}$ a set $\mathcal{P}_i$ of vertex-disjoint paths from X_i to X_{i+1} of size $|X_i|$. From these, for each i we get a set of $|X_i|$ vertex-disjoint rays in ω. Thus the size of X_i is at most k. On the other hand it is at least k as each ray R_j meets each set X_i.

Assume for contradiction that there is a vertex $v \in A_i \cap B_{i+1}$. Let R be a ray from v to ω inside B_{i+1}. Then R must meet X_i, contradicting the definition of B_{i+1}. Thus $A_i \cap B_{i+1}$ is empty.

Observe that $\bigcup \mathcal{P}_i \cup T_i$ is a connected subgraph of $A_{i+1} \cap B_i$ containing all vertices of X_i and X_{i+1}. For any vertex $v \in A_{i+1} \cap B_i$ there is a v–X_{i+1} path P in B_i. P meets B_{i+1} only in X_{i+1}. So P is included in $A_{i+1} \cap B_i$. Thus $A_{i+1} \cap B_i$ is connected. The remaining conditions are clear. □

Remark 4. *Every infinite subsequence of a sequence capturing ω also captures ω.* □

The following is obvious:

Remark 5. *Let G be a graph and $v, w \in V(G)$ If G contains arbitrarily many edge-disjoint v–w paths, then it contains infinitely many edge-disjoint v–w paths.* □

We will need the following special case of the theorem of Andreae mentioned in the Introduction.

Theorem 6 (Andreae [1])**.** *Let G be a graph and $v \in V(G)$. If there are arbitrarily many edge-disjoint rays all starting at v, then there are infinitely many edge-disjoint rays all starting at v.*

3 Known cases

Many special cases of Theorem 1 are already known or easy to prove. For example Halin showed the following.

Theorem 7 (Halin)**.** *Let G be a graph and ω an end of G. If ω contains arbitrarily many vertex-disjoint rays, then G has a half-grid as a minor.*

Corollary 8. *Any graph with an end of infinite vertex-degree has infinitely many edge-disjoint double rays.* □

Another simple case is the case where the graph has infinitely many ends.

Lemma 9. *A tree with infinitely many ends contains infinitely many edge-disjoint double rays.*

Proof. It suffices to show that every tree T with infinitely many ends contains a double ray such that removing its edges leaves a component containing infinitely many ends, since then one can pick those double rays recursively.

There is a vertex $v \in V(T)$ such that $T - v$ has at least 3 components C_1, C_2, C_3 that each have at least one end, as T contains more than 2 ends. Let

e_i be the edge vw_i with $w_i \in C_i$ for $i \in \{1, 2, 3\}$. The graph $T \setminus \{e_1, e_2, e_3\}$ has precisely 4 components (C_1, C_2, C_3 and the one containing v), one of which, D say, has infinitely many ends. By symmetry we may assume that D is neither C_1 nor C_2. There is a double ray R all whose edges are contained in $C_1 \cup C_2 \cup \{e_1, e_2\}$. Removing the edges of R leaves the component D, which has infinitely many ends. □

Corollary 10. *Any connected graph with infinitely many ends has infinitely many edge-disjoint double rays.* □

4 The 'two ended' case

Using the results of Section 3 it is enough to show that any graph with only finitely many ends, each of which is thin, has infinitely many edge-disjoint double rays as soon as it has arbitrarily many edge-disjoint double rays. Any double ray in such a graph has to join a pair of ends (not necessarily distinct), and there are only finitely many such pairs. So if there are arbitrarily many edge-disjoint double rays, then there is a pair of ends such that there are arbitrarily many edge-disjoint double rays joining those two ends. In this section we deal with the case where these two ends are different, and in Section 5 we deal with the case that they are the same. We start with two preparatory lemmas.

Lemma 11. *Let G be a graph with a thin end ω, and let $\mathcal{R} \subseteq \omega$ be an infinite set. Then there is an infinite subset of $\mathcal{R}$ such that any two of its members intersect in infinitely many vertices.*

Proof. We define an auxilliary graph H with $V(H) = \mathcal{R}$ and an edge between two rays if and only if they intersect in infinitely many vertices. By Ramsey's Theorem either H contains an infinite clique or an infinite independent set of vertices. Let us show that there cannot be an infinite independent set in H. Let k be the vertex-degree of ω: we shall show that H does not have an independent set of size $k+1$. Suppose for a contradiction that $X \subseteq \mathcal{R}$ is a set of $k+1$ rays that is independent in H. Since any two rays in X meet in only finitely many vertices, each ray in X contains a tail that is disjoint to all the other rays in X. The set of these $k+1$ vertex-disjoint tails witnesses that ω has vertex-degree at least $k+1$, a contradiction. Thus there is an infinite clique $K \subseteq H$, which is the desired infinite subset. □

Lemma 12. *Let G be a graph consisting of the union of a set $\mathcal{R}$ of infinitely many edge-disjoint rays of which any pair intersect in infinitely many vertices. Let $X \subseteq V(G)$ be an infinite set of vertices, then there are infinitely many edge-disjoint rays in G all starting in different vertices of X.*

Proof. If there are infinitely many rays in $\mathcal{R}$ each of which contains a different vertex from X, then suitable tails of these rays give the desired rays. Otherwise there is a ray $R \in \mathcal{R}$ meeting X infinitely often. In this case, we choose the desired rays recursively such that each contains a tail from some ray in $\mathcal{R} - R$.

Having chosen finitely many such rays, we can always pick another: we start at some point in X on R which is beyond all the (finitely many) edges on R used so far. We follow R until we reach a vertex of some ray R' in $\mathcal{R} - R$ whose tail has not been used yet, then we follow R'. □

Lemma 13. *Let G be a graph with only finitely many ends, all of which are thin. Let ω_1, ω_2 be distinct ends of G. If G contains arbitrarily many edge-disjoint double rays each of which converges to both ω_1 and ω_2, then G contains infinitely many edge-disjoint double rays each of which converges to both ω_1 and ω_2.*

Proof. For each pair of ends, there is a finite set separating them. The finite union of these finite sets is a finite set $S \subseteq V(G)$ separating any two ends of G. For $i = 1, 2$ let C_i be the component of $G - S$ containing ω_i.

There are arbitrarily many edge-disjoint double rays from ω_1 to ω_2 that have a common last vertex v_1 in S before staying in C_1 and also a common last vertex v_2 in S before staying in C_2. Note that v_1 may be equal to v_2. There are arbitrarily many edge-disjoint rays in $C_1 + v_1$ all starting in v_1. By Theorem 6 there is a countable infinite set $\mathcal{R}_1 = \{R_1^i \mid i \in \mathbb{N}\}$ of edge-disjoint rays each included in $C_1 + v_1$ and starting in v_1. By replacing $\mathcal{R}_1$ with an infinite subset of itself, if necessary, we may assume by Lemma 11 that any two members of $\mathcal{R}_1$ intersect in infinitely many vertices. Similarly, there is a countable infinite set $\mathcal{R}_2 = \{R_2^i \mid i \in \mathbb{N}\}$ of edge-disjoint rays each included in $C_2 + v_2$ and starting in v_2 such that any two members of $\mathcal{R}_2$ intersect in infinitely many vertices.

Let us subdivide all edges in $\bigcup \mathcal{R}_1$ and call the set of subdivision vertices X_1. Similarly, we subdivide all edges in $\bigcup \mathcal{R}_2$ and call the set of subdivision vertices X_2. Below we shall find double rays in the subdivided graph, which immediately give rise to the desired double rays in G.

Suppose for a contradiction that there is a finite set F of edges separating X_1 from X_2. Then v_i has to be on the same side of that separation as X_i as there are infinitely many v_i–X_i edges. So F separates v_1 from v_2, which contradicts the fact that there are arbitrarily many edge-disjoint double rays containing both v_1 and v_2. By Remark 5 there is a set $\mathcal{P}$ of infinitely many edge-disjoint X_1–X_2 paths. As all vertices in X_1 and X_2 have degree 2, and by taking an infinite subset if necessary, we may assume that each end-vertex of a path in $\mathcal{P}$ lies on no other path in $\mathcal{P}$.

By Lemma 12 there is an infinite set Y_1 of start-vertices of paths in $\mathcal{P}$ together with an infinite set $\mathcal{R}_1'$ of edge-disjoint rays with distinct start-vertices whose set of start-vertices is precisely Y_1. Moreover, we can ensure that each ray in $\mathcal{R}_1'$ is included in $\bigcup \mathcal{R}_1$. Let Y_2 be the set of end-vertices in X_2 of those paths in $\mathcal{P}$ that start in Y_1. Applying Lemma 12 again, we obtain an infinite set $Z_2 \subseteq Y_2$ together with an infinite set $\mathcal{R}_2'$ of edge-disjoint rays included in $\bigcup \mathcal{R}_2$ with distinct start-vertices whose set of start-vertices is precisely Z_2.

For each path P in $\mathcal{P}$ ending in Z_2, there is a double ray in the union of P and the two rays from $\mathcal{R}_1'$ and $\mathcal{R}_2'$ that P meets in its end-vertices. By construction, all these infinitely many double rays are edge-disjoint. Each of those double rays converges to both ω_1 and ω_2, since each ω_i is the only end in C_i. □

Remark 14. *Instead of subdividing edges we also could have worked in the line graph of G. Indeed, there are infinitely many vertex-disjoint paths in the line graph from $\bigcup \mathcal{R}_1$ to $\bigcup \mathcal{R}_2$.*

5 The 'one ended' case

We are now going to look at graphs G that contain a thin end ω such that there are arbitrarily many edge-disjoint double rays converging only to the end ω. The aim of this section is to prove the following lemma, and to deduce Theorem 1.

Lemma 15. *Let G be a countable graph and let ω be a thin end of G. Assume there are arbitrarily many edge-disjoint double rays all of whose rays converge to ω. Then G has infinitely many edge-disjoint double rays.*

We promise that the assumption of countability will not cause problems later.

5.1 Reduction to the locally finite case

A key notion for this section is that of a 2-ray. A *2-ray* is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path.

In order to deduce that G has infinitely many edge-disjoint double rays, we will only need that G has arbitrarily many edge-disjoint 2-rays. In this subsection, we illustrate one advantage of 2-rays, namely that we may reduce to the case where G is locally finite.

Lemma 16. *Let G be a countable graph with a thin end ω. Assume there is a countable infinite set $\mathcal{R}$ of rays all of which converge to ω.*

Then there is a locally finite subgraph H of G with a single end which is thin such that the graph H includes a tail of any $R \in \mathcal{R}$.

Proof. Let $(R_i \mid i \in \mathbb{N})$ be an enumeration of $\mathcal{R}$. Let $(v_i \mid i \in \mathbb{N})$ be an enumeration of the vertices of G. Let U_i be the unique component of $G \setminus \{v_1, \ldots, v_i\}$ including a tail of each ray in ω.

For $i \in \mathbb{N}$, we pick a tail R'_i of R_i in U_i. Let $H_1 = \bigcup_{i \in \mathbb{N}} R'_i$. Making use of H_1, we shall construct the desired subgraph H. Before that, we shall collect some properties of H_1.

As every vertex of G lies in only finitely many of the U_i, the graph H_1 is locally finite. Each ray in H_1 converges to ω in G since $H_1 \setminus U_i$ is finite for every $i \in \mathbb{N}$. Let Ψ be the set of ends of H_1. Since ω is thin, Ψ has to be finite: $\Psi = \{\omega_1, \ldots, \omega_n\}$. For each $i \leq n$, we pick a ray $S_i \subseteq H_1$ converging to ω_i.

Now we are in a position to construct H. For any $i > 1$, the rays S_1 and S_i are joined by an infinite set $\mathcal{P}_i$ of vertex-disjoint paths in G. We obtain H from

H_1 by adding all paths in the sets $\mathcal{P}_i$. Since H_1 is locally finite, H is locally finite.

It remains to show that every ray R in H is equivalent to S_1. If R contains infinitely many edges from the $\mathcal{P}_i$, then there is a single $\mathcal{P}_i$ which R meets infinitely, and thus R is equivalent to S_1. Thus we may assume that a tail of R is a ray in H_1. So it converges to some $\omega_i \in \Psi$. Since S_i and S_1 are equivalent, R and S_1 are equivalent, which completes the proof. □

Corollary 17. *Let G be a countable graph with a thin end ω and arbitrarily many edge-disjoint 2-rays of which all the constituent rays converge to ω. Then there is a locally finite subgraph H of G with a single end, which is thin, such that H has arbitrarily many edge-disjoint 2-rays.*

Proof. By Lemma 16 there is a locally finite graph $H \subseteq G$ with a single end such that a tail of each of the constituent rays of the arbitrarily many 2-rays is included in H. □

5.2 Double rays versus 2-rays

A connected subgraph of a graph G including a vertex set $S \subseteq V(G)$ is a *connector* of S in G.

Lemma 18. *Let G be a connected graph and S a finite set of vertices of G. Let $\mathcal{H}$ be a set of edge-disjoint subgraphs H of G such that each connected component of H meets S. Then there is a finite connector T of S, such that at most $2|S|-2$ graphs from $\mathcal{H}$ contain edges of T.*

Proof. By replacing $\mathcal{H}$ with the set of connected components of graphs in $\mathcal{H}$, if necessary, we may assume that each member of $\mathcal{H}$ is connected. We construct graphs T_i recursively for $0 \leq i < |S|$ such that each T_i is finite and has at most $|S| - i$ components, at most $2i$ graphs from $\mathcal{H}$ contain edges of T_i, and each component of T_i meets S. Let $T_0 = (S, \emptyset)$ be the graph with vertex set S and no edges. Assume that T_i has been defined.

If T_i is connected let $T_{i+1} = T_i$. For a component C of T_i, let C' be the graph obtained from C by adding all graphs from $\mathcal{H}$ that meet C.

As G is connected, there is a path P (possibly trivial) in G joining two of these subgraphs C_1' and C_2' say. And by taking the length of P minimal, we may assume that P does not contain any edge from any $H \in \mathcal{H}$. Then we can extend P to a C_1–C_2 path Q by adding edges from at most two subgraphs from $\mathcal{H}$ — one included in C_1' and the other in C_2'. We obtain T_{i+1} from T_i by adding Q.

$T = T_{|S|-1}$ has at most one component and thus is connected. And at most $2|S| - 2$ many graphs from $\mathcal{H}$ contain edges of T. Thus T is as desired. □

Let d, d' be 2-rays. d is a *tail* of d' if each ray of d is a tail of a ray of d'. A set D' is a *tailor* of a set D of 2-rays if each element of D' is a tail of some element of D but no 2-ray in D includes more than one 2-ray in D'.

Lemma 19. *Let G be a locally finite graph with a single end ω, which is thin. Assume that G contains an infinite set $D = \{d_1, d_2, \dots\}$ of edge-disjoint 2-rays.*

Then G contains an infinite tailor D' of D and a sequence $((A_i, B_i))_{i\in\mathbb{N}}$ capturing ω (see Definition 2) such that there is a family of vertex-disjoint connectors T_i of $A_i \cap B_i$ contained in $A_{i+1} \cap B_i$, each of which is edge-disjoint from each member of D'.

Proof. Let k be the vertex-degree of ω. By Lemma 3 there is a sequence $((A'_i, B'_i))_{i\in\mathbb{N}}$ capturing ω. By replacing each 2-ray in D with a tail of itself if necessary, we may assume that for all $(r, s) \in D$ and $i \in \mathbb{N}$ either both r and s meet A'_i or none meets A'_i. By Lemma 18 there is a finite connector T'_i of $A'_i \cap B'_i$ in the connected graph B'_i which meets in an edge at most $2k-2$ of the 2-rays of D that have a vertex in A'_i.

Thus, there are at most $2k-2$ 2-rays in D that meet all but finitely many of the T'_i in an edge. By throwing away these finitely many 2-rays in D we may assume that each 2-ray in D is edge-disjoint from infinitely many of the T'_i. So we can recursively build a sequence $N_1, N_2, \dots$ of infinite sets of natural numbers such that $N_i \supseteq N_{i+1}$, the first i elements of N_i are all contained in N_{i+1}, and d_i only meets finitely many of the T'_j with $j \in N_i$ in an edge. Then $N = \bigcap_{i\in\mathbb{N}} N_i$ is infinite and has the property that each d_i only meets finitely many of the T'_j with $j \in N$ in an edge. Thus there is an infinite tailor D' of D such that no 2-ray from D' meets any T'_j for $j \in N$ in an edge.

We recursively define a sequence $n_1, n_2, \dots$ of natural numbers by taking $n_i \in N$ sufficiently large that B'_{n_i} does not meet T'_{n_j} for any $j < i$. Taking $(A_i, B_i) = (A'_{n_i}, B'_{n_i})$ and $T_i = T'_{n_i}$ gives the desired sequences. □

Lemma 20. *If a locally finite graph G with a single end ω which is thin contains infinitely many edge-disjoint 2-rays, then G contains infinitely many edge-disjoint double rays.*

Proof. Applying Lemma 19 we get an infinite set D of edge-disjoint 2-rays, a sequence $((A_i, B_i))_{i\in\mathbb{N}}$ capturing ω, and connectors T_i of $A_i \cap B_i$ for each $i \in \mathbb{N}$ such that the T_i are vertex-disjoint from each other and edge-disjoint from all members of D.

We shall construct the desired set of infinitely many edge-disjoint double rays as a nested union of sets D_i. We construct the D_i recursively. Assume that a set D_i of i edge-disjoint double rays has been defined such that each of its members is included in the union of a single 2-ray from D and one connector T_j. Let $d_{i+1} \in D$ be a 2-ray distinct from the finitely many 2-rays used so far. Let C_{i+1} be one of the infinitely many connectors that is different from all the finitely many connectors used so far and that meets both rays of d_{i+1}. Clearly, $d_{i+1} \cup C_{i+1}$ includes a double ray R_{i+1}. Let $D_{i+1} = D_i \cup \{R_{i+1}\}$. The union $\bigcup_{i\in\mathbb{N}} D_i$ is an infinite set of edge-disjoint double rays as desired. □

5.3 Shapes and allowed shapes

Let G be a graph and (A, B) a separation of G. A *shape* for (A, B) is a word $v_1x_1v_2x_2 \ldots x_{n-1}v_n$ with $v_i \in A \cap B$ and $x_i \in \{l, r\}$ such that no vertex appears twice. We call the v_i the *vertices* of the shape. Every ray R *induces* a shape $\sigma = \sigma_R(A, B)$ on every separation (A, B) of finite order in the following way: Let $<_R$ be the *natural order* on $V(R)$ induced by the ray, where $v <_R w$ if w lies in the unique infinite component of $R - v$. The vertices of σ are those vertices of R that lie in $A \cap B$ and they appear in σ in the order given by $<_R$. For v_i, v_{i+1} the path v_iRv_{i+1} has edges only in A or only in B but not in both. In the first case we put l between v_i and v_{i+1} and in the second case we put r between v_i and v_{i+1}.

Let $(A_1, B_1), (A_2, B_2)$ be separations with $A_1 \cap B_2 = \emptyset$ and thus also $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. Let σ_i be a nonempty shape for (A_i, B_i). The word $\tau = v_1x_1v_2 \ldots x_{n-1}v_n$ is an *allowed shape linking* σ_1 *to* σ_2 with *vertices* $v_1 \ldots v_n$ if the following holds.

- v is a vertex of τ if and only if it is a vertex of σ_1 or σ_2,
- if v appears before w in σ_i, then v appears before w in τ,
- v_1 is the initial vertex of σ_1 and v_n is the terminal vertex of σ_2,
- $x_i \in \{l, m, r\}$,
- the subword vlw appears in τ if and only if it appears in σ_1,
- the subword vrw appears in τ if and only if it appears in σ_2,
- $v_i \neq v_j$ for $i \neq j$.

Each ray R defines a word $\tau = \tau_R[(A_1, B_1), (A_2, B_2)] = v_1x_1v_2 \ldots x_{n-1}v_n$ with vertices v_i and $x_i \in \{l, m, r\}$ as follows. The vertices of τ are those vertices of R that lie in $A_1 \cap B_1$ or $A_2 \cap B_2$ and they appear in τ in the order given by $<_R$. For v_i, v_{i+1} the path v_iRv_{i+1} has edges either only in A_1, only in $A_2 \cap B_1$, or only in B_2. In the first case we set $x_i = l$ and τ contains the subword v_ilv_{i+1}. In the second case we set $x_i = m$ and τ contains the subword v_imv_{i+1}. In the third case we set $x_i = r$ and τ contains the subword v_irv_{i+1}.

For a ray R to induce an allowed shape $\tau_R[(A_1, B_1), (A_2, B_2)]$ we need at least that R starts in A_2. However, each ray in ω has a tail such that whenever it meets an A_i it also starts in that A_i. Let us call such rays *lefty*. A 2-ray is *lefty* if both its rays are.

Remark 21. *Let (A_1, B_1), and (A_2, B_2) be two separations of finite order with $A_1 \subseteq A_2$, and $B_2 \subseteq B_1$. For every lefty ray R meeting A_1, the word $\tau_R[(A_1, B_1), (A_2, B_2)]$ is an allowed shape linking $\sigma_R(A_1, B_1)$ and $\sigma_R(A_2, B_2)$.* □

From now on let us fix a locally finite graph G with a thin end ω of vertex-degree k. And let $((A_i, B_i))_{i \in \mathbb{N}}$ be a sequence capturing ω such that each member has order k.

A 2-*shape* for a separation (A, B) is a pair of shapes for (A, B). Every 2-ray induces a 2-shape coordinatewise in the obvious way. Similarly, an *allowed* 2-*shape* is a pair of allowed shapes.

Clearly, there is a global constant $c_1 \in \mathbb{N}$ depending only on k such that there are at most c_1 distinct 2-shapes for each separation (A_i, B_i). Similarly, there is a global constant $c_2 \in \mathbb{N}$ depending only on k such that for all $i, j \in \mathbb{N}$ there are at most c_2 distinct allowed 2-shapes linking a 2-shape for (A_i, B_i) with a 2-shape for (A_j, B_j).

For most of the remainder of this subsection we assume that for every $i \in \mathbb{N}$ there is a set D_i consisting of at least $c_1 \cdot c_2 \cdot i$ edge-disjoint 2-rays in G. Our aim will be to show that in these circumstances there must be infinitely many edge-disjoint 2-rays.

By taking a tailor if necessary, we may assume that every 2-ray in each D_i is lefty.

Lemma 22. *There is an infinite set $J \subseteq \mathbb{N}$ and, for each $i \in \mathbb{N}$, a tailor D'_i of D_i of cardinality $c_2 \cdot i$ such that for all $i \in \mathbb{N}$ and $j \in J$ all 2-rays in D'_i induce the same 2-shape $\sigma[i, j]$ on (A_j, B_j).*

Proof. We recursively build infinite sets $J_i \subseteq \mathbb{N}$ and tailors D'_i of D_i such that for all $k \leq i$ and $j \in J_i$ all 2-rays in D'_k induce the same 2-shape on (A_j, B_j). For all $i \geq 1$, we shall ensure that J_i is an infinite subset of J_{i-1} and that the $i - 1$ smallest members of J_i and J_{i-1} are the same. We shall take J to be the intersection of all the J_i.

Let $J_0 = \mathbb{N}$ and let D'_0 be the empty set. Now, for some $i \geq 1$, assume that sets J_k and D'_k have been defined for all $k < i$. By replacing 2-rays in D_i by their tails, if necessary, we may assume that each 2-ray in D_i avoids A_ℓ, where ℓ is the $(i-1)$st smallest value of J_{i-1}. As D_i contains $c_1 \cdot c_2 \cdot i$ many 2-rays, for each $j \in J_{i-1}$ there is a set $S_j \subseteq D_i$ of size at least $c_2 \cdot i$ such that each 2-ray in S_j induces the same 2-shape on (A_j, B_j). As there are only finitely many possible choices for S_j, there is an infinite subset J_i of J_{i-1} on which S_j is constant. For D'_i we pick this value of S_j. Since each $d \in D'_i$ induces the empty 2-shape on each (A_k, B_k) with $k \leq \ell$ we may assume that the first $i - 1$ elements of J_{i-1} are also included in J_i.

It is immediate that the set $J = \bigcap_{i \in \mathbb{N}} J_i$ and the D'_i have the desired property. □

Lemma 23. *There are two strictly increasing sequences $(n_i)_{i \in \mathbb{N}}$ and $(j_i)_{i \in \mathbb{N}}$ with $n_i \in \mathbb{N}$ and $j_i \in J$ for all $i \in \mathbb{N}$ such that $\sigma[n_i, j_i] = \sigma[n_{i+1}, j_i]$ and $\sigma[n_i, j_i]$ is not empty.*

Proof. Let H be the graph on $\mathbb{N}$ with an edge $vw \in E(H)$ if and only if there are infinitely many elements $j \in J$ such that $\sigma[v, j] = \sigma[w, j]$.

As there are at most c_1 distinct 2-shapes for any separator (A_i, B_i), there is no independent set of size $c_1 + 1$ in H and thus no infinite one. Thus, by Ramsey's theorem, there is an infinite clique in H. We may assume without loss of generality that H itself is a clique by moving to a subsequence of the D'_i if necessary. With this assumption we simply pick $n_i = i$.

Now we pick the j_i recursively. Assume that j_i has been chosen. As i and $i+1$ are adjacent in H, there are infinitely many indicies $\ell \in \mathbb{N}$ such that $\sigma[i, \ell] = \sigma[i+1, \ell]$. In particular, there is such an $\ell > j_i$ such that $\sigma[i+1, \ell]$ is not empty. We pick j_{i+1} to be one of those ℓ.

Clearly, $(j_i)_{i\in\mathbb{N}}$ is an increasing sequence and $\sigma[i, j_i] = \sigma[i+1, j_i]$ as well as $\sigma[i, j_i]$ is non-empty for all $i \in \mathbb{N}$, which completes the proof. □

By moving to a subsequence of (D'_i) and $((A_j, B_j))$, if necessary, we may assume by Lemma 22 and Lemma 23 that for all $i, j \in \mathbb{N}$ all $d \in D'_i$ induce the same 2-shape $\sigma[i, j]$ on (A_j, B_j), and that $\sigma[i, i] = \sigma[i+1, i]$, and that $\sigma[i, i]$ is non-empty.

Lemma 24. *For all $i \in \mathbb{N}$ there is $D''_i \subseteq D'_i$ such that $|D''_i| = i$, and all $d \in D''_i$ induce the same allowed 2-shape $\tau[i]$ that links $\sigma[i, i]$ and $\sigma[i, i+1]$.*

Proof. Note that it is in this proof that we need all the 2-rays in D''_i to be lefty as they need to induce an allowed 2-shape that links $\sigma[i, i]$ and $\sigma[i, i+1]$ as soon as they contain a vertex from A_i. As $|D'_i| \geq i \cdot c_2$ and as there are at most c_2 many distinct allowed 2-shapes that link $\sigma[i, i]$ and $\sigma[i, i+1]$ there is $D''_i \subseteq D'_i$ with $|D''_i| = i$ such that all $d \in D''_i$ induce the same allowed 2-shape. □

We enumerate the elements of D''_j as follows: $d^j_1, d^j_2, \ldots, d^j_j$. Let (s^j_i, t^j_i) be a representation of d^j_i. Let $S^j_i = s^j_i \cap A_{j+1} \cap B_j$, and let $\mathcal{S}_i = \bigcup_{j\geq i} S^j_i$. Similarly, let $T^j_i = t^j_i \cap A_{j+1} \cap B_j$, and let $\mathcal{T}_i = \bigcup_{j\geq i} T^j_i$.

Clearly, $\mathcal{S}_i$ and $\mathcal{T}_i$ are vertex-disjoint and any two graphs in $\bigcup_{i\in\mathbb{N}}\{\mathcal{S}_i, \mathcal{T}_i\}$ are edge-disjoint. We shall find a ray R_i in each of the $\mathcal{S}_i$ and a ray R'_i in each of the $\mathcal{T}_i$. The infinitely many pairs (R_i, R'_i) will then be edge-disjoint 2-rays, as desired.

Lemma 25. *Each vertex v of $\mathcal{S}_i$ has degree at most 2. If v has degree 1 it is contained in $A_i \cap B_i$.*

Proof. Clearly, each vertex v of $\mathcal{S}_i$ that does not lie in any separator $A_j \cap B_j$ has degree 2, as it is contained in precisely one S^j_i, and all the leaves of S^j_i lie in $A_j \cap B_j$ and $A_{j+1} \cap B_{j+1}$ as d^j_i is lefty. Indeed, in S^j_i it is an inner vertex of a path and thus has degree 2 in there. If v lies in $A_i \cap B_i$ it has degree at most 2, as it is only a vertex of S^j_i for one value of j, namely $j = i$.

Hence, we may assume that $v \in A_j \cap B_j$ for some $j > i$. Thus, $\sigma[j, j]$ contains v and $l : \sigma[j, j] : r$ contains precisely one of the four following subwords:

$$lvl, lvr, rvl, rvr$$

(Here we use the notation $p : q$ to denote the concatenation of the word p with the word q.) In the first case $\tau[j-1]$ contains mvm as a subword and $\tau[j]$ has no m adjacent to v. Then S_i^{j-1} contains precisely 2 edges adjacent to v and S_i^j has no such edge. The fourth case is the first one with l and r and j and $j-1$ interchanged.

In the second and third cases, each of $\tau[j-1]$ and $\tau[j]$ has precisely one m adjacent to v. So both S_i^{j-1} and S_i^j contain precisely 1 edge adjacent to v.

As v appears only as a vertex of S_i^ℓ for $\ell = j$ or $\ell = j-1$, the degree of v in $\mathcal{S}_i$ is 2. ☐

Lemma 26. *There are an odd number of vertices in $\mathcal{S}_i$ of degree* 1*.*

Proof. By Lemma 25 we have that each vertex of degree 1 lies in $A_i \cap B_i$. Let v be a vertex in $A_i \cap B_i$. Then, $\sigma[i,i]$ contains v and $l : \sigma[i,i] : r$ contains precisely one of the four following subwords:

$$lvl, lvr, rvl, rvr$$

In the first and fourth case v has even degree. It has degree 1 otherwise. As $l : \sigma[i,i] : r$ starts with l and ends with r, the word lvr appear precisely once more than the word rvl. Indeed, between two occurrences of lvr there must be one of rvl and vice versa. Thus, there are an odd number of vertices with degree 1 in $\mathcal{S}_i$. ☐

Lemma 27. *$\mathcal{S}_i$ includes a ray.*

Proof. By Lemma 25 every vertex of $\mathcal{S}_i$ has degree at most 2 and thus every component of $\mathcal{S}_i$ has at most two vertices of degree 1. By Lemma 26 $\mathcal{S}_i$ has a component C that contains an odd number of vertices with degree 1. Thus C has precisely one vertex of degree 1 and all its other vertices have degree 2, thus C is a ray. ☐

Corollary 28. *G contains infinitely many edge-disjoint* 2*-rays.*

Proof. By symmetry, Lemma 27 is also true with $\mathcal{T}_i$ in place of $\mathcal{S}_i$. Thus $\mathcal{S}_i \cup \mathcal{T}_i$ includes a 2-ray X_i. The X_i are edge-disjoint by construction. ☐

Recall that Lemma 15 states that a countable graph with a thin end ω and arbitrarily many edge-disjoint double rays all whose subrays converge to ω, also has infinitely many edge-disjoint double rays. We are now in a position to prove this lemma.

Proof of Lemma 15. By Lemma 20 it suffices to show that G contains a subgraph H with a single end which is thin such that H has infinitely many edge-disjoint 2-rays. By Corollary 17, G has a subgraph H with a single end which is thin such that H has arbitrarily many edge-disjoint 2-rays. But then by the argument above H contains infinitely many edge-disjoint 2-rays, as required. ☐

With these tools at hand, the remaining proof of Theorem 1 is easy. Let us collect the results proved so far to show that each graph with arbitrarily many edge-disjoint double rays also has infinitely many edge-disjoint double rays.

Proof of Theorem 1. Let G be a graph that has a set D_i of i edge-disjoint double rays for each $i \in \mathbb{N}$. Clearly, G has infinitely many edge-disjoint double rays if its subgraph $\bigcup_{i\in\mathbb{N}} D_i$ does, and thus we may assume without loss of generality that $G = \bigcup_{i\in\mathbb{N}} D_i$. In particular, G is countable.

By Corollary 10 we may assume that each connected component of G includes only finitely many ends. As each component includes a double ray we may assume that G has only finitely many components. Thus, there is one component containing arbitrarily many edge-disjoint double rays, and thus we may assume that G is connected.

By Corollary 8 we may assume that all ends of G are thin. Thus, as mentioned at the start of Section 4, there is a pair of ends (ω, ω') of G (not necessarily distinct) such that G contains arbitrarily many edge-disjoint double rays each of which converges precisely to ω and ω'. This completes the proof as, by Lemma 13 G has infinitely many edge-disjoint double rays if ω and ω' are distinct and by Lemma 15 G has infinitely many edge-disjoint double rays if $\omega = \omega'$. □

6 Outlook and open problems

We will say that a graph H is *edge-ubiquitous* if every graph having arbitrarily many edge-disjoint H also has infinitely many edge-disjoint H.

Thus Theorem 1 can be stated as follows: the double ray is edge-ubiquitous. Andreae's Theorem implies that the ray is edge-ubiquitous. And clearly, every finite graph is edge-ubiquitous.

We could ask which other graphs are edge-ubiquitous. It follows from our result that the 2-ray is edge-ubiquitous. Let G be a graph in which there are arbitrarily many edge-disjoint 2-rays. Let $v * G$ be the graph obtained from G by adding a vertex v adjacent to all vertices of G. Then $v * G$ has arbitrarily many edge-disjoint double rays, and thus infinitely many edge-disjoint double rays. Each of these double rays uses v at most once and thus includes a 2-ray of G.

The vertex-disjoint union of k rays is called a *k-ray*. The k-ray is edge-ubiquitous. This can be proved with an argument similar to that for Theorem 1: Let G be a graph with arbitrarily many edge-disjoint k-rays. The same argument as in Corollaries 10 and 8 shows that we may assume that G has only finitely many ends, each of which is thin. By removing a finite set of vertices if necessary we may assume that each component of G has at most one end, which is thin. Now we can find numbers k_C indexed by the components C of G and summing to k such that each component C has arbitrarily many edge-disjoint k_C-rays. Hence, we may assume that G has only a single end, which is thin. By Lemma 16 we may assume that G is locally finite.

In this case, we use an argument as in Subsection 5.3. It is necessary to use k-shapes instead of 2-shapes but other than that we can use the same combinatorial principle. If C_1 and C_2 are finite sets, a (C_1, C_2)*-shaping* is a pair (c_1, c_2) where c_1 is a partial colouring of $\mathbb{N}$ with colours from C_1 which is defined at all but finitely many numbers and c_2 is a colouring of $\mathbb{N}^{(2)}$ with colours from C_2 (in our argument above, C_1 would be the set of all k-shapes and C_2 would be the set of all allowed k-shapes for all pairs of k-shapes).

Lemma 29. *Let $D_1, D_2, \ldots$ be a sequence of sets of (C_1, C_2)-shapings where D_i has size i. Then there are strictly increasing sequences $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ and subsets $S_n \subseteq D_{i_n}$ with $|S_n| \geq n$ such that*

- *for any $n \in \mathbb{N}$ all the values of $c_1(j_n)$ for the shapings $(c_1, c_2) \in S_{n-1} \cup S_n$ are equal (in particular, they are all defined).*
- *for any $n \in N$, all the values of $c_2(j_n, j_{n+1})$ for the shapings $(c_1, c_2) \in S_n$ are equal.*

Lemma 29 can be proved by the same method with which we constructed the sets D_i'' from the sets D_i. The advantage of Lemma 29 is that it can not only be applied to 2-rays but also to more complicated graphs like k-rays.

A *talon* is a tree with a single vertex of degree 3 where all the other vertices have degree 2. An argument as in Subsection 5.2 can be used to deduce that talons are edge-ubiquitous from the fact that 3-rays are. However, we do not know whether the graph in Figure 2 is edge-ubiquitous.

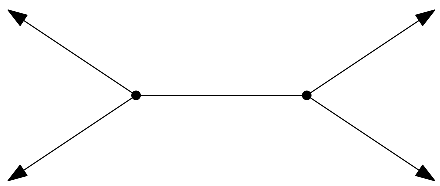

Figure 2: A graph obtained from 2 disjoint double rays, joined by a single edge. Is this graph edge-ubiquitous?

We finish with the following open problem.

Problem 30. *Is the directed analogue of Theorem 1 true? More precisely: Is it true that if a directed graph has arbitrarily many edge-disjoint directed double rays, then it has infinitely many edge-disjoint directed double rays?*

It should be noted that if true the directed analogue would be a common generalization of Theorem 1 and the fact that double rays are ubiquitous with respect to the subgraph relation.

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