Rudimentary recursion and provident sets

A. R. D. Mathias and N. J. Bowler

ERMIT, Université de la Réunion — *Fachbereich Mathematik, Universität Hamburg*

Abstract This paper, a contribution to "micro set theory", is the study promised by the first author in [M4], but improved and extended by work of the second. We use the *rudimentarily recursive* (set theoretic) functions and the slightly larger collection of *gentle* functions to develop the theory of *provident* sets, which are transitive models of PROVI, a very weak subsystem of KP which nevertheless supports not only familiar recursive definitions, such as rank, transitive closure and ordinal addition—though not ordinal multiplication—but also the fundamental definitions of forcing; for a sequel, *Provident sets and rudimentary set forcing* [MB], will show that forcing can be done over any transitive set or class M which is provident, with a poset that is a member of M; that the Forcing Theorem will hold for $\dot{\Delta}_0$ sentences of the forcing language; and that the generic extension will be provident.

Contents

Further details We include in §2 a example of a unary function with Δ_0 graph and of finite rank-bounded growth that is not rudimentary, and in §3 we use the $\mathbb{T}$ of [M3] to improve some arguments of Gandy.

In section 4 we find that the collection of rud rec functions is not closed under composition; but the slightly larger collection of *gentle* functions, namely composites $H \circ F$ where H is rudimentary and F is rud rec, ***is*** closed under composition, which collection is therefore henceforth emphasized. We consider recursions from an additional predicate and show that a function that is gentle in a gentle predicate is gentle.

In section 5, we advance to include recursions from parameters and find a single rudimentary recursion, with parameter, to instances of which all others reduce; which fact is central to our study in section 6 of those sets, which we call *provident*, which are non-empty, transitive and closed under all rudimentarily recursive functions, allowing parameters from within the set in question. We obtain various characterizations of provident sets; we take a first look at functions obtained by iterated recursions; and we show that the Gödel and Jensen segments $L_{\omega\nu}$ and J_ν are provident if and only if $\omega\nu$ is indecomposable.

In one model given in section 8, the failure is of Scott's celebrated trick for defining cardinal number; in section 9, we show, as promised in [M3], that the weak system MW supports a truth definition for $\dot{\Delta}_0$ formulæ; and we close with notes on the origins of the paper and its sequel, acknowledgments and references.

0: Introduction

What is the minimal context in which set forcing works well? It has long been known that the full power of ZF is not needed; but the results of [M4] show that forcing over models of set theories which, even if strong in other ways, offer no support for set-theoretic recursion can go pathologically wrong: for example, forcing using the trivial partial order can make non-trivial changes in the model.

So let us ask a more specific question: how much set-theoretic recursion is needed to do set forcing ? Again, an upper bound has long been known, as Kripke-Platek set theory, KP, is certainly strong enough to allow recursive definitions of the right sort, such as defining the interpretation of names; the validity of such definitions follows easily from the Σ_1 recursion theorem which proves that, in KP, if G is a total Σ_1 function then so is the function F given by the recursion

$$F(x) = G(F \restriction x),$$

and further F is provably equal to a Σ_1 function.

What we find is that even Σ_1 recursion is much stronger than needed for set forcing, and that a coherent and sufficiently strong recursion theory emerges if as our starting point we restrict attention to the above recursions when the defining function G is not merely Σ_1 but actually rudimentary in the sense of Jensen [J2]. In such cases we shall speak of F as given by a *rudimentary recursion*, or, more briefly, that F *is rud rec*.

We find that a smoother theory results if we move to a slightly larger class, which we call the gentle functions. With our application to the theory of forcing in mind, we then examine recursions with parameters, of the form

$$F(x) = G(p, F \restriction x)$$

where p is some set, yielding corresponding notions of p-rud rec and p-gentle functions.

We emphasize that in this paper we confine ourselves to developing our particular restricted form of set-theoretic recursion; we defer to the sequel [MB] the application of our theory to the treatment of forcing, as we hope that the general theory presented here will have other applications.

The time has come to look at some examples.

Some rudimentary recursions

0·0 Example The definition of *rank*:

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid_y y \in x\}$$

0·1 Example The definition of *transitive closure*:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in x\}$$

0·2 Example Let $\mathcal{S}(x)$ be the set of finite subsets of x. *Restricted to ordinals*, this has a rudimentarily recursive definition:

$$\mathcal{S}(0) = \{\varnothing\}; \quad \mathcal{S}(\zeta + 1) = \mathcal{S}(\zeta) \cup \{x \cup \{\zeta\} \mid_x x \in \mathcal{S}(\zeta)\}; \quad \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu).$$

0·3 Example Jensen in [J2] gives a single rudimentary function that can be used to generate the constructible universe. We recall below the definition of a unary rudimentary function $\mathbb{T}$, introduced in *Weak Systems* [M3], such that the following rudimentary recursion on On, the class of von Neumann ordinals,

$$T_0 = \varnothing; \qquad T_{\nu+1} = \mathbb{T}(T_\nu); \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

which can be said in one breath as

$$T_\zeta = \bigcup_{\nu<\zeta} \mathbb{T}(T_\nu)$$

generates L and the Jensen hierarchy in that $L = \bigcup_{\nu \in ON} T_\nu$, and $J_\nu = T_{\omega\nu}$.

0·4 Remark If we recursively define $T(x) = \bigcup_{y\in x} \mathbb{T}(T(y))$, then $T(x)$ always equals $T_{\varrho(x)}$. More generally, if G is a (class) (rudimentary) function such that $\forall u\; u \subseteq G(u)$, and we define $E_0 = \varnothing$, $E_{\nu+1} = G(E_\nu)$, $E_\lambda = \bigcup_{\nu<\lambda} E_\nu$ by recursion on the ordinals, and $F(x) = \bigcup_{y\in x} G(F(y))$ by set recursion, then for all x, $F(x) = E_{\varrho(x)}$.

0·5 Example To form $L(d)$, the constructible closure of d, a transitive set, requires a rud recursion in the parameter d: define

$$D(x) = d \cup \bigcup_{y\in x} \mathbb{T}(D(y)).$$

Then $D(x) = D_{\varrho(x)}$ where $D_0 = d$; $D_{\nu+1} = \mathbb{T}(D_\nu)$; $D_\lambda = \bigcup{}_{\nu<\lambda} D_\nu$, which is the usual ordinal recursion for this purpose. $L(d) = \bigcup_x D(x) = \bigcup_\nu D_\nu$.

In fact, for the purposes of the present paper a different recursion proves desirable, and will be introduced in due course.

0·6 Example Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member y of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} \;=_{\mathrm{df}}\; \{(\mathbf{1}^{\mathbb{P}},\, \hat{x}) \mid_x x \in y\}$$

That is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbf{1}^{\mathbb{P}}, F\restriction a)$$

where G is the rudimentary function $(\mathbf{1}^{\mathbb{P}},\, a) \mapsto \{\mathbf{1}^{\mathbb{P}}\} \times \mathrm{Im}(a)$: though it would be a simple matter to specify that $\mathbf{1}^{\mathbb{P}}$ is always to be some hereditarily finite set, for example 1, when G could be rewritten as a pure rud function.

0·7 Example If M is an (intransitive) elementary submodel of a transitive set or class, then the Mostowski collapsing isomorphism ϖ_M is given by the recursion

$$\varpi_M(x) = \{\varpi_M(y) \mid_y y \in x \cap M\}$$

so that, in some sense, ϖ_M is rudimentarily recursive in the predicate M.

0·8 Example If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model M, and we follow Shoenfield in treating all members of M as $\mathbb{P}$-names, the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$\mathrm{val}_{\mathcal{G}}(b) \;=_{\mathrm{df}}\; \{\mathrm{val}_{\mathcal{G}}(a) \mid_a \exists p_{\in \mathcal{G}}\, (p,\, a) \in b\}$$

The generic extension $M[\mathcal{G}]$ is then be defined as $\{\mathrm{val}_{\mathcal{G}}(a) \mid_a a \in M\}$.

0·9 Remark Note that the definition of the forcing relation $\Vdash$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has interesting properties.

0·10 Remark Close scrutiny reveals that the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ combines two functions, which we might call transforming and collapsing, and when considering forcing in certain contexts, to be explored in a companion paper *Provident sets and rudimentary set forcing*, [MB], there are grounds for separating the two functions. For example, if $\mathcal{G}$ is $(M, \mathbb{P})$-generic, one might first define for $x \in M$

$$\tilde{\pi}(x) = \{(\mathbf{1}^{\mathbb{P}}, \tilde{\pi}(a)) \mid_{p,a} (p,\, a) \in x \;\&\; p \in \mathcal{G}\},$$

thus transforming $\mathbb{P}$-names to 2-names;‡ and then one would collapse the class of pure 2-names, to obtain the desired generic extension, by setting for $x \in \mathrm{Im}\,(\tilde{\pi})$,

$$\varpi(x) = \{\varpi(y) \mid_y (1^{\mathbb{P}},\, y) \in x\},$$

which of course is the inverse of the function $x \mapsto \hat{x}$ when the latter is taken to be defined on $M[\mathcal{G}]$. Both recursions are rudimentary in appropriate parameters or classes.

0·11 EXAMPLE The relation $x \in^\star y$, meaning x is in the transitive closure of y, is given by a rud recursion on the second variable y, the first variable x remaining free:

$$x \in^\star y \Longleftrightarrow x \in y \vee \exists z_{\in y}\, x \in^\star z$$

0·12 EXAMPLE Ordinal addition is given by the recursion

$$A(\alpha, 0) = \alpha;\; A(\alpha, \beta+1) = A(\alpha,\beta) + 1;\; A(\alpha, \lambda) = \bigcup_{\nu<\lambda} A(\alpha, \nu)$$

That is again a rud recursion on the second variable, the first remaining free.

0·13 EXAMPLE The function $\zeta \mapsto 2{\cdot}\zeta$ is given by a rudimentary recursion. Nevertheless it is not rudimentary, for the rud closure of $\{\omega\}$ has ω as a member but, by Gandy [G], not $\mathit{EVEN} =_{\mathrm{df}} \{2 \cdot n \mid_n n \in \omega\}$.

0·14 EXAMPLE In *Weak Systems* [M3], §12, a transitive model of ZC is given in which TCo fails. Thus tcl though rud rec, cannot be rud. Note that the rank of $\mathrm{tcl}(x)$ always equals that of x.

0·15 REMARK *Weak Systems* also contains an example showing that the rank function cannot be rudimentary. We shall review this argument in §8.

0·16 REMARK Corollary 14·5 of *Weak Systems* shows that J_2 is not the rud closure of $J_1 \cup \{\omega\}$, J_1 not being a member of that latter set; but J_2 is the rud rec closure of $J_1 \cup \{\omega\}$; indeed of $\omega + 1$.

0·17 REMARK J_2 has recently been the object of study by Nik Weaver: *Analysis in J_2* [W].

0·18 REMARK The function $\beta \mapsto \beta + \omega$, simple though it be, is *not* given by a pure rudimentary recursion, as we shall show in §6; still less are the other functions of ordinal arithmetic; nor is the J hierarchy. The reason is that any rud function G only raises rank by a finite amount k, a uniform bound for all arguments; which we may call the *rudimentary constant* of G; from that it will follow that for a pure rud rec function F, for each argument x, $\varrho(F(x)) < \varrho(x) + \omega$.

0·19 REMARK It follows from the developments of §6 that, if F is rudimentary recursive then, for x of limit rank, $\varrho(F \restriction x) = \varrho(x)$. When a parameter p is involved, the same equality will hold for x of limit rank at least the ordinal product $\varrho(p) \cdot \omega$.

0·20 REMARK The function g given by the recursion

$$g(0) = 1; \qquad g(\nu+1) = f(g(\nu)); \qquad g(\lambda) = \sup g``\lambda$$

where $f(\xi) = 2 \cdot \xi$ is given by a rud-rec recursion, but not by a rud recursion, as its rate of growth for finite arguments is too great. We shall explore iterated recursions of that sort in §6.

0·21 REMARK The function $x \mapsto \mathcal{S}(x)$ of Example 0·2 is not given by a pure rud rec function, as we shall see below by estimating the rate of growth of its cardinality for $x \in \mathbf{HF}$. But we could define it by a recursion with parameter ω by remarking that for k a positive integer,

$$[a]^{k+1} = \Big\{x \cup \{y\} \;\Big|_{x,y}\; x \in [a]^k \;\&\; y \in \bigcup [a]^k\Big\} \smallsetminus [a]^k.$$

---

‡ Historical influences are slightly confusing our notation here, as 2 is the complete Boolean algebra naturally associated to the partial order whose sole member is $\mathbb{1}$.

0·22 Remark The functions $x \mapsto \mathcal{S}(x)$ and $\beta \mapsto \beta + \omega$ are examples of recursions of a kind we call Type III, to be defined in §5.

Provident sets

Returning to the question of a minimal context for forcing, we suggest that such a context is given by any transitive set which is closed under rudimentary and (parametrised) rudimentary recursive functions. We call such sets *provident*.

0·23 Remark We will show in §7 that it is enough to require closure under a particular finite basis of rudimentary recursive functions. This allows a finite axiomatisation of the notion of providence.

0·24 Remark Natural examples of provident sets abound: in §6 we give a very general notion of hierarchy such that the νth stage in any such hierarchy is provident whenever ν is an indecomposable ordinal: in particular, that will hold for the L and J hierarchies. We shall show in Proposition 6·40 that provident sets are closed under recursions of Type III.

0·25 Remark We expect that the notion of providence will be useful for exploring the fine structure of such hierarchies.

0·26 Remark We introduce two related notions: $\varnothing$-providence involves closure only under pure, rather than parametrised, rudimentary recursive functions. Limit providence involves closure under functions produced by iterated recursion, such as that in Remark 0·19. The phenomena of finite axiomatisability and regular presence in natural hierarchies also hold for these collections of sets.

0·27 Remark In §7 we will explain a simple construction that gives, for any set x, a minimal provident set $\mathrm{Prov}(x)$ including x. We call $\mathrm{Prov}(x)$ the *provident closure* of x. Provident closures allow the following simple formulation of the relationship between providence and forcing, which will be proved in [MB]:

$$M[\mathcal{G}] = \mathrm{Prov}(M \cup \{\mathcal{G}\}) \text{ when } M \text{ is provident.}$$

Here M and $\mathcal{G}$ are as in Example 0.8.

0·28 Remark The first author has in [M1] and [M2] given examples of the weakness for recursive definitions of the unimproved set theories of Zermelo and Mac Lane; further examples of his are given in §8, including a case where the addition of a Cohen generic real goes awry. In [M4] it is shown how passage to the provident closure of transitive models of those theories preserves the theories but adds the capacity for rudimentary recursion and therefore for doing set forcing.

In [M2] it was shown that passage to what in [M4] is called the lune of such models again preserves the theory (Zermelo or Mac Lane as the case may be) but adds the capacity for Σ_1 recursion.

In the next section we rapidly review some weak systems; greater detail will be found in [M3] and in the projected book [M8].

1: A rapid development of weak set theory

1·0 We regard set theory as formalised in a syntax with a class-forming operator and both restricted and unrestricted quantifiers. We have two two-place relation symbols $\in$ and $=$, propositional connectives $\neg$, &, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$, unrestricted quantifiers $\forall$, $\exists$, restricted quantifiers $\forall_r$, $\exists_r$, a class forming operator $\curlyvee$ and a supply of variables.

1·1 Our collection of *well-formed formulæ* is defined thus: atomic wffs are

$$x \in y, \qquad x = y$$

and if Φ and Ψ are well-formed, so are $\&\Phi\Psi$, $\vee\ \Phi\Psi$, $\neg\,\Phi$, $\forall x\Phi$, $\exists x\Phi$, $y \in \curlyvee x\Phi$, $\forall_r xy\Phi$, and $\exists_r xy\Phi$, where in the last two, x and y are **distinct** variables, so that restricted quantifiers $Q_r xy$ bind x but not y, in harmony with the axioms, given below, that express their intended meaning. The expressions $\forall_r xx\Phi$ and $\exists_r xx\Phi$ are ill-formed.

We write $\forall x_{\in y}\,\Phi$ for $\forall_r xy\Phi$, $\exists x_{\in y}\,\Phi$ for $\exists_r xy$ and $\{y \mid \Phi\}$ for $\curlyvee y\,\Phi$. Officially we use Polish notation and write $\&\Phi\Psi$; unofficially we use brackets, writing $(\Phi\ \&\ \Psi)$. Similarly we shall often adopt conventional ways of indicating negation, such as $\notin$ and $\neq$.

1·2 A string $\curlyvee x\,\Phi$, where Φ is a wff, is a *class*. Here are five examples:

$$\begin{aligned} V &=_{\rm df} \{x \mid x = x\} \\ \varnothing &=_{\rm df} \{x \mid x \neq x\} \\ \{x, y\} &=_{\rm df} \{z \mid z = x \vee z = y\} \\ x \setminus y &=_{\rm df} \{z \mid z \in x\ \&\ z \notin y\} \\ \bigcup x &=_{\rm df} \{z \mid \exists y_{\in x}\, z \in y\} \end{aligned}$$

Since $\varnothing$ is the smallest von Neumann ordinal, we shall also set $0 =_{\rm df} \emptyset =_{\rm df} \varnothing$, and will tend to use the notation 0 when we are thinking of this set in its ordinal capacity, $\varnothing$ when thinking of it as the empty set, and $\emptyset$ when thinking of it as the sequence of length 0. In the review of set-theoretic notation which we now give, we are liable to omit definitions of familiar extensions such as writing $\{x\}$ for $\{x, x\}$.

1·3 We denote by $[\Phi\,{}^{y}_{x}]$ the result of substituting the variable x for the free occurrences of the variable y in the formula Φ, bound occurrences of x in Φ being first changed to an as yet unused variable. Less formally, we permit ourselves informally to indicate the result of substituting one variable for another by such usages as $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$.

We progressively extend our notation to permit more liberal use of classes. Thus if Φ is a wff, t a variable or a class, and B a class, then

$$\begin{aligned} \exists z_{\in B}\ \Phi &\Longleftrightarrow_{\rm df} \exists z\,(z \in B\ \&\ \Phi) \\ \forall z_{\in B}\ \Phi &\Longleftrightarrow_{\rm df} \forall z\,(z \in B \implies \Phi) \\ t = B &\Longleftrightarrow_{\rm df} \forall x\,(x \in t \Longleftrightarrow x \in B) \\ B = t &\Longleftrightarrow_{\rm df} \forall x\,(x \in B \Longleftrightarrow x \in t) \\ B \in t &\Longleftrightarrow_{\rm df} \exists y_{\in t}\, y = B \end{aligned}$$

The first two would normally be used only when z is a variable not occurring in B, otherwise nonsense might result. In the last three x and y are presumed to be new variables occurring in neither B nor t.

1·4 Definition Let x be a variable, B a class and Φ a wff. Then $[\Phi\,{}^{x}_{B}]$ is the result of

i) changing all bound occurrences of variables in Φ to occurrences of variables not occurring in B or free in Φ;
ii) replacing all free occurrences of x in the new formula by B;
iii) expanding occurrences of the strings "$B \in t$", "$B = t$", "$t = B$", "$\forall y_{\in B}$" and "$\exists y_{\in B}$" according to the definitions above.

Similarly one may define $[A\,{}^{x}_{B}]$ for A a class. Expressions such as $[\Phi\,{}^{A}_{B}]$ are not defined.

Axioms of logic

All our systems of set theory will have among their axioms those of classical propositional and predicate logic, these two schemes of axioms relating restricted quantifiers to unrestricted ones,

$$\forall x_{\in y}\, \Phi \iff \forall x(x \in y \implies \Phi)$$
$$\exists x_{\in y}\, \Phi \iff \exists x(x \in y \;\&\; \Phi)$$

and the Church conversion scheme

$$x \in \{y \mid \Phi\} \iff [\Phi\ {}^{y}_{x}]$$

by which all occurrences of the class-forming operator are in principle eliminable.

The system S_0

S_0	Extensionality:	$(\forall w_{\in x}\ w \in y \,\&\, \forall w_{\in y}\ w \in x) \implies x = y$
	Empty Set:	$\varnothing \in V$
	Pair:	$\{x, y\} \in V$
	Difference:	$x \setminus y \in V$
	Union:	$\bigcup x \in V$

1·5 DEFINITION We define a Δ_0 formula or a Δ_0 class to be one containing no unrestricted quantifiers; a Π_1 formula is one of the form $\forall x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_1 formula is one of the form $\exists x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_2 formula is one of the form $\exists y\mathfrak{B}$ where $\mathfrak{B}$ is Π_1; and so on.

1·6 DEFINITION $\mathsf{Foundation}$, the axiom of (set) foundation, is $x \neq \varnothing \implies \exists y_{\in x}\, x \cap y = \varnothing$.

S_0' $\quad \mathsf{S}_0 + \mathsf{Foundation}$

1·7 DEFINITION If S is any system of set theory containing S_0 we say that a class A or a wff Φ is Δ_0^{S} iff there is a Δ_0 class B or a Δ_0 wff Ψ such that $\vdash_{\mathsf{S}} A = B$ or $\vdash_{\mathsf{S}} \Phi \iff \Psi$ respectively.

1·8 DEFINITION A class A is S-*suitable* if $\vdash_{\mathsf{S}} A \in V$ and for each Δ_0 wff Ψ and variable w **not occurring freely in** A, $\forall w_{\in A}\ \Psi$ is Δ_0^{S}.

1·9 REMARK If S is a subsystem of T, then all S-suitable classes are T-suitable.

This notion is important in building a calculus of Δ_0 wffs, which we now do.

1·10 PROPOSITION *If Φ and Ψ are Δ_0^{S}, so are $\exists w_{\in z}\,\Phi$, $\forall w_{\in z}\,\Phi$, where w and z are distinct variables, $(\Phi \,\&\, \Psi)$, $\neg\Phi$ and $x \in \{y \mid \Phi\}$.*

1·11 PROPOSITION *Let A be S-suitable.*
(i) if Φ is Δ_0^{S}, so is $\exists w_{\in A}\,\Phi$, provided w is not free in A;
(ii) $w \in A$, $w = A$, $A \in w$ are Δ_0^{S}, even if w occurs in A;
(iii) if Φ is Δ_0, $[\Phi\ {}^{x}_{A}]$ is Δ_0^{S};
(iv) if Φ is Δ_0^{S}, so is $[\Phi\ {}^{x}_{A}]$.

It is necessary to prove (iii) before (iv), since a subformula of a Δ_0^{S} formula need not be Δ_0^{S}.

1·12 PROPOSITION *If A and B are S-suitable, so is $[B\ {}^{x}_{A}]$.*

1·13 PROPOSITION *The classes $\varnothing$, $\{x, y\}$, $\{x\}$, $x \setminus y$ and $\bigcup x$ are S_0-suitable.*

Note that if we define

$$x \cup y =_{\mathrm{df}} \bigcup\{x, y\}$$
$$x \cap y =_{\mathrm{df}} x \setminus (x \setminus y),$$

we have $\vdash_{\mathsf{S}_0} x \cup y \in V \,\&\, x \cap y \in V$; indeed both $x \cup y$ and $x \cap y$ are S_0-suitable. We would wish to define

$$\bigcap x =_{\mathrm{df}} \{z \mid \forall y_{\in x}\, z \in y\};$$

but as $\vdash_{\mathsf{S}_0} \bigcap\varnothing = V$, and (by Russell) $\nvdash_{\mathsf{S}_0} V \in V$, $\bigcap x$ cannot be S_0-suitable. We therefore make an additional definition:

1·14 Definition $\bigcap' x =_{\mathrm{df}} \bigcup x \cap \bigcap x$,

which will prove to be suitable in our next system ReS_0. For now, we can prove that $\bigcap x$ is nearly suitable:

1·15 Proposition *If* $\vdash_{\mathsf{S}} A \neq \varnothing$ *and* A *is* S*-suitable, then so is* $\bigcap A$.

Descriptions are defined so that should the defining clause not have exactly one witness, the description is taken to mean the empty set:

1·16 Definition $\iota x\Phi =_{\mathrm{df}} \bigcup\{x \mid \{x\} = \{x \mid \Phi\}\}$

The next proposition echoes the recursion-theoretic concept of a "bounded" search.

1·17 Proposition *Let* A *be* S*-suitable,* Φ Δ_0^{S} *and* x *a variable not free in* A*. If* $\vdash_{\mathcal{S}} \Phi \implies x \in A$*, then* $\iota x\, \Phi$ *is* S*-suitable.*

Ordered pairs

Following Kuratowski, we introduce a pairing function, in the definition of which we exploit our new freedom to compose suitable classes:

1·18 Definition $(x, y)_2 =_{\mathrm{df}} \{\{x\}, \{x, y\}\}$

1·19 Proposition $(x, y)_2$ *is* S_0*-suitable.*

1·20 Lemma *"w is a singleton", "w is an un-ordered pair", and "w is an ordered pair" are all* $\Delta_0^{\mathsf{S}_0}$

Now we define the un-pairing functions:

1·21 Definition $(x)_\ell =_{\mathrm{df}} \iota y(x$ is an ordered pair and $\bigcup\bigcap x = y)$

1·22 Proposition $(x)_\ell$ *is* S_0*-suitable.*

1·23 Definition $(x)_r =_{\mathrm{df}} \iota y(x$ an ordered pair and **either** $\bigcup x = \bigcap x \,\&\, y = \bigcup\bigcap x$ **or** $\bigcup x \neq \bigcap x \,\&\, y = \bigcup(\bigcup x \setminus \bigcap x))$

1·24 Proposition $(x)_r$ *is* S_0*-suitable.*

Foundation, ordinals and the axiom of infinity

With $\mathsf{Foundation}$ added, the formulation of "ordinal" becomes Δ_0 and much of the elementary theory of ordinals can then be developed in S_0'. In this paper we shall usually be assuming the scheme of Foundation for Π_1 classes, which of course implies $\mathsf{Foundation}$.

Although in one model that we shall mention, we must use a different formulation, we shall usually take the axiom of infinity in the form $\omega \in V$, ω being defined as the class of all von Neumann ordinals such that they and all their predecessors are either 0 or successor ordinals.

ReS_0	S_0 plus the Δ_0 separation scheme: $x \cap A \in V$ for A a Δ_0 class.
ReS	ReS_0 plus the scheme of Π_1 foundation: $A \neq \varnothing \implies \exists x_{\in A}\, x \cap A = \varnothing$ for A a Π_1 class.
ReSI	$\mathsf{ReS} + \omega \in V$.

Digression: models with failures of Δ_0 separation.

We digress to construct, in some conveniently strong system, a transitive model of the system S_0 in which an instance of Δ_0 separation fails. Recall that a set u is *transitive* if $\bigcup u \subseteq u$.

Let θ be a limit ordinal, for example ω^2. A θ-interval is a set $\{\alpha \in ON | \beta \leq \alpha < \gamma\}$ with $\beta, \gamma < \theta$. Let K_0^θ be the set of all finite unions of θ-intervals. Note that K_0^θ is already a transitive model of all the axioms of S_0 except pairing, and that it is closed under finite unions. To get something which in addition models the pairing axiom, we define a sequence of sets K_n^θ, where K_{n+1}^θ is the set of all sets of the form $k \cup l$ with $k \in K_n^\theta$ and l a finite subset of K_n^θ, and we define $K^\theta = \bigcup_{n\in\omega} K_n^\theta$.

1·25 PROPOSITION *i) Each K_n^θ is a transitive set.*

ii) K^θ is transitive and $K^\theta \cap ON = \theta$

iii) The axioms of extensionality, infinity, pairing, union and difference are all true in K^θ.

1·26 PROPOSITION *If θ is a limit ordinal at least ω^2 then the set of limit ordinals less than θ is not a member of K^θ, and accordingly Δ_0 separation fails there.*

Proof : It is sufficient to note that every element of K^θ is a union of an element of K_0^θ with a finite set. $\dashv$ (1·26)

The definition of cartesian product

We introduce, successively, ordered $\mathfrak{k}$-tuples:

$$\begin{aligned}
(y_1, y_2, y_3)_3 &=_{\text{df}} (y_1, (y_2, y_3)_2)_2 \\
(y_1, y_2, y_3, y_4)_4 &=_{\text{df}} (y_1, (y_2, y_3, y_4)_3)_2 \\
(y_1, y_2, y_3, y_4, y_5)_5 &=_{\text{df}} (y_1, (y_2, y_3, y_4, y_5)_4)_2
\end{aligned}$$

and so on, and we may verify that all those are S_0-suitable.

1·27 REMARK Thus all Kuratowski $\mathfrak{k}$-tuples are generated from the single binary function $\{x, y\}$.

1·28 DEFINITION $x \times y =_{\text{df}} \{z \mid \exists a_{\in x} \exists b_{\in y}\, z = (a, b)_2\}$

It would be more convenient to formulate such a definition in this way:

$$x \times y =_{\text{df}} \{(a, b)_2 \mid a \in x \;\&\; b \in y\}$$

That is, though, ambiguous: where the context demands, we may remove the ambiguity by listing the variables to be quantified beside the | sign. Thus

$$\{(a, b)_2 \mid_b a \in x \;\&\; b \in y\}$$

would mean $\{a\} \times y$ if a is in x, and the empty set otherwise. Hence we make the following

1·29 DEFINITION Let A be a class; then

$$\{A \mid_{x_1 \ldots x_n} \Phi\} =_{\text{df}} \{y \mid \exists x_1 \ldots \exists x_n\, y = A \;\&\; \Phi\}.$$

1·30 REMARK Each $\bigcup^{\mathfrak{l}} x$ is S_0-suitable, where we define inductively $\bigcup^{\mathfrak{k}+1} x =_{\text{df}} \bigcup(\bigcup^{\mathfrak{k}} x)$. S_0 easily proves that if $x = (y, z)_2$, then $y \in \bigcup^2 x$ and $z \in \bigcup^2 x$; hence, using these S_0-suitable restrictions, one verifies easily that if $\mathfrak{A}$ is Δ_0 then the class $\{(y_1, y_2, \ldots, y_{\mathfrak{k}})_{\mathfrak{k}} \mid_{y_1, y_2, \ldots, y_{\mathfrak{k}}} \mathfrak{A}\}$ of $\mathfrak{k}$-tuples is equal, provably in S_0, to a Δ_0 class. But in general, if A is S-suitable and Φ is Δ_0, $\{A \mid_{x,y} \Phi(x, y)\}$ might not be a Δ_0^{S} class.

1·31 REMARK In both Models 1 and 2 of [M3, §4], ReSI holds but $\omega \times \omega$ is not a set.

Relations and functions

We may now develop the usual theory of relations, $\mathfrak{k}$-ary functions and so on: we treat functions as a subclass of their image $\times$ their domain. In discussing relations we shall often write Rxy to mean $(x, y)_2 \in R$, though this notation is perhaps too perilous to adopt in a general definition.

1·32 DEFINITION Let R be a variable or class: write

$$Rel(R) \iff_{\mathrm{df}} R = \{(x,y)_2 \mid Rxy\}.$$

We distinguish two relations by special, if inelegant, names:

1·33 DEFINITION

$$\begin{aligned} id &=_{\mathrm{df}} \{(x,x)_2 \mid x \in V\} \\ eps &=_{\mathrm{df}} \{(x,y)_2 \mid x \in y\} \end{aligned}$$

Let F be a set or class.

1·34 DEFINITION $Fn(F) \iff_{\mathrm{df}} Rel(F)\ \&\ \forall x \forall y \forall z (Fxz\ \&\ Fyz \implies x = y)$

Note that we are following a convention in which $(x,y)_2 \in F$ corresponds to the statment $x = F(y)$, rather than $y = F(x)$.

1·35 DEFINITION

$$\begin{aligned} F(t_1, \ldots t_n) &=_{\mathrm{df}} \iota x\, (x t_1 \ldots t_n)_{n+1} \in F \\ \langle t \mid_{x_1, \ldots x_n} \Phi \rangle &=_{\mathrm{df}} \{(t, x_1, \ldots x_n)_{n+1} \mid \Phi\} \end{aligned}$$

1·36 PROPOSITION *$f(x)$ is ReS_0-suitable.*

1·37 DEFINITION For R and t sets or classes, set

$$\begin{aligned} R``t &=_{\mathrm{df}} \{y \mid \exists x\, (Ryx\ \&\ x \in t)\} \\ R \restriction t &=_{\mathrm{df}} \{(x,y)_2 \mid Rxy\ \&\ y \in t\} \\ R^{-1} &=_{\mathrm{df}} \{(x,y)_2 \mid Ryx\} \\ \mathrm{Dom}\,(R) &=_{\mathrm{df}} R^{-1}``V \\ \mathrm{Im}\,(R) &=_{\mathrm{df}} R``V \\ \mathrm{Field}\,(R) &=_{\mathrm{df}} \mathrm{Dom}\,(R) \cup \mathrm{Im}\,(R) \end{aligned}$$

1·38 REMARK Note that by our system of definitions, $f(x)$ is always defined, with default value $\varnothing$; hence $\exists y\, y = f(x)$ is **not** equivalent to $x \in Dom(f)$. We shall occasionally write $f(x) \downarrow$ for the latter.

Our definition of *well-founded relation* includes the concept of being "set-like":

1·39 DEFINITION $Wf(R) \iff_{\mathrm{df}} Rel(R)\ \&\ \forall x \exists y (x \in y\ \&\ R``y \subseteq y)\ \&\ \forall x (x = 0 \vee \exists y_{\in x}\, x \cap R``\{y\} = 0)$

Relativisation of a formula to a class

1·40 DEFINITION Let M be a class. For each formula Φ of the language of set theory, we define $(\Phi)^M$, the *relativisation of* Φ *to* M by recursion on the length of Φ.

1·41 DEFINITION $(x \in y)^M$ is $x \in y$; $(x = y)^M$ is $x = y$; $(\neg\Phi)^M$ is $\neg(\Phi)^M$; $(\Phi\,\&\,\Psi)^M$ is $((\Phi)^M\,\&\,(\Psi)^M)$; $(\Phi \implies \Psi)^M$ is $((\Phi)^M \implies (\Psi)^M)$; and similarly for the other propositional connectives;
$(\forall x\, \Phi)^M$ is $\forall x_{\in M}\ (\Phi)^M$; $(\forall x_{\in y}\ \Phi)^M$ is $\forall x_{\in y \cap M}\ (\Phi)^M$; $(\exists x\, \Phi)^M$ is $\exists x_{\in M}\ (\Phi)^M$; $(\exists x_{\in y}\ \Phi)^M$ is $\exists x_{\in y \cap M}\ (\Phi)^M$; $(x \in \{y \mid \Phi\})^M$ is $x \in \{y \mid y \in M\,\&\,(\Phi)^M\}$.

We also define the relativisation of a class by:

1·41 DEFINITION $(\{y \mid \Phi\})^M$ is $\{y \mid y \in M\,\&\,(\Phi)^M\}$.

The systems DB, BS and MW

The next system, which we call DB for "Devlin Basic", adds the existence of cartesian product to ReS_0, but as it thereby becomes finitely axiomatisable, we give it officially as that finite axiomatisation.

DB_0 The system of which the set-theoretic axioms are Extensionality and the following nine set-existence axioms:

$$\begin{array}{lll} \varnothing \in V & \bigcup x \in V & a \cap \{(x,y)_2 \mid_{x,y} x \in y\} \in V \\ \{x,\, y\} \in V & \mathrm{Dom}\,(x) \in V & \{(y,x,z)_3 \mid_{x,y,z} (x,y,z)_3 \in b\} \in V \\ x \smallsetminus y \in V & x \times y \in V & \{(y,z,x)_3 \mid_{x,y,z} (x,y,z)_3 \in c\} \in V \end{array}$$

1·42 REMARK All those nine are theorems of ReS_0 + cartesian product.

1·43 THEOREM (Bernays) *All instances of Δ_0 separation are provable in the system* DB_0

1·44 DEFINITION We shall call a function of the form $x \mapsto x \cap A$, where A is a class, a *separator*, or a Δ_0-*separator* if A is a Δ_0 class.

DB DB_0 plus Π_1 foundation.

$\mathsf{DB}_0\mathsf{I}$ DB_0 plus $\omega \in V$.

DBI DB + $\omega \in V$.

1·45 PROPOSITION ($\mathsf{DB}_0\mathsf{I}$) $[\omega]^1$ *and* $[\omega]^2$ *exist.*

Proof : $\omega \in V$ is an axiom of $\mathsf{DB}_0\mathsf{I}$. By the definition of ordered pair, $[\omega]^1 \cup [\omega]^2 \subseteq \bigcup(\omega \times \omega)$, and the result follows by Δ_0 separation. ⊣ (1·45)

If we add the axiom of infinity plus the scheme of foundation for all classes to DB we obtain the system BS as formulated on page 36 of Devlin's book *Constructibility*:

BS ReS_0 + Cartesian product + full foundation + $\omega \in V$.

The system BS is used extensively by Devlin in his study [De] of constructibility: for each limit ordinal ζ the set L_ζ in Gödel's constructible hierarchy models BS. But counterexamples of Solovay show that it is not quite strong enough for its intended tasks, one of which was to give a definition of the truth predicate $\models_u \varphi$ where u is a set and φ is a sentence of an appropriate object language. To decide whether an existential statement $\bigvee \mathfrak{x}\vartheta(\mathfrak{x})$ is true in a model $\mathfrak{M}$ (here the symbol $\bigvee$ is the existential quantifier of the object language), one considers the set $S_\vartheta =_{\mathrm{df}} \{\vartheta[\underline{a}] \mid a \in \mathfrak{M}\}$ of substitution instances of ϑ, where $\underline{a}$ is the constant of the relevant language interpreted by the element a.

1·46 DEFINITION For each $\mathfrak{k} > 0$, we write $[\omega]^{\mathfrak{k}}$ for the class of subsets of ω of size $\mathfrak{k}$.

Now Model 6 of [M3, §5], where the defects of BS are discussed in detail, shows that although BS can prove the existence of $[\omega]^1$ and $[\omega]^2$ it cannot prove the existence of $[\omega]^3$, or indeed any $[\omega]^{\mathfrak{k}}$ for $\mathfrak{k} > 2$. Thus BS is unable to form the set S_ϑ and hence cannot define $\models$. The following strengthening suffices:

MW DBI + $\forall a \forall k_{\in\omega}\, [a]^k \in V$

That the truth relation $\models_u \varphi$ is provably in MW, Δ_1-definable was shown in [M3, §10]. §9 of this paper will give a new proof of that, and also of the corresponding result for $\overset{0}{\models}$, which has this interesting consequence:

1·47 THEOREM MW *is finitely axiomatisable, modulo one subtlety.*

Proof : We already know that DB_0 is; to that we have added an axiom of infinity, the axiom just given, and the scheme of Π_1 foundation. The subtlety is this: we use the truth definition for $\dot{\Delta}_0$ wffs: what are they ? Here we are quantifying in the language of discourse, not in the metalanguage, so we are getting slightly more than the scheme, but only in non-standard models will we be able to tell the difference. We invite the reader to complete the proof by using $\overset{0}{\models}$ to formulate $\dot{\Pi}_1$ foundation. $\dashv$ (1·47)

1·48 Remark In the transitive Model 7 of [M3, §5] MW is true but for some element a, $\{\bigcup x \mid_x x \in a\}$ is absent.

The system GJ

1·49 We now reach a system of the greatest importance in the study of constructibility, which was discovered independently by Gandy [G] and by Jensen [J2]. The transitive models of this system are precisely the transitive sets closed under a certain collection $\mathcal{R}$ of functions, which we have yet to define. The members of this collection were called *basic* by Gandy and *rudimentary* by Jensen; the second adjective has been generally adopted in the literature, and is customarily shortened to *rud.* We follow that usage, and shall define a subcollection $\mathcal{B}$ of $\mathcal{R}$, calling the members of $\mathcal{B}$ *basic* functions. The transitive sets closed under the members of $\mathcal{B}$ are the transitive models of DB.

GJ_0 $\quad \mathsf{DB}_0 + \{x``\{w\} \mid w \in y\} \in V$

GJ $\quad \mathsf{GJ}_0$ + the scheme of Π_1 foundation.

GJI $\quad \mathsf{GJ} + \omega \in V$.

1·50 Proposition *The class* $\{x``\{w\} \mid w \in y\}$ *is* GJ_0*-suitable.*

In the next section we shall prove the important *eyebrow principle* that if F is a rudimentary function so is $F``$. For its proof we shall introduce *companions* and establish the Gandy–Jensen Lemma.

1·51 Remark An application of that principle is that $a \in V \implies \{\bigcup x \mid_x x \in a\} \in V$ is provable in GJ_0.

Again using the eyebrow principle, we may prove the following scheme of theorems:

1·52 Proposition (GJ_0) *For each set* a, $[a]^{\mathfrak{k}}$ *exists.*

Proof : $[a]^0 = \{\varnothing\} \in V$. $[a]^1 = A_0``a \in V$. $[a]^{\mathfrak{k}+1} = \big\{s \cup \{x\} \bigm| (s,x)_2 \in ([a]^{\mathfrak{k}} \times a) \cap \{(s,x)_2 \mid x \notin s\}\big\}$, which is in $V$, being of the form $h``b$ for some set b and rudimentary function h. $\dashv$ (1·52)

That scheme becomes a single theorem once the right instances of Π_1 foundation are available:

1·53 Theorem (GJ) $\forall a \forall k_{\in\omega}\ [a]^k \in V$.

Proof : Once we know Theorem 2·93 of [M3], which runs:

Theorem (GJ) $\forall a \forall m_{\in\omega}\ {}^m a \in V$.

where ${}^m a$ is the set of functions from m to a, and which is proved by using Π_1 foundation to find for given a the least counterexample m, we may again invoke the eyebrow principle to obtain the desired result, since

$$[a]^k = \{\mathrm{Im}\,(f) \mid_f f \in {}^m a \;\&\; f \text{ is injective}\}. \qquad \dashv (1{\cdot}53)$$

1·54 Corollary MW *is a subsystem of* GJI.

1·55 Remark §6 of [M3] recalls the result of Gandy [G] that GJI does not prove the existence of $\mathcal{S}(\omega)$. Thus by Theorem 1·53 the function $(a,k)_2 \mapsto [a]^k$ is not rudimentary.

2: Review of the elementary theory of rudimentary functions

We introduce the rudimentary functions $R_0, \ldots R_8$ and certain auxiliary functions $A_0 \ldots A_{14}$ generated by them under composition: this is not the shortest possible list, but one that conveniently extends the list, given in the axioms of DB_0, that generates the Δ_0 separators.

$R_0(x,y) = \{x,y\}$
  $A_0(x) = \{x\} \,[= R_0(x,x)]$
  $A_1(x,y) = (x,y)_2 \,[= R_0(A_0(x), R_0(x,y))]$
  $A_2(x,y,z) = \{x, (y,z)_2\}$
  $A_3(x,y,z) = (x,y,z)_3 \,[= A_1(x, A_1(y,z))]$
$R_1(x,y) = x \setminus y$
  $A_4(x,y) = x \cap y \,[= x \setminus (x \setminus y)]$
  $A_5(x) = \varnothing \,[= x \setminus x]$
  $A_6(x) = x \,[= x \setminus \varnothing]$
$R_2(x) = \bigcup x$
$R_3(x) = \mathrm{Dom}\,(x)$
$R_4(x,y) = x \times y$
$R_5(x) = x \cap \{(a,b)_2 \mid_{a,b} a \in b\}$
  $A_7(x) = \mathit{eps} \restriction x \,[= R_5(\bigcup x \times x)]$
$R_6(x) = \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}$
$R_7(x) = \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}$
  $A_8(x) = \{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \,[= R_6(R_7(R_7(x)))]$
  $A_9(x) = x^{-1} \,[= \mathrm{Dom}\,(\{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in \{\varnothing\} \times x\}) = R_3(A_8(R_4(A_0(A_5(x)), x)))]$
  $A_{10}(x) = \mathrm{Im}\,(x) \,[= \mathrm{Dom}\,(x^{-1})]$
  $A_{11}(x,y) = \mathit{eps} \cap (x \times y) \,[= R_5(x \times y)]$
  $A_{12}(x,y) = \{w \mid_w x \in w \in y\} \,[= \mathrm{Dom}\,(A_{11}(\{x\} \times y))]$
  $A_{13}(x,y) = \mathit{id} \cap (x \times y)$
  $A_{14}(x,y) = x``\{y\} \,[= \mathrm{Dom}\,((x \cap ([\bigcup\bigcup x] \times \{y\}))^{-1})]$
$R_8(x,y) = \{x``\{w\} \mid_w w \in y\}$

Separators, basic functions and Δ_0 branching

2·0 Definition Let $\mathcal{R}$, the collection of *rudimentary functions*, be the closure of $R_0 \ldots R_8$ under composition. Let $\mathcal{B}$, the collection of *basic functions*, be the closure of $R_0 \ldots R_7$ under composition.

2·1 Proposition *(i) For each Δ_0 class A the map $x \mapsto x \cap A$ is in $\mathcal{B}$.*

(ii) It is a theorem of MW *that for each $\dot\Delta_0$ wff φ the map $a \mapsto a \cap \{x \mid \overset{0}{\models} \varphi[x]\}$ is in $\mathcal{B}$.*

The difference between the two results lies in the quantification, which in part (i) is in the metalanguage and in part (ii) in the language of discourse. So really we have cheated in not specifying in which language $\mathcal{B}$ is being defined. A similar ambiguity is inherent in our definition of $\mathcal{R}$.

Proposition 2.1 implies that branching over Δ_0 choices can be coded by rudimentary functions.

2·2 Proposition *For each Δ_0 class A the map* $x,y,z \mapsto \begin{cases} x & \text{if } z \in A \\ y & \text{otherwise.} \end{cases}$ *is rudimentary.*

Proof : The map can be expressed as

$$x,y,z \mapsto \mathrm{Dom}\,(x \times (\{z\} \cap A) \cup \mathrm{Dom}\,(y \times (\{z\} \cap (V \setminus A))$$

⊣ (2·2)

Resolution of a question of Sy Friedman

We may now answer a question of Sy Friedman, whether a unary function F with a Δ_0 graph and such that for some $k \in \omega$ and all x, $\varrho(F(x)) \leqslant \varrho(x) + k$ is necessarily rudimentary.

Write **HF** for the class of hereditarily finite sets, defined as the union of all finite transitive sets.

2·3 LEMMA *Let* $k \in \omega$*; then for any* x, $x \subseteq V_{k+1} \iff \bigcup x \subseteq V_k$*; hence* $x \subseteq V_{k+1} \iff \bigcup^{k+1} x \subseteq V_0 = \varnothing$.

2·4 LEMMA *The predicate* $a = \mathbf{HF}$ *is* Δ_0.

Proof : To say $a = \mathbf{HF}$, say that $\varnothing \in a$, that a is transitive and closed under $\bigcup$ and (unordered) pairing, that no member of a is a limit ordinal, and if $b \in a$ then there is an $f \in a$ with domain a successor ordinal $\ell + 1$ such that $f(0) = b$, for every $k < \ell$, $f(k+1) = \bigcup f(k)$ and $f(\ell) = \varnothing$. ⊣

2·5 PROPOSITION **HF** *is not a member of the rud closure of* $\{\omega\}$.

Proof : otherwise $\mathcal{S}(\omega)$ would be, contradicting the result of Gandy mentioned in Remark 1.55. ⊣ (2·5)

Now let F be $\{(x, y)_2 \mid_{x,y} (y = \omega \;\&\; x = \mathbf{HF}) \vee (y \neq \omega \;\&\; x = \varnothing)\}$. Then F is a function, its graph is Δ_0 and for any y, $\varrho(F(y)) \leqslant \varrho(y)$. But F is not rudimentary, for $F(\omega) = \mathbf{HF}$.

Companions for rudimentary functions

The collection of functions in $\mathcal{R}$ is closed under formation of images: by which is meant that if F is in $\mathcal{R}$ so is $x \mapsto F``x$. To prove that, we introduce the notion of a *companion*—we will actually have two such notions—and establish the Gandy–Jensen Lemma.

Let S be some system of set theory extending DB_0, and let G and F be Δ_0 classes such that S proves that both G and F are total functions.

2·6 DEFINITION G is a *1-companion* of F in S if G is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\in G(\vec{u})$$

2·7 DEFINITION H is a *2-companion* of F in S if H is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\subseteq H(\vec{u})$$

where $\vec{x} \in \vec{u}$ abbreviates $x_1 \in u_1 \;\&\; \ldots x_n \in u_n$ for an appropriate n.

2·8 PROPOSITION *If* G^1 *is a 1-companion of* G *in* S *and* H^1 *is a 1-companion of* H *in* S*, then* $G^1 \circ H^1$ *is a 1-companion of* $G \circ H$ *in* S.

The function $F``$, if available in S, is the best 1-companion of F in S, and in favourable cases separators may be used to reduce a given 1-companion F^1 of F to that one, since

$$\vdash_{\mathsf{S}} F``a = F^1(a) \cap \{y \mid \exists x_{\in a}\, y = F(x)\}$$

so that if F is given by an S-suitable term,

$$\vdash_{\mathsf{S}} y = F(x) \iff \forall w_{\in y}\, w \in F(x) \;\&\; \forall w_{\in F(x)}\, w \in y$$

2·9 PROPOSITION *Each of the functions* $R_0, \ldots, R_7$ *and* A_{14} *has a 2-companion in* DB.

Proof :
R_0: $a \in x \,\&\, b \in y \implies \{a, b\} \subseteq x \cup y = \bigcup\{x,\, y\}$.
R_1: $a \in x \,\&\, b \in y \implies a \setminus b \subseteq a \subseteq \bigcup x$.
R_2: $a \in x \implies \bigcup a \subseteq \bigcup\bigcup x$.
R_3: $a \in x \implies \mathrm{Dom}\,(a) \subseteq \bigcup\bigcup x$.
R_4: $a \in x \,\&\, b \in y \implies a \times b \subseteq \bigcup x \times \bigcup y$.
R_5: $t \in x \implies t \cap \{(a, b)_2 \mid a \in b\} \subseteq t \subseteq \bigcup x$.
R_6: $t \in x \implies \{(b, a, c)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(\bigcup x)) \times (\mathrm{Im}\,(\bigcup x) \times \mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)))$,
[by reasoning similar to that given below for R_7.]
R_7: $t \in x \implies \{(b, c, a)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(\bigcup x)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)) \times \mathrm{Im}\,(\bigcup x))$.

To see this, note that $\{(b, c, a)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(t)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(t)) \times \mathrm{Im}\,(t))$, and apply these principles: $t \in x \implies t \subseteq \bigcup x$; $t \subseteq s \implies \mathrm{Dom}\,(t) \subseteq \mathrm{Dom}\,(s)$; $t \subseteq s \implies \mathrm{Im}\,(t) \subseteq \mathrm{Im}\,(s)$; and $t \subseteq s \,\&\, v \subseteq u \implies t \times v \subseteq s \times u$.

A_{14}: $a \in x \;\&\; b \in y \implies a``\{b\} \subseteq \mathrm{Im}\,(\bigcup x)$. $\dashv$ (2·9)

2·10 Remark The above 2-companions are generated by four functions, namely, Im , Dom , $\bigcup$ and $\times$. We can get that down to two, $\bigcup$ and $\times$, by using the above principles. For u transitive, a single generator, the function $u \mapsto u^\star \;=_{\mathrm{df}}\; u \cup [u]^{\leqslant 2} \cup (u \times u)$ is enough.

2·11 Proposition *If F has a 1-companion F^1 then $\bigcup F^1$ is a 2-companion of F.*

2·12 Proposition *If G has a 2-companion G^2 and H has a 1-companion H^1, then $G^2 \circ H^1$ is a 2-companion of $G \circ H$.*

The Gandy–Jensen Lemma

The Gandy–Jensen Lemma is the core of the proof that $\mathcal{R}$ is closed under formation of images. Versions of it are to be found in the papers of Gandy [G] and Jensen [J2]. We discuss it only for 1-ary functions. The extension to n-ary functions poses no problems.

2·13 The Gandy–Jensen Lemma *Let S be a system extending DB_0. Suppose that H is a 2-companion of F in S, and that '$a \in F(b)$' is Δ_0^{S}. Then F is generated by composition from H and members of $\mathcal{B}$, and so is S-suitable; if in addition S extends GJ, then $\vdash_{\mathsf{S}} F``x \in V$ and $F``$ (as a function) is generated by H and members of $\mathcal{R}$ and (as a term) is S-suitable and is a 1-companion of F in S.*

Proof : We have

$$\vdash_{\mathsf{S}} x \in u \implies F(x) \subseteq H(u).$$

Working in S, form

$$h(u) \;=_{\mathrm{df}}\; \bigl(H(u) \times u\bigr) \cap \{(a,b)_2 \mid_{a,b} b \in u \;\&\; a \in F(b)\}.$$

Since "$a \in F(b)$" is Δ_0^{S} and for each Δ_0 A, the function $x \mapsto x \cap A$ is in $\mathcal{B}$ and is DB-suitable, we have that h is S-suitable, and is generated by H and functions in $\mathcal{B}$.

Now note that for $b \in u$, $F(b) = h(u)``\{b\} = A_{14}(h(u), b)$, so $F$ is built from $H$ and functions in $\mathcal{B}$; if $R_8$ is available in the system $\mathsf{S}$, we may argue further that $F``u = R_8(h(u), u)$ so $F``$ is built from $H$ and rudimentary functions, and is thus $\mathsf{S}$-suitable; hence $\vdash_{\mathsf{S}} F``u \in V$, and the function $F``$ now forms a 1-companion of F in S. $\dashv$ (2·13)

2·14 Proposition *R_8 has a 2-companion in GJ.*

Proof : By the Gandy–Jensen lemma, $A_{14}``$ is GJ-suitable, and so

$$a \in x \;\&\; b \in y \implies R_8(a,b) = \{A_{14}(a,w) \mid w \in b\} = A_{14}``(\{a\} \times b) \subseteq A_{14}``(x \times \bigcup y). \qquad \dashv (2·14)$$

2·15 Corollary *R_8 has a 1-companion in GJ.*

Proof : by the Gandy–Jensen Lemma. $\dashv$ (2·15)

2·16 Theorem *$\mathcal{R}$ is closed under formation of images and of unions of images.*

Proof : We have seen that each of $R_0, \ldots R_8$ has a 1-companion in GJ; the collection of functions possessing a 1-companion is closed under composition, and hence each function in $\mathcal{R}$ has a 1-companion in GJ; but if G is a 1-companion of F then $u \mapsto \bigcup(G(u))$ is a 2-companion of F. Hence each function F in $\mathcal{R}$ has a 2-companion in GJ; each such function is GJ-suitable, Proposition 1·11 proving the survival of suitability under composition, and so by the Gandy–Jensen lemma, $F``$ is in $\mathcal{R}$; composition with $\bigcup$ yields the last clause. $\dashv$ (2·16)

2·17 Remark Gandy shows in [G] that these three are equivalent: (i) F is rudimentary; (ii) "$a \in F(b)$" is Δ_0 and F has a 1-companion in GJ; (iii) "$a \in F(b)$" is Δ_0 and F has a 2-companion in GJ.

2·18 Remark Gandy in [G] and Jensen in [J2] supply other characterisations of $\mathcal{R}$ and other axiomatisations of GJ.

3: A single generating function for rud(u)

In developing further properties of the collection of rudimentary functions we shall use the function $\mathbb{T}$ introduced in Definition 2·73 of *Weak Systems*.

The function $\mathbb{T}$

3·0 DEFINITION

$$\begin{aligned}\mathbb{T}(u) =_{\rm df}\ & u \cup \{u\} \\ & \cup [u]^1 \cup [u]^2 \\ & \cup \{x \smallsetminus y \mid_{x,y} x, y \in u\} \\ & \cup \{\textstyle\bigcup x \mid_x x \in u\} \\ & \cup \{\mathrm{Dom}\,(x) \mid_x x \in u\} \\ & \cup \{u \cap (x \times y) \mid_{x,y} x, y \in u\} \\ & \cup \{x \cap \{(a,b)_2 \mid_{a,b} a \in b\} \mid_x x \in u\} \\ & \cup \{u \cap \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \mid_x x \in u\} \\ & \cup \{u \cap \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \mid_x x \in u\} \\ & \cup \{x``\{w\} \mid_{x,w} x \in u, w \in u\} \\ & \cup \Big\{u \cap \{x``\{w\} \mid_w w \in y\} \Big|_{x,y} x, y \in u\Big\}.\end{aligned}$$

3·1 REMARK The successive lines of the definition of $\mathbb{T}$, after the first, may be written more prosaically as $R_0``(u\times u)$, $R_1``(u\times u)$, $R_2``u$, $R_3``u$, $\{u\cap R_4(x,y) \mid_{x,y} x, y \in u\}$, $R_5``u$, $\{u\cap R_6(x) \mid_x x \in u\}$, $\{u\cap R_7(x) \mid_x x \in u\}$, $A_{14}``(u \times u)$ and $\{u\cap R_8(x,y) \mid_{x,y} x, y \in u\}$. It will be notationally convenient to treat all these functions as having three variables, so let us define $S_i(u; x, y) := R_i(x, y)$ for $i = 0, 1$; $S_i(u; x, y) := R_i(x)$ for $i = 2, 3, 5$; $S_i(u; x, y) := u \cap R_i(x, y)$ for $i = 4, 8$; $S_i(u; x, y) := u \cap R_i(x)$ for $i = 6, 7$; and $S_9(u; x, y) := A_{14}(x, y)$. Then each of those lines now takes the form $S_i``(\{u\} \times (u \times u))$ for some i.

We have proved the first clause of the following, and the others are easy.

3·2 PROPOSITION *$\mathbb{T}$ is rudimentary, $u \subseteq \mathbb{T}(u)$ and $u \in \mathbb{T}(u)$. Further, if u is transitive, then $\mathbb{T}(u)$ is a set of subsets of u, and hence $\mathbb{T}(u)$ is transitive.*

3·3 REMARK It will not in general be true that $u \subseteq v \implies \mathbb{T}(u) \subseteq \mathbb{T}(v)$, the problem being that $u \in \mathbb{T}(u)$, but if v is countably infinite, so is $\mathbb{T}(v)$ which therefore cannot contain all the subsets of v. Fortunately, $u \subseteq \mathbb{T}(u) \subseteq \mathbb{T}^2(u) \ldots$

3·4 LEMMA *If x and y are in u, then $R_0(x, y)$, $R_1(x, y)$, $R_2(x)$, $R_3(x)$, and $R_5(x)$ are all in $\mathbb{T}(u)$.*

In the next five results, it is supposed that u is transitive.

3·5 LEMMA *For x, y in u, $R_4(x, y) = x \times y \subseteq u \times u \subseteq \mathbb{T}^2(u)$.*

3·6 COROLLARY *For x, y in u, $R_4(x, y) \in \mathbb{T}^3(u)$.*

3·7 LEMMA *For a, b, c in u, $(a, c)_2 \in \mathbb{T}^2(u)$ and $(b, a, c)_3 \in \mathbb{T}^4(u)$.*

3·8 COROLLARY *For $x \in u$, $R_6(x)$ and $R_7(x)$ are in $\mathbb{T}^5(u)$.*

3·9 LEMMA *For x, $y \in u$, $R_8(x, y) \in \mathbb{T}^2(u)$.*

Proof : For x, w in u, $x``w \in \mathbb{T}(u)$, so $R_8(x, y) = \mathbb{T}(u) \cap \{x``w \mid_w w \in y\}$; $x, y \in \mathbb{T}(u)$, so $R_8(x, y) \in \mathbb{T}^2(u)$. ⊣ (3·9)

Those remarks, which were proved in *Weak Systems*, though regrettably without the requirement that u be transitive being clearly stated, and of which more general forms will be proved below, immediately yield:

3·10 PROPOSITION *If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.*

Proof : The stated property holds of the nine generating functions and is preserved under composition. ⊣ (3·10)

3·11 REMARK Strictly, we should give this as two different results, like Proposition 2.1, in one of which we quantify in the metalanguage (and so get a fact about each externally definable rudimentary function) and in the other of which we quantify internally, and so get a single fact about the internal set of all (codes for) rudimentary functions.

3·12 COROLLARY (Gandy; Jensen) *If F is rudimentary, then there is a finite ℓ such that the rank of the value is at most the maximum of the ranks of the arguments, plus ℓ.*

Proof : the function $\mathbb{T}$ increases rank by exactly 1. ⊣ (3·12)

3·13 COROLLARY *For any transitive u, $\bigcup_{n\in\omega}\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$ and models* TCo.

Functions rudimentary in a predicate

Let B be a unary predicate. The collection of functions *rudimentary in B* is that obtained by adding to the generators of $\mathcal{R}$ the function $x \mapsto x \cap B$.

To extend Proposition 3.10 to the collection of functions rudimentary in B, we introduce a function $\mathbb{T}^B$, rudimentary in B, given by

$$\mathbb{T}^B(u) = \mathbb{T}(u) \cup \{x \cap B \mid x \in \mathbb{T}(u)\}\,.$$

3·14 PROPOSITION *If $F(\vec{x})$ is a rudimentary function in B of several variables, then there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in (\mathbb{T}^B)^\ell(u)$.*

The intransitive case

The function $\mathbb{T}$ works very happily for transitive argument, but for intransitive argument it starts to create non-trivial problems. The aim, in the two cases, is not quite the same. The purpose of $\mathbb{T}$ is to proceed by rud steps from any transitive set u to $\mathrm{rud}(u)$, which will be of strictly greater rank; with an intransitive argument of limit rank, our first concern would be to fatten it to a transitive rud closed set, without raising rank. Here are two ways of doing so.

We introduce two functions, trud and krud.

3·15 DEFINITION $\mathrm{trud}(u) =_{\mathrm{df}} \bigcup\{F(\vec{x}) \mid F \text{ rud } \& \ \vec{x} \in u\}$.

That is a legitimate definition because we are quantifying over programs for rud functions; the axiom of infinity is at work here. Here $\vec{x}$ denotes a finite sequence of arguments of F, and we follow Devlin's convention that $\vec{x} \in u$ means that each argument is in u; if we wanted to say that the sequence is in u we would write $\langle\vec{x}\rangle \in u$.

3·16 PROPOSITION *For any set u, $\mathrm{trud}(u)$ is transitive, rud closed and includes u; and if A is transitive, rud closed and includes u, then $\mathrm{trud}(u) \subseteq A$. The rank of $\mathrm{trud}(u)$ will be the least limit ordinal greater than or equal to the rank of u.*

Proof : If $a \in b \in F(\vec{x})$, then $a \in \bigcup F(\vec{x}) \subseteq \mathrm{trud}(u)$, $\bigcup \circ F$ being rud; and so $\mathrm{trud}(u)$ is transitive.

If $G(\cdot,\cdot)$ is rud, $b_1 \in F_1(\vec{x})$, $b_2 \in F_2(\vec{y})$, then $G(b_1, b_2) \in G``H(\vec{x}, \vec{y})$ for some rud $H$; $G`` \circ H$ is rud, and so $G(b_1, b_2) \in \mathrm{trud}(u)$. Similarly for functions of a different number of variables.

If $a \in u$ then $a \in \{a\} \subseteq \mathrm{trud}(u)$.

If $\vec{x} \in u$ then $\vec{x} \in A$ as A includes u; then $F(\vec{x}) \in A$, A being rud closed; so $F(\vec{x}) \subseteq A$, as A is transitive. Thus $\mathrm{trud}(u) \subseteq A$. ⊣ (3·16)

The definition of trud can be given recursively.

3·17 DEFINITION $\mathbb{K}(u) = u \cup \bigcup u \cup \{R_i(x, y, z) \mid 0 \leqslant i \leqslant 8 \ \& \ x, y, z \in u \cup \bigcup u\}$.

That definition is intended for use even when u is intransitive. Note that $\mathbb{K}$ is rudimentary, and that it has the agreeable property that $u \subseteq v \implies \mathbb{K}(u) \subseteq \mathbb{K}(v)$.

3·18 DEFINITION $\mathbb{K}_0(u) = u$; $\mathbb{K}_{n+1}(u) = \mathbb{K}(\mathbb{K}_n(u))$; $\mathrm{krud}(u) = \bigcup_{n\in\omega}\mathbb{K}_n(u)$.

3·19 PROPOSITION *For any u, $\mathrm{krud}(u) = \mathrm{trud}(u)$.*

Proof : plainly krud(u) includes u, is transitive and is rud closed; so trud$(u) \subseteq$ krud(u).

If $u \subseteq A$ where A is transitive, rud closed and includes u then one verifies by an easy induction that each $\mathbb{K}_n(u) \subseteq A$. Hence krud$(u) \subseteq$ trud(u). ⊣ (3·19)

3·20 REMARK $\mathbb{K}$ has the property that for any rud function R there is a d such that $\mathbb{K}^d$ is a 1-companion of R.

Gandy reproved

The proofs of a couple of the very interesting results of Gandy's paper *Set theoretic functions* are unfortunately flawed, which may have resulted from Gandy encountering similar difficulties to those created by "the intransitive case". We shall give a brief review of the problems, and shall explain how to obtain proofs of those results which are right but not supported by Gandy's arguments as they stand. See especially Propositions 3·25 and 3·27 below.

The first problem is in his Lemma 1.5.3. on page 111. We will begin our discussion from his definition 1.5.2: he uses a bold-face $\mathbf{x}$ to denote the (meta) finite sequence $x_1, \ldots x_m$: cf the bottom of page 105. This usage is a little ambiguous; the letter m here may be a variable of the meta-language.

Let us for simplicity take the case $m = 1$, and write x for x_1. Then the first part of his Definition 1.5.2 runs

$$\mathrm{Cc}_0\{x\} = \{x\}; \mathrm{Cc}_{q+1}\{x\} = Cc_q\{x\} \cup \{\mathrm{Cc}_q\{x\}\} \cup \{\mathbf{F}_i uv : 1 \leqslant i \leqslant 9 \;\&\; u, v \in \mathrm{Cc}_q\{x\}\}.$$

For the purposes of this discussion, we shall take the letter q here to be a variable of the language of discourse.

3·21 PROPOSITION *For any $q \in \omega$ and any x, $\mathrm{Cc}_q\{x\}$ is a finite set.*

Proof : by induction on q. Indeed, for a given x, let n_q be the number of elements in $\mathrm{Cc}_q\{x\}$. Then $n_0 = 1$; $n_{q+1} \leqslant n_q + 1 + 9 \cdot {n_q}^2$. ⊣ (3·21)

3·22 Thus the second statement of part (ii) of Lemma 1.5.3 is false: if x is actually an infinite set, it cannot be a subset of any $\mathrm{Cc}_q\{x\}$.

Similarly, $\mathrm{Cc}\{x\}$ is defined as $\bigcup_{q\in\omega} \mathrm{Cc}_q\{x\}$; which will be a countable infinite set; so if x is uncountable, it cannot be a subset of $\mathrm{Cc}\{x\}$, even if it is transitive.

Lemma 1.5.4 is also incorrect—the difficulty is with step (C) of the proof. The 'only if' direction of Theorem 1.5.5, which relies on Lemma 1.5.4, is also wrong. Theorem 1.5.6 is false: $\mathrm{Bc}\{x\}$ is always transitive but $\mathrm{Cc}\{x\}$ need not be.

We now turn to the ways in which some of the correct results may be recoverd.

3·23 LEMMA *If u is a finite transitive set with $\overline{\overline{u}} = \ell$, then $\overline{\overline{\mathbb{T}(u)}} \leqslant \frac{1}{2}(2 + 13\ell + 9\ell^2)$.*

Proof : by inspection. ⊣ (3·23)

3·24 DEFINITION (Gandy) $\eta(x) =_{\mathrm{df}}$ the cardinal of the transitive closure of x.

3·25 PROPOSITION (Gandy) *If F is rud, then there is a k such that $\eta(F(\vec{x}))$ is less than $(\eta(\{\vec{x}\}) + 1)^k$.*

Here $\{\vec{x}\}$ for many variables means the set of them.

Proof : we know that there is an ℓ such that for u transitive and the arguments of F in u, $F(\vec{x}) \in \mathbb{T}^\ell(u)$. For u transitive, $\mathbb{T}(u)$ is transitive, and iterating the previous estimate, we find that there is a polynomial $Q(X)$ of degree 2^ℓ, (for example $13^{2^\ell - 1} X^{2^\ell}$) such that $x \in u$ implies that $\eta(F(\vec{x}))$ is at most $Q(\overline{\overline{u}} + 1)$. ⊣ (3·25)

3·26 REMARK We may now justify our earlier remark that there is no pure rud recursion for $\mathcal{S}(x)$ for x an arbitrary set. If we look at $\mathcal{S}(x)$ for $x \in \mathbf{HF}$, we see that $\mathcal{S}(V_n) = V_{n+1}$; if $\mathcal{S}(x)$ were pure rud rec, given by G, we would have

$$G(\mathcal{S} \restriction V_n) = V_{n+1}.$$

But if $\overline{\overline{V_n}} = N$, $\overline{\overline{V_{n+1}}} = 2^N$, whereas

$$\mathrm{tcl}(\mathcal{S} \restriction V_n) \subseteq \{(\mathcal{S}(x), x) \mid x \in V_n\} \cup \{\{\mathcal{S}(x)\} \mid x \in V_n\} \cup \{\{\mathcal{S}(x), x\} \mid x \in V_n\} \cup \{\mathcal{S}(x) \mid x \in V_n\} \cup V_n$$

which has cardinality at most $5N$; but for each k, $(5N)^k$ will be much less than 2^N for large N. $\dashv$ (3·26)

Gandy remarks on page 114 that there is a primitive recursive function which returns the value ω given any argument of infinite rank. Indeed the example he gives is rud rec: define

$$F(x) = \omega \cap \bigcup \{F(y) \cup \{F(y)\} \mid_y y \in x\},$$

which is rud rec as intersection with ω is given by a Δ_0 separator; and show first that if $x \in \mathbf{HF}$, then $F(x) = \varrho(x)$.

He then states as his Theorem 2.1.3 the following:

3·27 PROPOSITION (Gandy) *There is a set c of infinite rank such that for no rud function G is $G(c) = \omega$.*

Indeed there are many such sets, for it is possible to build a transitive model of Z not containing ω but containing sets of infinite rank. Such models are automatically rud closed, and absolute for rud functions. These constructions make heavy use of the power set axiom, and accordingly the details have been placed in a separate paper, [M7], though we shall sketch one such construction in §8.

4: The collections of pure rud rec and gentle functions

4·0 Definition (Mathias): By type I or pure rudimentary recursions we mean those given by a recursion equation of the form

$$F(x) = G(F \restriction x)$$

where G is a pure rud function with no hidden parameters. We call functions which may be defined in this way *rudimentary recursive*, or *rud rec*. For example, as was shown in the introduction, the rank function ϱ is rud rec. We will now explore the closure properties of rud rec functions.

4·1 Proposition *Every (unary) rud function is rud rec.*

Proof : If $F(\cdot)$ is unary and rud, let $G(f) =_{\text{df}} F(\text{Dom}\,(f))$; then G is rud and $\forall x\, F(x) = G(F \restriction x)$. Other rud functions can be transformed to unary functions by using the pairing and un-pairing functions, which are rudimentary. ⊣ (4·1)

4·2 Proposition *If F_1 and F_2 are rud rec, so is $x \mapsto (F_1(x), F_2(x))_2$.*

Proof : Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = \big(G_1(F_1 \restriction x), G_2(F_2 \restriction x)\big)_2$.

$K \restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$.

There are rud G_3 and G_4 such that $G_3(K \restriction x) = F_1 \restriction x$ and $G_4(K \restriction x) = F_2 \restriction x$. So

$$K(x) = \big(G_1(G_3(K \restriction x)), G_2(G_4(K \restriction x))\big)_2 = G_5(K \restriction x)$$

where $G_5(z) =_{\text{df}} \big(G_1(G_3(z)), G_2(G_4(z))\big)_2$. G_5 is rudimentary. ⊣ (4·2)

4·3 Proposition *Let G_1 and G_2 be rudimentary, and suppose that F_1 and F_2 are defined by the simultaneous recursion*

$$F_1(x) = G_1(F_1 \restriction x, F_2 \restriction x); \qquad F_2(x) = G_2(F_1 \restriction x, F_2 \restriction x).$$

Then the function $x \mapsto (F_1(x), F_2(x))_2$ is rud rec.

Proof : Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = (G_1(F_1 \restriction x, F_2 \restriction x), G_2(F_1 \restriction x, F_2 \restriction x))_2$.

$K \restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$.

There are rud G_3 and G_4 such that $G_3(K \restriction x) = F_1 \restriction x$ and $G_4(K \restriction x) = F_2 \restriction x$. So

$$K(x) = \big(G_1(G_3(K \restriction x), G_4(K \restriction x)), G_2(G_3(K \restriction x), G_4(K \restriction x))\big)_2 = G_6(K \restriction x)$$

where $G_6(z) =_{\text{df}} \big(G_1(G_3(z), G_4(z)), G_2(G_3(z), G_4(z))\big)_2$. G_6 is rudimentary. ⊣ (4·3)

4·4 Corollary *Let F be a rud rec function and H a rud function. Then $H \circ F$ is a projection of a rud rec function.*

Proof : Suppose that F is given by $F(x) = G(F \restriction x)$. Then F and $H \circ F$ are definable by the simultaneous recursion given by that equation and $H \circ F(x) = H(G(F \restriction x))$, and hence Proposition 4·3 applies. ⊣ (4·4)

Significantly, $H \circ F$ need not be rud rec:

4·5 Proposition (Bowler) *The function $H : x \mapsto \begin{cases} \omega & \text{if } \varrho(x) = \omega \\ 0 & \text{otherwise.} \end{cases}$ is a composite of a rud function with a rud rec fuction, but is not rud rec.*

Proof : H is the composite $\delta_\omega \circ \varrho$, where $\delta_\omega : x \mapsto \begin{cases} \omega & \text{if } x = \omega \\ 0 & \text{otherwise.} \end{cases}$ is rudimentary by Proposition 2.2. For any unary rud G and ℓ as in Proposition 3·10, (in fact $\ell = c_G$ as in Definition 6·9) and any transitive x,

$$\bigcup{}^{\ell+1} G(x) \subseteq \bigcup{}^{\ell+1} G''(x \cup \{x\}) \subseteq \bigcup{}^{\ell+1} \mathbb{T}^\ell(x \cup \{x\}) = \bigcup (x \cup \{x\}) = x.$$

Suppose that H were rud rec, given by G_0 say. Let $\ell = c_G$ where G is the rud function $G : y \mapsto G_0(\{0\} \times y)$. Let Z be the transitive set, of rank ω, of Zermelo integers: $Z = \{s^n(\varnothing) \mid n \in \omega\}$, where $s : x \mapsto \{x\}$. Then

$$\omega = \bigcup{}^{\ell+1}\omega = \bigcup{}^{\ell+1} H(Z) = \bigcup{}^{\ell+1}(G_0(H \restriction Z)) = \bigcup{}^{\ell+1}(G_0(\{0\} \times Z)) = \bigcup{}^{\ell+1}(G(Z)) \subseteq Z\text{—a falsehood !}$$

⊣ (4·5)

We therefore turn our attention to a collection of functions with better closure properties: those of the form $G \circ F$ with G rud and F rud rec. We call such functions *gentle*. Our first concern will be to show that, unlike the collection of rud rec functions, the collection of gentle functions is closed under composition.

4·6 LEMMA *Let F be rud rec, given by $F(x) = G(F \restriction x)$ where G is rud. Then there is a rud function H_G obtainable uniformly from G such that for every u, and every $v \subseteq \mathcal{P}u$, $F \restriction v = H_G(v, F \restriction u)$.*

Proof : For $x \in v$, $F \restriction x = (F \restriction u) \restriction x$. Let $\phi(f, x) = (G(f \restriction x), x)_2$. Then ϕ is rud, and

$$F \restriction v = \{\phi(F \restriction u, x) \mid_x x \in v\} = H_G(v, F \restriction u)$$

where H_G is rud. ⊣ (4·6)

4·7 COROLLARY *Let F be rud rec, given by $F(x) = G(F \restriction x)$ where G is rud. Then there is a rud function $H_G^{\mathbb{T}}$ obtainable uniformly from G such that for every transitive u, $F \restriction \mathbb{T}(u) = H_G^{\mathbb{T}}(F \restriction u)$.*

Proof : We take $H_G^{\mathbb{T}}(f) = H_G(\mathbb{T}(\mathrm{Dom}\,(f)), f)$. ⊣ (4·7)

4·8 PROPOSITION (Bowler) *If F_1 and F_2 are rud rec, then $F_1 \circ F_2$ is gentle.*

Proof : Let F_1 be given by $F_1(x) = G_1(F_1 \restriction x)$ and F_2 by $F_2(x) = G_2(F_2 \restriction x)$.

We say that a set is *sufficient for x* if it is the restriction of F_1 to a transitive set u containing $(F_2(x), x)_2$. We proceed by showing that there is a function F, definable by mutual rudimentary recursion with F_2 as in Proposition 4.3, with the property that for any x $F(x)$ is sufficient for x. For that, we must find a rudimentary function G in two variables such that, for any function f with domain x and sending each y in x to a set sufficient for y, $G(F_2 \restriction x, f)$ is sufficient for x.

Suppose we have such an f, with $f(y) = F_1 \restriction u(y)$ for each $y \in x$ (here $u(y)$ is defined to be the domain of $f(y)$). Then $\bigcup \mathrm{Im}\,(f) = F_1 \restriction u$, with $u = \bigcup_{y \in x} u(y)$ a transitive set of which $F_2 \restriction x$ is a subset. Thus $H_{G_1}(u \cup \{F_2 \restriction x\}, F_1 \restriction u) = F_1 \restriction (u \cup \{F_2 \restriction x\})$ is a restriction of F_1 to a transitive set containing $F_2 \restriction x$. The rudimentary function $K : f \mapsto (G_2(f), \mathrm{Dom}\,(f))_2$ has the property that for any x we have $K(F_2 \restriction x) = (F_2(x), x)_2$. So if we choose ℓ as in Proposition 3.10 for this K, then $(H_{G_1}^{\mathbb{T}})^\ell(F_1 \restriction (u \cup \{F_2 \restriction x\})) = F_1 \restriction \mathbb{T}^\ell(u \cup \{F_2 \restriction x\})$ is sufficient for x. So the rudimentary function $G : f, f' \mapsto (H_{G_1}^{\mathbb{T}})^\ell(H_{G_1}(\mathrm{Dom}\,(\bigcup \mathrm{Im}\,(f')) \cup \{f\}, \bigcup \mathrm{Im}\,(f')))$ has the property stated above: the function F defined by $F(x) = G(F_2 \restriction x, F \restriction x)$ sends each x to something sufficient for x.

By Proposition 4.3, $x \mapsto (F_2(x), F(x))_2$ is rud rec. Thus $F_1 \circ F_2$ is gentle, as it can be obtained by precomposing this rud rec function with the rudimentary function $q \mapsto \mathrm{right}(q)(\mathrm{left}(q))$. ⊣ (4·8)

4·9 THEOREM (Bowler) *Any composite of gentle functions is gentle.*

Proof : Suppose that H_1 and H_2 are gentle, with H_i given by $H_i = G_i \circ F_i$ with G_i rud and F_i rud rec. Then by Propositions 4.1 and 4.8 $F_1 \circ G_2$ is gentle—say it is given by $G \circ F$ with G rud and F rud rec. By Proposition 4.8 again, $F \circ F_2$ is gentle—say it is given by $G' \circ F'$ with G rud and F rud rec. Thus $H_2 \circ H_1 = G_2 \circ F_2 \circ G_1 \circ F_1 = G_2 \circ G \circ F \circ F_1 = (G_2 \circ G \circ G') \circ F'$ is gentle. ⊣ (4·9)

The collection of gentle functions has other good closure properties: for example, by Proposition 4.2 if H_1 and H_2 are gentle then so is $x \mapsto (H_1(x), H_2(x))_2$.

4·10 PROPOSITION *If F is rud rec, so is $x \mapsto F \restriction x$.*

Proof : Let F be given by G, and let $H(x) = F \restriction x$. Then

$$\begin{aligned} H(x) &= F \restriction x \\ &= \{(F(a), a)_2 \mid_a a \in x\} \\ &= \{\big(G(F \restriction a), a\big)_2 \mid_a a \in x\} \\ &= \{\big(G(H(a)), a\big)_2 \mid_a a \in x\} \\ &= G_2(H \restriction x) \end{aligned}$$

where, setting G_1 to be the rud function $x \mapsto \big(G(\mathrm{left}(x)), \mathrm{right}(x)\big)_2$, we take $G_2(x) \;=_{\mathrm{df}}\; G_1``x$. ⊣ (4·10)

4·11 COROLLARY *If H is gentle, then $H``$, being equal to $Im \circ (H \restriction)$, is also gentle.*

Thus any gentle function has a gentle 1-companion. It is also clear that any gentle function has a gentle 2-companion, obtained by precomposing this 1-companion with $\bigcup$.

Gentle predicates

4·12 Proposition *Let B be a predicate. The following are equivalent:*

i. The characteristic function of B is gentle.

ii. The function $x \mapsto x \cap B$ is gentle.

Proof : $(i) \Rightarrow (ii)$ is immediate from Proposition 4.10, and $(ii) \Rightarrow (i)$ follows from Proposition 4.9 and the fact that $x \in B$ iff $\{x\} \cap B \neq \varnothing$. ⊣ (4·12)

We call predicates with those properties *gentle*. There is a variant of Corollary 4.7 for $\mathbb{T}^B$ with B gentle.

4·13 Lemma *If B is a gentle predicate, with the function $x \mapsto x \cap B$ given by $H \circ F$ with H rudimentary and F rud rec, then there is a rudimentary function $G^{\mathbb{T}^B}$ such that, for any transitive set u, $G^{\mathbb{T}^B}(F \restriction u) = F \restriction \mathbb{T}^B(u)$.*

Proof : Let F be given by $F(x) = G(F \restriction x)$. We take the function $f \mapsto H_G(\mathbb{T}(\mathrm{Dom}\,(f)) \cup \mathrm{Im}\,(H \circ f), f)$. ⊣ (4·13)

4·14 Lemma *If B is a gentle predicate, with the function $x \mapsto x \cap B$ given by $H \circ F$, H rud and F rud rec, and G is a unary function which is rudimentary in B, then there is a binary rudimentary function $\hat{G}$ such that $\hat{G}(x, y) = G(x)$ whenever y is a restriction of F to a transitive set containing x.*

Proof : Using Proposition 3.13, we can find some ℓ such that, for any transitive set u containing x and any subterm G' of some term representating G as a function rudimentary in B, $G'(x) \in (\mathbb{T}^B)^\ell(u)$. Thus for y as in the statement, $f = (G^{\mathbb{T}^B})^\ell(y)$ is a restriction of F to a transitive set containing all of the $G'(x)$. Thus $G(x)$ can be obtained by using $H \circ f$ in place of $z \mapsto z \cap B$ in the term defining G.

Variants

There are some obvious variations on the definition of rudimentary recursion, which we now show do not give more general collections of functions. For example, we could vary the relation used in the recursion.

4·15 Proposition *Let F be defined by $F(x) = G(x, F \restriction \mathrm{tcl}(x))$, where G is rudimentary. Define H by $H(x) = F \restriction \mathrm{tcl}(\{x\})$. Then H is rud rec and therefore F is gentle.*

Proof : $F \restriction \mathrm{tcl}(x) = \bigcup_{y \in x} H(y)$, so

$$\begin{aligned} H(x) &= \{(F(x), x)_2\} \cup \bigcup_{y \in x} H(y) \\ &= \{\big(G(x, \bigcup_{y \in x} H(y)), x\big)_2\} \cup \bigcup_{y \in x} H(y) \\ &= G_1(H \restriction x) \end{aligned}$$

where $G_1(h) = \{\big(G(\mathrm{Dom}\,(h), \bigcup \mathrm{Im}\,(h)), \mathrm{Dom}\,(h)\big)_2\} \cup \bigcup \mathrm{Im}\,(h)$, so that G_1 is rudimentary and H is rud rec. Then $F(x) = [H(x)](x)$, the evaluation of $H(x)$ at argument x, and is thus a trivial rud function of x and $H(x)$. ⊣ (4·15)

4·16 Corollary *Functions defined by recursions of the form $F(x) = G(x, F \restriction \bigcup\bigcup x)$ are thus gentle.*

4·17 Remark Recursions of that kind occur in the definition of forcing.

We could also restrict the domain of the recursion, for example to the ordinals.

4·18 Remark For G a rudimentary function, define $G'(f) = G(f) \cap \{z \mid \mathrm{Dom}\, f \in On\}$. Then G' is rudimentary, by Proposition 2·1; and if we recursively define $F(x) = G'(F \restriction x)$, then F is rudimentarily recursive and

$$F(x) = \begin{cases} G(F \restriction x) & \text{if } x \in On \\ \varnothing & \text{otherwise.} \end{cases}$$

We could also consider gentle functions of more than one variable—for example, any gentle function H can be considered as giving the function $x, y \mapsto H((x, y)_2)$ of two variables. Gentle functions in multiple variables are still closed under composition. We could also consider functions defined by mutual recursions—but as Proposition 4.3 shows, that does not take us outside the collection of gentle functions.

The final variant we shall consider is rudimentary recursion in a predicate. We call a function *rud rec in B* if it is of the form

$$F(x) = G(F \restriction x)$$

where G is rud in B. We say K is *gentle in B* iff it is of the form $H \circ F$ with H rud in B and F rud rec in B. It is clear that rudimentary recursion in arbitrary predicates is more general than pure rudimentary recursion. However, rudimentary recursion in gentle predicates is not.

4·19 THEOREM (Bowler) *Let F_2 be a gentle function in a gentle predicate B. Then F_2 is gentle.*

Proof : Suppose that $x \mapsto x \cap B$ is given by $H_1 \circ F_1$, with F_1 given by $F_1(x) = G_1(F_1 \restriction x)$, where G_1 and H_1 are rud. Since any gentle function in B is a composite of rud rec functions in B and any composite of gentle functions is gentle, we may suppose without loss of genrality that F_2 is rud rec in B, given by $F_2(x) = G_2(F_2 \restriction x)$, where G_2 is rud in B.

We say that a set is *sufficient for x* if it is the restriction of F_1 to a transitive set u containing $(F_2(x), x)_2$. We proceed, as in the proof of Proposition 4.8, by showing that there is a function F, definable by mutual rudimentary recursion with F_2 as in Proposition 4.3, with the property that for any x $F(x)$ is sufficient for x. As in that proof (but using Lemma 4.13 in place of Corollary 4.7), we can find a rudimentary function G in two variables such that, for any function f with domain x and sending each y in x to a set sufficient for y, $G(F_2 \restriction x, f)$ is sufficient for x. It also follows from this construction that $F_2 \restriction x$ is in the domain of $G(F_2 \restriction x, f)$. Thus $F_2(x) = \hat{G}_2(F_2 \restriction x, G(F_2 \restriction x, f))$. Therefore there is an F which is definable together with F_2 by the simultaneous rudimentary recursion

$$F(x) = G(F_2 \restriction x, F \restriction x); \qquad F_2(x) = \hat{G}_2(F_2 \restriction x, G(F_2 \restriction x, F \restriction x)).$$

and so by Proposition 4.3 F_2 is gentle. ⊣ (4·19)

An illusory recursion

Just to warn the reader:

4·20 PROPOSITION *There are rud functions G and H such that for any function F, $F(x) = G(F \restriction H(x))$.*

The Scott–McCarty definition of ordered pair

We draw attention to the paper [SMcC] which proposes a new definition of *ordered pair* that does not raise rank: their definition is rudimentarily recursive.

5: Rudimentary recursion from parameters

5·0 We have defined functions of type I, or pure rud rec functions to be those given by a recursion equation of the form

$$F(x) = G(F\restriction x)$$

where G is a pure rud function with no hidden parameters.

5·1 Definition (Mathias): For recursions involving parameters, the following definition seems the most satisfactory, which we call type II.

$$F(x) = G(p, F\restriction x)$$

Here G is a pure rud function of two variables and p is some set. We shall call such an F *p-rud rec* or *a function of Type II*. Similarly, we call F *p-gentle* if it is a composite of a rudimentary function with a p-rud rec function.

5·2 It might be asked whether a simpler kind of recursion, which we might call type II', will suffice. Let us say that F is *rud rec from* p, where p is some set, if there are G_0 and G, pure rud functions of one variable, such that

$$F(x) = \begin{cases} G_0(p) & \text{if } x = \varnothing; \\ G(F\restriction x) & \text{if } x \neq \varnothing. \end{cases}$$

For such an F and for any rudimentary function H we shall say $H \circ F$ is *gentle from* p.

Thus in type II recursion the parameter p may be re-used throughout the recursion, whereas in type II', use of the parameter p occurs only at the beginning.

5·3 A delicate distinction has to be made here. The two collections of functions given by recursions of type II and of type II' from a given parameter are not the same: for example, for p of infinite rank, the function $F : x \mapsto p \times x$ is p-rud rec but not rud rec from p, since there is no rud G with $p \times \{\varnothing\} = F(\{\varnothing\}) = G(F\restriction \{\varnothing\}) = G(\{(\varnothing, \varnothing)_2\})$. The closure property given in Proposition 4·10 holds for the collection of p-rud rec functions, by essentially the same proof, but fails for the collection of functions rud rec from p, since if K is the constant function with value ω, K is rud rec from ω, but $x \mapsto K\restriction x = \{\omega\} \times x$ is not. It is for such reasons that we have preferred type II to type II'.

But when we pass to the associated gentle collections, we may breathe again, as that distinction no longer applies:

5·4 Proposition (Bowler) *A function F is p-gentle iff it is gentle from p.*

Proof : The 'only if' direction is clear from the definitions and from Proposition 2.2. For the 'if' direction, note that without loss of generality F is p-rud rec, given by $F(x) = G(p, F\restriction x)$. Let $K : x \mapsto (p, F(x))_2$. There is a rudimentary function G_1 such that for any x we have $G_1(K\restriction x) = F\restriction x$, and so

$$K(x) = \begin{cases} (p, G(p, \varnothing))_2 & \text{if } x = \varnothing; \\ (\bigcup \mathrm{Im}\,(\mathrm{Im}\,(K\restriction x)), G\,(\bigcup \mathrm{Im}\,(\mathrm{Im}\,(K\restriction x)), G_1(K\restriction x)))_2 & \text{if } x \neq \varnothing. \end{cases}$$

Thus K is rud rec from p and so F is gentle from p. ⊣ (5·4)

5·5 Essentially the same arguments as in the last section show that the p-gentle functions have good closure properties. For example, if F is p-gentle then so is $x \mapsto F\restriction x$. However, it is not true that any composite of p-gentle functions is p-gentle: for example, the function $x \mapsto \omega + x$ is ω-gentle, but its composite with itself is not. This composite is, however, $\omega + \omega$-gentle and there is a similar phenomenon in general.

5·6 Proposition (Bowler) *Let F_1 be p_1-rud rec and F_2 be p_2-rud rec. Then $F_1 \circ F_2$ is $(p_1, F_1\restriction \mathrm{tcl}\{p_2\})_2$-gentle.*

5·7 Proposition (Bowler) *Let B be a p_1-gentle predicate, with $x \mapsto x \cap B$ represented as $H_1 \circ F_1$, and let F_2 be p_2-gentle in B. Then F_2 is $(p_1, F_1\restriction \mathrm{tcl}\{p_2\})_2$-gentle.*

The proofs are like those in the last section. Apart from these two cases, the results of the last section transfer directly to p-gentle functions, and we may refer to them in future as if they were stated in those terms. Specifically, in Propositions 4·1, 4·2, 4·3, 4·12 and 4·15; Lemma 4·6; Corollaries 4·7 and 4·16 and Remarks 4·17 and 4·18, we may replace rud, rud rec and gentle respectively by p-rud (that is, of the form $x \mapsto G(p,x)$ with G rud), p-rud rec and p-gentle.

5·8 Remark Type II recursions will underlie our discussion of rudimentary forcing in the sequel, with the poset $\mathbb{P}$ of conditions as an ever-present parameter.

5·9 Remark The first Jensen fragment after J_1 that is closed under functions of Type II is J_ω, as given J_k we could set $f(0) = J_k; f(n+1) = \mathbb{T}(f(n)); f(\lambda) = \bigcup f``\lambda$, and then $f(\omega) = J_{k+1}$.

5·10 Finally, we ask what happens to type II if we turn the parameter back into a variable and consider recursion equations of the following form

$$F(v,x) = G(v, F \restriction (\{v\} \times x))$$

which we shall call type III, though in this paper we shall say little about them.

5·11 Remark The recursion here is on the second variable, in harmony with the form of the definition of ordinal addition as given in Example 0·12.

5·12 Proposition *For each fixed v the map $x \mapsto F(v,x)$ is rud recursive of type II, in the parameter v.*

Proof : Let $E(x) = F(v,x)$. Then $E \restriction x = \{(F(v,b),b)_2 \mid_b b \in x\}$ whereas

$$\begin{aligned} F \restriction (\{v\} \times x) &= \{\big(F(v,b),(v,b)_2\big)_2 \mid_b b \in x\} \\ &= \{\big(E(b),(v,b)_2\big)_2 \mid_b b \in x\} \\ &= H(v, E \restriction x) \end{aligned}$$

for a certain rud function H; so $E(x) = G(v, H(v, E \restriction x)) = G_1(v, E \restriction x)$, for some rud function G_1.

$\dashv$ (5·12)

5·13 Remark Since x is recoverable by a rud function from $F \restriction (\{v\} \times x)$, as the domain of its domain, no new functions result from equations of the form

$$F(v,x) = H(v,x,F \restriction (\{v\} \times x)).$$

6: Provident sets

6·0 Definition (Mathias): A set A is *p-provident*, where p is a set, if it is non-empty, transitive, closed under pairing and for all p-rud rec F (or equivalently all p-gentle F) and all x in A, $F(x) \in A$.

6·1 Remark If A is p-provident, $p \in A$.

6·2 Example We shall see that the Jensen fragment J_ν is $\varnothing$-provident for all $\nu \geqslant 1$.

6·3 Theorem *Any directed union of $\varnothing$-provident sets is $\varnothing$-provident. Explicitly, if $\mathcal{A}$ is a nonempty set of $\varnothing$-provident sets such that for any $A, B \in \mathcal{A}$ there is $C \in \mathcal{A}$ with $A \cup B \subseteq C$, then $\bigcup \mathcal{A}$ is $\varnothing$-provident.*

Proof : $\bigcup \mathcal{A}$ is nonempty since $\mathcal{A}$ and all $A \in \mathcal{A}$ are nonempty. It is transitive since each $A \in \mathcal{A}$ is. For any $x, y \in \bigcup \mathcal{A}$, we can find $A, B \in \mathcal{A}$ with $x \in A$ and $y \in B$, and we can find $C \in \mathcal{A}$ with $A \cup B \subseteq C$. Since C is provident, $\{x, y\} \in C \subseteq \bigcup \mathcal{A}$. Finally, $\bigcup \mathcal{A}$ is closed under $\varnothing$-rud rec functions since each such function is unary. ⊣ (6·3)

6·4 Definition (Mathias): A is *provident* if it is p-provident for every $p \in A$.

6·5 Remark The only provident set not containing an infinite set is **HF**.

6·6 Remark For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x, y\}$ is x-rud rec, being given by the recursion $F(y) = \{x, \operatorname{Dom} F \restriction y\}$. But the union of two sets each closed under $\varnothing$-rud rec functions might not be closed under pairing, though as rud rec functions are unary, that union would be closed under $\varnothing$-rud rec functions: for example, let a and b be mutually Cohen-generic subsets of ω and consider the model $J_2(a) \cup J_2(b)$.

6·7 Theorem *Any directed union of provident sets is provident.*

Ranks of provident sets

If A is an $\varnothing$-provident set, then for $\nu < \varrho(A)$ we have $\nu = \varrho(x)$ for some $x \in A$ and so $\nu \in A$. Thus $\varrho(A) = On \cap A$. Since the function $\nu \mapsto \nu + 1$ is rudimentary, we can deduce that $\varrho(A)$ is a limit ordinal. If A is provident, then since the function $\nu \mapsto \mu + \nu$ is μ-rud rec, $\varrho(A)$ is closed under addition.

6·8 Definition An ordinal θ is *indecomposable* iff for any $\mu, \nu < \theta$ we have $\mu + \nu < \theta$.

6·9 Remark The discussion above shows that the rank of any provident set is an infinite indecomposable ordinal.

6·10 Lemma *A positive ordinal θ is indecomposable iff it is of the form ω^α for some α.*

Proof : For $\mu, \nu < \omega^0$, we have $\mu = \nu = 0$, so $\mu + \nu = 0 < \omega^0$. If $\alpha = \beta + 1$ then for $\mu, \nu < \omega^\alpha$ we can choose $m, n < \omega$ with $\mu \le \omega^\beta.m$ and $\nu \le \omega^\beta.n$, so that $\mu + \nu \le \omega^\beta(m + n) < \omega^\alpha$. If α is a limit, then for $\mu, \nu < \omega^\alpha$ we can choose $\kappa < \alpha$ with $\mu, \nu < \omega^\kappa$ and so $\mu + \nu < \omega^\kappa < \omega^\alpha$.

Conversely, suppose that θ is positive and indecomposable. Let β be minimal such that $\omega^\beta > \theta$. Since exponentiation by ω is continuous, β must be a successor: say $\beta = \alpha + 1$. Now choose $n < \omega$ maximal so that $\omega^\alpha.n \le \theta$. If $n \ne 1$, then the identity $\omega^\alpha.(n-1) + \omega^\alpha = \omega^\alpha.n$ contradicts indecomposability of θ, so we must have $n = 1$. Let $\theta = \omega^\alpha + \gamma$. Since $n = 1$, $\gamma < \omega^\alpha$ and so since θ is indecomposable we must have $\gamma = 0$. Thus $\theta = \omega^\alpha$, as required. ⊣ (6·10)

A typical provident set is $J_{\omega^\nu}(a)$ provided ω^ν is greater than the rank of the transitive set a. But it proves desirable to alter the customary definition of $L(a)$ mentioned in Example 0·5.

Bounding rudimentary functions in a finite progress

6·11 Definition Let ξ be an ordinal or On. A *ξ-progress* is a sequence $\langle P_\nu \mid \nu < \xi \rangle$ of transitive sets such that for each ν with $\nu + 1 < \xi$, $\mathbb{T}(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu \subseteq P_\lambda$; the progress is *strict* if for each ν with $\nu + 1 < \xi$, $P_{\nu+1} \subseteq \mathcal{P}(P_\nu)$; and *continuous* if $P_0 = \varnothing$ and for each limit $\lambda \leqslant \xi$, $P_\lambda = \bigcup_{\nu<\lambda} P_\nu$.

6·12 Proposition *If the progress is strict and continuous then for each $\nu < \xi$, $\varrho(P_\nu) = \nu$.*

Proof : for transitive u, $\varrho(\mathbb{T}(u)) = \varrho(u) + 1 = \varrho(\mathcal{P}(u))$. ⊣ (6·12)

6·13 Theorem *Let R be a rudimentary function of n variables. There is a $c_R \in \omega$ such that for every $(c_R + 1)$-progress $P_0, P_1, \ldots, P_{c_R}$, $R``P_0^n \subseteq P_{c_R}$.*

6·14 DEFINITION We call c_R the *rudimentary constant* of R. For $R : a \mapsto a \cap \{x \mid \models^0 \varphi(x, b)\}$ with φ a $\dot{\Delta}_0$ formula, we also call c_R the *separational delay.*

6·15 REMARK More precisely, there is a recursive function sending a program for R to a bound; but the function sending a program for R to the minimal bound is not recursive.

We prove the theorem in a series of lemmata.

6·16 LEMMA *If x and y are in P_ν then* $\{x, y\} \in P_{\nu+1}$, $x \smallsetminus y \in P_{\nu+1}$, $\bigcup x \in P_{\nu+1}$ *and* $\mathrm{Dom}\,(x) \in P_{\nu+1}$.

Proof : Immediate from lines 2, 3, 4 and 5 of the definition of $\mathbb{T}$. ⊣ (6·16)

6·17 LEMMA $x, y \in P_\zeta \implies x \times y \in P_{\zeta+3}$.

Proof : If x and y are in P_ν then both $\{x\}$ and $\{x, y\}$ are in $P_{\nu+1}$; so $\{\{x\}, \{x, y\}\}$ are in $P_{\nu+2}$; P_ν being transitive, we may infer that if $a \in x$ and $b \in x$, then $(a, b)_2$ is in $P_{\nu+2}$; thus $x \times y \subseteq P_{\nu+2}$, which, since $P_\nu \subseteq P_{\nu+2}$, implies that $x \times y \in P_{\nu+3}$. ⊣ (6·17)

6·18 LEMMA $x, y \in P_\zeta \implies R_5(x, y) \in P_{\zeta+1}$.

6·19 LEMMA $a, b, c \in P_\zeta \implies [(a, c)_2 \in P_{\zeta+2} \;\&\; (b, a, c)_3 \in P_{\zeta+4}]$.

6·20 LEMMA $x \in P_\zeta \implies R_6(x) \in P_{\zeta+5}$.

6·21 LEMMA $x \in P_\zeta \implies R_7(x) \in P_{\zeta+5}$.

6·22 LEMMA $x, w \in P_\zeta \implies x``\{w\} \in P_{\zeta+1}$.

6·23 LEMMA $x, y \in P_\zeta \implies R_8(x, y) \in P_{\zeta+2}$.

Proof of Theorem 6·13: The lemmata show that for $i = 0, \dots 8$, we may take c_{R_i} to be 1, 1, 1, 1, 3, 1, 5, 5, 2 respectively. The theorem now follows by remarking that if S and T_i are rudimentary and for all x, $Q(\vec{x}) = S(T_0(\vec{x}), \dots, T_k(\vec{x}))$, we may take $c_Q = c_S + \max_i c_{T_i}$. ⊣ (6·13)

6·24 COROLLARY *If $\langle P_\nu \mid \nu < \xi\rangle$ is a ξ-progress, then at each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu$ is rud closed.*

The canonical progress towards a given transitive set

6·25 Let c be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$, which we shall use tbelow as a direct recursive definition.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using c as a parameter we define a sequence of pairs $((c_\nu, P^c_\nu))_\nu$ by a rud recursion on ν. Each P^c_ν will be of rank ν; we shall use the function $\mathbb{T}$, but we shall also "feed" stages of c into the process.

The sequence $(P^c_\nu)_\nu$ forms a strict continuous progress; such is the importance of this definition in the sequel that we shall call it the *canonical progress* towards, to, or through c, the choice of preposition depending on the length of the sequence as compared to the rank of c.

6·26 DEFINITION

$$\begin{array}{lllllllll} c_0 & = & \varnothing & \quad c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & \quad c_\lambda & = & \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 & = & \varnothing & \quad P^c_{\nu+1} & = & \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1} & \quad P^c_\lambda & = & \bigcup_{\nu<\lambda} P^c_\nu \end{array}$$

6·27 LEMMA *Each P^c_ν is transitive. $P^c_\nu \subseteq P^c_{\nu+1}$. $P^c_\nu \in P^c_{\nu+1}$; and so for $\nu < \zeta$, $P^c_\nu \subseteq P^c_\zeta$ and $P^c_\nu \in P^c_\zeta$.*

6·28 REMARK $c_\nu = c \cap P^c_\nu$; $\varrho(P^c_\nu) = \nu$.

6·29 REMARK P^c_ν may be defined by a single rud recursion on ordinals:

$$P^c_0 = \varnothing; \qquad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\}); \qquad P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

With that definition, one should then verify by induction that for each ν, $c \cap P^c_\nu = c \cap \{x \mid \varrho(x) < \nu\}$, and thence that the two definitions agree.

6·30 REMARK Each P^c_λ is rud closed, for λ a limit ordinal, by Theorem 6·13.

6·31 Remark $P^c_\omega = V_\omega$: for each $P^c_n \subseteq V_n$ and so $P^c_\omega \subseteq V_\omega$; equality will follow from the fact that P^c_ω is a non-empty rud closed set, by the previous remark.

Bounding rudimentary recursive functions in a progress

To see how quickly progresses tend to become closed under p-rud rec functions, we must recall the notion of an F-attempt.

6·32 Definition Let F be the p-rud rec function defined by $F(x) = G(p, F \restriction x)$. A set f is an *F-attempt* iff it satisfies

$$Fn(f) \ \& \ \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \ \& \ \forall x {\in} \mathrm{Dom}\,(f) \ f(x) = G(p, f \restriction x).$$

We say that an F-attempt f *attains* x iff $x \in \mathrm{Dom}\,(f)$.

Note that that is Δ_0 in f and p; we will denote the separational delay of that predicate by s_F.

6·33 Proposition *Let F be a p-rud rec function. Then there is a natural number c_F such that for any set x and any $(c_F + 1)$-progress $P_0, P_1, \ldots, P_{c_F}$ such that P_0 contains p and x and contains, for each $y \in x$, an F-attempt attaining y, the set P_{c_F} contains an F-attempt attaining x.*

Proof : Given such a progress, we have that $f_0 = \bigcup\{f \in P_0 \mid f$ is an F-attempt$\}$ is an F-attempt attaining every $y \in x$ and contained in P_{s_F+2}. Now $F \restriction x = f_0 \restriction x$, and so $f_0 \cup \{(G(p, f_0 \restriction x), x)_2\}$ is an F-attempt attaining x. It is therefore enough to take $c_F = s_F + c_R + 2$, where R is the rudimentary function $(x, p, f) \mapsto f \cup \{(G(p, f \restriction x), x)_2\}$. ⊣ (6·33)

6·34 Theorem *Let F be a p-rud rec function, x a set, and $\langle P_\nu \mid \nu < \xi \rangle$ a ξ-progress with $\xi > c_F \cdot (\varrho(x)+1)$, and p and x in P_0. Then $P_{c_F \cdot (\varrho(x)+1)}$ contains an F-attempt attaining x.*

Proof : By induction on $\varrho(x)$. For each $y_0 \in x$, we have $\varrho(y_0) < \varrho(x)$ and so $P_{\bigcup_{y \in x} c_F \cdot (\varrho(y)+1)}$ contains an F-attempt attaining y_0. Thus $P_{\bigcup_{y \in x} c_F . (\varrho(y)+1) + c_F}$ contains an F-attempt attaining x, which is the desired result, as $\bigcup_{y \in x} c_F . (\varrho(y) + 1) + c_F = c_F . (\bigcup_{y \in x} (\varrho(y) + 1) + 1) = c_F . (\varrho(x) + 1)$. ⊣ (6·34)

6·35 Theorem *Let $\langle P_\nu \mid \nu \leqslant \theta \rangle$ a strict continuous $(\theta+1)$-progress. Then P_θ is provident iff θ is an infinite indecomposable ordinal.*

Proof : The 'only if' direction is immediate from Remark 6·9. For the 'if' direction, let $x, p \in P_\theta$; choose $\nu < \theta$ with $x, p \in P_\nu$. Let F be p-rud rec. Then $\varrho(x) < \nu$ and so $F(x) \in P_{\nu + c_F \cdot \nu} \subseteq P_\theta$. ⊣ (6·35)

In fact, by an identical induction to that above, we can get a slightly sharper bound on how soon these attempts appear in a strict continuous progress.

6·36 Theorem *Let $\langle P_\mu \mid \mu < \xi \rangle$ be a strict continuous ξ-progress with $\kappa + c_F . \nu < \xi$, and let $p \in P_\kappa$ and $x \in P_\nu$. Then there is an F-attempt attaining x in $P_{\kappa + c_F \cdot \nu}$.*

6·37 Theorem *Let $\langle P_\nu \mid \nu \leq \theta \rangle$ a strict continuous $(\theta+1)$-progress. Then P_θ is $\varnothing$-provident iff θ is a limit ordinal.*

Proof : Like that of Theorem 6·35, but using the better bound of Theorem 6·36 ⊣ (6·37)

In fact, something a little more general is true.

6·38 Theorem *Let $\langle P_i \mid i < \omega \rangle$ a strict ω-progress with P_0 p-provident. Then $\bigcup_{i<\omega} P_i$ is also p-provident.*

Proof : Let F be p-rud rec. As in Proposition 4·15, $x \mapsto F \restriction \mathrm{tcl}\{x\}$ is p-gentle and so for any $x \in P_0$ there is an F-attempt attaining x in P_0. Then by induction on i, with $i < \omega$, using Proposition 6·33, we obtain that for any $x \in P_i$ there is an F-attempt attaining x in $P_{c_F . i}$, and in particular that $F(x) \in P_{c_F . i}$. ⊣ (6·38)

6·39 Proposition *Let c be a transitive set and θ an infinite indecomposable ordinal. Then*

$$P^c_\theta = P^{c_\theta}_\theta = \bigcup_{\lambda < \theta} P^{c_\lambda}_\theta.$$

Proof : If $x \in P^c_\theta$, then for some $\lambda < \theta$, $x \in P^c_\lambda = P^{c_\lambda}_\lambda \subseteq P^{c_\lambda}_\theta$.

Conversely, if $\lambda < \theta$, c_λ is in P^c_θ, which we know to be provident, and the map $\nu \mapsto P^{c_\lambda}_\nu$ is given by a c_λ-rudimentary recursion, and so each $P^{c_\lambda}_\nu$, for $\nu < \theta$, is in P^c_θ; thus $P^{c_\lambda}_\theta \subseteq P^c_\theta$. ⊣ (6·39)

6·40 PROPOSITION *A transitive set A is provident iff it contains the graph of the restriction of F to $X \times X$ for any $X \in A$ and any F which is recursive of Type III.*

Proof : It is clear that if A contains all these graphs then it is provident.

Conversely, suppose A is provident of rank θ and $X \in A$. If $A = \mathbf{HF}$, the result is clear, so we assume $\theta > \omega$. Let F be defined by $F(v, x) = G(v, F \restriction (\{v\} \times x))$. Then for each $v \in X$ we have by Theorem 6·36 that there is an $F(v, -)$-attempt attaining X in $P^X_{\varrho(X).2+\omega}$. The graph in question is then given by

$$[P^X_{\varrho(x).2+\omega} \times (X \times X)] \cap \bigcup \{(y, (v, x)_2)_2 \mid \exists f_{\in P^X_{\varrho(x).2+\omega}}\ f \text{ is an } F(v, -)\text{-attempt attaining } x \ \&\ f(x) = y\}$$

⊣ (6·40)

Iterated recursion

In fact, we can obtain similar bounds on the growth of functions obtained by recursing rud rec functions, or by recursing functions obtained in that way, etc. More precisely:

6·41 DEFINITION A unary class function $F : V \to V$ is p-rud $[\mathrm{rec}]^0$ iff it is rud. F is p-rud $[\mathrm{rec}]^{n+1}$ iff there is a p-rud $[\mathrm{rec}]^n$ function G such that for all x we have $F(x) = G((p, F \restriction x)_2)$. F is p-rud $[\mathrm{rec}]^{<\omega}$ iff it is p-rud $[\mathrm{rec}]^n$ for some $n < \omega$.

Thus F is p-rud $[\mathrm{rec}]^1$ iff it is p-rud rec.

6·42 REMARK That is more powerful than rudimentary recursion, but it is still fairly weak. For example, as we shall see in Corollary 6·53, for no p is $\nu \mapsto \nu + \omega$ p-rud $[\mathrm{rec}]^{<\omega}$, in contrast to the fact that $\nu \mapsto \alpha + \nu$ is α-rud rec for each α.

6·43 REMARK Provident sets need not be closed under p-rud $[\mathrm{rec}]^n$ functions for $n > 1$. For example, the ordinal function $x \mapsto \omega + x$ is ω-rud rec, and so the ordinal function $F : x \mapsto \omega^2 \cap (\omega.x)$ obtained from it by recursion is ω-rud $[\mathrm{rec}]^2$. But $P^\varnothing_{\omega^2}$ is not closed under F, since $F(\omega) = \omega^2$.

However, we shall find that to check whether a provident set is closed under such recursions it is enough to know the rank of that provident set. To prove that, we shall consider bounds on the growth of such functions; and for that we must first consider a notion of limitation for ordinal functions.

6·44 DEFINITION For λ an ordinal, we say that an ordinal function $l : On \to On$ is *λ-restrained* iff for all ordinals ν we have $l(\nu) < \lambda + \nu + \omega$.

6·45 LEMMA *If l is λ-restrained, then it is λ'-restrained for any $\lambda' \geq \lambda$.*

6·46 LEMMA *If l_1 and l_2 are λ-restrained, then so is $\nu \mapsto l_1(\nu) \cup l_2(\nu)$.*

6·47 LEMMA *If l_1 is λ_1-restrained and l_2 is λ_2-restrained then $l_1 \circ l_2$ is $\lambda_1 + \lambda_2$-restrained.*

Proof : For any ν, we can pick $m \in \omega$ with $l_2(\nu) \leq \lambda_2 + \nu + m$ and so $l_1(l_2(\nu)) < \lambda_1 + l_2(\nu) + \omega \leq \lambda_1 + \lambda_2 + \nu + m + \omega = \lambda_1 + \lambda_2 + \nu + \omega$. ⊣ (6·47)

6·48 LEMMA *If l is λ-restrained and increasing then the function l' defined by $l'(\nu) = l\left(\bigcup_{\mu<\nu} l'(\mu)\right)$ is $\lambda.\omega$-restrained and increasing.*

Proof : l' is clearly increasing. $l'(0) = l(0) < \lambda + \omega \leq \omega.\lambda + 0 + \omega$. For ν a successor, say $\nu = \mu + 1$, we can pick $m < \omega$ such that $l'(\mu) \leq \lambda.\omega + \mu + m$, so that $l'(\nu) = l(l'(\mu)) \leq l(\lambda.\omega + \mu + m) < \lambda + \lambda.\omega + \mu + m + \omega = \lambda.\omega + \nu + \omega$. Finally, for ν a limit, for every $\mu < \nu$ we have $l'(\mu) < \lambda.\omega + \mu + \omega \leq \lambda.\omega + \nu$ and so $\bigcup_{\mu<\nu} l'(\mu) \leq \lambda.\omega + \nu$, and so $l'(\nu) \leq l(\lambda + \nu) < \lambda + \lambda.\omega + \nu + \omega = \lambda.\omega + \nu + \omega$. ⊣ (6·48)

6·49 DEFINITION For ordinals κ and λ and a set p, a unary class function F is (p, κ, λ)-restrained if there are an increasing λ-restrained ordinal function l and a rudimentary function H such that for any strict continuous progress P with $p \in P_\kappa$ and $x \in P_\nu$ and any $\alpha \geq l(\nu)$ we have $H(p, P_\alpha, x) = F(x)$.

That notion is designed to make the following true:

6·50 LEMMA *Any p-rud rec function is (p, κ, κ)-restrained for every κ.*

Proof : Let $H : p, P, x \mapsto [\bigcup\{f \in P \mid f \text{ is an } F\text{-attempt}\}](x)$. Let $l : \nu \mapsto \kappa + c_F.\nu$. By Proposition 6·12 and Theorem 6·36, if P, ν and x are as in Definition 6·49, and $\alpha \geq l(\nu)$, then P_α contains an F-attempt attaining x, and so $H(p, P_\alpha, x) = F(x)$. ⊣ (6·50)

We now mimic the argument used to obtain Theorem 6·36, to show that functions obtained by recursion from restrained functions are still restrained.

6·51 THEOREM (Bowler) *Suppose that F is defined by $F(x) = G((p, F \restriction x)_2)$, where G is (p, κ, λ)-restrained. Then F is $(p, \kappa, (\lambda + \kappa).\omega)$-restrained.*

Proof : Let H and l witness the fact that G is (p, κ, λ)-restrained, as in Definition 6·49. We say that f is an *F-attempt using P* if

$$Fn(f) \;\&\; \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \;\&\; \forall y_{\in \mathrm{Dom}\,(f)}\, f(y) = H(p, P, (p, f \restriction y)_2).$$

We shall refer to this Δ_0 formula again, so we denote it by $A(p, P, f)$. Let $K : p, P, x \mapsto [\bigcup\{f \in P \mid A(p, P, f)\}](x)$. We say that x is *attained by P* iff there is some $f \in P$ such that $A(p, P, f)$ and $x \in \mathrm{Dom}\,(f)$, and for every $y \in \mathrm{tcl}\{x\}$ we have $H(p, P, (p, f \restriction y)_2) = G((p, f \restriction y)_2)$. Thus if x is attained by P then $K(p, P, x) = F(x)$.

Next we define a sequence of variously restrained ordinal functions which will help us restrain the growth of F. Let $l_1 : \nu \mapsto \kappa + \nu + c_{R_1} + 1$, where R_1 is the rudimentary function $P, p \mapsto \bigcup\{f \in P \mid A(p, P, f)\}$. Let $l_2 : \nu \mapsto l_1(\nu) + c_{R_2}$, where R_2 is the rudimentary function $p, f, x \mapsto (p, f \restriction x)_2$. Let $l_3 = l \circ l_2$. Let $l_4 : \nu \mapsto l_3(\nu) + c_{R_5}$, where R_4 is the rudimentary function $p, P, f, x \mapsto H(p, P, (p, f \restriction x)_2)$. Let $l_5 : \nu \mapsto (l_4(\nu) \cup c_{\mathrm{tcl}}.\nu) + c_{R_5}$, where R_5 is the rudimentary function $f, t, v, x \mapsto f \restriction t \cup \{(v, x)_2\}$. Finally, let l_6 be defined by $l_6(\nu) = l_5(\bigcup_{\mu<\nu} l_6(\mu))$. Each of the l_i is increasing. Evidently l_1 and l_2 are κ-restrained. Thus by Lemma 6.40, l_3 is $\lambda + \kappa$-restrained and therefore so are l_4 and l_5. Therefore by Lemma 6.41, l_6 is $(\lambda + \kappa).\omega$-restrained.

Now suppose that we have a strictly continuous progress P with $p \in P_\kappa$. We shall show by induction on ν that for $x \in P_\nu$ and $\alpha \geq l_6(\nu)$ x is attained by P_α. Let

$$f_0 = \bigcup\{f \in P_{\bigcup_{\mu<\nu} l_6(\mu)} \mid A(p, P_{\cup_{\mu<\nu} l_6(\mu)}, f)\},$$

which is in $P_{l_1(\bigcup_{\mu<\nu} l_6(\mu))}$. By our induction hypothesis, for any $y \in x$ we have $f_0(y) = K(p, P_{\bigcup_{\mu<\nu} l_6(\mu)}, y) = F(y)$. Thus $(p, F \restriction x)_2 = (p, f_0 \restriction x)_2 \in P_{l_2(\bigcup_{\mu<\nu} l_6(\mu))}$. Therefore for $\alpha \geq l_3\left(\bigcup_{\mu<\nu} l_6(\mu)\right)$, $F(x) = G((p, F \restriction x)_2) = H(p, P_\alpha, (p, f_0 \restriction x)_2)$, and in particular $F(x) \in P_{l_4(\bigcup_{\mu<\nu} l_6(\mu))}$. Thus $f_1 = f_0 \restriction \mathrm{tcl}(x) \cup \{(F(x), x)_2\} \in P_{l_5(\bigcup_{\mu<\nu} l_6(\mu))} = P_{l_6(\nu)}$ and $A(p, P, f_1)$, so x is attained by any $\alpha \geq l_6(\nu)$.

Thus K and l_6 witness that F is $(p, \kappa, (\lambda + \kappa).\omega)$-restrained. ⊣ (6·51)

6·52 THEOREM *Any p-rud [rec]^n function is $(p, \kappa, \kappa.(\omega^{n-1} + n - 1))$-restrained for every κ.*

Proof : By induction on n. The base case follows from Lemma 6·50, and for the induction step let F be p-rud [rec]$^{n+1}$, given by $F(x) = G((p, F \restriction x)_2)$ for some p-rud [rec]^n function G. Then by the induction hypothesis G is $(p, \kappa, \kappa.(\omega^{n-1}+n-1))$-restrained, and so by Theorem 6·51 F is $(p, \kappa, (\kappa.(\omega^{n-1}+n-1)+\kappa).\omega))$-restrained, which is the desired result as $(\kappa.(\omega^{n-1} + n - 1) + \kappa).\omega = \kappa.(\omega^{n-1} + n).\omega \leq \kappa.(\omega^n + n)$. ⊣ (6·52)

6·53 COROLLARY *For no p is $F : \nu \mapsto \nu + \omega$ is p-rud [rec]$^{<\omega}$.*

Proof : Suppose for a contradiction that F is p-rud [rec]^n for some p and n. Let $c = \mathrm{tcl}\{p\}$, $\kappa = \varrho(p) + 1$ and $\lambda = \kappa.(\omega^{n-1} + n - 1)$. Then F is (p, κ, λ)-restrained: let l and H witness that. As $p \in P^c_\kappa$, we must have $\lambda.\omega + \omega = F(\lambda.\omega) \in P^c_{l(\lambda.\omega)+c_H}$, which is the desired contradiction as $\varrho(P^c_{l(\lambda.\omega)+c_H}) = l(\lambda.\omega) + c_H < \lambda + \lambda.\omega + \omega = \lambda.\omega + \omega$.

6·54 THEOREM (Bowler) *Let A be a provident set other than* **HF**. *The following are equivalent:*

i. $\varrho(A)$ is of the form ω^α for some ordinal α which is a limit.

ii. A is closed under p-rud [rec]^2 functions for $p \in A$.

iii. A is closed under p-rud [rec]$^{<\omega}$ functions for $p \in A$.

6·55 DEFINITION We call such sets *limit provident sets.*

Proof : It is clear that $(iii) \Rightarrow (ii)$. To see that $(ii) \Rightarrow (i)$, we know by Remark 6·9 and Lemma 6·10 that there is some $\alpha > 0$ with $\varrho(A) = \omega^\alpha$. Suppose for a contradiction that α is a successor ordinal, say $\alpha = \beta + 1$. Then $\nu \mapsto \omega^\beta + \nu$ is ω^β-rud rec and so $F : \nu \mapsto \omega^\alpha \cap \omega^\beta.\nu$ is ω^β-rud $[\mathrm{rec}]^2$, which contradicts (ii), as $\omega \in A$ and $\omega^\beta \in A$ but $F(\omega) = \omega^\alpha \notin A$.

For $(i) \Rightarrow (iii)$, let $p, x \in A$ and let F be p-rud $[\mathrm{rec}]^n$. Let $\kappa = \varrho(\{p, x\})$. By Theorem 6.45, F is $(p, \kappa, \kappa.(\omega^{n-1}+n-1))$-restrained: let H and l witness this. We have $x, p \in P_\kappa^{\{p,x\}}$, and so $F(x) \in P^{\{p,x\}}_{l(\kappa)+c_H+1}$. Since $\kappa < \omega^\alpha$, we can pick some $\beta < \alpha$ with $\kappa \leq \omega^\beta$, and so $l(\kappa)+c_H+1 \leq \kappa.(\omega^{n-1}+n-1)+\kappa+\omega+c_H+1 \leq \omega^\beta(\omega^{n-1}.\omega) = \omega^{\beta+n} < \omega^\alpha$ and so, since A is provident, $P^{\{p,x\}}_{l(\kappa)+c_H+1} \in A$ and so $F(x) \in A$ as required.

$\dashv$ (6·55)

6·56 THEOREM *Any directed union of limit provident sets is limit provident.*

Proof : The rank of such a directed union is a union of ordinals of the form ω^α with α limit, and so is of the same form. $\dashv$ (6·56)

6·57 THEOREM *Let $\langle P_\nu \mid \nu \leqslant \theta \rangle$ a strict continuous $(\theta + 1)$-progress. Then P_θ is limit provident iff θ is of the form ω^α with α a limit ordinal.*

Proof : Immediate from the definition, Theorem 6·35 and Proposition 6·12. $\dashv$ (6·57)

The constructible hierarchy

Since the L_ν in the constructible hierarchy form a strict continuous progress, we know by Theorems 6.35, 6.37 and 6.57 that L_ν is $\varnothing$-provident iff ν is a limit, provident iff ν is infinite and indecomposable, and limit provident iff ν is of the form ω^α with α a limit ordinal. We can say something similar about the Jensen hierarchy, since it is clear by induction that $J_\nu = P^{\varnothing}_{\omega.\nu}$ for any ν. Thus (since the function $\nu \mapsto \omega.\nu$ is injective), J_ν is $\varnothing$-provident iff $\nu > 0$, provident iff ν is nonzero and indecomposable, and limit provident iff ν is of the form ω^α with α a limit ordinal.

The situation is only marginally less pleasant for hierarchies relative to a parameter. Thus, since $\varrho(L_\nu(A)) = \varrho(A)+1+\nu$, we have by Theorem 6.34 that $L_\nu(A)$ is provident iff ν is an infinite indecomposable ordinal greater than $\varrho(A)$, and is limit provident iff ν is greater than $\varrho(A)$ and is of the form ω^α with α a limit. An induction argument using Theorem 6.38 then shows that $L_\nu(A)$ is $\varnothing$-provident for ν a limit ordinal at least as big as the least indecomposable ordinal greater than $\varrho(A)$. However, there is no reason to expect $L_\nu(A)$ to be $\varnothing$-provident any earlier, as we shall soon illustrate. The $L_\nu[A]$ behave even better: since they form a strict continuous progress, the same remarks apply to them as to the L_ν.

Before discussing the Jensen hierarchy, we construct a refinement: we will not discuss the various other refinements of this hierarchy here.

6·58 Definition Let c be a transitive set; define

$$T_0(c) = c;\ T_{\nu+1}(c) = \mathbb{T}(T_\nu(c));\ T_\lambda(c) = \bigcup_{\nu<\lambda} T_\nu(c);\ L(c) = \bigcup_{\nu} T_\nu(c).$$

Since $\varrho(T_\nu(c)) = \varrho(c)+\nu$, we have by Theorem 6.34 that $T_\nu(A)$ is provident iff ν is an infinite indecomposable ordinal greater than $\varrho(c)$, and is limit provident iff ν is greater than $\varrho(c)$ and is of the form ω^α with α a limit. An induction argument using Theorem 6.38 then shows that $T_\nu(A)$ is $\varnothing$-provident for ν a limit ordinal at least as big as the least indecomposable ordinal greater than $\varrho(A)$. Again, there is no reason to expect $T_\nu(A)$ to be $\varnothing$-provident any earlier. We therefore prefer the strict continuous progress P^c_ν to $T_\nu(c)$. With the T^c_ν in hand, we turn to the Jensen hierarchy: by an induction argument, for any ordinal ν we have $J_\nu(c) = T_{\omega.\nu}(c)$. Thus $J_\nu(c)$ is provident iff ν is nonzero and indecomposable with $\omega.\nu > \varrho(c)$. It is limit provident iff ν is bigger than $\varrho(c)$ and of the form ω^α with α a limit ordinal. Finally, $J_\nu(c)$ is $\varnothing$-provident if $\nu > 0$ and $\omega.\nu$ is at least as big as the least indecomposable ordinal greater than $\varrho(c)$, but just as before there is no reason to expect it to be $\varnothing$-provident before this.

6·59 Example Let D be the set $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, ...\}$ of Zermelo natural numbers. Then, by induction on ν, $L_\nu(D) \cap ON \subseteq 2+\nu$, whereas $\varrho(L_\nu(D)) = \omega+1+\nu$. Thus $L_\nu(D)$ cannot be closed under the rank function for $\nu < \omega^2$, and in particular $L_\nu(D)$ is $\varnothing$-provident only at limit ordinals at least as big as ω^2. A similar argument shows that $J_\nu(D)$ is $\varnothing$-provident only for $\nu \geqslant \omega$.

Provident closures

7·0 THEOREM *Suppose that M is a non-empty set. Let θ be the least indecomposable ordinal not less than $\varrho(M)$. Set*

$$Prov(M) =_{\rm df} \bigcup\{P_\theta^{\mathrm{tcl}(s)} \mid s \in \mathcal{S}(M)\}.$$

Then $Prov(M)$ is provident and includes M, and if P is any other such, $Prov(M) \subseteq P$.

7·1 DEFINITION We call $\mathrm{Prov}(M)$ the *provident closure* of M.

Proof : $\mathrm{Prov}(M)$ is provident by Theorems 6.7 and 6.35. Suppose that P is provident and $M \subseteq P$. Then $\mathcal{S}(M) \subseteq P$; $\theta \leqslant On \cap P$; for each $s \in \mathcal{S}(M)$, $\mathrm{tcl}(s) \in P$, and for $\nu < \theta$, $P_\nu^{\mathrm{tcl}(s)} \in P$, and so $\mathrm{Prov}(M) \subseteq P$. ⊣ (7·1)

7·2 THEOREM *Suppose that M is a non-empty set. Let θ be the least ordinal not less than $\varrho(M)$ and of the form ω^α with α a limit ordinal. Set*

$$LProv(M) =_{\rm df} \bigcup\{P_\theta^{\mathrm{tcl}(s)} \mid s \in \mathcal{S}(M)\}.$$

Then $LProv(M)$ is limit provident and includes M, and if P is any other such, $LProv(M) \subseteq P$.

7·3 DEFINITION We call $\mathrm{LProv}(M)$ the *limit provident closure of M.*

The theories PROV, PROVI and LPROV

Theorem 7·0 implies that there is a finitely axiomatisable set theory (which we call PROV) of which the transitive models are the provident sets.

Let PROV be the following axioms

(7·3·0) extensionality

(7·3·1) the ten axioms of GJ_0, as given in Section 1.

$\varnothing \in V$	$\bigcup x \in V$	$a \cap \{(x,y)_2 \mid x \in y\} \in V$
$\{x,\, y\} \in V$	$\mathrm{Dom}\,(x) \in V$	$\{(y,x,z)_3 \mid (x,y,z)_3 \in b\} \in V$
$x \smallsetminus y \in V$	$x \times y \in V$	$\{(y,z,x)_3 \mid (x,y,z)_3 \in c\} \in V$

(R_8) $$\{x``\{w\} \mid w \in y\} \in V$$

(7·3·2) `each set is in the domain of an attempt at the rank function`;

(which implies both TCo and set foundation)

(7·3·3) `any two ordinals are in the domain of an attempt at ordinal addition`

(7·3·4) `for each transitive` $\mathfrak{c}$ `each ordinal is in the domain of an attempt`
`at the sequence` $\langle \mathsf{P}_\nu^{\mathfrak{c}} \mid \nu \mathrel{\epsilon} ON\rangle$;

We write PROVI for PROV $+\ \omega \in V$.

Theorem 7.0 will suffice to prove that the transitive models of PROV are the provident sets; the reasoning in this paper has been mainly semantic, but experience of the weak systems in [M3] suggest that if one wished to use PROV for syntactical reasoning, it would be desirable to enhance it by adding the axiom of infinity and the scheme of Π_1 foundation. With a little extra work one could show that that too is finitely axiomatisable, by using the predicate $\stackrel{0}{\models}$.

We write LPROV for the theory obtained from PROVI by adding the axiom

(7·3·1) `for each ordinal` α, ω `is in the domain of an attempt at the function` F `given by the recursion` $F(\nu) = \alpha + \bigcup_{\mu<\nu} F(\nu)$

Note that a formal statement of this axiom should include the postulation of a sufficiently long attempt at the function $\nu \mapsto \alpha + \nu$.

Then the transitive models of LPROV are the limit provident sets.

The theory $\varnothing$-PROV

Next, we will obtain a finite theory whose transitive models are the $\varnothing$-provident sets. For this, we will present a finite collection of rud rec functions which capture all rud rec functions, in the same sense that the canonical progresses (P^c_ν) capture all parametrised rud rec functions.

7·4 DEFINITION Let $Y_1 : x \mapsto \{Y_1(y) \mid y \in x\} \cup \{x\}$, $Y_2 : x \mapsto \{\{Y_2(y)\} \mid y \in x\} \cup x$ and $Y_3 : x \mapsto \bigcup_{y\in x} \mathbb{T}(Y_3(y)) \cup x$.

Each of these is rud rec, and $Y_3(x)$ is transitive for any x.

7·5 LEMMA *For any $n < \omega$ and any x we have $x \in Y_2^n(Y_1(x))$.*

Proof : By induction on n. The case $n = 0$ is immediate from the definition of Y_1, and for the induction step, for any x we have $x \in Y_2^n(Y_1(x)) \subseteq Y_2(Y_2^n(Y_1(x)))$. ⊣ (7·5)

7·6 DEFINITION We define the relations $\in_n$ for each $n < \omega$ by $x \in_0 y$ iff $x = y$, and $x \in_{n+1} y$ iff there is z with $x \in_n z \in y$. Thus $y \in_n x$ iff there is a sequence $y = y_0 \in y_1 \in y_2 \in ... \in y_n = x$.

7·7 LEMMA *For any $n < \omega$ and any $y \in x$ we have $Y_2^n(y) \in_{2^n} Y_2^n(x)$.*

Proof : By induction on n. The case $n = 0$ holds by definition. For the induction step, we have $Y_2(y) \in \{Y_2(y)\} \in Y_2(x)$ and so by the induction hypothesis $Y_2^n(Y_2(y)) \in_{2^n} Y_2^n(\{Y_2(y)\}) \in_{2^n} Y_2^n(Y_2(x))$, so that $Y_2^{n+1}(y) \in_{2^{n+1}} Y_2^{n+1}(x)$. ⊣ (7·7)

7·8 THEOREM (Bowler) *Let F be rud rec, and let $n < \omega$ with $2^n \geq c_F$. Then for any x, and any $y \in x$, $Y_3(Y_2^n(Y_1(x)))$ contains an F-attempt attaining y.*

Proof : By induction on $\varrho(x)$. Let $y \in x$. Then $Y_1(y) \in Y_1(x)$ and so by Lemma 7.7 $Y_2^n(Y_1(y)) \in_{2^n} Y_2^n(Y_1(x))$ and so we can find a sequence $Y_2^n(Y_1(y)) = y_0 \in y_1 \in \ldots \in y_{2^n} = Y_2^n(Y_1(x))$. Define a $2^n + 1$-progress P by $P_i = Y_3(y_i)$. By the induction hypothesis P_0 contains, for each $z \in y$, an F-attempt attaining z, and by Lemma 7.5 and the definition of Y_3 we have $y \in P_0$. So by Proposition 6.33 we know that P_{c_F} contains an F-attempt attaining y, which F-attempt must then be contained in each P_j with $j \geq c_F$, and in particular in $P_{2^n} = Y_3(Y_2^n(Y_1(x)))$. ⊣ (7·8)

Thus if we let $\varnothing$-PROV be the following axioms then the transitive models of $\varnothing$-PROV are the $\varnothing$-provident sets:

(7·8·0) extensionality

(7·8·1) the ten axioms of GJ_0, as given in Section 1.

(7·8·2) `each set is in the domain of an attempt at each of` Y_1, Y_2 `and` Y_3.

In fact, we need only add a very simple kind of parametrised recursion to obtain a theory equivalent to PROV. The recursive definition of ordinal addition makes sense even if the first input is not an ordinal: for any set x define $x + \beta$ by recursion on the ordinal β, as $x + 0 = x$ and $x + \beta = \bigcup_{\gamma<\beta}((x + \gamma) \cup \{x + \gamma\})$ for $\beta > 0$. We get a theory whose transitive models are provident sets by adding the following axiom to $\varnothing$-PROV:

(7·8·2) `for any set` x, `each ordinal is in an attempt at the function` $\beta \mapsto x + \beta$.

This works because for any set x, the sequence $(Y_3(x + \beta))_\beta$ is a progress, so we can get any value of any parametrised rud rec function using Theorem 6.34.

8: Models of stunted growth

We have mentioned "Model $\mathbf{M}_{13,\lambda}$" studied in *Weak Systems*, which is supertransitive and a proper class but which contains only the ordinals $< \lambda$; so in that model rank is stunted.

8·0 ASIDE Consider that model in the special case $\lambda = \omega$; ω is not a member of $\mathbf{M}_{13,\omega}$, which is otherwise a model of Z, save for the axiom of infinity in its customary form. But that axiom is not used in defining the finite basis of rudimentary functions; so $\mathbf{M}_{13,\omega}$ is rud closed; and therefore ω is not of the form $F(x)$ for any rud function F and $x \in \mathbf{M}_{13,\omega}$.

That is the promised sketch of the argument for Gandy's Theorem 2.1.3. It also demonstrates the claim in Remark 0·15 that the rank function is not rudimentary. Note that in Model $\mathbf{M}_{13,\lambda}$, TCo holds; by supertransitivity, the actual transitive closure of each member of the model is a member of the model.

We may generalise the idea behind model $\mathbf{M}_{13}$ thus:

8·1 DEFINITION Suppose that $F : On \longrightarrow V$ is a function such that for $\mu < \nu$ we have $F(\mu) \in F(\nu)$. For limit λ, set

$$A_{F,\lambda} =_{\rm df} \big\{u \bigm| \bigcup u \subseteq u \ \&\ \sup\{\nu \in On \mid F(\nu) \in u\} < \lambda\big\}; \quad M_{F,\lambda} = \bigcup A_{F,\lambda}.$$

8·2 PROPOSITION *If there is $\nu < \lambda$ with $F(\nu) \notin \omega$ then $M_{F,\lambda}$ is a supertransitive model of Z for which $F(\xi) \in M_{F,\lambda} \iff \xi < \lambda$. For any F and λ, the model $M_{F,\lambda}$ will be a proper class.*

Proof : as in Section 7 of [M3]. The union of two members of $A_{F,\lambda}$ is in $A_{F,\lambda}$, and if $u \in A_{F,\lambda}$, so is $\mathcal{P}(u)$; so that $M_{F,\lambda}$ will be a supertransitive model of Z. If $F \restriction \lambda$ only takes ordinals as values, the argument in M2 shows that $M_{F,\lambda}$ will contain sets of all ranks. Otherwise, there is some $\eta < \lambda$ such that $F(\xi)$ is not an ordinal for $\xi \geqslant \eta$, and in that case $M_{F,\lambda}$ will contain all ordinals. ⊣ (8·2)

8·3 DEFINITION For limit λ, set $\mathbf{A}_{17,\lambda} =_{\rm df} \big\{u \bigm| \bigcup u \subseteq u \ \&\ \sup\{\nu \mid P^{\varnothing}_{\nu} \in u\} < \lambda\big\}; \quad \mathbf{M}_{17,\lambda} =_{\rm df} \bigcup \mathbf{A}_{17,\lambda}$.

8·4 PROPOSITION $\mathbf{M}_{17,\lambda}$ *is a supertransitive proper class, containing all ordinals but the T hierarchy only up to λ but no further. In this model the rud recursion defining rank is total but that defining the growth of the Jensen auxiliary hierarchy stops prematurely.*

8·5 PROPOSITION *There is a supertransitive class model $\mathbf{M}_{18,\lambda}$ of* Z *which contains a Cohen generic real c, and all constructible sets, but such that neither $L_{\omega+\omega}(c)$ nor $P^c_{\omega+\omega}$ is in $\mathbf{M}_{18,\lambda}$.*

Proof : This time take $\lambda = \omega + \omega$ and $F(\zeta) = P^c_\zeta$ and $\mathbf{M}_{18,\lambda} = M_{F,\lambda}$. $c \in P^c_\zeta$ whenever $\zeta \geqslant \omega + 1$, so that each $L_\eta \in A_{F,\lambda}$ and $L \subseteq M_{F,\lambda}$. ⊣ (8·5)

In the above model the Jensen hierarchy exists for all ordinals, but the same hierarchy relativised to c is defined before but not at level $\omega + \omega$.

8·6 REMARK We have seen that in the model $\mathbf{K}$, which should have been called $\mathbf{M}_{16}$, of section 12 of [M3], the definition of tcl is stunted, and therefore also the definition of rank, for if every set is a member of the domain of some attempt at ϱ, that domain will be a transitive set; so TCo holds, and hence tcl may be recovered using the full strength of the axioms of Z.

$\mathbf{M}_{13}$ is a model of ZC in which rank is stunted but tcl not; $\mathbf{M}_{17}$ is a model of ZC in which the Jensen hierarchy is stunted but tcl and rank not; $\mathbf{M}_{18}$ is a model of ZC in which the relative Gödel and Jensen hierarchies $L_\nu(c)$ and $J_\nu(c)$ are stunted but the hierarchies L_ν and J_ν and tcl and rank are not. So there is a certain ordering to some rudimentary recursions; but we have seen in Section 7 that there is, in a sense, a finite basis to the collection of rud recursions.

Failure of Scott's trick in a model of Zermelo

We record here another variant of the above construction.

8·7 DEFINITION Let $A = \leqslant_R$ be a well-ordering, viewed as a binary relation $\leqslant_R$ on the set $\{x \mid (x,x)_2 \in A\}$. For such A, define $I(A)$ to be the class of well-orderings isomorphic to A, and, in imitation of Scott's celebrated trick for reducing equivalence classes to equivalence sets, let $ST(A)$ be the class $\{B \in I(A) \mid \forall C_{\in I(A)}\, \varrho(C) \geqslant \varrho(B)\}$, the class of wellorderings of minimal rank isomorphic to A.

The following shows that Z is too weak a set theory for Scott's trick to work.

8·8 THEOREM *Let $\kappa = \beth_\kappa$ be a beth fixed point. Let A_κ be the epsilon relation restricted to κ; thus a well-ordering of length κ. There is a supertransitive, proper class, model $\mathbf{M}_{19}$ of* Z *containing all ordinals and the well-ordering A_κ, in which every set has a rank, but in which $(ST(A_\kappa))^{\mathbf{M}_{19}}$, though a definable class of the model, is not a set.*

Proof : Take $F(\nu) = V_\nu$ and $\mathbf{M}_{19} = M_{F,\kappa}$. $V_\nu \in \mathbf{M}_{19} \iff \nu < \kappa$. As κ is a beth fixed point, $V_\kappa = H_\kappa$, so that all well-orderings of length κ in the universe must be of rank at least κ. Thus $A_\kappa \in I(A_\kappa)$. Let B_ξ be the well-ordering $\{(b_\nu, b_\xi)_2 \mid_{\nu,\xi} \nu \leqslant \xi < \kappa\}$ where for $\zeta \leqslant \xi$, $b_\zeta = V_\zeta$, and for $\xi < \zeta < \kappa$, $b_\zeta = \zeta$.

Then each $B_\xi \in \mathbf{M}_{19}$, being obtained from $V_{\xi+1}$ and A_κ by rudimentary operations. Further each B_ξ is of rank κ, and thus is in $ST(A_\kappa)$, even as defined in $\mathbf{M}_{19}$. Thus the class $ST(A_\kappa)$ cannot be a set in $\mathbf{M}_{19}$, as $\bigcup^4 ST(A_\kappa) = V_\kappa$.

9: The truth predicate for $\dot\Delta_0$ sentences

In [M3, section 10, culminating in page 211] it is shown that the truth predicate $\models_u \varphi$ is, provably in MW, Δ_1-definable.

It was promised in [M3] that the proof of Devlin VI.1.14 would be reworked in the present paper. We shall instead present a proof of the following sharper result.

9·0 Theorem (Mathias) *Truth for $\dot\Delta_0$ sentences is uniformly Δ_1 for transitive models of* MW.

Proof : We must begin by introducing some notation for $\dot\Delta_0$ formulae, in order to maintain the distinction between formulae in the object language and those in the language of discourse. Thus we denote conjunction, disjunction, and negation in the object language by $\wedge$, $\vee$ and $\neg$ rather than & , $\vee$ and $\neg$. We denote unversal and existential quantification by $\bigwedge$ and $\bigvee$, and we denote the equality and containment relations by $=$ and ϵ. We shall typically denote variables in the object language by lowercase letters in the Fraktur font, such as $\mathfrak{x}$ or $\mathfrak{y}$, and formulae in the object language with variant forms of lowercase greek letters, such as ϑ or φ. The notation for restricted quantifiers in the object language is also new: for example, instead of $\forall x_{\in y}$, we write $\bigwedge\mathfrak{x}\colon\epsilon\,\mathfrak{y}$. For any set a, the object language contains a name $\mathring{a}$ for a.

Let M be a transitive model of MW, and φ a $\dot\Delta_0$ sentence of $\mathcal{L}_M$. It suffices to find a Σ_1 definition of $\models_M \varphi$, for if a truth predicate for a class of sentences that is closed under negation is Σ_1 it will automatically be Π_1, since $\models \varphi \iff \neg \models \neg\varphi$.

We have a sentence φ; let k be its length; let N_φ be the finite set comprising those members of M of which the names occur in φ; let q_φ be the number of occurrences of quantifiers in φ.

Step 1: we rewrite φ by de-nesting restricted quantifiers: for example,

$$\text{replace } \bigwedge\mathfrak{x}_{\epsilon\mathring{a}}\bigvee\mathfrak{y}_{\epsilon\mathfrak{x}}\,\vartheta \text{ by } \bigwedge\mathfrak{x}_{\epsilon\mathring{a}}\bigvee\mathfrak{y}_{\epsilon\mathring{c}}\,[\mathfrak{y}\ \epsilon\ \mathfrak{x} \wedge \vartheta], \text{ where } c = \bigcup a.$$

We thus reach within q_φ steps a formula φ' in which all quantifiers are restricted by *free terms*, each of the form <name of> $\bigcup^m a$, where $a \in N_\varphi$ and $m < q_\varphi$. As the Axiom of Union is among those of MW, each such $\bigcup^m a$ will be in M. Let F_φ be the finite set comprising those members of M of which the names occur in φ'.

A similar process is described in some detail in section 8 of [M3], though there, but not here, the formalism admits limited quantifiers as well as restricted ones.

Step 2: using the usual procedures of predicate logic, we rewrite φ' in prenex form, thus reaching a sentence φ^* in which a string of quantifiers, all restricted by free terms, precedes a quantifier-free formula ϑ.

These two steps may be achieved by primitive recursive processes applied to the formulæ in question.

We must now show that M contains a set which contains all the constants that will occur in substitution instances of subformulæ of φ^*: but such a set will be $P_\varphi =_{\mathrm{df}} F_\varphi \cup \bigcup F_\varphi$

Let S_φ be the class of quantifier-free sentences, of length no longer than k, in which the only names occurring are those of members of P_φ, That, provably in MW, will be a set.

Step 3: we show that truth for members of S_φ is uniformly Σ_1 for transitive models of MW.

Specifically, we show that there is an *evaluation* $g_\varphi : S_\varphi \longrightarrow 2$, that is, a function which obeys the rules for evaluation of Boolean combinations of atomic statements. These rules are:

$$g(\mathring{x} = \mathring{y}) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$g(\mathring{x}\ \epsilon\ \mathring{y}) = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{if } x \notin y \end{cases}$$

$$g(\neg\vartheta) = 1 - g(\vartheta)$$

$$g(\vartheta_1 \wedge \vartheta_2) = \inf\{g(\vartheta_1), g(\vartheta_2)\}$$

$$g(\vartheta_1 \vee \vartheta_2) = \sup\{g(\vartheta_1), g(\vartheta_2)\}$$

and similarly for other propositional connectives if they have also been taken as primitive.

We saw in [M3] that a statement of the form $\vartheta = \vartheta_1 \wedge \vartheta_2$ is not Δ_0 but will be Δ_0 in any sufficiently long attempt at addition. As the sentences to be considered are all of length not exceeding that of φ, a single sufficiently long attempt, α, will exist, and we shall be able to express the above rules for g as a statement that is $\Delta_0(\alpha, g, S_\varphi)$.

Thus the desired Σ_1 truth predicate for sentences ϑ in S_φ will be of the form

$$\exists\alpha,\ \text{a sufficiently long attempt at addition, } \exists g : S_\varphi \longrightarrow 2,\ g \text{ an evaluation, with } g(\vartheta) = 1.$$

Step 4: we show how to reduce the computation of truth of φ^* to that of numerous substitution instances in S_φ.

9·1 Remark This step would be possible even if we had not done Steps One and Two, but would be more complicated to express.

Suppose that φ^* has $n+1$ quantifiers, so that there are $a_0, \ldots, a_n$ in M such that $\varphi_\emptyset$ is

$$\mathcal{Q}_0 \mathfrak{x}_{0\,\epsilon\,\mathring{a}_0} \mathcal{Q}_1 \mathfrak{x}_{1\,\epsilon\,\mathring{a}_1} \ldots \mathcal{Q}_n \mathfrak{x}_{n\,\epsilon\,\mathring{a}_n} \vartheta$$

where $\vartheta \in S_\varphi$ but may contain other names besides those shown. n is not greater than k.

We consider the tree T of all finite sequences $\langle c_0, \ldots c_\ell\rangle$ of members of P_φ where $\ell \leqslant n$ and for each i, $c_i \in a_i$. Provably in MW, T is a set. We write $\emptyset$ for the empty sequence.

We define for each $t \in T$ a sentence φ_t by recursion on the length of t.

Let $\varphi_\emptyset = \varphi^*$.

Once we have defined φ_t then for $c \in a_{\ell h(t)}$ we define $\varphi_{t^\frown\langle c\rangle}$ to be $\mathrm{Subst}(\varphi_t, \mathfrak{x}_{\ell h(t)}, \mathring{c})$.

Let $T_\varphi = \{\varphi_t \mid t \in T\}$, a tree of sentences.

Let g_φ be the evaluation defined on S_φ in Step 3. Extend it to T_φ by a reverse induction: if $\ell h(t) = n+1$, φ_t will be a member of S_φ, and so $g_\varphi(\varphi_t)$ has been defined in Step 3. If $g_\varphi(\varphi_u)$ has been defined for all $u \in T$ of length $\ell h(t)+1$, then define

$$g_\varphi(\varphi_t) = \begin{cases} \sup\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigvee \\ \inf\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigwedge \end{cases}$$

So $\overset{0}{\models} \varphi \Longleftrightarrow g_\varphi(\varphi_\emptyset) = 1$.

We have finally to observe that as M models MW, then for φ a $\dot\Delta_0$ sentence of $\mathcal{L}_M$, all the above sets and functions, in particular P_φ, S_φ, T_φ and g_φ are in M; so the desired Σ_1 formula simply says that there exist sets and functions which obey the rules imposed on them and which lead to the evaluation of φ. ⊣ (9·1)

The same argument with very few changes will give a less laborious proof of the result proved in section 10 of [M3]:

9·2 Theorem *The truth predicate $\models_u \varphi$, for u a transitive set and φ an arbitrary sentence of $\mathcal{L}_u$, is Δ_1^{MW}.*

Proof : Immediate from Theorem 9.0, since the process of replacing each unbounded quantifier $\bigwedge\mathfrak{x}$ or $\bigvee\mathfrak{x}$ by the corresponding bounded quantifier $\bigwedge\mathfrak{x}_{\epsilon\,\mathring{u}}$ or $\bigvee\mathfrak{x}_{\epsilon\,\mathring{u}}$ is primitive recursive. ⊣ (9·2)

9·3 Remark A similar argument shows that the predicate $\models_A \varphi$ is Δ_1^{MW}, where φ is a sentence of some language (not necessarily the language of sets) and A is a small internal structure for that language, where the functions and relations of the language are all coded in the usual way by sets.

Notes and acknowledgments

The first author's interest in the problems addressed in this paper and its sequel was fired in 1987 by Stanley's call, in his review [St] of Devlin's treatise [De], for a development of constructibility that would meet Devlin's unachieved aims. Subsequently, in Barcelona in the mid 90s, the author became greatly interested in the problem of finding the weakest system of set theory that will support a recognisable theory of forcing. Over the following decade he accumulated assorted observations about weak systems, which during the set theory year, 2003-4, at the CRM at Barcelona, grew into a coherent apparatus [M3] for addressing the problems with Devlin's book, and which sowed the seeds of the theory of rudimentary recursion and the sense that the search for the correct minimal theory for a development of forcing was getting warm.

He began a series of draft papers, called rudrec or fifo, with a draft number; rudrec4, dated October 2004, gives the definition of *rudimentarily recursive* and the beginning of a discussion of forcing in that context, and asks for an example of a function that, in today's terminology, is gentle but not rud rec.

In February 2006, the author was encouraged by a correspondence [S] with Dana Scott who had found the Gandy–Jensen theory of rudimentary functions useful in the study of certain problems in formal geometry, and who was relieved to find that [M3] had, as he put it, "rescued" Devlin's book.

Gradually the theory of rudimentary recursion matured; by November 2007 the idea of what is now called a provident set was there, and a scenario for a proof that a generic extension of a provident set would be provident. That scenario ran into difficulties but the proof was saved by the idea of construction from a "dynamic" predicate: one defined by simultaneous recursion with a particular strict continuous progress, as in [M5, Definition 8·5]. Without this notion, the proof as it stood would have needed the ground model to be limit provident, not just provident.

Progressively more mature versions of this material were presented in the ERMIT seminar in Réunion, where they benefitted from the comments of Dr Olivier Esser; in talks in January 2008 at University College, London and at Oberwolfach, following which the term "provident" was adopted; in a talk in May 2008 at Leeds; and in October 2008 at Lisbon, in talks based on rudrec36 and fifo27. At the Zermelo centenary meeting at Brussels in late October 2008, the complementary theme of the inadequacy of Zermelo's sytem for forcing was discussed, as was the compensatory use of the passage to the provident closure or to the lune [M4].

In March 2009 copies of rudrec39 and fifo29 were sent to an Editor of *Fundamenta Mathematicæ*, who declined the first as being on a topic unsuited to his pages but asked for formal submission of the second, on forcing over provident sets, to be made: accordingly on May 5, 2009, fifo31 [M6] was submitted, with rudrec41 [M5] attached for the assistance of the referee.

In July 2009 the material was presented in condensed form in talks at Oxford and at Bedlewo, and an extended abstract prepared for the website of the latter meeting.

The first author received the referee's conditionally favourable report from *Fundamenta* on July 5th 2010, a week or so after receiving the kind invitation of Professor Martin Hyland to give a course of twenty four lectures to a graduate and post-doctoral audience at Cambridge (U.K.) in the Michaelmas Term, 2010. He accepted with pleasure this invaluable opportunity of testing in detail his approach to constructibility and forcing *via* weak systems, rudimentary recursion and provident sets.

In the Cambridge audience was the second author, who promptly found the counterexample given in Proposition 4·5 to the question whether the composition of two rud rec functions is rud rec, and who went on to prove Proposition 4·8 and Theorem 4·9. The elegance and strength of his notion of a gentle function have subsequently been confirmed in his Theorem 4·19; and comparison with [M5] will show how his ideas have interacted with those of the first author, in some cases, such as those of "dynamic predicate" and "function of uniform affine growth", supplanting them, and in others, clarifying and developing them and where necessary giving them appropriate concrete form and generality.

Besides those mentioned above, the first author thanks Carlos Montenegro in Bogotà, and in Barcelona James Harris and the seminar of Joan Bagaria, for their patience in listening to immature versions of these ideas; and for their steadfast encouragement in his study of weak systems, Kai Hauser, Ronald Jensen and Colin McLarty.

R E F E R E N C E S

[De] K. Devlin, *Constructibility*, Perspectives in Mathematical Logic, *Springer-Verlag, Berlin*, 1984.

[Do] A. J. Dodd, *The Core Model*, London Mathematical Society Lecture Note Series, 61, Cambridge University Press, 1982. MR **84a:**03062.

[G] R. O. Gandy, Set-theoretic functions for elementary syntax, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part II, ed. T. Jech, American Mathematical Society, 1974, 103–126.

[J1] R. B. Jensen, Stufen der konstruktiblen Hierarchie. Habilitationsschrift, Bonn, 1967 (?)

[J2] R. B. Jensen, The fine structure of the constructible hierarchy, with a section by Jack Silver, *Annals of Mathematical Logic,* **4** (1972) 229–308; erratum ibid **4** (1972) 443.

[JK] R. B. Jensen and C. Karp, Primitive recursive set functions, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part I, ed. D. Scott, American Mathematical Society, 1971, 143–176.

[M1] A. R. D. Mathias, Slim models of Zermelo Set Theory, *Journal of Symbolic Logic* **66** (2001) 487–496.

[M2] A. R. D. Mathias, The Strength of Mac Lane Set Theory, *Annals of Pure and Applied Logic,* **110** (2001) 107–234.

[M3] A. R. D. Mathias, Weak systems of Gandy, Jensen and Devlin, in *Set Theory: Centre de Recerca Matemàtica, Barcelona 2003-4* , edited by Joan Bagaria and Stevo Todorcevic, Trends in Mathematics, Birkhäuser Verlag, Basel, 2006, 149–224.

[M4] A. R. D. Mathias, Set forcing over models of Zermelo or Mac Lane, in R. Hinnion and T. Libert (eds), "One hundred years of axiomatic set theory", Cahiers du Centre de logique, Vol. 17, Academia-Bruylant, Louvain-la-Neuve, 2010, 41–66.

[M5] A. R. D. Mathias, `http://personnel.univ-reunion.fr/ardm/rudrec41.pdf`, preprint dated 3.v.2009.

[M6] A. R. D. Mathias, `http://www.dpmms.cam.ac.uk/~ardm/fifo31.pdf`, preprint dated 3.v.2009.

[M7] A. R. D. Mathias, Unordered pairs in the set theory of Bourbaki 1949, *Archiv für Mathematik* **94** (2010) 1–10.

[M8] A. R. D. Mathias, *Provident set theory, in preparation.*

[MB] A. R. D. Mathias and N. J. Bowler, Provident sets and rudimentary set forcing, *Fundamenta Mathematicæ, to appear.*

[S] Dana Scott, email correspondence on formal geometry and rudimentary functions

[SMcC] Dana Scott and Dominic McCarty, Reconsidering Ordered Pairs, *Bulletin of Symbolic Logic* **14**, (2008) 379–97.

[St] Stanley, Lee, review of [De], *Journal of Symbolic Logic* **53**(1987) 864–8.

[W] Weaver, Nik, Analysis in J_2, arXiv:math.LO/0509245.