Bounding connected tree-width

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Abstract

Diestel and Müller showed that the *connected tree-width* of a graph G, i. e., the minimum width of any tree-decomposition with connected parts, can be bounded in terms of the tree-width of G and the largest length of a geodesic cycle in G. We improve their bound to one that is of correct order of magnitude. Finally, we construct a graph whose connected tree-width exceeds the connected order of any of its brambles. This disproves a conjecture by Diestel and Müller asserting an analogue of tree-width duality.

1 Introduction

Intuitively, a tree-decomposition $(T,(V_t)_{t\in T})$ of a graph G can be regarded as giving a bird's-eye view on the global structure of the graph, represented by T, while each part represents local information about the graph. But this interpretation can be misleading: the tree-decomposition may have disconnected parts, containing vertices which lie at great distance in G, and so this intuitively appealing distinction between local and global structure can not be maintained.

This can be remedied if we require every part to be connected. We call such a tree-decomposition *connected*. Jegou and Terrioux [4, 5] pointed out that the efficiency of algorithmic methods based on tree-decompositions for solving constraint satisfaction problems can be improved when using connected tree-decompositions.

The *connected tree-width* ctw(G) is defined accordingly as the minimum width of a connected tree-decomposition of the graph G. Trivially, the connected tree-width of a graph is at least as large as its tree-width and, as Jegou and Terrioux [5] observed, long cycles are examples of graphs of small tree-width but large connected tree-width. Diestel and Müller [2] showed that, more generally, the existence of long *geodesic* cycles, that is, cycles in a graph G that contain a shortest path in G between any two of their vertices, raises the connected tree-width. Furthermore, they proved that these two obstructions to small connected tree-width, namely, large tree-width and long geodesic cycles, are essentially the only obstructions:

Theorem 1 ([2, Theorem 1.1])**.** *There is a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that the connected tree-width of any graph of tree-width k and without geodesic cycles of length greater than ℓ is at most $f(k, \ell)$.*

They also showed that $f(k, \ell) = \mathcal{O}(k^3\ell)$. In fact, their proof does not only work with geodesic cycles, but with any collection of cycles that generate the cycle space of the graph G. Given a graph G, we define $\ell(G)$ to be the smallest natural number ℓ such that the cycles of length at most ℓ generate the cycle space of G. Our main result improves the bound of Diestel and Müller significantly:

Theorem 2. *Let G be a graph containing a cycle. Then the connected tree-width of G is at most* $\mathrm{tw(G)}(\ell(\mathrm{G}) - 2)$.

(Observe that every forest satisfies $\mathrm{ctw(G)} = \mathrm{tw(G)} \leq 1$.) Theorem 2 will be proved in Sections 2–4. In Section 6 we discuss an example that demonstrates that this bound is best possible up to a constant factor.

Note that $\ell(G)$ can differ arbitrarily from the length of a longest geodesic cycle: consider e. g. an $(n \times n)$-grid where every edge except for those on the boundary is subdivided once. Then the boundary is a geodesic cycle of length $4(n-1)$, while the cycle space is generated by the collection of 'squares', each of length at most 8. It is no coincidence that the graph in this example has large tree-width, as the following unexpected consequence of our inquiry shows:

Corollary 3. *Every graph G containing a geodesic cycle of length k has tree-width at least $k/\ell(G)$.*

The *tree-width duality theorem* of Seymour and Thomas [6] asserts that a graph has tree-width less than k if and only if it has no bramble of order at least k. Diestel and Müller [2] conjectured that a similar duality holds for connected tree-width and the maximum *connected order* of a bramble: the minimum size of a connected vertex set meeting every element of the bramble. We disprove their conjecture by giving an infinite family of counterexamples in Section 7.

Since every tree-decomposition has, for every bramble of the graph, a part covering it, Theorem 2 immediately yields an upper bound on the connected order of any bramble of the graph. In Section 5, we apply the techniques and results from previous sections to strengthen this bound:

Theorem 4. *Let G be a graph containing a cycle. Then the connected order of any bramble of G is at most* $\mathrm{tw(G)}\lfloor \frac{\ell(\mathrm{G})}{2} \rfloor + 1$.

2 Definitions and notation

For a tree T with root r, we call s a *descendant* of t and t an *ancestor* of s if t lies on the unique path from r to s. If additionally $st \in E(T)$, we call s a *child* of t and t the *parent* of s. We write T_t for the subtree of descendants of t. Recall that a *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for every node of T, such that:

(T1) $V(G) = \bigcup_{t \in T} V_t$,

(T2) for every edge e of G there exists a $t \in T$ with $e \subseteq V_t$,

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1–t_3 path in T.

The sets V_t in such a tree-decomposition are its *parts*. For $A \subseteq V(T)$ we write $V_A := \bigcup_{t \in A} V_t$. The *width* of $(T, \mathcal{V})$ is $\max_{t \in T}(|V_t| - 1)$ and the *tree-width* tw(G) of G is the minimum width of any of its tree-decompositions.

In our proof of Theorem 2 we will make use of an explicit procedure that transforms a given tree-decomposition into a connected tree-decomposition by iteratively adding paths to a disconnected part of the decomposition. For this to work efficiently, we will restrict ourselves to paths of a particular kind.

Let $(T, \mathcal{V})$ be a *rooted* tree-decomposition of G, i. e. T is rooted, and $t \in T$. A path P in G is *t-admissible* if it lies entirely in V_{T_t}, joins different components of V_t and is shortest possible with these properties. Note that t-admissible paths have precisely two vertices in V_t:

Lemma 5. *Let $(T, \mathcal{V})$ be a rooted tree-decomposition of a graph G, $t \in T$ and P a t-admissible path. Then there is a unique child s of t such that all internal vertices of P lie in $V_{T_s} \setminus V_t$.* □

In general, t-admissible paths need not exist. However, as we shall see, we can easily confine ourselves to tree-decompositions that always have t-admissible paths.

We call a tree-decomposition $(T, \mathcal{V})$ *stable* if for every edge $t_1 t_2 \in E(T)$ of T, both V_{T_1} and V_{T_2} are connected in G, where T_i, for $i = 1, 2$, is the component of $T - t_1 t_2$ containing t_i. (Later, we will use this naming convention without further mention.)

Lemma 6. *Let $(T, \mathcal{V})$ be a rooted stable tree-decomposition of a connected graph G. Then every $t \in T$ with disconnected V_t has a t-admissible path.* □

Stable tree-decompositions were also studied in [3], where they are called *connected tree-decompositions*. In that article, an explicit algorithm is presented that turns a tree-decomposition of a connected graph into a stable tree-decomposition without increasing its width. For our purposes it suffices to know that every connected graph has a stable tree-decomposition of minimum width. This can also be deduced from [2, Corollary 3.5].

Proposition 7. *Every connected graph G has a stable tree-decomposition of width* tw(G). □

If we add a t-admissible path P to a part V_t in order to join two of its components, we might not obtain a tree-decomposition. The following lemma shows how it can be patched.

Lemma 8. *Let $(T, \mathcal{V})$ be a rooted tree-decomposition of a graph G, $t \in T$ and P a t-admissible path. For $u \in T$ let*

$$W_u := \begin{cases} V_u \cup (V(P) \cap V_{T_u}), & \text{if } u \in T_t, \\ V_u, & \text{if } u \notin T_t. \end{cases} \tag{$*$}$$

Then $(T, \mathcal{W})$ is a tree-decomposition of G. For all $u \in T$, every component of W_u contains a vertex of V_u. If $(T, \mathcal{V})$ is stable, so is $(T, \mathcal{W})$.

Proof. Since $V_u \subseteq W_u$ for all $u \in T$, every vertex and every edge of G is contained in some part W_u.

Let I be the set of internal vertices of P. By Lemma 5 there is a unique child s of t such that $I \subseteq V_{T_s} \setminus V_t$. For $x \notin I$, the set of parts containing x has not changed. For $x \in I$, the set $A_x := \{u \in T \colon x \in V_u\}$ induces a subtree of T_s and $x \in W_u$ if and only if $u \in A_x$ or u lies on the path joining t to A_x. So $\{u \colon x \in W_u\}$ is also a subtree of T.

Note that every component of $P \cap V_{T_u}$ is a path with ends in V_u. Therefore every $x \in W_u \setminus V_u$ is joined to two vertices in V_u and thus every component of W_u contains vertices from V_u.

Suppose now $(T, \mathcal{V})$ is stable, let $t_1t_2 \in E(T)$ and $i \in \{1, 2\}$. Then V_{T_i} is connected. For $x \in W_{T_i} \setminus V_{T_i}$ there is a $u \in T_i \cap T_t$ with $x \in W_u \setminus V_u$. But then, by the above, W_u contains a path joining x to V_{T_i}. As $V_{T_i} \subseteq W_{T_i}$, also W_{T_i} is connected. □

3 The construction

We now describe a construction that turns a stable tree-decomposition $(T, \mathcal{V})$ of a connected graph into a connected tree-decomposition. First, choose a root r for T and keep it fixed. It will be crucial to our analysis that the nodes of T are processed in the induced order of the tree, i. e. we enumerate the nodes $t_1, t_2, \ldots$ so that each node precedes its descendants and we process the nodes in this order.

Initially we set $W_t = V_t$ for all $t \in T$. Throughout the construction, we maintain the invariant that $(T, \mathcal{W})$ is a stable tree-decomposition *extending* $(T, \mathcal{V})$, by which we mean that they are tree-decompositions over the same rooted tree, satisfying $V_t \subseteq W_t$ for all $t \in T$.

When processing a node $t \in T$ with disconnected part W_t, we use the stability of $(T, \mathcal{W})$ to find a t-admissible path by Lemma 6 and update $\mathcal{W}$ as in $(*)$. By Lemma 8, this does not violate stability and it clearly reduces the number of components of W_t by one. We iterate this until W_t is connected. Once that is achieved, we continue with the next node in our enumeration.

Observe that each 'update' only affects descendants of the current node. Once a node $t \in T$ has been processed, so have all of its ancestors. Hence, no further changes are made to W_t afterwards. In particular, W_t remains connected. It thus follows that, when every node has been processed, the resulting tree-decomposition is indeed connected.

In order to control the size of each part W_u, we will use a *bookkeeping graph* Q_u to keep track of what we have added. Initially, Q_u is the empty graph on V_u, and in each step Q_u is a graph on the vertices of W_u. Whenever something is added to W_u, we are considering a t-admissible path P for some ancestor t of u and P contains vertices of W_{T_u}. Every component of $P \cap W_{T_u}$ is a path with ends (and possibly also some internal vertices) in W_u. We then add $P \cap W_{T_u}$ to Q_u, that is, we add all the vertices not contained in W_u and all the edges of $P \cap W_{T_u}$.

Lemma 9. *During every step of the procedure, Q_u is acyclic.*

Proof. This is certainly true initially. Suppose now that at some step a cycle is formed in Q_u. By definition, it must be that an ancestor t of u is being processed and a t-admissible path P is added such that two vertices $a, b \in W_u$ which were already connected in Q_u lie in the same component of $P \cap W_{T_u}$.

The vertices a, b being connected in Q_u by a path $a = a_0 a_1 \dots a_n = b$ means that there have been, for every $0 \leq j \leq n-1$, ancestors t_j of u that added paths P_j such that a_j, a_{j+1} were consecutive vertices on a segment S_j of $P_j \cap W_{T_u}$. By the order in which the nodes are processed and by $(*)$, these t_j are also ancestors of t. Therefore when P_j was added to W_{t_j}, the segment S_j was contained in a segment of $P_j \cap W_{T_t}$, since $W_{T_t} \supseteq W_{T_u}$. Therefore, at the time P is added to W_t, all these segments are contained in W_t and, in particular, $a, b \in W_t$. By Lemma 5, P does not have internal vertices in W_t so that a and b must in fact be the ends of P. But W_t already contains a walk from a to b, consisting of the segments S_j, so that the two do not lie in different components of W_t, contradicting the t-admissibility of P. □

We now show how the sparse structure of Q_u reflects the efficiency of our procedure.

Lemma 10. *The number of components of Q_u never increases. Whenever something is added to W_u, the number of components of Q_u decreases.*

Proof. Suppose that in an iteration a change is made to Q_u. Then an ancestor t of u is being processed and the chosen path P meets W_{T_u}. Every component of $P \cap W_{T_u}$ is a path with both ends in W_u. Therefore, every newly introduced vertex is joined to a vertex in Q_u and no new components are created.

If a vertex from $P \cap (W_{T_u} \setminus W_u)$ is added to W_u, the segment containing it has length at least two and has two ends $a, b \in Q_u$. By Lemma 9, Q_u must remain acyclic, so that a and b in fact lie in different components of Q_u, which are now joined. □

The previous lemma allows us to control the number of iterations that affect a fixed node $t \in T$. The second key ingredient for the proof of Theorem 2 will be to bound the length of each of the paths used, see Section 4.

Proposition 11. *Let G be a connected graph, $(T, \mathcal{V})$ a rooted stable tree-decomposition of G. For $t \in T$ let $m_t \geq 1$ be such that for every stable tree-decomposition $(T, \mathcal{W})$ extending $(T, \mathcal{V})$ and every ancestor t' of t, the length of a t'-admissible path in $(T, \mathcal{W})$ does not exceed m_t. Then the construction produces a connected tree-decomposition $(T, \mathcal{U})$ in which for all $t \in T$*

$$|U_t| \leq m_t(|V_t| - 1) + 1.$$

Proof. We have already shown that $(T, \mathcal{U})$ is connected. By Lemma 10, every time something was added to W_t, the number of components of Q_t decreased and it never increased. Since initially Q_t had precisely $|V_t|$ components, this can only have happened at most $|V_t| - 1$ times. In each such iteration we added some internal vertices of a t'-admissible path in a stable tree-decomposition extending $(T, \mathcal{V})$ for some ancestor t' of t, thus at most $m_t - 1$ vertices. In total, we have

$$|U_t| \leq |V_t| + (m_t - 1)(|V_t| - 1) = m_t(|V_t| - 1) + 1. \qquad \square$$

4 Bounding the length of admissible paths

We will now use ideas from [2] to bound the length of t-admissible paths in stable tree-decompositions. Together with Proposition 11, this will imply our main result.

Lemma 12. *Let G be a graph and Γ a set of cycles that generates its cycle space. Let $(T, \mathcal{V})$ be a stable tree-decomposition of G and $t_1 t_2 \in E(T)$. Suppose that $V_{t_1} \cap V_{t_2}$ meets two distinct components of V_{t_1}. Then there is a cycle $C \in \Gamma$ such that some component of $C \cap V_{T_2}$ meets V_{t_1} in two distinct components.*

Proof. As V_{T_2} is connected, we can choose a shortest path P in V_{T_2} joining two components of V_{t_1}. Let $x, y \in V_{t_1}$ be its ends and note that all internal vertices of P lie in $V_{T_2} \setminus V_{t_1}$. As V_{T_1} is connected as well, we also find a path $Q \subseteq V_{T_1}$ joining x and y, which is internally disjoint from P. By assumption, there is a subset $\mathcal{C}$ of Γ such that $P + Q = \bigoplus \mathcal{C}$. We subdivide $\mathcal{C}$ as follows: $\mathcal{C}_1$ comprises all those cycles which are entirely contained in $V_{T_1} \setminus V_{T_2}$, $\mathcal{C}_2$ those in $V_{T_2} \setminus V_{T_1}$ and $\mathcal{C}_X$ those that meet $X := V_{t_1} \cap V_{t_2}$.

Assume now for a contradiction that for every $C \in \mathcal{C}_X$ and every component S of $C \cap V_{T_2}$ there is a unique component D_S of V_{t_1} met by S. Note that S is a cycle if $C \subseteq V_{T_2}$ and a path with ends in X otherwise. Either way, the number of edges of S between X and $V_{T_2} \setminus X$, denoted by $|E_S(X, V_{T_2} \setminus X)|$, is always even. It thus follows that for any component D of V_{t_1}

$$|E_C(D, V_{T_2} \setminus X)| = \sum_{S \subseteq C \cap V_{T_2}} |E_S(D, V_{T_2} \setminus X)| = \sum_{S\colon D_S = D} |E_S(X, V_{T_2} \setminus X)|$$

is even. But then also the number of edges in $\bigoplus \mathcal{C}_X$ between D and $V_{T_2} \setminus X$ is even. Since the edges of $\bigoplus \mathcal{C}_1$ and $\bigoplus \mathcal{C}_2$ do not contain vertices from X, we have

$$E_{\bigoplus \mathcal{C}_X}(X, V_{T_2} \setminus X) = E_{P+Q}(X, V_{T_2} \setminus X) = \{xx', yy'\},$$

where x' and y' are the neighbours of x and y on P, respectively. Due to parity, x and y need to lie in the same component of V_{t_1}, contrary to definition. □

Proof of Theorem 2. Since both parameters appearing in the bound do not increase when passing to a component of G and as we can combine connected tree-decompositions of the components to obtain a connected tree-decomposition of G, it suffices to consider the case that G is connected.

We use Lemma 12 to bound the length of t-admissible paths in any stable tree-decomposition of G. Let $\ell = \ell(G)$ and Γ be the set of all cycles of length at most ℓ, which by definition generates the cycle space of G. Let $(T, \mathcal{W})$ be a rooted stable tree-decomposition, $t \in T$ and P a t-admissible path. By Lemma 5 there is a child s of t such that all internal vertices of P lie in $V_{T_s} \setminus V_t$. By Lemma 12 we find a cycle $C \in \Gamma$ and a path $S \subseteq C \cap V_{T_s}$ joining distinct components of V_{t_1}. Since $S \subseteq V_{T_t}$ and P was chosen to be a shortest such path, we have $|P| \leq |S|$. The ends of S lie in distinct components of V_{t_1} and are therefore, in particular, not adjacent, so that overall

$$|V(P)| \leq |V(S)| \leq |V(C)| - 1 \leq \ell - 1.$$

By Proposition 7, G has a stable tree-decomposition $(T, \mathcal{V})$ of width tw(G). Proposition 11 then guarantees that we find a connected tree-decomposition of width at most $(\ell - 2)\mathrm{tw(G)}$. □

5 Brambles

Recall that a *bramble* is a collection of connected vertex sets of a given graph such that the union of any two of them is again connected. A *cover* of a bramble is a set of vertices that meets every element of the bramble. The aim of this section is to derive a strengthened upper bound on the *connected order* of a bramble, the minimum size of a connected cover.

Lemma 13. *Suppose $(T, \mathcal{V})$ is a tree-decomposition of a graph G and $k \in \mathbb{N}$ an integer such that for every $t \in T$ there is a connected set of size at most $k+1$ containing V_t. Then G has no bramble of connected order greater than $k+1$.*

Proof. Let $\mathcal{B}$ be a bramble of G. By a standard argument, see e. g. the proof of [1, Theorem 12.3.9], one of the parts V_t of $(T, \mathcal{V})$ covers $\mathcal{B}$ and thus so does any connected set containing V_t. □

Let us call the smallest integer k such that there is a tree-decomposition satisfying the hypothesis of Lemma 13 the *weak connected tree-width* wctw(G) of the graph G. Clearly wctw(G) $\leq$ ctw(G), as any connected tree-decomposition of minimum width satisfies the hypothesis. Theorem 4 follows directly from Lemma 13 and the following.

Theorem 14. *Let G be a graph containing a cycle. Then*

$$\mathrm{wctw(G)} \leq \mathrm{tw(G)} \lfloor \frac{\ell(\mathrm{G})}{2} \rfloor.$$

Proof. It suffices to consider the case where G is connected, since all three parameters involved are simply their respective maxima over the components of G. Let $\ell = \ell(G)$ and Γ be the set of all cycles of G of length at most ℓ, which by definition generates the cycle space of G. By Proposition 7, G has a stable tree-decomposition (T, V) of width tw(G). We now show that every part V_t of $(T, \mathcal{V})$ is contained in a connected set of size at most $(|V_t| - 1)\lfloor \frac{\ell}{2} \rfloor + 1$.

Let now $t \in T$ be fixed. Root T at t and apply the construction from Section 3. As t does not have any ancestors other than itself, the statement follows from Proposition 11 once we have verified that all t-admissible paths in a stable tree-decomposition $(T, \mathcal{W})$ extending $(T, \mathcal{V})$ have length at most $\ell/2$. So let $(T, \mathcal{W})$ be a stable tree-decomposition of G extending $(T, \mathcal{V})$ and let P be a t-admissible path. By Lemma 5, all its internal vertices lie in $W_{T_s} \setminus W_t$ for some child s of t. By Lemma 12 we find a cycle $C \in \Gamma$ that meets W_t in two vertices x, y from distinct components of W_t. Either segment of C between x and y lies in W_{T_t} and joins two components of W_t, so by minimality P has length at most $\lfloor \ell/2 \rfloor$. $\square$

Diestel and Müller [2] showed that if a graph G contains a geodesic cycle of length k, then G has a bramble of connected order $\lceil k/2 \rceil + 1$. Combined with the upper bound of Theorem 4, this implies Corollary 3.

6 A graph of large connected tree-width

In this section we discuss an example that shows that our upper bound on connected tree-width is tight up to a constant factor. Given $n, k \in \mathbb{N}$, $n \geq 3$, obtain G from the complete graph on n vertices by subdividing every edge with k newly introduced vertices. As subdivision does not alter tree-width, we have tw(G) = n − 1. The cycle space of G is generated by the collection of all subdivisions of triangles of the underlying complete graph, so $\ell(G) = 3(k+1)$. We will now show that the connected tree-width of G is precisely $r := (n-1)(k+1) - \lfloor (k+1)/2 \rfloor$. The bound of Theorem 2 is therefore asymptotically tight up to a factor of 3.

Let $A \subseteq V(G)$ denote the set of vertices of degree $n - 1$. The graph G thus consists of A and, for any two $a, b \in A$, a path P_{ab} of length $k + 1$ between them. We first describe a bramble that cannot be covered with any connected set of size at most r. The lower bound on the connected tree-width of G then follows from Lemma 13. Any connected set X consists of some vertices from A, its *branchvertices*, some *internal vertices* X^0 on paths joining two of its branchvertices and possibly some *additional vertices*. Any connected set with j branchvertices must have at least $(j-1)k$ internal vertices, resulting in a minimum size of $(j-1)(k+1)+1$. Let X be a connected set of size at most r. Then, by the above, X cannot contain all the vertices of A and, moreover, all vertices of $A \setminus X$ lie in the same component $C(X)$ of $G - X$: If $a, b \in A \setminus X$, then by connectedness either $X \cap P_{ab} = \emptyset$ or $X \subseteq P_{ab}$, in which case a and b can be joined through some other $c \in A$. Let $\mathcal{B}$ be the collection of all these components $C(X)$ for $X \subseteq V$ connected of size at most r.

Clearly, $\mathcal{B}$ can not be covered by any connected set of size at most r, so it only remains to verify that $\mathcal{B}$ is indeed a bramble. Let $X_1, X_2 \subseteq V$ be two connected sets of size at most r, containing j_1, j_2 vertices of A, respectively. Suppose that $C(X_1)$ and $C(X_2)$ did not touch. Then for every pair (a, b) with $a \in A \setminus X_1$, $b \in A \setminus X_2$, the sets X_1 and X_2 must have a common vertex on P_{ab}. By definition, all these are additional vertices for both sets, so

$$\begin{aligned} |X_1| + |X_2| &\geq |X_1^0| + |X_2^0| + (n - j_1)(n - j_2)(k + 1) \\ &= (k + 1)(n^2 - (n - 1)(j_1 + j_2) + j_1 j_2 - 2) + 2. \end{aligned}$$

This expression, seen as a function of j_1 and j_2, assumes its minimum for $j_1 = j_2 = n - 1$. We thus conclude $|X_1| + |X_2| \geq 2(k + 1)(n - 1) - k + 1$, hence the larger of the two sets has size at least $r + 1$, a contradiction.

We now describe a connected tree-decomposition of width r. Fix two $a, b \in A$ and let $A^- := A \setminus \{a, b\}$. Let T be a star with root s and leaves $t, u_1, \ldots, u_m$ with $m = \binom{n-2}{2}$. Each V_{u_i} consists of a different path P_{cd} with $c, d \in A^-$. Let V_s consist of the union of all P_{bc} with $c \in A^-$ and the first $\lceil (k + 1)/2 \rceil$ vertices from P_{ba}. Define V_t similarly. This tree-decomposition has the desired width.

7 A counterexample for duality

In this section, we present a graph whose connected tree-width is larger than the largest connected order of any of its brambles. Hence, we disprove the duality conjecture of Diestel and Müller [2] for connected tree-width.

Let $n \geq 4$ be an integer. For $i = 0, 1, 2$, let $P_i = x_1^i \ldots x_{2n}^i$ be three pairwise disjoint paths and $Q = y_1 \ldots y_{4n}$ another path disjoint from each P_i. Between every two vertices x_j^i, y_k we add a new internally disjoint path $P_{j,k}^i$ of length $5n$, except for $k = n + j$, where they have length n. Let G' be the resulting graph. Let G be the disjoint union of G' with a cycle C of length $16n + 2$, where we choose two antipodal vertices a, b, i. e. vertices of C with $d_C(a, b) = 8n + 1$, and add the edges ax_1^0, ay_1 and bx_{2n}^0, by_{4n}. Figure 1 shows the graph G without P_2 and its attachment paths to Q.

We claim that the connected order of any of its brambles is at most $5n + 3$ and that its connected tree-width is at least $6n$. Thus, up to additive constants, these parameters differ at least by a factor of $6/5$.

We will now give a tree-decomposition demonstrating that wctw(G) $\leq$ 5n+2, which is sufficient to prove the upper bound on the connected order of any bramble by Lemma 13. Start with $V_{t_0} := V(Q) \cup \{a, b\}$, which is connected and of size $4n + 2$. Clearly, $G - V_{t_0}$ consists of five components: each of the P_i along with their attachments to V_{t_0} and the two arcs of C. Accordingly, we add five branches to t_0, each decomposing one of the components, as follows.

For $i = 0, 1, 2$, attach a path $t_1^i \ldots t_{2n-1}^i$ to t_0 and put $V_{t_j^i} = V_{t_0} \cup \{x_j^i, x_{j+1}^i\}$. Each of these is contained in a connected set of size $5n + 3$, as x_j^i is joined to Q

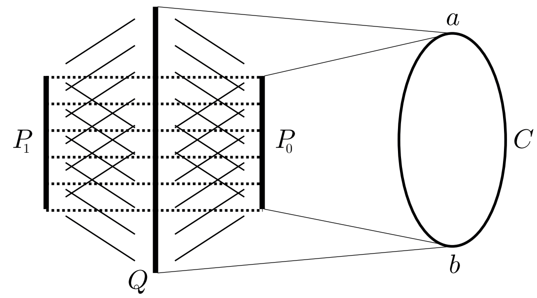


Figure 1: The graph G without P_2 and its attachment paths.

by a path of length n. To each t^i_j attach $4n$ leaves, each consisting of some $P^i_{j,k}$, $k \in [4n]$, which obviously does not exceed the prescribed size. To t^i_{2n-1} we add another $4n$ leaves consisting of all the $P^i_{j+1,k}$. To decompose C, we attach two more paths $s^1_0 s^1_1$ and $s^2_0 s^2_1$ to t_0, one for each arc S^j of $C - \{a, b\}$. For $j = 1, 2$, $V_{s^j_0}$ contains $\{a, b\}$ and the $3n$ vertices of S^j which lie closest to a, while $V_{s^j_1}$ contains b and its closest $5n + 1$ vertices on S^j. Both of these sets are contained in connected sets of size $5n + 2$. Figure 2 shows our decomposition tree.

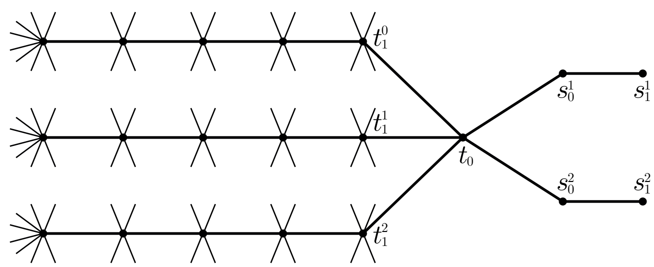


Figure 2: The decomposition tree of G.

To show ctw(G) $\geq$ 6n, let us assume for a contradiction that G had a connected tree-decomposition $(T, \mathcal{V})$ of width less than $6n$. We shall show that some part V_t contains Q and some other part $V_{t'}$ contains P_0. To see that some part contains Q, we define a bramble as follows: For For all i, j, k, let $B^i_{j,k}$ be the union of all the paths from x^i_j to Q with all end vertices except for y_k deleted. It is easy so see that the collection $\mathcal{B}_1$ of all these sets $B^i_{j,k}$ is a bramble. Therefore, some part V_t of $(T, \mathcal{V})$ must cover $\mathcal{B}_1$. If some vertex $y_k \in Q$ is not included in V_t, then V_t must contain at least one vertex from each of the $6n$ pairwise disjoint sets $B^i_{j,k} \setminus \{y_k\}$ with $i \in \{0, 1, 2\}$ and $j \in [2n]$. Since no such selection of vertices is connected without the addition of further vertices, this contradicts our assumption that $|V_t| \leq 6n$.

We now show that some part contains P_0. Let C' be the cycle of length

$12n+2$ consisting of one of the a–b paths on C together with Q and let $\mathcal{B}_2$ be the bramble consisting of all segments of C' of length $6n+1$. Again, there must be a part $V_{t'}$ covering $\mathcal{B}_2$. Assume for a contradiction that some vertex $x_j^0 \in P_0$ was not contained in $V_{t'}$. Observe, crucially, that C' is geodesic in $G' := G - x_j^0$ and hence $\mathcal{B}_2$ has connected order $6n+2$ in G' (see [2, Lemma 7.1]). As $V_{t'}$ is a cover of $\mathcal{B}_2$ in G', it follows that $V_{t'} \geq 6n+2$, which is a contradiction.

So we have found parts $V_t, V_{t'}$ containing Q and P_0, respectively. Choose two such parts at minimum distance in T. Note first that $t \neq t'$, because $P_0 \cup Q$ has size $6n$ and we need at least one further vertex, for example a, to connect these two paths. We now distinguish two cases. Suppose first that another node s of T lies between V_t and $V_{t'}$. By our choice of t, t', there must be some $x_j^0, y_k \notin V_s$. But V_s separates V_t and $V_{t'}$, so it must contain some vertex from $P_{j,k}^0$. Being connected, V_s is actually contained in this path. But then it cannot separate any other two vertices of P and Q, which is a contradiction. Suppose now that t and t' are neighbours in T. Pick any $x_p^0 \in P_0 \setminus V_t$ and $y_q \in Q \setminus V_{t'}$. Since $V_t \cap V_{t'}$ separates the two, it contains some vertex of $P_{p,q}^0$, and thus at least one of V_t, $V_{t'}$ contains at least half the vertices of $P_{p,q}^0$. We may assume that this applies to Q; the other case follows symmetrically. For every $1 \leq j \leq 2n$ consider $R_j := \bigcup_{k=1}^{4n} P_{j,k}^0 \setminus Q$, the subdivision of a star with root x_j^0. These are pairwise disjoint and disjoint from Q, and since V_t contains at least $n/2 \geq 2$ vertices from R_p, there is some $m \in [2n]$ with $V_t \cap R_m = \emptyset$, by our assumption on the width. As $V_t \cap V_{t'}$ separates x_m^0 from V_t, we must have $Q \subseteq V_{t'}$, contradicting $t \neq t'$.

8 Concluding Remarks and open problems

Define the *connected bramble number* cbn(G) of a graph G to be the maximum connected order of any bramble in G. In Section 5 we observed that

$$\mathrm{cbn(G)} - 1 \leq \mathrm{wctw(G)} \leq \mathrm{ctw(G)} \qquad (\dagger)$$

holds for any graph G. Diestel and Müller [2] conjectured that $\mathrm{cbn(G)} - 1 = \mathrm{ctw(G)}$, but our example in Section 7 shows that the second of the two inequalities in (†) cannot be replaced by an equality. We do suspect, however, that the first inequality is in fact an equality:

Problem 1. Let k be a positive integer. A graph G has a tree-decomposition in which every part is contained in a connected set of at most k vertices if and only if every bramble of G can be covered by a connected set of size at most k.

It seems that neither the proof techniques of ordinary tree-width duality nor the ideas underlying our counterexample to connected tree-width duality are apt to solve this problem; hence we are confident that an inquiry into this problem is going to provide new ideas and insights.

The second problem concerns the second inequality of (†). The proof of [2, Theorem 1.2] combined with the improved bound of Theorem 2 shows that

$\mathrm{ctw(G)} \leq 2(\mathrm{cbn(G)}-1)(\mathrm{cbn(G)}-2)$, unless G is a forest in which case $\mathrm{ctw(G)} = \mathrm{tw(G)}$. This implies a locality principle for connected tree-width: if there is a tree-decomposition in which every part, individually, can be wrapped in a connected set of size at most k, then there is a tree-decomposition with connected parts of size at most $2(k-1)(k-2)$. It would be interesting to get a better understanding of this dependency.

Problem 2. Is there a constant $\alpha > 0$ such that for every graph G

$$\mathrm{ctw(G)} \leq \alpha\, \mathrm{wctw(G)}?$$

Our example in Section 7 shows that this is not true for any $\alpha < 6/5$.

References

[1] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.

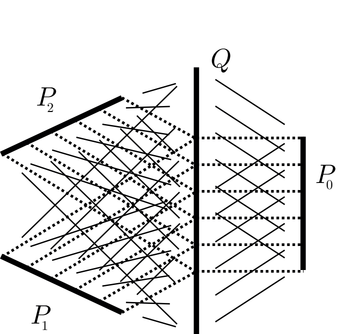
[2] R. Diestel and M. Müller. Connected tree-width. arXiv:1211.7353, 2014.

[3] P. Fraigniaud and N. Nisse. Connected treewidth and connected graph searching. In *Proceedings of the 7th Latin American Conference on Theoretical Informatics*, LATIN'06, pages 479–490. Springer-Verlag, 2006.

[4] P. Jegou and C. Terrioux. Bag-connected tree-width: a new parameter for graph decomposition. `http://www.cs.uic.edu/pub/Isaim2014/WebPreferences/ISAIM2014_Jegou_Terrioux.pdf`, 2014.

[5] P. Jegou and C. Terrioux. Tree-decompositions with connected clusters for solving constraint networks. In B. O'Sullivan, editor, *CP 2014*, volume 8656, pages 407–403. Springer Lecture Notes in Computer Science, 2014.

[6] P. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. *J. Combin. Theory (Series B)*, 58(1):22–33, 1993.



Q
P_2
P_0
P_1