# TOWARDS AN UNDERSTANDING OF RAMIFIED EXTENSIONS OF STRUCTURED RING SPECTRA

BJØRN IAN DUNDAS, AYELET LINDENSTRAUSS, AND BIRGIT RICHTER

Abstract. We determine second order topological Hochschild homology of the $p$-local integers. For the tamely ramified extension of the map from the connective Adams summand to $p$-local complex topological K-theory we determine the relative topological Hochschild homology and show that it detects the tame ramification of this extension. We show that the complexification map from connective topological real to complex K-theory shows features of a wildly ramified extension. We also determine relative topological Hochschild homology for some quotient maps with commutative quotients.

## 1. Introduction

Let $A$ be a commutative ring spectrum and let $B$ be a commutative $A$-algebra with an action of a finite group $G$ via maps of commutative $A$-algebras. Then the extension $A \to B$ is called unramified [R08, (4.1.2)], if the map

$$h\colon B \wedge_A B \to \prod_G B$$

is an equivalence. Here, $h$ can be informally described as $b_1 \otimes b_2 \mapsto (b_1 g(b_2))_{g\in G}$. Rognes shows [R08, 9.2.6, proof of 9.1.2] that the unramified condition ensures that the map from $B$ to relative topological Hochschild homology of $B$ over $A$, $\mathsf{THH}^A(B)$, is a weak equivalence. Thus the failure of the map

$$B \to \mathsf{THH}^A(B)$$

to being a weak equivalence is a measure for the ramification of the extension $A \to B$. It also makes sense to study $\mathsf{THH}^A(B)$ in more general situations, for instance in the absence of a group action.

For $A$ and $B$ as above we denote by $\mathsf{THH}^{[n],A}(B)$ the higher order topological Hochschild homology of $B$ as a commutative $A$-algebra, *i.e.*,

$$\mathsf{THH}^{[n],A}(B) = B \otimes \mathbb{S}^n$$

where $(-)\otimes\mathbb{S}^n$ denotes the tensor with the $n$-sphere in the category of commutative $A$-algebras. This can be viewed as the realization of the simplicial commutative $A$-algebra whose $q$-simplices are given by

$$\coprod_{x\in\mathbb{S}^n_q} B,$$

where the coproduct is the smash product over $A$.

Higher $\mathsf{THH}$ also detects ramification [DLR∞], but with coefficients in the residue field we don't see a difference between tame and wild ramification in higher $\mathsf{THH}$. We offer some partial results towards calculations of higher $\mathsf{THH}$ with unreduced coefficients. We calculate second order $\mathsf{THH}$ of the $p$-local integers:

$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n, x_n^p = p x_{n+1}.$$

See Theorem 2.1.

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We study the examples of the connective covers of the Galois extensions [R08] $KO \to KU$ and $L_p \to KU_p$. In the latter case, the connective cover behaves like an extension of the corresponding rings of integers. We test ramification with relative (higher) topological Hochschild homology and for $\ell \to ku_{(p)}$ we detect tame ramification (see Theorem 4.1): $\mathsf{THH}^{\ell}_*(ku_{(p)})$ is a square zero extension of $\pi_* ku_{(p)}$ of bounded $u$-exponent. The complexification map $c\colon ko \to ku$ however, shows features in its relative $\mathsf{THH}$ that are similar to the behavior of a wildly ramified extension of number rings, *e.g.*, the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$.

Working with structured ring spectra means working in a derived setting, so quotient maps can be thought of as extensions. We offer some calculations of relative $\mathsf{THH}$ in situations where we kill generators of homotopy groups. We consider a version of $ku/(p, v_1)$ and quotients of the form $R/x$ where $x$ is a regular element in $\pi_*(R)$ where $R$ is a commutative ring spectrum such that $R/x$ is still commutative.

## 2. Second order $\mathsf{THH}$ of the $p$-local integers

This section consists of a proof of the following somewhat surprising result. In the context of the current paper, this calculation is a starting point for comparing with future calculations for other rings of integers. See Remark 2.4 for a discussion of the fact that the answer agrees with topological Hochschild cohomology.

**Theorem 2.1.** *For all primes $p$:*
$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n, x_n^p - p x_{n+1}$$
*with $|x_n| = 2p^n$.*

The entire section is devoted to proving this result For all primes $p$ the cofiber sequence

(1) $$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p) \xrightarrow{\delta} \Sigma\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$$

is a sequence of $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$-modules; in particular, $\delta$ is a module map. Furthermore, from [DLR∞] we have that

(2) $$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(y) \otimes \Lambda_{\mathbb{F}_p}(z),$$

where $|y| = 2p$ and $|z| = 2p+1$. We denote the generator $\gamma_{p^i}(y)$ in the divided power algebra $\Gamma_{\mathbb{F}_p}(y)$ in degree $2p^{i+1}$ by $y_{p^i}$ and if $t = t_0 + t_1 p + \cdots + t_n p^n$ is the $p$-adic expansion of $t$, then we set $y_t = y_1^{t_0} y_p^{t_1} \ldots y_{p^n}^{t_n}$ with $y_t^p = 0$.

By the $\mathsf{Tor}$ spectral sequence,
$$\mathsf{Tor}^{\mathsf{THH}_*(\mathbb{Z}_{(p)})}_{*,*}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \Rightarrow \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$$
we know that $\mathsf{THH}^{[2]}_s(\mathbb{Z}_{(p)})$ is finite $p$-torsion for positive $s$ because
$$\mathsf{THH}_*(\mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_{(p)}, & * = 0, \\ \mathbb{Z}_{(p)}/i, & * = 2i-1, \\ 0, & \text{otherwise.} \end{cases}$$
By (2) and using the notation introduced below it, this implies that there are integers $a_1, a_2, \ldots$ such that
$$\mathsf{THH}^{[2]}_s(\mathbb{Z}_{(p)}) \begin{cases} 0, & 2p \not| s, \\ \mathbb{Z}/p^{a_t}\{\tilde{y}_t\}, & s = 2pt, \end{cases}$$
where the $\tilde{y}_t$ are generators of the given cyclic groups which are sent to the corresponding generators $y_t$ in $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p)$. We will show

**Lemma 2.2.** *The function $a\colon \mathbb{N} \to \mathbb{N}$ factors over the $p$-adic valuation $v\colon \mathbb{N} \to \mathbb{N}$, $a_t = b_{v(t)}$, with $b\colon \mathbb{N} \to \mathbb{N}$ a* strictly *increasing function with positive values and $b_0 = 1$.*

*Proof.* To this end we use induction on the following statement $\mathbf{P(n)}$ for positive integers $n$. The generators $\tilde{y}_t$ are chosen inductively.

$\mathbf{P(n)}$: For positive integers $s$, $t$ such that $v(s)$, $v(t)$ are less than $n$ the following properties hold:

(1) If $v(s) = v(t)$, then $a_s = a_t$,
(2) If $v(s) > v(t)$, then $a_s > a_t$,
(3) If $s = s_0 + s_1 p + \cdots + s_{n-1} p^{n-1}$ is the $p$-adic expansion of $s$ (so that $0 \leqslant s_0, \ldots, s_n < p$), then $\tilde{y}_s = \tilde{y}_1{}^{s_0} \ldots \tilde{y}_{p^{n-1}}^{s_{n-1}}$,
(4) If $n > v$, then $\tilde{y}_{p^v} = p^{a_{p^v} - a_{p^{v-1}}} \tilde{y}_{p^{v-1}}^p$.

We will repeatedly be considering the cofiber sequence (1). In homotopy, the maps are trivial except in degrees of the form $2pt$ (for varying $t$) in which case they are

$$0 \longrightarrow \mathbb{F}_p\{zy_{t-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_t\} \longrightarrow 0$$

forcing all the $a_t$ to be positive. For any generator $w$, $\mathbb{F}_p\{w\}$ denotes the graded vector space generated by $w$. Here $r$ is multiplicative and $\delta$ is a $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$-module map. By the surjectivity of $r$ we have that the $y_t$'s can be lifted to integral classes.

*Establishing* $\mathbf{P(1)}$. Let $t = 1$. The sequence

$$0 \longrightarrow \mathbb{F}_p\{z\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2p}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2p}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1\} \longrightarrow 0.$$

shows that $a_1 > 0$ and by adjusting $z$ up to a unit we may choose $\tilde{y}_1$ so that $\delta(z) = p^{a_1 - 1}\tilde{y}_1$ and $r(\tilde{y}_1) = y_1$. In the Tor-spectral sequence we only get a $\mathbb{Z}/p\mathbb{Z}$ in bidegree $(1, 2p-1)$ which survives and shows that $a_1 = 1$, and so $\delta(z) = \tilde{y}_1$.

If $1 < t < p$ the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{t-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1^t\} \longrightarrow 0$$

gives that $\delta(zy_1^{t-1}) = \delta(z)\cdot\tilde{y}_1^{t-1} = \tilde{y}_1^t \neq 0$, $p\tilde{y}_1^t = 0$ and $r(\tilde{y}_1^t) = y_1^t \neq 0$. The last point shows that $\tilde{y}_1^t$ is not divisible by $p$ and hence we can choose it as our generator: $\tilde{y}_t = \tilde{y}_1^t$, and furthermore, this generator is killed by $p$, so $a_t = 1$.

If $t = t_0 + t_1 p$ with $0 < t_0 < p$, then the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{t_0-1}y_{t_1p}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1^{t_0}y_{t_1p}\} \longrightarrow 0$$

gives that $\delta(zy_1^{t_0-1}y_{t_1p}) = \delta(z)\cdot\tilde{y}_1^{t_0-1}\tilde{y}_{t_1p} = \tilde{y}_1^{t_0}\tilde{y}_{t_1p} \neq 0$, $p\tilde{y}_1^{t_0}\tilde{y}_{t_1p} = 0$ and $r(\tilde{y}_1^{t_0}\tilde{y}_{t_1p}) = y_1{}^{t_0}y_{t_1p} \neq 0$ for any choice of a lift $\tilde{y}_{t_1p}$. The last point shows that $\tilde{y}_1^{t_0}\tilde{y}_{t_1p}$ is not divisible by $p$ and hence we can choose it as our generator: $\tilde{y}_t = \tilde{y}_1^{t_0}\tilde{y}_{t_1p}$, and furthermore, this generator is killed by $p$, so $a_t = 1$.

Note that we may reconsider our choice of $\tilde{y}_{t_1p}$ later, and so the exact choice of $\tilde{y}_t$ may still change within these bounds, but the choices of $\tilde{y}_1, \ldots, \tilde{y}_{p-1}$ remain fixed from now on. Hence $\mathbf{P(1)(1)} - \mathbf{P(1)(3)}$ are established and as $\mathbf{P(1)(4)}$ is vacuous we have shown $\mathbf{P(1)}$.

*Establishing* $\mathbf{P(n+1)}$. Now, assume $\mathbf{P(n)}$. First, consider the case $t = p^n$. For $\mathbf{P(n+1)(4)}$ we only have to show that

$$\tilde{y}_{p^n} = p^{a_{p^n} - a_{p^{n-1}}}\tilde{y}_{p^{n-1}}^p,$$

and that $a_{p^n} > a_{p^{n-1}}$. Consider the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{p-1} \ldots y_{p^{n-1}}^{p-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2p^n}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2p^n}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_{p^n}\} \longrightarrow 0.$$

Firstly, by induction we have that

$$\begin{aligned}\delta(zy_1^{p-1}\dots y_{p^{n-1}}^{p-1}) &= \tilde{y}_1\tilde{y}_1^{p-1}\dots\tilde{y}_{p^{n-1}}^{p-1}\\ &= p^{a_p-a_1}\tilde{y}_p\tilde{y}_p^{p-1}\dots\tilde{y}_{p^{n-1}}^{p-1}\\ &= p^{a_p-a_1}p^{a_{p^2}-a_p}\tilde{y}_{p^2}\tilde{y}_{p^2}^{p-1}\dots\tilde{y}_{p^{n-1}}^{p-1} = \dots\\ &= p^{a_{p^{n-1}}-1}\tilde{y}_{p^{n-1}}^{p}\neq 0.\end{aligned}$$

Secondly,

$$p\delta(zy_1^{p-1}\cdots y_{p^{n-1}}^{p-1}) = pp^{a_{p^{n-1}}-1}\tilde{y}_{p^{n-1}}^{p} = p^{a_{p^{n-1}}}\tilde{y}_{p^{n-1}}^{p} = 0.$$

Together this shows that (up to a unit) $\delta(zy_1^{p-1}\cdots y_{p^{n-1}}^{p-1}) = p^{a_{p^n}-1}\tilde{y}_{p^n}$, and that $\tilde{y}_{p^{n-1}}^{p} = p^{a_{p^n}-a_{p^{n-1}}}\tilde{y}_{p^n}$, and since $y_{p^{n-1}}^{p}=0$ that $a_{p^n} > a_{p^{n-1}}$.

Now, for $\mathbf{P(n+1)(1)}$ and $\mathbf{P(n+1)(2)}$, consider a general $t$ with $v(t)=n$ and write $t = t_np^n + sp^{n+1}$ with $0<t_n<p$. The short exact sequence

$$\mathbb{F}_p\{zy_1^{p-1}\dots y_{p^{n-1}}^{p-1}y_{p^n}^{t_n-1}y_{sp^{n+1}}\} \xrightarrow{\delta} \mathsf{THH}_{2pt}^{[2]}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}_{2pt}^{[2]}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_{p^n}^{t_n}y_{sp^{n+1}}\}$$

gives that

$$\begin{aligned}\delta(zy_1^{p-1}\dots y_{p^{n-1}}^{p-1}y_{p^n}^{t_n-1}y_{sp^{n+1}}) &= \tilde{y}_1\tilde{y}_1^{p-1}\dots\tilde{y}_{p^{n-1}}^{p-1}\tilde{y}_{p^n}^{t_n-1}\tilde{y}_{sp^{n+1}}\\ &= p^{a_{p^n}-1}\tilde{y}_{p^n}^{t_n}\tilde{y}_{sp^{n+1}}\neq 0,\end{aligned}$$

but $p\delta(zy_1^{p-1}\dots y_{p^{n-1}}^{p-1}y_{p^n}^{t_n-1}y_{sp^{n+1}}) = p^{a_{p^n}}\tilde{y}_{p^n}^{t_n}\tilde{y}_{sp^{n+1}} = 0$ and $r(\tilde{y}_{p^n}^{t_n}\tilde{y}_{sp^{n+1}}) = y_{p^n}^{t_n}y_{sp^{n+1}}\neq 0$. Again, the last point shows that $\tilde{y}_{p^n}^{t_n}\tilde{y}_{sp^{n+1}}$ is not divisible by $p$, and so we may choose $\tilde{y}_t = \tilde{y}_{p^n}^{t_n}\tilde{y}_{sp^{n+1}}$, and furthermore that this generator is annihilated by $p^{a_{p^n}}$, but not by $p^{a_{p^n}-1}$, so that $a_t = a_{p^n}$.

Lastly, by $\mathbf{P(n)(3)}$, we have that if $s = s_0 + s_1p + \dots + s_{n-1}p^{n-1}$ is the $p$-adic expansion of $s$, then $\tilde{y}_s = \tilde{y}_1^{s_0}\dots\tilde{y}_{p^{n-1}}^{s_{n-1}}$. If $t = s + s_np^n$, then $r(\tilde{y}_s\tilde{y}_{p^n}^{s_n}) = y_t$, so we can choose $\tilde{y}_t = \tilde{y}_s\tilde{y}_{p^n}^{s_n}$ as desired in $\mathbf{P(n+1)(3)}$. $\square$

*Background on Bocksteins.* Let $(C_*,\partial)$ be a complex of free abelian groups and assume $\alpha\in C_n$ has the property that $\alpha\otimes 1$ is a cycle in $C_*\otimes\mathbb{F}_p$. That the Bockstein $\beta_{i-1}[\alpha\otimes 1]$ is defined and equal to zero for some $i\geqslant 2$, means that there exist $\gamma\in C_n$ and cycle $\delta\in C_{n-1}$ so that $\partial(\alpha+p\gamma) = p^i\delta$, and in that case, $\beta_i[\alpha\otimes 1] = [\delta\otimes 1]$.

Assume we have a short exact sequence of complexes of free abelian groups $0\to B_*\to C_*\to A_*\to 0$. Choosing a section in each degree, we may assume $C_n = A_n\oplus B_n$ for all $n$. Suppose we have $a\in A_n$ and $b\in B_n$ so that $[a+b]$ represents a cycle in $C_*\otimes\mathbb{F}_p$ with $\beta_{i-1}([(a+b)\otimes 1]) = [0]\in H_{n-1}(C_*\otimes\mathbb{F}_p)$. As above, there exist $c\in A_n$, $d\in B_n$, $e\in A_{n-1}$, $f\in B_{n-1}$ with $e+f\in C_{n-1}$ a cycle with $\partial(a+b+p(c+d)) = p^i(e+f)$, and in that case $\beta_i([(a+b)\otimes 1]) = [(e+f)\otimes 1]$. Then if $[e\otimes 1]\neq[0]\in H_{n-1}(A_*\otimes\mathbb{F}_p)$, we get that $\beta_i([(a+b)\otimes 1])\neq[0]$, since $[(e+f)\otimes 1]\mapsto[e\otimes 1]\neq[0]$ by the homomorphism induced by the projection $C_*\to A_*$.

More generally, consider a filtered complex $C_*$ of free abelian groups. Assume we have a chain $a\in E(C_*)_{s,t}^0$ in the associated spectral sequence such that $[a\otimes 1]$ survives to $E(C_*\otimes\mathbb{F}_p)_{s,t}^\infty$ in the mod $p$ spectral sequence. If we know that the class $[(a+b)\otimes 1]\in H_{s+t}(C_*\otimes\mathbb{F}_p)$ with $b\in F_{s-1}(C_*)$ which $[a\otimes 1]$ represents in $E(C_*\otimes\mathbb{F}_p)_{s,t}^\infty$ satisfies $\beta_{i-1}([(a+b)\otimes 1]) = [0]\in H_{s+t-1}(C_*\otimes\mathbb{F}_p)$, but that $d^0(a\otimes 1) = p^i(e\otimes 1)$ and $[e\otimes 1]\neq[0]\in E(C_*\otimes\mathbb{F}_p)_{s,t-1}^1$, then $\beta_i([(a+b)\otimes 1])\neq[0]$.

*The p-order of the multiplicative generators.* We will calculate $\mathsf{THH}_*^{[2]}(\mathbb{Z}_{(p)})$ by studying its Hurewicz image in $H_*(\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)});\mathbb{F}_p)$, using the model

$$\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)}) \simeq B(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)}).$$

We use the filtration by simplicial skeleta. We denote $H_*(H\mathbb{Z}_{(p)};\mathbb{F}_p)$ by $\bar{\mathcal{A}}$, and by Bökstedt,

$$H_*(\mathsf{THH}(\mathbb{Z}_{(p)});\mathbb{F}_p) \cong \bar{\mathcal{A}} \otimes \mathbb{F}_p[x_{2p}] \otimes \Lambda[x_{2p-1}],$$

where the augmentation $\mathsf{THH}(\mathbb{Z}_{(p)}) \to H\mathbb{Z}_{(p)}$ induces the projection $\bar{\mathcal{A}}\otimes\mathbb{F}_p[x_{2p}]\otimes\Lambda[x_{2p-1}] \to \bar{\mathcal{A}}$ sending $x_{2p}$ and $x_{2p-1}$ to zero. We get that

$$E^1_{*,*} \cong B(\bar{\mathcal{A}}, \bar{\mathcal{A}} \otimes \mathbb{F}_p[x_{2p}] \otimes \Lambda[x_{2p-1}], \bar{\mathcal{A}})$$

is isomorphic to

$$B(\bar{\mathcal{A}}, \bar{\mathcal{A}}, \bar{\mathcal{A}}) \otimes B(\mathbb{F}_p, \mathbb{F}_p[x_{2p}], \mathbb{F}_p) \otimes B(\mathbb{F}_p, \Lambda[x_{2p-1}], \mathbb{F}_p),$$

and so its homology is

$$E^2_{*,*} \cong \bar{\mathcal{A}} \otimes \Lambda[y_{2p+1}] \otimes \Gamma[y_{2p}]$$

with $y_{2p+1} = 1 \otimes x_{2p} \otimes 1$ and $y_{2p} = 1 \otimes x_{2p-1}$ and $y_{2p}^{(a)} = 1 \otimes x_{2p-1}^{\otimes a} \otimes 1$.

The dimensions in each total degree in the $E^2$-term account for $p$-torsion of rank 1 in each positive dimension divisible by $2p$, and from knowing $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)};\mathbb{F}_p)$ [DLR∞, Theorem 3.1] we get that this agrees with the abutment of the spectral sequence, so it has to collapse at $E^2$.

We use this to prove Theorem 2.1. By Lemma 2.2, the only remaining problem is to determine the order of the $p$-torison in each dimension divisible by $2p$.

**Lemma 2.3.** *The $p$-torsion in $\mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)})$ is precisely $\mathbb{Z}_{(p)}/pt \cong \mathbb{Z}_{(p)}/p^{v_p(t)+1}$.*

*Proof.* We know from Lemma 2.2 that for $t = p^a m$, $(p,m) = 1$, in dimension $2pt$ the order of the torsion is divisible by $p^{a+1}$. We will use the general observation about Bocksteins above for

$$C_* = C_*(\Omega^\infty(\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)}));\mathbb{Z}) = C_*(\Omega^\infty B(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z}),$$

filtered by simplicial skeleta of the bar construction, to get that the torsion is exactly $p^{a+1}$.

Fixing a $t$, we have two quasi-isomorphisms (letting $s$ vary)

$$\begin{aligned}
&C_s(\Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z})\\
\to &C_{s+t}(\Delta^t \times \Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)}), \partial\Delta^t \times \Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z})\\
\to &E^0_{t,s}
\end{aligned}$$

and we call their composition $\varphi$.

We know by Bökstedt that additively $\mathsf{THH}(\mathbb{Z}_{(p)}) \simeq H\mathbb{Z}_{(p)} \vee \Sigma^{2p-1}H\mathbb{F}_p \vee \cdots$, so we can map

$$\begin{aligned}
&S^0 \wedge K(\mathbb{F}_p, 2p-1)^{\wedge t} \wedge S^0 = S^0 \wedge (\Omega^\infty(\Sigma^{2p-1}H\mathbb{F}_p))^{\wedge t} \wedge S^0\\
\to &\Omega^\infty H\mathbb{Z}_{(p)} \wedge (\Omega^\infty(\mathsf{THH}(\mathbb{Z}_{(p)}))^{\wedge t} \wedge \Omega^\infty H\mathbb{Z}_{(p)} \to \Omega^\infty(H\mathbb{Z}_{(p)} \wedge (\mathsf{THH}(\mathbb{Z}_{(p)})^{\wedge t} \wedge H\mathbb{Z}_{(p)}).
\end{aligned}$$

We call this composition $\psi$. It induces

$$\psi_* : \ C_*(K(\mathbb{F}_p, 2p-1)^{\wedge t};\mathbb{Z}) \to C_*(\Omega^\infty(H\mathbb{Z}_{(p)} \wedge \mathsf{THH}(\mathbb{Z}_{(p)})^{\wedge t} \wedge H\mathbb{Z}_{(p)});\mathbb{Z}),$$

so composing we get a map of complexes

$$\varphi \circ \psi_* : \ C_*(K(\mathbb{F}_p, 2p-1)^{\wedge t};\mathbb{Z}) \to E^0_{t,*}.$$

On the Eilenberg Mac Lane space $K(\mathbb{F}_p, 2p-1)$, we have a $2p$-chain with integer coefficients $\tilde{x}_{2p}$ so that $[\tilde{x}_{2p}]$ (mod $p$) generates $H_{2p}(K(\mathbb{F}_p, 2p-1);\mathbb{F}_p) \cong \mathbb{F}_p$ and $\partial\tilde{x}_{2p} = p\tilde{x}_{2p-1}$ for a chain $\tilde{x}_{2p-1}$ so that $[\tilde{x}_{2p-1}]$ (mod $p$) generates $H_{2p-1}(K(\mathbb{F}_p, 2p-1);\mathbb{F}_p) \cong \mathbb{F}_p$. For these elements, $\beta_1([\tilde{x}_{2p}]) = [\tilde{x}_{2p-1}]$. Note that these elements map to generators of the stable homology in the correct dimensions. Thus, $\varphi \circ \psi_*(\tilde{x}_{2p}) \otimes 1$ can be taken as a representative of $x_{2p}$, and $\varphi \circ \psi_*(\tilde{x}_{2p-1}) \otimes 1$ can be taken as a representative of $x_{2p-1}$, and we still have $d^0(\varphi \circ \psi_*(\tilde{x}_{2p})) = p\varphi \circ \psi_*(\tilde{x}_{2p-1})$ in $E^0_{1,*}$. And more generally, for any $a, b \geqslant 0$, in $E^0_{a+b+1,*}$ we also have

$$d^0(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge a} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge b})) = p\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge(a+b+1)}).$$

We know that the class $(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge a} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge b})) \otimes 1$ represents the class $1 \otimes x_{2p-1}^{\otimes a} \otimes x_{2p} \otimes x_{2p-1}^{\otimes b} \otimes 1$ which survives to $E^2_{a+b+1,*}$ and therefore to $E^\infty_{a+b+1,*}$, and similarly for $(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge(a+b+1)})) \otimes 1$ and $1 \otimes x_{2p-1}^{\otimes a+b+1} \otimes 1$.

And so, if $t = p^a m$ with $(p, m) = 1$,

$$d^0(\sum_{i=0}^{t-1}(-1)^i(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge i}\wedge\tilde{x}_{2p}\wedge\tilde{x}_{2p-1}^{\wedge t-1-i}))\otimes 1 = pt\cdot(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge(t)}))\otimes 1 = p^{a+1}m\cdot(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge(t)}))\otimes 1.$$

The mod $p$ homology class which is the image under the Hurewicz map of $z\gamma_{t-1}(y)$ can be expressed as

$$(1 \otimes x_{2p} \otimes 1)(1 \otimes x_{2p-1}^{\otimes n-1} \otimes 1)$$

via the bar construction and it is represented by $(\sum_{i=0}^{t-1}(-1)^i(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge i} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge t-1-i}) \otimes 1$. From Lemma 2.2 we have a lower bound on the order of the torsion and hence $\beta_a(z\gamma_{t-1}(y)) = [0]$ and by the $d^0$ calculation above $\beta_{a+1}(z\gamma_{t-1}(y)) = \gamma_t(y)$ up to a unit.

This result is a result on stable mod $p$ homology rather than on stable mod $p$ homotopy, but since we are applying it to the images under the Hurewicz map of the two stable mod $p$ homotopy classes of an Eilenberg Mac Lane space of rank 1 $p$-torsion, the Bockstein operators have to do the same on the mod $p$ homotopy. □

*Proof of Theorem 2.1.* Set $x_n = \tilde{y}_{p^n}$. Then we get the $p$-order of these elements from Lemma 2.3 and we worked out the multiplicative relations in Lemma 2.2. □

**Remark 2.4.** Mike Hill noticed that $\mathsf{THH}_*^{[2]}(\mathbb{Z}_{(p)})$ is abstractly isomorphic to $\mathsf{THH}^*(\mathbb{Z}_{(p)})$: the calculation of $\mathsf{THH}^*(\mathbb{Z}_{(p)})$ is due to Franjou and Pirashvili [FP98]. We are not sure whether this is a coincidence or whether (for some commutative $S$-algebras) there is a duality between $\mathsf{THH}_*^{[2]}$ and topological Hochschild cohomology. Note, however, that $\mathsf{THH}_*^{[2]}(\mathbb{F}_p)$ is an exterior algebra over $\mathbb{F}_p$ on a class in degree three whereas $\mathsf{THH}^*(\mathbb{F}_p)$ is much larger:

$$\mathsf{THH}^*(\mathbb{F}_p) \cong \mathbb{F}_p[e_0, e_1, \ldots]/(e_0^p, e_1^p, \ldots), \quad |e_i| = 2p^i$$

[FLS94, 7.3], [Bö∞], so there is no isomorphism of these groups in general.

## 3. Greenlees' approach to THH

There is a relative version of the cofiber sequence from [G16, Lemma 7.1] already mentioned in [DLR∞]. We make it explicit for later use. Here and elsewhere $S$ denotes the sphere spectrum.

**Lemma 3.1.** *Let $R$ be a commutative $S$-algebra and let $C \to B \to k$ be a sequence of maps of commutative $R$-algebras. Then there is a cofiber sequence of commutative $k$-algebras*

$$B \wedge_C^L k \to \mathsf{THH}^R(C, k) \to \mathsf{THH}^R(B, k).$$

The proof is obtained from the one of [G16, Lemma 7.1] by replacing the sphere spectrum by $R$.

**Remark 3.2.** Note that there are two cofiber sequences for any such sequence $C \to B \to k$, because we can forget the commutative $R$-algebra structures on $C$ and $B$ and consider them as commutative $S$-algebras. This gives a commutative diagram of cofiber sequences

$$\begin{tikzcd} B \wedge_C k \arrow[r] \arrow[d, equal] & \mathsf{THH}(C,k) \arrow[r] \arrow[d] & \mathsf{THH}(B,k) \arrow[d] \\ B \wedge_C k \arrow[r] & \mathsf{THH}^R(C,k) \arrow[r] & \mathsf{THH}^R(B,k), \end{tikzcd}$$

so $B \wedge_C k$ measures the difference of the absolute and also of the relative $\mathsf{THH}$-terms of $C$ and $B$.

Let us abbreviate $B \wedge^L_C k$ by $A$. Lemma 3.1 provides an equivalence

$$\mathsf{THH}^R(B,k) \simeq \mathsf{THH}^R(C,k) \wedge^L_A k$$

and thus we get a spectral sequence whose $E^2$-term is

$$\mathsf{Tor}^{A_*}_{*,*}(\mathsf{THH}^R_*(C,k), k_*)$$

which converges to $\mathsf{THH}^R_*(B,k)$.

We will consider the following examples.

- Let $\ell$ denote the Adams summand of $p$-local connective topological complex K-theory, $ku_{(p)}$, for some odd prime $p$. For
$$R = \ell \to C = \ell \to B = ku_{(p)} \to k$$
with $k = H\mathbb{Z}_{(p)}$ or $k = H\mathbb{F}_p$ we obtain calculations for $\mathsf{THH}^\ell_*(ku_{(p)}, k)$. We determine $\mathsf{THH}^\ell_*(ku_{(p)})$ by different means.
- The complexification map from real to complex topological K-theory $c\colon ko \to ku$ is a map of commutative $S$-algebras. Wood's theorem displays the $ko$-module $ku$ as the cofiber of the Hopf map $\eta\colon \Sigma ko \to ko$. Consequently, the $ku$-module $ku \wedge_{ko} ku$ is the cofiber of $\eta\colon \Sigma ku \to ku$, and the resulting short exact sequences
$$0 \to \pi_{2m} ku \to \pi_{2m}(ku \wedge_{ko} ku) \to \pi_{2m-1}(\Sigma ku) \to 0$$
are split via the multiplication map on $ku$, because the map $ku \to ku \wedge_{ko} ku$ above is induced by the unit map of $ku$ as a commutative $ko$-algebra so we get
$$\pi_{2m}(ku \wedge_{ko} ku) \cong \pi_{2m} ku \oplus \pi_{2m-2}(ku).$$
We will determine the $ku_*$-algebra structure of $\pi_*(ku \wedge_{ko} ku)$ in Lemma 5.1. This is the input for the Tor-spectral sequence computing $\mathsf{THH}^{ko}_*(ku)$ and we will identify $\mathsf{THH}^{ko}_*(ku)$ in Theorem 5.2.

  We will also use the cofiber sequences of commutative $k$-algebras
$$ku \wedge_{ko} k \to ku \to \mathsf{THH}^{ko}(ku, k)$$
for $k = H\mathbb{Z}_{(2)}$ and $k = H\mathbb{F}_2$ and we will calculate $\mathsf{THH}$ of $ku$ over $ko$ with coefficients in $H\mathbb{Z}_{(2)}$ and $H\mathbb{F}_2$ (see Proposition 5.4).
- We propose $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ as a model for $ku/(p, v_1)$ and use the sequence
$$S \to H\mathbb{F}_p \to ku_{(p)} \wedge_\ell H\mathbb{F}_p \to H\mathbb{F}_p$$
for calculating its $\mathsf{THH}$ with coefficients in $H\mathbb{F}_p$ (Proposition 6.2).
- In Section 7 we determine relative topological Hochschild homology of quotient maps $R \to R/x$.

## 4. Relative $\mathsf{THH}$ of $ku_{(p)}$ as a commutative $\ell$-algebra

Let $p$ be an odd prime. On the level of coefficients, the map from the connective Adams summand to $p$-local connective topological complex K-theory is $\ell_* = \mathbb{Z}_{(p)}[v_1] \to \mathbb{Z}_{(p)}[u] = (ku_{(p)})_*$, $v_1 \mapsto u^{p-1}$. The corresponding $p$-complete periodic extension is a $C_{p-1}$-Galois extension [R08]. However, the connective extension is not unramified, but it is a topological analogue of a tamely ramified extension. Rognes defined a notion of $\mathsf{THH}$-étale extensions in [R08, 9.2.1]: A map of commutative $S$-algebras $A \to B$ is *formally* $\mathsf{THH}$*-étale*, if the canonical map from $B$ to $\mathsf{THH}^A(B)$ is an equivalence. For instance, Galois extensions are formally $\mathsf{THH}$-étale [R08, 9.2.6]. We will show that the map $\ell \to ku_{(p)}$ is not formally $\mathsf{THH}$-étale by determining $\mathsf{THH}^\ell(ku_{(p)})$. Rognes mentions in [R08, p. 59] that $ku_{(p)} \to \mathsf{THH}^\ell(ku_{(p)})$ is a $K(1)$-local equivalence and Sagave showed in [S] that the map $\ell \to ku_{(p)}$ is log-étale. Ausoni proved that the $p$-completed extension even satisfies Galois descent for $\mathsf{THH}$ and algebraic K-theory [Au05, Theorem 1.5]:

$$\mathsf{THH}(ku_p)^{hC_{p-1}} \simeq \mathsf{THH}(\ell_p), \quad K(ku_p)^{hC_{p-1}} \simeq K(\ell_p).$$

The tame ramification is visible in THH:

**Theorem 4.1.**

$$\mathsf{THH}^{\ell}_*(ku_{(p)}) \cong (ku_{(p)})_* \rtimes (ku_{(p)})_*\langle y_0, y_1, \ldots\rangle/u^{p-2},$$

*where* $(ku_{(p)})_* \rtimes M$ *denotes a square-zero extension of* $(ku_{(p)})_*$ *by a* $(ku_{(p)})_*$*-module* $M$*. The degree of* $y_i$ *is* $2pi+3$*.*

*Proof.* We can apply the Bökstedt spectral sequence with $\pi_*$ as the homology theory because $(ku_{(p)})_*$ is projective over $\ell_*$. The $E^2$-page consists of

$$E^2_{s,t} = \mathsf{HH}^{\ell_*}_{s,t}((ku_{(p)})_*, (ku_{(p)})_*).$$

As an $\ell_*$-algebra $(ku_{(p)})_*$ is isomorphic to $\ell_*[u]/(u^{p-1} - v_1)$. From [LL92] we know that we can use the following complex in order to calculate Hochschild homology:

$$\ldots \xrightarrow{\Delta(u)} \Sigma^{2p}(ku_{(p)})_* \xrightarrow{0} \Sigma^{2p-2}(ku_{(p)})_* \xrightarrow{\Delta(u)} \Sigma^{2}(ku_{(p)})_* \xrightarrow{0} (ku_{(p)})_*,$$

where $\Delta(u) = (p-1)u^{p-2}$. As $(p-1)$ and $v_1$ are units in $\ell_*$, this yields:

$$\mathsf{HH}^{\ell_*}_i((ku_{(p)})_*, (ku_{(p)})_*) = \begin{cases} (ku_{(p)})_*, & \text{if } i = 0, \\ \Sigma^{2mp-2m+2}(ku_{(p)})_*/u^{p-2}, & \text{if } i = 2m+1, m \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathsf{THH}^{\ell}(ku_{(p)})$ is an augmented commutative $ku_{(p)}$-algebra, we know that $ku_{(p)}$ splits off $\mathsf{THH}^{\ell}(ku_{(p)})$. Therefore the copy of the homotopy groups of $ku_{(p)}$ in the zero column of the spectral sequence has to survive and cannot be hit by any differentials. For degree reasons, there are no other possible non-trivial differentials and the spectral sequence collapses at the $E^2$-page.

In every fixed total degree there is only one term in the $E^2$-page contributing to this degree: If we consider an element $u^{k_1}$ in homological degree $2m_1+1$ and another element $u^{k_2}$ in homological degree $2m_2+1$ for $m_1 \neq m_2$, then their total degrees are $2m_1p + 2k_1 + 3$ and $2m_2p + 2k_2 + 3$. These degrees can only be equal if $2p(m_1 - m_2) = 2(k_2 - k_1)$. Thus $p$ has to divide $k_2 - k_1 \neq 0$. But $0 \leqslant k_1, k_2 \leqslant p-3$, so this cannot happen.

Thus there are no additive extensions and therefore additively we get the desired result.

As $\mathsf{THH}^{\ell}_*(ku_{(p)})$ is an augmented graded commutative $(ku_{(p)})_*$-algebra and as everything in the augmentation ideal is concentrated in odd degrees there cannot be any non-trivial multiplication of any two elements in the augmentation ideal.

The spectral sequence is a spectral sequence of $(ku_{(p)})_*$-modules and elements of the form $u^k \cdot \Sigma^{2mp-2m+2}u^m$ are cycles, thus the copy of $(ku_{(p)})_*$ in homological degree zero acts on $ku_{(p)})_*/u^{p-2}y_m$ in the standard way. □

**Remark 4.2.** For Galois extensions of non-connective commutative ring spectra we would like to have a good notion of rings of integers. In the above case $ku_{(p)}$ behaves like the ring of integers of $KU_{(p)}$, and similarly for the connective Adams summand. The result for relative THH corresponds to the one of ordinary rings of integers [LM00]. In other cases, taking the connective cover does not seem to give good results.

For coefficients in $H\mathbb{Z}_{(p)}$ and $H\mathbb{F}_p$ we obtain a rather different result.

**Proposition 4.3.**

$$\mathsf{THH}^{\ell}_*(ku_{(p)}, H\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u)$$

*and also*

$$\mathsf{THH}^{\ell}_*(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$

*Proof.* We consider the sequence of comutative $\ell$-algebras

$$R = \ell \to C = \ell \to B = ku_{(p)} \to k$$

with $k = H\mathbb{Z}_{(p)}$ and $k = H\mathbb{F}_p$. In both cases we obtain a cofiber sequence of commutative $k$-algebras [G16]

$$ku_{(p)} \wedge_\ell k \to k \to \mathsf{THH}^\ell(ku, k)$$

because $\mathsf{THH}^\ell(\ell, k) \simeq k$. We therefore get a Tor-spectral sequence

$$\mathsf{Tor}_{*,*}^{\pi_*(ku_{(p)}\wedge_\ell k)}(\pi_* k, \pi_* k) \Rightarrow \mathsf{THH}_*^\ell(ku_{(p)}, k).$$

For $k = H\mathbb{Z}_{(p)}$ homological algebra tells us that

$$\mathsf{Tor}_{*,*}^{\mathbb{Z}_{(p)}[u]/u^{p-1}}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u).$$

Here, $|\varepsilon u| = 3$ and $|\varphi^0 u| = 2p$. There are no differentials in this spectral sequence for degree reasons and there are no multiplicative extensions, hence we get the claim.

For $k = H\mathbb{F}_p$ the same method gives

$$\mathsf{THH}_*^\ell(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$

□

## 5. Relative THH of the complexification map

The graded commutative ring $ko_*$ is $\mathbb{Z}[\eta, y, w]/\langle 2\eta, \eta y, \eta^3, y^2 - 4w\rangle$ with $|\eta| = 1$, $|y| = 4$ and $w$ is the Bott class in degree 8. The complexification map $c\colon ko \to ku$ induces a map $c_*\colon ko_* \to ku_* = \mathbb{Z}[u]$ and it sends $\eta$ to zero, $y$ to $2u^2$ and the Bott class $w$ to $u^4$.

Note that the homotopy fixed points of $ku$ with respect to complex conjugation are not equivalent to $ko$. The homotopy fixed points spectral sequence yields generators in negative degrees in the homotopy groups of $ku^{hC_2}$ [R08, 5.3].

**Lemma 5.1.** *As a graded commutative augmented $ku_*$-algebra*

$$(ku \wedge_{ko} ku)_* \cong ku_*[\tilde{u}]/\tilde{u}^2 - u^2$$

*with $|\tilde{u}| = 2$.*

*Proof.* As we saw in the introduction, Wood's theorem gives that $(ku \wedge_{ko} ku)_*$ is additively isomorphic to $ku_* \oplus \pi_*(\Sigma^2 ku)$. The $\mathsf{Tor}$ spectral sequence

$$\mathsf{Tor}_{*,*}^{ko_*}(ku_*, ku_*) \Rightarrow (ku \wedge_{ko} ku)_*$$

allows us to determine the multiplicative structure.

The tensor product $ku_* \otimes_{ko_*} ku_* \cong ku_*[\tilde{u}]/(2\tilde{u}^2 - 2u^2, \tilde{u}^4 - u^4)$ has three generators in degree four: $u^2, u\tilde{u}, \tilde{u}^2$. The element $\tilde{u}^2 - u^2$ is 2-torsion, but there is no 2-torison in the abutment $(ku \wedge_{ko} ku)_*$. Hence this class has to die via a differential in the spectral sequence.

□

**Theorem 5.2.** *The Tor spectral sequence*

$$E^2_{*,*} = \mathsf{Tor}_{*,*}^{(ku\wedge_{ko}ku)_*}(ku_*, ku_*) \Rightarrow \mathsf{THH}_*^{ko}(ku)$$

*collapses at the $E^2$-page and $\mathsf{THH}_*^{ko}(ku)$ is a square zero extension of $ku_*$:*

$$\mathsf{THH}_*^{ko}(ku) \cong ku_* \rtimes ku_*/2u\langle y_0, y_1, \ldots\rangle$$

*with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.*

*Proof.* Lemma 5.1 implies that the $E^2$-term of the Tor spectral sequence is

$$E^2_{*,*} = \mathsf{Tor}^{(ku\wedge_{ko}ku)_*}_{*,*}(ku_*, ku_*) = \mathsf{Tor}^{ku_*[\tilde{u}]/\tilde{u}^2-u^2}_{*,*}(ku_*, ku_*),$$

where $\varepsilon\colon ku_*[\tilde{u}]/\tilde{u}^2 - u^2 \to ku_*$, $\varepsilon(\tilde{u}) = u$ gives the module structure of $ku_*$ over $(ku \wedge_{ko} ku)_*$. We have a periodic free resolution of $ku_*$ as a module over $ku_*[\tilde{u}]/(\tilde{u}^2 - u^2)$

$$\ldots \xrightarrow{u-\tilde{u}} \Sigma^4 ku_*[\tilde{u}]/(\tilde{u}^2 - u^2) \xrightarrow{u+\tilde{u}} \Sigma^2 ku_*[\tilde{u}]/(\tilde{u}^2 - u^2) \xrightarrow{u-\tilde{u}} ku_*[\tilde{u}]/(\tilde{u}^2 - u^2).$$

Tensoring this down to $ku_*$ yields

$$\ldots \xrightarrow{0} \Sigma^4 ku_* \xrightarrow{2u} \Sigma^2 ku_* \xrightarrow{0} ku_*.$$

As $ku_*$ splits off $\mathsf{THH}^{ko}_*(ku)$ the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

For the $E^\infty$-term we therefore get $E^\infty_{0,*} \cong ku_*$, $E^\infty_{2j,*} = 0$ for $j > 0$, and $E^\infty_{2j+1,*} \cong (ku_*/2u)\langle y_j\rangle$ for $y_j$ in bidegree $(2j + 1, 4j + 2)$ if $j > 0$. Thus we have multiple contributions when the odd total degree is greater than or equal to 9; we claim that the additive extensions are all trivial.

The spectral sequence is one of $ku_*$-algebras, so in particular, one of $ku_*$-modules. In total degree 9 we only have the generators $y_1 \in E^\infty_{3,6}$ generating a copy of $\mathbb{Z}$ and $u^3y_0 \in E^\infty_{1,8}$ generating a copy of $\mathbb{Z}/2\mathbb{Z}$. Since the only extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$ is the trivial one, we conclude that

$$\mathsf{THH}^{ko}_9(ku) \cong \mathbb{Z}/2\mathbb{Z}\langle u^3y_0\rangle \oplus \mathbb{Z}\langle y_1\rangle$$

and moreover, since the image of $\mathsf{THH}^{ko}_9(ku)$ under the multiplication by powers of $u$ gives $F_3(\mathsf{THH}^{ko}_{9+2i}(ku))$ for all $i \geqslant 1$, that

$$F_3(\mathsf{THH}^{ko}_{9+2i}(ku)) \cong \mathbb{Z}/2\mathbb{Z}\langle u^{3+i}y_0\rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle u^iy_1\rangle$$

for all such $i$, concluding the calculation of $\mathsf{THH}^{ko}_{11}(ku)$ and $\mathsf{THH}^{ko}_{13}(ku)$. In total degree 15, we also get $y_2 \in E^\infty_{5,10}$, but since the only extension of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$ is the trivial one, we conclude that

$$\mathsf{THH}^{ko}_{15}(ku) \cong \mathbb{Z}/2\mathbb{Z}\langle u^6y_0\rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle u^3y_1\rangle \oplus \mathbb{Z}\langle y_2\rangle$$

and similarly that $F_5(\mathsf{THH}^{ko}_{15+2i}(ku))$ splits as a direct sum of all the $E^\infty$ contributions in filtration degree less than or equal to 5 for all $i \geqslant 1$, and we continue inductively.

Since the generators $y_i$ over $ku_*$ are all in odd degree, and their products cannot hit the direct summand $ku_*$ in filtration degree zero, their products are all zero.

□

**Remark 5.3.** The relative $\mathsf{THH}$-groups above are similar to the Hochschild homology groups of the Gaussian integers:

$$\mathsf{HH}^{\mathbb{Z}}_*(\mathbb{Z}[i]) \cong \mathsf{THH}^{H\mathbb{Z}}_*(H\mathbb{Z}[i]) = \begin{cases} \mathbb{Z}[i], & \text{for } * = 0,\\ \mathbb{Z}[i]/2i, & \text{for odd } *,\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\mathsf{HH}^{\mathbb{Z}}_*(\mathbb{Z}[i]) \cong \mathbb{Z}[i] \rtimes (\mathbb{Z}[i]/2i)\langle y_j, j \geqslant 0\rangle$$

with $|y_j| = 2j + 1$. Hence we might view $ko \to ku$ as being wildly ramified.

We consider the sequence of commutative $ko$-algebras $R = ko \to C = ko \to B = ku$ with $k = H\mathbb{F}_2$ or $k = H\mathbb{Z}_{(2)}$ and, (since $\mathsf{THH}^{ko}(ko, k) \simeq k$), we get cofiber sequences of commutative $k$-algebras

$$ku \wedge_{ko} k \to k \to \mathsf{THH}^{ko}(ku, k).$$

This yields a Tor-spectral sequence

$$E^2_{s,t} = \mathsf{Tor}^{\pi_*(ku\wedge_{ko}k)}_{s,t}(k_*, k_*) \Rightarrow \mathsf{THH}^{ko}_{s+t}(ku, k). \tag{3}$$

Wood's cofiber sequence identifies $ku$ as the cone on $\eta$: $\Sigma ko \to ko$. Thus we get a cofiber sequence

$$\Sigma k \to k \to ku \wedge_{ko} k$$

and $\pi_*(ku \wedge_{ko} k) \cong \pi_*(k \vee \Sigma^2 k) \cong \Lambda_{\pi_* k}(x_2)$ where $x_2$ is a generator of degree two.

For $k = H\mathbb{F}_2$ and $H\mathbb{Z}_{(2)}$ we can deduce with [DLR∞, 2.1] that as a commutative augmented $k$-algebra $ku \wedge_{ko} k$ is weakly equivalent to the square-zero extension $k \vee \Sigma^2 k$. Thus

$$\mathsf{THH}^{ko}(ku, k) \simeq k \wedge_{k \vee \Sigma^2 k} k$$

and the spectral sequence (3) reduces to

$$E^2_{s,t} = \mathsf{Tor}^{\pi_* k[x_2]/x_2^2}_{s,t}(\pi_* k, \pi_* k) \Rightarrow \mathsf{THH}^{ko}_{s+t}(ku, k).$$

But $\mathsf{Tor}^{\pi_* k[x_2]/x_2^2}_{s,t}(\pi_* k, \pi_* k) \cong \Lambda_{\pi_* k}(\varepsilon x_2) \otimes \Gamma_{\pi_* k}(\varphi^0 x_2)$ with $|\varepsilon x_2| = 3$, $|\varphi^0 x_2| = 6$, and we know from [BLPRZ15] combined with the methods from [DLR∞, Section 3] that there cannot be any differentials in this spectral sequence. Hence we obtain

**Proposition 5.4.**

$$\mathsf{THH}^{ko}_*(ku, H\mathbb{Z}_{(2)}) \cong \Lambda_{\mathbb{Z}_{(2)}}(\varepsilon x_2) \otimes \Gamma_{\mathbb{Z}_{(2)}}(\varphi^0 x_2)$$

*and also*

$$\mathsf{THH}^{ko}_*(ku, H\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\varepsilon x_2) \otimes \Gamma_{\mathbb{F}_2}(\varphi^0 x_2)$$

**Remark 5.5.** To the eyes of $\mathsf{THH}$ with coefficients in $H\mathbb{F}_p$ coefficients (for $p = 2$ resp. $p = 3$) the extensions $ko \to ku$ and $\ell \to ku_{(3)}$ show a similar behaviour. This is analogous to the algebraic case: Hochschild homology homology of the 2-local Gaussian integers with coefficients in $\mathbb{F}_2$ is isomorphic to $\Lambda_{\mathbb{F}_2}(x_1) \otimes \Gamma_{\mathbb{F}_2}(x_2)$ and $HH^{\mathbb{Z}_{(3)}}_*(\mathbb{Z}_{(3)}[\sqrt{3}], \mathbb{F}_3) \cong \Lambda_{\mathbb{F}_3}(x_1) \otimes \Gamma_{\mathbb{F}_3}(x_2)$. Thus Hochschild homology (and also higher Hochschild homology) with reduced coefficients doesn't distinguish tame from wild ramification either.

## 6. $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ as a model for $ku/(p, v_1)$

John Greenlees asks in [G16, Example 8.4] for a commutative $S$-algebra model of $ku/(p, v_1)$. We suggest $ku/(p, v_1) = ku_{(p)} \wedge_\ell H\mathbb{F}_p$ which is a commutative $S$-algebra (even an augmented commutative $H\mathbb{F}_p$-algebra, which might not be what Greenlees had in mind) and satisfies $\pi_*(ku_{(p)} \wedge_\ell H\mathbb{F}_p) \cong \mathbb{F}_p[u]/u^{p-1}$.

**Remark 6.1.** Alternatively one could consider $ku/(p, v_1)$ defined by an iterated cofiber sequence. This is an $A_\infty$-ring spectrum [A08, 3.7], hence an associative $S$-algebra, but we cannot expect any decent level of commutativity without the price of getting something of the homotopy type of a generalized Eilenberg-Mac Lane spectrum: if $ku(p, v_1)$ were a pseudo-$H_2$ spectrum, then it automatically splits as a wedge of suspensions of $H\mathbb{F}_p$'s [BMMS86, III.4.1]. In particular, an $E_\infty$-structure (*i.e.*, a commutative $S$-algebra structure) would lead to such a splitting.

We use Greenlees' cofiber sequence [G16, 7.1] in order to determine $\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p)$.

**Proposition 6.2.** *Topological Hochschild homology of $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ with coefficients in $H\mathbb{F}_p$ is*

$$\mathsf{THH}_*(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u)$$

*where $\mathbb{F}_p[\mu] = \mathsf{THH}_*(H\mathbb{F}_p)$.*

*Proof.* We consider the sequence of commutative $S$-algebras

$$S \to \ell \wedge_\ell H\mathbb{F}_p \simeq H\mathbb{F}_p \to ku_{(p)} \wedge_\ell H\mathbb{F}_p \to H\mathbb{F}_p = k,$$

and, since $(ku_{(p)} \wedge_\ell H\mathbb{F}_p) \wedge_{(\ell \wedge_\ell H\mathbb{F}_p)} H\mathbb{F}_p \simeq ku_{(p)} \wedge_\ell H\mathbb{F}_p$, we get an equivalence

$$\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge^L_{ku_{(p)} \wedge_\ell H\mathbb{F}_p} \mathsf{THH}(H\mathbb{F}_p).$$

Therefore, the $\mathsf{Tor}$-spectral sequence has $E^2$-term

$$\mathsf{Tor}^{\mathbb{F}_p[u]/u^{p-1}}_{*,*}(\mathbb{F}_p, \mathsf{THH}_*(H\mathbb{F}_p)).$$

We use the standard periodic resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[u]/u^{p-1}$. As $\mathsf{THH}(H\mathbb{F}_p)$ has the same chromatic type as $H\mathbb{F}_p$, $u$ acts by zero on $\mathsf{THH}_*(H\mathbb{F}_p) = \mathbb{F}_p[\mu]$ and hence the $E^2$-term is isomorphic to

$$\mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$

As $\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p)$ is an augmented commutative $\mathsf{THH}(H\mathbb{F}_p)$-algebra, the $\mathbb{F}_p[\mu]$-factor splits off and hence there cannot be any differentials and multiplicative extensions. □

## 7. Killing regular generators in $\pi_* R$

Killing regular elements in the homotopy groups of a commutative $S$-algebra rarely gives rise to commutative quotients. However, there are some important examples for which we *do* get commutative quotients whose relative $\mathsf{THH}$ can be calculated.

**Proposition 7.1.** *Let $R$ be a connective commutative $S$-algebra whose coefficients $\pi_* R$ are concentrated in even degrees, with a nonzero divisor $x$ of positive degree. If the canonical map $R \to R/x$ is a morphism of commutative $S$-algebras, then the Tor spectral sequence*

$$\mathsf{Tor}_{*,*}^{\pi_*(R/x \wedge_R R/x)}(R_*, R_*) \Rightarrow \mathsf{THH}_*^R(R/x)$$

*collapses at the $E^2$-term. Its $E^\infty$-term is isomorphic to $\Gamma_{\pi_* R/x}(\rho^0 \varepsilon x)$ with $|\rho^0 \varepsilon x| = |x| + 2$ and there are no additive extensions.*

*Proof.* The defining cofiber sequence

$$\Sigma^{|x|} R \xrightarrow{x} R \longrightarrow R/x$$

gives, via a $\mathsf{Tor}$-spectral sequence, that

$$\pi_*(R/x \wedge_R R/x) \cong \Lambda_{\pi_*(R)/x}(\varepsilon x)$$

with $|\varepsilon x| = |x| + 1$. In the spectral sequence for $\mathsf{THH}$ we have as an $E^2$-term

$$\mathsf{Tor}_{*,*}^{\Lambda_{\pi_*(R)/x}(\varepsilon x)}(\pi_* R/x, \pi_* R/x).$$

We consider the periodic resolution of $\pi_* R/x$

$$\ldots \xrightarrow{\varepsilon x} \Sigma^{2|x|+2} \Lambda_{\pi_* R/x}(\varepsilon x) \xrightarrow{\varepsilon x} \Sigma^{|x|+1} \Lambda_{\pi_* R/x}(\varepsilon x) \xrightarrow{\varepsilon x} \Lambda_{\pi_* R/x}(\varepsilon x)$$

and tensor it down to $\pi_* R/x$. As $\pi_* R/x$ is concentrated in even degrees, the multiplication by $\varepsilon x$ induces the trivial map and hence our Tor-terms are the homology of the complex

$$\ldots \xrightarrow{\varepsilon x = 0} \Sigma^{2|x|+2} \pi_* R/x \xrightarrow{\varepsilon x = 0} \Sigma^{|x|+1} \pi_* R/x \xrightarrow{\varepsilon x = 0} \pi_* R/x$$

and this gives a divided power algebra $\Gamma_{\pi_* R/x}(\rho^0 \varepsilon x)$ with a generator $\rho^0 \varepsilon x$ in degree $|x|+2$. We have to show that there are no non-trivial differentials and no extension problems. The spectral sequence is a spectral sequence of $\pi_* R/x$-algebras because $R/x$ is assumed to be a commutative $R$-algebra, hence $\mathsf{THH}^R(R/x)$ is a commutative $R/x$-algebra.

As we assumed that $x$ has positive degree, we can split $\Gamma_{\pi_* R/x}(\rho^0 \varepsilon x)$ as $\pi_* R/x \otimes_{\pi_0 R} \Gamma_{\pi_0 R}(\rho^0 \varepsilon x)$. The $\pi_* R/x$-module generators are the $\pi_0 R$-module generators in $\Gamma_{\pi_0 R}(\rho^0 \varepsilon x)$. These generators sit in bidegrees of the form $(n, n(|x|+1))$. A differential $d^r$ on a generator in bidegree $(n, n(|x|+1))$ is in bidegree $(n - r, n(|x|+1) + r - 1)$. A general element in the spectral sequence come from a product of powers of generators times an element from $R_*/x$, hence we get that a potential target has a bidegree of the form

$$(\sum_i u_i n_i, (\sum_i u_i n_i)(|x|+1) + 2m).$$

Comparing components of the bidegree gives the two equations

$$n - r = \sum_i u_i n_i \text{ and } n(|x|+1) + r - 1 = (|x|+1)(\sum_i u_i n_i) + 2m.$$

We rewrite the second equation as

$$2m + 1 = (n - \sum_i u_i n_i)(|x| + 1) + r.$$

Using that $n - \sum_i u_i n_i$ is $r$ yields $2m + 1 = r(|x| + 2)$, but the degree of $x$ is even, so there can be no non-trivial differentials in this spectral sequence.

We do not have additive extensions because the $E^\infty$-term is free over $\pi_* R/x$. Thus as an $\pi_* R/x$-module we get that $\mathsf{THH}^R_*(R/x)$ is isomorphic to $\pi_* R/x \otimes_{\pi_0 R} \Gamma_{\pi_0 R}(\rho^0 \varepsilon x)$. □

**Corollary 7.2.** *If in addition to the assumptions in Proposition 7.1 we have that $R/x$ is an Eilenberg-MacLane spectrum of a commutative ring $k$, then*

$$\mathsf{THH}^R(Hk, Hk) \simeq Hk \wedge_{Hk \vee \Sigma^{|x|+1} Hk} Hk$$

*as augmented commutative $Hk$-algebras. In particular,*

$$\mathsf{THH}^R_*(Hk) \cong \Gamma_k(\rho^0 \varepsilon x)$$

*with $|\rho^0 \varepsilon x| = |x| + 2$*

*Proof.* Greenlees' cofiber sequence identifies $\mathsf{THH}^R(Hk)$ as

$$Hk \wedge^L_{Hk \wedge_R Hk} Hk$$

using the sequence of commutative ring spectra $R = R \to Hk = Hk$. The homotopy groups of $Hk \wedge_R Hk$ are isomorphic to $\Lambda_k(\varepsilon x)$ with $|\varepsilon x| = |x| + 1$. Hence we know from [DLR∞, Proposition 2.1] that

$$Hk \wedge_R Hk \sim Hk \vee \Sigma^{|x|+1} Hk$$

with the square zero multiplication as augmented commutative $Hk$-algebras. Therefore we get the first claim. This also shows that $\mathsf{THH}^R(Hk)$ can be modeled as the two-sided bar construction

$$B^{Hk}(Hk, Hk \vee \Sigma^{|x|+1} Hk, Hk)$$

and by [DLR∞] we know that its homotopy groups are the homology groups of the algebraic bar construction $B^k(k, \Lambda(\varepsilon x), k)$. We know from [BLPRZ15, Proposition 2.3] that there is a quasi-isomorphism between $\Gamma_k(\rho^0 \varepsilon x)$ (with zero differential) and the differential graded commutative algebra associated to $B^k(k, \Lambda(\varepsilon x), k)$. □

**Remark 7.3.** Despite the fact that the result for $\mathsf{THH}$ in the above case looks like the examples from [DLR∞] where we could iterate the calculation and obtain higher $\mathsf{THH}$, in the above cases we have to consider $B^R(Hk, \mathsf{THH}^R(Hk), Hk)$ for determining $\mathsf{THH}^{R,[2]}_*(Hk)$, but here, the bar construction is relative to $R$ (and not $Hk$), so the iteration method from [DLR∞] does not apply.

**Proposition 7.4.** *Assume in addition to the requirements of Proposition 7.1 that there is a regular sequence $(x, y_1, \ldots, y_n)$ in $\pi_* R$ such that $R/(x, y_1, \ldots, y_n)$ is $Hk$ for some field $k$. Then*

$$\mathsf{THH}^R_*(R/x, Hk) \cong \Gamma_k(\rho^0 \varepsilon x)$$

*with $|\rho^0 \varepsilon x| = |x| + 2$.*

*Proof.* We consider the sequence of commutative $S$-algebras

$$R \to R \to R/x \to Hk.$$

Then $\pi_*(Hk \wedge_R R/x) \cong \Lambda_k(\varepsilon x)$ and as before we can conclude with [DLR∞, 2.1] that $Hk \wedge_R R/x$ is equivalent to the square zero extension $Hk \vee \Sigma^{|\varepsilon x|} Hk$ in the homotopy category of commutative augmented $Hk$-algebras.

Greenlees' cofiber sequence identifies $\mathsf{THH}^R(R/x, Hk)$ as

$$Hk \wedge^L_{Hk \vee \Sigma^{|\varepsilon x|} Hk} Hk$$

and we know from [DLR∞, BLPRZ15] that this gives $\mathsf{THH}^R_*(R/x, Hk) \cong \Gamma_k(\rho^0 \varepsilon x)$. □

**Examples 7.5.** We end the section with some examples.

(1) Let $R$ be an Eilenberg-MacLane spectrum $HA$ with $A$ a commutative ring and let $x$ be regular in $A$. Then $\mathsf{THH}^{HA}_*(HA/x)$ is isomorphic to Shukla-homology of $A/x$ over $A$, $SH^A_*(A/x)$. In this case we obtain
$$\mathsf{THH}^{HA}_*(HA/x) \cong SH^A_*(A/x) \cong \Gamma_{A/x}(\rho^0\varepsilon x)$$
with $|\rho^0\varepsilon x| = 2$. An explicit generator of $SH^A_{2m}(A/x)$ is given by
$$\sum_{i=0}^{m}(-1)^i\tau^{\otimes i} \otimes 1 \otimes \tau^{\otimes m-i}.$$
Here, we consider the resolution of $A/x$ that is of the form $(A[\tau]/\tau^2, d(\tau) = x)$.

(2) Recall that the connective covers of the Morava $E$-theories, $e_n$, have coefficients
$$\pi_*(e_n) \cong W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u]$$
with $|u| = 2$, where $W\mathbb{F}_{p^n}$ denotes the Witt vectors over $\mathbb{F}_{p^n}$ and where the $u_i$ are generators in degree zero. Hence $\pi_0(e_n) = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]]$. The quotient $e_n/u = HW\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]]$ is a commutative $S$-algebra and the map $e_n \to e_n/u$ can be realized as a map of commutative $S$-algebras.

The residue field $H\mathbb{F}_{p^n}$ is the quotient $e_n/(u, u_1, \ldots, u_{n-1}, p)$ and Corollary 7.2 and Proposition 7.4 calculate $\mathsf{THH}^{e_n}_*(e_n/u, e_n/u)$ and $\mathsf{THH}^{e_n,[m]}_*(e_n/u, H\mathbb{F}_{p^n})$ for all $m \geqslant 1$.

(3) Lawson and Naumann show in [LN12] that $BP\langle 2\rangle$ at the prime two is a commutative $S$-algebra by identifying it with the 2-localized connective spectrum of topological modular forms together with a level three structure, $\mathsf{tmf}_1(3)_{(2)}$. They prove in [LN14, section 5] that there is a map of commutative $S$-algebras $\varrho\colon \mathsf{tmf}_1(3)_{(2)} \to ku_{(2)}$ and there is a complex orientation of $\mathsf{tmf}_1(3)_{(2)}$ such that the effect of $\varrho$ on homotopy groups is as follows [LN14, section 5]:
$$\pi_*(\mathsf{tmf}_1(3)_{(2)}) = \mathbb{Z}_{(2)}[a_1, a_3] \to \mathbb{Z}_{(2)}[u], \quad a_1 \mapsto u, \quad a_3 \mapsto 0.$$
Here the degree of $a_i$ is $2i$.

With Propositions 7.1 and 7.4 we can determine $\mathsf{THH}^{\mathsf{tmf}_1(3)_{(2)}}_*(ku_{(2)})$ additively and relative $\mathsf{THH}$ of $ku_{(2)} \cong \mathsf{tmf}_1(3)_{(2)}/a_3$ with respect to $\mathsf{tmf}_1(3)_{(2)}$ and with coefficients in $H\mathbb{F}_2 = \mathsf{tmf}_1(3)_{(2)}/(a_3, a_1, 2)$.

(4) The discretization map from $ku$ to $H\mathbb{Z} = ku/u$ gives rise to another example of a regular quotient with a commutative $S$-algebra structure with residue field $H\mathbb{F}_p = ku/(u, p)$ for any prime $p$, and so does the map from the connective Adams summand $\ell$ to $H\mathbb{Z}_{(p)} = \ell/v_1$ with residue field $H\mathbb{F}_p = \ell/(v_1, p)$.

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Department of Mathematics, University of Bergen, Postboks 7800, 5020 Bergen, Norway
*E-mail address*: `dundas@math.uib.no`
*URL*: `http://folk.uib.no/nmabd/`

Mathematics Department, Indiana University, 236 Rawles Hall, Bloomington, IN 47405, USA
*E-mail address*: `alindens@indiana.edu`
*URL*: `http://pages.iu.edu/~alindens/`

Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
*E-mail address*: `richter@math.uni-hamburg.de`
*URL*: `http://www.math.uni-hamburg.de/home/richter/`