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CONSISTENT SYSTEMS OF CORRELATORS IN NON-SEMISIMPLE CONFORMAL FIELD THEORY

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Abstract

Based on the modular functor associated with a – not necessarily semisimple – finite non-degenerate ribbon category $\mathcal{D}$, we present a definition of a consistent system of bulk field correlators of a conformal field theory which comprises invariance under mapping class group actions and compatibility with the sewing of surfaces. We show that when restricting to surfaces of genus zero such systems are in bijection with commutative symmetric Frobenius algebras in $\mathcal{D}$, while for surfaces of any genus they are in bijection with modular Frobenius algebras in $\mathcal{D}$. This extends structures familiar from rational conformal field theories to rigid logarithmic conformal field theories.

1 Introduction and main result

A crucial task in any quantum field theory is to establish the existence of a consistent system of correlators for the fields of the theory. In two-dimensional conformal field theories these correlators are specific elements in suitable spaces of conformal blocks, characterized by the fact that they satisfy various consistency conditions. The spaces of conformal blocks of a conformal field theory can be explored from several different mathematical points of view. The approach relevant to the present paper describes them as finite-dimensional vector spaces that carry projective representations of mapping class groups of surfaces with marked points and are compatible with the sewing of surfaces. (The fact that these spaces are finite-dimensional is a non-trivial and useful finiteness property, which is fulfilled in many interesting classes of conformal field theories.) These finite-dimensional vector spaces are constructed in terms of morphism spaces of a braided monoidal category $\mathcal{D}$ [Ly2, BK2]; we refer to the data coming with the spaces of conformal blocks as the *monodromy data* based on $\mathcal{D}$.

Specifically, we consider local conformal field theories on closed oriented surfaces. For these, the fields are called bulk fields, and a consistent choice of bulk fields provides an object of $\mathcal{D}$, which we denote by F and to which for brevity we refer as the *bulk object*.

In this paper we give a precise mathematical realization of the consistent choices of bulk object and of systems of correlators of bulk fields for a conformal field theory corresponding to a given category $\mathcal{D}$. A novelty of our approach is that $\mathcal{D}$ does not have to be semisimple, whereby it becomes relevant for important applications of non-semisimple conformal field theories like e.g. the theory of critical dense polymers. (Because of the analytic properties of their conformal blocks, such theories go under the name of logarithmic conformal field theories.) After developing a precise definition of the notion of consistency of a system of bulk field correlators (see Definition 3.16), we prove:

Theorems 4.6 and 4.8.

(i) *Let $\mathcal{D}$ be a finite ribbon category and F an object of $\mathcal{D}$. The consistent systems of genus-zero bulk field correlators for monodromy data based on $\mathcal{D}$ and with bulk object F are in bijection with structures of a commutative symmetric Frobenius algebra on F.*

(ii) *Let $\mathcal{D}$ be a modular finite ribbon category and F an object of $\mathcal{D}$. The consistent systems of bulk field correlators for monodromy data based on $\mathcal{D}$ and with bulk object F are in bijection with structures of a modular Frobenius algebra (in the sense of Definition* 4.7) on F.

Consistency conditions for correlators have been discussed extensively in the conformal field theory literature (see e.g. [FriS, So, Le]). They amount to requiring that the correlator assigned to a surface is invariant under the action of the mapping class group of the surface and that upon sewing of surfaces, correlators are mapped to correlators [FFFS]. (In addition, a non-degeneracy requirement must be imposed on the two-point correlator on the sphere.) An insight on which the present paper builds is that these requirements can be implemented with the help of the structural morphisms of the coends that in the construction of [Ly2] afford the sewing of spaces of conformal blocks.

The notion of a modular finite ribbon category is recalled in Section 2.3; the category H-mod of finite-dimensional modules over any finite-dimensional factorizable ribbon Hopf algebra H belongs to this class of categories. Modular Frobenius algebras are introduced in Definition

4.7; it is known [FSS1] that in case $\mathcal{D}$ is the enveloping category $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ of a category $\mathcal{C} = H$-mod of the type just mentioned, every ribbon automorphism of the identity functor of $\mathcal{C}$ gives rise to a modular Frobenius algebra in $\mathcal{D}$. We expect that this continues to hold for the enveloping category of any modular finite ribbon category $\mathcal{C}$, but the methods of [FSS1] which are based on Hopf algebra technology are insufficient to show this. Also, even when restricting to the case $\mathcal{C} = H$-mod it is not known whether those methods can be used to address the compatibility of correlators with sewing, while they do allow one [FSS2] to establish invariance under the action of mapping class groups. For $\mathcal{D}$ the enveloping category of a *semisimple* modular finite ribbon category $\mathcal{C}$, modular Frobenius algebras in $\mathcal{D}$ are obtained by applying a center construction [FrFRS, Da] to the strongly separable symmetric Frobenius algebras in $\mathcal{C}$ that were considered in [FRS, FFRS1]. In the present paper we rely on finiteness properties that are shared by these examples, but we neither have to require semisimplicity nor equivalence to the representation category of a Hopf algebra.

Before giving more details, let us put our results into context. While quantum field theory is studied in many different frameworks, a minimal consensus would be close to the following: To establish the existence of a quantum field theory or, at least, of the subsector of bulk fields of the theory, we must specify a vector space of such fields as well as, for a suitable category of manifolds with marked points, correlators for those fields. The correlators are required to be local in a suitable sense, and correlators on different manifolds must fit together.

This is far too vague for being implementable in practice, but for classes of theories that enjoy suitable finiteness properties and share certain symmetries the situation is more amenable. Symmetries strongly constrain the possible correlators: they give rise to differential equations for them, so-called Ward identities. The spaces of solutions to those equations – called spaces of conformal blocks in the case of conformal field theories – are thus spaces of candidates for correlators. In the present paper we impose the finiteness condition that the spaces of solutions to the Ward identities, at any genus and for any number of field insertions, are finite-dimensional.

One can then impose the relevant locality requirements and relations between correlators on different manifolds on the so obtained spaces of candidates for correlators and thereby specify the actual correlators as particular vectors in these spaces.

As we show in this paper, this idea can be fully realized in a large class of two-dimensional conformal field theories, including non-semisimple theories. We do not work with vector spaces obtained as solutions to differential equations (but, prompted by that interpretation, still use the term monodromy data to refer to the corresponding information). Instead, we obtain an equivalent system of vector spaces algebraically, as morphism spaces of a suitable $\Bbbk$-linear monoidal category $\mathcal{D}$, which is possible if $\mathcal{D}$ is a modular braided finite tensor category. This class of categories includes all semisimple modular tensor categories, such as those for the rational conformal field theories of Wess-Zumino-Witten type, for which the differential equations obeyed by the conformal blocks are known explicitly and include in particular the equation expressing the flatness of the Knizhnik-Zamolodchikov connection (see [EFK] for a review). For our construction $\mathcal{D}$ does, however, not have to be semisimple, and thus our analysis covers theories well beyond the subclass of rational conformal field theories. Also, albeit for us $\mathcal{D}$ matters only as a category endowed with appropriate structure, it is adequate to think of $\mathcal{D}$ as the representation category of an algebraic structure – a conformal vertex algebra, or a conformal net of observables – that formalizes the physical notion of a chiral algebra of a two-dimensional conformal field theory.

The space F of bulk fields we are looking for carries a representation of the chiral algebra (or of the tensor product of two copies of the chiral algebra, which in physics are sometimes called left- and right-movers, respectively) and is thus an object of the category $\mathcal{D}$. In this paper we identify the additional structure on F that is necessary and sufficient for obtaining a consistent system of correlators. The existence of such structure imposes restrictions on the category $\mathcal{D}$.

Let us now describe the contents of this paper more explicitly. We consider a symmetric monoidal category $\mathcal{S}urf$ of oriented smooth surfaces with boundary, with the monoidal structure given by disjoint union. As morphisms of $\mathcal{S}urf$ we take orientation preserving diffeomorphisms combined with sewings $E \mapsto \cup E$, i.e. (compare Figure 1(ii) below) with compositions of surfaces via identification of boundary circles. The boundary circles are the recipients of insertions of (incoming or outgoing) bulk fields, which are described as an object F of the category $\mathcal{D}$. To each surface $E \in \mathcal{S}urf$ we assign, in a two-step procedure, a bulk field correlator as a vector $\mathrm{v}(E)$ in a vector space $\mathrm{Bl}(E)$. The vector spaces $\mathrm{Bl}(E)$, which are the conformal block spaces, are obtained as morphism spaces of the $\Bbbk$-linear category $\mathcal{D}$; this first step is well known, albeit (see Section 2) it still needs to be adapted to fit our purposes.

Each space $\mathrm{Bl}(E)$ carries a projective representation of the mapping class group of E, and the collection of these spaces has locality properties. The latter are encoded in linear maps between spaces of conformal blocks assigned to surfaces of different topology that are related by sewing. The novel main part of our construction is then to find a system of vectors $\mathrm{v}(E) \in \mathrm{Bl}(E)$ that are invariant under the mapping class group action and such that sewing $E \mapsto \cup E$ transports $\mathrm{v}(E)$ to $\mathrm{v}(\cup E)$ – provided that such a set of vectors exists at all, a requirement that puts restrictions on the allowed categories $\mathcal{D}$ and bulk objects F in $\mathcal{D}$. A crucial idea of the construction is to start from simple building pieces not only for the spaces of conformal blocks, but correspondingly also for vectors in those spaces as building blocks of the correlators.

Our choice of morphisms of the category $\mathcal{S}urf$ allows us to treat compatibility with sewing and mapping class group invariance on the same footing. Concretely, we can describe the conformal blocks $\mathrm{Bl}(E)$ as the vector spaces that a symmetric monoidal functor Bl assigns to the objects of $\mathcal{S}urf$, and a consistent system $\{\mathrm{v}(E)\}$ of bulk field correlators as a monoidal natural transformation

$$\mathrm{v}: \quad \Delta_{\Bbbk} \Longrightarrow \mathrm{Bl}$$

between a constant functor $\Delta_{\Bbbk}$, with value the ground field, from $\mathcal{S}urf$ to $\mathcal{V}ect$ and the block functor Bl.

At the level of the vector spaces of conformal blocks, locality can be implemented by using pair-of-pants decompositions of the objects $E \in \mathcal{S}urf$. Moreover, such decompositions also allow one to get control over the action of the mapping class group, at the expense of endowing the surfaces with further structure, namely a certain embedded graph called a *marking* [BK1]. With these extra structures of pair-of-pants decompositions and markings, we get a symmetric monoidal category $m\mathcal{S}urf$ of *marked surfaces* together with a forgetful functor U: $m\mathcal{S}urf \rightarrow \mathcal{S}urf$. For the construction of the functor Bl it proves to be convenient to make explicit use of the extra structure of $m\mathcal{S}urf$, whereby at first one arrives at a similar symmetric monoidal functor $\widetilde{\mathrm{Bl}}$: $m\mathcal{S}urf \rightarrow \mathcal{V}ect$. (Actually, to account for the framing anomaly, one must work with central extensions of the categories $\mathcal{S}urf$ and $m\mathcal{S}urf$, see Section 3.2; for brevity we suppress this issue in this introductory exposition.)

Our strategy for proving our main result is now the following: First, we construct the monoidal functor Bl as a right Kan extension

$$\begin{tikzcd} m\mathcal{S}urf \arrow[r, "\widetilde{\mathrm{Bl}}"] \arrow[d, "U"'] & \mathcal{V}ect \\ \mathcal{S}urf \arrow[ur, dashed, "\mathrm{Bl}"'] \arrow[u, Rightarrow, shorten=3mm] & \end{tikzcd}$$

of $\widetilde{\mathrm{Bl}}$ along the forgetful functor U. Hereby we can reduce the existence of a monoidal natural transformation $\mathrm{v}\colon \Delta_{\Bbbk} \Rightarrow \mathrm{Bl}$ to the existence of an analogous natural transformation

$$\widetilde{\mathrm{v}}: \quad \widetilde{\Delta}_{\Bbbk} \Longrightarrow \widetilde{\mathrm{Bl}}$$

from a constant functor $\widetilde{\Delta}_{\Bbbk}\colon m\mathcal{S}urf \to \mathcal{V}ect$ to the functor $\widetilde{\mathrm{Bl}}$. Second, we establish necessary and sufficient conditions for the existence of such a natural transformation $\widetilde{\mathrm{v}}$. The latter is achieved with the help of the description of the morphisms of $m\mathcal{S}urf$ through generators and relations (a variant of the so-called Lego-Teichmüller game [HT, BK1]), which in particular allows us to express the required restrictions on the bulk object F in terms of a triple of vectors in specific morphism spaces involving F, namely in $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F^{\otimes 3})$, $\mathrm{Hom}_{\mathcal{D}}(F, \mathbf{1})$ and $\mathrm{Hom}_{\mathcal{D}}(F, F^{\vee})$, respectively. It turns out that once the problem is reformulated in terms of this triple of morphisms, it can be solved, and in particular the required additional structure of F is expressible in terms of the chosen triple of morphisms. It is in fact an old idea in conformal field theory that correlators on arbitrary surfaces are determined by a small set of correlators in low genus, subject to only a few consistency constraints. Our results make this idea precise for theories that are not required to be semisimple.

The rest of this paper is organized as follows. Section 2 is preparatory; it collects pertinent information about the Lego-Teichmüller game [BK1] and about the construction of conformal blocks, adapting the results of [Ly2] such that they can be combined with the framework of [BK1] that was designed for finite semisimple categories. Specifically, we explain the notions of (extended) surfaces and marked surfaces and the groupoid of fine markings on a surface, and describe the conformal blocks for a finite ribbon category $\mathcal{D}$ as $\Bbbk$-linear functors from tensor powers of $\mathcal{D}$ to $\mathcal{V}ect$. In Section 3 we combine, for a choice of object $F \in \mathcal{D}$, the conformal block functors for all surfaces to a monoidal functor $\mathrm{Bl}^{(F)}$ from (a central extension of) a symmetric monoidal category $\mathcal{S}urf$ of surfaces with values in vector spaces. Invariance of the correlators under the action of mapping class groups and compatibility with sewing are then formulated as a monoidal natural transformation from a constant functor $\Delta_{\Bbbk}$ to the functor $\mathrm{Bl}^{(F)}$. For arriving at this formulation we first work with a monoidal functor $\widetilde{\mathrm{Bl}}^{(F)}$ having as domain a category $m\mathcal{S}urf$ of marked surfaces and then obtain $\mathrm{Bl}^{(F)}$ via a Kan extension. Finally in Section 4 we show that, subject to a non-degeneracy condition, the existence of a monoidal natural transformation $\Delta_{\Bbbk} \Rightarrow \mathrm{Bl}^{(F)}$ is equivalent to F having the properties stated in Theorems 4.6 and 4.8.

2 Conformal blocks

In this section we collect pertinent background information on marked surfaces, finite ribbon categories and conformal blocks. The expert reader may wish to have just a quick glance at this part, e.g. at Definitions 2.1, 2.3 and 2.4, at the figures 1 and 2, at the list of elementary moves (M1) – (M5) in Section 2.2, and at the formulas (2.9) and (2.10) for the conformal blocks. It should be appreciated that (2.9) and (2.10) do not require semisimplicity.

2.1 Marked surfaces

By a *surface* E we mean a compact oriented smooth two-dimensional manifold, possibly disconnected and possibly with boundary. We endow each connected component α of the boundary ∂E with the structure of an oriented 1-manifold. If the 1-orientation on α coincides with the one induced by the 2-orientation of E, we refer to α as an *outgoing* boundary circle, otherwise as an *incoming* one, and denote the union of all outgoing and all incoming boundary circles by $\partial_{\text{out}}E$ and $\partial_{\text{in}}E$. respectively. We call $(E, \partial_{\text{in}}E, \partial_{\text{out}}E)$ a *surface with oriented boundary.* The surfaces of our interest are endowed with additional structure on the boundary (compare [BK1, Def. 2.1] for surfaces with non-oriented boundary circles):

Definition 2.1. An *extended surface* $E = (E, \partial_{\text{in}}E, \partial_{\text{out}}E, \{p_\alpha\})$ is a surface $(E, \partial_{\text{in}}E, \partial_{\text{out}}E)$ with oriented boundary together with a marked point p_α on each connected component of ∂E.

Definition 2.2. The *mapping class group* $\mathrm{Map}(E)$ of an extended surface E is the group of homotopy classes of orientation preserving diffeomorphisms $E \to E$ that map $\partial_{\text{in}}E$ to $\partial_{\text{in}}E$ (and thus $\partial_{\text{out}}E$ to $\partial_{\text{out}}E$) and map marked points to marked points.

The *genus* $g(E)$ of an extended surface is the genus of the closed surface that is obtained from E by gluing disks to all boundary components. An extended surface can be glued, or sewn: for $(\alpha, \beta) \in \pi_0(\partial_{\text{in}}E) \times \pi_0(\partial_{\text{out}}E)$, the sewn surface $\cup_{\alpha,\beta}E$ is the extended surface obtained by identifying the boundary components α and β in such a way that their orientations match and that their marked points get identified. (In principle we must glue along collars for $\cup_{\alpha,\beta}E$ to be smooth; but for our purposes this is inessential, compare e.g. [Ko, Thm. 1.3.12].)

A *cut* on an extended surface E is a smooth simple closed curve in the interior of E together with a distinguished point on the curve. A *cut system* [HT] on E is a finite set C of disjoint cuts, such that each connected component of $E \setminus C$ has genus 0. Given a cut system C, let $\mathrm{cut}_C(E)$ denote the closed manifold obtained from the disjoint union of all components of $E \setminus C$ by suitably adding two copies of each cut in C. This manifold acquires the structure of an extended surface by endowing the two components of $\partial\mathrm{cut}_C(E)$ that come from a cut $c \in C$ with the 1-orientation induced by the 2-orientation of $\mathrm{cut}_C(E)$ and by taking as marked points on them the points that come from the distinguished point of c. This is illustrated in Figure 1. Note that cutting and sewing are inverse operations: for any cut c on E, sewing the cut surface $\mathrm{cut}_{\{c\}}(E)$ along the two boundary components stemming from the cut c gives back E.

We need to refine this notion of surface further. Similarly as in [BK1, Sect. 2.3] we first introduce specific reference surfaces. For $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{\pm 1\}^n$, let the *standard sphere* $S^\circ_{n;\varepsilon}$ be the extended surface obtained by removing from the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with standard orientation n open disks D_α of radius 1/3 centered at the points $1, 2, \dots, n$, with the component $(\partial S^\circ_{n;\varepsilon})_\alpha = \partial D_\alpha$ of the boundary outgoing if $\varepsilon_\alpha = +1$ and incoming if $\varepsilon_\alpha = -1$, and with the marked points being $\{\alpha - \mathrm{i}/3 \,|\, \alpha = 1, 2, \dots, n\}$. The *standard marking* Γ°_n on $S^\circ_{n;\varepsilon}$ is the following graph on $S^\circ_{n;\varepsilon}$: The vertices are the marked points $\alpha - \mathrm{i}/3$, for $\alpha = 1, 2, , \dots, n$, together with a vertex at $-2\mathrm{i} \in S^\circ_{n;\varepsilon}$, called the *internal vertex*; the edges are the n straight

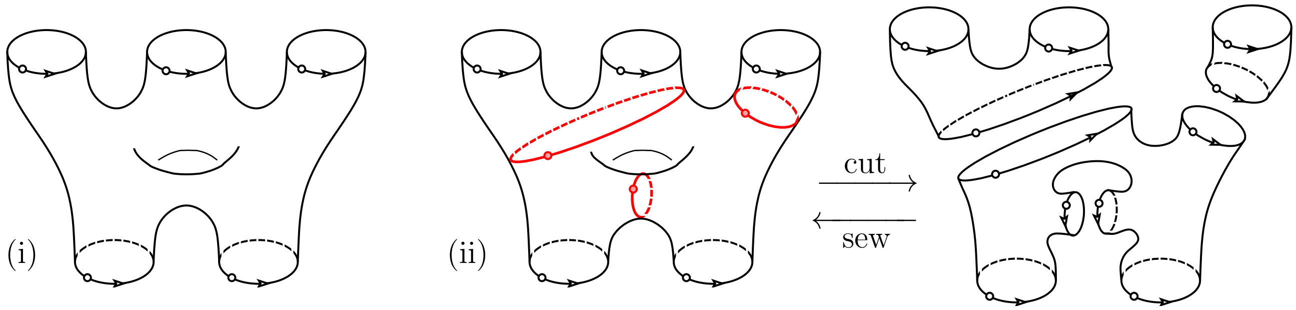

Figure 1: (i) An extended surface E of genus 1 with three outgoing and two incoming boundary circles. (ii) A (non-fine) cut system C on E (left) and the cut surface $\mathfrak{cut}_C(E)$, having spheres with 3, 2 and 6 boundary circles as connected components (right).

lines e_α connecting the marked points α with the internal vertex. The set of edges of Γ_n° is ordered according to the standard order on $\mathbb{N}$; the edge connecting the internal vertex to the left-most marked point $1-\mathrm{i}/3$ is called the *distinguished edge* and denoted by $e_\uparrow$, and $1-\mathrm{i}/3$ is called the *distinguished vertex*. We can now give

Definition 2.3. [BK1, Def. 3.3 & Sect. 3.6]
(i) A *marking without cuts* on a connected extended surface E of genus zero is a graph Γ on E for which there exists an orientation preserving diffeomorphism $\varphi\colon E \to S^\circ_{n;\varepsilon}$, for some appropriate $n \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}^n$, such that $\Gamma = \varphi^{-1}(\Gamma_n^\circ)$.
(ii) A *marked surface* (E, C, Γ) is an extended surface E endowed with the structure of a *marking*, i.e. with a cut system C and a graph Γ that provide a marking without cuts on every connected component of $\mathfrak{cut}_C(E)$.
(iii) Two markings (C, Γ) and (C', Γ') on E are called *isotopic* iff there exists an isotopy $f\colon E \times [0,1] \to E$ such that $(f(C,t), f(\Gamma,t))$ furnishes a marking on E for every $t \in [0,1]$ as well as $f(-,0) = id_E$ and $f(C,1) = C'$, $f(\Gamma,1) = \Gamma'$.
(iv) For $(\alpha,\beta) \in \pi_0(\partial_{\mathrm{in}}E) \times \pi_0(\partial_{\mathrm{out}}E)$, the *sewn surface* $(E', C', \Gamma') = \cup_{\alpha,\beta}(E, C, \Gamma)$ is the marked surface with underlying surface $E' = \cup_{\alpha,\beta}E$ whose cut system C' is given by the union of the cut system C of (E, C, Γ) and the image of α and β, and whose graph Γ' is obtained from Γ by gluing and by taking the image of the marked point on α and β as an additional vertex (compare [BK1, Fig. 6]).

In the sequel we often suppress the cut system in our notation and just write (E, Γ) for a marked surface, and refer to an isotopy class of markings just as a marking. It will be sufficient to work with the subclass of *fine* markings:

Definition 2.4. A *fine cut system* on a surface E is a cut system C for which every connected component of $\mathfrak{cut}_C(E)$ is a sphere with at most three holes. A *fine marking* (C, Γ) of E is a marking for which the cut system C is fine.

As an illustration, Figure 2 shows a fine cut system and a fine marking on the surface from Figure 1.

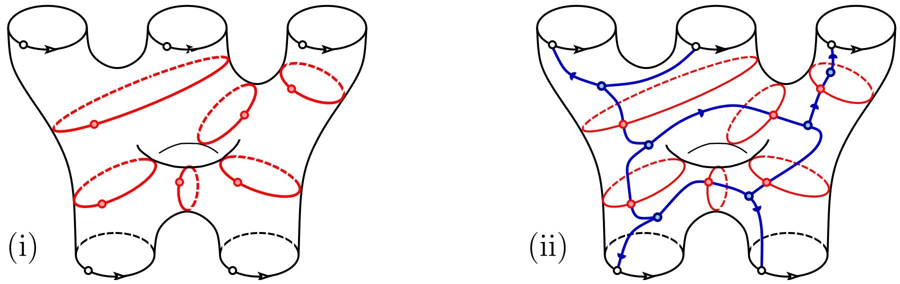

Figure 2: (i) A fine cut system C on the surface E from Figure 1. (The resulting cut surface is the disjoint union of one 2-holed and five 3-holed spheres.) (ii) A fine marking (C,Γ) on E. The distinguished edge of the restriction of the graph Γ to each connected component of the cut surface is accentuated by a small triangular flag.

2.2 Fine markings

To any extended surface E one can associate a groupoid describing the set of isotopy classes of markings on E and their relations [HLS, FuG, BK1]. The (isotopy classes of) fine markings on E form the objects of an equivalent groupoid $\mathcal{FM}(E)$; its morphisms are (classes of) finite sequences of moves that change a fine marking of E to another one. A geometric realization of $\mathcal{FM}(E)$ is furnished by a graph with vertices given by the objects of $\mathcal{FM}(E)$. This graph is connected and simply connected, and accordingly $\mathcal{FM}(E)$ can be presented by generators and relations (see [BK1, Thm. 5.1] and [BP]). The generators, to be called elementary moves, of $\mathcal{FM}(E)$ and the relations satisfied by them can be taken as follows [BK1, Sect. 5]. There are five types of elementary moves:

(M1) The *Z-move* Z of a sphere E with two or three holes. This move maps the graph Γ on E to the graph Γ' that coincides with Γ as an unordered graph and whose distinguished edge is the one adjacent to the distinguished edge of Γ in clockwise direction, [1] and which keeps the cyclic ordering of the edges.

(M2) The *B-move* B of a sphere E with three holes and without cuts [BK1, Fig. 10]. For the case that $E = S_3^\circ$ is the 3-holed standard sphere with standard marking, the move B results in a marking that can also be obtained by performing the following braiding diffeomorphism: move the boundary circles centered at $1 \in \mathbb{C}$ and $2 \in \mathbb{C}$ by an angle π clockwise around $\frac{3}{2} \in \mathbb{C}$ such that each of them is mapped to the previous position of the other one, while the third boundary circle is kept in place.

(M3) The *F-move* F of a sphere $E = (E, \{c\}, \Gamma)$ with at most three holes, with a single cut c such that one of the two components of $\mathfrak{cut}_{\{c\}}(E)$ is a sphere with one or two holes, and a graph Γ such that the edges of Γ ending at the distinguished point $p_c \in c$ are the distinguished edge on one of the components of $\mathfrak{cut}_{\{c\}}(E)$ and the 'last' edge on the other component. $\mathrm{F} \equiv \mathrm{F}_c$ removes the cut c and contracts the edges of Γ ending at p_c to a point (compare [BK1, Fig. 9]).

(M4) The *A-move* A of a sphere E with four holes and a single cut c and a graph Γ such that the edges of Γ ending at the distinguished point of c are the 'last' edges of the graphs on

[1] Our convention differs from the one in [BK1], where the distinguished edge is changed in counter-clockwise direction instead.

both components of $\mathfrak{cut}_{\{c\}}(E)$. $\mathrm{A} \equiv \mathrm{A}_c \equiv \mathrm{A}_{c,c'}$ replaces c by another cut c' not isotopic to c such that $\mathfrak{cut}_{\{c'\}}(E)$ consists of two three-holed spheres; for details see [BK1, Fig. 20].

(M5) The *S-move* S of a one-holed torus T with a single cut. $\mathrm{S} \equiv \mathrm{S}_{c_1,c_2}$ maps the marking $(\{c_1\}, \Gamma_1)$ of T to $(\{c_2\}, \Gamma_2)$, with $\{c_1, c_2\}$ a symplectic homology basis of $H_1(T, \mathbb{Z})$, and with Γ_1 and Γ_2 graphs having a common single vertex in the interior of T and two edges $\{e_\uparrow, e_1'\}$ and $\{e_\uparrow, e_2'\}$, respectively. The common distinguished edge $e_\uparrow$ connects the interior vertex with the boundary circle, while e_1' and e_2' are loops homotopic to c_2 and c_1, respectively [BK1, Fig. 16].

And the relations among these moves can be taken to be the following equalities between isotopy classes of compositions of moves (throughout, as in [BK1], some easily reconstructed intermediate Z-moves are omitted in order to improve readability):

(W1) Commutativity of moves in different connected components of E.

(W2) The cylinder axiom: Given the standard cylinder $S = (S^\circ_{1,1}, \emptyset, \Gamma^\circ_{1,1})$ with standard marking and with one incoming and one outgoing boundary component, a gluing $\cup_{\gamma,\beta}$ of S to a surface (E, C, Γ) (with $\beta \in \pi_0(\partial S)$ and $\gamma \in \pi_0(\partial E)$) and any move $\mathrm{m}\colon (E, \Gamma) \to (E, \Gamma')$, one has

$$\psi \circ \mathrm{F}_\gamma \circ (\mathrm{m} \cup_{\gamma,\beta} id) = \mathrm{m} \circ \psi \circ \mathrm{F}_\gamma$$

with ψ a morphism which amounts to a compression of $(E, \Gamma) \cup_{\gamma,\beta} (S^\circ_{1,1}, \emptyset, \Gamma^\circ_{1,1})$ to (E, Γ) (compare [BK1, Fig. 12]).

(W3) For any sphere with $n \in \{2, 3\}$ holes, the Z-move obeys $\mathrm{Z}^n = id$.

(W4) Compatibility of F- and Z-moves: For an F-move of a surface E with a single cut c such that $\mathfrak{cut}_{\{c\}}(E) = E_1 \sqcup E_2$ with E_1 the component containing the distinguished edge ending on c and $n_1 := |\pi_0(\partial E_1)|$ one has $\mathrm{Z}^{1-n_1} \circ \mathrm{F} = \mathrm{F} \circ (\mathrm{Z} \sqcup \mathrm{Z}^{-1})$.

(W5) Compatibility of B- and Z-moves: For $(E, \emptyset, \Gamma)$ a cylinder with a marking without cuts one has $\mathrm{Z} \circ \mathrm{B} = \mathrm{B} \circ \mathrm{Z}$.

(W6) Commutativity of F-moves involving a cylinder: If (C, Γ) is a fine marking on E with cut system $C = \{c, d\}$ such that one of the components of $\mathfrak{cut}_C(E)$ is a cylinder, then $\mathrm{F}_c \circ \mathrm{F}_d = \mathrm{F}_d \circ \mathrm{F}_c$.

(W7) Involutivity of the A-move: $\mathrm{A}^2 = id$.

(W8) The triangle axiom: For a marking of a 3-holed sphere with cut system $C = \{c, d\}$ such that when cutting along d one of the resulting connected components is a one-holed sphere, one has $\mathrm{F}_{c'} \circ \mathrm{F}_d \circ \mathrm{A} = \mathrm{F}_c \circ \mathrm{F}_d$, where c' is the cut created by the A-move (for details see [BK1, Fig. 29]).

(W9) The pentagon axiom for the A-move, an analogue of the pentagon identity for the associator of a monoidal category: For a fine marking (C, Γ) of a 5-holed sphere with $C = \{c, d\}$ and Γ a multiperipheral graph, one has $\mathrm{A}_{c'} \circ \mathrm{A}_d \circ \mathrm{A}_c = \mathrm{A}_c \circ \mathrm{A}_d$, where c' is the cut that is created by the A-move A_c (compare [BK1, Fig. 30]).

(W10) Two hexagon axioms for the B- and A-moves, analogues of the hexagon identities for the braiding and associator of a braided monoidal category: For S a sphere with four holes, labeled α, β, γ and δ, and a fine marking $(\{c\}, \Gamma)$ on S with Γ a multiperipheral graph whose distinguished edges end on the boundary component α and on the cut c, respectively, one has $\mathrm{B}_{\alpha,\gamma} \circ \mathrm{A}_c \circ \mathrm{B}_{\alpha,\beta} = \mathrm{A}_{c'} \circ \mathrm{B}_{\alpha,c'} \circ \mathrm{A}_c$ together with the equality obtained by replacing all B-moves by their inverses, where again c' is the cut created by the A-move A_c (compare [BK1, Fig. 31]).

(W11) The first of the two $\mathrm{SL}(2,\mathbb{Z})$-relations for a 1-holed torus T: For any marking with a single cut c on T one has $\mathrm{B}_{c,\alpha} \circ \mathrm{Z} = \mathrm{S}^2$, where α is the boundary circle of T and T_c is the *Dehn move* around the cut c, i.e. is a specific composition [BK1, Ex. 4.15 & 4.17] of B-, Z- and F-moves.

(W12) The second of the $\mathrm{SL}(2,\mathbb{Z})$-relations for a 1-holed torus T: With the same notations as in (W11) one has $(\mathrm{S} \circ \mathrm{T}_c)^3 = \mathrm{S}^2$.

(W13) For a 2-holed torus T with boundary circles α and β and a specific marking on T with cut system consisting of two cuts c and d (for details see [BK1, App. B]), the equality $\mathrm{Z} \circ \mathrm{B}_{\alpha,\beta} \circ \mathrm{A}_{c,c'} \circ \mathrm{A}_{d,d'} = \mathrm{S}^{-1}_{c'',d'} \circ \mathrm{A}_{d'',c'} \circ \mathrm{T}_{c''} \circ \mathrm{T}^{-1}_{d''} \circ \mathrm{A}_{d,d''} \circ \mathrm{S}^{-1}_{c,c''}$.

2.3 Finite ribbon categories and coends

Let $\Bbbk$ denote an algebraically closed field, and $\mathcal{V}ect$ the category of finite-dimensional vector spaces over $\Bbbk$. A finite tensor category [EO] is a $\Bbbk$-linear abelian rigid monoidal category such that all morphism spaces are finite-dimensional over $\Bbbk$, there are up to isomorphism finitely many simple objects, each of them has a projective cover, every object has finite length, and the monoidal unit $\mathbf{1}$ is simple. A ribbon category is a rigid braided monoidal category endowed with a compatible twist (or balancing) or, equivalently, a braided pivotal category. By a *finite ribbon category* we mean a finite tensor category that is also ribbon. For any finite tensor category $\mathcal{D}$ the coend

$$K := \int^{X\in\mathcal{D}} X \otimes X^\vee \tag{2.1}$$

exists as a coalgebra in $\mathcal{D}$, and if $\mathcal{D}$ is in addition braided, then K carries a natural structure of a Hopf algebra endowed with a non-zero left integral as well as with a Hopf pairing ϖ_K. A finite ribbon category $\mathcal{D}$ is called *modular* iff the pairing ϖ_K is non-degenerate [KL]. Modularity of $\mathcal{D}$ is in particular equivalent [Sh2] to the property that the functor from the enveloping category $\mathcal{D} \boxtimes \mathcal{D}^{\mathrm{rev}}$ to the center $\mathcal{Z}(\mathcal{D})$ that maps $X \boxtimes Y$ to $X \otimes Y$ endowed with half-braiding $(c_{X,-} \otimes id_Y) \circ (id_X \otimes c^{-1}_{-,Y})$ is a braided equivalence. If $\mathcal{D}$ is modular, then the integral of K is two-sided. A semisimple modular finite ribbon category is the same as a modular tensor category in the conventional (see e.g. [BK2]) sense.

Various other coends exist in the situation of our interest as well. For $\Bbbk$-linear categories we have [Ly2, Lemma B.1]

$$\int^{Y\in\mathcal{D}} G(Y) \otimes_\Bbbk \mathrm{Hom}_\mathcal{D}(Y,U) = G(U) \tag{2.2}$$

for any left exact $\Bbbk$-linear functor $G\colon \mathcal{D} \to \mathcal{V}ect$. The component i_Y of the dinatural family of this coend is the linear map $g \otimes h \mapsto G(h)(g)$.

Conformal blocks are functors from some Deligne power of $\mathcal{D}$ to $\mathcal{V}ect$. In the context of conformal field theory it is essential to consider also coends taken in the corresponding category of left exact functors. Following [Ly2] we denote such coends by the symbol $\oint$. For finite tensor categories one finds for any $U, V \in \mathcal{D}$ that [Ly2, Sect. 8.2]

$$\oint^{X\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(U, V \otimes X \otimes X^{\vee}) = \mathrm{Hom}_{\mathcal{D}}(U, V \otimes K)\,, \tag{2.3}$$

with $K \in \mathcal{D}$ the coend (2.1), and with dinatural family given by post-composition with $id_V \otimes \imath^K$, where $\imath^K$ is the dinatural family for the coend K.

In the sequel, $\mathcal{D}$ will be a finite ribbon category and, unless noted otherwise, it will be modular.

2.4 Conformal block functors for modular finite ribbon categories

It is known that, given a modular finite ribbon category $\mathcal{D}$, one can assign to any extended surface E with p incoming and q outgoing boundary circles a left-exact functor from the Deligne product $\mathcal{D}^{\boxtimes q} \boxtimes (\mathcal{D}^{\mathrm{op}})^{\boxtimes p}$ to $\mathcal{V}ect$, in such a way that the so obtained vector spaces are morphism spaces of $\mathcal{D}$. We will think of a functor with domain $\mathcal{D}^{\mathrm{op}}$ as a contravariant functor with domain $\mathcal{D}$, and correspondingly work with functors

$$\mathrm{Bl}_E : \quad \mathcal{D}^{\boxtimes(p+q)} \longrightarrow \mathcal{V}ect \tag{2.4}$$

with covariance properties understood. We refer to these as *conformal block functors*. For compatibility with the symmetric monoidal structure of the category of surfaces and of $\mathcal{V}ect$ we put $\mathrm{Bl}(E \sqcup E') := \mathrm{Bl}(E) \otimes_{\Bbbk} \mathrm{Bl}(E')$ and $\mathrm{Bl}(\emptyset) := \Bbbk$. Also, it suffices to define Bl_E for all ordered $(p{+}q)$-tuples of objects of $\mathcal{D}$, corresponding to objects $X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_{p+q}$ of $\mathcal{D}^{\boxtimes(p+q)}$. We think of the entry X_α of the tuple as labeling the boundary circle $\alpha \in \pi_0(\partial E)$. If $\mathcal{D}$ is semisimple, the conformal block functors constitute part of the three-dimensional topological field theory that is associated with $\mathcal{D}$ [Tu]; in the general case no three-dimensional topological field theory exists.

To construct the functors auxiliary structure on the surfaces is needed in intermediate steps [Ly2, Ly1]. Fine markings, as introduced in Definition 2.4, provide such an auxiliary structure. Accordingly we consider a functor

$$\widetilde{\mathrm{Bl}}_{E,\Gamma} : \quad \mathcal{D}^{\boxtimes(p+q)} \longrightarrow \mathcal{V}ect \tag{2.5}$$

for every finely marked surface (E, Γ). Henceforth we will deal exclusively with markings that are fine; accordingly we often refer to them just as markings.

In view of their role in quantum field theory, the functors $\widetilde{\mathrm{Bl}}_{E,\Gamma}$ should be compatible with the following idea of locality: We demand that for any cut system C on E the functor $\widetilde{\mathrm{Bl}}_{E,\Gamma}$ is expressible through the functors $\widetilde{\mathrm{Bl}}_{E_i,\Gamma_i}$ for the connected components of the cut surface $\mathrm{cut}_C(E)$. This implies that the functor for any surface can be obtained by implementing the sewing of spheres that have at most three holes and markings without cuts. Moreover, the functors for the latter elementary world sheets should be expressible in terms of the tensor product of $\mathcal{D}$ and the Hom functor; thereby they will in particular be left exact and allow for an interpretation in terms of intermediate states that fit into representations of some symmetry structure.

Implementing sewing on conformal blocks is achieved with the help of suitable coend constructions [Ly2] that keep us within the class of left exact functors. In more detail, we proceed as follows.

(1) *Spheres with at most three holes.*
Let $(E^0_{p|3-p}, \Gamma)$ be a marked surface of genus zero with 3 holes, p of which are incoming, endowed with a marking without cuts. Denote by $\bar\varphi$ the cyclic permutation of the edges of the standard marking on the three-holed standard sphere $S^\circ_{3;\varepsilon}$ that is induced by the orientation preserving diffeomorphism $\varphi\colon E^0_{p|3-p} \to S^\circ_{3;\varepsilon}$ from Definition 2.3(i). Further, write $X^\varepsilon_\alpha := X_\alpha$ for $\varepsilon_\alpha = +1$ (i.e. α an outgoing boundary circle) and $X^\varepsilon_\alpha := X^\vee_\alpha$ for $\varepsilon_\alpha = -1$ (α incoming). We then define $\widetilde{\mathrm{Bl}}_{E^0_{p|3-p},\Gamma}$ as the left exact functor given by

$$\widetilde{\mathrm{Bl}}_{E^0_{p|3-p},\Gamma}(X_1,X_2,X_3) := \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, X^\varepsilon_{\bar\varphi^{-1}(1)} \otimes X^\varepsilon_{\bar\varphi^{-1}(2)} \otimes X^\varepsilon_{\bar\varphi^{-1}(3)}) \,. \tag{2.6}$$

For spheres with $n<3$ holes we define $\widetilde{\mathrm{Bl}}$ analogously, with the covariant argument of the Hom functor being a tensor product having n factors.

(2) *Spheres with any number of holes.*
Let (E,Γ) be a connected marked surface of genus zero. Denote by (E_l,Γ_l), $l=1,2,\dots,\ell$, the connected components of the cut surface $\mathrm{cut}_C(E)$. The marking being fine, each of the surfaces E_l is a sphere with at most three holes. In agreement with the general remarks above, we define a left exact functor $\widetilde{\mathrm{Bl}}_{E,\Gamma}$ as a suitable coend over the tensor product of the conformal block functors for the surfaces (E_l,Γ_l).

To give this prescription explicitly, we need additional notation. For every cut $c_k \in C$, label the two corresponding boundary circles of $\mathrm{cut}_C(E)$ by an object $Y_k \in \mathcal{D}$. Denote by $X_{l;i}$, $i \in \{1,2,\dots,n_l\}$, the labels of the boundary circles of the component E_l that come from the boundary of E and by $\tilde Y_{l;j}$, $j \in \{1,2,\dots,m_l\}$, the labels of those which come from cuts of E (such that each of the objects Y_k appears precisely twice in the list of all $\tilde Y_{l;j}$, for two distinct values of l). By a slight abuse of notation write $(X_{l;1},\dots,X_{l;n_l},\tilde Y_{l;1},\dots,\tilde Y_{l;m_l})$ for the tuple of objects of $\mathcal{D}$ that label the boundary circles of E_l, ordered according to the ordering of the edges of Γ_l. (For an illustration of these conventions, in the case of a genus-one surface, see Figure 3.) We then set

$$\widetilde{\mathrm{Bl}}_{E,\Gamma}(X_{1;1},\dots,X_{\ell;n_\ell}) := \int^{Y_1\boxtimes\cdots\boxtimes Y_{|C|}\,\in\,\mathcal{D}^{\boxtimes|C|}} \bigotimes_{l=0}^{\ell} \widetilde{\mathrm{Bl}}_{E_l,\Gamma_l}(X_{l;1},\dots,X_{l;n_l},\tilde Y_{l;1},\dots,\tilde Y_{l;m_l}) \,. \tag{2.7}$$

This indeed furnishes a left exact functor, which can be seen as follows. By invoking the Fubini theorem we can rewrite the right hand side as an iterated coend

$$\widetilde{\mathrm{Bl}}_{E,\Gamma}(X_{1;1},\dots,X_{\ell;n_\ell}) = \int^{Y_1\in\mathcal{D}}\!\!\int^{Y_2\in\mathcal{D}}\!\!\cdots\int^{Y_{|C|}\in\mathcal{D}} \bigotimes_{l=0}^{\ell} \widetilde{\mathrm{Bl}}_{E_l,\Gamma_l}(X_{l;1},\dots,X_{l;n_l},\tilde Y_{l;1},\dots,\tilde Y_{l;m_l}) \,.$$

(Up to unique natural isomorphism the ordering of the cuts in C does not matter; owing to the symmetry of $\mathcal{V}ect$, neither does the ordering of the components of $\mathrm{cut}_C(E)$.) For each of the iterated coends we can then invoke, consecutively, the property (2.2) of the Hom functor, combined, if needed, with the duality of $\mathcal{D}$. Doing so, for any marked sphere (E,Γ) we obtain a canonical isomorphism $\widetilde{\mathrm{Bl}}_{E,\Gamma}(-,\dots,-) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, -^{\varepsilon} \otimes \cdots \otimes -^{\varepsilon})$ of left exact functors. (It is thus appropriate to think of the coend prescription above as a means to ensure compatibility of the conformal blocks at genus zero with the tensor product.)

(3) *Disconnected surfaces.*
The functor for the disjoint union of two marked surfaces (E_1, Γ_1) and (E_2, Γ_2) is defined to be the tensor product

$$\widetilde{\mathrm{Bl}}_{(E_1,\Gamma_1)\sqcup(E_2,\Gamma_2)} := \widetilde{\mathrm{Bl}}_{(E_1,\Gamma_1)} \otimes_{\Bbbk} \widetilde{\mathrm{Bl}}_{((E_2,\Gamma_2)}\,, \tag{2.8}$$

and we set $\widetilde{\mathrm{Bl}}_{\emptyset} := \Bbbk$.

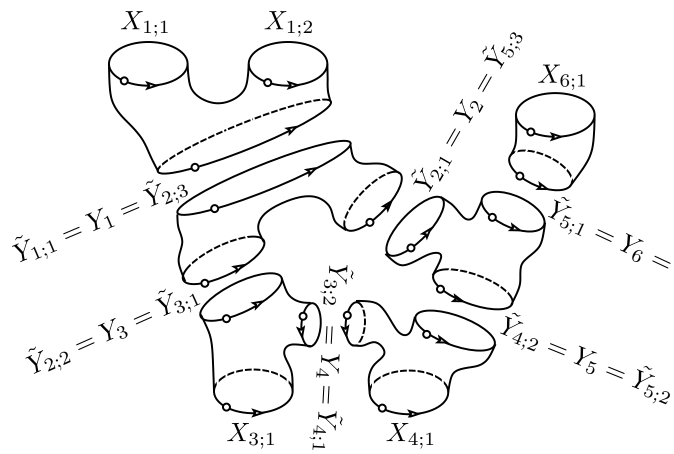

Figure 3: A labeling of the boundary circles of the components (E_l, Γ_l) by objects $X_{l;i}$ and $\tilde{Y}_{l;j}$ as described in the text before (2.7), for the cut surface that results from the marking shown in Figure 2.

(4) *Higher genus surfaces.*
Let now (E, Γ) be a marked surface of arbitrary genus g. Then we define the functor $\widetilde{\mathrm{Bl}}_{E,\Gamma}$ by

$$\widetilde{\mathrm{Bl}}_{E,\Gamma}(X_{1;1}, \dots, X_{\ell;n_\ell}) := \oint^{Y_1\boxtimes\cdots\boxtimes Y_{|C|}\in\mathcal{D}^{\boxtimes|C|}} \bigotimes_{l=0}^{\ell} \widetilde{\mathrm{Bl}}_{E_l,\Gamma_l}(X_{l;1}, \dots, X_{l;n_l}, \tilde{Y}_{l;1}, \dots, \tilde{Y}_{l;m_l})\,. \tag{2.9}$$

Here the conventions concerning the objects $X_{l;i}$ and $\tilde{Y}_{l;j}$ are the same as above, while when taking coends we now need to work explicitly with coends $\oint$ in categories of left exact functors. (Left-exactness, and thus representability, would no longer necessarily be preserved when taking coends in the category of all functors from the appropriate Deligne power of $\mathcal{D}$ to $\mathcal{V}ect$.) A variant of the Fubini theorem [Ly2, Thm. B.2] allows us to rewrite $\widetilde{\mathrm{Bl}}_{E,\Gamma}$, similarly as in the case of spheres, as an iterated coend:

$$\widetilde{\mathrm{Bl}}_{E,\Gamma}(X_{1;1}, \dots, X_{\ell;n_\ell}) = \oint^{Y_1\in\mathcal{D}} \cdots \oint^{Y_g\in\mathcal{D}} \int^{Y_{g+1}\in\mathcal{D}} \cdots \int^{Y_{|C|}\in\mathcal{D}} \bigotimes_{l=0}^{\ell} \widetilde{\mathrm{Bl}}_{E_l,\Gamma_l}(X_{l;1}, \dots, X_{l;n_l}, \tilde{Y}_{l;1}, \dots, \tilde{Y}_{l;m_l})\,.$$

Here the subset $\{c_1, c_2, \dots, c_g\} \subset C$ consisting of those cuts that correspond to the objects Y_k with $k \in \{1, 2, \dots, g\}$ needs to be selected in such a way that the corresponding cut surface $\mathrm{cut}_{\{c_1,\dots c_g\}}(E)$ has genus zero. Given this description of $\widetilde{\mathrm{Bl}}_{E,\Gamma}$, by recalling the genus-0 result

(2.6) and invoking iteratively the relations (2.2) and (2.3) one obtains for any connected marked surface $(E,\varGamma)$ a distinguished isomorphism

$$\widetilde{\mathrm{Bl}}_{E,\varGamma}(-,\dots,-) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, -^{\varepsilon} \otimes \cdots \otimes -^{\varepsilon} \otimes K^{\otimes g}) \tag{2.10}$$

of left exact functors, with $K \in \mathcal{D}$ the coend (2.1).

Remark 2.5. As we are working with coends in functor categories, the prescription for higher genus applies directly to surfaces for which each connected component has non-empty boundary. But once $\widetilde{\mathrm{Bl}}_{E,\varGamma}$ is defined for all such surfaces, we can define $\widetilde{\mathrm{Bl}}_{E,\varGamma}$ for a surface E with empty boundary as the vector space $\widetilde{\mathrm{Bl}}_{E',\varGamma'}(\mathbf{1})$ with $\mathbf{1}$ the monoidal unit of $\mathcal{D}$ and E' obtained from E by removing a disk.

For any extended surface E there is a functor U_E from the groupoid $\mathfrak{FM}(E)$ of fine markings on E to the category $E/\!/\mathrm{Map}(E)$ with a single object E and with morphisms given by the mapping class group $\mathrm{Map}(E)$ that forgets the structure of a marking (similar to the functor U that will be introduced in Definition 3.3 below). One can construct the conformal block functor Bl_E (2.4) from the functors $\widetilde{\mathrm{Bl}}_{E,\varGamma}$ by a right Kan extension along U_E. The so obtained conformal block functors obey analogues of (2.8) and (2.10). We do not give any further details of the construction of Bl_E because they will not be needed in the sequel.

Remark 2.6. For any modular finite ribbon category $\mathcal{D}$ the conformal blocks obtained by the prescription above provide equivalent representations of the mapping class groups as those obtained in [Ly2] by a different construction. The variant presented above is tailored to our goal of determining consistent systems of correlators in the sense of Definition 3.16. Apart from the precise treatment of boundary circles (as either incoming or outgoing), for the case that $\mathcal{D}$ is a semisimple modular tensor category it reduces to the construction in [BK1].

The correlators we are looking for are elements of very specific conformal blocks spaces $\mathrm{Bl}_E(X_1,...,X_n)$: those for which each of the arguments X_i is one and the same object of $\mathcal{D}$, namely the bulk object F. Nevertheless we had to discuss also block spaces with generic arguments X_i: Compatibility with the sewing of surfaces is part of the consistency requirements to be imposed on correlators. As we will see in Section 3.3, to formulate sewing we need certain structure morphisms of coends, and to get these morphisms we need to consider blocks for arbitrary insertions.

3 Consistency conditions for correlators

The purpose of this section is to give a concise definition of the notion of a consistent system of bulk field correlators for local conformal field theories on oriented surfaces. In short, the correlators must be invariant under the action of mapping class groups on conformal blocks, must be compatible with the sewing of surfaces, and must obey a non-degeneracy condition. To formalize these conditions, as compared to Section 2 we change our perspective in two respects: First, in Section 2.4 we described conformal blocks for (extended) surfaces E with arbitrary objects of $\mathcal{D}$ associated to the boundary circles of E, and thereby dealt with a system of functors (2.4) of conformal blocks. In contrast, for the correlators we need to associate to every boundary circle of E one and the same object (respectively its dual, in the case of an incoming boundary circle), the *bulk object*. In case the category $\mathcal{D}$ has a representation theoretic interpretation, the vector space underlying this object is the space of states that is related to bulk fields under a field-state correspondence.

Accordingly we now select one specific object F of $\mathcal{D}$ as a (candidate) bulk object. Thus for each surface E we are now dealing with a *vector space* of conformal blocks, endowed with an action of the mapping class group $\mathrm{Map}(E)$. The correlator for E is a vector in this space; it is required to be invariant under the $\mathrm{Map}(E)$-action. Second, previously we treated one surface at a time, e.g. associated, in Section 2.2, a groupoid $\mathcal{FM}(E)$ separately to each surface. In contrast, the sewing constraints require the system of correlators to be compatible with sewings that connect correlators on different surfaces. Accordingly we now study all surfaces together, and in particular treat the morphisms in all the groupoids $\mathcal{FM}(E)$ as well as sewings of surfaces on the same footing. (Since sewing involves a sum over intermediate states, and thus a coend, the considerations in Section 2 are necessary for this approach.)

A major step in this section will therefore be to construct, for a chosen object F of $\mathcal{D}$, a symmetric monoidal functor $\mathrm{Bl}^{(F)}$, to be called the *F-pinned block functor*, or just pinned block functor, from a suitable category of surfaces to $\mathcal{V}ect$. We will then see that the consistency conditions for correlators with F as the (candidate) bulk object can be neatly summarized as the requirement that these vectors define a monoidal natural transformation, satisfying a simple non-degeneracy condition, from a certain trivial functor to the pinned block functor $\mathrm{Bl}^{(F)}$. Our first task will be to introduce the relevant categories of surfaces and marked surfaces. This is initiated in Section 3.1 and completed in Section 3.2, where we accommodate the fact that the mapping class groups act on conformal blocks only projectively.

3.1 The categories of surfaces and of marked surfaces

The two geometric categories of our interest have (extended) surfaces, respectively marked surfaces, as objects. Their morphisms are generated by two types of special morphisms: automorphisms respectively moves of a given surface on the one hand, and sewings of surfaces, as described in Definition 2.3(iv), on the other.

Recall that a move of a marked surface is a morphism of the groupoid $\mathcal{FM}(E)$, and that it can be presented as a finite sequence of the elementary moves listed in Section 2.2, According to Definition 2.2, the elements of the mapping class group $\mathrm{Map}(E)$ preserve the orientation of each boundary circle and thus map the subsets of incoming and outgoing boundary circles to themselves. $\mathcal{FM}(E)$ is in fact also intimately related to a larger group $\mathrm{MAP}(E) \supset \mathrm{Map}(E)$, defined analogously as $\mathrm{Map}(E)$, but without the restriction to preserve the orientations of boundary circles [BK1]: First, the set of morphisms of $\mathcal{FM}(E)$ is invariant under the obvious action of $\mathrm{MAP}(E)$. Moreover, for every $\phi \in \mathrm{MAP}(E)$ there is a move $\mathrm{m} = \mathrm{m}(\phi)$ in $\mathcal{FM}(E)$ that maps a given fine marking (C, Γ) of E to $(\phi(C), \phi(\Gamma))$ while, conversely, any finite sequence $\mathrm{m}\colon (E, \Gamma) \to (E', \Gamma')$ of elementary moves in $\mathcal{FM}(E)$ not involving the F-move (which changes the number of cuts) uniquely determines an element ϕ_{m} of $\mathrm{MAP}(E)$ such that the action of ϕ_{m} on a fine marking of E reproduces the effect of the move m. Recall e.g. that the B-move affects the markings in the same way as a certain braiding diffeomorphism.

We call a morphism m of $\mathcal{FM}(E)$ an *admissible move* iff the associated element ϕ_{m} of the group $\mathrm{MAP}(E)$ is contained in the subgroup $\mathrm{Map}(E)$, i.e. is a mapping class in the sense of Definition 2.2.

Definition 3.1.
(i) The *category $\mathcal{S}urf$ of surfaces* is the monoidal category having extended surfaces E as objects and whose morphisms are generated by mapping classes $\varphi \in \mathrm{Map}(E)$ and by all possible sewings $E \to \cup E$ for any extended surface E.
(ii) The *category m$\mathcal{S}urf$ of marked surfaces* is the monoidal category having marked surfaces (E, Γ) with fine marking as objects and whose morphisms are generated by admissible moves of the groupoids $\mathcal{FM}(E)$ for all surfaces $E \in \mathcal{S}urf$ and by all possible sewings $(E, \Gamma) \to \cup(E, \Gamma)$ of marked surfaces.

In both categories the tensor product is given by disjoint union and $\emptyset$ is the monoidal unit; both $\mathcal{S}urf$ and m$\mathcal{S}urf$ are symmetric monoidal. In the definition we have suppressed the description of the relations among the generators. Besides the relations in the individual mapping class groups $\mathrm{Map}(E)$, respectively those among the admissible moves of the individual groupoids $\mathcal{FM}(E)$, they consist of obvious compatibility relations between sewings and mapping classes, respectively between sewings and admissible moves. These relations are discussed in detail in [HLS]; we refrain from writing any explicit formulas (compare also [BK2, Rem. 5.6.4]).

Remark 3.2. In [BK2, Sect. 5.6] the Teichmüller tower of mapping class groups is studied. It has the same objects as $\mathcal{S}urf$, but only mapping classes are taken as morphisms, while sewings are regarded as an additional structure on the category. For the purpose of describing conformal field theory correlators it is very convenient to take, as in [Ly2], also sewings as morphisms. These are non-invertible; note that we do not introduce morphisms for the operation of cutting surfaces, which is inverse to the sewing operation.

Note that sewing is a local construction. As a consequence, we can restrict our attention to suitable *elementary* sewing morphisms involving only specific types of surfaces. For the sake of concreteness in explicit formulas it is, however, still convenient to treat two kinds of situations separately, namely (in the case of $\mathcal{S}urf$, and analogously for m$\mathcal{S}urf$) the following: Either a sewing

$$\mathrm{s}_{E_1,E_2}: \quad E^{g_1}_{p_1|q_1} \sqcup E^{g_2}_{p_2|q_2} \to E^{g_1+g_2}_{p_1+p_2-1|q_1+q_2-1} \tag{3.1}$$

of the disjoint union of two connected surfaces of genus g_1 and g_2 having $n_1 = p_1 + q_1$ and $n_2 = p_2 + q_2$ holes, respectively, to a connected surface of genus $g_1 + g_2$ with $n_1 + n_2 - 2$ holes; or else, a sewing

$$\mathrm{s}_E: \quad E^g_{p|q} \to E^{g+1}_{p-1|q-1} \tag{3.2}$$

of a connected surface of genus g with n holes to a connected surface of genus $g+1$ with $n-2$ holes.

While apart from the sewings, the morphisms of m$\mathcal{S}urf$ and $\mathcal{S}urf$ are quite different, owing to the relation between morphisms of $\mathcal{FM}(E)$ and the group $\mathrm{MAP}(E)$ there is nevertheless a natural functor from m$\mathcal{S}urf$ to $\mathcal{S}urf$ that is a kind of forgetful functor. Its action on objects and on sewings is the obvious one, namely forgetting the marking.

Definition 3.3. The *unmarking functor* U: m$\mathcal{S}urf \to \mathcal{S}urf$ is the symmetric monoidal functor that is uniquely determined by the following prescription.
(i) On objects and on sewings, U forgets the marking, i.e. $U(E, \Gamma) := E$ and

$$U\big((E, \Gamma) \to \cup(E, \Gamma)\big) := (E \to \cup E) \,.$$

(ii) The F-move $\mathrm{F}\colon (E,\Gamma)\to(E,\Gamma')$ is mapped to the identity, $U(\mathrm{F}) := id_E$.
(iii) The other elementary moves m of $\mathcal{FM}(E)$ – the Z-move, B-move, A-move and S-move – are mapped to the mapping class ϕ_{m} that reproduces the effect of m on the marking of E.

We would now like to fix an object F of $\mathcal{D}$ and construct the F-pinned block functor that provides the conformal block spaces for bulk fields. We will first obtain it as a monoidal functor $\widetilde{\mathrm{Bl}}^{(F)}$ from marked surfaces to $\mathcal{V}ect$ and then upon Kan extension along the unmarking functor get a monoidal functor $\mathrm{Bl}^{(F)}$ from (extended) surfaces to $\mathcal{V}ect$. Defining $\widetilde{\mathrm{Bl}}^{(F)}$ on an object (E,Γ) of $m\mathcal{S}urf$ is easy; we just apply the functor $\widetilde{\mathrm{Bl}}_{E,\Gamma}$ introduced in (2.5) to the object $F^{\boxtimes|\pi_0(\partial E)|} \in \mathcal{D}^{\boxtimes|\pi_0(\partial E)|}$:

Definition 3.4. To any object (E,Γ) of $m\mathcal{S}urf$, $\widetilde{\mathrm{Bl}}^{(F)}$ assigns the finite-dimensional vector space

$$\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma) := \widetilde{\mathrm{Bl}}_{E,\Gamma}(F,F,\dots,F) \tag{3.3}$$

with the appropriate number $|\pi_0(\partial E)|$ of arguments of $\widetilde{\mathrm{Bl}}_{E,\Gamma}$.

To define $\widetilde{\mathrm{Bl}}^{(F)}$ also on morphisms is somewhat less straightforward: there is an obstruction, known as the framing anomaly, which results from the fact that the action of $\mathrm{Map}(E)$ on the space $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ is projective. We deal with this obstruction by adequately extending the categories $m\mathcal{S}urf$ and $\mathcal{S}urf$.

3.2 Central extensions of categories of surfaces

As a first step of defining the pinned block functor $\widetilde{\mathrm{Bl}}^{(F)}$ on morphisms we consider moves of marked surfaces. This will in particular demonstrate the need to work with suitable extensions of the categories $m\mathcal{S}urf$ and $\mathcal{S}urf$. Besides these extensions our construction will involve the dinatural structure morphisms for the coends (2.2) and (2.3), as well as suitable specific families of bijective linear maps which correspond to the elementary moves of marked surfaces.

We start by introducing the latter families of linear maps. The prescription is entirely based on structural data of the finite ribbon category $\mathcal{D}$, including those which are captured by the Hopf algebra $K\in\mathcal{D}$:

Definition 3.5. Denote the braiding of $\mathcal{D}$ by c, the evaluation and coevaluation for the right duality of $\mathcal{D}$ by d and b, respectively, and the pivotal structure of $\mathcal{D}$ by π (i.e. $\pi_X\colon X\to X^{\vee\vee}$).
(i) For $X,Y\in\mathcal{D}$, the *Z-isomorphism* $\mathrm{Z}_{X,Y}\colon \mathrm{Hom}_{\mathcal{D}}(\mathbf{1},X\otimes Y)\to\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},Y\otimes X)$ is the linear map given by

$$\mathrm{Z}_{X,Y}(f) := (d_X\otimes id_{Y\otimes X})\circ(id_{X^\vee}\otimes f\otimes\pi_X^{-1})\circ b_{X^\vee}\,. \tag{3.4}$$

(ii) For $X,Y,U\in\mathcal{D}$, the *B-isomorphism* $\mathrm{B}_{X,Y,U}\colon \mathrm{Hom}_{\mathcal{D}}(\mathbf{1},X\otimes Y\otimes U)\to\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},Y\otimes X\otimes U)$ is the linear map given by

$$\mathrm{B}_{X,Y,U}(f) := (c_{X,Y}\otimes id_U)\circ f\,. \tag{3.5}$$

(iii) For $U,V\in\mathcal{D}$, the *F-isomorphism*

$$\mathrm{F}_{U,V}:\quad \int^{X\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1},X\otimes V)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},U\otimes X^\vee)\;\longrightarrow\;\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},U\otimes V)$$

is the linear map that on morphisms $f \otimes g$ in $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, X \otimes V) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes X^\vee)$ acts as

$$f \otimes g \longmapsto (id_U \otimes d_X \otimes id_V) \circ (g \otimes f)\,. \tag{3.6}$$

(iv) For $U, U', V, V' \in \mathcal{D}$, the *A-isomorphism* $\mathrm{A}_{U,U',V,V'}$ is the composition

$$\begin{aligned} &\int^{X\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes U' \otimes X) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, V \otimes V' \otimes X^\vee) \\ &\qquad \xrightarrow{=} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes U' \otimes V \otimes V') \xrightarrow{\mathrm{Z}_{U,U'\otimes V\otimes V'}} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U' \otimes V \otimes V' \otimes U) \\ &\qquad\qquad \xrightarrow{=} \int^{Y\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U' \otimes V \otimes Y) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, V' \otimes U \otimes Y^\vee)\,, \end{aligned} \tag{3.7}$$

where the equalities indicate the identifications of coends that result when applying formula (2.2) with G an appropriate Hom functor. [2]

(v) Denote by ε_K the counit and by Λ_K a non-zero two-sided integral of the Hopf algebra K, and recall that $\imath^K$ denotes the dinatural family of K as a coend. Define $\mathcal{Q}_K \in \mathrm{End}_{\mathcal{D}}(K \otimes K)$ by

$$\mathcal{Q}_K \circ (\imath^K_X \otimes \imath^K_Y) := (\imath^K_X \otimes \imath^K_Y) \circ \big(id_X \otimes (c_{Y,X^\vee} \circ c_{X^\vee,Y}) \otimes id_{Y^\vee} \big)$$

and set [Ly1]

$$S^K := (\varepsilon_K \otimes id_K) \circ \mathcal{Q}_K \circ (id_K \otimes \Lambda_K) \in \mathrm{End}_{\mathcal{D}}(K)\,. \tag{3.8}$$

Then for $U \in \mathcal{D}$, the *S-isomorphism* S_U is the linear endomorphism of $\oint^{X\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(U, X \otimes X^\vee) = \mathrm{Hom}_{\mathcal{D}}(U, K)$ that acts as post-composition by the isomorphism S^K.

It is known (see [Ly2, Sect. 8.8] and [Ly1, Thm 2.1.9]) that the normalization of the integral Λ_K can be chosen (uniquely up to sign) in such a way that the square of the endomorphism S^K in (3.8) equals the inverse of the antipode s_K of K,

$$\big(S^K\big)^2 = \mathrm{s}_K^{-1}. \tag{3.9}$$

In the sequel we take Λ_K to be normalized in this way.

We can now define $\widetilde{\mathrm{Bl}}^{(F)}$ on generating (and thus not necessarily admissible) moves. Let us first recall the notation $\varepsilon \in \{\pm 1\}$ indicating whether a boundary circle is outgoing ($\varepsilon = 1$) or incoming ($\varepsilon = -1$), and the corresponding notation X^ε standing for $X \in \mathcal{D}$ if $\varepsilon = 1$ and for $X^\vee$ if $\varepsilon = -1$. We supplement these conventions by setting

$$\jmath^\varepsilon_F := \begin{cases} id_{F^\vee \otimes F} \in \mathrm{End}_{\mathcal{D}}(F^\vee \otimes F) & \text{for } \varepsilon = 1\,, \\ \pi_F \otimes id_{F^\vee} \in \mathrm{Hom}_{\mathcal{D}}(F \otimes F^\vee, F^{\vee\vee} \otimes F^\vee) & \text{for } \varepsilon = -1\,, \end{cases}$$

with π the pivotal structure of $\mathcal{D}$.

---

[2] It is worth noting that the map (3.6) is nothing but the dinatural structure morphism for the coend in question, so that $\mathrm{F}_{U,V}$ is the identification of the vector space $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes V)$ as the coend $\int^{X\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, X \otimes V) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes X^\vee)$. Accordingly it is appropriate to write the first and third maps in the A-isomorphism (3.7) as equalities.

Definition 3.6. On generating moves $\mathrm{m}\colon (E,\Gamma) \to (E,\Gamma')$ of the groupoid $\mathcal{FM}(E)$ the assignment $\widetilde{\mathrm{Bl}}^{(F)}$ is defined as follows: To each elementary move assign the specific linear isomorphism $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma) \to \widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma')$ from Definition 3.5 that bears the same name as the move and for which all objects of $\mathcal{D}$ involved are given by either F or $F^\vee$. Explicitly, if all boundary circles of E are outgoing, then

$$\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{B}) := \mathrm{B}_{F,F,F}\,, \qquad \widetilde{\mathrm{Bl}}^{(F)}(\mathrm{A}) := \mathrm{A}_{F,F,F,F} \qquad \text{and} \qquad \widetilde{\mathrm{Bl}}^{(F)}(\mathrm{S}) := \mathrm{S}_F\,,$$

as well as

$$\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{Z}) := \mathrm{Z}_{F,F\otimes F} \qquad \text{and} \qquad \widetilde{\mathrm{Bl}}^{(F)}(\mathrm{F}) := \mathrm{F}_{F,F\otimes F}$$

in case E has three holes, and similarly if E has less than three holes. If any of the boundary circles of E are incoming, the appropriate occurrences of F are to be replaced by $F^\vee$.

For an arbitrary move m one might wish to define $\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{m})$ to be the composition of isomorphisms from Definition 3.6 according to the expression of m as a sequence of elementary moves. However, this works directly only if those isomorphisms respect the relations among elementary moves. We first note

Lemma 3.7. *Applying* $\widetilde{\mathrm{Bl}}^{(F)}$ *to any of the relations* (W1) – (W11) *and* (W13) *listed in Section* 2.2 *yields an equality of linear isomorphisms.*

Proof. For (W1) and (W2) the claim follows directly from the definitions, while for (W3) and (W4) one has to make use of the defining properties of the pivotal structure. As another example, the proof for (W7) follows by rewriting the Z-isomorphism as

$$\mathrm{Z}_{X,Y}(f) = (id_Y \otimes \theta_X) \circ c_{X,Y} \circ f = (id_Y \otimes \theta_X^{-1}) \circ c_{Y,X}^{-1} \circ f\,, \tag{3.10}$$

which uses the relation between the pivotal structure and the twist and braiding of the ribbon category $\mathcal{D}$. Let us give some details for the case of (W11), i.e. the genus-1 relation $\mathrm{B}_{c,\alpha} \circ \mathrm{Z} = \mathrm{S}^2$, with α the boundary of a one-holed torus that is endowed with a cut system consisting of a single cut c. To prove the assertion we have to deal with morphisms involving the Hopf algebra K, which by the coend property of K amount to *families* of morphisms. Specifically, one finds that the family realizing the move $\mathrm{B}_{c,\alpha} \circ \mathrm{Z}$ consists of the linear maps

$$f \longmapsto (id_U \otimes id_{X^\vee} \otimes \pi_X^{-1}) \circ \mathrm{Z}_{X,U\otimes X^\vee} \circ (B_{U,X}(f) \otimes id_{X^\vee}) \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes X^\vee \otimes X^{\vee\vee})$$

with $f \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes X \otimes X^\vee)$, for all $X \in \mathcal{D}$. Using (3.10) one can see that these give the endomorphism $f \mapsto (id_U \otimes \mathrm{s}_K^{-1}) \circ f$ of $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes K)$, with s_K the antipode of the Hopf algebra K. Hence invoking the equality (3.9) (and thereby adopting the corresponding choice of normalization of Λ_K), indeed post-composition with $id_U \otimes \mathrm{s}_K^{-1}$ implements the square of the S-isomorphism on $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, U \otimes K)$, as required to realize the relation (W11). Finally we mention that in the proof for (W13), a crucial additional ingredient is to invoke an isomorphism of the form $\oint^{Y\in\mathcal{D}}\!\int^{X\in\mathcal{D}} G(U;X,X,Y,Y) \cong \oint^{X\in\mathcal{D}}\!\int^{Y\in\mathcal{D}} G(U;X,X,Y,Y)$, which exists and is uniquely determined as a consequence of the Fubini theorem. $\square$

It remains to examine the relation (W12), i.e. the modular group relation $(\mathrm{S} \circ \mathrm{T}_c)^3 = \mathrm{S}^2$, where T_c is the Dehn move around the single cut c of a one-holed torus. Again we deal with morphisms involving the coend K and thus work with families of morphisms. We find that the family realizing the Dehn move T_c is

$$f \longmapsto \mathrm{Z}_{U,X\otimes X^\vee}^{-1} \circ \big(\mathrm{B}_{X,X^\vee\otimes U}^{-1} \circ Z_{X,X^\vee\otimes U}\big) \circ \mathrm{Z}_{U,X\otimes X^\vee} \circ f$$

with $f \in \mathrm{Hom}_\mathcal{D}(\mathbf{1}, U \otimes X \otimes X^\vee)$, for all $X \in \mathcal{D}$. Upon again invoking (3.10), this amounts to the linear endomorphism of $\mathrm{Hom}_\mathcal{D}(\mathbf{1}, U \otimes K)$ that is given by post-composition with $id_U \otimes T^K$, where $T^K \in \mathrm{End}_\mathcal{D}(K)$ is determined by

$$T^K \circ \imath_X^K = \imath_X^K \circ (\theta_X \otimes id_{X^\vee})$$

with θ the twist of $\mathcal{D}$. Now the morphisms S^K and T^K obey [Ly1, Thm. 2.1.9] the modular group relation up to a scalar factor,

$$\big(S^K \circ T^K\big)^3 = \zeta\, \big(S^K\big)^2 \tag{3.11}$$

with $\zeta := \varepsilon_K \circ T^K \circ \Lambda_K \in \Bbbk^\times$. The number ζ, which via its dependence on the integral Λ_K is determined up to sign, is called the *central charge* or *framing anomaly*. Its presence in (3.11) obstructs a linear realization of the morphisms of *m𝒮urf*. In terms of the category *𝒮urf* we then get projective rather than genuine representations of mapping class groups.

Remark 3.8. At genus one the projective action of the mapping class group can actually be reduced to a genuine linear action upon redefining the action of the Dehn twist T_c [At]. However, this would not lead to genuine actions at higher genus, compare Remark 3.1.9 of [BK2].

We can trade these projective realizations for linear ones by considering suitable central extensions of categories of surfaces [Se, §4]. Analogously as in [Ly2, Sect. 7] (compare also [BK2, Sect. 5.7]), in terms of generators and relations for morphisms of marked surfaces the required central extension is implemented as follows. First, introduce for each connected surface E with marking $\mathit{\Gamma}$ a new invertible generator $\mathrm{C}_{(E,\mathit{\Gamma})}$. Second, require that these generators are compatible with (not necessarily admissible) moves m, in the sense that

$$\mathrm{m} \circ \mathrm{C}_{(E,\mathit{\Gamma})} = \mathrm{C}_{(E,\mathit{\Gamma}')} \circ \mathrm{m} \tag{3.12}$$

for any move $\mathrm{m}\colon (E,\mathit{\Gamma}) \to (E,\mathit{\Gamma}')$, and keep all relations (W1) – (W11) and (W13) among the generating moves, while replacing the modular group relation (W12) by

$$(\mathrm{W12})^{\mathrm{C}}: \qquad (\mathrm{S} \circ \mathrm{T}_c)^3 = \mathrm{C} \circ \mathrm{S}^2$$

with $\mathrm{C} = \mathrm{C}_{(T,\mathit{\Gamma})}$ the new generating morphism for a one-holed torus T with marking $\mathit{\Gamma}$ as described for (W12) in Section 2.2. And third, impose $\mathrm{C}_\emptyset = id_\emptyset$ as well as the relation

$$\mathrm{s}_{E_1,E_2} \circ \big(\mathrm{C}^{\,k_1}_{(E_1,\mathit{\Gamma}_1)} \sqcup \mathrm{C}^{\,k_2}_{(E_2,\mathit{\Gamma}_2)}\big) = \mathrm{C}^{\,k_1+k_2}_{(E,\mathit{\Gamma})} \circ \mathrm{s}_{E_1,E_2} \tag{3.13}$$

for any $k_1, k_2 \in \mathbb{Z}$ and any sewing $\mathrm{s}_{E_1,E_2}\colon (E_1,\mathit{\Gamma}_1) \sqcup (E_2,\mathit{\Gamma}_2) \to (E,\mathit{\Gamma})$ of the type (3.1) among connected surfaces. Thus we introduce, similarly as in [Ly2, Def. 7.2]:

Definition 3.9. The *central extension* $m\mathcal{S}urf^{\mathrm{C}}$ *of the category m𝒮urf of marked surfaces* is the category with the same objects as *m𝒮urf* and with morphisms generated by the morphisms of *m𝒮urf* together with the morphisms $\mathrm{C}_{(E,\mathit{\Gamma})}$ for all connected marked surfaces $(E,\mathit{\Gamma})$, subject to the following relations: those obtained from the relations among morphisms of *m𝒮urf* when replacing (W12) by $(\mathrm{W12})^{\mathrm{C}}$; the relations (3.12) for all admissible moves m; $\mathrm{C}_\emptyset = id_\emptyset$; and (3.13) for any integers k_1, k_2 and any sewing of the type (3.1).

To see how to extend the definition of $\widetilde{\mathrm{Bl}}^{(F)}$ to the morphisms $\mathrm{C}_{(E,\Gamma)}$, recall that $\widetilde{\mathrm{Bl}}^{(F)}$ maps the moves S and T_c to post-composition by endomorphisms of the Hopf algebra K. This must then likewise apply to the morphism C in $(\mathrm{W12})^{\mathrm{C}}$. To ensure compatibility with (3.11) we set $\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{C}) := (id_{F^\varepsilon} \otimes \mathrm{C}^K)_*$ with

$$\mathrm{C}^K := \zeta \, id_K \,,$$

where $\zeta \in \Bbbk^\times$ is the scalar appearing in (3.11). To also account for the commutation relations (3.13), we generalize this prescription to arbitrary marked surfaces (E, Γ), to which $\widetilde{\mathrm{Bl}}^{(F)}$ assigns the space $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, X \otimes K^{\otimes g})$ (with X an appropriate tensor product of factors F and $F^\vee$), by

$$\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{C}_{(E,\Gamma)}) := \big(id_X \otimes (\mathrm{C}^K)^{\otimes g}\big)_* \,. \tag{3.14}$$

Next recall the unmarking functor $U \colon m\mathcal{S}urf \to \mathcal{S}urf$ introduced in Definition 3.3. Analogously as for marked surfaces we can centrally extend $\mathcal{S}urf$ to a category $\mathcal{S}urf^{\mathrm{C}}$ that has a new central automorphism C_E for every connected surface E, and because of the relations (3.12) we can immediately extend U to a functor from $m\mathcal{S}urf^{\mathrm{C}}$ to $\mathcal{S}urf^{\mathrm{C}}$ by the prescription $\mathrm{C}_{(E,\Gamma)} \mapsto \mathrm{C}_E$, i.e. by simply forgetting the marking. This allows us to give

Definition 3.10. Let p be the natural projection of $m\mathcal{S}urf^{\mathrm{C}}$ to $m\mathcal{S}urf$ that exists by Definition 3.9. The *central extension* $\mathcal{S}urf^{\mathrm{C}}$ *of the category* $\mathcal{S}urf$ *of surfaces* is the central extension of $\mathcal{S}urf$ that has the same objects as $\mathcal{S}urf$ and whose morphisms are determined by commutativity of the diagram

$$\begin{tikzcd} m\mathcal{S}urf^{\mathrm{C}} \arrow[r, "p"] \arrow[d, "U^{\mathrm{C}}"'] & m\mathcal{S}urf \arrow[d, "U"] \\ \mathcal{S}urf^{\mathrm{C}} \arrow[r] & \mathcal{S}urf \end{tikzcd}$$

where U^{C} is the functor that acts as U on all objects and on all morphisms of $m\mathcal{S}urf^{\mathrm{C}}$ that are morphisms of $m\mathcal{S}urf$, and by by mapping central elements to central elements.

The morphisms of $\mathcal{S}urf^{\mathrm{C}}$ are thus generated by sewings and by elements of central extensions $\mathrm{Map}^{\mathrm{C}}(E)$ of the mapping class groups $\mathrm{Map}(E)$. These groups $\mathrm{Map}^{\mathrm{C}}(E)$ have e.g. been described in [MR, Ge].

Remark 3.11. Note that no relations are imposed for sewings s_E of the type (3.2). Accordingly a redefinition of the action of the Dehn twist T_c as mentioned in Remark 3.8 cannot lead to genuine actions of (non-extended) higher genus mapping class groups, albeit it does so at genus one.

3.3 The pinned block functor

We finally turn our attention to sewings. Recall that the construction in Section 2.4 gives the conformal blocks as coends. We define the functor $\widetilde{\mathrm{Bl}}^{(F)}$ on sewings with the help of the corresponding dinatural structure morphisms of the coends (2.7). For the sake of giving explicit formulas we invoke a Fubini theorem to write these coends as iterated coends, again as in Section 2.4. This requires to treat the elementary sewings of the form (3.1) and (3.2) separately, even though sewing is a local operation: in the case of (3.1) we deal with a coend to which the formula (2.2) applies, while for a sewing as in (3.2) the result (2.3) is relevant. This leads us to

Definition 3.12. On elementary sewings $\widetilde{\mathrm{Bl}}^{(F)}$ acts as follows:

To a sewing $\mathrm{s}_{E_1,E_2}\colon (E_1,\Gamma_1)\sqcup(E_2,\Gamma_2)\to(E,\Gamma)=(E_1,\Gamma_1)\cup_{\beta,\gamma}(E_2,\Gamma_2)$ of the type (3.1), for which we have

$$\begin{aligned}\widetilde{\mathrm{Bl}}^{(F)}(E_1,\Gamma_1)\sqcup(E_2,\Gamma_2)) &= \widetilde{\mathrm{Bl}}^{(F)}(E_1,\Gamma_1)\otimes_{\Bbbk}\widetilde{\mathrm{Bl}}^{(F)}(E_2,\Gamma_2)\\ &= \mathrm{Hom}_{\mathcal{D}}(\mathbf{1},U\otimes F^{\varepsilon}\otimes X)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},V\otimes F^{-\varepsilon}\otimes Y)\end{aligned}$$

with objects $U,V,X,Y\in\mathcal{D}$ which are appropriate tensor products of F, $F^{\vee}$ and K, and with appropriate $\varepsilon\in\{\pm1\}$, assign the linear map $\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_{E_1,E_2})$ that maps a linear map $f_1\otimes f_2$ in $\widetilde{\mathrm{Bl}}^{(F)}(E_1,\Gamma_1)\otimes_{\Bbbk}\widetilde{\mathrm{Bl}}^{(F)}(E_2,\Gamma_2)$ to

$$\begin{aligned}&\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_{E_1,E_2})\,(f_1{\otimes}f_2)\\ &\qquad := \big[\, id_U\otimes(d_{F^{-\varepsilon}}\circ\jmath_F^{\varepsilon})\otimes id_{Y\otimes V\otimes X}\big]\circ\big[\, id_{U\otimes F^{\varepsilon}}\otimes \mathrm{Z}_{V.F^{-\varepsilon}\otimes Y}(f_2)\otimes id_X\big]\circ f_1\,.\end{aligned}\tag{3.15}$$

To a sewing $\mathrm{s}_E\colon (E,\Gamma)\to(E',\Gamma')=\cup_{\beta,\gamma}(E,\Gamma)$ of the type (3.2), for which $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)=\mathrm{Hom}_{\mathcal{D}}$ $(\mathbf{1},U\otimes F^{\varepsilon}\otimes V\otimes F^{-\varepsilon}\otimes W)$ with appropriate $U,V,W\in\mathcal{D}$ and $\varepsilon\in\{\pm1\}$, assign the linear map $\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_E)$ that acts as

$$\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_E)\,(f) := \big[\, id_{U\otimes V}\otimes(\imath^K_{F^{\varepsilon}}\circ\jmath_F^{\varepsilon})\otimes id_W\big]\circ\big[\, id_U\otimes c_{F^{\varepsilon},V}\otimes id_{F^{-\varepsilon}\otimes W}\big]\circ f \tag{3.16}$$

on $f\in\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$.

Note that the linear map $\phi=\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_{E_1,E_2})(f_1{\otimes}f_2)$ is by definition an element of the space $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},U\otimes Y\otimes V\otimes X)$. This morphism space is isomorphic to the space of blocks for the marked surface (E,Γ), and is equal to the space $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma')$ of blocks for some marking Γ' on E. In the sequel we tacitly identify ϕ with $\psi\circ\phi\in\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$, where $\psi\colon\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma')\to\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ is the distinguished isomorphism assigned by Definition 3.6 to the move that transforms Γ' to Γ. Also recall that there is a Fubini theorem which provides unique isomorphisms between different realizations of $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ as morphism spaces of $\mathcal{D}$. By adequately composing with such isomorphisms as well as suitable Z-isomorphisms one can restrict to the case $V=\mathbf{1}$ in the definition of $\widetilde{\mathrm{Bl}}^{(F)}(\mathrm{s}_E)$.

We are now ready to state

Proposition 3.13. *Define* $\widetilde{\mathrm{Bl}}^{(F)}\colon \mathrm{m}\mathcal{S}urf^{\mathrm{C}}\to\mathcal{V}ect$ *as follows.*

(i) $\widetilde{\mathrm{Bl}}^{(F)}$ *acts on objects of the category* $\mathrm{m}\mathcal{S}\mathrm{urf}^{\mathrm{C}}$ *as prescribed in Definition* 3.4.

(ii) $\widetilde{\mathrm{Bl}}^{(F)}$ *acts on moves as prescribed in Definition* 3.6*, on elementary sewings as in Definition* 3.12*, and on the morphisms* $\mathrm{C}_{(E,\Gamma)}$ *as in formula* (3.14)*.*

(iii) *To any presentation of a morphism of* $\mathrm{m}\mathcal{S}\mathrm{urf}^{\mathrm{C}}$ *as a word in the morphisms from* (ii)*,* $\widetilde{\mathrm{Bl}}^{(F)}$ *assigns the corresponding composition of linear maps.*

Then $\widetilde{\mathrm{Bl}}^{(F)}$ *constitutes a symmetric monoidal functor from* $\mathrm{m}\mathcal{S}\mathrm{urf}^{\mathrm{C}}$ *to* $\mathcal{V}ect$*.*

Proof. To any presentation of a morphism $\mathrm{m}\colon(E_1,\Gamma_1)\to(E_2,\Gamma_2)$ of $\mathrm{m}\mathcal{S}urf^{\mathrm{C}}$, $\widetilde{\mathrm{Bl}}^{(F)}$ assigns a linear map $\widetilde{\mathrm{Bl}}^{(F)}(E_1,\Gamma_1)\to\widetilde{\mathrm{Bl}}^{(F)}(E_2,\Gamma_2)$. To prove that $\widetilde{\mathrm{Bl}}^{(F)}$ is a well-defined, we must show that this linear map in fact depends only on m, but not on the chosen presentation of m or, equivalently, that the prescriptions (ii) for specific types of morphisms respect all relations.

Then $\widetilde{\mathrm{Bl}}^{(F)}$ is in particular also compatible with the composition of morphisms.

Compatibility with all relations among (not necessarily admissible) moves except those involving the relation $(\mathrm{W12})^{\mathrm{C}}$ holds by Lemma 3.7. Compatibility with $(\mathrm{W12})^{\mathrm{C}}$ holds as well: the linear maps obtained by applying $\widetilde{\mathrm{Bl}}^{(F)}$ to the left and right hand sides of $(\mathrm{W12})^{\mathrm{C}}$ are equal owing to (3.11) and (3.14). Compatibility with all relations among sewings is guaranteed by Fubini theorems. Finally, the fact that cutting and sewing are inverse operations allows one to reduce any relation between moves and an elementary sewing to a relation among moves of the sewn surface.

The symmetric monoidal structure is implied by (2.8): we have $\widetilde{\mathrm{Bl}}^{(F)}(\emptyset) = \Bbbk$ as well as

$$\widetilde{\mathrm{Bl}}^{(F)}\big((E_1,\Gamma_1) \sqcup (E_2,\Gamma_2)\big) = \widetilde{\mathrm{Bl}}^{(F)}(E_1,\Gamma_1) \otimes_\Bbbk \widetilde{\mathrm{Bl}}^{(F)}(E_2,\Gamma_2)\,,$$

and analogously for morphisms. □

Remark 3.14. Proposition 3.13 may be seen as a rephrasing of Theorem 8.1 of [Ly2] in the framework of finely marked surfaces used here. From the results of [Ly2] it then follows further that there is a unique left exact symmetric monoidal functor from $m\mathcal{S}urf^{\mathrm{C}}$ to $\mathcal{V}ect$ satisfying the requirements (i) and (ii) of the proposition.

Now recall that our goal is to study correlators of a conformal field theory. These are assigned to (extended) surfaces, rather than to marked surfaces. Accordingly we are really looking for a symmetric monoidal functor $\mathrm{Bl}^{(F)}\colon \mathcal{S}urf^{\mathrm{C}} \to \mathcal{V}ect$, rather than the functor $\widetilde{\mathrm{Bl}}^{(F)}$ constructed above for marked surfaces. But we can obtain $\mathrm{Bl}^{(F)}$ easily from $\widetilde{\mathrm{Bl}}^{(F)}$, namely as a Kan extension:

Proposition 3.15. (i) *The right Kan extension*

$$\begin{tikzcd} m\mathcal{S}urf^{\mathrm{C}} \arrow[r, "\widetilde{\mathrm{Bl}}^{(F)}"] \arrow[d, "U^{\mathrm{C}}"'] & \mathcal{V}ect \\ \mathcal{S}urf^{\mathrm{C}} \arrow[u, Rightarrow, shorten=8pt] \arrow[ur, dashed, "\mathrm{Bl}^{(F)}"'] & \end{tikzcd} \tag{3.17}$$

of $\widetilde{\mathrm{Bl}}^{(F)}$ *along the unmarking functor* U^{C} *exists.*

(ii) *The so defined functor* $\mathrm{Bl}^{(F)}\colon \mathcal{S}urf^{\mathrm{C}} \to \mathcal{V}ect$ *has a natural symmetric monoidal structure.*

Proof. (i) For each $E \in \mathcal{S}urf^{\mathrm{C}}$ consider the natural projection $Q_E\colon (f,(E',\Gamma')) \mapsto (E',\Gamma')$ from the comma category $(E \downarrow U^{\mathrm{C}})$ to $m\mathcal{S}urf^{\mathrm{C}}$. The limit of the functor $\widetilde{\mathrm{Bl}}^{(F)} \circ Q_E\colon (E \downarrow U^{\mathrm{C}}) \to \mathcal{V}ect$ not only exists (as for any functor to $\mathcal{V}ect$), but it can also be realized concretely. Indeed, since up to unique isomorphism $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ only depends on the underlying surface E of (E,Γ), as a vector space the limit can be realized, up to distinguished isomorphism, simply by $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma_E)$ for any reference choice of fine marking Γ_E on E. It follows [Ma, Thm. X.3.1] that the right Kan extension of $\widetilde{\mathrm{Bl}}^{(F)}$ along U^{C} exists and is realized by taking the limit of $\widetilde{\mathrm{Bl}}^{(F)} \circ Q_E$ at each object and each morphism of $\mathcal{S}urf^{\mathrm{C}}$.

(ii) We are free to choose the auxiliary markings Γ_E arbitrarily for all connected surfaces E and to set $\Gamma_{E\sqcup E'} := \Gamma_E \sqcup \Gamma_{E'}$. Then we have

$$\begin{aligned} \mathrm{Bl}^{(F)}(E \sqcup E') &= \widetilde{\mathrm{Bl}}^{(F)}(E \sqcup E', \Gamma_{E\sqcup E'}) = \widetilde{\mathrm{Bl}}^{(F)}((E,\Gamma_E) \sqcup (E',\Gamma_{E'})) \\ &= \widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma_E) \otimes_\Bbbk \widetilde{\mathrm{Bl}}^{(F)}(E',\Gamma_{E'}) = \mathrm{Bl}^{(F)}(E) \otimes_\Bbbk \mathrm{Bl}^{(F)}(E')\,. \end{aligned}$$

Thus indeed $\mathrm{Bl}^{(F)}$ is (strict) symmetric monoidal.
(With the same prescription of auxiliary data the natural transformation $\psi\colon \mathrm{Bl}^{(F)} \circ U^{\mathrm{C}} \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$ that is part of the Kan extension is monoidal as well: By construction the linear map $\psi_{(E,\Gamma)}$ is nothing but the distinguished isomorphism $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma_E) \xrightarrow{\cong} \widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$. It thus follows directly that $\psi_{(E\sqcup E',\Gamma\sqcup\Gamma')} = \psi_{(E,\Gamma)} \otimes_{\Bbbk} \psi_{(E',\Gamma')}$.) $\square$

3.4 Systems of correlators as natural transformations

Having at hand the functor $\mathrm{Bl}^{(F)}$ it is easy to formulate the consistency conditions that have to be obeyed by a system of correlators: The correlator $\mathrm{v}_F(E)$ for a surface E is an element of the space $\mathrm{Bl}^{(F)}(E)$; it is required that $\mathrm{v}_F(E)$ is invariant under the centrally extended mapping class group of E,

$$\mathrm{Bl}^{(F)}(\phi)\left(\mathrm{v}_F(E)\right) = \mathrm{v}_F(E) \qquad \text{for any} \quad \phi \in \mathrm{Map}^{\mathrm{C}}(E)\,,$$

and that correlators for surfaces that are related by sewing are mapped to each other,

$$\mathrm{Bl}^{(F)}(\mathsf{s})\left(\mathrm{v}_F(E)\right) = \mathrm{v}_F(E') \qquad \text{for any sewing} \quad \mathsf{s}\colon E \to E'\,.$$

In addition, to exclude degenerate solutions, we require that the endomorphism of F that is provided by the correlator for a cylinder $E^0_{1|1}$, i.e. for a sphere with one ingoing and one outgoing boundary circle, is invertible. (In the conformal field theory literature, this condition is known as non-degeneracy of the two-point function of bulk fields.)

Let us rewrite these conditions more compactly. We introduce the constant symmetric monoidal functor that assigns the ground field to each object and the identity morphism to each morphism,

$$\begin{array}{rrcl} \Delta_{\Bbbk}: & \mathcal{S}urf^{\mathrm{C}} & \longrightarrow & \mathcal{V}ect \\ & E & \longmapsto & \Bbbk \\ & E \xrightarrow{\varphi} E' & \longmapsto & id_{\Bbbk} \end{array}$$

We can then give

Definition 3.16. Let $\mathcal{D}$ be a modular finite ribbon category and $F \in \mathcal{D}$ an object. A *consistent system of bulk field correlators* for monodromy data based on $\mathcal{D}$ and with bulk object F is a monoidal natural transformation

$$\mathrm{v}_F: \quad \Delta_{\Bbbk} \Rightarrow \mathrm{Bl}^{(F)} \tag{3.18}$$

for which the linear map $(id_F \otimes d_F) \circ (\mathrm{v}_F(E^0_{1|1}) \otimes id_F) \in \mathrm{End}_{\mathcal{D}}(F)$ is invertible.

Here and below we identify the linear map $\mathrm{v}_F(E)$ with its value at $1 \in \Bbbk = \Delta_{\Bbbk}(E)$.

As in the construction of the pinned bock functor, for examining such natural transformations it is advantageous to also work with marked surfaces. Thus we introduce the constant functor $\widetilde{\Delta}_{\Bbbk}\colon m\mathcal{S}urf^{\mathrm{C}} \to \mathcal{V}ect$ that again maps every object to $\Bbbk$ and every morphism to $id_{\Bbbk}$, and consider monoidal natural transformations

$$\widetilde{\mathrm{v}}_F: \quad \widetilde{\Delta}_{\Bbbk} \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}. \tag{3.19}$$

Once we are given such a monoidal natural transformation $\widetilde{\mathrm{v}}_F$, we can obtain a corresponding consistent system v_F of correlators by invoking the defining universal property of the Kan extension (3.17). Indeed, the two constant functors $\Delta_\Bbbk$ and $\widetilde{\Delta}_\Bbbk$ are related by a right Kan extension

$$\begin{tikzcd} m\mathcal{S}urf^{\mathrm{C}} \arrow[r, "\widetilde{\Delta}_\Bbbk"] \arrow[d, "U^{\mathrm{C}}"'] & \mathcal{V}ect \\ \mathcal{S}urf^{\mathrm{C}} \arrow[ur, dashed, "\Delta_\Bbbk"'] \arrow[u, phantom, shift left=4ex, "\Rightarrow id"] & \end{tikzcd}$$

with trivial natural transformation. When composed with a natural transformation $\widetilde{\mathrm{v}}_F \colon \widetilde{\Delta}_\Bbbk \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$, this gives the diagram

$$\begin{tikzcd} m\mathcal{S}urf^{\mathrm{C}} \arrow[r, "\widetilde{\mathrm{Bl}}^{(F)}"] \arrow[d, "U^{\mathrm{C}}"'] & \mathcal{V}ect \\ \mathcal{S}urf^{\mathrm{C}} \arrow[ur, bend right, "\Delta_\Bbbk"'] \arrow[ur, phantom, shift left=3ex, "\Rightarrow \widetilde{\mathrm{v}}_F"] & \end{tikzcd}$$

By the universal property of $\mathrm{Bl}^{(F)}$ as a right Kan extension there then exists, uniquely up to unique natural isomorphism, a natural transformation $\mathrm{v}_F \colon \Delta_\Bbbk \Rightarrow \mathrm{Bl}^{(F)}$ such that $\widetilde{\mathrm{v}}_F$ is given by the composition

$$\begin{tikzcd} m\mathcal{S}urf^{\mathrm{C}} \arrow[r, "\widetilde{\mathrm{Bl}}^{(F)}"] \arrow[d, "U^{\mathrm{C}}"'] & \mathcal{V}ect \\ \mathcal{S}urf^{\mathrm{C}} \arrow[ur, "\mathrm{Bl}^{(F)}"] \arrow[ur, bend right=50, "\Delta_\Bbbk"'] \arrow[ur, phantom, shift left=4ex, "\Rightarrow \psi"] \arrow[ur, phantom, shift right=4ex, "\Uparrow \mathrm{v}_F"] & \end{tikzcd} \tag{3.20}$$

with ψ the natural transformation that is part of the Kan extension (3.17). Also, if the natural transformation $\widetilde{\mathrm{v}}_F$ is monoidal, then so is v_F, and if $\widetilde{\mathrm{v}}_F(E^0_{1|1}, \mathit{\Gamma})$ for any marking $\mathit{\Gamma}$ is invertible, then so is $\mathrm{v}_F(E^0_{1|1})$.

We will refer, analogously as in Definition 3.16, to a natural transformation $\widetilde{\mathrm{v}}_F \colon \widetilde{\Delta}_\Bbbk \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$ as a *consistent system of correlators on marked surfaces* or, slightly abusing terminology, just as a consistent system of correlators.

4 Consistent systems of correlators

We are now in a position to address our primary goal: to formulate necessary and sufficient conditions for the existence of a consistent system of bulk field correlators with bulk object F, expressed as a monoidal natural transformation (3.18). To achieve this, taking advantage of the results of Section 3, we work with the category $m\mathcal{S}urf^{\mathrm{C}}$ of marked surfaces instead of $\mathcal{S}urf^{\mathrm{C}}$, and thus consider instead of (3.18) a monoidal natural transformation $\widetilde{\mathrm{v}}_F$ (3.19). That $\widetilde{\mathrm{v}}_F$ is a natural transformation from $\widetilde{\Delta}_\Bbbk$ to $\widetilde{\mathrm{Bl}}^{(F)}$ means that the square

$$\begin{tikzcd} \Bbbk \arrow[r, "id_\Bbbk"] \arrow[d, "\widetilde{\mathrm{v}}_F(E,\mathit{\Gamma})"'] & \Bbbk \arrow[d, "\widetilde{\mathrm{v}}_F(E',\mathit{\Gamma}')"] \\ \widetilde{\mathrm{Bl}}^{(F)}(E,\mathit{\Gamma}) \arrow[r, "\widetilde{\mathrm{Bl}}^{(F)}(f)"] & \mathrm{Bl}^{(F)}(E',\mathit{\Gamma}') \end{tikzcd} \tag{4.1}$$

commutes for any morphism f of $m\mathcal{S}urf^{\mathcal{C}}$. Moreover, by the construction of the functor $\widetilde{\mathrm{Bl}}^{(F)}$, such a natural transformation is already completely characterized by commutativity of this square for all generating morphisms f (see Section 2.2 and formulas (3.1) and (3.2)) of $m\mathcal{S}urf^{\mathcal{C}}$.

4.1 Elementary correlators

For analyzing the condition (4.1) it is worth recalling the basic idea of the construction of conformal blocks in Section 2.4: First one defines conformal block functors for spheres with at most three holes, and then these are used as building blocks to obtain the functors for arbitrary surfaces with the help of coend constructions. Inspired by this idea, here we start by selecting vectors in the morphism spaces $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ for E a sphere with at most three holes and then obtain vectors in $\widetilde{\mathrm{Bl}}^{(F)}(E,\Gamma)$ for any marked surface (E,Γ) which are uniquely determined by the requirement to furnish a monoidal natural transformation $\widetilde{\mathrm{v}}_F\colon \widetilde{\Delta}_{\Bbbk} \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$.

In fact, denoting by $E^0_{p|q}$ a sphere with p incoming and q outgoing boundary circles, we have

Proposition 4.1. *A monoidal natural transformation $\widetilde{\mathrm{v}}_F\colon \widetilde{\Delta}_{\Bbbk} \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$ is completely determined by its values on the spheres $E^0_{0|3}$, $E^0_{1|0}$ and $E^0_{2|0}$ with any choice of marking without cuts.*

Proof. Commutativity of (4.1) for $E=E'$ on these three surfaces and for f the Z-isomorphism (3.4) amounts to the statement that the values of $\widetilde{\mathrm{v}}_F$ on these surfaces indeed do not depend on a choice of marking without cuts on them. Further, that the natural transformation $\widetilde{\mathrm{v}}_F$ is (strictly) monoidal simply means that

$$\widetilde{\mathrm{v}}_F\big((E_1,\Gamma_1)\sqcup(E_2,\Gamma_2)\big) = \widetilde{\mathrm{v}}_F(E_1,\Gamma_1)\otimes_{\Bbbk}\widetilde{\mathrm{v}}_F(E_2,\Gamma_2)\,.$$

As a consequence we can restrict our attention to connected surfaces.
Thus let (E,C,Γ) be any connected marked surface with non-empty fine cut system C and $c\in C$ any of the cuts. Then by compatibility with sewing, the correlator for (E,C,Γ) must coincide with the vector obtained from the correlator for the cut surface $\mathfrak{cut}_{\{c\}}(E)$ by applying the linear map that by Definition 3.12 is associated to the sewing $\mathsf{s}\colon \mathfrak{cut}_{\{c\}}(E)\to E$:

$$\widetilde{\mathrm{v}}_F(E,C,\Gamma) = \widetilde{\mathrm{Bl}}^{(F)}(\mathsf{s})\circ\widetilde{\mathrm{v}}_F(\mathfrak{cut}_{\{c\}}(E)) \tag{4.2}$$

(in the notation we suppress the marking of $\mathfrak{cut}_{\{c\}}(E)$, which is completely determined by the one of E). Repeating this prescription for every cut in C, $\widetilde{\mathrm{v}}_F(E,C,\Gamma)$ gets expressed through the correlators for spheres that have at most three holes and markings without cuts. To complete the proof we need to show that all the latter correlators can, in turn, be expressed through the three correlators $\widetilde{\mathrm{v}}_F(E^0_{0|3})$, $\widetilde{\mathrm{v}}_F(E^0_{1|0})$ and $\widetilde{\mathrm{v}}_F(E^0_{2|0})$. This will be done separately in Lemma 4.2 below. □

Note that a priori the so defined vectors could depend on the order in which the cuts are removed. Recall, however, that according to Definition 3.12 the linear maps $\widetilde{\mathrm{Bl}}^{(F)}(\mathsf{s})$ are given by the dinatural structure morphisms of coends (see formulas (3.15) and (3.16)). Independence of the ordering is thus in fact guaranteed by the Fubini theorem.

To formulate the postponed part of the proof it will be convenient to work with the three morphisms

$$\begin{aligned} \omega_F &:= \widetilde{\mathrm{v}}_F(E^0_{0|3}) \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F\otimes F\otimes F) \\ \varepsilon_F &:= d_F \circ (\widetilde{\mathrm{v}}_F(E^0_{1|0}) \otimes id_F) \in \mathrm{Hom}_{\mathcal{D}}(F, \mathbf{1}) \qquad \text{and} \\ \Phi_F &:= (id_{F^\vee} \otimes d_F) \circ (\widetilde{\mathrm{v}}_F(E^0_{2|0}) \otimes id_F) \in \mathrm{Hom}_{\mathcal{D}}(F, F^\vee)\,. \end{aligned} \tag{4.3}$$

Lemma 4.2. *Let $\widetilde{\mathrm{v}}_F$ be a natural transformation $\widetilde{\mathrm{v}}_F\colon \widetilde{\Delta}_{\Bbbk} \Rightarrow \widetilde{\mathrm{Bl}}^{(F)}$. For E any sphere with at most three holes and Γ any fine marking without cuts on E, $\widetilde{\mathrm{v}}_F(E,\Gamma)$ can be expressed through the three morphisms* (4.3). *Specifically,*

(i) *To $(E^0_{0|2},\Gamma)$, $\widetilde{\mathrm{v}}_F$ assigns the vector*

$$(\varepsilon_F \otimes id_F \otimes id_F)\circ\omega_F = (id_F\otimes\varepsilon_F\otimes id_F)\circ\omega_F = (id_F\otimes id_F\otimes\varepsilon_F)\circ\omega_F \tag{4.4}$$

in $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F\otimes F)$.

(ii) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{0|1},\Gamma) = (\varepsilon_F\otimes\varepsilon_F\otimes id_F)\circ\omega_F$, *and similarly as in* (4.4) *one gets the same vector in* $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},F)$ *if the occurrence of id_F in this expression is interchanged with any occurrence of ε_F.*

(iii) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{0|0},\emptyset) = (\varepsilon_F\otimes\varepsilon_F\otimes\varepsilon_F)\circ\omega_F$.

(iv) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{1|1},\Gamma)$ *is given by either* $(id_F\otimes\varepsilon_F\otimes\Phi_F)\circ\omega_F$ *or* $(\Phi_F\otimes\varepsilon_F\otimes id_F)\circ\omega_F$, *depending on whether the boundary circle on which the distinguished first edge of Γ ends is outgoing or incoming.*

(v) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{1|2},\Gamma)$ *is given by either* $(\Phi_F\otimes id_F\otimes id_F)\circ\omega_F$ *or* $(id_F\otimes\Phi_F\otimes id_F)\circ\omega_F$ *or* $(id_F\otimes id_F \otimes\Phi_F)\circ\omega_F$, *depending on whether the edge that ends on the incoming boundary circle is the first, second or last of the three edges of Γ.*

(vi) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{2|1},\Gamma)$ *is given by either* $(id_F\otimes\Phi_F\otimes\Phi_F)\circ\omega_F$ *or* $(\Phi_F\otimes id_F\otimes\Phi_F)\circ\omega_F$ *or* $(\Phi_F\otimes\Phi_F \otimes id_F)\circ\omega_F$, *depending on whether the edge that ends on the outgoing boundary circle is the first, second or last of the three edges of Γ.*

(vii) $\widetilde{\mathrm{Bl}}^{(F)}(E^0_{3|0},\emptyset) = (\Phi_F\otimes\Phi_F\otimes\Phi_F)\circ\omega_F$.

Proof. (i) Both the equalities (4.4) and the assertion that $\widetilde{\mathrm{v}}_F(E^0_{0|2},\Gamma)$ is given, for any Γ, by this vector follow by combining invariance under the Z-isomorphism and compatibility with sewings $\mathsf{s}\colon E^0_{0|3}\sqcup E^0_{1|0}\to E^0_{0|2}$. To see this, one notes that according to Definition 3.12, $\widetilde{\mathrm{Bl}}^{(F)}(\mathsf{s})$ acts on $\widetilde{\mathrm{v}}_F(E^0_{0|3})\otimes\widetilde{\mathrm{v}}_F(E^0_{1|0})$ as post-composition by $id_F\otimes[d_{F^\vee}\circ(\pi_F\otimes id_F)]$, and by the defining properties of the pivotal structure we have $d_{F^\vee}\circ(\pi_F\otimes\widetilde{\mathrm{v}}_F(E^0_{1|0})) = \varepsilon_F$.
Note that in the first place sewing $E^0_{0|3}$ to $E^0_{1|0}$ yields a marking on $E^0_{0|2}$ that has a cut. But by compatibility with the F-move (M3) this cut can be omitted without changing the value of $\widetilde{\mathrm{v}}_F(E^0_{0|2})$.
(ii) is shown in the same way as (i), considering instead sewings $E^0_{0|2}\sqcup E^0_{1|0}\to E^0_{0|1}$.
(iii) follows from (ii) by compatibility with the sewing $E^0_{0|1}\sqcup E^0_{1|0}\to E^0_{0|0}$.
(iv) – (vii) follow in the same way as (i) when considering sewings $E^0_{0|2}\sqcup E^0_{2|0}\to E^0_{1|1}$, respectively $E^0_{0|3}\sqcup E^0_{2|0}\to E^0_{1|2}$, respectively $E^0_{1|2}\sqcup E^0_{2|0}\to E^0_{2|1}$, respectively $E^0_{2|1}\sqcup E^0_{2|0}\to E^0_{3|0}$. □

4.2 Correlators on surfaces of genus zero

In view of Proposition 4.1 it is not hard to examine the implications of the covariance requirement (4.1). In particular, commutativity of (4.1) for $f = \mathsf{s}$ a sewing amounts to well-definedness of the prescription given in the proof of Proposition 4.1 which, as already noted, is ensured by the Fubini theorem. We are thus left with the case that f is an elementary move of one of the groupoids $\mathcal{FM}(E)$ – or rather, to be precise, either an admissible elementary move or a simple combination of non-admissible elementary moves that is admissible. In addition we must take care of the non-degeneracy requirement in Definition 3.16.

For the time being we restrict our attention to surfaces of genus zero. Recall that the functor $\widetilde{\mathrm{Bl}}^{(F)}$ maps the elementary moves to the isomorphisms (3.4) (Z-isomorphism), (3.5) (B-isomorphism), (3.6) (F-isomorphism) and (3.7) (A-isomorphism). We first show

Lemma 4.3. *Let* $\widetilde{\mathrm{v}}_F$ *be a consistent system of correlators on marked surfaces. Then*

(i) *The morphisms* ω_F *and* ε_F *defined in* (4.4) *are non-zero.*

(ii) *The morphism* Φ_F *defined in* (4.4) *is an isomorphism.*

(iii) $\omega_F \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F^{\otimes 3})$ *is invariant under any braiding of the three* F*-strands.*

(iv) *The object* $F \in \mathcal{D}$ *is self-dual and has trivial twist.*

Proof. On the cylinder $E^0_{1|1}$ select a marking Γ without cuts that has trivial winding with respect to the non-contractible cycle of the cylinder and for which the distinguished edge ends on the outgoing boundary circle. Abbreviate $j_F := (id_F \otimes d_F) \circ [\widetilde{\mathrm{v}}_F(E^0_{1|1}, \Gamma) \otimes id_F] \in \mathrm{End}_{\mathcal{D}}(F)$. The non-degeneracy condition demands that j_F is invertible. On the other hand, gluing two such cylinders and invoking invariance under the F-isomorphism shows that j_F is an idempotent; thus $j_F = id_F$. This is equivalent to $\widetilde{\mathrm{v}}_F(E^0_{1|1}, \Gamma) = b_F$, from which it follows in particular that ω_F and ε_F are non-zero, thus proving (i), and that the morphism $\Phi_F^- := (d_F \otimes \varepsilon_F \otimes id_F) \circ (id_{F^\vee} \otimes \omega_F)$ $\in \mathrm{Hom}_{\mathcal{D}}(F^\vee, F)$ is a right-inverse of Φ_F.
Next we note, using naturality of the braiding and the relation between the pivotal structure and the twist, that for any marking Γ on $E^0_{0|2}$ we have

$$\mathrm{Z}_{F,F}\big(\widetilde{\mathrm{v}}_F(E^0_{0|2}, \Gamma)\big) = (id_F \otimes \theta_F) \circ \big(\mathrm{B}_{F,F,\mathbf{1}}\big(\widetilde{\mathrm{v}}_F(E^0_{0|2}, \Gamma)\big)\big) .$$

Invariance of $\widetilde{\mathrm{v}}_F(E^0_{0|2}, \Gamma)$ under both the Z- and the B-isomorphism thus implies that F has trivial twist, $\theta_F = id_F$. Using this result, it follows further that $\mathrm{Z}_{F,F\otimes F}$ and $\mathrm{B}_{F,F,F}$ generate an action of the braid group B_3 on the three tensor factors F in the codomain of ω. Invariance of $\widetilde{\mathrm{v}}_F(E^0_{0|3}, \Gamma)$ under the Z- and the B-isomorphisms thus proves the claim (iii). Combining the results obtained so far one easily shows that the morphism Φ_F^- is also a left-inverse of Φ_F, thus completing the proof of (ii). Finally, (ii) implies that F is self-dual. □

One can further check that

$$(d_F \otimes \varepsilon_F \otimes id_F) \circ (id_{F^\vee} \otimes \omega_F) \equiv \Phi_F^{-1} = \big(id_F \otimes \varepsilon_F \otimes [d_{F^\vee} \circ (\pi_F \otimes id_{F^\vee})]\big) \circ (\omega_F \otimes id_{F^\vee}) ,$$

which in turn implies

$$\Phi_F^\vee \circ \pi_F = \Phi_F .$$

This identity (which was in fact already used implicitly in the statement of Lemma 4.2 (v) and (vi) above) means that the object F is not only self-dual, but also has a Frobenius-Schur indicator equal to 1.

To proceed we introduce the specific expressions

$$
\begin{aligned}
&\Delta_F := \big(id_{F^{\otimes 2}} \otimes [d_F \circ (\Phi_F \otimes id_F)]\,\big) \circ (\omega_F \otimes id_F) \;\in \mathrm{Hom}_{\mathcal{D}}(F, F \otimes F)\,, \\
&\eta_F := (\varepsilon_F \otimes \varepsilon_F \otimes id_F) \circ \omega_F \;\in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F) \qquad \text{and} \qquad\qquad (4.5)\\
&m_F := \big(id_F \otimes [d_{F^{\otimes 2}} \circ (\Phi_F \otimes \Phi_F \otimes id_{F^{\otimes 2}})]\,\big) \circ (\omega_F \otimes id_{F^{\otimes 2}}) \;\in \mathrm{Hom}_{\mathcal{D}}(F \otimes F, F)
\end{aligned}
$$

in the morphisms used so far. We have

Proposition 4.4. *Let $\widetilde{\mathrm{v}}_F$ be a consistent system of correlators with bulk state space F. Then the morphisms $(m_F, \eta_F, \Delta_F, \varepsilon_F)$ endow the object F with the structure of a Frobenius algebra in $\mathcal{D}$.*

Proof. Invariance of $\widetilde{\mathrm{v}}_F(E^0_{0|4}, \Gamma)$, for a suitable choice of marking Γ, under the A-isomorphism amounts to the equality

$$
\begin{aligned}
&\big(id_{F^{\otimes 2}} \otimes [d_F \circ (\Phi_F \otimes id_F)] \otimes id_{F^{\otimes 2}}\big) \circ (\omega_F \otimes \omega_F) \\
&\qquad\qquad = \big(id_{F^{\otimes 3}} \otimes [d_F \circ (\Phi_F \otimes id_F)] \otimes id_F\big) \circ (id_F \otimes \omega_F \otimes id_{F^{\otimes 2}}) \circ \omega_F\,.
\end{aligned}
$$

When expressed in terms of Δ_F, this is nothing but coassociativity. The counit properties of ε_F are equivalent to the result obtained in the proof of Lemma 4.3 that the morphism j_F defined there equals id_F. This shows that $(F, \Delta_F, \varepsilon_F)$ is a coalgebra.

That (F, m_F, η_F) is an algebra follows by dualizing these considerations after noticing that

$$
m_F^\vee = (\Phi_F \otimes \Phi_F) \circ \Delta_F \circ \Phi_F^{-1} \qquad \text{and} \qquad \eta_F^\vee = \varepsilon_F \circ \Phi_F^{-1}. \tag{4.6}
$$

Finally, using invariance of ω_F under the B-isomorphism one sees that the self-duality morphism Φ_F and associativity of m_F are compatible in the sense that $d_F \circ (\Phi_F \otimes id_F) \circ (m_F \otimes m_F) = d_F \circ \big(\Phi_F \otimes [m_F \circ (id_F \otimes m_F)]\,\big)$. When combined with the formula (4.6) for $m_F^\vee$, this implies the Frobenius property $(m_F \otimes id_F) \circ (id_F \otimes \Delta_F) = (id_F \otimes m_F) \circ (\Delta_F \otimes id_F)$. $\square$

Also note that $\Phi_F = ((\varepsilon_F \circ m_F) \otimes id_{F^\vee}) \circ (id_F \otimes b_F)$. Further properties of F now follow easily:

Proposition 4.5. *Let $\widetilde{\mathrm{v}}_F$ be a consistent system of correlators with bulk state space F. Then the Frobenius algebra $(F, m_F, \eta_F, \Delta_F, \varepsilon_F)$ is (co)commutative and symmetric.*

Proof. Cocommutativity of Δ_F is an immediate consequence of the invariance of ω_F under the B-isomorphism. For a Frobenius algebra, commutativity is equivalent to cocommutativity. (In the case at hand, commutativity of m_F also follows directly by dualizing after invoking (4.6).) Commutativity together with triviality of the twist imply that the Frobenius form $\varepsilon_F \circ m_F$ is symmetric. (Conversely, $\theta_F = id_F$ follows by combining commutativity and symmetry.) $\square$

It is not hard to check that demanding that $\widetilde{\mathrm{v}}_F$ is a consistent system of correlators on marked surfaces of *genus zero* does not lead to any further constraints. Moreover, by reading some of the arguments backwards it is evident that the restrictions on the object F obtained above are indeed also sufficient. Recalling in addition that $\widetilde{\mathrm{v}}_F$ gives us a monoidal natural transformation $\mathrm{v}_F\colon \Delta_{\Bbbk} \Rightarrow \mathrm{Bl}^{(F)}$ via (3.20), we can state the first part of our main result, which describes the bulk fields of a conformal field theory that is defined on surfaces of genus zero:

Theorem 4.6. *For $\mathcal{D}$ a finite ribbon category, the consistent systems of bulk field correlators on surfaces of genus zero with (genus-zero) monodromy data based on $\mathcal{D}$ and with bulk object $F \in \mathcal{D}$ are in bijection with structures of a commutative symmetric Frobenius algebra on F.*

4.3 Higher genus correlators

To extend our findings to surfaces of any genus we need to analyze invariance under the S-isomorphism. Recall that for $U \in \mathcal{D}$ the S-isomorphism S_U is the linear endomorphism of $\mathrm{Hom}_{\mathcal{D}}(U,K)$ given by post-composition with the isomorphism $S^K \in \mathrm{End}_{\mathcal{D}}(K)$ defined in (3.8). We are thus dealing with correlators for genus-1 surfaces. Indeed invariance under the S-isomorphism boils down to invariance of the correlator of a one-holed torus $E^1_{1|0}$ with some choice of marking Γ, or rather, to fit with the conventions chosen for the S-isomorphism, of the combination

$$\widetilde{\mathrm{v}}^1_{1|0} := (id_K \otimes d_F) \circ (\widetilde{\mathrm{v}}_F(E^1_{1|0}, \Gamma) \otimes id_F) \;\in \mathrm{Hom}_{\mathcal{D}}(F,K)\,.$$

In short, a system of correlators that is consistent at genus zero can be consistently extended to higher genus if and only the equality

$$S^K \circ \widetilde{\mathrm{v}}^1_{1|0} = \widetilde{\mathrm{v}}^1_{1|0} \tag{4.7}$$

holds.

Via the prescription (4.2) the vector $\widetilde{\mathrm{v}}^1_{1|0}$ is expressed through the correlator of a three-holed sphere obtained as a cut surface, $E^0_{2|1} = \mathrm{cut}_{\{c\}}(E^1_{1|0})$. Further, without loss of generality we may assume that the fine cut system on the torus $E^1_{1|0}$ is minimal and thus consists of a single cut, and take the graph Γ on the torus in such a way that the resulting graph Γ_c on the cut surface is the one for which $\widetilde{\mathrm{v}}_F(E^0_{2|1}, \Gamma_c) = (id_F \otimes \Phi_F \otimes \Phi_F) \circ \omega_F$. Doing so we arrive at

$$\widetilde{\mathrm{v}}^1_{1|0} = (\imath^K_F \otimes d_F) \circ (id_F \otimes \Phi_F \otimes \Phi_F \otimes id_F) \circ (\omega_F \otimes id_F)$$

with $\imath^K$ the dinatural transformation of the coend K. By comparison with the definition of the coproduct Δ_F in (4.5), this can also be written as $\widetilde{\mathrm{v}}^1_{1|0} = \imath^K_F \circ (id_F \otimes \Phi_F) \circ \Delta_F$, so that we may think of invariance under the S-isomorphism as a compatibility property of the coproduct of F with the structural morphism $\imath^K_F$ of the coend K. We accommodate this observation by

Definition 4.7. A commutative symmetric Frobenius algebra $(X, m, \eta, \Delta, \varepsilon)$ in a modular finite ribbon category $\mathcal{D}$ is called *modular* iff it satisfies

$$S^K \circ [\imath^K_X \circ (id_X \otimes \Phi) \circ \Delta] = \imath^K_X \circ (id_X \otimes \Phi) \circ \Delta\,,$$

with $\Phi = ((\varepsilon \circ m) \otimes id_{X^\vee}) \circ (id_X \otimes b_X)$.

We have thus arrived at the second part of our main result:

Theorem 4.8. *For $\mathcal{D}$ a modular finite ribbon category, the consistent systems of bulk field correlators with monodromy data based on $\mathcal{D}$ and with bulk object $F \in \mathcal{D}$ are in bijection with structures of a modular Frobenius algebra on F.*

It is worth noting that by combining 4.2 with the explicit form of $\widetilde{\mathrm{Bl}}^{(F)}(\mathsf{s})$ for the relevant sewings s (see Definition 3.12), every correlator $\mathrm{v}_F(E)$ can be expressed in closed form in terms of the three morphisms (4.3), the duality and pivotality morphisms for F and the morphism $\imath^K_F$. This expression takes a particularly suggestive form when using the duality to rewrite the correlator for $E^g_{p|q}$ as a morphism $\mathrm{v}^g_{p|q} \in \mathrm{Hom}_{\mathcal{D}}(F^{\otimes p}, F^{\otimes q} \otimes K^{\otimes g})$ and using the following abbreviations: Write

$$\tau_F := (m_F \otimes \imath^K_F) \circ \big(id_F \otimes [(\Phi_F^{-1} \otimes \pi_F^{-1}) \circ b_{F^\vee}] \otimes \Phi_F\big) \circ \Delta_F \;\in \mathrm{Hom}(F, F \otimes K)\;:$$

set $m_F^{(0)} := \eta_F$, $m_F^{(1)} := id_F$, $m_F^{(2)} := m_F$ as well as $\Delta_F^{(0)} := \varepsilon_F$, $\Delta_F^{(1)} := id_F$, $\Delta_F^{(2)} := \Delta_F$, and similarly $\tau_F^{(0)} := id_F$ and $\tau_F^{(1)} := \tau_F$; then define recursively

$$m_F^{(n)} := m_F \circ (m_F^{(n-1)} \otimes id_F)\,, \qquad \Delta_F^{(n)} := (\Delta_F^{(n-1)} \otimes id_F) \circ \Delta_F \qquad \text{and}$$
$$\tau_F^{(n-1)} := (\tau_F^{(n-2)} \otimes id_K) \circ \tau_F$$

for $n \geq 3$. Then we have

Proposition 4.9. *Let* v_F *be a consistent system of bulk field correlators with bulk object* F*. Then the correlator for a genus-*g *surface with* p *incoming and* q *outgoing boundary circles is given by*

$$\mathrm{v}_{p|q}^{g} = (\Delta_F^{(q)} \otimes id_{K^{\otimes g}}) \circ \tau_F^{(g)} \circ m_F^{(p)}. \tag{4.8}$$

Remark 4.10.

(i) For any finite ribbon category the monoidal unit $\mathbf{1}$ carries a trivial structure of a Frobenius algebra, which is commutative and symmetric. It thus provides a consistent system of bulk field correlators at genus zero, albeit a rather boring one. In fact, in this case the expression (4.8) reduces to $\mathrm{v}_{p|q}^{g} = \tau_{\mathbf{1}}^{(g)} = \eta_K^{\otimes g}$. In particular, $S^K \circ \mathrm{v}_{1|0}^{1} = S^K \circ \eta_K = \Lambda_K$, implying that $\mathbf{1}$ is *not* modular, unless $\mathcal{D} \simeq \mathcal{V}ect$.

(ii) Existence of any modular Frobenius algebra in a modular finite ribbon category is far from guaranteed. In case $\mathcal{D}$ is semisimple, it follows from Corollary 4.1(i) of [DMNO] that a modular Frobenius algebra in $\mathcal{D}$ exists iff $\mathcal{D}$ is a Drinfeld center.

(iii) If $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ is the enveloping category of a semisimple modular tensor category $\mathcal{C}$, consistent systems of correlators (both for bulk fields and for boundary fields, and also for surfaces with a network of topological defect lines) can be constructed with the help of the three-dimensional topological field theory based on $\mathcal{C}$ [FRS, FFRS1]. The corresponding modular Frobenius algebras in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ are related by a center construction [FrFRS, Da] to the strongly separable symmetric Frobenius algebras in $\mathcal{C}$ used in [FRS, FFRS1]. Theorem 4.8 reproduces in this case the classification results in [FFRS2] and [KoLR].

(iv) For any finite-dimensional factorizable ribbon Hopf algebra H, the category H-mod of finite-dimensional H-modules is a modular finite ribbon category. Further [FSS1], for any ribbon automorphism ω of H the coend $F_\omega := \int^{M \in H\text{-mod}} \bar\omega(M) \boxtimes M^\vee$, with $\bar\omega$ the automorphism of the identity functor induced by ω, carries a structure of a modular Frobenius algebra in the enveloping category H-mod $\boxtimes$ H-mod$^{\mathrm{rev}}$. The formula (4.8) for the associated correlators has in these cases been given in [FSS2, Rem. 3.3] (see also [FSS2, Eq. (3.5)] for a graphical description). It is worth noting that for non-semisimple H the objects $F_\omega \in H\text{-mod} \boxtimes H\text{-mod}^{\mathrm{rev}}$ are neither semisimple nor projective; we expect that this is a generic feature when $\mathcal{D}$ is non-semisimple.

(v) For $\mathcal{C}$ any finite tensor category, denote by R the right adjoint of the forgetful functor from the Drinfeld center $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$. If $\mathcal{C}$ is unimodular, then $R(\mathbf{1})$ is a commutative symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$ [Sh1, Thm. 6.1]. For $\mathcal{C} = H$-mod as in (iv), $R(\mathbf{1})$ is the image of F_{id} under the equivalence $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$. It is thus natural to conjecture that $R(\mathbf{1})$ is in fact a modular Frobenius algebra for any modular finite ribbon category $\mathcal{C}$. If $\mathcal{C}$ is semisimple, then by Proposition 4.8 of [DMNO] $R(\mathbf{1})$ is a Lagrangian algebra in the sense of [DMNO, Def. 4.2].

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