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Sharp thresholds for Ramsey properties of strictly balanced nearly bipartite graphs

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SHARP THRESHOLDS FOR RAMSEY PROPERTIES OF STRICTLY BALANCED NEARLY BIPARTITE GRAPHS

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Abstract. For a given graph F we consider the family of (finite) graphs G with the *Ramsey property* for F, that is the set of such graphs G with the property that every two-colouring of the edges of G yields a monochromatic copy of F. For F being a triangle Friedgut, Rödl, Ruciński, and Tetali (2004) established the sharp threshold for the Ramsey property in random graphs. We obtained a simpler proof of this result which extends to a more general class of graphs F including all cycles.

The proof is based on Friedgut's criteria (1999) for sharp thresholds, and on the recently developed *container method* for independent sets in hypergraphs by Saxton and Thomason, and Balogh, Morris and Samotij. The proof builds on some recent work of Friedgut et al. who established a similar result for van der Waerden's theorem.

1. Introduction

A common theme in extremal and probabilistic combinatorics in recent years concerns the transfer of classical results to sparse random structures. Prime examples include Ramsey's theorem, Turán's theorem, and Szemerédi's theorem (see, e.g., [2, 11, 19, 21]). Here we often want to replace the complete graph K_n or the set of integers $[n] = \{1, \dots, n\}$ (implicitly appearing in the classical results mentioned above) by a random graph $G(n, p)$ or a random subset of $[n]$.

For example, in the context of Ramsey's theorem for a given number of colours k and a graph F, one may consider the class $\mathcal{A}$ consisting of all graphs G with the property that every k-colouring of its edges yields a monochromatic copy of F. Then one may consider the following question: *When does the binomial random graph $G(n, p)$ satisfy $\mathcal{A}$ asymptotically almost surely (a.a.s.)?* More precisely, for which $p = p(n)$ we have $\lim_{n\to\infty} \mathbb{P}(G(n, p) \in \mathcal{A}) = 1$? It turns out that for many natural graph properties there exists a threshold function $\hat{p} = \hat{p}(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = \begin{cases} 0, & \text{if } p = o(\hat{p}), \\ 1, & \text{if } p = \omega(\hat{p}). \end{cases} \tag{1}$$

After establishing the threshold for a given property $\mathcal{A}$, one may study more closely how quickly the transition from a.a.s. not having $\mathcal{A}$ to a.a.s. having $\mathcal{A}$ occurs. If for all $\varepsilon > 0$ it is possible to replace $p = o(\hat{p})$ in (1) by $p \leqslant (1-\varepsilon)\hat{p}$ and $p = \omega(\hat{p})$ by $p \geqslant (1+\varepsilon)\hat{p}$, then the threshold is called *sharp* and otherwise we refer to it as a *coarse* threshold.

In that direction only a few results are known. In [6] Friedgut presents a characterization of coarse thresholds in a general setting. In case of random graphs it roughly says, that a threshold is

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coarse if and only if it is correlated to a local property. For example the graph property "$G(n,p)$ contains a triangle" depends on local events and has a coarse threshold while the graph property "$G(n,p)$ is connected" is a global property and has in fact a sharp threshold.

Friedgut's work (refined by Bourgain) yields a tool to verify sharp thresholds by contradiction. Supposing to the contrary that the threshold in question would be coarse, one may use the characterization of Friedgut to deduce additional structural properties (see e.g. Theorem 4) which might be used to derive a contradiction.

There are some results in this area based on this approach. For example, it was shown in [9] that the Ramsey-type property "in every vertex colouring of $G(n,p)$ with two colours there is a monochromatic triangle" has a sharp threshold (see also [9] for some extensions).

Regarding Ramsey-type properties concerning edge colourings the applicability of Friedgut's criterion seems more involved. In that direction it was shown by Friedgut, Rödl, Ruciński, and Tetali in [10] that the Ramsey property for the triangle and two colours has a sharp threshold. More recently, Friedgut, Hàn, Person and Schacht [8] studied van der Waerden's property in random subsets of $\mathbb{Z}/n\mathbb{Z}$ and established a sharp threshold for this property. Essentially the same proof can be used to deduce the sharpness of the threshold for the Ramsey properties for strictly balanced (see (2) below) k-partite k-uniform hypergraphs and, hence, in particular for even cycles in graphs and two colours.

We extend this research to non-bipartite graphs. In particular, we obtain a shorter proof of the triangle result from [10]. We will use the arrow notation from Ramsey theory. For two graphs G and F we write $G \to (F)^e_r$ if for all edge colourings of G with r colours there exists a monochromatic copy of F. If, on the other hand, there is an r-colouring of $E(G)$ with no monochromatic copy of F, then we write $G \nrightarrow (F)^e_r$. Our first result establishes the sharp threshold when F is a cycle.

Theorem 1. *For a cycle C_k of length $k \geqslant 3$ there exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (C_k)^e_2) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-(k-2)/(k-1)}, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-(k-2)/(k-1)}. \end{cases}$$

In fact, we shall show the sharpness of the threshold for a more general class of graphs than cycles. For a graph $F = (V,E)$ we write $v(F) = |V(F)|$ and $e(F) = |E(F)|$. For graphs F with at least one edge let the 2-*density* $m_2(F)$ be defined by

$$m_2(F) = \max\{d_2(F')\colon\ F' \subseteq F \text{ and } e(F') \geqslant 1\}$$

where

$$d_2(F') = \begin{cases} \frac{e(F')-1}{v(F')-2}, & \text{if } v(F') > 2, \\ 1, & \text{if } F' = K_2. \end{cases} \tag{2}$$

If $d_2(F) = m_2(F)$, then we call F *balanced*. Moreover, F is *strictly balanced* if in addition $d_2(F') < m_2(F)$ for all proper subgraphs $F' \subsetneq F$ with at least one edge. We say a graph F is *nearly bipartite* if $e(F) \geqslant 2$ and there is a bipartite graph F' and some edge e such that $F = F' + e = (V(F'), E(F') \cup \{e\})$. Note that this definition includes all bipartite graphs with at least two edges. Since for every $k \geqslant 3$ the cycle C_k of length k is strictly balanced and nearly bipartite, the following result includes Theorem 1 as a special case.

Theorem 2. *For all strictly balanced and nearly bipartite graphs F there exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)_2^e) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-1/m_2(F)}, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-1/m_2(F)}. \end{cases}$$

We remark that here we defined $d_2(K_2) = 1$ in (2). As a consequence it follows that

$$m_2(F) > 1 \tag{3}$$

for every strictly balanced and nearly bipartite graph F, since every nearly bipartite graph is required to have at least two edges by definition. In fact, this is the reason why we restrict ourselves in the definition of nearly bipartite to graphs with at least two edges.

The proof of Theorem 2 refines ideas from the work in [8] and also uses Friedgut's criterion for coarse thresholds [6] and the recent *hypergraph container* results of Balogh, Morris, and Samotij [1] and Saxton and Thomason [20]. In Section 2 we will introduce these tools and in addition we will state the two main technical lemmas, Lemmas 7 and 8, which we will need in the proof of the main result. Section 3 is devoted to the proof of Theorem 2 based on these tools. In Section 4 and Section 5 we then prove Lemmas 7 and 8, respectively. We close with a few remarks concerning possible generalisations of Theorem 2 and related open questions.

2. Main tools and outline of the proof

In this section we introduce the necessary tools for the proof of the main result. We use the following notation: For a graph B and $n \geqslant v(B)$ we define $\Psi_{B,n}$ as the set of all embeddings of B into the complete graph K_n. So $\Psi_{B,n}$ corresponds to the unlabelled copies of B in K_n and, clearly, $|\Psi_{B,n}| = \Theta(n^{v(B)})$.

The starting point of the proof is the Rödl-Ruciński theorem (stated below) which establishes that $n^{-1/m_2(F)}$ is the threshold for the property $G(n,p) \to (F)_2^e$ for most graphs F. In view of Theorem 2 we restrict our discussion below to two colours and to strictly balanced and nearly bipartite graphs F. In particular, owing to (3) we have $m_2(F) > 1$ and exclude all forests (some forests exhibit a slightly different behaviour in this context see [15, Theorem 8.1] for details).

Theorem 3 (Rödl & Ruciński (special case))**.** *For all strictly balanced and nearly bipartite graphs F, the function $\hat{p} = \hat{p}(n) = n^{-1/m_2(F)}$ is the threshold for the property $G(n,p) \to (F)_2^e$. In fact, there exist constants $C_1 \geqslant C_0 > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)_2^e) = \begin{cases} 0, & \text{if } p \leqslant C_0 n^{-1/m_2(F)}, \\ 1, & \text{if } p \geqslant C_1 n^{-1/m_2(F)}. \end{cases}$$

□

We will strengthen Theorem 3 and show that these thresholds are sharp. For that we will appeal to Bourgain's refinement [6, Appendix] of Friedgut's criterion for coarse thresholds which will be introduced in Section 2.1. Then we present a recent structural result on independent sets in hypergraphs which plays a crucial rôle in our proof. In Section 2.3 we introduce two somewhat technical probabilistic lemmas needed for the proof of Theorem 2. Section 2.4 establishes the connection between independent sets in hypergraphs and colourings of the edges of the random graph without monochromatic copies of the given graph F considered in our setting.

2.1. **Bourgain's criterion.** In the appendix of Friedgut's original work [6] Bourgain established a closely related criterion for coarse thresholds, which is better suited for our application. The following version of Bourgain's criterion appeared in [7, Theorem 2.4].

Theorem 4 (Bourgain's criterion). *Let $\mathcal{A}$ be a monotone graph property with a coarse threshold. Then there exist $p = p(n)$, constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$, $\tau > 0$, and a graph B satisfying*

(i) $\alpha < \mathbb{P}(G(n,p) \in \mathcal{A}) < 1 - 3\alpha$ *and*
(ii) $\mathbb{P}(B \subseteq G(n,p)) > \tau$

such that for every graph property $\mathcal{G}$ with a.a.s. $G(n,p) \in \mathcal{G}$ there exist infinitely many $n \in \mathbb{N}$ and for each such n a graph $Z \in \mathcal{G}$ on n vertices such that the following holds.

(1) $\mathbb{P}(Z \cup h(B) \in \mathcal{A}) > 1 - \alpha$ *, where $h \in \Psi_{B,n}$ is chosen uniformly at random,*
(2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \in \mathcal{A}) < 1 - 2\alpha$*,*

where the random graph $G(n, \varepsilon p)$ and Z have the same vertex set. □

Note that the $\mathbb{P}(\cdot)$ in *(i)* (and *(ii)*), in (1), and in (2) concern different probability spaces. While in *(i)* and *(ii)* it concerns the random graph $G(n,p)$ we consider h chosen uniformly at random in (1) and the random graph $G(n, \varepsilon p)$ in (2). Below we reformulate Theorem 4 suited for our application.

Corollary 5. *Let F be a strictly balanced and nearly bipartite graph. Assume that the property $G \to (F)^e_2$ does not have a sharp threshold. Then there exists a function $p(n) = c(n)n^{-1/m_2(F)}$ with $C_0 < c(n) < C_1$ for some $C_0, C_1 > 0$, there are constants $\frac{1}{3} > \alpha > 0$ and $\varepsilon > 0$, and there is a graph B with $B \nrightarrow (F)^e_2$ such that for infinitely many $n \in \mathbb{N}$ and for every family of graphs $\mathcal{G}$ on n vertices with a.a.s. $G(n,p) \in \mathcal{G}$ there exists a $Z \in \mathcal{G}$ such that the following hold*

(1) $\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha$*, with $h \in \Psi_{B,n}$ chosen uniformly at random,*
(2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \to (F)^e_2) < 1 - 2\alpha$*.*

Corollary 5 is just a reformulation of Theorem 4 in our context. We give the details below.

Proof of Corollary 5. Note that conclusions (1) and (2) of Corollary 5 are identical to (1) and (2) of Theorem 4 for the monotone graph property $\mathcal{A} = \{G \colon G \to (F)^e_2\}$. Owing to Theorem 3 we infer that because of *(i)* in Theorem 4 the probability $p(n)$ must satisfy $p(n) = c(n)n^{-1/m_2(F)}$ where $C_0 < c(n) < C_1$ for constants C_0, C_1 given by Theorem 3. It is only left to show that $B \nrightarrow (F)^e_2$ is a consequence of *(ii)* of Theorem 4.

For that we recall that it was shown in [18, Theorem 6] that if $B \to (F)^e_2$ then $m(B) = \frac{e(B)}{v(B)} > m_2(F)$. Thus a standard application of Markov's inequality yields $\mathbb{P}(H \subseteq G(n,p)) = o(1)$ for every H with $H \to (F)^e_2$ and $p = \Theta(n^{-1/m_2(F)})$. Consequently the graph B provided by Theorem 4 must satisfy $B \nrightarrow (F)^e_2$, due to *(ii)* of Theorem 4. □

2.2. **Hypergraph containers.** We shall also use a recent result concerning independent sets in hypergraphs, which was obtained independently by Saxton and Thomason [20] and Balogh, Morris, and Samotij [1]. Here we will use the version from [20].

Let $\mathcal{H}$ be an ℓ-uniform hypergraph on $m = |V(\mathcal{H})|$ vertices. For a subset $\sigma \subset V(\mathcal{H})$ we define its *degree* by

$$d(\sigma) = |\{e \in E(\mathcal{H}) \colon \sigma \subseteq e\}| .$$

For a vertex $v \in V$ and an integer j with $2 \leqslant j \leqslant \ell$ we consider the maximum degree over all j-element sets σ containing v

$$d^{(j)}(v) = \max\{d(\sigma) \colon v \in \sigma \subset V(\mathcal{H}) \text{ and } |\sigma| = j\} .$$

We denote by $d = \ell|E(\mathcal{H})|/m > 0$ the average degree of $\mathcal{H}$ and, following the notation of [20], for $\tau > 0$ and $j = 2, \dots, \ell$ we set

$$\delta_j = \frac{1}{\tau^{j-1} m d} \sum_{v \in V(\mathcal{H})} d^{(j)}(v)$$

and

$$\delta(\mathcal{H}, \tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j .$$

We write $\mathcal{P}(X)$ for the power set of X and denote by $\mathcal{P}^s(X) = \mathcal{P}(X) \times \cdots \times \mathcal{P}(X)$ the s-fold cross product of $\mathcal{P}(X)$.

Theorem 6 (Saxton & Thomason)**.** *Let $\mathcal{H}$ be an ℓ-uniform hypergraph on the vertex set $[m]$ and let $0 < \varepsilon < \frac{1}{2}$. Suppose that for $\tau > 0$ we have $\delta(\mathcal{H}, \tau) \leqslant \varepsilon/12\ell!$ and $\tau \leqslant 1/144\ell!^2\ell$. Then there exist a constant $c = c(\ell)$ and a collection $\mathcal{J} \subset \mathcal{P}([m])$ such that the following holds*

- (*a*) *for every independent set I in $\mathcal{H}$ there exists $T = (T_1, \dots, T_s) \in \mathcal{P}^s(I)$ with $|T_i| \leqslant c\tau m$, $s \leqslant c\log(1/\varepsilon)$ and there exists a $J = J(T) \in \mathcal{J}$ only depending on T such that $I \subseteq J(T) \in \mathcal{J}$,*
- (*b*) *$e(\mathcal{H}[J]) \leqslant \varepsilon e(\mathcal{H})$ for all $J \in \mathcal{J}$ and*
- (*c*) *$\log|\mathcal{J}| \leqslant c\tau\log(1/\tau)\log(1/\varepsilon)m$.* □

We will apply Theorem 6 to an auxiliary hypergraph described in the following section.

2.3. **Main probabilistic lemmas.** The hypergraph $\mathcal{H}$ to which we will apply Theorem 6 depends on the graph $Z \in \mathcal{G}$ which will be provided by Bourgain's criterion (Corollary 5) applied for the strictly balanced, nearly bipartite graph F. For the verification of the assumptions of Theorem 6 we will restrict the family $\mathcal{G}$ containing Z. Recall that $\mathcal{G}$ can be chosen to be any graph property which is satisfied a.a.s. by $G(n,p)$ for any p with $p = \Theta(n^{-1/m_2(F)})$. In what follows we discuss the restrictions for the family $\mathcal{G}$ (see Lemmas 7 and 8 below) and for that we introduce the required notation.

Let Z and B be two subgraphs of the complete graph K_n. We say $z \in E(Z)$ *focuses* on $b \in E(B)$ if there exists a copy of F in $Z \cup B$ which contains z and b. We set

$$M(Z, B) = \{z \in E(Z) \colon \text{there is } b \in E(B) \text{ such that } z \text{ focuses on } b\} .$$

The pair (Z, B) is called *interactive* if $E(Z) \cap E(B) = \varnothing$, $Z \nrightarrow (F)^e_2$, and $B \nrightarrow (F)^e_2$, but $Z \cup B \rightarrow (F)^e_2$. For a collection $\Xi \subset \Psi_{B,n}$ of embeddings of B into K_n the pair (Z, Ξ) is called *interactive* if $(Z, h(B))$ is interactive for all $h \in \Xi$. Furthermore, a pair (Z, Ξ) is *regular* if for all $h \in \Xi$ every $z \in E(Z)$ focuses on at most one $b \in E(h(B))$. We call $h \in \Psi_{B,n}$ *regular* w.r.t. Z if $(Z, \{h\})$ is regular. The hypergraphs $\mathcal{H}$ considered here are defined in terms of regular pairs (Z, Ξ).

For a pair (Z, Ξ) with $Z \subseteq K_n$ and $\Xi \subseteq \Psi_{B,n}$ we define the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi)$ with vertex set

$$V(\mathcal{H}) = E(Z)$$

and edge set

$$E(\mathcal{H}) = \{M(Z, h(B))\colon\ h \in \Xi\}\,.$$

For our presentation it will be useful to consider orderings of the edges of the involved graphs and "order consistent" embeddings. For that we fix an arbitrary ordering of $E(K_n)$ and an ordering of $E(B)$. For an interactive and regular pair (Z, Ξ) and $h \in \Xi$ we say that $z \in M(Z, h(B)) = \{e_1, \dots, e_\ell\}$ with $e_1 < e_2 < \dots < e_\ell$ has *index* i if $z = e_i$. Furthermore, we call (Z, Ξ) and $\mathcal{H}(Z, \Xi)$ *index consistent* if for all $z \in E(Z)$ and all $h, h' \in \Xi$ with $z \in M(Z, h(B)) \cap M(Z, h'(B))$ the indices of z in $M(Z, h(B))$ and in $M(Z, h'(B))$ are the same. Let $b_1 < \dots < b_{e(B)}$ be the ordering of the edges of B, then the *profile* of $M(Z, h(B))$ is the function $\pi\colon [|M(Z, h(B))|] \to [e(B)]$ defined by $\pi(i) = j$ if and only if e_i focuses on $h(b_j)$. Since the pair (Z, Ξ) is regular, for each edge of $\mathcal{H}$ each e_i focuses on maximal one $h(b_j)$ and, hence, the profile is well defined. We say (Z, Ξ) has profile π if all edges $M(Z, h(B))$ for $h \in \Xi$ have profile π. Note that in this case all sets $M(Z, h(B))$ have the same cardinality and $|M(Z, h(B))|$ is called the *length* of the profile π.

Having established this notation we now state the following technical lemma which gives one part of the graph property $\mathcal{G}$ for application of Corollary 5. Moreover, we shall also apply Theorem 6 which results in useful properties of the hypergraph $\mathcal{H}(Z, \Xi)$ for $Z \in \mathcal{G}$ and some appropriately chosen $\Xi \subseteq \Psi_{B,n}$.

Lemma 7. *For all constants $C_1 > C_0 > 0$, $\frac{1}{3} > \alpha > 0$ and graphs F and B, where F is strictly balanced and nearly bipartite, there exist $\alpha', \beta, \gamma > 0$ and $L \in \mathbb{N}$ such that for every $p = c(n)n^{-1/m_2(F)}$ with $C_0 \leqslant c(n) \leqslant C_1$ a.a.s. $Z \in G(n, p)$ satisfies the following. If*

$$\mathbb{P}(Z \cup h(B) \to (F)_2^e) > 1 - \alpha$$

then there exists $\Xi_{B,n} \subseteq \Psi_{B,n}$ with $|\Xi_{B,n}| \geqslant \alpha' n^2$ and $Z \cup h(B) \to (F)_2^e$ for all $h \in \Xi_{B,n}$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ is index consistent for some profile π of length $\ell \leqslant L$ and there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ satisfying

(1) $\log |\mathcal{C}| \leqslant e(Z)^{1-\gamma}$,
(2) $|C| \geqslant \beta e(Z)$ *for all* $C \in \mathcal{C}$ *and*
(3) every hitting set A of $\mathcal{H}$ contains a $C \in \mathcal{C}$, i.e., for every $A \subseteq V(\mathcal{H})$ with $e \cap A \neq \varnothing$ for all $e \in E(\mathcal{H})$ there exists $C \in \mathcal{C}$ with $C \subseteq A$.

We need another restriction on the family $\mathcal{G}$ which is satisfied a.a.s. by $G(n, p)$. For a nearly bipartite graph $F = F' + e$ we consider those pairs of vertices in K_n which complete a copy of the bipartite subgraph F' in a given subgraph of $G(n, p)$ to a full copy of F in K_n. Hence, for a graph $G \subseteq K_n$ we define the *basegraph* $\mathrm{Base}_F(G) \subseteq K_n$ with edge set

$$\big\{\{x, y\}\colon\ \exists F' \subseteq G' \text{ such that } F' + \{x, y\} \text{ forms a copy of } F\big\}\,.$$

We require that for every relatively dense subgraph G' of $G(n, p)$ the basegraph spans many copies of F itself. More precisely, for a graph G on n vertices and a nearly bipartite graph $F = F' + e$ and

$\lambda, \eta > 0$ we say G has the property $T(\lambda, \eta, F)$ if for every subgraph $G' \subset G$ with $e(G') \geqslant \lambda e(G)$ we have that the basegraph $\mathrm{Base}_F(G')$ contains at least $\eta n^{v(F)}$ copies of F.

Lemma 8 gives the second restriction for the family $\mathcal{G}$ for our application of Corollary 5. Assuming that there is no copy of F in the bigger colour class of Z, Lemma 8 will be helpful to find a copy of F in the intersection of $Z \cap G(n, \varepsilon p)$ with the other colour class.

Lemma 8. *For all $\lambda > 0$, $C_1 > C_0 > 0$ and every strictly balanced and nearly bipartite graph F there exists $\eta > 0$ such that for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ the random graph $G(n, p)$ a.a.s. satisfies $T(\lambda, \eta, F)$.*

2.4. **Colourings and hitting sets.** In this section we establish the connection between hitting sets of the hypergraph $\mathcal{H}(Z, \Xi)$ and F-free colourings of Z.

Recall that the definition of an interactive pair (Z, Ξ) says that for every embedding $h \in \Xi \subseteq \Psi_{B,n}$ the graphs Z and $h(B)$ are edge disjoint and $Z \nrightarrow (F)_2^e$ and $B \nrightarrow (F)_2^e$ but $Z \cup h(B) \rightarrow (F)_2^e$. Let $b_1, \ldots, b_K$ be an enumeration of $E(B)$ and fix an F-free colouring $\sigma\colon E(B) \rightarrow \{\text{red,blue}\}$. We copy this colouring for every $h \in \Xi$ by setting $\sigma_h\colon E(h(B)) \rightarrow \{\text{red,blue}\}$ with $\sigma_h(h(b_i)) = \sigma(b_i)$ for all $i = 1, \ldots, K$. Furthermore, let φ be an arbitrary F-free colouring of Z.

Since $Z \cup h(B) \rightarrow (F)_2^e$, the joint colouring of $Z \cup h(B)$ given by φ and σ_h yields a monochromatic copy of F and this copy must contain edges of both graphs, of Z and of $h(B)$. Thus each edge $M(Z, h(B))$ of the hypergraph $\mathcal{H}(Z, \Xi)$ contains an $e \in E(Z)$ which focuses on some $h(b)$ with $b \in E(B)$, where we have $\varphi(e) = \sigma_h(h(b)) = \sigma(b)$. We say such an edge $e \in E(Z)$ (resp. vertex $e \in V(\mathcal{H})$) is *activated* by φ, σ, and h. We define the set of activated vertices by

$$A_\varphi^\sigma = A_\varphi^\sigma(Z, \Xi) = \bigcup_{h \in \Xi} \{e \in E(Z)\colon\ e \text{ is activated by } \sigma,\ \varphi \text{ and } h\} \subseteq V(\mathcal{H})\,. \tag{4}$$

Note that by definition for an interactive pair (Z, Ξ) every edge $M(Z, h(B))$ of $\mathcal{H}(Z, \Xi)$ contains an activated vertex and, hence, the set of activated vertices A_φ^σ is a hitting set of $\mathcal{H}(Z, \Xi)$. In what follows we will use different colourings φ of Z but we will always restrict to the same colouring σ of B.

Suppose now in addition that we have a fixed ordering of $E(Z)$ and as above let $E(B) = \{b_1, \ldots, b_K\}$. Further suppose that the interactive pair (Z, Ξ) is also index consistent with profile π of length ℓ. In particular, the hypergraph $\mathcal{H}(Z, \Xi)$ is ℓ-uniform.

It also follows from the definitions that for $z \in A_\varphi^\sigma \cap A_{\varphi'}^\sigma$ for two colourings φ and φ' we have $\varphi(z) = \varphi'(z)$. In fact, for $z \in A_\varphi^\sigma$ there exist an $h \in \Xi$ such that z is activated by σ, φ and h. Let i be the index of z in $M(Z, h(B))$, then z focuses on $h(b_{\pi(i)})$ and, therefore, $\varphi(z) = \sigma(b_{\pi(i)})$. Consequently, repeating the same argument for φ', we have $\varphi'(z) = \sigma(b_{\pi(i)}) = \varphi(z)$. We summarise these observations in the following fact.

Fact 9. *Let (Z, Ξ) be a an interactive, regular and index consistent pair with profile π and let σ be an F-free colouring of $E(B)$ and φ be an F-free colouring of $E(Z)$. Then*

(A1) *$A_\varphi^\sigma(Z, \Xi)$ is a hitting set of $\mathcal{H}(Z, \Xi)$ and*
(A2) *for all F-free colourings φ' of $E(Z)$ and for all $z \in A_\varphi^\sigma \cap A_{\varphi'}^\sigma$ we have $\varphi(z) = \varphi'(z)$.*

Now we are prepared to give the proof of the main theorem based on the lemmas and theorems of this section.

3. Proof of the main theorem

3.1. **Outline of the proof.** The starting point of the proof is Bourgain's criterion (see Corollary 5) applied to the contradictory assumption, that the Ramsey property $G \to (F)^e_2$ for a given strictly balanced and nearly bipartite graph F has a coarse threshold. For that we define a family of graphs $\mathcal{G}$ having "useful" properties and Lemma 7 and Lemma 8 show that a.a.s. $G(n,p)$ displays these properties. Then Bourgain's criterion asserts for infinitely many $n \in \mathbb{N}$ the existence of an n-vertex graph $Z \in \mathcal{G}$, a graph B (called *booster*), constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$ and a family of embeddings $\Psi'_{B,n} \subseteq \Psi_{B,n}$ with $Z \cup h(B) \to (F)^e_2$ for all $h \in \Psi'_{B,n}$ and $|\Psi'_{B,n}| \geqslant (1-\alpha)|\Psi_{B,n}|$, but $\mathbb{P}(Z \cup G(n,\varepsilon p) \to (F)^e_2) < 1 - 2\alpha$. The goal is to find a contradiction to the last fact by showing $\mathbb{P}(Z \cup G(n,\varepsilon p) \to (F)^e_2) = 1 - o(1)$.

Let Φ be the set of all F-free colourings of Z. We have to show that for any $\varphi \in \Phi$ the probability to extend φ to an F-free colouring of $Z \cup G(n,\varepsilon p)$ is very small. We are able to show that this probability is of order $\exp(-\Omega(pn^2))$. Now we would like to use a union bound for all $\varphi \in \Phi$. However, we have only little control over $|\Phi|$ and the trivial upper bound $2^{\Theta(pn^2)}$ is too high to combine it with the bound from above $\exp(-\Omega(pn^2))$ to obtain for $\mathbb{P}(Z \cup G(n,\varepsilon p) \nrightarrow (F)^e_2)$ a bound of order $o(1)$ by the union bound.

Instead we shall find a partition of Φ into $2^{o(pn^2)}$ classes such that two colourings from the same partition class always agree on a large subset of Z. These subsets are called *cores*. Then we will show that the colouring of φ restricted to the associated core implies that φ is only with probability at most $\exp(-\Omega(pn^2))$ extendible to an F-free colouring of $Z \cup G(n,\varepsilon p)$. This allows us to use a union bound over all partition classes to get the desired upper bound on $\mathbb{P}(Z \cup G(n,\varepsilon p) \nrightarrow (F)^e_2)$ of order $o(1)$.

For the definition of the cores we will appeal to the hypergraph $\mathcal{H} = \mathcal{H}(Z,\Xi)$ which was defined in Section 2.3. Recall that $V(\mathcal{H}) = e(Z)$ and hyperedges of $\mathcal{H}$ correspond to embeddings of B in K_n, which are given by a carefully chosen subset $\Xi \subseteq \Psi'_{B,n}$. In fact, we shall select $\Xi \subseteq \Psi'_{B,n}$ in such a way, that we can apply the structural result on independent sets of hypergraphs by Saxton and Thomason [20] to $\mathcal{H}$ (see Lemma 7). In fact, the cores then correspond to the complements of the almost independent sets from $\mathcal{J}$ given by the Saxton-Thomason theorem (Theorem 6). This yields a small family $\mathcal{C}$ of subsets of $V(\mathcal{H})$, that means of size $2^{o(pn^2)}$, such that the elements $C \in \mathcal{C}$ are not too small and every hitting set of $\mathcal{H}$ contains at least one element from $\mathcal{C}$.

We then associate every F-free colouring φ of Z with a hitting set A^σ_φ of $\mathcal{H}$ (for some F-free colouring σ of B, see part (A1) of Fact 9) and thus we can associate to each such colouring φ a core $C \in \mathcal{C}$ contained in A^σ_φ. This allows us to define the desired partition of the set of colourings Φ using the "small" family of cores $\mathcal{C}$. Finally, we use the union bound to estimate the probability that there is an F-free colouring of Z that can be extended to an F-free colouring of $Z \cup G(n,\varepsilon p)$ by $o(1)$, which contradicts $\mathbb{P}(Z \cup G(n,\varepsilon p) \to (F)^e_2) < 1 - 2\alpha$. Below we give the details of this proof.

3.2. **Details of the proof.**

Proof of Theorem 2. Let $F = F' + \{a_1, a_2\}$ be a strictly balanced, nearly bipartite graph with F' being bipartite and assume for a contradiction that the property $G \to (F)^e_2$ does not have a sharp threshold.

We apply Corollary 5 and obtain a function $p(n) = c(n)n^{-1/m_2(F)}$ such that $C_0 < c(n) < C_1$ for some $C_1 > C_0 > 0$, constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$ and a graph B with $B \nrightarrow (F)^e_2$.

For these parameters we apply Lemma 7 and obtain constants $\alpha', \beta, \gamma > 0$ and $L \in \mathbb{N}$. Set $\lambda = \beta/2$ and apply Lemma 8, which yields $\eta > 0$. Then let $\mathcal{G}_n$ be the family of graphs G on n vertices that satisfy the conclusions of Lemma 7 and Lemma 8 for the chosen parameters and $\frac{1}{4}pn^2 \leqslant e(G) \leqslant pn^2$. Since these properties hold a.a.s. in $G(n,p)$, it follows from Corollary 5, that there are infinitely many $n \in \mathbb{N}$ for which there is some $Z \in \mathcal{G}_n$ satisfying

(R1) $\mathbb{P}(Z \cup h(B) \rightarrow (F)^e_2) > 1 - \alpha$, with $h \in \Psi_{B,n}$ chosen uniformly at random,
(R2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \rightarrow (F)^e_2) < 1 - 2\alpha$

as well as by Lemma 8

(T) Z has the property $T(\lambda, \eta, F)$

and

(Z) $\frac{1}{4}pn^2 \leqslant e(Z) \leqslant pn^2$.

Owing to $Z \in \mathcal{G}_n$ and (R1) we can use Lemma 7 to find some $\Xi_{B,n} \subseteq \Psi_{B,n}$ of size at least $\alpha' n^2$ with $Z \cup h(B) \rightarrow (F)^e_2$ for all $h \in \Xi_{B,n}$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ is index consistent with a profile π of length $\ell \leqslant L$ and such that there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ with

(C1) $\log |\mathcal{C}| \leqslant e(Z)^{1-\gamma}$,
(C2) $|C| \geqslant \beta e(Z)$ for all $C \in \mathcal{C}$ and
(C3) every hitting set A of $\mathcal{H}$ contains a set $C \in \mathcal{C}$.

Our proof is by contradiction and we shall establish such a contradiction to the assertion (R2).

Let Φ be the set of all F-free edge colourings of $E(Z)$ and pick an arbitrary F-free colouring σ of B. We want to split Φ into "few" classes. For this we use the correspondence between any colouring $\varphi \in \Phi$ and the hitting set $A^\sigma_\varphi = A^\sigma_\varphi(Z, \Xi_{B,n})$ of $\mathcal{H}$ given by part (A1) of Fact 9. Moreover, for $C \in \mathcal{C}$ we define

$$\Phi_C = \{\varphi \in \Phi \colon\ C \subseteq A^\sigma_\varphi\}\,.$$

Then $\Phi = \bigcup_{C \in \mathcal{C}} \Phi_C$ (not necessarily disjoint) since by (C3) for every $\varphi \in \Phi$ the hitting set A^σ_φ contains some $C \in \mathcal{C}$ and hence $\varphi \in \Phi_C$.

Part (A2) of Fact 9 asserts that $\varphi(z) = \varphi'(z)$ for all $z \in A^\sigma_\varphi \cap A^\sigma_{\varphi'}$ and any colourings $\varphi, \varphi' \in \Phi$. In other words, all colourings in Φ_C agree on C and, hence, there exists a monochromatic subset $R_C \subseteq C$ (w.l.o.g. coloured red) of size at least $|C|/2 \geqslant \beta e(Z)/2 = \lambda e(Z)$ (see (C2) and the choice of λ).

For the desired contradiction we add $G(n, \varepsilon p)$ to Z. We have to show that

$$\mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)^e_2) = o(1)\,.$$

For this purpose we find for all F-free colourings φ of Z an upper bound for the probability that φ is extendible to an F-free colouring of $Z \cup G(n, \varepsilon p)$. For φ we use only the colouring on the associated core $C \subseteq A^\sigma_\varphi$, instead of the colouring on all edges of Z. In this way we can deal with all embeddings $\varphi \in \Phi_C$ at once since they coincide on C.

Since the red colour class R_C contains at least $\lambda e(Z)$ edges it follows from property (T), that there are at least $\eta n^{v(F)}$ copies of F in the basegraph $\mathrm{Base}_F(R_C)$ of R_C w.r.t. F. In an F-free colouring of $Z \cup G(n, \varepsilon p)$ all edges in

$$U_C = G(n, \varepsilon p) \cap \mathrm{Base}_F(R_C)$$

have to be coloured blue since every edge in $\mathrm{Base}_F(R_C)$ completes a red copy of F' in R_C to a copy of F. Consequently, φ cannot be extended to an F-free colouring of $Z \cup G(n, \varepsilon p)$ if U_C spans a copy of F. However, since $\mathrm{Base}_F(R_C)$ contains $\Omega(n^{v(F)})$ copies of F and $p = \Omega(n^{-1/m_2(F)})$ it follows from Janson's inequality [13] (see also [14]) that it is very unlikely that U_C is F-free. In fact, a standard application of Janson's inequality asserts that there exists some $\gamma' = \gamma'(\varepsilon, \eta, C_0, C_1, F)$ such that

$$\mathbb{P}(F \nsubseteq G(n, \varepsilon p) \cap \mathrm{Base}_F(R_C)) = \mathbb{P}(F \nsubseteq U_C) \leqslant \exp\left(-\gamma' n^{2-\frac{1}{m_2(F)}}\right). \tag{5}$$

We then deduce the desired contradiction to (R2) by

$$\begin{aligned} \mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)^e_2) &\leqslant |\mathcal{C}| \cdot \mathbb{P}(\exists \varphi \in \Phi_C \colon \varphi \text{ is extendible to } U_C) \\ &\overset{(C1)}{\leqslant} \exp\left(e(Z)^{1-\gamma}\right) \cdot \mathbb{P}(F \nsubseteq U_C) \\ &\overset{(Z)}{\leqslant} \exp\left((pn^2)^{1-\gamma}\right) \cdot \mathbb{P}(F \nsubseteq U_C) \\ &\overset{(5)}{\leqslant} \exp\left((C_1 n^{2-\frac{1}{m_2(F)}})^{1-\gamma}\right) \cdot \exp\left(-\gamma' n^{2-\frac{1}{m_2(F)}}\right) \\ &< \alpha, \end{aligned}$$

for sufficiently large n, since $\gamma > 0$ and C_1, γ, and γ' are constants independent of n. This concludes the proof of Theorem 2. $\square$

4. Proof of Lemma 7

The key tool to prove Lemma 7 is the *container theorem* of Saxton and Thomason (see Theorem 6). We shall apply Theorem 6 to the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$. In order to satisfy the assumptions of Theorem 6 we may enforce some properties on the typical graph Z and the family of embeddings $\Xi_{B,n}$. Firstly in Section 4.1 we will formulate some properties on Z that hold a.a.s. for $G(n, p)$ and which will turn out to be useful for locating a suitable family of embeddings $\Xi_{B,n} \subseteq \Psi_{B,n}$ (see Section 4.2). In Section 4.3 we finally check that for those choices the assumptions of Theorem 6 are satisfied by the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$.

4.1. Some typical properties of $G(n, p)$.

Theorem 5 yields a family of embeddings of B into K_n. We restrict ourselves to "regular" embeddings with foresight to the later parts of the proof. Actually we want that for every edge $e \in E(Z)$ and every embedding h there is only one $b \in E(B)$ such that e focuses on $h(b)$. In addition there should be exactly one copy of F that contains e and $h(b)$ if e focuses on $h(b)$. There are three ways such that this fails.

Definition 10. *Let F, B, Z be graphs with $Z \subseteq K_n$. An embedding $h \in \Psi_{B,n}$ is bad (with respect to F and Z) if one of the following holds*

(B1) *either there is a copy F_1 of F in $Z \cup h(B)$ that contains at least one edge of $E(Z) \setminus E(h(B))$ and at least two edges of $E(h(B))$,*

(B2) *or there are distinct copies F_1 and F_2 of F in $Z \cup h(B)$ and edges e, $f_1 \neq f_2$ with $e \in E(Z) \setminus E(h(B))$ and $e \in E(F_1) \cap E(F_2)$, $f_1, f_2 \in E(h(B))$ such that $f_1 \in E(F_1)$ and $f_2 \in E(F_2)$*

(B3) *or there are distinct copies F_1 and F_2 of F in $Z \cup h(B)$ and edges e, f with $e \in E(Z) \backslash E(h(B))$ and $e \in E(F_1) \cap E(F_2)$, $f \in E(h(B))$ and $f \in E(F_1) \cap E(F_2)$.*

Note that (B3) would be a special case of (B2) if we would not require $f_1 \neq f_2$ there. However, for the later discussion it is better to distinguish these cases.

Fact 11. *For F, B and Z let $\Xi_{B,n} \subseteq \Psi_{B,n}$ be a family of embeddings such that for all $h \in \Xi_{B,n}$ properties (B1) and (B2) are not satisfied. Then clearly the pair $(Z, \Xi_{B,n})$ is regular.*

We shall show that for the random graph $Z = G(n,p)$ only a few embeddings $h \in \Psi_{B,n}$ are bad (see (Z5) in Definition 12 and Lemma 13 below), which enables us to focus on regular pairs $(Z, \Xi_{B,n})$. Moreover, we shall restrict to typical graphs Z, which render a few more somewhat technical properties such as containing roughly the expected number of some special subgraphs. We discuss those below.

Let $\mathcal{F}_-$ be the family of subgraphs of F obtained by removing some edge and for a graph G we denote by $\mathcal{F}_-(G)$ the copies of a member of $\mathcal{F}_-$ in G. Furthermore, for an edge $e \in E(G)$ let $\mathcal{F}_-(G,e)$ be those copies in $\mathcal{F}_-(G)$ that contain e. For two edges $e_1, e_2 \in E(G)$ let $\mathcal{F}_-(G, e_1, e_2) \subseteq \mathcal{F}_-(G, e_1) \times \mathcal{F}_-(G, e_2)$ be the set of pairs of copies (F_1, F_2) such that F_1 and F_2 intersect in at least two vertices $\{x_1, x_2, \dots, x_s\}$, F_i together with $e = \{x_1, x_2\}$ is isomorphic to F for $i = 1, 2$ and $e(F_1 \cap F_2) = 0$. For $s \geqslant 2$ let $\mathcal{F}_{-,s}(G, e_1, e_2) \subseteq \mathcal{F}_-(G, e_1, e_2)$ be the set of pairs as in $\mathcal{F}_-(G, e_1, e_2)$ such that F_1 and F_2 intersect in exactly s vertices.

These concepts lead to the following definition of "good" graphs Z, where we impose that the sizes of the introduced families defined above are close to the respective expectation in $G(n,p)$. Then Lemma 13 states that a.a.s. $G(n,p)$ is indeed good for the right choice of parameters.

Definition 12. *For graphs F and B and constants $D > 0$, $\zeta > 0$, $\delta > 0$ and $p \in (0,1)$ we consider the set of graphs $\mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ on n vertices that is given by $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ if and only if*

(Z1) $\frac{1}{4}pn^2 \leqslant e(Z) \leqslant pn^2$,
(Z2) $|\mathcal{F}_-(Z)| \leqslant Dn^2$,
(Z3) $|\mathcal{F}_-(Z,e)| \leqslant \frac{D}{p}$ *for all* $e \in E(Z)$,
(Z4) $|\mathcal{F}_-(Z, e_1, e_2)| \leqslant \frac{D}{pn^\delta}$ *for all but at most* $\frac{Dpn^2}{n^\delta}$ *pairs of distinct edges* $e_1, e_2 \in E(Z)$ *and*
(Z5) $|\{h \in \Psi_{B,n} \colon h$ *is bad w.r.t.* F *and* $Z\}| \leqslant \frac{|\Psi_{B,n}|}{n^\zeta}$.

The following Lemma shows that a.a.s. $G(n,p) \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ for D sufficiently large and ζ and δ sufficiently small (in fact, our choice of δ will imply $pn^\delta \to 0$).

Lemma 13. *For every strictly balanced and nearly bipartite graph F, for every graph B, and for all constants $C_1 \geqslant C_0 > 0$ there are constants $D > 0$, $\zeta > 0$, and δ with $0 < \delta \leqslant \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$ such that for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ a.a.s. $G(n,p) \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$.*

We will split the proof into two parts: First we consider (Z1)-(Z4) which deals with subgraphs of Z (Lemma 14), and then we deal with the bad embeddings considered in (Z5) (Lemma 16).

Lemma 14. *Let $C_1 \geqslant C_0 > 0$, F be a strictly balanced and nearly bipartite graph and $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$. Then there exist constants $D > 0$ and δ with $0 < \delta < \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$ such* *that a.a.s. $G(n,p)$ satisfies the properties (Z1), (Z2), (Z3), and (Z4) with the parameters p, D, and δ and for the graph F.*

For the proof of Lemma 14 we note that property (Z1) follows directly from the concentration of the binomial distribution and (Z2) follows (with slightly adjusted D) from (Z1) and (Z3). The proof of (Z3) will make use of Spencer's extension lemma (Theorem 15 stated below). Finally, (Z4) follows from a standard second moment argument. Below we introduce the necessary notation for the statement of Theorem 15.

For a graph H and an ordered subset $R = (x_1, \dots, x_r)$ of $V(H)$ the pair (R, H) is called *rooted graph* with *roots* R. For an induced subgraph $H' = H[S]$ of H with $\{x_1, \dots, x_r\} \subsetneq S \subseteq V(H)$ we say (R, H') is a rooted subgraph of (R, H). We define the density of a rooted graph (R, H) by

$$\operatorname{dens}(R,H) = \frac{e(H) - e(H[R])}{v(H) - |R|}\,.$$

Let $V(H)\backslash\{x_1, \dots, x_r\} = \{y_1, \dots, y_\nu\}$ for some $\nu \geqslant 1$. For a graph G with distinct vertices $(x'_1, \dots, x'_r)$ an ordered tuple $(y'_1, \dots, y'_\nu)$ is called an (R,H)-*extension of* $(x'_1, , \dots, x'_r)$ if

- the y'_i are distinct from each other and from the x'_j,
- $\{x'_i, y'_j\} \in E(G)$ whenever $\{x_i, y_j\} \in E(H)$ and
- $\{y'_i, y'_j\} \in E(G)$ whenever $\{y_i, y_j\} \in E(H)$.

The number of (R,H)-extensions $(y'_1, \dots, y'_\nu)$ is denoted by $N(x'_1, \dots, x'_r)$. Finally, we define $\operatorname{mad}(R,H)$ as the maximal average degree of a rooted graph (R,H) by

$$\operatorname{mad}(R,H) = \max\{\operatorname{dens}(R,H')\colon\ (R,H') \text{ is rooted subgraph of } (R,H)\}\,.$$

Theorem 15 ([23, Theorem 3])**.** *Let (R,H) be an arbitrary rooted graph and let $\varepsilon > 0$. Then there exist t such that if $p \geqslant n^{-1/\operatorname{mad}(R,H)}(\log n)^{1/t}$ then a.a.s. in $G(n,p)$*

$$(1-\varepsilon)\mathbb{E}[N(\boldsymbol{x}')] < N(\boldsymbol{x}') < (1+\varepsilon)\mathbb{E}[N(\boldsymbol{x}')]$$

for all $\boldsymbol{x}' = (x'_1, \dots, x'_r)$ chosen from $[n]$. □

Proof of Lemma 14. (Z1) This follows from an application of Chernoff's inequality.

(Z2) As already mentioned this property follows (with slightly adjusted D) from (Z1) and (Z3). However, here is a standard direct proof based on the subgraph containment threshold in random graphs.

For $F_- \in \mathcal{F}_-$ and $e \in \binom{[n]}{2}$ let X_e be the random variable that counts the number of copies of F_- that build a copy of F by adding e and let X be the random variable that counts the number of copies of F_- contained in $G(n,p)$. Since $p = \Theta(n^{-1/m_2(F)})$ and since F is strictly balanced, the expectation of X satisfies

$$\mathbb{E}[X] = \Theta\Bigg(\sum_{e \in \binom{[n]}{2}} \mathbb{E}[X_e]\Bigg) = \Theta\left(\binom{n}{2} n^{v(F)-2} p^{e(F)-1}\right) = \Theta(n^2)\,.$$

Since F is strictly balanced we can also use [15, Remark 3.7] and obtain that X converges to $\mathbb{E}[X]$ in probability and, in particular, $\mathbb{P}(X \geqslant 2\mathbb{E}[X]) \to 0$ for $n \to \infty$. Summing over all $F_- \in \mathcal{F}_-$ yields the claim.

(Z3) Consider a graph $F_- \in \mathcal{F}_-$ and remove some edge $\{x_1, x_2\}$ from F_- and call the resulting graph F_{-2}. For $e \in \binom{[n]}{2}$ let X_e be the random variable that counts the number of copies of F_{-2} that build a copy of F_- by adding e and let X be the random variable that counts the number of copies of F_{-2} contained in $G(n,p)$.

Now we can use Spencer's extension lemma (Theorem 15). We consider the rooted graph $((x_1,x_2),F_-)$. Let $\hat{F}$ be an induced subgraph of F_- such that $((x_1,x_2),\hat{F})$ is a rooted subgraph of $((x_1,x_2),F_-)$ which maximizes the density $\operatorname{dens}((x_1,x_2),\hat{F})$. Since $F \supsetneq F_- \supseteq \hat{F}$ is strictly balanced we have

$$m_2(F) > d_2(\hat{F}) = \frac{e(\hat{F})-1}{v(\hat{F})-2} = \operatorname{dens}((x_1,x_2),\hat{F}) = \operatorname{mad}((x_1,x_2),F_-)\,.$$

Consequently, Theorem 15 applied with $\varepsilon = 1$ implies a.a.s.

$$N(x_1',x_2') \leqslant 2\mathbb{E}(X_e) = O(p^{e(F)-2}n^{v(F)-2})$$

for every $x_1' \neq x_2' \in [n]$. Owing to $p = \Theta(n^{-1/m_2(F)})$ and the (strict) balancedness of F we have that $p^{e(F)}n^{v(F)} = \Theta(pn^2)$ and, consequently, for sufficiently large D the claim follows by summing over all choices of $F_- \in \mathcal{F}_-$ and $\{x_1,x_2\} \in E(F_-)$.

(Z4) We show that this property holds a.a.s. for

$$\delta = \frac{1}{6}\min\left\{\tfrac{1}{m_2(F)}, 1 - \tfrac{1}{m_2(F)}\right\} \tag{6}$$

and some $D > 0$ independent of n. In the proof below we distinguish several cases. In the first case we only look at configurations from $\mathcal{F}_{-,2}(G(n,p),e_1,e_2)$. Afterwards we consider configurations from $\mathcal{F}_{-,s}(G(n,p),e_1,e_2)$ for $s > 2$.

Case 1: $s = 2$**.** For two pairs $e_1 \neq e_2 \in \binom{[n]}{2}$ let X_{e_1,e_2} be the random variable given by $|\mathcal{F}_{-,2}(G(n,p),e_1,e_2)|$ and denote by v_1 and u_1 the vertices of e_1 and by v_2 and u_2 the vertices of e_2. We want to use Chebyshev's Inequality to obtain the claimed bound for most pairs. Therefore we have to estimate the expectation and variance of X_{e_1,e_2}. We distinguish between the cases $e_1 \cap e_2 = \varnothing$ and $|e_1 \cap e_2| = 1$.

First let $e_1 \cap e_2 = \varnothing$. Since $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ and F is strictly balanced we have $n^{v(F)}p^{e(F)} = \Theta(pn^2)$ and

$$n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}\,. \tag{7}$$

For $F_0 \subseteq F$ with $v(F_0) \geqslant 2$ it follows from F being strictly balanced that there is some $d > 0$ only depending on F and C_0 such that

$$n^{v(F_0)}p^{e(F_0)} \geqslant dpn^2\,. \tag{8}$$

The expectation of X_{e_1,e_2} is

$$\begin{aligned}\mathbb{E}[X_{e_1,e_2}] &\leqslant e(F)^4 n^{2v(F)-6}p^{2e(F)-4}\\ &\overset{(7)}{\leqslant} e(F)^4 C_1^{2e(F)-2}n^{-2}p^{-2}\end{aligned} \tag{9}$$

and $\mathbb{E}[X_{e_1,e_2}] \to 0$ for n tending to infinity since $p = \Theta(n^{-1/m_2(F)})$ and $m_2(F) > 1$.

Now we estimate the variance of X_{e_1,e_2}. We will show

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \frac{c}{n^2p^2}\left(1+\frac{1}{np^2}\right)$$

for some constant $c > 0$ depending only on F, C_0 and C_1. For this purpose let (F_a, F_b) and (F_c, F_d) be two different pairs of graphs that contribute to the number $|\mathcal{F}_{-,2}(G(n,p), e_1, e_2)|$ with $F_a \cap F_b = \{x_1, x_2\}$, $F_c \cap F_d = \{y_1, y_2\}$, $F_a \cap F_c \supseteq e_1$ and $F_b \cap F_d \supseteq e_2$. We denote by $\mathcal{F}^2_{e_1,e_2}$ the family of isomorphism types of possible pairs $((F_a, F_b), (F_c, F_d))$ such that the conditions above are satisfied.

For $P = ((F_a, F_b), (F_c, F_d)) \in \mathcal{F}^2_{e_1,e_2}$ let $\mathcal{S}_P$ be the set of subsets of $[n]$ of size $v(F_a \cup F_b \cup F_c \cup F_d)$ that contain the vertices of the edges e_1 and e_2. For $S \in \mathcal{S}_P$ let 1_S be the indicator random variable for the event "there exists a copy of P in $G(n,p)$ on the vertex set S". Then

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \mathbb{E}[X_{e_1,e_2}] + \sum_{P\in\mathcal{F}^2}\sum_{S\in\mathcal{S}_P}\mathbb{P}(1_S = 1) \tag{10}$$

For the estimation of $\sum_{P\in\mathcal{F}^2}\sum_{S\in\mathcal{S}_P}\mathbb{P}(1_S = 1)$ we use the following notation. For $\alpha, \beta \in \{a, b, c, d\}$ and $\square \in \{\cup, \cap\}$ we set

$$v_{\alpha\square\beta} = v(F_\alpha \square F_\beta) \quad \text{and} \quad e_{\alpha\square\beta} = e(F_\alpha \square F_\beta)\,,$$

where $F_\alpha \cap F_\beta$ and $F_\alpha \cup F_\beta$ denotes the normal union and intersection of two graphs. Moreover, we can extend this to longer expressions of unions and intersections, like $v_{(\alpha\cap\beta)\cup\gamma}$, and we will make use of this short hand notation in the calculations below. We also set

$$v_{\alpha\setminus\beta} = v_\alpha - v_{\alpha\cap\beta} \qquad \text{and} \qquad e_{\alpha\setminus\beta} = e_\alpha - e_{\alpha\cap\beta} \tag{11}$$

and note that $e_{\alpha\setminus\beta}$ denotes the number of edges in F_α with at least one vertex outside $V(F_\alpha) \cap V(F_\beta)$.

We estimate $\sum_{P\in\mathcal{F}^2}\sum_{S\in\mathcal{S}_P}\mathbb{P}(1_S = 1)$ by counting the number of choices for the vertices of the desired configuration and determine the number of needed edges. Recalling that every $P \in \mathcal{F}^2_{e_1,e_2}$ corresponds to $((F_a, F_b), (F_c, F_d))$ we count those by first choosing (F_a, F_b), then F_c and then F_d and deal with the vertices and edges that are counted several times by looking at the intersections between the different copies of F.

$$\begin{aligned}
&\sum_{P\in\mathcal{F}^2}\sum_{S\in\mathcal{S}_P}\mathbb{P}(1_S = 1)\\
&\leqslant \sum_{P\in\mathcal{F}^2}(4v(F))!\cdot n^{2v(F)-6}p^{2e(F)-4}\cdot n^{v_{c\setminus(a\cup b)}}p^{e_{c\setminus(a\cup b)}}\cdot n^{v_{d\setminus(a\cup b\cup c)}}p^{e_{d\setminus(a\cup b\cup c)}} &(12)\\
&\overset{(11)}{\leqslant} (4v(F))!\sum_{P\in\mathcal{F}^2} n^{4v(F)-6}p^{4e(F)-8}\cdot n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}}\cdot n^{-v_{d\cap(a\cup b\cup c)}}p^{-e_{d\cap(a\cup b\cup c)}}\\
&= (4v(F))!\sum_{P\in\mathcal{F}^2} n^2p^{-4}(n^{v(F)-2}p^{e(F)-1})^4 n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}}n^{-v_{d\cap(a\cup b\cup c)}}p^{-e_{d\cap(a\cup b\cup c)}}\\
&\overset{(7)}{\leqslant} C\sum_{P\in\mathcal{F}^2} n^2p^{-4}\cdot n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}}\cdot n^{-v_{d\cap(a\cup b\cup c)}}p^{-e_{d\cap(a\cup b\cup c)}}\,, &(13)
\end{aligned}$$

where $C > 0$ is a constant depending only on F and C_1. For the estimation of

$$f_P(n,p) := n^2p^{-4} \cdot n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}} \cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-e_{(a\cup b\cup c)\cap d}} \tag{14}$$

we distinguish several cases depending on the structure of P.

First we look at summands in (13) with $\{x_1, x_2\} \subseteq V(F_c)$. Clearly, then we have $\{x_1, x_2, v_1, u_1\} \subseteq V(F_a \cap F_c)$ and since $F_a \cap F_c \subseteq F_a \subset F$ we know $F_0 := (F_a \cap F_c) + \{x_1, x_2\} + e_1 \subseteq F$. Therefore,

$$\frac{1}{n^{v_{a\cap c}}p^{e_{a\cap c}}} = \frac{p^2}{n^{v(F_0)}p^{e(F_0)}} \overset{(8)}{\leqslant} \frac{p^2}{dpn^2} = \frac{p}{dn^2}\,.$$

Similarly, $(F_b \cap F_c) + \{x_1, x_2\} \subseteq F$ and $((F_a \cup F_b \cup F_c) \cap F_d) + \{y_1, y_2\} + e_2 \subseteq F$. The same argument yields

$$\frac{1}{n^{v_{b\cap c}}p^{e_{b\cap c}}} \leqslant \frac{1}{dn^2} \quad \text{and} \quad \frac{1}{n^{v_{(a\cup b\cup c)\cap d}}p^{e_{(a\cup b\cup c)\cap d}}} \leqslant \frac{p}{dn^2}\,.$$

Applying these bounds and the facts that $v_{a\cap b\cap c} \leqslant 2$ and $e_{a\cap b\cap c} = 0$ to (14) yields

$$\begin{aligned} f_P(n,p) &= n^2p^{-4} \cdot n^{-v_{a\cap c}}p^{-e_{a\cap c}} \cdot n^{-v_{b\cap c}}p^{-e_{b\cap c}} \cdot n^{v_{a\cap b\cap c}} \cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-e_{(a\cup b\cup c)\cap d}} \\ &\leqslant n^2p^{-4} \cdot \frac{p}{dn^2} \cdot \frac{1}{dn^2} \cdot n^2 \cdot \frac{p}{dn^2} \\ &= \frac{1}{d^3p^2n^2}\,. \end{aligned} \tag{15}$$

By symmetry we obtain the same estimate in the case that $\{x_1, x_2\} \subseteq V(F_d)$ and in the remaining case we may assume

(I) $|V(F_c) \cap \{x_1, x_2\}| \leqslant 1$ and $|V(F_d) \cap \{x_1, x_2\}| \leqslant 1$.

Next we consider those summands in (13) with (I) and $v_{b\cap c} \geqslant 2$. From (I) it follows that $v_{a\cap b\cap c} \leqslant 1$. We proceed in a similar way as above. This time we use that $(F_a \cap F_c) + e_1 \subseteq F$ and similarly that $((F_a \cup F_b \cup F_c) \cap F_d) + e_2 + \{y_1, y_2\} \subseteq F$ and, therefore,

$$\frac{1}{n^{v_{a\cap c}}p^{e_{a\cap c}}} \overset{(8)}{\leqslant} \frac{1}{dn^2} \quad \text{and} \quad \frac{1}{n^{v_{(a\cup b\cup c)\cap d}}p^{e_{(a\cup b\cup c)\cap d}}} \overset{(8)}{\leqslant} \frac{p}{dn^2}\,.$$

Moreover, since we assume $v_{b\cap c} \geqslant 2$ we can apply (8) with $F_0 = F_b \cap F_c$

$$\frac{1}{n^{v_{b\cap c}}p^{e_{b\cap c}}} \leqslant \frac{1}{dpn^2}\,.$$

Combining these bounds with (14) and $v_{a\cap b\cap c} \leqslant 1$ and $e_{a\cap b\cap c} = 0$ yields

$$\begin{aligned} f_P(n,p) &\leqslant n^2p^{-4} \cdot n^{-v_{a\cap c}}p^{-e_{a\cap c}} \cdot n^{-v_{b\cap c}}p^{-e_{b\cap c}} \cdot n \cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-(a\cup b\cup c)\cap d} \\ &\leqslant n^2p^{-4} \cdot \frac{1}{dn^2} \cdot \frac{1}{dpn^2} \cdot n \cdot \frac{p}{dn^2} \\ &= \frac{1}{d^3p^4n^3}\,. \end{aligned} \tag{16}$$

Next we consider the subcase of (I) when

$$v_{b\cap c} = 1 \quad \text{and} \quad V(F_c) \cap \{x_1, x_2\} = \varnothing\,.$$

Then we have $e_{b\cap c}=0$ and $v_{a\cap b\cap c}=0$. Since $(F_a\cap F_c)+e_1\subseteq F$ and $((F_a\cup F_b\cup F_c)\cap F_d)+e_2+\{y_1,y_2\}\subseteq F$ we get

$$\frac{1}{n^{v_{a\cap c}}p^{e_{a\cap c}}}\overset{(8)}{\leqslant}\frac{1}{dn^2}\quad\text{and}\quad\frac{1}{n^{v_{(a\cup b\cup c)\cap d}}p^{e_{(a\cup b\cup c)\cap d}}}\overset{(8)}{\leqslant}\frac{p}{dn^2}\,.$$

Consequently, in this case we have

$$\begin{aligned}f_P(n,p)&=n^2p^{-4}\cdot n^{-v_{a\cap c}-v_{b\cap c}+v_{a\cap b\cap c}-v_{(a\cup b\cup c)\cap d}}p^{-e_{a\cap c}-e_{b\cap c}+e_{a\cap b\cap c}-e_{(a\cup b\cup c)\cap d}}\\&\leqslant n^2p^{-4}\cdot n^{-v_{a\cap c}}p^{-e_{a\cap c}}\cdot n^{-1}\cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-(a\cup b\cup c)\cap d}\\&\leqslant n^2p^{-4}\cdot\frac{1}{dn^2}\cdot n^{-1}\cdot\frac{p}{dn^2}\\&=\frac{1}{d^2p^3n^3}\,.\end{aligned}\tag{17}$$

For the last remaining cases we consider summands in (13) with (I) and either

(A) $v_{b\cap c}=1$ and $V(F_c)\cap\{x_1,x_2\}\neq\varnothing$ (and, hence, $V(F_b)\cap V(F_c)\subsetneq\{x_1,x_2\}$) or
(B) $v_{b\cap c}=0$.

In both cases together with (I) we get

$$v_{b\cap(a\cup c)\cap d}=|\{x_1,x_2\}\cap V(F_d)|\leqslant 1\,.\tag{18}$$

Based on (18) we can treat both subcases in same way. This time we consider $((F_a\cup F_b)\cap F_c)+e_1\subseteq F$, $(F_b\cap F_d)+e_2\subseteq F$ and $((F_a\cup F_c)\cap F_d)+\{y_1,y_2\}\subseteq F$ and get

$$\frac{1}{n^{v_{(a\cup b)\cap c}}p^{e_{(a\cup b)\cap c}}}\overset{(8)}{\leqslant}\frac{1}{dn^2}\,,\ \frac{1}{n^{v_{b\cap d}}p^{e_{b\cap d}}}\overset{(8)}{\leqslant}\frac{1}{dn^2}\ \text{and}\ \frac{1}{n^{v_{(a\cup c)\cap d}}p^{e_{(a\cup c)\cap d}}}\overset{(8)}{\leqslant}\frac{1}{dn^2}\,,$$

which leads to

$$\begin{aligned}f_P(n,p)\ &=\ n^2p^{-4}\cdot n^{-v_{(a\cup b)\cap c}-v_{b\cap d}-v_{(a\cup c)\cap d}+v_{b\cap(a\cup c)\cap d}}\\&\qquad\cdot p^{-e_{(a\cup b)\cap c}-e_{b\cap d}-e_{(a\cup c)\cap d}+e_{b\cap(a\cup c)\cap d}}\\&\overset{(18)}{\leqslant}\ n^2p^{-4}\cdot n^{-v_{(a\cup b)\cap c}}p^{-e_{(a\cup b)\cap c}}\cdot n^{-v_{b\cap d}}p^{-e_{b\cap d}}\cdot n^{-v_{(a\cup c)\cap d}}p^{-(a\cup c)\cap d}\cdot n\\&\leqslant\ n^2p^{-4}\cdot\left(\frac{1}{dn^2}\right)^3\cdot n\\&=\ \frac{1}{d^3p^4n^3}\,.\end{aligned}\tag{19}$$

Using the bounds from (15), (16), (17) and (19) and $pn\to\infty$ for $n\to\infty$ we summarize that there are constants $c',c>0$ only depending on F,C_0 and C_1 such that for sufficiently large n

$$f_P(n,p)\leqslant c'\left(\frac{1}{p^2n^2}+\frac{1}{p^4n^3}\right)\,.$$

Since the sum in (13) has finitely many summands, together with (10) and (13) it follows that

$$\operatorname{Var}(X_{e_1,e_2})\leqslant\frac{c}{p^2n^2}\left(1+\frac{1}{p^2n}\right)\,.\tag{20}$$

Recall that we want to show that there are at most $Dpn^2n^{-\delta}$ pairs of edges e_1,e_2 in $G(n,p)$ so that $X_{e_1,e_2}>Dp^{-1}n^{-\delta}$ for some constant $D>0$ independent of n and $\delta>0$ chosen in (6). For this purpose we use Markov's Inequality and Chebyshev's Inequality. Let $t = p^{-1}n^{-\delta}$, then Chebyshev's Inequality tells us

$$\mathbb{P}(X_{e_1,e_2} \geqslant \mathbb{E}[X_{e_1,e_2}] + t) \leqslant \frac{\operatorname{Var}(X_{e_1,e_2})}{t^2}.$$

Let X be the number of pairs $(e_1, e_2) \in \binom{E(Z)}{2}$ which satisfy $X_{e_1,e_2} \geqslant 2p^{-1}n^{-\delta}$ and $e_1 \cap e_2 = \varnothing$. Since $\mathbb{E}[X_{e_1,e_2}] \leqslant t$ we have

$$\begin{aligned} \mathbb{E}[X] &\leqslant \binom{pn^2}{2}\mathbb{P}(X_{e_1,e_2} \geqslant \mathbb{E}[X_{e_1,e_2}] + t) & (21) \\ &\leqslant \frac{p^2n^4}{2} \cdot \frac{cp^2n^{2\delta}}{p^2n^2}\left(1 + \frac{1}{np^2}\right) \\ &= \frac{1}{2}cp^2n^{2+2\delta}\left(1 + \frac{1}{np^2}\right). & (22) \end{aligned}$$

We distinguish the cases $n^{-1}p^{-2} > 1$ and $n^{-1}p^{-2} \leqslant 1$. For $n^{-1}p^{-2} > 1$ we have for sufficiently large n

$$\mathbb{E}[X] \leqslant \frac{cp^2n^{2+2\delta}}{np^2} \leqslant cn^{1+2\delta} \leqslant pn^{2-2\delta},$$

where the last inequality follows from our choice of $\delta < \frac{1}{4}(1 - \frac{1}{m_2(F)})$.

For the case $n^{-1}p^{-2} \leqslant 1$ we have for sufficiently large n

$$\mathbb{E}[X] \leqslant cp^2n^{2+2\delta} \leqslant pn^{2-2\delta}$$

where the last inequality follows by the choice of $\delta < \frac{1}{4m_2(F)}$.

Consequently $\mathbb{E}[X] \leqslant pn^{2-2\delta}$ and by Markov's Inequality

$$\mathbb{P}(X > pn^{2-\delta}) \leqslant \frac{\mathbb{E}[X]}{pn^{2-\delta}} \leqslant n^{-\delta}$$

thus a.a.s. $X \leqslant pn^{2-\delta}$.

For sufficiently large n this finishes the case $e_1 \cap e_2 = \varnothing$. It remains the case when $|e_1 \cap e_2| = 1$.

Now let $e_1, e_2 \in \binom{[n]}{2}$ with $|e_1 \cap e_2| = 1$. We repeat essentially the same calculations of the first case $e_1 \cap e_2 = \varnothing$ with the following differences.

- For the expectation of X_{e_1,e_2} in (9) we get
$$\mathbb{E}[X_{e_1,e_2}] = O\left(\frac{1}{np^2}\right).$$
- For the variance we will show
$$\operatorname{Var}(X_{e_1,e_2}) \leqslant \frac{c}{np^2}\left(1 + \frac{1}{np^2}\right).$$

  In the calculation of the variance there is essentially one difference compared to the case $e_1 \cap e_2 = \varnothing$. In (12) we get
$$v_{a\cup b} - |\{x_1, x_2\} \cup \{v_1, u_1\} \cup \{v_2, u_2\}| \leqslant 2v(F) - 5$$
  instead of $2v(F) - 6$ which leads to an additional n factor. This n factor carries over to
$$f_P(n,p) := n^3p^{-4} \cdot n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}} \cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-e_{(a\cup b\cup c)\cap d}} \tag{23}$$

in (14).

For the following case distinction we repeat in the case $\{x_1, x_2\} \subseteq V(F_c)$ the calculation, but keep the additional n factor. Consequently we get in (15)

$$f_P(n,p) = O\left(\frac{1}{p^2 n}\right).$$

Similarly we get with the additional n factor in (16)

$$f_P(n,p) = O\left(\frac{1}{p^4 n^2}\right).$$

The case $v_{b\cap c} = 1$ and $V(F_c) \cap \{x_1, x_2\} = \varnothing$ disappears since F_b and F_c intersect at least in $e_1 \cap e_2 \subseteq \{x_1, x_2\}$. For the same reason the case $v_{b\cap c} = 0$ disappears. For the last remaining case in (19) we get again the same bound with an additional factor of n

$$f_P(n,p) = O\left(\frac{1}{p^4 n^2}\right).$$

Consequently

$$\operatorname{Var}(X_{e_1,e_2}) \leqslant \frac{c}{np^2}\left(1 + \frac{1}{np^2}\right).$$

- The expectation still satisfies $\mathbb{E}[X_{e_1,e_2}] \leqslant t$ for the same choice of $t = p^{-1}n^{-\delta}$. This follows since $\mathbb{E}[X_{e_1,e_2}] = O(\frac{1}{np^2})$, $t = \frac{1}{pn^\delta}$ and $\delta < 1 - \frac{1}{m_2(F)}$.
- Let X' be the number of pairs $(e_1, e_2) \in \binom{E(Z)}{2}$ with $|e_1 \cap e_2| = 1$ and $X_{e_1,e_2} \geqslant 2p^{-1}n^{-\delta}$. We know by the condition $|e_1 \cap e_2| = 1$ that $X' \leqslant 2p^2n^3$, thus we get with X' instead of X in (21) a factor of $2p^2n^3$ instead of $\binom{pn^2}{2}$ which results in a factor of n^{-1} compared to the first case. Consequently the n^{-1} factor cancels with the n factor above which leads to the same order of magnitude in (22). Then the rest of the proof is the same as in the first case.

Setting $D' \geqslant 2$ sufficiently large such that $2p^{-1}n^{-\delta} \leqslant \frac{D'pn^2}{n^\delta}$ then yields

$$|\mathcal{F}_{-,2}(Z, e_1, e_2)| \leqslant \frac{D'}{pn^\delta} \tag{24}$$

for all but at most $\frac{D'pn^2}{n^\delta}$ pairs of edges $e_1, e_2 \in E(Z)$.

Case 2: $s > 2$. In this case we consider configurations from $\mathcal{F}_{-,s}(G(n,p), e_1, e_2)$ with $s > 2$. For two pairs $e_1 \neq e_2 \in \binom{[n]}{2}$ let Y_{e_1,e_2} be the random variable given by $|\mathcal{F}_{-,s}(G(n,p), e_1, e_2)|$. Here it is sufficient to use Markov's inequality instead of Chebyshev's inequality which will allow us to avoid the calculation of the variance, but we still have to distinguish the cases $e_1 \cap e_2 = \varnothing$ and $|e_1 \cap e_2| = 1$.

For the first case let $e_1 \cap e_2 = \varnothing$. The expectation of Y_{e_1,e_2} is

$$\begin{aligned}
\mathbb{E}[Y_{e_1,e_2}] &\leqslant e(F)^4 n^{2v(F)-4-s} v(F)^s p^{2e(F)-4} \\
&\overset{(7)}{\leqslant} e(F)^4 v(F)^s C_1^{2e(F)-2} n^{-s} p^{-2} \\
&\leqslant C' n^{-3} p^{-2}
\end{aligned}$$

with $C' = e(F)^4 v(F)^s C_1^{2e(F)-2}$. We use Markov's inequality and get

$$\mathbb{P}\left(Y_{e_1,e_2} \geqslant \frac{1}{pn^\delta}\right) \leqslant C' n^{-3} p^{-2} \cdot pn^\delta = C' p^{-1} n^{-3+\delta}.$$

Let Y be the number of pairs $e_1, e_2 \in E(Z)$ with $e_1 \cap e_2 = \varnothing$ and $Y_{e_1,e_2} \geqslant p^{-1}n^{-\delta}$. Then

$$\mathbb{E}[Y] \leqslant \binom{pn^2}{2} C' n^{-3+\delta} p^{-1} \leqslant \frac{C' pn^{1+\delta}}{2}$$

and a second use of Markov's inequality yields

$$\mathbb{P}(Y \geqslant pn^{2-\delta}) \leqslant \frac{C' pn^{1+\delta}}{2pn^{2-\delta}} = o(1)$$

where the last inequality follows from our choice $\delta < 1/2$ and for sufficiently large n.

We repeat the same proof for the case $|e_1 \cap e_2| = 1$ with the following differences.

- $\mathbb{E}[Y_{e_1,e_2}] \leqslant C'' n^{-2} p^{-2}$ for some $C'' > 0$.
- $\mathbb{P}\left(Y_{e_1,e_2} \geqslant \frac{1}{pn^\delta}\right) \leqslant C'' p^{-1} n^{-2+\delta}$.
- $\mathbb{E}[Y] \leqslant 2p^2 n^3 C'' p^{-1} n^{-2+\delta} \leqslant 2C'' pn^{1+\delta}$.
- $\mathbb{P}(Y \geqslant pn^{2-\delta}) \leqslant \frac{2C'' pn^{1+\delta}}{pn^{2-\delta}} = o(1)$.

Consequently for all $s > 3$ we have $|\mathcal{F}_{-,s}(G(n,p), e_1, e_2)| \leqslant p^{-1}n^{-\delta}$ for all but at most $pn^{2-\delta}$ pairs of edges $e_1, e_2 \in E(Z)$. Together with (24) this concludes the proof of (Z4) and finishes the proof of Lemma 14. ∎

The next lemma deals with property (Z5), which concerns the number of bad embeddings as defined in Definition 10.

Lemma 16. *For all graphs B and all strictly balanced and nearly bipartite graphs F, for all constants $C_1 \geqslant C_0 > 0$ and for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ there exists $\zeta > 0$ such that a.a.s. $G(n,p)$ satisfies (Z5).*

Proof of Lemma 16. We shall show that there exist a $\xi > 0$ such that for any given $h \in \Psi_{B,n}$ we have for sufficiently large n

$$\mathbb{P}(h \text{ is bad w.r.t. } F \text{ and } G(n,p)) \leqslant n^{-\xi}.$$

Then the lemma follows from Markov's inequality with $\zeta = \xi/2$.

Let $h \in \Psi_{B,n}$ be fixed. We first consider the case that h is bad w.r.t. F and $G(n,p)$ because of (B1). Since F is strictly balanced, for all proper subgraphs $F_0 \subsetneq F$ with $e(F_0) \geqslant 2$ we have

$$\begin{aligned} p^{e(F_0)} n^{v(F_0)} &= pn^2 \cdot p^{e(F_0)-1} n^{v(F_0)-2} \\ &\geqslant pn^2 \cdot C_0^{e(F_0)-1} n^{-\frac{1}{m_2(F)}(e(F_0)-1)+v(F_0)-2} \\ &= pn^2 \cdot C_0^{e(F_0)-1} n^{(e(F_0)-1)\left(\frac{v(F_0)-2}{e(F_0)-1} - \frac{1}{m_2(F)}\right)} \\ &= pn^2 \cdot C_0^{e(F_0)-1} n^{(e(F_0)-1)\left(\frac{1}{d_2(F_0)} - \frac{1}{d_2(F)}\right)} \\ &\geqslant pn^2 \cdot n^{\xi'} \end{aligned} \tag{25}$$

for some $\xi' > 0$. We bound the probability for h being bad because of case (B1) by estimating the number of configurations leading to this event. In this case F_0 stands for the part of F that is contained in $h(B)$ and hence consists of at least two edges. Using again $n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}$ yields

$$
\begin{aligned}
&\mathbb{P}(h \text{ is bad because of case (B1)}) \\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0) \geqslant 2} v(B)^{v(F_0)} n^{v(F)-v(F_0)} p^{e(F)-e(F_0)} \\
&\qquad \overset{(25)}{\leqslant} \sum_{F_0 \subsetneq F, e(F_0) \geqslant 2} v(B)^{v(F_0)} C_1^{e(F)-1} n^{-\xi'} \\
&\qquad \leqslant n^{-\xi_1}
\end{aligned}
$$

for some $\xi_1 > 0$ and sufficiently large n.

When we address the case (B2) we can assume that h is not bad because of case (B1). Hence, it suffices to consider copies F_1 and F_2 of F intersecting $h(B)$ in precisely one edge and $F_0 := F_1 \cap F_2$ having no edge in $h(B)$. Again we will use $n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}$ and that $n^{v(F_0)}p^{e(F_0)} \geqslant dpn^2$ for $F_0 \subsetneq F$ with $e(F_0) \geqslant 1$ for some $d > 0$ only depending on F and C_0 (cf. (8)). Note that two fixed edges of $h(B)$ determine at least three vertices of $F_1 \cup F_2$.

$$
\begin{aligned}
&\mathbb{P}(h \text{ is bad because of case (B2) and not of case (B1)}) \\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0) \geqslant 1} v(B)^4 n^{2v(F)-v(F_0)-3} p^{2e(F)-e(F_0)-2} \\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0) \geqslant 1} v(B)^4 C_1^{2e(F)-2} \frac{n}{p^{e(F_0)} n^{v(F_0)}} \\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0) \geqslant 1} v(B)^4 C_1^{2e(F)-2} \frac{1}{dpn} \\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0) \geqslant 1} v(B)^4 C_1^{2e(F)-3} n^{-(1-\frac{1}{m_2(F)})} \\
&\qquad \leqslant n^{-\xi_2}
\end{aligned}
$$

for some $\xi_2 > 0$ since $m_2(F) > 1$.

For case (B3) we assume that h is not bad because of case (B1) or case (B2). Again we bound the probability by the expected number of options to obtain a configuration as in (B3). In this case F_0 stands for the intersection of two different copies of F and includes at least two edges, e and f from (B3), where f is also contained in $h(B)$.

$$
\begin{aligned}
&\mathbb{P}(h \text{ is bad because of case (B3) and not of case (B1) or (B2)})\\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0)\geqslant 2} v(B)^2 n^{2v(F)-v(F_0)-2} p^{2e(F)-e(F_0)-1}\\
&\qquad \leqslant \sum_{F_0 \subsetneq F, e(F_0)\geqslant 2} v(B)^2 C_1^{2e(F)-2} \cdot pn^2 \cdot \frac{1}{p^{e(F_0)} n^{v(F_0)}}\\
&\qquad \overset{(25)}{\leqslant} n^{-\xi_3}
\end{aligned}
$$

for some $\xi_3 > 0$ and, hence, $\mathbb{P}(h \text{ is bad}) = n^{-\xi}$ for any $0 < \xi < \min\{\xi_1, \xi_2, \xi_3\}$ and sufficiently large n. □

4.2. **Restricting embeddings of B.** In this section we focus on restricting the family $\Psi_{B,n}$ of all embeddings B in K_n to a suitable subset $\Xi_{B,n}$ so that we can apply Theorem 6 for the proof of Lemma 7. In particular, our choice of $\Xi_{B,n}$ will ensure conditions on the maximum degree and maximum pair degree of $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$. For the control of the pair degree of $\mathcal{H}$ the following definition will be useful.

Definition 17. *For a pair of edges $e_1, e_2 \in E(Z)$ and an embedding $h \in \Xi_{B,n} \subseteq \Psi_{B,n}$ we write $e_1 \approx_h e_2$ if e_1 and e_2 both focus on $h(B)$. Moreover, if e_1 and e_2 focus jointly on only one edge of $h(B)$, then we write $e_1 \sim_h e_2$. We denote by $c_{\Xi_{B,n}}(e_1, e_2)$ the number of $h \in \Xi_{B,n}$ such that $e_1 \approx_h e_2$.*

In the next definition and lemma we define the properties of the desired family of embeddings.

Definition 18. *Let F, B be graphs and let $\alpha > 0$. We call a family $\Xi_{B,n} \subseteq \Psi_{B,n}$ of embeddings of B into K_n α-normal if the following conditions are satisfied.*

- (N1) $|\Xi_{B,n}| \geqslant \alpha n^2$ *and*
- (N2) $|V(h(B)) \cap V(h'(B))| \leqslant 1$ *for all* $h \neq h' \in \Xi_{B,n}$.

Lemma 19. *Let F and B be graphs. For all $\frac{1}{3} > \alpha > 0$, $D > 0$, $1 > \zeta > 0$, $\min\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\} > \delta > 0$ and $C_1 > C_0 > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ the following holds. If $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ and*

$$\mathbb{P}(Z \cup h(B) \to (F)_2^e) > 1 - \alpha$$

where $h \in \Psi_{B,n}$ chosen uniformly at random then there exists $\Xi^0_{B,n} \subseteq \Psi_{B,n}$ such that

- (Ξ1) $\Xi^0_{B,n}$ *is $\tilde{\alpha}$-normal for some* $\tilde{\alpha} = \tilde{\alpha}(B) = \frac{1}{13 v(B)^4 v(B)!} > 0$,
- (Ξ2) $Z \cup h(B) \to (F)_2^e$ *for all* $h \in \Xi^0_{B,n}$,
- (Ξ3) *for all pairs* $\{e_1, e_2\} \in \binom{E(Z)}{2}$ *we have* $c_{\Xi^0_{B,n}}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$,
- (Ξ4) *h is not bad w.r.t. F and Z for all $h \in \Xi^0_{B,n}$ (see Definition 10), and*
- (Ξ5) *for all $h \in \Xi^0_{B,n}$ we have $E(h(B)) \cap E(Z) = \varnothing$.*

We say a family $\Xi^0_{B,n}$ is $(\tilde{\alpha}, Z)$-normal if it satisfies all the conditions (Ξ1), (Ξ2), (Ξ3), (Ξ4), and (Ξ5) for a given $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$.

Proof of Lemma 19. Given F, B and the constants as above we set

$$\tilde{\alpha} = \frac{1}{13v(B)^4 v(B)!}\,.$$

Let $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ and suppose $\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1-\alpha$.

For the construction of $\Xi^0_{B,n}$ we start with the family $\Psi_{B,n}$ and remove embeddings that do not satisfy property (Ξ*2*), embeddings that do not satisfy property (Ξ*4*) and embeddings that will later lead to problems for (Ξ*3*). After that we choose at random $2\tilde{\alpha}n^2$ embeddings which will induce property (Ξ*3*) and show that after deleting the embeddings that intersect in more than one vertex we keep $C\tilde{\alpha}n^2$ with $C > 1$. Afterwards we remove embeddings not satisfying (Ξ*5*). Since $e(Z) = \Theta(pn^2)$ we keep at least $(C\tilde{\alpha} - o(1))n^2 > \tilde{\alpha}n^2$ embeddings h, which finishes the proof.

Since $\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1-\alpha > 2/3$ there is a family $\Psi^1_{B,n} \subseteq \Psi_{B,n}$ of embeddings of B of size $\frac{2}{3}|\Psi_{B,n}|$ such that $Z \cup h(B) \to (F)^e_2$ for all $h \in \Psi^1_{B,n}$, i.e., $\Psi^1_{B,n}$ satisfies (Ξ*2*).

Moreover, since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ there are at most $n^{-\zeta}|\Psi_{B,n}|$ embeddings that are bad w.r.t. F and Z. We remove those bad embeddings from $\Psi^1_{B,n}$. In this way for sufficiently large n we obtain a family $\Psi^2_{B,n} \subseteq \Psi^1_{B,n}$ of size at least $\frac{1}{2}|\Psi_{B,n}|$ that contains no bad embedding and, therefore, $\Psi^2_{B,n}$ satisfies (Ξ*4*).

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ there are at most $\frac{Dpn^2}{n^\delta}$ pairs of distinct edges $e_1, e_2 \in E(Z)$ such that $|\mathcal{F}_-(Z,e_1,e_2)| > \frac{D}{pn^\delta}$. For those pairs of edges e_1, e_2 we delete all embeddings $h \in \Psi^2_{B,n}$ with $e_1 \sim_h e_2$. Since $|\mathcal{F}_-(Z,e)| \leqslant \frac{D}{p}$ for all $e \in E(Z)$ for $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ we delete at most

$$\frac{Dpn^2}{n^\delta} \cdot \frac{D}{p} n^{v(B)-2} = \frac{D^2 n^{v(B)}}{n^\delta} = o(|\Psi_{B,n}|)$$

embeddings from $\Psi^2_{B,n}$. So we get for sufficiently large n a family $\Psi^3_{B,n} \subseteq \Psi^2_{B,n}$ of size at least $\frac{1}{3}|\Psi_{B,n}|$ such that for all distinct e_1, $e_2 \in E(Z)$ we have

(F1) if $e_1 \sim_h e_2$ for some $h \in \Psi^3_{B,n}$, then $|\mathcal{F}_-(Z,e_1,e_2)| \leqslant \frac{D}{pn^\delta}$.

Next we will select a subset $\Psi^4_{B,n} \subseteq \Psi^3_{B,n}$, which allows us to bound $c_{\Psi^4_{B,n}}(e_1,e_2)$ for every pair of edges of Z. For this purpose for

$$\varepsilon = 2\tilde{\alpha} = \frac{2}{13v(B)^4 v(B)!}$$

we select with repetition εn^2 times an element of $\Psi^3_{B,n}$, where we assume for simplicity that εn^2 is an integer. Then for each selection S we can define a family of embeddings $\Psi_S \subseteq \Psi^3_{B,n}$ by taking all embeddings that were chosen at least once in S. We will show that the random selection S a.a.s. satisfies that $c_{\Psi_S}(e_1,e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all $e_1, e_2 \in E(Z)$ and that with probability less than $\frac{1}{2}$ there are more than $\frac{\varepsilon}{2}n^2$ single embeddings that share at least two vertices with some other embedding in the selection.

First we show that a.a.s. $c_{\Psi_S}(e_1,e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all $e_1, e_2 \in E(Z)$. For $i = 1, \dots, \varepsilon n^2$ let $X_{e_1,e_2,i}$ be the indicator random variable for the event "$e_1 \approx_{h_i} e_2$", where h_i denotes the embedding $h \in \Psi^3_{B,n}$ chosen in the ith step. Since there are no bad embeddings w.r.t. F and Z in $\Psi^3_{B,n}$ we know that if e focuses on $h(B)$ then e focuses on exactly one edge in $E(h(B))$ (see property (B1) in Definition 10).

Hence, for $e_1 \approx_h e_2$ we may consider the following two cases. Either $e_1 \sim_h e_2$ or e_1 and e_2 focus on two different edges in $h(B)$.

For the first case we shall use (F1) and $|\Psi^3_{B,n}| \geqslant \frac{1}{3}\binom{n}{v(B)}$ to bound the probability that $e_1 \sim_{h_i} e_2$. In fact,

$$\begin{aligned}\mathbb{P}(e_1 \sim_{h_i} e_2) &\leqslant \frac{D}{pn^\delta} \cdot v(F)^2 \cdot \frac{v(B)(v(B)-1)\cdot(n-2)\cdots(n-v(B)+1)}{|\Psi^3_{B,n}|} \\ &\leqslant \frac{3Dv(F)^2v(B)^2v(B)!}{pn^{2+\delta}}\,.\end{aligned}$$

For the second case we shall use (Z3) of Definition 12 for the upper bound on $|\mathcal{F}_-(Z,e)| \leqslant \frac{D}{p}$. This and the fact that two edges fix at least three vertices yield

$$\begin{aligned}&\mathbb{P}(e_1 \approx_{h_i} e_2 \text{ and not } e_1 \sim_{h_i} e_2) \\ &\quad\leqslant |\mathcal{F}_-(Z,e)|^2 \cdot v(F)^4 \cdot \frac{v(B)(v(B)-1)(v(B)-2)\cdot(n-3)\cdots(n-v(B)+1)}{|\Psi^3_{B,n}|} \\ &\quad\leqslant \frac{3D^2v(F)^4v(B)^3v(B)!}{p^2n^3}\,.\end{aligned}$$

Consequently

$$\mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant 3Dv(F)^2v(B)^2v(B)!\left(\frac{1}{pn^{2+\delta}} + \frac{Dv(F)^2v(B)}{p^2n^3}\right)\,. \tag{26}$$

Since $\delta < 1 - \frac{1}{m_2(F)}$ we infer $n^\delta < n^{1-\frac{1}{m_2(F)}}$ for sufficiently large n. Therefore the right hand side of (26) is of order $\Theta(\frac{1}{pn^{2+\delta}})$ and we can bound

$$\mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant \frac{D_0}{pn^{2+\delta}}\,.$$

where $D_0 = 4Dv(F)^2v(B)^2v(B)!$. For the expected number of connections we get

$$\mathbb{E}[c_{\Psi_S}(e_1,e_2)] \leqslant \sum_{i=1}^{\varepsilon n^2} \mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant \frac{\varepsilon D_0}{pn^\delta}\,.$$

Consequently, Chernoff's Inequality yields

$$\mathbb{P}\left(c_{\Psi_S}(e_1,e_2) \geqslant \frac{3}{2}\cdot\frac{\varepsilon D_0}{pn^\delta}\right) \leqslant \exp\left(-\frac{1}{12}\cdot\frac{\varepsilon D_0}{pn^\delta}\right)\,.$$

Note that $\frac{1}{pn^\delta} > n^\beta$ for some $\beta > 0$ since $\delta < \frac{1}{m_2(F)}$, hence, we can apply the union bound for all pairs of edges $e_1, e_2 \in E(Z)$ and get that a.a.s.

$$c_{\Psi_S}(e_1,e_2) \leqslant \frac{3\varepsilon D_0}{2pn^\delta} \leqslant \frac{1}{pn^{\delta/2}}\,.$$

Finally we verify that most pairs of selected embeddings intersect in at most one vertex. In fact, for $i = 1, \ldots, \varepsilon n^2$ let 1_{h_i} be the indicator random variable for the event "there is $j \in [\varepsilon n^2]\backslash\{i\}$ such that $v(h_i(B) \cap h_j(B)) \geqslant 2$" and set $Y = \sum_{i=1}^{\varepsilon n^2} 1_{h_i}$. Then

$$\mathbb{E}[1_{h_1}] \leqslant (\varepsilon n^2 - 1)\frac{\binom{v(B)}{2}\cdot v(B)(v(B)-1)\cdot(n-2)\cdots(n-v(B)+1)}{|\Psi^3_{B,n}|} \leqslant D_1\varepsilon$$

for some constant $D_1 = D_1(B)$ with $0 < D_1 < \frac{3}{2}v(B)^4 v(B)!$ independent of ε. Hence,

$$\mathbb{E}[Y] \leqslant \varepsilon n^2 D_1 \varepsilon = D_1 \varepsilon^2 n^2$$

and by Markov's Inequality we get

$$\mathbb{P}(Y > 2\mathbb{E}[Y]) \leqslant \frac{1}{2},$$

so there is a selection S of εn^2 embeddings such that $Y \leqslant 2D_1\varepsilon^2 n^2$ as well as $c_{\Psi_S}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all pairs of edges. For this choice of S we can simply delete all those embeddings h_i that intersect with some other embedding h_j in at least two vertices. We call the remaining family $\Psi^4_{B,n}$. Using $D_1 \leqslant 3v(B)^4 v(B)!/2$ and the definition $\varepsilon = 2\tilde{\alpha} = \frac{2}{13v(B)^4 v(B)!}$ yields

$$|\Psi^4_{B,n}| \geqslant \varepsilon n^2 - 2D_1\varepsilon^2 n^2 \geqslant C\tilde{\alpha} n^2$$

for some $C > 1$ and, hence, $\Psi^4_{B,n}$ satisfies (Ξ*1*)–(Ξ*4*).

To achieve (Ξ*5*) we make use of $e(Z) \leqslant pn^2$ (see (Z1) of Definition 12). Since no two embeddings from $\Psi^4_{B,n}$ share an edge, we may remove all embeddings from $\Psi^4_{B,n}$ which share at least one edge with Z and this results in the desired family $\Xi^0_{B,n} \subseteq \Psi^4_{B,n}$ of size at least $\tilde{\alpha} n^2$, which finishes the proof. □

For Lemma 7 we have to show that there is a family of embeddings $\Xi_{B,n}$ such that the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$ is index consistent with a profile π. Lemma 20 will ensure this.

Lemma 20. *For all constants $1 > \tilde{\alpha} > 0$ and $D > 0$, for all graphs F and B with F being strictly balanced and nearly bipartite and with $E(B) = \{e_1, \ldots, e_K\}$, there exist $\alpha' > 0$ and $L \in \mathbb{N}$ such that every graph Z on n vertices with a fixed ordering of its edge set and the property*

(Z) $|\mathcal{F}_-(Z)| \leqslant Dn^2$

satisfies the following.

For every $(\tilde{\alpha}, Z)$-normal family $\Xi^0_{B,n}$ there is an (α', Z)-normal family $\Xi_{B,n} \subset \Xi^0_{B,n}$ and there is a profile π of length at most L such that $(Z, \Xi_{B,n})$ is index consistent with profile π.

Note that it is rather unlikely that M_h and $M_{h'}$ of $\mathcal{H}$ are equal for distinct $h, h' \in \Xi^0_{B,n}$ and, hence, Lemma 20 follows by a simple average argument. We will use Lemma 20 for $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ which satisfies (Z) by (Z2) from Definition 12.

Proof of Lemma 20. Let $1 > \tilde{\alpha} > 0$, $D > 0$, F and B be given. We define

$$L = (e(F) - 1)\frac{2}{\tilde{\alpha}}v(F)^2 D \qquad \text{and} \qquad \alpha' = \frac{\tilde{\alpha}}{2L(KL)^L}.$$

Given some Z satisfying (Z) and an $(\tilde{\alpha}, Z)$-normal family $\Xi^0_{B,n} \subseteq \Psi_{B,n}$ we will restrict $\Xi^0_{B,n}$ to the promised set $\Xi_{B,n}$ with the desired properties.

Note that the family $\Xi_{B,n} \subseteq \Xi^0_{B,n}$ inherits the properties (Ξ*2*)–(Ξ*5*) from the $(\tilde{\alpha}, Z)$-normality of $\Xi^0_{B,n}$ since they are independent of $\tilde{\alpha}$. Consequently, to establish that $\Xi_{B,n}$ is indeed (α', Z)-normal, we only have to focus on (Ξ*1*). Since again property (N2) of Definition 18 is inherited from the normality of $\Xi^0_{B,n}$, it suffices to show that $|\Xi_{B,n}| \geqslant \alpha' n^2$.

Because of (Z) we know that Z contains at most Dn^2 copies of some $F' \subseteq F$ with $e(F') = e(F) - 1$. Also due to $\Xi^0_{B,n}$ being $(\tilde{\alpha}, Z)$-normal there are no bad embeddings w.r.t. F and Z in $\Xi^0_{B,n}$ and thus by Fact 11 the pair $(Z, \Xi^0_{B,n})$ is regular. In particular, for every $h \in \Xi^0_{B,n}$ we have that every edge $e \in M_h$ focuses on exactly one $b \in E(h(B))$. Furthermore, since every $h \in \Xi^0_{B,n}$ also does not satisfy (B3) of Definition 10, each $e \in M_h$ focuses on one $b \in E(h(B))$ in only one way, i.e. there is only one copy of F in $Z \cup h(B)$ containing b and e. Therefore, $\ell_h = |M_h|$ is a multiple of $e(F) - 1$ and each M_h gives rise to $\ell_h/(e(F) - 1)$ copies of some F' in Z obtained from F by removing some edge. Clearly, each such $(e(F) - 1)$-element subset of M_h might be completed to a copy of F in at most $\binom{v(F)}{2} - e(F) + 1 < v(F)^2$ ways.

Applying the upper bound on the number of copies of F with one edge removed from (Z) yields

$$\sum_{h \in \Xi^0_{B,n}} \frac{\ell_h}{e(F) - 1} \leqslant v(F)^2 \cdot Dn^2 \,.$$

So there are at most $\widetilde{\alpha} n^2/2$ embeddings $h \in \Xi^0_{B,n}$ with $\ell_h > L$, and, consequently, at least $\widetilde{\alpha} n^2/2$ embeddings $h \in \Xi^0_{B,n}$ with $\ell_h \leqslant L$. Since there are at most K^ℓ different profiles of length ℓ, there must be a profile π of length $\ell \leqslant L$ and a subset $\Xi'_{B,n} \subseteq \Xi^0_{B,n}$ with

$$|\Xi'_{B,n}| \geqslant \frac{1}{LK^L} \cdot \frac{\widetilde{\alpha}}{2} n^2$$

such that $(Z, \Xi'_{B,n})$ has profile π.

Next we apply another averaging argument to achieve index consistency. We consider some partition $Z_1 \dot\cup \ldots \dot\cup Z_\ell$ of Z into ℓ classes chosen uniformly at random. Recall that we ordered the edges of Z. For $h \in \Xi'_{B,n}$ consider $M_h = (z_1, \ldots, z_\ell)$ with the ordering of Z inherited. We include h in $\Xi_{B,n}$ if $z_i \in Z_i$ for all $i = 1, \ldots, \ell$. Clearly $\mathbb{P}(h \in \Xi_{B,n}) = \frac{1}{\ell^\ell}$ and $\mathbb{E}[|\Xi_{B,n}|] = \frac{|\Xi'_{B,n}|}{\ell^\ell}$, that means there is an $\Xi_{B,n} \subseteq \Xi'_{B,n}$ with $|\Xi_{B,n}| \geqslant |\Xi'_{B,n}|/\ell^\ell \geqslant \frac{1}{L^L} \frac{\widetilde{\alpha} n^2}{2L(K)^L} = \alpha' n^2$. Now let $h, h' \in \Xi_{B,n}$ and let $z \in M_h \cap M_{h'}$. Since $z \in Z_j$ for some partition class Z_j we know that z has index j in both M_h and $M_{h'}$. Therefore $(Z, \Xi_{B,n})$ is index consistent which finishes the proof. □

4.3. **Proof of Lemma 7.** Finally we prove Lemma 7. The previous lemmas will be utilised to show that the hypergraph $\mathcal{H}(Z, \Xi)$ satisfies the conditions of Theorem 6 of Saxton and Thomason about independent sets in hypergraphs.

Proof of Lemma 7. Let constants $C_1 > C_0 > 0$, $\frac{1}{3} > \alpha > 0$ and graphs F and B with F being strictly balanced and nearly bipartite be given.

First we fix all constants used in the proof. For the given graphs F and B and the given constants C_1 and C_0 Lemma 13 yields constants $D > 0$, $\zeta > 0$, and δ with $0 < \delta < \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$. Similarly Lemma 20 applied to F, B, D and

$$\widetilde{\alpha} = \tfrac{1}{13 v(B)^4 v(B)!}$$

yields α' and L. Fixing an auxiliary constant

$$k = \binom{L}{e(F) - 1} \binom{v(F)}{2}$$

allows us to set

$$\beta = \frac{\alpha'}{Dkv(F)^2} \quad \text{and} \quad \gamma = \frac{\delta}{10L} \,. \tag{27}$$

We shall show that α', β, γ, and L defined this way have the desired property. For that let $p = p(n) = c(n)n^{-1/m_2(F)}$ for some $c(n)$ satisfying $C_0 \leqslant c(n) \leqslant C_1$. We shall show that $G(n,p)$ a.a.s. satisfies the property of Lemma 7. Hence, in view of Lemma 13 we may assume that the graphs Z considered in Lemma 7 are from the set $\mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$. Moreover, let n be sufficiently large, so that Lemma 19 applied with F, B, α, D, ζ, δ, C_1 and C_0 holds for n.

Now let $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ such that for $h \in \Psi_{B,n}$ chosen uniformly at random we have

$$\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha.$$

Then Lemma 19 yields an $(\tilde{\alpha}, Z)$-normal family of embeddings $\Xi^0_{B,n} \subseteq \Psi_{B,n}$, i.e., the family $\Xi^0_{B,n}$ satisfies properties $(\Xi 1)$–$(\Xi 5)$ of Lemma 19 for the parameters chosen above.

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ it satisfies property (Z2) of Definition 12 and, hence, Z satisfies in particular assumption (Z) of Lemma 20. Consequently, Lemma 20 yields an (α', Z)-normal family $\Xi_{B,n} \subseteq \Xi^0_{B,n}$ and a profile π of length $\ell \leqslant L$ such that the pair $(Z, \Xi_{B,n})$ is index consistent for π.

Next we consider the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ defined by

$$V(\mathcal{H}) = E(Z) \quad \text{and} \quad E(\mathcal{H}) = \{M(Z, h(B))\colon\ h \in \Xi_{B,n}\}\,,$$

where

$$M(Z,h(B)) = \{z \in E(Z)\colon \text{ there is } b \in E(h(B)) \text{ such that } z \text{ focuses on } b\}\,.$$

Clearly, $\mathcal{H}$ is an ℓ-uniform hypergraph on $m = e(Z)$ vertices. Below we show that $\mathcal{H}$ satisfies the assumptions of Theorem 6 for

$$\varepsilon = \tfrac{1}{4} \qquad \text{and} \qquad \tau = n^{-\frac{\delta}{4(\ell-1)}}\,.$$

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ it displays properties (Z1)–(Z5) of Definition 12. In particular, (Z1) guarantees

$$\frac{1}{4}pn^2 \leqslant e(Z) = m \leqslant pn^2 < n^2\,. \tag{28}$$

Now we bound $e(\mathcal{H})$. Since $\Xi_{B,n}$ is α'-normal, it follows from (N1) and (N2) of Definition 18 that $\alpha' n^2 \leqslant |\Xi_{B,n}| \leqslant n^2$ and, consequently, we have $e(\mathcal{H}) \leqslant n^2$. On the other hand, for any hyperedge M_h of size ℓ there are at most $\binom{\ell}{e(F)-1}$ different copies of some $F' \subseteq F$ with $e(F') = e(F) - 1$ in M_h and each such copy can be extended to F by at most $\binom{v(F)}{2}$ different boosters since all boosters are edge disjoint. Consequently, M_h could be the hyperedge for at most $\binom{\ell}{e(F)-1}\binom{v(F)}{2} \leqslant k$ different embeddings $h \in \Xi_{B,n}$ and, therefore, we have

$$\frac{\alpha' n^2}{k} \leqslant e(\mathcal{H}) \leqslant n^2\,. \tag{29}$$

Hence, for the average degree of $\mathcal{H}$ we obtain

$$d(\mathcal{H}) = \ell \cdot \frac{e(\mathcal{H})}{v(\mathcal{H})} \geqslant \ell \cdot \frac{\alpha' n^2}{k} \cdot \frac{1}{pn^2} = \frac{\ell\alpha'}{kp}\,.$$

We denote by $\Delta_1(\mathcal{H}) = \max_{v \in V(\mathcal{H})} |\{e \in E(\mathcal{H}) : e \text{ contains } v\}|$ the maximum vertex degree and by $\Delta_2(\mathcal{H}) = \max_{(v,v') \in \binom{V(\mathcal{H})}{2}} |\{e \in E(\mathcal{H}) : e \text{ contains } v \text{ and } v'\}|$ the maximum codegree of $\mathcal{H}$ and below we will bound $\Delta_1(\mathcal{H})$ and $\Delta_2(\mathcal{H})$.

We start with $\Delta_1(\mathcal{H})$. Suppose $e \in M(Z, h(B))$ for some $h \in \Xi_{B,n}$. Since $\Xi_{B,n}$ contains no bad embeddings w.r.t. F and Z and $E(h(B)) \cap E(Z) = \varnothing$ there exists a unique copy $F_- \in \mathcal{F}_-(Z, e)$ with $e \in E(F_-)$ and $f \in h(B)$ such that $F_- + f$ forms a copy of F. Moreover, since every two distinct embeddings h, $h' \in \Xi_{B,n}$ intersect in at most one vertex the degree of e in $\mathcal{H}$ is bounded by $|\mathcal{F}_-(Z,e)| \cdot \binom{v(F)}{2}$. Consequently, it follows from property (Z3) given by $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ that

$$\Delta_1(\mathcal{H}) \leqslant \frac{D}{p} \cdot \binom{v(F)}{2}.$$

For $\Delta_2(\mathcal{H})$ we have to look at pairs of edges of Z. Two edges $e_1, e_2 \in E(Z)$ are both contained in $M(Z, h(B))$ if and only if $e_1 \approx_h e_2$. By ($\Xi 3$) we know $c_{\Xi_{B,n}}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$, so

$$\Delta_2(\mathcal{H}) \leqslant \frac{1}{pn^{\frac{\delta}{2}}}.$$

Note that $\frac{1}{pn^{\frac{\delta}{2}}} \to \infty$ for $n \to \infty$ since $\delta \leqslant \frac{1}{m_2(F)}$.

In order to verify the assumptions of Theorem 6 we estimate $\delta(\mathcal{H}, \tau)$ for ε and τ defined above. Indeed we have

$$\begin{aligned}
\delta(\mathcal{H},\tau) &= 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \sum_{v \in V(\mathcal{H})} d^{(j)}(v) \\
&\leqslant 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \cdot m \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} \frac{1}{\tau^{\ell-1} d(\mathcal{H})} \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1} \cdot \ell \cdot n^{\frac{\delta}{4}} \cdot \frac{kp}{\ell\alpha'} \cdot \frac{1}{pn^{\frac{\delta}{2}}} \\
&= 2^{\binom{\ell}{2}-1} \cdot \frac{k}{\alpha'} \cdot \frac{1}{n^{\frac{\delta}{4}}} \\
&\leqslant \frac{\varepsilon}{12\ell!},
\end{aligned}$$

where the last inequality holds for sufficiently large n.

By Theorem 6 we get some constant $c = c(\ell)$ and a family $\mathcal{J} \subset \mathcal{P}(V(\mathcal{H}))$ satisfying (a), (b) and (c) from Theorem 6. We define

$$\mathcal{C} = \{C \subset V(\mathcal{H}) \colon\ C = V(\mathcal{H}) \backslash J \text{ for one } J \in \mathcal{J}\}.$$

Below we show that $\mathcal{C}$ has the desired properties (1), (2) and (3) of Lemma 7.

(1) follows from (c) since $|\mathcal{C}| = |\mathcal{J}|$ and

$$|\mathcal{J}| \leqslant c\tau \log(1/\tau) \log(1/\varepsilon) m \leqslant m \cdot n^{-\frac{\delta}{4(\ell-1)}} c \log(1/\tau) \log(1/\varepsilon) \leqslant m^{1-\gamma},$$

where the last inequality follows for sufficiently large n from

$$m^{\gamma} \overset{(28)}{<} n^{2\gamma} \overset{(27)}{=} n^{\frac{\delta}{5\ell}},$$

since $c = c(\ell)$ and $\log(1/\varepsilon)$ are constants independent of n and $\log(1/\tau) < \log n$.

(2) follows from (*b*). Assume for a contradiction that there is $C \in \mathcal{C}$ with $|C| < \beta m$ and let $J = V \backslash C \in \mathcal{J}$. Then we count the number of hyperedges of $\mathcal{H}$.

$$\begin{aligned} e(\mathcal{H}) &\leqslant e(\mathcal{H}[V \backslash C]) + |C| \cdot \Delta_1(\mathcal{H}) \\ &< e(\mathcal{H}[J]) + \beta m \cdot \frac{D}{p}\binom{v(F)}{2} \\ &\overset{(28)}{\leqslant} \varepsilon e(\mathcal{H}) + \beta D \binom{v(F)}{2} n^2 \\ &\overset{(29)}{\leqslant} \varepsilon e(\mathcal{H}) + \frac{\beta D k}{\alpha'}\binom{v(F)}{2} e(\mathcal{H}) \\ &= \left(\varepsilon + \frac{\beta D k}{\alpha'}\binom{v(F)}{2}\right) e(\mathcal{H}) \\ &\overset{(27)}{<} e(\mathcal{H}) \end{aligned}$$

with a contradiction, so $|C| \geqslant \beta m$ for all $C \in \mathcal{C}$.

(3) For a hitting set A of $\mathcal{H}$ consider the independent set $I = V \backslash A$. Hence by (*a*) of Theorem 6 there exists $J \in \mathcal{J}$ such that $I \subseteq J$ and, therefore, $A \supseteq V \backslash J = C$ which is an element of $\mathcal{C}$. □

5. Proof of Lemma 8

The proof of Lemma 8 follows the proof in [10, Lemma 2.3] and is based on an application of the regularity method for subgraphs of sparse random graphs which we introduce first.

Let $\varepsilon > 0$, $p \in (0, 1]$ and $H = (V, E)$ be a graph. For X, $Y \subset V$ non-empty and disjoint let

$$d_{H,p}(X, Y) = \frac{e(X, Y)}{p|X||Y|}$$

and we say (X, Y) is (ε, p)-regular if

$$|d_{H,p}(X, Y) - d_{H,p}(X', Y')| < \varepsilon$$

for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geqslant \varepsilon|X|$ and $|Y'| \geqslant \varepsilon|Y|$. We will use the sparse regularity lemma in the following form (see, e.g., [16]).

Lemma 21. *For all $\varepsilon > 0$, $t_0 \in \mathbb{N}$ there exists an integer T_0 such that for every function $p = p(n) \gg 1/n$ a.a.s. $G \in G(n, p)$ has the following property. Every subgraph $H = (V, E)$ of G with $|V| = n$ vertices admits a partition $V = V_1 \dot\cup \ldots \dot\cup V_t$ satisfying*

(*i*) $t_0 \leqslant t \leqslant T_0$,
(*ii*) $|V_1| \leqslant \cdots \leqslant |V_t| \leqslant |V_1| + 1$ *and*
(*iii*) *all but at most εt^2 pairs (V_i, V_j) with $i \neq j$ are (ε, p)-regular.* □

For a partition $\mathcal{P}$ as in the last lemma we call the graph $R = R(\mathcal{P}, d, \varepsilon)$ with vertex set $V(R) = \{V_1, \ldots, V_t\}$ and edges

$$\{V_i, V_j\} \in E(R) \iff (V_i, V_j) \text{ is } (\varepsilon, p)\text{-regular with } d_{H,p}(V_i, V_j) \geqslant d$$

the reduced graph w.r.t. $\mathcal{P}$, d, and ε.

The next lemma is a counting lemma for subgraphs of random graphs from [1, 3, 20].

Lemma 22. *For every graph F with vertex set $V(F) = [\ell]$ and $d > 0$ there exist $\varepsilon > 0$ and $\xi > 0$ such that for every $\eta > 0$ there exists $C > 0$ such that for $p > Cn^{-1/m_2(F)}$ a.a.s. $G \in G(n,p)$ satisfies the following.*

Let $H = (V_1 \dot\cup \ldots \dot\cup V_\ell, E_H)$ be an ℓ-partite (not necessarily induced) subgraph of G with vertex classes of size at least ηn and with the property that for every edge $\{i,j\} \in E(F)$ the pair (V_i, V_j) in H is (ε, p)-regular with density $d_{H,p}(V_i, V_j) \geqslant d$. Then the number of partite copies of F in H is at least

$$\xi p^{e(F)} \prod_{i=1}^{\ell} |V_i| \,,$$

where partite copy means that there is a graph homomorphism $\varphi\colon F \to H$ with $\varphi(i) \in V_i$. □

The next lemma bounds the number of edges between large sets of vertices of $G(n,p)$ as well as the number of copies of some bipartite graphs $F^\star$ with two vertices from a prescribed set W.

Lemma 23. *Let $F^\star$ be a bipartite graph with two marked vertices $a_1, a_2 \in V(F^\star)$ from the same colour class. For all $(\log n)/n \leqslant p = p(n) < 1$ the random graph $G \in G(n,p)$ satisfies a.a.s. the following properties.*

(*A*) *For all disjoint subset U, $W \subseteq V(G)$ with $|U|, |W| \geqslant n/\log\log n$ we have*

$$p|U|^2/3 < e_G(U) < p|U|^2 \quad \text{and} \quad p|U||W|/2 < e_G(U,W) < 2p|U||W| \,.$$

(*B*) *For all subsets $W \subset V(G)$ there exists a set of edges $E_0 \subseteq E(G)$ with $|E_0| = n\log n$ such that there are at most $2p^{e(F^\star)} n^{v(F^\star)-2}|W|^2$ many copies $\varphi(F^\star)$ of $F^\star$ in the graph $(V(G), E(G)\backslash E_0)$ with $V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}$.*

The proof of (*A*) follows directly from Chernoff's inequality and the proof of (*B*) is based on the so-called *deletion method* in form of the following lemma.

Lemma 24. [15, Lemma 2.51] *Let Γ be a set, $S \subseteq [\Gamma]^s$ and $0 < p < 1$. Then for every $k > 0$ with probability at least $1 - \exp(-\frac{k}{2s})$ there exists a set $E_0 \subset \Gamma_p$ of size k such that $\Gamma_p \backslash E_0$ contains at most 2μ sets from S where μ is the expected number of sets from S contained in Γ_p.* □

Proof of Lemma 23. Since part (*A*) follows from Chernoff's inequality, we will only focus on property (*B*), which is a direct consequence of Lemma 24.

In fact, let V be a set of n vertices, $W \subset V$ and a bipartite graph $F^\star$ with two marked vertices $a_1, a_2 \in V(F^\star)$ from the same colour class be given. We use Lemma 24 with $\Gamma = \binom{V}{2}$, $s = e(F^\star)$,

$$S = \big\{\text{copies } \varphi(F^\star) \text{ of } F^\star \text{ in } (V,\Gamma) \text{ with } V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}\big\} \,,$$

p, and $k = n\log n$. In particular, $\Gamma_p = G(n,p)$ in our setup here. With probability at least $1 - \exp\big(-\frac{n\log n}{2e(F^\star)}\big)$ there exists a set $E_0 \subseteq E(G(n,p))$ of size at most $n\log n$ such that there are at most

$$2\mu \leqslant 2p^{e(F^\star)} n^{v(F^\star)-2}|W|^2$$

many copies $\varphi(F^\star)$ with $V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}$ in $(V, E(G(n,p))\backslash E_0)$. The lemma then follows from the union bound applied for all 2^n possible choices $W \subset V$. □

Finally, we can prove Lemma 8. Let G be a typical graph in $G(n,p)$ and let H be a subgraph of G with $|E(H)| \geqslant \lambda |E(G)|$. First we apply the sparse regularity lemma (Lemma 21) to H. Since H is relatively dense in $G(n,p)$ we infer that the corresponding reduced graph R (for suitable chosen parameters) has many, i.e. $\Omega(|V(R)|^2)$ edges. So we can find many large complete bipartite graphs in R. We conclude that there is some partition class $V_i \in V(R)$ contained in many complete bipartite graphs.

We analyse the graph $G_0 = \mathrm{Base}_H(F)[V_i]$ on the vertex set V_i with edges being those pairs in $\binom{V_i}{2}$ that complete a copy of the bipartite graph $F' \subseteq F' + e = F$ in H to a copy of F. We say that G_0 is (ϱ, d)-dense if for all $W \subseteq V(G_0)$ with $|W| \geqslant \varrho |V_i|$ we have $e_{G_0}(W) \geqslant d\binom{|W|}{2}$. It is well known that sufficiently large (ϱ, d)-dense graphs contain any fixed subgraph (see e.g. [19]).

Lemma 25. *For all $d > 0$ and F there exist ϱ, $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that for every (ϱ, d)-dense graph G_0 with $n = v(G_0) \geqslant n_0$ we have that G_0 contains at least $c_0 n^{v(F)}$ copies of F.* □

To show the (ϱ, d)-denseness of G_0 we consider $W \subseteq V_i$ with $|W| \geqslant \varrho |V_i|$. Then by Lemma 22 we will find many copies of F' in H where the missing edge has to be in $\binom{W}{2}$. Together with an upper bound for the number of graphs that are combinations of two different copies of F' ((B) of Lemma 23) we ensure that not too many copies of F' are completed to F by the same pair in W. Thus there are many edges in $\mathrm{Base}_H(F)[W]$ and G_0 is (ϱ, d)-dense.

Proof of Lemma 8. Let $\lambda > 0, C_1 > C_0 > 0$ and let F be a strictly balanced nearly bipartite graph such that $F = F' + \{a_1, a_2\}$, where F' is bipartite with partition classes $A = \{a_1, \dots, a_a\}$ and $B = \{b_1, \dots, b_b\}$.

The Sparse Counting Lemma (Lemma 22) applied with F' and $d_{\mathrm{CL}} = \lambda/4$ yields constants $\varepsilon_{\mathrm{CL}} > 0$ and $\xi_{\mathrm{CL}} > 0$. Since we don't know whether the given constant C_0 is at least 1 or not, we find it convenient to fix an auxiliary constant

$$C_0' = \min\{1, C_0^{e(F)-1}\}\,. \tag{30}$$

Furthermore, we set

$$d = \frac{\left(\frac{\lambda}{6}\right)^{2(a-1)b} \cdot \xi_{\mathrm{CL}}^2 \cdot C_0^{2(e(F)-1)} \cdot C_0'}{64(v(F)+1)^{v(F)} \cdot C_1^{2(e(F)-1)}}\,. \tag{31}$$

Next we appeal to Lemma 25. For F and for this choice of d this lemma yields constants ϱ, $c_0 > 0$ and $n_0 \in \mathbb{N}$. Furthermore, set

$$\varepsilon = \min\left\{\frac{\varrho \varepsilon_{\mathrm{CL}}}{4}, \frac{\lambda}{48}\right\} \qquad \text{and} \qquad t_0 \geqslant \frac{48}{\lambda}\,. \tag{32}$$

Lemma 21 applied with ε and t_0 yields $T_0 \in \mathbb{N}$ and Lemma 22 applied with $\eta_{\mathrm{CL}} = \varrho/(2T_0)$ yields C_{CL}. Finally, we fix the promised

$$\eta = c_0 T_0^{-v(F)}$$

and let $C_0 n^{-1/m_2(F)} \leqslant p = p(n) \leqslant C_1 n^{-1/m_2(F)}$. For later reference we note that due to the balancedness of F we have

$$p^{e(F)} n^{v(F)} \leqslant C_1^{e(F)-1} p n^2 \tag{33}$$

and owing to the choice of C_0' in (30) we have

$$p^{e(F_1)}n^{v(F_1)} \geqslant C_0' pn^2 \tag{34}$$

for every subgraph $F_1 \subseteq F$ with $e(F_1) \geqslant 1$.

Since we have to show that $G(n,p)$ a.a.s. satisfies $T(\lambda, \eta, F)$ we can assume that n is arbitrarily large. Consider any $G \in G(n,p)$ that satisfies the properties of Lemma 21 and Lemma 22, as well as property *(A)* and property *(B)* of Lemma 23 for all bipartite graphs $F^\star$ such that $F^\star$ is the union of two different copies $\varphi_1(F')$ and $\varphi_2(F')$ of F' with $\{\varphi_1(a_1), \varphi_1(a_2)\} = \{\varphi_2(a_1), \varphi_2(a_2)\}$. In other words, for the rest of the proof we consider a fixed graph G to which we can apply the Lemmas 21–23 and we will show that such a G satisfies $T(\lambda, \eta, F)$. For that let $H \subseteq G$ with

$$e(H) \geqslant \lambda e(G) > \frac{1}{3}\lambda pn^2$$

where the second inequality follows from property *(A)* of Lemma 23.

Lemma 21 applied to H yields a partition $\mathcal{P}$ of the vertices $V = V_1 \dot\cup \ldots \dot\cup V_t$ with at least $(1-\varepsilon)\binom{t}{2}$ many (ε, p)-regular pairs for some t with $t_0 \leqslant t \leqslant T_0$. We assume w.l.o.g. that t divides n. We infer that there are at least $\frac{\lambda}{6}\binom{t}{2}$ regular pairs with edge density at least $\frac{\lambda}{4}p$ since otherwise we could bound the number of edges of H by

$$\begin{aligned} e(H) &\leqslant \frac{\lambda}{6}\binom{t}{2} \cdot 2p\left(\frac{n}{t}\right)^2 + \binom{t}{2} \cdot \frac{\lambda}{4}p\left(\frac{n}{t}\right)^2 + \varepsilon\binom{t}{2} \cdot 2p\left(\frac{n}{t}\right)^2 + t\binom{\frac{n}{t}}{2} \cdot 2p \\ &\leqslant \frac{1}{2}pn^2\left(\frac{\lambda}{3} + \frac{\lambda}{4} + 2\varepsilon + \frac{2}{t}\right) \\ &\overset{(32)}{\leqslant} \frac{1}{3}\lambda pn^2\,, \end{aligned}$$

which would contradict the derived lower bound $e(H) > \frac{1}{3}\lambda pn^2$.

Let $R = R(\mathcal{P}, d_{\mathrm{CL}}, \varepsilon)$ be the reduced graph w.r.t. to the partition $\mathcal{P}$ and relative density $d_{\mathrm{CL}} = \frac{\lambda}{4}$. In particular R has exactly $t \geqslant t_0$ vertices and at least $\frac{\lambda}{6}\binom{t}{2}$ edges. It follows from the theorem of Kövari, Sós and Turán [17] (see also [4]) that there are at least γt^{a+b-1} copies of the complete bipartite graph $K_{a-1,b}$ in R where

$$\gamma = \gamma(F, \lambda) = \frac{1}{2}\left(\frac{\lambda}{6}\right)^{(a-1)b}. \tag{35}$$

Hence, there is a partition class V_{a_0} of $\mathcal{P}$ such that V_{a_0} is contained in at least γt^{a+b-2} copies of $K_{a-1,b}$ in R where V_{a_0} is always contained in partition class A of $K_{a-1,b}$ for these copies.

Our goal is to show that the graph G_0 induced by $\mathrm{Base}_F(H)$ on V_{a_0} is (ϱ, d)-dense, which due to our choice of c_0 and η above leads to $c_0(n/t)^{v(F)} > \eta n^{v(F)}$ copies of F in G_0 (see Lemma 25). So let $W \subseteq V_{a_0}$ with $|W| \geqslant \varrho|V_{a_0}|$ and fix some partition $W = W_1 \dot\cup W_2$ with $|W_1| = |W_2| = |W|/2$ (for simplicity, we may assume that $|W|$ is even). Note that for any j for which (V_{a_0}, V_j) is (ε, p)-regular we still have that (W_1, V_j) and (W_2, V_j) are $(2\varepsilon/\varrho, p)$-regular.

We will ensure many copies of F' with $a_1 \in W_1$ and $a_2 \in W_2$ which forces edges in $G_0 = \mathrm{Base}_F(H)[V_{a_0}]$. However, we have to make sure that not too many copies force the same edge in G_0. For this purpose we delete some edges by *(B)* of Lemma 23 to restrict the number of graphs $F^\star$ that are unions of two different copies of F' that force the same edge in G_0.

Let $\varphi_1(F'), \varphi_2(F')$ be two copies of F' with $\varphi_1(\{a_1, a_2\}) = \varphi_2(\{a_1, a_2\})$ and let $F^\star = \varphi_1(F') \cup \varphi_2(F')$. We find by (B) of Lemma 23 at most $n \log n$ edges $E_{F^\star}$ such that there are at most

$$2p^{e(F^\star)} n^{v(F^\star)-2} |W|^2 \tag{36}$$

copies of $F^\star$ in $(V(H), E(H) \backslash E_{F^\star})$ with $\varphi_1(a_1), \varphi_1(a_2) \in W_1 \cup W_2$. We repeat this argument for all possible graphs $F^\star$ that can be created this way and we denote by $\mathcal{F}^\star$ the family of those graphs. Since there are at most $2(a+1)^{a-2}(b+1)^b$ such graphs $F^\star$, in total we delete at most

$$2(a+1)^{a-2}(b+1)^b n \log n = o(pn^2)$$

edges of H, i.e., for $H' = H - \bigcup_{F^\star \in \mathcal{F}^\star} E_{F^\star}$ we have

$$e(H') \geqslant (1 - o(1)) e(H).$$

In particular, for sufficiently large n the density and the regularity of the pairs in the partition $\mathcal{P}$ is not affected much and (δ, p)-regular pairs in H are still $(2\delta, p)$-regular in H'.

Lemma 22 yields many copies of F' in H'. In fact, since $m_2(F') < m_2(F)$ we get

$$p \geqslant C_0 n^{-\frac{1}{m_2(F)}} > C_{\mathrm{CL}} n^{-\frac{1}{m_2(F')}}\,.$$

For any copy of $K_{a-1,b}$ in the reduced graph R that contains V_{a_0} among the $a-1$ classes of the bipartition of $K_{a-1,b}$ Lemma 22 yields at least

$$\xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W_1||W_2| = \frac{1}{4} \xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W|^2$$

partite copies of F' in H' with $a_1 \in W_1$ and $a_2 \in W_2$. Repeating this for the γt^{a+b-2} different copies of $K_{a-1,b}$ in R that contain V_{a_0} in the described way, in total we obtain at least

$$\gamma t^{v(F)-2} \cdot \frac{1}{4} \xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W|^2 = \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot p^{e(F)-1} n^{v(F)-2} |W|^2 \geqslant \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot C_0^{e(F)-1} |W|^2 \tag{37}$$

copies of F' in H' with $a_1 \in W_1$ and $a_2 \in W_2$.

For a pair of vertices $e \in \binom{W}{2}$ we define

$$x_e = \left|\left\{\varphi(F') \text{ copy of } F' \text{ in } H' \colon e = \{\varphi(a_1), \varphi(a_2)\}\right\}\right|\,.$$

By (37) we know that

$$\sum_{e \in \binom{W}{2}} x_e \geqslant \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot C_0^{e(F)-1} |W|^2\,. \tag{38}$$

Let $\mathcal{W}_{>0} = \left\{e \in \binom{W}{2} : x_e \neq 0\right\}$ and $N = |\mathcal{W}_{>0}|$. Since this N corresponds to the number of edges in $\mathrm{Base}_{H'}(F)[W] \subseteq \mathrm{Base}_H(F)[W]$ we shall show that $N \geqslant d\binom{|W|}{2}$. For this purpose we use (38) and an upper bound for $\sum_{e \in \binom{W}{2}} x_e^2$ that follows from (36). In fact,

$$\sum_{e \in \binom{W}{2}} x_e^2 \overset{(36)}{\leqslant} |\mathcal{F}^\star| \cdot 2p^{e(\hat{F})} n^{v(\hat{F})-2} |W|^2 \tag{39}$$

where $\hat{F}$ is a graph in $\mathcal{F}^\star$ that maximises the value of $p^{e(F^\star)} n^{v(F^\star)-2}$ for $F^\star \in \mathcal{F}^\star$. We will show that $p^{e(\hat{F})} n^{v(\hat{F})-2}$ is bounded by a constant only depending on C_0, C_1 and F. In fact, for $F^\star =$

$\varphi_1(F') \cup \varphi_2(F') \in \mathcal{F}^\star$ let $F_0 = \varphi_1(F') \cap \varphi_2(F')$ and $e = \{\varphi_1(a_1), \varphi_1(a_2)\}$. In particular, $F_0 + e \subseteq F$ and we have

$$p^{e(F^\star)} n^{v(F^\star)-2} = \frac{p^{e(F^\star+e)} n^{v(F^\star+e)}}{pn^2} = \frac{\left(p^{e(F)} n^{v(F)}\right)^2}{p^{e(F_0+e)} n^{v(F_0+e)} \cdot pn^2} \overset{(33)}{\leqslant} \frac{C_1^{2e(F)-2} pn^2}{p^{e(F_0+e)} n^{v(F_0+e)}} \overset{(34)}{\leqslant} \frac{C_1^{2e(F)-2}}{C_0'} .$$

Combining (39) with the simple upper bound $|\mathcal{F}^\star| \leqslant (v(F)+1)^{v(F)}$ and the last inequality yields

$$\sum_{e \in \binom{W}{2}} x_e^2 \leqslant 2(v(F)+1)^{v(F)} \frac{C_1^{2(e(F)-1)}}{C_0'} |W|^2 . \tag{40}$$

Finally, we establish the (ϱ, d)-denseness of G_0. In fact, from the Cauchy-Schwarz inequality we know

$$\Bigg(\sum_{e \in \binom{W}{2}} x_e\Bigg)^2 = \Bigg(\sum_{e \in \mathcal{W}_{>0}} x_e\Bigg)^2 \leqslant N \cdot \sum_{e \in \mathcal{W}_{>0}} x_e^2 = N \cdot \sum_{e \in \binom{W}{2}} x_e^2 .$$

and, consequently,

$$\begin{aligned} N &\geqslant \frac{\left(\sum_{e \in \binom{W}{2}} x_e\right)^2}{\sum_{e \in \binom{W}{2}} x_e^2} \\ &\overset{(38),(40)}{\geqslant} \frac{\left(\gamma \xi_{\mathrm{CL}} C_0^{e(F)-1} |W|^2/4\right)^2}{2(v(F)+1)^{v(F)} C_1^{2(e(F)-1)} |W|^2 / C_0'} \\ &> \frac{\gamma^2 \xi_{\mathrm{CL}}^2 C_0^{2(e(F)-1)} C_0'}{16(v(F)+1)^{v(F)} C_1^{2(e(F)-1)}} \cdot \binom{|W|}{2} \\ &\overset{(31),(35)}{=} d \cdot \binom{|W|}{2} . \end{aligned}$$

Recalling that $W \subseteq V_{a_0}$ with $|W| \geqslant \varrho |V_{a_0}|$ was arbitrary, implies that G_0 is (ϱ, d)-dense which finishes the proof. □

6. Concluding remarks

6.1. Ramsey properties for $\mathbb{Z}/n\mathbb{Z}$. The methods used here can be adjusted to obtain the sharpness for some cases of Rado's theorem for two colours in $\mathbb{Z}/n\mathbb{Z}$. For van der Waerden's theorem such a result appeared in [8] and, in fact, the work presented here relied on some of those ideas. However, the approach in [8] made use of the fact that the corresponding extremal problem (known as Szemerédi's theorem) has density 0, which limits the approach to so-called *density regular systems* (see, e.g., [5]). Maybe the simplest regular, but not density regular, instance of Rado's theorem is the well known result of Schur [22], which asserts for finite colourings of $\mathbb{Z}/n\mathbb{Z}$ the existence of a monochromatic solution for the equation $x + y = z$ for sufficiently large n. The threshold for this property appeared in [12] for two colours and in [2, 11] for an arbitrary number of colours. The sharpness for two colours is based on some of the ideas used in [8] and the work here, will appear in the PhD thesis of the second author.

6.2. **Ramsey properties of nearly partite hypergraphs.** Instead of nearly bipartite graphs one may consider *nearly k-partite k-uniform hypergraphs*, i.e., k-uniform hypergraphs with vertex partition $V_1 \dot\cup \dots \dot\cup V_k$ and the property that at most one hyperedge is contained in V_1 and the remaining hyperedges contain exactly one vertex from each vertex class. Again one may require additional *balancedness* assumptions (similar as in Theorem 2). However, for the proof of a lemma corresponding to Lemma 8 one would need a sparse version of the so-called weak regularity lemma for hypergraphs and a corresponding embedding/counting lemma for subhypergraphs of random hypergraphs (see, e.g., [3, Section 5.1]). For the more relaxed version of nearly partite, which would allow the additional hyperedge to span across more than one vertex class, one would likely need sparse analogues of the strong hypergraph regularity method for subhypergraphs of random hypergraphs.

6.3. **Ramsey properties for more general graphs and more colours.** It would be very interesting to extend Theorem 2 to more general graphs F. The class of nearly bipartite graphs contains the triangle K_3 and an extension for all cliques would be desirable. The main obstacle seems to establish a suitable analogue of Lemma 8 for this case.

Another limitation is the restriction to two colours only. The Rödl-Ruciński theorem [19] applies, up to very few exceptions (see, e.g., [15, Section 8.1]), to arbitrary graphs and any number of colours $r \geqslant 2$. However, besides for the case of trees (see [9]), all known sharpness results address only the two-colour case and extending these results to more than two colours appears an interesting open problem in the area.

Finally, we mention that due to Friedgut's criterion the $c = c(n)$ in Theorem 2 is bounded by constants, but it may depend on n. It seems plausible, that a strengthening of Theorem 2 for some constant c independent of n also holds. However, this would likely require a very different approach to these problems.

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