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THE CHROMATIC NUMBER OF FINITE TYPE-GRAPHS

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Abstract. By a *finite type-graph* we mean a graph whose set of vertices is the set of all k-subsets of $[n] = \{1, 2, \ldots, n\}$ for some integers $n \geqslant k \geqslant 1$, and in which two such sets are adjacent if and only if they realize a certain order type specified in advance. Examples of such graphs have been investigated in a great variety of contexts in the literature with particular attention being paid to their chromatic number. In recent joint work with Tomasz Łuczak, two of the authors embarked on a systematic study of the chromatic numbers of such type-graphs, formulated a general conjecture determining this number up to a multiplicative factor, and proved various results of this kind. In this article we fully prove this conjecture.

§1. Motivation

Our goal in this article is to analyze the asymptotic behaviour of the chromatic number of certain finite graphs, that are called type-graphs in the sequel. In general the vertex set of such a graphs is, for some positive integers $n \geqslant k$, the collection of all k-element subsets of the set $[n] = \{1, 2, \ldots, n\}$. Whether two such subsets are to be connected by an edge or not is decided solely in terms of the mutual position of their elements or, equivalently, it only depends on the order type that this pair of sets realizes. Before defining these type-graphs accurately, we would like to fix some notation concerning order types of pairs of ordered sets. In particular we shall encode such order types as finite sequences consisting of ones, twos, and threes. At first sight, allowing rational numbers in the definition that follows might look unnecessarily general, but it will turn out to be useful at a later occasion.

Definition 1.1. Let X and Y be two finite sets of rational numbers with $|X \cup Y| = \ell$ and $X \cup Y = \{z_1, z_2, \ldots, z_\ell\}$, these elements being listed in increasing order. We say that the *order type* of the pair (X, Y) is the sequence $\tau = (\tau_1, \ldots, \tau_\ell)$ and set $\tau(X, Y) = \tau$ if for every $i \in [\ell]$

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we have

$$\tau_i = \begin{cases} 1 & \text{if } z_i \in X \backslash Y\,, \\ 2 & \text{if } z_i \in Y \backslash X\,, \\ 3 & \text{if } z_i \in X \cap Y\,. \end{cases}$$

For example, given $X = \{1, 2, 3, 5\}$ and $Y = \{3, 4, 5\}$ we get $\tau(X, Y) = 11323$. Clearly for any finite sequence τ consisting of ones, twos, and threes there are two finite subsets X and Y of $\mathbb{Q}$ with $\tau = \tau(X, Y)$ and in fact one may even find such sets with $X, Y \subseteq \mathbb{N}$.

The case most relevant for the definition of type-graphs below is $|X| = |Y|$.

Definition 1.2. Consider two nonnegative integers k and ℓ. By a *type* of *width* k and *length* ℓ we mean the order type of a pair (X, Y) with $X, Y \subseteq \mathbb{Q}$, $|X| = |Y| = k$, and $|X \cup Y| = \ell$.

So $\tau = 123312$ is a type of width 4 and length 6 that is realized, e.g., by $X = \{1, 3, 4, 7\}$ and $Y = \{2, 3, 4, 9\}$. It is not hard to observe that in any type of width k and length ℓ there appear $\ell - k$ ones, $\ell - k$ twos, and $2k - \ell$ threes. As a degenerate case we regard the empty sequence $\varnothing$ as an *empty type* of width and length 0. A type is said to be *trivial* if it consists of threes only, or in other words if its width equals its length.

Now we are prepared to define the main objects under consideration in this article.

Definition 1.3. For a nontrivial type τ of width k and an integer $n \geqslant k$, the *type-graph* $G(n, \tau)$ is the graph with vertex set $\binom{[n]}{k}$ in which two vertices X and Y are declared to be adjacent if and only if we have $\tau(X, Y) = \tau$ or $\tau(Y, X) = \tau$.

Such graphs and their chromatic numbers have been studied in numerous articles. For example, it is known that the *shift graph* $G(n, 132)$ has chromatic number $\lceil \log(n) \rceil$, where the base of the logarithm is 2. It is straightforward to check that these shift graphs are triangle-free, and thus they provide explicit examples of triangle-free graphs with arbitrarily large chromatic number. More generally, Erdős and Hajnal [3] considered the type-graph $G(n, \sigma_k)$ with

$$\sigma_k = 1\,\underbrace{3 \ldots 3}_{k-1}\,2\,, \tag{1.1}$$

and the infinite analogues of this graph that naturally arise when one replaces the finite number n by an arbitrary cardinal number. Concerning the chromatic number of the finite graphs $G(n, \sigma_k)$ they obtained the following result that we will apply later.

Theorem 1.4 (Erdős and Hajnal)**.** *For any integer $k \geqslant 2$ we have*

$$\chi\big(G(n, \sigma_k)\big) = \big(1 + o(1)\big) \cdot \log_{(k-1)}(n)$$

as n tends to infinity.

Here for any $t \in \mathbb{N}$ and any sufficiently large $n \in \mathbb{N}$, we denote the t-fold iterated base 2 logarithm of n by $\log_{(t)}(n)$. Strictly speaking Erdős and Hajnal did mainly focus on the case where n is infinite, see [3, Lemma 2], but their method of proof applies to finite values of n as well. The thus adapted proof may be found with more details in [2] or [7]. In the latter reference, the alternative language of ordered Ramsey theory is used. We note that the infinite case of Theorem 1.4 has applications to the computation of infinite Ramsey numbers [3, Theorem 1] and refer the reader interested in further applications of infinite type-graphs to [9], [4], [6], and [5].

Another interesting consequence of Theorem 1.4 is that it provides us with explicit examples of graphs having large chromatic number and large odd girth. In fact, any odd cycle contained in $G(n, \sigma_k)$ has at least the length $2k + 1$. This line of thought was substantially continued by Nešetřil and Rödl, who used unions of general type-graphs in some of their early work on structural Ramsey theory, see e.g. [8].

The problem of determining the chromatic number of general finite type-graphs was recently approached in joint work of Łuzcak and two of the current authors [1]. The last section of that article contains a conjecture, restated as Theorem 1.8 below, that predicts this number asymptotically up to a constant multiplicative factor. In particular this conjecture implies that for each nontrivial type τ there exists a nonnegative integer β with $\chi\big(G(n, \tau)\big) = \Theta\big(\log_{(\beta)}(n)\big)$ as n tends to infinity. When intending to calculate β from τ the first thing one has to do is to express τ as a product of as many other types as possible. The next two definitions help us to talk about this process:

Definition 1.5. Given two finite sequences $\tau = (\tau_1, \dots, \tau_\ell)$ and $\tau' = (\tau'_1, \dots, \tau'_{\ell'})$ we write $\tau\tau'$ for their *concatenation* $(\tau_1, \dots, \tau_\ell, \tau'_1, \dots, \tau'_{\ell'})$.

Definition 1.6. A nonempty type is said to be *irreducible* if it cannot be written as the concatenation of two nonempty types.

It should be clear that each nonempty type τ can be written in a unique manner as the concatenation of several irreducible types. In fact, one finds this unique *factorization* of τ by keeping track of the numbers of ones and twos already encountered while reading τ from left to right, and starting a new factor at every moment where these two numbers are equal. As it will turn out, most of our work concerning $\chi\big(G(n,\tau)\big)$ addresses the irreducible case. Once it is solved, the reducible case reduces to that case.

In the next section, we describe an algorithm which partitions any given irreducible type τ into so-called blocks. Notice that if τ is trivial, i.e., a string of threes, we must have $\tau = 3$ and in this case the number of blocks is going to be 1. On the other hand, any nontrivial irreducible type is going to be partitioned into at least 2 blocks.

Our main result on irreducible types states:

Theorem 1.7. *If τ is a nontrivial irreducible type of width k with b blocks, then*

$$(1+o(1))\log_{(b-2)}\left(\tfrac{n}{k}\right) \leqslant \chi\big(G(n,\tau)\big) \leqslant \big(2^{(b-2)^2}+o(1)\big)\log_{(b-2)}(n)$$

and hence

$$\chi\big(G(n,\tau)\big) = \Theta\big(\log_{(b-2)}(n)\big)\,.$$

More generally we shall obtain the following:

Theorem 1.8. *Let $\tau = \varrho_1\varrho_2\cdot\ldots\cdot\varrho_t$ be the factorization of an arbitrary nontrivial type τ into irreducible types. Suppose that ϱ_i has b_i blocks for $i \in [t]$, and set $b^* = \max(b_1,\ldots,b_t)$. Then we have*

$$\chi\big(G(n,\tau)\big) = \Theta\big(\log_{(b^*-2)}(n)\big)\,.$$

The rest of this article is structured as follows: In Section 2 we describe the block algorithm and thus clarify the meaning of our main results. Then the next two sections are dedicated to the proofs of the lower and upper bounds appearing in Theorem 1.7. Finally, in Section 5 we will deduce Theorem 1.8 by means of a product argument.

§2. The block algorithm

In this section we describe an algorithm partitioning the terms of any irreducible type τ into blocks of consecutive terms. We will call this algorithm the *block algorithm* and the partition it produces will be referred to as the *block decomposition* of τ.

As said above, if τ is trivial we have $\tau = 3$ by irreducibility. In this special case we regard τ as consisting of one block only, namely τ itself. If $\tau \neq 3$, then the first digit of τ is either a one or a two, because otherwise we could write $\tau = 3\varrho$ for some type $\varrho \neq \varnothing$, contrary to the irreducibility of τ. We call τ *primary* if it starts with a one and *secondary* if it starts with a two.

Given a subsequence B of a type τ that consists of consecutive terms, we write $\mathbf{1}(B)$ for the total number of ones and threes occurring in B, and $\mathbf{2}(B)$ for the total number of twos and threes in B.

Now we are ready to explain how the block algorithm is applied to any primary irreducible type τ. Processing τ from left to right we are to perform the following steps:

(*i*) The first block B_1 consists of all the initial ones appearing in τ.

(*ii*) In general, if the block B_i has just been constructed, the next block B_{i+1} consists of the next consecutive digits of τ such that $\mathbf{2}(B_{i+1}) = \mathbf{1}(B_i)$ and such that subject to this condition the block B_{i+1} is as long as possible.

(*iii*) The algorithm stops when all the terms of τ have been placed in a block.

E.g., for the type $\tau = 1121112121212222$ we get $B_1 = 11$, $B_2 = 211121$, $B_3 = 212122$, and finally $B_4 = 22$. One may use appropriate spacing to make the outcome of the block algorithm notationally visible and write, for instance,

$$\tau = 11 \;\; 211121 \;\; 212122 \;\; 22\,.$$

Similarly the type 131122311222 decomposes into

$$1 \;\; 311 \;\; 22311 \;\; 222$$

and for the type $\sigma_4 = 13332$ that we have already encountered in (1.1) the algorithm produces

$$\sigma_4 = 1 \;\; 3 \;\; 3 \;\; 3 \;\; 2\,.$$

Fact 2.1. *When applied to a primary irreducible type τ the block algorithm does indeed provide a factorization $\tau = B_1 B_2 \cdot \ldots \cdot B_b$ of τ into some nonempty blocks $B_1, \ldots, B_b$, where $b \geqslant 2$. Moreover, we have $\mathbf{1}(B_b) = 0$.*

Proof. Since τ starts with a one, rule (*i*) gives us a first block $B_1 \neq \varnothing$. Now let i be the largest integer for which the block algorithm produces in its first i steps some nonempty blocks $B_1, \ldots, B_i$. This happens by an initial application of (*i*) followed by $i-1$ applications of (*ii*). Let C denote the finite sequence satisfying

$$\tau = B_1 \cdot \ldots \cdot B_i C\,. \tag{2.1}$$

We intend to show that either $C = \varnothing$ so that the algorithm stops, or $0 < \mathbf{1}(B_i) \leqslant \mathbf{2}(C)$, meaning that the algorithm produces a further nonempty block B_{i+1}. The latter alternative, however, would contradict the maximality of i.

Recall that by construction we have $\mathbf{2}(B_1) = 0$ and $\mathbf{1}(B_j) = \mathbf{2}(B_{j+1})$ for all $j \in [i-1]$. This yields

$$\mathbf{1}(B_1 \cdot \ldots \cdot B_{i-1}) = \mathbf{2}(B_1 \cdot \ldots \cdot B_i) \tag{2.2}$$

and in combination with (2.1) and $\mathbf{1}(\tau) = \mathbf{2}(\tau)$ it follows that we have $\mathbf{1}(B_i) \leqslant \mathbf{1}(B_iC) = \mathbf{2}(C)$. So if $\mathbf{1}(B_i) > 0$ we could use *(ii)* once more to obtain the next nonempty block B_{i+1}, contrary to the maximality of i.

Thus we must have $\mathbf{1}(B_i) = 0$ and (2.2) entails that $B_1 \cdot \ldots \cdot B_i$ is a type. By (2.1) and the irreducibility of τ it follows that $C = \varnothing$, meaning that the algorithm stops with a final application of rule *(iii)*. Now $b = i$, the moreover-part was obtained at the beginning of this paragraph, and $b \geqslant 2$ is clear. □

So far we have only talked about primary types. For dealing with secondary types we use the following symmetry: If τ denotes any finite sequence of ones, twos, and threes, we write τ' for the sequence obtained from τ by replacing all ones by twos and vice versa. Evidently if τ is a secondary irreducible type, then τ' is a primary irreducible type and thus we already know how to find its block decomposition $\tau' = B_1B_2 \cdot \ldots \cdot B_b$. Now we have $\tau = B_1'B_2' \cdot \ldots \cdot B_b'$ and we define this to be the block decomposition of τ. In particular, τ and τ' have the same numbers of blocks.

Notice that if $\tau(X, Y) = \varrho$ holds for some finite sets $X, Y \subseteq \mathbb{Q}$, then $\tau(Y, X) = \varrho'$ follows. In particular, for any type τ the two type-graphs $G(n, \tau)$ and $G(n, \tau')$ are the same and thus it suffices to prove Theorem 1.7 for primary τ.

We conclude this section with two statements concerning irreducible types and the block algorithm that will be employed in Section 4.

Lemma 2.2. *Suppose that τ is a primary irreducible type of width k and that $X, Y \subseteq \mathbb{Q}$ are two finite sets with $\tau = \tau(X, Y)$. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, the elements being listed in increasing order. Then we have*

(a) $x_i < y_i$ for all $i \in [k]$
(b) and $x_{i+1} \leqslant y_i$ for all $i \in [k-1]$.

Proof. Let $\tau = (\tau_1, \ldots, \tau_\ell)$, where ℓ denotes the length of τ. We contend that

$$\text{if } i \in [k-1] \text{ and } x_i \leqslant y_i, \text{ then } x_{i+1} \leqslant y_i. \tag{2.3}$$

To show this, let y_i be the m-th element in the increasing enumeration of $X \cup Y$. In view of $1 \leqslant i < k$ we have $1 \leqslant m < \ell$ and thus $(\tau_1, \ldots, \tau_m)$ cannot be a type due to the irreducibility of τ. This in turn yields $|X \cap (-\infty, y_i]| \neq |Y \cap (-\infty, y_i]| = i$. But assuming $x_i \leqslant y_i$ the number $|X \cap (-\infty, y_i]|$ is at least i, so that altogether it must be at least $i+1$, which means that $x_{i+1} \leqslant y_i$. This proves (2.3).

Next we show *(a)* by induction on i. The base case $x_1 < y_1$ follows from τ being primary. For the induction step we suppose that $x_i < y_i$ holds for some $i < k$. Then (2.3) entails $x_{i+1} \leqslant y_i < y_{i+1}$, which concludes the argument.

Finally *(b)* is an immediate consequence of (2.3) and *(a)*. □

We now come to the only place in the proof of Theorem 1.7 where the demand from the second rule of the block algorithms that the blocks should end with as many ones as possible is utilized. The purpose of the following lemma is that, roughly speaking, it tells us how the "blocks" of two finite sets X and Y realizing an irreducible type τ overlap each other. This will be useful in Subsection 4.1 for embedding $G(n, \tau)$ into an auxiliary graph whose chromatic number is easier to bound from above.

Lemma 2.3. *Let $\tau = B_1 B_2 \cdot \ldots \cdot B_b$ be the block decomposition of some primary irreducible type whose width is k and set $s(i) = \mathbf{2}(B_1 \cdot \ldots \cdot B_i)$ for all $i \in [b]$. Then for any two sets X and Y satisfying $\tau = \tau(X, Y)$, say $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ with the elements listed in increasing order, we have $x_{s(i+1)} < y_{s(i)+1} \leqslant x_{s(i+1)+1}$ for all $i \in [b-2]$.*

Proof. Let $X \cup Y = \{z_1, \ldots, z_\ell\}$, the elements again being listed in increasing order. Fix any $i \in [b-2]$ and set $\beta = \sum_{j=1}^{i} |B_j|$. By rule *(ii)* of the block algorithm the block B_{i+1} cannot start with a one and thus we have $z_{\beta+1} \in Y$. In combination with

$$s(i) = \mathbf{2}(B_1 \cdot \ldots \cdot B_i) = |Y \cap (-\infty, z_\beta]|$$

this yields

$$y_{s(i)+1} = z_{\beta+1} \,. \tag{2.4}$$

Similarly we have

$$s(i+1) = \mathbf{2}(B_1 \cdot \ldots \cdot B_{i+1}) = \mathbf{1}(B_1 \cdot \ldots \cdot B_i) = |X \cap (-\infty, z_\beta]|$$

and thus $x_{s(i+1)} \leqslant z_\beta$ as well as $z_{\beta+1} \leqslant x_{s(i+1)+1}$. The desired conclusion follows from these two estimates and (2.4). □

§3. The lower bound – uncolourability

In this section we shall prove the lower bound from Theorem 1.7. So we intend to show that a certain graph $G(n,\tau)$ cannot be coloured with a certain "small" number of colours. Recall that for any graph H and any natural number r, the statement $\chi(H) > r$ means the same as saying that there is no graph homomorphism from H to the r-clique K_r. Thus one strategy to prove such an uncolourability statement is to exhibit a homomorphism from some auxiliary graph G to H, with $\chi(G) > r$ already being known. So in the light of Theorem 1.4 our task reduces to:

Proposition 3.1. *For every nontrivial irreducible type τ of width k with b blocks and every integer $n \geqslant b$ there is a graph homomorphism*

$$\varphi\colon G(n,\sigma_{b-1}) \longrightarrow G(kn,\tau)\,.$$

For the construction of such a homomorphism, we will make use of the following

Fact 3.2. *If B denotes a finite sequence of ones, twos, and threes, and $Y \subseteq \mathbb{Q}$ has size $\mathbf{2}(B)$, then there is a set $X \subseteq \mathbb{Q}$ with $\tau(X,Y) = B$.*

This can easily be shown by induction on the number of ones appearing in B and we leave the details to the reader.

Proof of Proposition 3.1. As said above we may assume that τ is primary. Let

$$\tau = B_1 B_2 \cdot \ldots \cdot B_b$$

be the block decomposition of τ. We commence by defining recursively an auxiliary sequence $R_0, R_1, \ldots, R_b$ of finite subsets of $\mathbb{Q}$ with

$$|R_{i-1}| = \mathbf{2}(B_i) \qquad \text{for all } i \in [b]\,. \tag{3.1}$$

Since B_1 consists exclusively of ones, such a sequence needs to start with $R_0 = \varnothing$. Once R_{i-1} has been defined for some $i \in [b]$, we use Fact 3.2 to obtain a set $R_i \subseteq \mathbb{Q}$ satisfying $\tau(R_i, R_{i-1}) = B_i$. Notice that for $i < b$ this yields $|R_i| = \mathbf{1}(B_i) = \mathbf{2}(B_{i+1})$, so that the construction may be continued. We also get $|R_b| = \mathbf{1}(B_b) = 0$ and hence $R_b = \varnothing$ from Fact 2.1.

In view of (3.1) we have

$$\sum_{i=0}^{b-1} |R_i| = \sum_{i=1}^{b} \mathbf{2}(B_i) = \mathbf{2}(\tau) = k \tag{3.2}$$

and thus there exist k rational numbers $\alpha_1 < \ldots < \alpha_k$ with

$$\bigcup_{0\leqslant i<b} R_i \subseteq \{\alpha_1, \ldots, \alpha_k\}\,.$$

Pulling this situation back to $[k]$ we define $R_i^* = \{j \in [k] \,|\, \alpha_j \in R_i\}$ for all $i \in [b-1]$ as well as $R_0^* = R_b^* = \varnothing$. The main properties of these sets are

$$R_i^* \subseteq [k] \quad \text{and} \quad \tau(R_i^*, R_{i-1}^*) = B_i \quad \text{for all } i \in [b]\,. \tag{3.3}$$

Now we are ready to define the requested map

$$\varphi\colon \binom{[n]}{b-1} \longrightarrow \binom{[kn]}{k}\,.$$

Given any integers h_i for $i \in [b-1]$ with $1 \leqslant h_1 < \ldots < h_{b-1} \leqslant n$ we set

$$\varphi\big(\{h_1, \ldots, h_{b-1}\}\big) = \bigcup_{i\in[b-1]} \big\{(h_i - 1)k + j \,|\, j \in R_i^*\big\}\,.$$

Due to $R_i^* \subseteq [k]$ the right-hand side of this formula is indeed a subset of $[kn]$ and by (3.2) its size is k. It remains to check that φ maps edges of $G(n, \sigma_{b-1})$ to edges of $G(kn, \tau)$. To this end let any integers h_i for $i \in [b]$ with $1 \leqslant h_1 < \ldots < h_b \leqslant n$ be given. Then by (3.3) we have

$$\begin{aligned}\tau\Big(\varphi\big(\{h_1, \ldots, h_{b-1}\}\big), \varphi\big(\{h_2, \ldots, h_b\}\big)\Big) &= \tau(R_1^*, R_0^*) \cdot \tau(R_2^*, R_1^*) \cdot \ldots \cdot \tau(R_b^*, R_{b-1}^*) \\ &= B_1 B_2 \cdot \ldots \cdot B_b = \tau\,,\end{aligned}$$

as desired. □

§4. The upper bound – constructing colourings

This entire section is dedicated to the proof of the upper bound from Theorem 1.7. The strategy we use is to embed the type-graph $G(n, \tau)$ into some other graph $G_{b-1}(n)$ that depends solely on b and n but not on τ itself. Thereby the task we are to perform gets reduced to the problem of colouring these auxiliary graphs with "few" colours and it seems that this new problem is more susceptible to an inductive treatment than the old one.

4.1. **Embedding type-graphs.** We begin by defining the auxiliary graphs $G_b(n)$ mentioned above.

Definition 4.1. For any positive integers b and n we set

$$W_b(n) = \{(x_1, \ldots, x_{2b-1}) \mid 1 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{2b-1} \leqslant n\}$$

and

$$V_b(n) = \{(x_1, \ldots, x_{2b-1}) \in W_b(n) \mid x_1 < x_3 < \ldots < x_{2b-1}\}\,.$$

By $G_b(n)$ we mean the graph with vertex set $V_b(n)$ in which an unordered pair $e \subseteq V_b(n)$ is declared to be an edge if we can write $e = \{\vec{x}, \vec{y}\}$, $\vec{x} = (x_1, \ldots, x_{2b-1})$, and $\vec{y} = (y_1, \ldots, y_{2b-1})$ such that

(i) $x_1 < y_1 \leqslant x_3 < y_3 \leqslant \ldots \leqslant x_{2b-1} < y_{2b-1}$
(ii) and $x_{j+1} \leqslant y_j$ for $j \in [2b-2]$.

It should perhaps be observed that the conditions (i) and (ii) from this definition do not determine uniquely how the elements of the multiset $\{x_1, \ldots, x_{2b-1}\} \cup \{y_1, \ldots, y_{2b-1}\}$ are ordered. This makes it more plausible, of course, that many type-graphs embed homomorphically into $G_b(n)$ and in fact we have

Theorem 4.2. *For any nontrivial irreducible type τ with $b \geqslant 2$ blocks and every positive integer n there is a graph homomorphism $\varphi\colon G(n, \tau) \longrightarrow G_{b-1}(n)$.*

Proof. As usual we may assume that τ is a primary type of width k, say. Let $\tau = B_1 B_2 \cdot \ldots \cdot B_b$ be its block decomposition and define $s(i) = \mathbf{2}(B_1 \cdot \ldots \cdot B_i)$ for any $i \in [b]$. Since

$$0 = s(1) < s(2) < \ldots < s(b) = k\,,$$

there is a map

$$\varphi\colon \binom{[n]}{k} \longrightarrow V_{b-1}(n)$$

given by

$$\varphi\big(\{x_1, \ldots, x_k\}\big) = \big(x_{s(1)+1}, x_{s(2)}, x_{s(2)+1}, \ldots, x_{s(b-1)}, x_{s(b-1)+1}\big)\,,$$

whenever $1 \leqslant x_1 < \ldots < x_k \leqslant n$. So roughly speaking φ remembers where the "blocks" of such a set $\{x_1, \ldots, x_k\}$ start and end and forgets everything else.

It remains to verify that φ sends edges of $G(n, \tau)$ to edges of $G_{b-1}(n)$. For this purpose let any two vertices X and Y of $G(n, \tau)$ with $\tau(X, Y) = \tau$ be given and write $X = \{x_1, \ldots, x_k\}$ as well as $Y = \{y_1, \ldots, y_k\}$, listing the elements in increasing order. We need to show that $\{\varphi(X), \varphi(Y)\}$ is an edge of $G_{b-1}(n)$, i.e., that the clauses (i) and (ii) from Definition 4.1 are satisfied.

Now by Lemma 2.2 *(a)* we have in particular $x_{s(i)+1} < y_{s(i)+1}$ for all $i \in [b-1]$ and Lemma 2.3 tells us that $y_{s(i)+1} \leqslant x_{s(i+1)+1}$ holds for all $i \in [b-2]$. Both statements together yield condition *(i)* from Definition 4.1.

For the verification of *(ii)* we consider the cases that the index j appearing there is odd or even separately. To deal with the case where j is odd we need to check that $x_{s(i+1)} \leqslant y_{s(i)+1}$ holds for all $i \in [b-2]$ and Lemma 2.3 informs us that this is indeed true. For even j we need that $x_{s(i+1)+1} \leqslant y_{s(i+1)}$ holds for all $i \in [b-2]$ and this was obtained in Lemma 2.2 *(b)*. □

Now it is clear that in order to complete the proof of Theorem 1.7 we just need to establish the following result.

Theorem 4.3. *For every positive integer b we have*

$$\chi\big(G_b(n)\big) \leqslant \big(2^{(b-1)^2} + o(1)\big) \log_{(b-1)}(n)\,.$$

Throughout the rest of this section we deal with the proof this theorem. We will proceed by induction on b, considering the base cases $b = 1$ and $b = 2$ separately. The main idea for the induction step is to relate the graphs $G_b(2^n)$ to $G_{b-1}(n)$ to each other. Roughly speaking, we will show that for any $b \geqslant 3$ the vertex set of the graph $G_b(2^n)$ may be split into about 2^{2b-3} pieces, each of which induces a graph that embeds homomorphically into $G_{b-1}(n)$. For the construction of half of these homomorphisms it will be helpful to bear the following symmetry in mind.

Fact 4.4. *For any positive integers b and n the bijection $\eta\colon V_b(2^n) \longrightarrow V_b(2^n)$ given by*

$$(x_1, x_2, \dots, x_{2b-1}) \longmapsto \big((2^n+1) - x_{2b-1}, (2^n+1) - x_{2b-2}, \dots, (2^n+1) - x_1\big)$$

is an automorphism of $G_b(2^n)$.

We leave the easy proof of this assertion to the reader.

4.2. **Colouring the auxiliary graphs $G_b(n)$.** Clearly the graph $G_1(n)$ is nothing else than a clique with n vertices. Thus we have

$$\chi\big(G_1(n)\big) = n \qquad \text{for every positive integer } n\,. \tag{4.1}$$

The case $b = 2$ of Theorem 4.3 is technically a lot easier than the general case and thus we would like to treat is separately.

Lemma 4.5. *We have $\chi\big(G_2(n)\big) \leqslant 2\lceil\log(n)\rceil - 1$ for all integers $n \geqslant 2$.*

Proof. Clearly it suffices to show $\chi\big(G_2(2^k)\big) \leqslant 2k-1$ for all positive integers k and we shall do so by induction on k. The base case $k=1$ poses no difficulty because the graph $G_2(2)$ just consists of two isolated vertices. To handle the induction step it is enough to show

$$\chi\big(G_2(2m)\big) \leqslant \chi\big(G_2(m)\big) + 2 \quad \text{for all } m \geqslant 2\,. \tag{4.2}$$

Bearing this goal in mind we partition the vertex set of $G_2(2m)$ into the four classes

$$\begin{aligned} A &= \big\{(x,y,z) \in V_2(2m) \,|\, z \leqslant m\big\}\,, \\ B &= \big\{(x,y,z) \in V_2(2m) \,|\, y \leqslant m < z\big\}\,, \\ C &= \big\{(x,y,z) \in V_2(2m) \,|\, x \leqslant m < y\big\}\,, \\ \text{and} \qquad D &= \big\{(x,y,z) \in V_2(2m) \,|\, m < x\big\}\,. \end{aligned}$$

We also identify subsets of $V_2(2m)$ with the subgraphs of $G_2(2m)$ that they induce. Evidently A is the same as $G_2(m)$, the map $(x,y,z) \longmapsto (x+m, y+m, z+m)$ provides an isomorphism between A and D, and there are no edges between A and D. Therefore $A \cup D$ is a disjoint union of two copies of $G_2(m)$ and we have $\chi(A\cup D) = \chi\big(G_2(m)\big)$. Moreover, using condition *(ii)* from Definition 4.1 it is easy to check that the sets B and C are independent. This concludes the proof of (4.2) and, thus, the proof of Lemma 4.5. □

Before we proceed to the colouring of $G_b(2^n)$ for $b \geqslant 3$ we introduce some auxiliary functions.

Lemma 4.6. *Given any integers x and y with $1 \leqslant x < y$ there exist a positive integer f and an odd positive integer q such that*

$$(q-1)\cdot 2^{f-1} < x \leqslant q \cdot 2^{f-1} < y \leqslant (q+1)\cdot 2^{f-1}\,.$$

Moreover, f and q are uniquely determined by x and y so that we may write $f = f(x,y)$ as well as $q = q(x,y)$.

Proof. Let us first prove the existence of f and q. To this end, we pick an integer n with $y \leqslant 2^n$. Then we expand $x-1$ and $y-1$ in the binary system using n digits and allowing leading zeros. Say that this yields $x - 1 = x_{n-1}\dots x_1x_0$ and $y-1 = y_{n-1}\dots y_1 y_0$. Next we compare these expansions from left to right and let $x_{f-1} \neq y_{f-1}$ be the first place where they differ. Notice that $x<y$ entails $x_{f-1} = 0$ and $y_{f-1} = 1$. Finally we let q be the number with binary representation $q = x_{n-1}\dots x_f 1$.

So formally we have

$$x-1=\sum_{i=0}^{n-1} x_i\cdot 2^i\,,\quad y-1=\sum_{i=0}^{n-1} y_i\cdot 2^i\,,\quad q=1+\sum_{i=1}^{n-f} x_{f+i-1}2^i$$

and $x_j = y_j$ for $j \in [f, n-1]$. Clearly, q is odd and

$$(q-1)\cdot 2^{f-1} \leqslant x-1 < q\cdot 2^{f-1} \leqslant y-1 < (q+1)\cdot 2^{f-1}\,,$$

wherefore f and q are as desired.

	2^{n-1}	2^{n-2}	$\dots$	2^f	2^{f-1}	2^{f-2}	$\dots$	1
$x-1$	x_{n-1}	x_{n-2}	$\dots$	x_f	0	x_{f-2}	$\dots$	x_0
$y-1$	x_{n-1}	x_{n-2}	$\dots$	x_f	1	y_{f-2}	$\dots$	y_0
$q\cdot 2^{f-1}$	x_{n-1}	x_{n-2}	$\dots$	x_f	1	0	$\dots$	0

The uniqueness of f and q may likewise by shown by studying the binary expansions of $x-1$ and $y-1$. An alternative argument proceeds as follows:

Given x and y, let (f, q) and (f', q') be two pairs with the requested properties. Due to symmetry we may suppose $f \leqslant f'$. Now we have $(q-1)\cdot 2^{f-1} < x \leqslant q'\cdot 2^{f'-1}$ and consequently $q \leqslant q'\cdot 2^{f'-f}$. Similarly $q'\cdot 2^{f'-1} < y \leqslant (q+1)2^{f-1}$ yields $q'\cdot 2^{f'-f} \leqslant q$. The combination of both estimates reveals $q = q'\cdot 2^{f'-f}$ but, since q is odd, this if only possible if $f = f'$ and $q = q'$. ☐

We would like to point out that the uniqueness of f and q is several times going to be essential in the arguments that follow. By redoing the above proof of this uniqueness more carefully one can show the following monotonicity property of the function f.

Lemma 4.7. *For any three positive integers x, y, and z such that $x < y \leqslant z$ the inequality $f(x, y) \leqslant f(x, z)$ holds.*

Proof. For brevity we set $f = f(x, y)$, $q = q(x, y)$, $f' = f(x, z)$, and $q' = q(x, z)$. Arguing indirectly we assume $f' < f$. Now $(q'-1)\cdot 2^{f'-1} < x \leqslant q\cdot 2^{f-1}$ entails $q' \leqslant q\cdot 2^{f-f'}$ and similarly $q\cdot 2^{f-1} \leqslant y \leqslant z < (q'+1)\cdot 2^{f'-1}$ leads to $q\cdot 2^{f-f'} \leqslant q'$. Hence we must have $q' = q\cdot 2^{f-f'}$, contrary to the fact that q' is odd. ☐

The following will be a standard argument later on.

Lemma 4.8. *For any positive integers $x < y \leqslant z$ and f with $f = f(x, y) = f(x, z)$ we have*

$$(q-1)\cdot 2^{f-1} < x \leqslant q\cdot 2^{f-1} < y \leqslant z \leqslant (q+1)\cdot 2^{f-1}\,,$$

where $q = q(x, y) = q(x, z)$.

Proof. Define $q = q(x, y)$. Lemma 4.6 gives

$$(q-1) \cdot 2^{f-1} < x \leqslant q \cdot 2^{f-1} < y \leqslant (q+1) \cdot 2^{f-1}$$

and thus $q \cdot 2^{f-1}$ is the least multiple of 2^{f-1} which is at least x. Due to $f = f(x, z)$ this yields $q(x, z) = q$ and hence $z \leqslant (q+1) \cdot 2^{f-1}$. □

Next we record another property of f that shall be utilized later.

Lemma 4.9. *If four positive integers t, x, y, and z satisfy $t \leqslant x < y \leqslant z$ and $f(x, y) = f(x, z)$, then $f(t, y) = f(t, z)$ holds as well.*

Proof. Setting $f = f(t, z)$ and $q = q(t, z)$ we get

$$(q-1) \cdot 2^{f-1} < t \leqslant q \cdot 2^{f-1} < z \leqslant (q+1) \cdot 2^{f-1}$$

from the definition of these quantities.

Of course the claim would easily follow from $q \cdot 2^{f-1} < y$. So from now on we may assume $y \leqslant q \cdot 2^{f-1}$ towards contradiction. This yields

$$(q-1) \cdot 2^{f-1} < t \leqslant x < y \leqslant q \cdot 2^{f-1} < z \leqslant (q+1) \cdot 2^{f-1},$$

and in particular we obtain $f(x, z) = f$ but $f(x, y) \neq f$, thus reaching a contradiction. □

To conclude our dicussion of the auxiliary functions f and q we state how they interact with the map η introduced in Fact 4.4.

Fact 4.10. *For any integers x and y with $1 \leqslant x < y \leqslant 2^n$ we have*

$$\begin{aligned} f(x, y) &\in [n], \\ f(2^n + 1 - y, 2^n + 1 - x) &= f(x, y), \\ \textit{and} \quad q(2^n + 1 - y, 2^n + 1 - x) &= 2^{n+1-f} - q(x, y). \end{aligned}$$

Again we leave the straightforward verification to the reader. We may now return to the problem of colouring the graphs $G_b(2^n)$.

Proposition 4.11. *We have*

$$\chi\big(G_b(2^n)\big) \leqslant (2b-6) + 2^{2b-3}\,\chi\big(G_{b-1}(n)\big)$$

for any integers $n \geqslant b \geqslant 3$.

Proof. For any vertex $\vec{x} = (x_1, x_2, \ldots, x_{2b-1})$ of $G_b(2^n)$ we use the abbreviations

$$\begin{aligned} f(\vec{x}) &= f(x_1, x_{2b-1})\,, \\ q(\vec{x}) &= q(x_1, x_{2b-1})\,, \\ T^-(\vec{x}) &= \big(q(\vec{x}) - 1\big) \cdot 2^{f(\vec{x})-1}\,, \\ T(\vec{x}) &= q(\vec{x}) \cdot 2^{f(\vec{x})-1}\,, \\ \text{and} \qquad T^+(\vec{x}) &= \big(q(\vec{x}) + 1\big) \cdot 2^{f(\vec{x})-1}\,. \end{aligned}$$

Recall that by Lemma 4.6 we have

$$T^-(\vec{x}) < x_1 \leqslant T(\vec{x}) < x_{2b-1} \leqslant T^+(\vec{x}) \tag{4.3}$$

for any such vertex $\vec{x}$ and in the first steps of the current proof we will distinguish these vertices according to the position of their other entries x_i with respect to $T(\vec{x})$. To begin with, we partition $V_b(2^n)$ into three sets,

$$V_b(2^n) = A \cup B \cup C\,, \tag{4.4}$$

that are defined by

$$\begin{aligned} A &= \big\{\vec{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, x_{2b-3} \leqslant T(\vec{x})\big\}\,, \\ B &= \big\{\vec{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, x_3 \leqslant T(\vec{x}) < x_{2b-3}\big\}\,, \\ \text{and} \qquad C &= \big\{\vec{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, T(\vec{x}) < x_3\big\}\,. \end{aligned}$$

Again we identify subsets of $V_b(2^n)$ with the corresponding induced subgraphs of $G_b(2^n)$. We will use different colours for these three sets and commence by colouring B. This set may be partitioned further into

$$B = B_3 \cup B_4 \cup \ldots \cup B_{2b-4}\,,$$

where

$$B_i = \big\{\vec{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, x_i \leqslant T(\vec{x}) < x_{i+1}\big\}$$

for any integer index $i \in [3, 2b-4]$. We claim that each of these $2b-6$ sets is independent. To show this suppose that $\{\vec{x}, \vec{y}\}$ was an edge of $G_b(2^n)$ with $\vec{x}, \vec{y} \in B_i$ for some $i \in [3, 2b-4]$. Let the notation be as in Definition 4.1. By $\vec{x} \in B$ and the inequalities 4.1(i) and (4.3) we have

$$T^-(\vec{x}) < x_1 < y_1 \leqslant x_3 \leqslant T(\vec{x}) < x_{2b-3} < y_{2b-3} \leqslant x_{2b-1} \leqslant T^+(\vec{x})\,,$$

whence $f(y_1, y_{2b-3}) = f(\vec{x})$ and $q(y_1, y_{2b-3}) = q(\vec{x})$. Due to $\vec{y} \in B$ this yields $f(\vec{y}) = f(\vec{x})$ and $q(\vec{y}) = q(\vec{x})$. For this reason $\vec{x}, \vec{y} \in B_i$ implies $y_i \leqslant T(\vec{y}) = T(\vec{x}) < x_{i+1}$, contrary to 4.1*(ii)*. So the sets B_i are indeed independent and we obtain

$$\chi(B) \leqslant 2b - 6\,. \tag{4.5}$$

This accounts for the summand $2b-6$ on the right-hand side of our claim and we may proceed with analyzing A and C. Using Fact 4.10 it is not hard to check that the map η from Fact 4.4 constitutes an isomorphism between A and C, wherefore

$$\chi(A) = \chi(C)\,. \tag{4.6}$$

Now by (4.4), (4.5), and (4.6) we have

$$\chi\big(G_b(2^n)\big) \leqslant \chi(A) + \chi(B) + \chi(C) \leqslant (2b-6) + 2\,\chi(A)$$

and thus to finish the current proof we just need to show

$$\chi(A) \leqslant 2^{2b-4}\,\chi\big(G_{b-1}(n)\big)\,. \tag{4.7}$$

The main idea for proving this is to split A into at most 2^{2b-4} further sets, each of which is either independent or has the property of being homomorphically mapped into $G_{b-1}(n)$ by a certain function φ that is to be introduced next. Observe that by the first statement from Fact 4.10 and by Lemma 4.7 there is a map $\varphi\colon A \longrightarrow W_{b-1}(n)$ defined by

$$\varphi(x_1, x_2, \dots, x_{2b-1}) = \big(f(x_1, x_3), f(x_1, x_4), \dots, f(x_1, x_{2b-1})\big)$$

for any $(x_1, x_2, \dots, x_{2b-1}) \in A$. We call two vertices $\vec{x} = (x_1, \dots, x_{2b-1})$ and $\vec{y} = (y_1, \dots, y_{2b-1})$ from A equivalent and write $\vec{x} \sim \vec{y}$ if for any integer $i \in [3, 2b-2]$ we have

$$f(x_1, x_i) = f(x_1, x_{i+1}) \quad\Longleftrightarrow\quad f(y_1, y_i) = f(y_1, y_{i+1})\,.$$

It is plain that equivalence is an equivalence relation and that the number of its equivalence classes is at most 2^{2b-4}. Thus to conclude the proof of (4.7) we just need to verify the following statement:

$$\text{If } \vec{x}, \vec{y} \in A,\ \vec{x} \sim \vec{y}, \text{ and } \{\vec{x}, \vec{y}\} \in E\big(G_b(2^n)\big), \text{ then } \{\varphi(\vec{x}), \varphi(\vec{y})\} \in E\big(G_{b-1}(n)\big)\,. \tag{4.8}$$

So let any two equivalent vertices $\vec{x}$ and $\vec{y}$ from A be given and suppose that they are connected by an edge of $G_b(2^n)$, the notation for this being as in Definition 4.1. For any $i \in [2b-3]$ we set

$$\alpha_i = f(x_1, x_{i+2}) \quad \text{and} \quad \beta_i = f(x_1, y_{i+2})\,. \tag{4.9}$$

Notice that there is no misprint in the last formula – it is true that $\beta_i = f(y_1, y_{i+2})$ holds as well, and actually this fact is very relevant to our main concern, but it will only be shown at a rather late moment of our argument.

Combining the assumption that $\{\vec{x}, \vec{y}\}$ be an edge of $G_b(2^n)$ with Lemma 4.7 we infer

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_3 \leqslant \beta_3 \leqslant \ldots \leqslant \alpha_{2b-3} \leqslant \beta_{2b-3} \tag{4.10}$$

as well as

$$\alpha_{j+1} \leqslant \beta_j \text{ for } j \in [2b-4]\,. \tag{4.11}$$

Next we would like to show

$$\alpha_{2b-3} < \beta_{2b-3}\,. \tag{4.12}$$

Assume contrariwise that $\alpha_{2b-3} = \beta_{2b-3}$, i.e., $f(\vec{x}) = f(x_1, y_{2b-1})$. Lemma 4.8 yields

$$T^-(\vec{x}) < x_1 \leqslant T(\vec{x}) < x_{2b-1} < y_{2b-1} \leqslant T^+(\vec{x})\,,$$

so in combination with $\{\vec{x}, \vec{y}\}$ being an edge and with $\vec{x} \in A$ we obtain

$$T^-(\vec{x}) < x_1 < y_1 \leqslant x_{2b-3} \leqslant T(\vec{x}) \leqslant x_{2b-1} \leqslant y_{2b-2} \leqslant y_{2b-1} \leqslant T^+(\vec{x})\,.$$

It follows that $T(\vec{y}) = T(\vec{x})$ and $f(y_1, y_{2b-2}) = f(y_1, y_{2b-1}) = f(\vec{x})$. Using $\vec{x} \sim \vec{y}$ we may deduce $f(x_1, x_{2b-2}) = f(x_1, x_{2b-1})$. Now Lemma 4.8 shows that $q(x_1, x_{2b-2}) = q(x_1, x_{2b-1})$ holds as well and consequently we have $T(\vec{x}) < x_{2b-2} \leqslant y_{2b-3}$. Thus we get a contradiction to $\vec{y} \in A$, whereby (4.12) is proved.

Extending this result we contend that more generally we have

$$\alpha_i < \beta_i \quad \text{for all } i \in [2b-3]\,. \tag{4.13}$$

Arguing indirectly again, we let i denote the largest counterexample to this claim. Notice that (4.12) tells us $i \leqslant 2b-4$. Set $q = q(x_1, x_{i+2})$, $T^- = (q-1)\cdot 2^{\alpha_i - 1}$, $T = q \cdot 2^{\alpha_i-1}$, and $T^+ = (q+1)\cdot 2^{\alpha_i-1}$. Due to Lemma 4.8 our indirect assumption $\alpha_i = \beta_i$ entails

$$T^- < x_1 \leqslant T < x_{i+2} \leqslant y_{i+2} \leqslant T^+\,,$$

which in combination with $x_{i+2} \leqslant x_{i+3} \leqslant y_{i+2}$ shows $f(x_1, x_{i+2}) = f(x_1, x_{i+3})$. Now $\vec{x} \sim \vec{y}$ discloses $f(y_1, y_{i+2}) = f(y_1, y_{i+3})$ and by Lemma 4.9 it follows that $f(x_1, y_{i+2}) = f(x_1, y_{i+3})$. Using Lemma 4.8 again we obtain

$$T^- < x_1 \leqslant T < x_{i+3} \leqslant y_{i+3} \leqslant T^+$$

and thus $\alpha_{i+1} = \beta_{i+1}$, contrary to the maximality of i. Thereby (4.13) is proved as well.

Now we are ready to confirm the alternative definition of β_i announced above. That is, for any $i \in [2b-3]$ we claim

$$\beta_i = f(x_1, y_{i+2}) = f(y_1, y_{i+2})\,. \tag{4.14}$$

To see this, set $q = q(x_1, y_{i+2})$, $S^- = (q-1)\cdot 2^{\beta_i - 1}$, $S = q\cdot 2^{\beta_i-1}$, and $S^+ = (q+1)\cdot 2^{\beta_i-1}$. Now

$$S^- < x_1 \leqslant S < y_{i+2} \leqslant S^+$$

and $x_3 < y_3 \leqslant y_{i+2}$. Hence $S < x_3$ would entail

$$S^- < x_1 \leqslant S < x_3 \leqslant S^+$$

and, consequently, $\alpha_1 = f(x_1, x_3) = \beta_i \geqslant \beta_1$, which contradicts the case $i = 1$ of (4.13). This proves $x_1 < y_1 \leqslant x_3 \leqslant S$, which in turn establishes (4.14).

Putting everything together, the equations (4.9) and (4.14) yield

$$\varphi(\vec{x}) = (\alpha_1, \alpha_2, \dots, \alpha_{2b-3}) \quad \text{and} \quad \varphi(\vec{y}) = (\beta_1, \beta_2, \dots, \beta_{2b-3})$$

and by (4.13) we may strengthen (4.10) to

$$\alpha_1 < \beta_1 \leqslant \alpha_3 < \beta_3 \leqslant \dots \leqslant \alpha_{2b-3} < \beta_{2b-3}\,.$$

In particular this shows that $\varphi(\vec{x})$ and $\varphi(\vec{y})$ are indeed vertices of $G_{b-1}(n)$ and together with (4.11) it further shows that these two vertices are adjacent. This concludes the proof of (4.8) and, hence, the proof of Proposition 4.11. □

Let us now summarize why all the work performed in this subsection demonstrates Theorem 4.3.

Proof of Theorem 4.3. We argue by induction on b. The base cases $b = 1$ and $b = 2$ have been dealt with in (4.1) and Lemma 4.5 respectively. In the light of Proposition 4.11 the induction step is easy. □

Finally we would like to emphasize again that the combination of Proposition 3.1, Theorem 4.2, and Theorem 4.3 implies Theorem 1.7.

§5. Reducible types

Having thus said everything we want to say about the chromatic number of irreducible type-graphs, we devote the present section to the proof of Theorem 1.8. So we consider any nontrivial type τ and let $\tau = \varrho_1\varrho_2 \cdot \ldots \cdot \varrho_t$ be its factorization into irreducible types. For each $i \in [t]$ the number of blocks into which ϱ_i decomposes is denoted by b_i and we set $b^* = \max(b_1, \dots, b_t)$. Finally, let k be the width of τ and let ϱ_i have width k_i for $i \in [t]$.

The notation introduced up to this moment will be used throughout this section without being repeated in the numbered statements that will occur.

Recall that our goal is to show

$$\chi\big(G(n,\tau)\big) = \Theta\big(\log_{(b^*-2)}(n)\big)\,.$$

Here we have $b^* \geqslant 2$ because otherwise each factor ϱ_i of τ would have to be equal to 3, meaning that τ were trivial. Again we treat the lower bound and the upper bound separately, but this time the latter is easier, so we start with it.

Fact 5.1. *For every $i \in [t]$ and every integer $n \geqslant k$ there is a graph homomorphism*

$$\varphi_i \colon G(n,\tau) \longrightarrow G(n,\varrho_i)\,.$$

Proof. Set $r = \mathbf{1}(\varrho_1 \cdot \ldots \cdot \varrho_{i-1})$ and $s = \mathbf{1}(\varrho_1 \cdot \ldots \cdot \varrho_i)$. Clearly ϱ_i has width $k_i = s - r$, and, since $\varrho_1, \dots, \varrho_i$ are types, we also have $r = \mathbf{2}(\varrho_1 \cdot \ldots \cdot \varrho_{i-1})$ and $s = \mathbf{2}(\varrho_1 \cdot \ldots \cdot \varrho_i)$. Now it easy to confirm that the map

$$\varphi_i \colon \binom{[n]}{k} \longrightarrow \binom{[n]}{k_i}$$

given by

$$\varphi\big(\{x_1, \dots, x_k\}\big) = \{x_{r+1}, \dots, x_s\}$$

whenever $1 \leqslant x_1 < x_2 < \ldots < x_k \leqslant n$ is as desired. □

Applying this in particular to some index $i^* \in [t]$ with $b_{i^*} = b^*$ we may deduce the following by means of Theorem 1.7.

Fact 5.2. *As n tends to infinity we have*

$$\chi\big(G(n,\tau)\big) \leqslant \big(2^{(b^*-2)^2} + o(1)\big)\log_{(b^*-2)}(n)\,. \tag{5.1}$$

In the other direction, we will use Proposition 3.1 to embed the generalized shift graph $G\big(n,\sigma_{b^*-1}\big)$ homomorphically into $G(kn,\tau)$.

Fact 5.3. *For every integer $n \geqslant b^*$ there is a graph homomorphism*

$$\psi \colon G(n,\sigma_{b^*-1}) \longrightarrow G(kn,\tau)$$

and, consequently, we have

$$(1+o(1))\log_{(b^*-2)}\big(\tfrac{n}{k}\big) \leqslant \chi\big(G(n,\tau)\big)\,. \tag{5.2}$$

Proof. Let $I = \{i \in [t] \mid \varrho_i \neq 3\}$ and write $c_i = \sum_{j=1}^{i} k_j$ for every integer $i \in [0, t]$. Recall that we know from Proposition 3.1 that for every index $i \in I$ there exists a homomorphism $\psi_i \colon G(n, \sigma_{b_i-1}) \longrightarrow G(k_i n, \varrho_i)$. Utilizing these, we define for each $i \in [t]$ a map

$$\widehat{\psi}_i \colon \binom{[n]}{b^* - 1} \longrightarrow \binom{[c_{i-1}n + 1, c_i n]}{k_i}$$

by stipulating

$$\widehat{\psi}_i\big(\{h_1, \dots, h_{b^*-1}\}\big) = \begin{cases} c_{i-1}n + \psi_i\big(\{h_1, \dots, h_{b_i-1}\}\big) & \text{if } i \in I, \\ \{c_i n\} & \text{if } i \notin I, \end{cases}$$

whenever $1 \leqslant h_1 < \ldots < h_{b-1} \leqslant n$, where the addition of a number to a set in the upper case is to be performed "elementwise". We leave it to the reader to check that the map

$$\psi \colon \binom{[n]}{b^* - 1} \longrightarrow \binom{[kn]}{k}$$

given by

$$\psi(X) = \bigcup_{i \in [t]} \widehat{\psi}_i(X)$$

for all $X \in \binom{[n]}{b^*-1}$ is indeed a homomorphism from $G(n, \sigma_{b^*-1})$ to $G(kn, \tau)$.

Formula (5.2) follows from the mere existence of ψ and from Theorem 1.4. $\square$

Owing to (5.1) and (5.2) the proof of Theorem 1.8 is complete.

6. References

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