Tree sets

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Abstract

We study an abstract notion of tree structure which generalizes tree-decompositions of graphs and matroids. Unlike tree-decompositions, which are too closely linked to graph-theoretical trees, these *tree sets* can provide a suitable formalization of tree structure also for infinite graphs, matroids, or set partitions, as well as for other discrete structures such as order trees.

In this first of two papers we introduce tree sets, establish their relation to graph and order trees, and show how, when they describe separations of finite sets, graphs or matroids, they correspond to tree-decompositions.

1 Introduction

There are a number of concepts in combinatorics that express the tree-likeness of discrete[1] structures. Among these are:

- graph trees
- order trees
- nested subsets, or bipartitions, of a set.

Other notions of tree-likeness, such as tree-decompositions of graphs or matroids, are modelled on these.

All these notions of tree-likeness work well in their own contexts, but sometimes less well outside them:

- graph trees need vertices, which in some desired applications – even as close as matroids – may not exist;
- order trees need additional poset structure which is more restrictive than the tree-likeness it implies;[2]
- nested sets of bipartitions require a ground set that can be partitioned, which does not exist, say, in the case of tree-decompositions of a graph;
- tree-decompositions of infinite graphs, which are modelled on graph trees, cannot describe separations that are limits of other separations, because graph trees do not have edges that are limits of other edges.

The purpose of this paper is to study an abstract notion of tree structure which is general enough to describe all these examples, yet substantial enough that each of these instances, in their relevant context, can be recovered from it.

[1] There are also non-discrete such concepts, such as $\mathbb{R}$-trees, which are not our topic here.

[2] Finite order trees, for example, correspond to rooted graph trees, but there is nothing in their definition from which we can abstract so that what remains corresponds to the underlying unrooted graph tree.

We shall introduce this abstract notion of 'tree sets' formally in Section 3. It builds on an equally abstract notion of 'separation systems', to be introduced in Section 2. These latter are an abstraction of oriented separations in graphs, or of oriented bipartitions of a set, which just remember how these separations are partially ordered and that reversing their orientation reverses this ordering. Formally, a *separation system* will just be a poset with an order-reversing involution, and *tree sets* will essentially be nested separation systems: those in which every element or its inverse is comparable with every other element or its inverse.

Tree-likeness has been modelled in many ways [10], and even the idea to formalize tree-likeness in this abstract way is not new. Abstract nested separation systems as above were introduced by Dunwoody [7, 8] under the name of 'protrees' as an abstract structure for groups to act on, and used by Hundertmark [9] as a basis for structure trees of graphs and matroids that can separate their tangles and related substructures. They have not, however, been studied systematically, which is our purpose in this paper.

Although studying abstract tree sets may seem amply justified by their ubiquity in different contexts, there are two concrete applications that I would like to point out. The first of these is to order trees. These are often used in infinite combinatorics to capture tree structure wherever it arises. The reason they can do this better than graph-theoretical trees is that they may contain limit points, as those tree-like structures to be captured frequently do. However, order trees come with more information than is needed just to capture tree structure, which can make their use cumbersome. For a finite tree structure, for example, they correspond to a graph-theoretical tree together with the choice of a leaf as the root. A change of root will change the induced tree order but not the underlying graph tree, which already captures that finite tree structure.

It turns out that abstract tree sets can provide an analogue of this also for infinite order trees: these will correspond precisely to the *consistently oriented* tree sets. Just as different choices of a root turn the same graph-theoretical tree into related but different order trees, different consistent orientations of an abstract trees set yield related but different order trees. Order trees thus appear as a category of 'pointed tree sets' not only when their tree structure is represented by a graph but always, also when it has limit points. It thus becomes possible to 'forget' the ordering of an order tree but retain more than a set: the set plus exactly the information that makes it tree-like.

The application of tree sets that originally motivated this paper was one to graphs, as follows. In graph minor theory there are duality theorems saying that a given finite graph either has a highly connected substructure, such as a tangle or bramble, or if not then this is witnessed by a tree-decomposition which immediately shows that such a highly connected substructure cannot exist, because 'there is no room for it'. The edges of the decomposition tree then correspond to separations of this graph that form a tree set in our sense.

Conversely, if a tree set of separations of a graph or matroid is finite, it is always induced by a tree-decomposition in this way. Thus, finite tree-decompositions and tree sets of separations amount to the same thing. Infinite tree sets of separations are equally useful. But they need not come from tree-decompositions, since nested separations can have limits but edges in trees do not. This is why width duality theorems for infinite structures, such as those in [1], require tree sets of separations, rather than tree-decompositions, to express the tree structures that witness the absence of highly connected substructures such as tangles. Several of the result we shall prove in this paper will be needed in [1].

Separation systems of sets can be used not only for tree sets that witness the absence of some highly connected substructure in a graph or matroid. They can also be used to identify those structures: by their consistent orientations 'towards' them [6]. It turns out that, if we substitute these orientations of separation systems of sets for the substructures they point to, we can prove theorems which assert a duality between the existence of such a substructure and the existence of a tree set of separations precluding it entirely within the setting of *abstract* separation systems (posets with an order-reversing involution), not just those of sets. We thus obtain a wealth of duality theorems for potentially very abstract combinatorial structures all based on the abstract separation systems and tree sets studied in this paper.

Orientations of separation systems, and in particular of tree sets, will be our topic in Section 4. In Sections 5, 6 and 7 we show how abstract tree sets can be used to describe the tree structures of our earlier examples: of graph-theoretical trees, of order trees, of nested sets of bipartitions of a set, and of tree-decompositions of graphs and matroids. We shall also see how these representations of tree sets can be recovered from the tree sets they represent. Where relevant we shall point out how, conversely, abstract tree sets describe tree-like structures in these contexts that do not come from such examples: where tree sets provide not just a convenient common language for different kinds of tree structures but define new ones, including ones that are needed for applications in traditional settings such as graphs and matroids [1].

Any terminology used but not defined in this paper can be found in [3].

2 Separations

A *separation of a set* V is a set $\{A, B\}$ such that $A \cup B = V$.[3] The ordered pairs (A, B) and (B, A) are its *orientations*. The *oriented separations* of V are the orientations of its separations. Mapping every oriented separation (A, B) to its *inverse* (B, A) is an involution that reverses the partial ordering

$$(A, B) \le (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D,$$

since the above is equivalent to $(D, C) \le (B, A)$. Informally, we think of (A, B) as *pointing towards* B and *away from* A. Similarly, if $(A, B) \le (C, D)$, then (A, B) *points towards* $\{C, D\}$, while (C, D) *points away from* $\{A, B\}$.

More generally, a *separation system* $(\vec{S}, \le, ^*)$ is a partially ordered set $\vec{S}$ with an order-reversing involution *. Its elements are called *oriented separations*. An *isomorphism* between two separation systems is a bijection between their underlying sets that respects both their partial orderings and their involutions.

When a given element of $\vec{S}$ is denoted as $\vec{s}$, its *inverse* $\vec{s}^{\,*}$ will be denoted as $\overleftarrow{s}$, and vice versa. The assumption that * be *order-reversing* means that, for all $\vec{r}, \vec{s} \in \vec{S}$,

$$\vec{r} \le \vec{s} \;\Leftrightarrow\; \overleftarrow{r} \ge \overleftarrow{s}\,. \tag{1}$$

[3]We can make further requirements here that depend on some structure on V which $\{A, B\}$ is meant to separate. If V is the vertex set of a graph G, for example, we usually require that G has no edge between $A \smallsetminus B$ and $B \smallsetminus A$. But such restrictions will depend on the context and are not needed here; in fact, even the separations of a *set* V defined here is just an example of the more abstract 'separations' we are about to introduce.

A *separation* is a set of the form $\{\vec{s}, \overleftarrow{s}\}$, and then denoted by s. We call $\vec{s}$ and $\overleftarrow{s}$ the *orientations* of s. The set of all such sets $\{\vec{s}, \overleftarrow{s}\} \subseteq \vec{S}$ will be denoted by S. If $\vec{s} = \overleftarrow{s}$, we call both $\vec{s}$ and s *degenerate*.

When a separation is introduced ahead of its elements and denoted by a single letter s, we shall use $\vec{s}$ and $\overleftarrow{s}$ (arbitrarily) to refer to its elements. Given a set S' of separations, we write $\vec{S'} := \bigcup S' \subseteq \vec{S}$ for the set of all the orientations of its elements. With the ordering and involution induced from $\vec{S}$, this is again a separation system.[4]

Separations of sets, and their orientations, are clearly an instance of this abstract setup if we identify $\{A, B\}$ with $\{(A, B), (B, A)\}$.

If there are binary operations $\vee$ and $\wedge$ on our separation system $\vec{S}$ such that $\vec{r} \vee \vec{s}$ is the supremum and $\vec{r} \wedge \vec{s}$ the infimum of $\vec{r}$ and $\vec{s}$ in $\vec{S}$, we call $(\vec{S}, \le, {}^*, \vee, \wedge)$ a *universe* of (oriented) separations.

The oriented separations of a set V form such a universe: if $\vec{r} = (A, B)$ and $\vec{s} = (C, D)$, say, then $\vec{r} \vee \vec{s} := (A \cup C, B \cap D)$ and $\vec{r} \wedge \vec{s} := (A \cap C, B \cup D)$ are again oriented separations of V, and are the supremum and infimum of $\vec{r}$ and $\vec{s}$, respectively. Similarly, the oriented separations of a graph form a universe of separations. The oriented separations of order $< k$ of a graph, however, for any fixed k, form a separation system inside this universe that may not itself be a universe with respect to $\vee$ and $\wedge$ as defined above.

A separation $\vec{r} \in \vec{S}$ is *trivial in* $\vec{S}$, and $\overleftarrow{r}$ is *co-trivial*, if there exists $s \in S$ such that $\vec{r} < \vec{s}$ as well as $\vec{r} < \overleftarrow{s}$. We call such an s a *witness* of $\vec{r}$ and its triviality. If neither orientation of r is trivial, we call r *nontrivial*.

Note that if $\vec{r}$ is trivial in $\vec{S}$ then so is every $\vec{r'} \le \vec{r}$. If $\vec{r}$ is trivial, witnessed by s, then $\vec{r} < \vec{s} < \overleftarrow{r}$ by (1). Hence if $\vec{r}$ is trivial, then $\overleftarrow{r}$ cannot be trivial. In particular, degenerate separations are nontrivial.

Lemma 2.1. *If S is finite, then every trivial separation in $\vec{S}$ has a nontrivial witness. In particular, if S is non-empty it has a nontrivial element.*

Proof. Any trivial $\vec{r} \in \vec{S}$ lies below a maximal trivial $\vec{r'} \in \vec{S}$. If $s \in S$ witnesses the triviality of $\vec{r'}$, it also witnesses that of $\vec{r}$. By the maximality of $\vec{r'}$, neither orientation of s is trivial. □

There can also be separations $\vec{s}$ with $\vec{s} < \overleftarrow{s}$ that are not trivial. But anything smaller than these is again trivial: if $\vec{r} < \vec{s} \le \overleftarrow{s}$, then s witnesses the triviality of $\vec{r}$. Separations $\vec{s}$ such that $\vec{s} \le \overleftarrow{s}$, trivial or not, will be called *small*; note that, by (1), if $\vec{s}$ is small then so is every $\vec{s'} \le \vec{s}$.

Small separations that are not trivial directly precede their inverses in $\le$:

Lemma 2.2. *If $\vec{r}$ is small and $\vec{r} < \vec{s} < \overleftarrow{r}$ for some $\vec{s}$, then $\vec{r}$ is trivial.*

Proof. The second inequality is equivalent to $\vec{r} < \overleftarrow{s}$, by (1). □

The trivial oriented separations of a set V, for example, are those of the form $\vec{r} = (A, B)$ with $A \subseteq C \cap D$ and $B \supseteq C \cup D = V$ for some $s = \{C, D\} \neq r$ in the set S considered. The small separations (A, B) of V are all those with $B = V$.

[4]When we refer to oriented separations using explicit notation that indicates orientation, such as $\vec{s}$ or (A, B), we sometimes leave out the word 'oriented' to improve the flow of words. Thus, when we speak of a 'separation (A, B)', this will in fact be an oriented separation.

The orientations of two separations r, s can be related in four possible ways[5]: as $\vec{r} \le \vec{s}$ or $\vec{r} \ge \vec{s}$ or $\vec{r} \le \overleftarrow{s}$ or $\vec{r} \ge \overleftarrow{s}$. If r, s are distinct and nontrivial, no more than one of these relations can hold:

Lemma 2.3. *If $r, s \in S$ are distinct, and have orientations $\vec{r} \le \vec{s}$ such that neither $\vec{r}$ nor $\overleftarrow{s}$ is trivial in $\vec{S}$, then $\vec{r} \ngeq \vec{s}$ and $\vec{r} \nleq \overleftarrow{s}$ and $\vec{r} \ngeq \overleftarrow{s}$.*

Proof. If $\vec{r} \ge \vec{s}$, then $\vec{r} \le \vec{s} \le \vec{r}$ with equality, contradicting $r \ne s$.

If $\vec{r} \le \overleftarrow{s}$ then s witnesses that $\vec{r}$ is trivial, contradicting our assumption.

If $\vec{r} \ge \overleftarrow{s}$, then r witnesses that $\overleftarrow{s}$ is trivial, a contradiction. □

A set $O \subseteq \vec{S}$ of oriented separations is *antisymmetric* if $|O \cap \{\vec{s}, \overleftarrow{s}\}| \le 1$ for all $\vec{s} \in \vec{S}$. It is *consistent* if there are no distinct $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in O$.

3 Tree sets and stars

Two separations r, s are *nested* if they have comparable orientations; otherwise they *cross*. Two oriented separations $\vec{r}, \vec{s}$ are *nested* if r and s are nested.[6] We say that $\vec{r}$ *points towards* s, and $\overleftarrow{r}$ *points away from* s, if $\vec{r} \le \vec{s}$ or $\vec{r} \le \overleftarrow{s}$.

In this informal terminology, two oriented separations are nested if and only if they are either comparable or point towards each other or point away from each other. And a set $O \subseteq \vec{S}$ is consistent if and only if it does not contain orientations of distinct separations that point away from each other.

A set of separations is *nested* if every two of its elements are nested. A *tree set* is a nested separation system without trivial or degenerate elements. A tree set is *regular* if none of its elements is small.

For example, the set of orientations (u, v) of the edges uv of a tree T form a regular tree set with respect to the involution $(u, v) \mapsto (v, u)$ and the *natural partial ordering* on $\vec{E}(T)$: the ordering in which $(x, y) < (u, v)$ if $\{x, y\} \ne \{u, v\}$ and the unique $\{x, y\}$–$\{u, v\}$ path in T joins y to u.

Note that a degenerate separation s is never nested with another nontrivial separation r, since one of its orientations, $\vec{r}$ say, would satisfy $\vec{r} < \vec{s} = \overleftarrow{s}$ and hence be trivial. In particular, a nested set of separations has at most one degenerate element, and if it does then this is its only nontrivial element.

In every nested separation system $\vec{S}$, the set S' of nontrivial non-degenerate elements forms a tree set $\vec{S'} \subseteq \vec{S}$. By Lemma 2.1, S' is the unique largest tree set contained in $\vec{S}$. We shall say that $\vec{S'}$ is the tree set *induced by* $\vec{S}$.

A subset σ of a separation system $\vec{S}$ is a *star of separations* if its separations all point towards each other: if $\vec{r} \le \overleftarrow{s}$ for all distinct $\vec{r}, \vec{s} \in \sigma$. In particular, stars of separations are nested. They are also consistent: if distinct $\overleftarrow{r}, \vec{s}$ lie in the same star we cannot have $\vec{r} < \vec{s}$, since also $\vec{s} \le \vec{r}$ by the star property.

A star σ is *proper* if, for all distinct $\vec{r}, \vec{s} \in \sigma$, the relation $\vec{r} \le \overleftarrow{s}$ required by the definition of 'star' is the only one among the four possible relations between orientations of distinct r and s: if $\vec{r} \le \overleftarrow{s}$ but $\vec{r} \nleq \vec{s}$ and $\vec{r} \ngeq \vec{s}$ and $\vec{r} \ngeq \overleftarrow{s}$.

[5] Actually, in eight ways; but by (1), they come in equivalent pairs. The explicit list of relations stated here represents these four types, because every item involves $\vec{r}$, not $\overleftarrow{r}$.

[6] Terms introduced for unoriented separations may be used informally for oriented separations too if the meaning is obvious, and vice versa.

Lemma 3.1. (i) *A nested set of oriented separations is a proper star if and only if it is an antisymmetric, consistent antichain.*

(ii) *A star is proper if and only if it is an antisymmetric antichain.* □

Let us call a star $\sigma \in \vec{S}$ *proper in* $\vec{S}$ if it is proper and is not a singleton $\{\overleftarrow{s}\}$ with $\overleftarrow{s}$ co-trivial in $\vec{S}$. We shall call such stars *co-trivial singletons*. A star $\{\vec{s}\}$ with s degenerate is a *degenerate singleton*.

Non-singleton proper stars cannot contain co-trivial separations. In fact, they cannot contain any separation whose inverse is small:

Lemma 3.2. *If a proper star $\sigma \subseteq \vec{S}$ has an element $\vec{s}$ such that $\overleftarrow{s}$ is small, then it has no other elements. In particular, if σ is proper in $\vec{S}$ then none of its elements is co-trivial in $\vec{S}$.*

Proof. Suppose there exists some $\vec{r} \in \sigma \setminus \{\vec{s}\}$. Then $\vec{r} \le \overleftarrow{s} \le \vec{s}$, since $\overleftarrow{s}$ is small. Hence σ is not an antichain and thus not a proper star. □

The simplest example of an improper star is one violating antisymmetry, e.g., the star $\{\vec{s}, \overleftarrow{s}\}$ for non-degenerate s. Such a star may contain further separations $\vec{r}$, but note that these must be trivial, as witnessed by s. If a star is antisymmetric but fails to be an antichain, containing separations $\vec{r} < \vec{s}$ say, then again $\vec{r}$ must be trivial, since we also have $\vec{r} \le \overleftarrow{s}$ by the star property (and $\vec{r} \neq \overleftarrow{s}$ by antisymmetry, so $r \neq s$). Thus,

Lemma 3.3. *Any improper star in $\vec{S}$ that is not a co-trivial singleton and is not of the form $\{\vec{s}, \overleftarrow{s}\}$ contains a separation that is trivial in $\vec{S}$.* □

Our partial ordering on $\vec{S}$ also relates its subsets, and in particular its stars: for $\sigma, \tau \subseteq \vec{S}$ we write $\sigma \le \tau$ if for every $\vec{s} \in \sigma$ there exists some $\vec{t} \in \tau$ with $\vec{s} \le \vec{t}$. This relation is obviously reflexive and transitive, but in general it is not antisymmetric: if σ contains separations $\vec{s} < \vec{t}$, then for $\tau = \sigma \setminus \{\vec{s}\}$ we have $\sigma < \tau < \sigma$ (where $<$ denotes '$\le$ but not $=$'). However, it is antisymmetric on antichains, and thus in particular on proper stars:

Lemma 3.4. *The set of all proper stars $\sigma \subseteq \vec{S}$ is partially ordered by $\le$.*

Proof. We only have to show antisymmetry. If $\sigma \le \tau \le \sigma$, then for every $\vec{s} \in \sigma$ there are $\vec{t} \in \tau$ and $\vec{s'} \in \sigma$ such that $\vec{s} \le \vec{t} \le \vec{s'}$. If σ is an antichain this implies $\vec{s} = \vec{s'}$, and hence $\sigma \subseteq \tau$. Likewise $\tau \subseteq \sigma$, so $\sigma = \tau$. □

When we speak of *maximal proper stars in* a separation system $(\vec{S}, \le, {}^*)$, we shall always mean stars that are $\le$-maximal in the set of stars that are proper in $\vec{S}$. These stars σ need not be maximal among all the stars in $\vec{S}$, not even among the proper ones; for example, there may be a co-trivial singleton $\{\overleftarrow{r}\}$ in $\vec{S}$ (which is a proper star, though not proper in $\vec{S}$) such that $\vec{s} \le \overleftarrow{r}$ for all $\vec{s} \in \sigma$.

Lemma 3.2 tells us that proper stars in a separation system $\vec{S}$ cannot contain separations that are co-trivial in $\vec{S}$. Our next lemma says that maximal proper stars in nested separation systems $\vec{S}$ do not contain trivial separations either, and will thus lie in the tree set that $\vec{S}$ induces:

Lemma 3.5. *Maximal proper stars in a nested separation system $\vec{S}$ contain no separations that are trivial or co-trivial in $\vec{S}$.*

Proof. Let σ be a proper star in $\vec{S}$. By Lemma 3.2 it contains no co-trivial separations. We assume that σ has an element $\vec{r}$ that is trivial in $\vec{S}$, witnessed by $r' \in S$ say, and show that σ is not maximal among the proper stars in $\vec{S}$.

Not both orientations of r' can be trivial; let $\overleftarrow{r'}$ be one that is not. Let $\sigma' > \sigma$ be obtained from σ by replacing $\vec{r}$ with $\overrightarrow{r'}$ and deleting any $\vec{s} \in \sigma$ such that $\vec{s} < \overrightarrow{r'}$. Let us show that σ' is again a proper star in $\vec{S}$; this will show that σ was not maximal among these.

No $\vec{s} \in \sigma' \smallsetminus \{\overrightarrow{r'}\}$ can point away from r': that would imply either $\vec{r} < \overrightarrow{r'} \leq \vec{s}$ or $\vec{r} < \overleftarrow{r'} \leq \vec{s}$, contradicting the fact that $\vec{r}$ and $\vec{s}$ lie in the same proper star σ. Hence every such $\vec{s}$ points towards r', because S is nested. But $\vec{s} \not< \overrightarrow{r'}$ by definition of σ', so $\vec{s} \leq \overleftarrow{r'}$. Thus, σ' is indeed a proper star. It is also proper in $\vec{S}$, because $\overrightarrow{r'}$ was chosen not to be co-trivial in $\vec{S}$. □

4 Orientations of separation systems

An *orientation* of a separation system $\vec{S}$, or of a set S of separations, is a set $O \subseteq \vec{S}$ that contains for every $s \in S$ exactly one of its orientations $\vec{s}, \overleftarrow{s}$. A *partial orientation* of S is an orientation of a subset of S: an antisymmetric subset of $\vec{S}$.

We shall be interested particularly in consistent orientations of separation systems $\vec{S}$. If $\vec{S}$ comes from a concrete combinatorial structure which its elements separate, its consistent orientations can be thought of as pointing to the locations in this combinatorial structure which the separations in S separate from each other. Examples include all the classical highly connected substructures of graphs and matroids that have been studied in the context of width parameters, such as blocks, tangles or brambles [1, 2, 4, 6, 9]. They also include the vertices and the ends of a tree: these correspond to the consistent orientations of its edge set E, since these always point towards a unique vertex or end (see Section 5).

While these locations might originally be identified by concrete substructures of a structure which our separation system $\vec{S}$ separates, we now see that this is not in fact needed: they are all identified by certain consistent orientations of $\vec{S}$. These locations in various kinds of structures can therefore be described purely in terms of $\vec{S}$, with reference only to the axiomatic properties of $\vec{S}$.

This shift of paradigm enables us to treat all kinds of locations in discrete structures uniformly. For example, we can prove very general duality theorems asserting that a separation system $\vec{S}$ either admits certain consistent orientations (which for concrete choices of $\vec{S}$ might correspond to certain types of dense substructure of a given structure) or else contains a tree set whose consistent orientations all point to locations that are 'small' in a corresponding dual sense [6].

But our general framework also defines new kinds of 'locations' in discrete structures that had not previously been studied: every consistent orientation of a natural separation system of a given structure can in principle be thought of as such a 'location'.

To emphasise this point, and to support our intuition, we shall therefore think of the consistent orientations also of abstract separation systems $\vec{S}$ in this way: as regions of some unknown combinatorial structure which S 'separates', regions that are either too small or too highly connected to be split by the separations in S.

Every consistent orientation O of a separation system $\vec{S}$ is *closed down* in $\vec{S}$: if $\vec{r}, \vec{s} \in \vec{S}$ satisfy $\vec{s} \in O$ and $\vec{r} \leq \vec{s}$, we also have $\vec{r} \in O$, since otherwise $\overleftarrow{r} \in O$, contradicting the consistency of O. Conversely, every orientation of $\vec{S}$ that is closed down in the ordering of $\vec{S}$ is obviously consistent.

A consistent orientation of a set $\vec{S}$ of separations cannot contain any separations that are co-trivial in $\vec{S}$: if $\vec{r}$ is trivial, witnessed by s say, then s cannot be oriented consistently with $\overleftarrow{r}$. The following lemma provides a kind of converse to this observation:

Lemma 4.1. *Let S be a set of separations, and let P be a consistent partial orientation of S.*

- (i) *P extends to a consistent orientation O of S if and only if no element of P is co-trivial in $\vec{S}$.*
- (ii) *If $\vec{p}$ is maximal in P, then the O in* (i) *can be chosen with $\vec{p}$ maximal in O if and only if p is nontrivial in S.*
- (iii) *If S is nested, then the orientation O in* (ii) *is unique.*

Proof. (i) The forward implication follows from the fact that no consistent orientation of S can contain a co-trivial separation $\overleftarrow{r}$: if the triviality of $\vec{r}$ is witnessed by $s \in S$, then both $\{\overleftarrow{r}, \vec{s}\}$ and $\{\overleftarrow{r}, \overleftarrow{s}\}$ would be inconsistent.

For the backward implication, use Zorn's lemma to extend P to a maximal consistent partial orientation O of S. Suppose S has an element s such that neither $\vec{s}$ nor $\overleftarrow{s}$ lies in O. Then $O \cup \{\vec{s}\}$ is inconsistent, which means that there is some $\overleftarrow{r} \in O$ such that $\vec{r} < \vec{s}$. And $O \cup \{\overleftarrow{s}\}$ is inconsistent, so there is some $\vec{t} \in O$ such that $\vec{s} < \vec{t}$. But then $\vec{r} < \vec{t}$ with $\overleftarrow{r}, \vec{t} \in O$, contradicting the consistency of O.

(ii) The forward implication is again clear. Indeed, if $\vec{p} < \vec{s}$ and $\vec{p} < \overleftarrow{s}$, then $\vec{p}$ cannot be maximal in any orientation of a set containing s. Since, by (i), $\vec{p}$ cannot be co-trivial, this proves that p is nontrivial.

For the backward implication let P' be the set of separations $\vec{s} \in \vec{S}$ such that $\vec{s}$ points towards p but $\overleftarrow{s}$ does not. Neither $\vec{p}$ nor $\overleftarrow{p}$ lies in P'. Let us show that $P \cup P'$ is still consistent. If $\overleftarrow{s}, \vec{s'} \in P'$ are inconsistent, then they both point towards p and $\vec{s} < \vec{s'}$. But then $\vec{s}$ also points towards p, contradicting the definition of P'. If $\overleftarrow{r} \in P$ is inconsistent with $\vec{s} \in P'$, then $\vec{r} < \vec{s} \leq \vec{p}$ or $\vec{r} < \vec{s} \leq \overleftarrow{p}$. The first of these cases contradicts the consistency of P (which contains both $\overleftarrow{r}$ and $\vec{p}$). In the latter case we have $\vec{p} \leq \overleftarrow{s} < \overleftarrow{r} \in P$, contradicting the maximality of $\vec{p}$ in P. This completes the proof that $P \cup P'$ is consistent.

By assumption (i), no element of P is co-trivial in $\vec{S}$. Let us show that also no $\vec{s} \in P'$ is co-trivial in $\vec{S}$. If it is, then $\overleftarrow{s}$ is trivial, and hence $\vec{p} \not\leq \overleftarrow{s}$ since $\vec{p}$ is non-trivial by assumption. The fact that $\vec{s}$ points to p thus means that $\vec{s} < \vec{p}$ (since we cannot have $\vec{s} \leq \overleftarrow{p}$). But then our assumed co-triviality of $\vec{s}$ implies that $\vec{p}$ too is co-trivial. This contradicts the fact that $\vec{p} \in P$.

By (i), we can extend $P \cup P'$ to a consistent orientation O of S. To show that $\vec{p}$ is maximal in O, assume there exists $\vec{s} \in O$ such that $\vec{p} < \vec{s}$. Then $\overleftarrow{s}$ points towards p. But $\vec{s}$ does not: we cannot have $\vec{s} \leq \vec{p}$ (as $\vec{p} < \vec{s}$), we cannot have $\vec{s} = \overleftarrow{p}$ (since $\vec{p} \in O$ implies $\overleftarrow{p} \notin O$), and we cannot have $\vec{s} < \overleftarrow{p}$, since then $\vec{p} < \overleftarrow{s}$ as well as $\vec{p} < \vec{s}$, contradicting the non-triviality of $\vec{p}$. Hence $\overleftarrow{s} \in P'$. But then $O \supseteq P'$ contains $\overleftarrow{s}$ as well as $\vec{s}$, a contradiction.

(iii) Let $\vec{p}$ be as in (ii), and consider any $\vec{s} \in \vec{S}$ such that $\vec{s} < \vec{p}$. Then $\{\overleftarrow{s}, \vec{p}\}$ is inconsistent, so $\overleftarrow{s} \notin O$ and hence $\vec{s} \in O$. Now consider any $\vec{s} \in \vec{S}$ such that $\vec{p} < \vec{s}$. Then $\vec{s} \in O$ would contradict the maximality of $\vec{p}$ in O, so $\overleftarrow{s} \in O$. Hence no matter how the O in (ii) was chosen, it orients every $s \in S$ that is nested with p in the same way. □

In order to identify a consistent orientation of $\vec{S}$, we need to know only its maximal elements. Indeed, given a subset $\sigma \subseteq \vec{S}$, let us write

$$\lceil \sigma \rceil := \{\, \vec{r} \in \vec{S} \mid \exists \vec{s} \in \sigma \colon \vec{r} \le \vec{s} \,\}$$

for its *down-closure* in $\vec{S}$, and put

$$\overleftarrow{\sigma} := \{\, \overleftarrow{s} \mid \vec{s} \in \sigma \} \smallsetminus \sigma.$$

We can recover a finite consistent orientation of S from the set of its maximal elements. More generally:

Lemma 4.2. *Let O be a consistent orientation of a separation system $\vec{S}$ such that every element of O lies below some maximal element of O. Let σ be the set of maximal elements of O. Then $O = \lceil \sigma \rceil \smallsetminus \overleftarrow{\sigma}$. In particular, O is uniquely determined by σ.*

Proof. Every $\vec{r} \in O$ lies in $\lceil \sigma \rceil \smallsetminus \overleftarrow{\sigma}$, by definition of σ. Conversely if $\vec{r} \in \lceil \sigma \rceil \smallsetminus \overleftarrow{\sigma}$, then either $\vec{r} \in \sigma \subseteq O$ or $\vec{r} < \vec{s} \in \sigma \subseteq O$ for some $\vec{s} \neq \overleftarrow{r}$. In the latter case we have $\overleftarrow{r} \notin O$ since O is consistent, and hence again $\vec{r} \in O$. □

The consistent orientations of tree sets will be of particular interest to us.

Lemma 4.3. *The consistent orientations of a nested separation system $\vec{S}$ are precisely the consistent orientations of its induced tree set, together with all its trivial separations and the unique orientation of any degenerate separation in S.*

Proof. Let $\vec{S}'$ be the tree set that $\vec{S}$ induces, and let $\vec{R}$ be the set of trivial or degenerate separations in $\vec{S}$. Every consistent orientation O of $\vec{S}$ contains $\vec{R}$ by Lemma 4.1(i), and $O \cap \vec{S}'$ is a consistent orientation of $\vec{S}'$.

Conversely, if O' is a consistent orientation of $\vec{S}'$ then $O' \cup \vec{R}$ is a consistent orientation of $\vec{S}$. Indeed, no two elements of $\vec{R}$ can point away from each other, since this would make both their inverses trivial; but neither a trivial nor a degenerate separation has a trivial inverse. But neither can $\vec{r} \in \vec{R}$ and $\overleftarrow{s'} \in O'$ point away from each other, since this would make $\overleftarrow{s'} < \vec{r}$ trivial, contradicting the fact that $s' \in S'$ is nontrivial. Hence no two separations in $O' \cup \vec{R}$ point away from each other, which shows that $O' \cup \vec{R}$ is consistent. □

Let us say that a subset σ of a nested separation system $\vec{S}$ *splits* $\vec{S}$ if S has a consistent orientation O whose set of maximal elements is precisely σ and which satisfies $O \subseteq \lceil \sigma \rceil$. For example, the set $\vec{E}(T)$ of oriented edges of a tree T is split precisely by the sets of edges at a node t, oriented towards t (Proposition 5.1).

Splitting subsets contain no trivial elements:

Lemma 4.4. *Let $\vec{S}$ be a nested separation system.*

(i) *If S has a degenerate element s, then $\{\vec{s}\}$ is the unique splitting set in $\vec{S}$.*

(ii) *If S has no degenerate element, then the subsets splitting $\vec{S}$ are precisely the subsets that split the tree set which $\vec{S}$ induces.*

Proof. (i) If S has a degenerate element s, then s is its only nontrivial element, so tree set which $\vec{S}$ induces is empty. By Lemma 4.4, therefore, S has a unique consistent orientation O, which consists of all its trivial elements and $\vec{s} = \overleftarrow{s}$. Then $O = \lceil \vec{s} \rceil$ by Lemma 2.1, so $\{ \vec{s} \}$ splits $\vec{S}$, but no other subset of $\vec{S}$ does.

(ii) Consider any element $\vec{p}$ of a splitting subset σ of $\vec{S}$. Since $\vec{p}$ is maximal in a consistent orientation of S, Lemma 4.1(ii) applied with $P = \{ \vec{p} \}$ implies that $\vec{p}$ is not trivial in $\vec{S}$. Hence σ lies in the tree set $\vec{S'}$ that $\vec{S}$ induces, which it clearly also splits.

Conversely, if σ' splits $\vec{S'}$, witnessed by the orientation O' of $\vec{S'}$, say, then by Lemma 2.1 adding to O' the trivial separations of $\vec{S}$ extends it to a consistent orientation of $\vec{S}$ whose set of maximal elements is still σ', and which splits $\vec{S}$. □

In our earlier example, the subsets of $\vec{E}(T)$ that split it can be described without reference to orientations of $E(T)$ (unlike in the definition of 'split'): as we shall prove formally in Proposition 5.1, they are precisely the maximal proper stars in $\vec{E}(T)$, in the partial ordering of subsets of $\vec{E}(T)$ (cf. Lemma 3.4).

For arbitrary tree sets $\vec{S}$, this remains true with one curious exception. But this exception, too, can be described without reference to orientations of $\vec{S}$. Once more, consider the partial ordering $\le$ of the proper stars in $\vec{S}$ from Lemma 3.4.

Lemma 4.5. *A subset σ of a nested separation system $\vec{S}$ splits $\vec{S}$ if and only if either*

(i) *σ is a maximal proper star in $\vec{S}$; or*

(ii) *σ is a proper star in $\vec{S}$ that contains a small separation, and every $\sigma' > \sigma$ that is a proper star in $\vec{S}$ is of the form $\sigma' = \{ \overleftarrow{s} \}$ with $\vec{s} \in \sigma$ small.*

If S has no degenerate element, then σ lies in the tree set which $\vec{S}$ induces. If S has a degenerate element s, then $\sigma = \{ \vec{s} \}$.

Proof. If S has a degenerate element s then, by Lemma 4.4, $\sigma = \{ \vec{s} \}$ is the unique splitting subset of $\vec{S}$, while by Lemma 3.5 it is the unique maximal proper star in $\vec{S}$. Let us now assume that S has no degenerate element.

We assume first that σ splits $\vec{S}$, and show that it satisfies (i) or (ii). Let σ be the set of maximal elements of the consistent orientation O of S.

Clearly, σ is a proper star. It is also a proper star in $\vec{S}$, since O is consistent and so $\sigma \subseteq O$ cannot be a co-trivial singleton in $\vec{S}$ (cf. Lemma 4.1 (i)). Suppose $\sigma' > \sigma$ is another proper star in $\vec{S}$. We shall prove that unless we obtain a contradiction, which will establish (i), we have (ii) witnessed by σ'.

Since $\sigma' \not\le \sigma$ (cf. Lemma 3.4), there exisist $\vec{s'} \in \sigma'$ such that $\vec{s'} \not\le \vec{s}$ for all $\vec{s} \in \sigma$. Since every element of O lies below some element of σ, this means that $\vec{s'} \notin O$, and hence $\overleftarrow{s'} \in O$. Hence there exists $\vec{s} \in \sigma$, and as $\sigma \le \sigma'$ also some $\vec{s''} \in \sigma'$, such that $\overleftarrow{s'} \le \vec{s} \le \vec{s''}$. As $\vec{s'}, \vec{s''} \in \sigma'$, this contradicts the fact that σ' is a proper star, establishing (i) – as long as $\vec{s'} \neq \vec{s''}$.

If $\vec{s'} = \vec{s''}$, we have $\overleftarrow{s'} \le \vec{s} \le \vec{s'}$. Applying (1) to the second of these inequalities we obtain $\overleftarrow{s'} \le \overleftarrow{s}$ (as well as $\overleftarrow{s'} \le \vec{s}$, the first inequality). If $s \neq s'$ this means that $\overleftarrow{s'}$ is trivial, contradicting Lemma 3.2 for σ'. Hence $s = s'$. Since $\vec{s'} \neq \vec{s}$ by the choice of $\vec{s'}$, this means that $\vec{s'} = \overleftarrow{s}$. Our double inequality now yields $\vec{s} \le \overleftarrow{s}$, so $\vec{s}$ is small, and σ' has no other element than $\vec{s'} = \overleftarrow{s}$ by Lemma 3.2.

Conversely, assume that σ satisfies (i) or (ii). Like all stars, σ is consistent. By Lemmas 3.2 and 4.1 (i) we can extend σ to a consistent orientation O of S. We shall prove that $O \subseteq \lceil \sigma \rceil$: then σ clearly splits $\vec{S}$.

Suppose there exists $\vec{s'} \in O \setminus \lceil \sigma \rceil$. Let $\sigma' := (\sigma \cup \{\vec{s'}\}) \setminus (\lceil \vec{s'} \rceil \cap \sigma)$. Since σ' is contained in the consistent and antisymmetric set O, and is an antichain by definition, Lemma 3.1 implies that σ' is a proper star. As also $\sigma < \sigma'$ by definition of σ', we cannot have (i) and must have (ii) with $\sigma' = \{ \overleftarrow{s} \}$ for some $\vec{s} \in \sigma$. As both $\vec{s} \in \sigma$ and $\overleftarrow{s} \in \sigma'$ lie in O, which is antisymmetric, this means that s is degenerate. But then $\vec{s'} = \overleftarrow{s} = \vec{s} \in \sigma$, contrary to the choice of σ'.

The final statement of our lemma follows from Lemma 4.4. □

In view of Lemma 4.5, we shall call the subsets of $\vec{S}$ that split it the *splitting stars* of $\vec{S}$.

5 Tree sets from graph-theoretical trees

Recall that the set

$$\vec{E}(T) := \{ (x, y) : xy \in E(T) \}$$

of all *orientations* (x, y) of the edges $xy = \{x, y\}$ of a tree T form a regular tree set with respect to the involution $(x, y) \mapsto (y, x)$ and the *natural partial ordering* on $\vec{E}(T)$: the ordering in which $(x, y) < (u, v)$ if $\{x, y\} \neq \{u, v\}$ and the unique $\{x, y\}$–$\{u, v\}$ path in T joins y to u.

Since $\vec{E}(T)$ has no small elements, Lemma 4.5 says that the subsets splitting it are precisely its maximal proper stars. By Lemma 3.3, these are precisely its maximal stars not of the form $\{ \vec{s}, \overleftarrow{s} \}$, the stars

$$\vec{F}_t := \{(x, t) : xt \in E(T)\}$$

where t varies over the nodes of T. We shall call $\vec{F}_t$ the *oriented star at t* in T.

Let us prove this directly, without the unnecessary detour via those lemmas:

Proposition 5.1. *The subsets splitting the set $\vec{E}(T)$ of oriented edges of a tree T are precisely the sets of the form $\vec{F}_t$, where t ranges over the nodes of T.*

Proof. The down-closure in $\vec{E}(T)$ of a set $\vec{F}_t$ is clearly a consistent orientation of $E(T)$ whose set of maximal elements is precisely $\vec{F}_t$.

Conversely, let $\sigma \subseteq \vec{E}(T)$ split $\vec{E}(T)$. Then σ is the set of maximal elements of some consistent orientation O of $E(T)$, and $O \subseteq \lceil \sigma \rceil$. In particular, $\sigma \neq \emptyset$ unless $E(T) = \emptyset$ (in which case the assertion is true), so O has a maximal element (x, t). For every neighbour $y \neq x$ of t, the maximality of (x, t) in O implies that $(t, y) \notin O$ and hence $(y, t) \in O$.

Thus, $\vec{F}_t \subseteq O$. As O is closed down in $\vec{E}(T)$ and the down-closure of $\vec{F}_t$ in $\vec{E}(T)$ orients all of $E(T)$, this down-closure equals O and has $\vec{F}_t$ as its set of maximal elements, giving $\sigma = \vec{F}_t$ as desired. □

We remark that infinite trees can also admit consistent edge orientations that have no maximal elements, and hence do not define a splitting set: as soon as the tree contains a ray, we can orient all the edges of this ray forward and all other edges towards that ray.

Proposition 5.1 allows us to recover a tree T from the tree set $\vec{E}(T) =: \tau$ it defines. Indeed, given just τ, let V be the set of its splitting stars σ. Define a graph G on V by taking τ as its set of oriented edges, assigning to every edge $\vec{s}$ the splitting star of τ that contains $\vec{s}$ as its terminal node. These are well defined – i.e., every $\vec{s} \in \tau$ lies in a unique splitting star – by our assumption that τ is of the form $\vec{E}(T)$ and Proposition 5.1. Then, clearly, the map $t \mapsto \vec{F}_t$ is a graph isomorphism between T and G.

Our assumption above that τ is the set of oriented edges of some tree T cannot be omitted: an arbitrary tree set need not be realizable as the set of oriented edges of a tree. Finite tree sets are, and we shall prove this as part of Theorem 8.8. More generally, we have the following characterization:

Theorem 5.2. [5] *A tree set is isomorphic to the tree set $\vec{E}(T)$ of the oriented edges of a suitable tree T if and only if it contains no chain of order type $\omega + 1$.*

6 Tree sets from order trees

An *order tree*, for the purpose of this paper, is a poset $(T, \leq)$ in which the down set

$$\lceil \mathring{t} \rceil := \{ s \in T \mid s < t \}$$

below every element $t \in T$ is a chain. We do not require this chain to be well-ordered. To ensure that order trees induce order trees on the subsets of their ground set, we also do not require that every two elements have a common lower bound. Order trees that do have this property will be called *connected*.

Order trees are often used to describe the tree-likeness of other combinatorial structures. In such contexts it can be unfortunate that they come with more information than just this tree-likeness, and one has to find ways of 'forgetting' the irrelevant additional information.

Theorem 6.1 below offers a way to do this: it canonically splits the information inherent in an order tree into the 'tree part' represented by an unoriented tree set, and an 'orienting part' represented by an orientation of this tree set.

Let us show first how to extend an order tree $(T, \leq)$ canonically to a tree set. For every $t \in T$ we add an *inverse* t^*, choosing these different for distinct t and different from all elements of T, and relating them for all $s, t \in T$ as follows:

$$\begin{aligned} s^* < t^* \quad &:\Leftrightarrow \quad s > t \\ s^* < t \quad &:\Leftrightarrow \quad s, t \text{ are incomparable.} \end{aligned}$$

Let us check that this extends $(T, \leq)$ to a partial order on $T \cup T^*$, where $T^* = \{ t^* \mid t \in T \}$. Then this clearly becomes a regular tree set with involution * if we set $t^{**} := t$ for all $t \in T$.

The only non-trivial property to check is transitivity. For example, suppose that $r^* < s < t$ for some $r, s, t \in T$. The first inequality implies, by our definition of $<$, that r and s are incomparable in T. But then so are r and t (giving $r^* < t$ as desired): if $r < t$ then $r, s < t$, which makes r and s comparable (which they are not) since T is an order tree, while if $t < r$ then $s < t < r$ in T, again contradicting the incomparability of r and s. Similarly if $r^* < s^* < t$ then $t^* < s < r$, which as just seen implies $t^* < r$ and hence $r^* < t$.

Note that T^* is a consistent orientation of the tree set $T \cup T^*$, since $T \cap T^* = \emptyset$ and we never have $r < s^*$ for any $r^*, s^* \in T^*$.

If $(T, \leq)$ is connected then $(T \cup T^*, \leq, ^*)$, as defined above, is in fact the unique smallest regular tree set to which $(T, \leq)$ extends. Indeed, in any regular tree set τ extending T we must have $T^* \cap T = \emptyset$. For if $s, t \in T$ are comparable, with $s < t$ say, then $s^* = t$ would make s small, while if they are incomparable there will be an $r \in T$ below both (since T is connected), and $s^* = t$ would make $r < s, t$ trivial. But if τ is no larger than $T \cup T^*$ and this union is disjoint, then the only way in which $\leq$ on $\tau = T \cup T^*$ can extend $\leq$ on T while making τ into a tree set is the way we defined it. Indeed, if $s, t \in T$ are comparable, with $s > t$ say, we must have $s^* < t^*$ by (1). Whereas if they are incomparable, and $r \in T$ lies below both, we cannot have $s < t^*$, since this would imply $s < t^* < r^*$ and hence $r < s^*$ by (1), as well as $r < s$, making r trivial. Since τ is nested we must therefore have $s^* < t$, as we defined it.

Even if $(T, \leq)$ is not connected, $(T \cup T^*, \leq, ^*)$ as we defined it is the unique regular tree set τ that extends T in such a way that T^* becomes a consistent orientation of τ. Indeed, this assumption implies at once that τ is the disjoint union of T and T^*. And the consistency of T^* implies that $\leq$ on τ must be defined the way we did: for $s, t \in T$ we cannot have $s < t^*$, since that would make s^* inconsistent with t^*, so if s and t are incomparable we must have $s^* < t$.

As before in Section 5, we can recover every order tree from the consistently oriented tree set it defines in the way indicated above. But this time more is true: for every regular tree set $(\tau, \leq, ^*)$ and every consistent orientation O of τ there is an order tree T giving rise to τ and O in this way.

Indeed, let us show that $T = \{ \overleftarrow{s} \mid \vec{s} \in O \}$ is an order tree in the ordering induced from τ. To this end, consider $\vec{r}, \vec{s}, \vec{t} \in O$ with $\overleftarrow{r}, \overleftarrow{s} < \overleftarrow{t}$, and let us show that $\overleftarrow{r}, \overleftarrow{s}$ are comparable in τ. If not, then $\vec{r}$ is comparable with $\overleftarrow{s}$ (and $\overleftarrow{r}$ with $\vec{s}$), because r and s have comparable orientations since τ is a tree set. Since O is consistent we cannot have $\vec{r} > \overleftarrow{s}$, so $\vec{r} < \overleftarrow{s}$. But this implies that $\vec{t} < \vec{r} < \overleftarrow{s} < \overleftarrow{t}$ with a contradiction, since τ has no small elements.

If we now form $(T \cup T^*, \leq)$, as defined earlier, from the order tree T just obtained from O, we reobtain $(\tau, \leq)$: given incomparable $s, t \in T$, also $s^*, t^* \in O$ are incomparable in τ, which by the consistency of O means that s^* and t^* point towards each other. Hence $s^* < t$ (as well as $t^* < s$) in τ, as in the definition of our ordering on $T \cup T^*$.

We have proved the following:

Theorem 6.1. (i) *Every order tree $(T, \leq)$ extends to a regular tree set $(\tau, \leq, ^*)$ such that T^* is a consistent orientation of τ. This tree set is unique up to isomorphisms of separation systems fixing T pointwise.*

(ii) *For every consistent orientation O of a regular tree set $(\tau, \leq, ^*)$, the poset $(O^*, \leq)$ is an order tree.* □

7 Tree sets from nested subsets of a set

Let X be a non-empty set. The power set 2^X of X is a separation system with respect to inclusion and taking complements in X. It contains the empty set $\emptyset$ as a small element, but every nested subset of $2^X \setminus \{\emptyset\}$ is a regular tree set.

For compatibility with our earlier notion of set separations, let us refer to subsets A of X as special kinds of separations of X: those of the form $(A, X \setminus A)$. A *bipartition* of X, then, is an ordered pair (A, B) of disjoint non-empty subsets of X whose union is X. The bipartitions of X form a separation system $\vec{S}(X)$ with respect to their *natural ordering* $(A, B) \leq (C, D)$ defined by $A \subseteq C$ and the involution $(A, B) \mapsto (B, A)$. This separation system has no small elements, so every nested symmetric subset is a regular tree set.

Let us show that, conversely, every abstract regular tree set $(\tau, \le, {}^*)$ can be represented as a tree set of bipartitions of a set: that, given $(\tau, \le, {}^*)$, there exists a set X and a nested symmetric set $\vec{N}$ of bipartitions of X with a bijection $\tau \to \vec{N}$ that respects the orderings and involutions of the separation systems τ and $\vec{N}$.

Given τ, let $X = \mathcal{O}$ be the set of consistent orientations of τ. Every $\vec{s} \in \tau$ defines a bipartition (A, B) of X: into the set $A = \mathcal{O}(\overleftarrow{s})$ of consistent orientations of τ containing $\overleftarrow{s}$ and the set $B = \mathcal{O}(\vec{s})$ of those containing $\vec{s}$. Note that this is indeed a bipartition of $\mathcal{O}$; in particular, A and B are non-empty by Lemma 4.1 (i) applied to $\{\overleftarrow{s}\}$ and $\{\vec{s}\}$, respectively.

The map

$$f\colon \vec{s} \mapsto (\mathcal{O}(\overleftarrow{s}), \mathcal{O}(\vec{s}))$$

from τ to $\vec{S}(\mathcal{O})$ respects the involutions (obviously) and the partial orderings on τ and $\vec{S}(\mathcal{O})$. Indeed, if $\vec{r} < \vec{s}$ then no consistent orientation of τ containing $\overleftarrow{r}$ contains $\vec{s}$, so $\mathcal{O}(\overleftarrow{r}) \subseteq \mathcal{O}(\overleftarrow{s})$.[7] Conversely, let us show that if $\vec{r}, \vec{s} \in \tau$ are such that $\mathcal{O}(\overleftarrow{r}) \subseteq \mathcal{O}(\overleftarrow{s})$, equivalently $\mathcal{O}(\vec{s}) \subseteq \mathcal{O}(\vec{r})$, then $\vec{r} \le \vec{s}$. Since τ is nested, r and s have comparable orientations. We cannot have $\vec{s} > \overleftarrow{r}$, since this implies $\vec{r} \neq \vec{s}$ by the regularity of τ and hence makes $\{\vec{r}, \vec{s}\}$ inconsistent, which it cannot be since both $\vec{r}$ and $\vec{s}$ lie in every $O \in \mathcal{O}(\vec{s}) \subseteq \mathcal{O}(\vec{r})$, which exists by Lemma 4.1 (i) applied with $P = \{\vec{s}\}$. But neither can we have $\vec{s} < \vec{r}$ or $\vec{s} < \overleftarrow{r}$, since then $\{\overleftarrow{r}, \vec{s}\}$ is consistent and therefore, again by Lemma 4.1 (i), extends to a consistent orientation of τ that lies in $\mathcal{O}(\vec{s}) \smallsetminus \mathcal{O}(\vec{r})$, contradicting our assumption.

In order to see that f is injective, consider distinct $\vec{r}, \vec{s} \in \tau$. By renaming we may assume that $\vec{s} \not< \vec{r}$. Then $\{\vec{r}, \overleftarrow{s}\}$ is consistent and hence, by Lemma 4.1 (i), extends to a consistent orientation O of τ. As $O \in \mathcal{O}(\vec{r}) \smallsetminus \mathcal{O}(\vec{s})$, we have $\mathcal{O}(\vec{r}) \neq \mathcal{O}(\vec{s})$ and hence $f(\vec{r}) \neq f(\vec{s})$ as desired.

We have proved the following:

Theorem 7.1. (i) *Every nested set of bipartitions of some fixed set is a regular tree set.*

(ii) *Given any regular tree set τ, the map $f\colon \vec{s} \mapsto (\mathcal{O}(\overleftarrow{s}), \mathcal{O}(\vec{s}))$ from τ to the set $\vec{S}(\mathcal{O})$ of bipartitions of the set $\mathcal{O}$ of all consistent orientations of τ is an isomorphism of tree sets*[8] *between τ and its image $f(\tau)$ in $\vec{S}(\mathcal{O})$.*

By Theorem 7.1, every symmetric nested set $\vec{N}$ of bipartitions of a set X is a regular tree set τ, which we can in turn represent as a tree set $\vec{\mathcal{N}}$ of bipartitions of the set $\mathcal{O}$ of its consistent orientations. However, in the transition $\vec{N} \to \tau \to \vec{\mathcal{N}}$ we are likely to lose some information: we shall not be able to recover $\vec{N}$ from $\vec{\mathcal{N}}$, even up to a suitable bijection between X and $\mathcal{O}$.

For example, distinct $x, x' \in X$ may be indistinguishable by $\vec{N}$: there may be no partition in $\vec{N}$ that assigns x and x' to different partition classes. But distinct $O', O'' \in \mathcal{O}$ are always distinguished by a separation $(\mathcal{O}', \mathcal{O}'') \in \vec{S}(\mathcal{O})$ in the image $\vec{\mathcal{N}}$ of f. For since $O' \neq O''$ there exists $\vec{s} \in \tau$ with $\overleftarrow{s} \in O'$ and $\vec{s} \in O''$. Then $O' \in \mathcal{O}'$ but $O'' \in \mathcal{O}''$ for $(\mathcal{O}', \mathcal{O}'') = f(\vec{s})$.

[7] Note that we just used the regularity of τ: if $\vec{r}$ is small, we can have $\vec{r} < \vec{s} = \overleftarrow{r}$, in which case 'both' $\overleftarrow{r}$ and $\vec{s}$ can occur in the same consistent orientation of τ.

[8] i.e., an isomorphism of separation systems (which happen to be tree sets)

And there is another feature of $\vec{\mathcal{N}} = f(\tau)$ which $\vec{N}$ need not have: each of its consistent orientations – which are just the elements of $\mathcal{O}$ viewed in another way – is induced by one point O of $\mathcal{O}$, in the sense that it contains precisely those elements $(\mathcal{O}', \mathcal{O}'')$ of $\vec{\mathcal{N}}$ for which $O \in \mathcal{O}''$. By contrast, a consistent orientation of $\vec{N}$ need not have the form $\{ (A,B) \in \vec{S}(X) \mid x \in B \}$ for any particular $x \in X$.

For example, let X be the vertex set of a ray R, let $\vec{N}$ be the set of bipartitions of X corresponding to the oriented edges of R, and choose from every inverse pair of separations in $\vec{N}$ the separation that corresponds to the oriented edge of R which points towards its tail. This is a consistent orientation of $\vec{N}$ that does not have the above form. Or let $\vec{N}$ correspond to the three edges of a 3-star, orient every edge towards the centre of that star, and then consider the star of separations that this induces on the 3-set of only the leaves x_1, x_2, x_3. Once more, this is a consistent orientation O of $\vec{N}$ that is not of the above form, since no leaf x lies in $\{x_j, x_k\}$ for each of the three leaf partitions $(\{x_i\}, \{x_j, x_k\})$ in O.

However, if we assume for $\vec{N}$ these two properties that $\vec{\mathcal{N}}$ will invariably have, we can indeed recover it from $\vec{\mathcal{N}}$ in the best way possible, namely, up to a specified bijection between the ground sets involved:

Theorem 7.2. *Let $\vec{N}$ be a tree set of bipartitions of a set X such that*

- *for all distinct $x, y \in X$ there exists $(A,B) \in \vec{N}$ such that $x \in A$ and $y \in B$;*
- *for every consistent orientation O of $\vec{N}$ there exists an $x \in X$ such that $O = \{ (A,B) \in \vec{N} \mid x \in B \}$.*

Let τ be any tree set with an isomorphism $g \colon \vec{N} \to \tau$ of separation systems. Let $\mathcal{O}$ be the set of consistent orientations of τ, and let $f \colon \tau \to \vec{S}(\mathcal{O})$ be the map from Theorem 7.1 (ii)*. Then there is a bijection $h \colon X \to \mathcal{O}$ whose natural action on $\vec{N}$ equals $f \circ g$. In this way, $\vec{N}$ is canonically isomorphic as a tree set to the image $\vec{\mathcal{N}}$ of τ under f.*

Proof. Given $x \in X$, let $O_x = \{ (A,B) \in \vec{N} \mid x \in B \}$; this is clearly a consistent orientation of $\vec{N}$. Hence $h \colon x \mapsto g(O_x)$ is a well defined map from X to $\mathcal{O}$. It is injective by the first condition in the theorem, and surjective by the second. Its action on the subsets of X therefore maps partitions of X to partitions of $\mathcal{O}$. The induced action of h on $\vec{N}$ is easily seen to equal $f \circ g$, which is an isomorphism of tree sets by the choice of g and Theorem 7.1 (ii). □

Theorem 7.1 (ii) provides us with a standard representation of an abstract regular tree set τ as a tree set $\vec{\mathcal{N}}$ of bipartitions of a set, and Theorem 7.2 shows that this standard representation describes, up to isomorphisms of tree sets, all the representations of τ as a tree set $\vec{N}$ of bipartitions of a set X that is not unnecessarily large (ie, constains no two elements indistinguishable by the tree set) but large enough that every consistent orientation of $\vec{N}$ is induced by one of its elements.

When τ is finite, there is an equally standard, but 'smaller', alternative way to represent it as a tree set of bipartitions of a set. As an example, consider the tree set of the oriented edges of a finite tree T. Every edge of $\vec{e} \in \vec{E}(T)$ defines a bipartition (A,B) of its vertex set: into the set B of vertices of T to which $\vec{e}$ points and the set A of vertices to which $\overleftarrow{e}$ points. These bipartitions of $V(T)$ are nested, and the tree set they form is isomorphic to the tree set $\vec{E}(T)$

by this map $\vec{e} \mapsto (A,B)$. Now consider the bipartitions (A',B') which these (A,B) induce just on the set of leaves of T. These (A',B'), too, will be distinct for different edges $\vec{e} \in \vec{E}(T)$ as long as T has no vertex of degree 2, and they will be nested in the same way as the (A,B) that defined them. So these bipartitions of the leaves of T will still form a tree set isomorphic to $\vec{E}(T)$, by the map $\vec{e} \mapsto (A',B')$ indicated above. In particular, we can recover (A,B) from (A',B') from this isomorphism, as $(A',B') \mapsto \vec{e} \mapsto (A,B)$.

Let us generalize this observation to arbitrary finite regular tree sets τ. Let $\mathcal{O}'$ be the set of those consistent orientations of τ that have a greatest element.[9]

The $O \in \mathcal{O}'$ are precisely the down-closures in τ of its maximal elements. Indeed, if $\vec{s}$ is maximal in τ then $\lceil \vec{s} \rceil$ is a consistent orientation of all of τ: since τ is a nested, for every $\vec{r} \in \tau$ either $\vec{r}$ or $\overleftarrow{r}$ is comparable to $\vec{s}$, and since it cannot be bigger it must lie in its down-closure. As $\vec{s}$ is clearly the greatest element in its down-closure, we have $\lceil \vec{s} \rceil \in \mathcal{O}'$ as claimed. Conversely, the greatest element $\vec{s}$ of a consistent orientation O of τ clearly satisfies $\lceil \vec{s} \rceil = O$, and it is maximal in τ because τ is regular: if $\vec{s} < \vec{r}$ then $\vec{r} \notin O$ by the maximality of $\vec{s}$, so $\overleftarrow{r} \in O$ and hence $\overleftarrow{r} \le \vec{s} < \vec{r}$, making $\overleftarrow{r}$ small.

As τ is finite, each of its elements lies below some maximal one and therefore in some $O \in \mathcal{O}'$. In particular $\mathcal{O}' \neq \emptyset$ if $\tau \neq \emptyset$. Let us see when τ is represented canoncially by a tree set of bipartitions of $\mathcal{O}'$.

Write $\mathcal{O}'(\vec{s}) := \{\, O \in \mathcal{O}' \mid \vec{s} \in O \,\}$. As in Theorem 7.1 (ii), the map

$$f' \colon \vec{s} \mapsto (\mathcal{O}'(\overleftarrow{s}), \mathcal{O}'(\vec{s}))$$

from τ to the set $\vec{S}(\mathcal{O}')$ of bipartitions of $\mathcal{O}'$ is an isomorphism of tree sets between τ and its image $f'(\tau)$ in $\vec{S}(\mathcal{O}')$ whenever it is injective. Let us show that f' is injective if and only if τ has no splitting stars of order 2.[10]

Suppose first that $\sigma = \{\vec{r}, \vec{s}\}$ splits τ, with $\vec{r} \neq \vec{s}$. Then every $O \in \mathcal{O}'$ contains either $\vec{r}$ and $\overleftarrow{s}$ or $\vec{s}$ and $\overleftarrow{r}$: it cannot contain $\overleftarrow{\sigma}$ by consistency, and it cannot contain σ by definition of $\mathcal{O}'$, as $|\sigma| > 1$. Therefore $f'(\vec{r}) = f'(\overleftarrow{s})$ and $f'(\vec{s}) = f'(\overleftarrow{r})$, showing that f' is not injective.

Conversely, assume that τ contains no splitting star of order 2. To show that f' is injective, consider distinct $\vec{r}, \vec{s} \in \tau$. We shall find an $O \in \mathcal{O}'$ that contains one of these but not the other; then $f'(\vec{r}) \neq f'(\vec{s})$ by definition of f'. If $\{\vec{r}, \vec{s}\}$ is inconsistent, pick any $O \in \mathcal{O}'$ containing $\vec{r}$ (which we have seen always exists); then $\vec{s} \notin O$ by the consistency of O. If $\{\vec{r}, \vec{s}\}$ is a star, pick any $O \in \mathcal{O}'$ containing $\overleftarrow{s}$; this will also contain $\vec{r} \le \overleftarrow{s}$ but not $\vec{s}$. Finally, assume that $\vec{r} < \vec{s}$. Then $\{\vec{r}, \overleftarrow{s}\}$ is a consistent subset of τ which, by Lemma 4.1 extends to a unique consistent orientation O of τ in which $\vec{r}$ is maximal. The set σ of maximal elements of O contains $\vec{r}$ and some $\overleftarrow{s'} \ge \overleftarrow{s}$. This $\overleftarrow{s'}$ is not $\vec{r}$, as otherwise $\overleftarrow{s} < \overleftarrow{r}$ (by assumption) as well as $\overleftarrow{s} \le \overleftarrow{s'} = \vec{r}$, making $\overleftarrow{s}$ small (contradicting the regularity of τ). As $|\sigma| \neq 2$, there is a third element $\vec{r'} \in \sigma \setminus \{\vec{r}, \overleftarrow{s'}\}$. Pick $O' \in \mathcal{O}'$ so as to contain $\overleftarrow{r'}$. Then O' also contains $\vec{r} \le \overleftarrow{r'}$ and $\overleftarrow{s} \le \overleftarrow{s'} \le \overleftarrow{r'}$, and hence does not contain $\vec{s}$.

[9] In our example, these were the consistent orientations of $\vec{E}(T)$ that point to a leaf of T: every edge $\vec{e}$ with a leaf as its terminal vertex is the greatest element in the unique consistent orientation of $\vec{E}(T)$ to which $\{\vec{e}\}$ extends, and no other consistent orientation of $\vec{E}(T)$ has a greatest element, because the set of its maximal elements has the form $\vec{F}_t$ for some $t \in T$.

[10] In our example, f' is injective if and only if T has no vertex of degree 2.

We have thus proved the following variant of Theorem 7.1 for finite tree sets:

Theorem 7.3. *Let τ be any finite regular tree set. Let $\vec{S}(\mathcal{O}')$ be the separation system of the bipartitions of the set $\mathcal{O}'$ of all consistent orientations of τ that have a greatest element. Then the map $f'\colon \vec{s} \mapsto (\mathcal{O}'(\overleftarrow{s}), \mathcal{O}'(\vec{s}))$ from τ to $\vec{S}(\mathcal{O}')$ is an isomorphism of tree sets, between τ and its image $f'(\tau)$ in $\vec{S}(\mathcal{O})$, if and only if it is injective, which it is if and only if τ has no splitting star of order 2.*

What about an analogue of Theorem 7.2? As before, the tree set $f'(\tau)$ of bipartitions of $\mathcal{O}'$ will distinguish every two elements of $\mathcal{O}'$, so $\vec{N}$ must satisfy this for X if we wish to recover a copy of it on $\mathcal{O}'$. For every $O \in \mathcal{O}'$ there will be a unique maximal element $\vec{s}$ of τ such that $O = \lceil \vec{s} \rceil$, and conversely O will be the only consistent orientation of τ containing $\vec{s}$. If we want there to be a bijection $h'\colon X \to \mathcal{O}'$ as in Theorem 7.2, we must therefore ask the same of X and $\vec{N}$: that for every $x \in X$ there be a unique maximal element (A, B) of $\vec{N}$ such that $x \in B$, and that these (A, B) differ for different choices of x, i.e., that $B = \{x\}$. Thus, $\vec{N}$ has to contain all the separations $(X \setminus \{x\}, \{x\})$ for $x \in X$.

As these separations already distinguish every two elements of X, we shall no longer have to require this explicitly in order to make h' injective. Also, we do not have to require explicitly, in order to make h' surjective, that no consistent orientations of $\vec{N}$ other than those with a greatest element be of the form $O_x = \{\, (A, B) \in \vec{N} \mid x \in B \,\}$: since $(X \setminus \{x\}, \{x\}) \in \vec{N}$, this separation will lie in O_x and thus be its greatest element.

With these provisions, Theorem 7.2 adapts as follows:

Theorem 7.4. *Let $\vec{N}$ be a finite tree set of bipartitions of a set X that contains all the partitions $(X \setminus \{x\}, \{x\})$ with $x \in X$. Let τ be any tree set with an isomorphism $g\colon \vec{N} \to \tau$ of separation systems. Let $\mathcal{O}'$ be the set of consistent orientations of τ that have a greatest element, and let $f'\colon \tau \to \vec{S}(\mathcal{O}')$ be the map from Theorem 7.3. Then there is a bijection $h\colon X \to \mathcal{O}'$ whose natural action on $\vec{N}$ equals $h = f \circ g$. In this way, $\vec{N}$ is canonically isomorphic as a tree set to the image $\vec{\mathcal{N}}$ of τ under f.*

We omit the proof.

The sets X in Theorems 7.2 and 7.4 are, in a sense, maximal and minimal, respectively, for the existence of a tree set $\vec{N}$ of bipartitions of X that represents a given abstract finite tree set τ. While in Theorem 7.2 the set X has enough elements x to give every consistent orientation of $\vec{N}$ the form O_x, this is the case in Theorem 7.4 only for the orientations of $\vec{N}$ that have a greates element, where it cannot be avoided.

If desired, however, we can have any mixture of these extremes that we like. Indeed, starting with τ we can build X by assigning to every $O \in \mathcal{O}$ a set X_O that is either empty or a singleton $\{x_O\}$, making sure that $X_O \neq \emptyset$ if $O \in \mathcal{O}'$. Then for $X := \bigcup_{O \in \mathcal{O}} X_O$ a separation $\vec{s} \in \tau$ will be represented by the partition (A, B) of X in which $B = \bigcup \{\, X_O \mid \vec{s} \in O \,\}$ and $A = \bigcup \{\, X_O \mid \overleftarrow{s} \in O \,\}$. These ideas will be developed further in the next section.

8 Tree sets from tree-decompositions of graphs and matroids

In this section we clarify the relationship between finite tree-decompositions, the more general 'S-trees' introduced in [6], and nested separation systems and tree sets. Given a tree-decomposition of a finite graph or matroid X, the separations of X that correspond to the edges of the decomposition tree are always nested. If they form a tree set then, pathological exceptions aside, the decomposition can be recovered from it, and the purpose of this section is to show how.

The point of doing this is to establish that tree sets, which are more versatile for infinite combinatorial structures (even just for graphs) than tree-decompositions, are also at least as powerful as tree-decompositions when they are finite: if desired, we can construct from the separations in any finite tree set of separations of a graph or matroid a tree-decomposition of this graph or matroid whose tree edges correspond to precisely these separations.

Given a graph G and a family $\mathcal{V} = (V_t)_{t \in T}$ of subsets of its vertex set indexed by the node of a tree T, the pair $(\mathcal{V}, T)$ is called a *tree-decomposition* of G if G is the union of the subgraphs $G[V_t]$ induced by these subsets, and $V_t \cap V_{t''} \subseteq V_{t'}$ whenever t' lies on the t–t'' path in T. The *adhesion sets* $V_{t_1} \cap V_{t_2}$ of $(\mathcal{V}, T)$ corresponding to the edges $e = t_1 t_2$ of T then separate the sets $U_1 := \bigcup_{t \in T_1} V_t$ from $U_2 := \bigcup_{t \in T_2} V_t$ in G, where T_i is the component of $T - e$ containing t_i, for $i = 1, 2$; see [3]. These separations $\{U_1, U_2\}$ are the separations of G *associated with* $(T, \mathcal{V})$, and with the edges of T.

Tree-decompositions can be described entirely in terms of T and the oriented separations $\alpha(t_1, t_2) := (U_1, U_2)$ of G associated with its edges. Indeed, we can recover its parts V_t from these separations as the sets $V_t = \bigcap \{\, B \mid (A, B) \in \sigma_t \}$, where σ_t is the star of the separations $\alpha(x, t)$ with x adjacent to t in T. Our aim in this section is to see under what assumptions the tree-decomposition can be recovered not only from this nested set τ of separations together with the information of how it relates to T, but from the set τ alone.

In an intermediate step, let us use both T and the set of separations corresponding to its edges to view $(\mathcal{V}, T)$ in the following more general set-up from [6]. Let $\vec{S}$ be a separation system, and let $\mathcal{F} \subseteq 2^{\vec{S}}$. An *$S$-tree* is a pair (T, α) of a tree T and a function $\alpha \colon \vec{E}(T) \to \vec{S}$ such that

(i) for every edge xy of T, if $\alpha(x, y) = \vec{s}$ then $\alpha(y, x) = \overleftarrow{s}$.

(T, α) is an S-tree *over* $\mathcal{F} \subseteq 2^{\vec{S}}$ if, in addition,

(ii) for every[11] node t of T we have $\alpha(\vec{F}_t) \in \mathcal{F}$.

(As defined in Section 5, $\vec{F}_t$ is the oriented star at t in T.) We shall say that the set $\alpha(\vec{F}_t) \subseteq \vec{S}$ is *associated with* t in (T, α). The sets $\mathcal{F}$ we shall consider will all be *standard*, which means that they contain every co-trivial singleton $\{\overleftarrow{r}\}$ in $\vec{S}$.

Since tree-decompositions can be recovered from the S-trees they induce, as pointed out earlier, our remaining task is to see which S-trees can be recovered just from the set $\alpha(\vec{E}(T))$ of their separations. As it turns out, this will be possible once we have trimmed a given S-tree down to its 'essence', which is done in three steps.

[11] By definition [3], trees must have at least one node.

An S-tree (T, α) is *redundant* if it has a node t of T with distinct neighbours t', t'' such that $\alpha(t, t') = \alpha(t, t'')$; otherwise it is *irredundant*. Redundant S-trees can be pruned to irredundant ones over the same $\mathcal{F}$, simply by deleting those 'redundant' branches of the tree:

Lemma 8.1. *For every finite S-tree (T, α) over some $\mathcal{F} \subseteq 2^{\vec{S}}$ there is an irredundant S-tree (T', α') over $\mathcal{F}$ such that $T' \subseteq T$ and $\alpha' = \alpha \restriction \vec{E}(T')$.*

Proof. Let $t \in T$ have neighbours t', t'' witnessing the redundance of (T, α). Deleting from T the component C of $T - t$ that contains t'' turns (T, α) into an S-tree in which $\vec{F}_t$ has changed but $\alpha(\vec{F}_t)$ has not, and neither has $\alpha(\vec{F}_x)$ for any other node $x \in T - C$. So this is still an S-tree over $\mathcal{F}$. As T is finite, we obtain the desired S-tree (T', α') by iterating this step. □

An important example of S-trees are irredundant S-trees *over stars*: those over some $\mathcal{F}$ all of whose elements are stars of separations. In such an S-tree (T, α) the map α preserves the natural partial ordering on $\vec{E}(T)$ defined at the start of Section 5:

Lemma 8.2. *Let (T, α) be an irredundant S-tree over stars. Let $\vec{e}, \vec{f} \in \vec{E}(T)$.*

(i) *If $\vec{e} \le \vec{f}$ then $\alpha(\vec{e}) \le \alpha(\vec{f})$. In particular, the image of α in $\vec{S}$ is nested.*

(ii) *If $\alpha(\vec{e}) < \alpha(\vec{f})$ then $\vec{e} < \vec{f}$, unless either $\alpha(\vec{e}) = \alpha(\overleftarrow{f})$ is small, or $\alpha(\vec{e})$ or $\alpha(\overleftarrow{f})$ is trivial.*

Proof. (i) Assume first that e and f are adjacent; then $\vec{e}, \overleftarrow{f} \in \vec{F}_t$ for some $t \in T$. As (T, α) is irredundant we have $\alpha(\vec{e}) \neq \alpha(\overleftarrow{f})$, and hence $\alpha(\vec{e}) \le \alpha(\vec{f})$ since $\alpha(\vec{F}_t)$ is a star. By induction on the length of the e–f path in T this implies (i) also for nonadjacent e and f.

(ii) Suppose $\vec{e} \not< \vec{f}$. Since e and f are nested, we then have

$$\vec{e} \ge \vec{f} \quad \text{or} \quad \vec{e} \ge \overleftarrow{f} \quad \text{or} \quad \vec{e} \le \overleftarrow{f}.$$

If $\vec{e} \le \overleftarrow{f}$, we have $\alpha(\vec{e}) \le \alpha(\overleftarrow{f})$ by (i), while $\alpha(\overleftarrow{f}) < \alpha(\overleftarrow{e})$ by assumption and (1) (and the fact that α commutes with inversion). If even $\alpha(\vec{e}) < \alpha(\overleftarrow{f})$, then $\alpha(\vec{e})$ is trivial. Otherwise, $\alpha(\vec{e}) = \alpha(\overleftarrow{f}) < \alpha(\overleftarrow{e})$ is small.

Suppose next that $\vec{e} \ge \overleftarrow{f}$. Then $\alpha(\overleftarrow{f}) \le \alpha(\vec{e})$ by (i), while $\alpha(\overleftarrow{f}) < \alpha(\overleftarrow{e})$ by assumption. If even $\alpha(\overleftarrow{f}) < \alpha(\vec{e})$ then $\alpha(\overleftarrow{f})$ is trivial. Otherwise, $\alpha(\vec{e}) = \alpha(\overleftarrow{f}) < \alpha(\overleftarrow{e})$ is small.

Suppose finally that $\vec{e} \ge \vec{f}$. Then $\alpha(\vec{e}) < \alpha(\vec{f}) \le \alpha(\vec{e})$ by assumption and (i), a contradiction. □

By Lemma 8.2 (i), the separations in an irredundant S-tree over stars are nested. For redundant S-trees this need not be so: if $\alpha(\vec{e}) = \alpha(\vec{f})$ for $\vec{e}, \vec{f} \in \vec{F}(t)$, then separations $\alpha(\vec{e}')$ with $\vec{e}' < \vec{e}$ may cross separations $\alpha(\vec{f}')$ with $\vec{f}' < \vec{f}$. This is because we defined stars of separations as sets, not as multisets: for $\vec{e}$ and $\vec{f}$ as above we do not require that $\alpha(\vec{e}) \le \alpha(\overleftarrow{f})$ when we ask that $\alpha(\vec{F}_t)$ be a star, since $\alpha(\vec{e}) = \alpha(\vec{f})$ are not distinct elements of $\alpha(\vec{F}_t)$.

Lemma 8.2 (ii) is best possible in that all the cases mentioned can occur independently. We also need the inequalities to be strict, unless we assume that the S-tree is tight (see below).

Two edges of an irredundant S-tree over stars cannot have orientations pointing towards each other that map to the same separation, unless this is trivial:

Lemma 8.3. *Let (T,α) be an irredundant S-tree over a set $\mathcal{F}$ of stars. Let e, f be distinct edges of T with orientations $\vec{e} < \overleftarrow{f}$ such that $\alpha(\vec{e}) = \alpha(\vec{f}) =: \vec{r}$. Then $\vec{r}$ is trivial.*

Proof. If α maps all $\vec{e'}$ with $\vec{e} < \vec{e'} < \overleftarrow{f}$ to $\vec{r}$ or to $\overleftarrow{r}$, then the e–f path in T has a node with two incoming edges mapped to $\vec{r}$. This contradicts our assumption that (T,α) is irredundant. Hence there exists such an edge $\vec{e'}$ with $\alpha(\vec{e'}) = \vec{s}$ for some $s \neq r$. Lemma 8.2 implies that $\vec{r} = \alpha(\vec{e}) \leq \alpha(\vec{e'}) \leq \alpha(\overleftarrow{f}) = \overleftarrow{r}$, so $\vec{r} \leq \vec{s}$ as well as $\vec{r} \leq \overleftarrow{s}$ by (1). As $s \neq r$ these inequalities are strict, so s witnesses that $\vec{r}$ is trivial. □

Let us call an S-tree (T,α) *tight* if its sets $\alpha(\vec{F}_t)$ are antisymmetric. The name 'tight' reflects the fact that from any S-tree we can obtain a tight one over the same $\mathcal{F}$ by contracting edges:

Lemma 8.4. *For every finite S-tree (T,α) over some $\mathcal{F} \subseteq 2^{\vec{S}}$ there exists an irredundant and tight S-tree (T',α') over $\mathcal{F}$ such that T' is a minor of T and $\alpha' = \alpha \restriction \vec{E}(T')$.*

Proof. By Lemma 8.1 we may assume that (T,α) is irredundant. Consider any node t of T for which $\alpha(\vec{F}_t)$ is not antisymmetric. Then t has distinct neighbours t', t'' such that $\alpha(t',t) = \alpha(t,t'') =: \overleftarrow{s}$. Let T' be obtained from T by contracting one of these edges and any branches of T attached to t by edges other than these two.[12] Let $\alpha'(t',t'') := \vec{s}$ and $\alpha'(t'',t') := \overleftarrow{s}$, and otherwise let $\alpha' := \alpha \restriction \vec{E}(T')$. Then (T',α') is again an S-tree, whose sets $\vec{F}_t$ in T' are the same as they were in T, for every $t' \in T'$. In particular, (T',α') is still irredundant and an S-tree over $\mathcal{F}$. Iterate this step until the S-tree is tight. □

Let us call an S-tree (T,α) *essential* if it is irredundant, tight, and $\alpha(\vec{E}(T))$ contains no trivial or degenerate separation. Let the *essential core* of a set $\mathcal{F} \subseteq 2^{\vec{S}}$ be the set of all $F' \subseteq \vec{S}$ obtained from some $F \in \mathcal{F}$ by deleting all its trivial or degenerate elements. And call $\mathcal{F}$ *essential* if it equals its essential core.

An S-tree over stars can be made essential by first pruning it to make it irredundant (Lemma 8.1), then contracting the pruned tree to make it tight (Lemma 8.4), and finally deleting any edges mapping to trivial separations:

Lemma 8.5. *For every irredundant and tight finite S-tree (T,α) over a set $\mathcal{F}$ of stars there is an essential S-tree (T',α') over the essential core of $\mathcal{F}$ such that $T' \subseteq T$ and $\alpha' = \alpha \restriction \vec{E}(T')$.*

Proof. Recall that if $\vec{s} \in \vec{S}$ is trivial then so is every $\vec{r} \leq \vec{s}$. By Lemma 8.2, therefore, the set $\vec{F}$ of all edges $\vec{e} \in \vec{E}(T)$ such that $\alpha(\vec{e})$ is trivial is closed down in $\vec{E}(T)$. Hence the subgraph T' of T obtained by deleting each of these edges e together with the initial vertex of $\vec{e}$ is connected, and therefore a tree: it may be edgeless, but it will not be empty, since the target vertex of any maximal edge in $\vec{F}$ will be in T'.

[12] In other words: delete the component of $T - t't - tt''$ containing t, and join t' to t''. Then think of the edge $t't''$ as the old edge $t't$, so that $E(T') \subseteq E(T)$ as desired.

If there is no $\vec e \in \vec E(T')$ with $\alpha(\vec e)$ degenerate, then (T', α') has all the properties claimed. If there is, then e is the only edge of T'. Indeed, if $e = tt'$ and $f \neq e$ is another edge at t, with $\vec f$ pointing towards t say, then $\alpha(\vec f) \le \alpha(\overleftarrow e) = \alpha(\vec e)$ by Lemma 8.2, and the inequality is strict because (T, α) is tight. This would make $\alpha(\vec f)$ trivial, contradicting the definition of T'. But this means that $\alpha(\vec F_t) \in \mathcal F$ contains only trivial and denerate separations, and hence that $\emptyset$ lies in the core of $\mathcal F$. Choosing the one-node tree as T' now satisfies the assertion. □

Combining Lemmas 8.4 and 8.5, we obtain

Corollary 8.6. *For every finite S-tree (T, α) over a set $\mathcal F$ of stars there is an essential S-tree (T', α') over the essential core of $\mathcal F$ such that T' is a minor of T and $\alpha' = \alpha \restriction \vec E(T')$.* □

Lemma 8.7. *For every essential S-tree (T, α) over stars the map α is injective.*

Proof. Suppose there are distinct $\vec e, \vec f \in \vec E(T)$ with $\alpha(\vec e) = \alpha(\vec f) =: \vec s$. Then also $e \neq f$: otherwise $\vec e = \overleftarrow f$ making $\vec s$ degenerate, contrary to our assumptions.

Suppose first that $\vec e < \vec f$ in the natural order on $\vec E(T)$. By Lemma 8.2, every $\vec{e'} \in \vec E(T)$ with $\vec e \le \vec{e'} \le \vec f$ satisfies $\vec s = \alpha(\vec e) \le \alpha(\vec{e'}) \le \alpha(\vec f) = \vec s$, so $\alpha(\vec{e'}) = \vec s$. As $\vec s \neq \overleftarrow s$, this contradicts our assumption that (T, α) is tight, since $\vec s, \overleftarrow s \in \alpha(\vec F_t)$ for the terminal node t of $\vec e$.

Suppose now that $\vec e < \overleftarrow f$. Then $\vec s$ is trivial by Lemma 8.3, contradicting our assumption that (T, α) is essential.

Up to renaming $\vec e$ and $\vec f$ as $\overleftarrow e$ and $\overleftarrow f$, this covers all cases. □

As we have seen, a tree-decomposition $(\mathcal V, T)$ of a graph or matroid can be recaptured from the structure of T and the family $(\alpha(\vec e) \mid \vec e \in \vec E(T))$ of oriented separations it induces, i.e., from the S-tree (T, α). We can now show that if this S-tree is essential (which we may often assume, cf. Lemmas 8.1, 8.4 and 8.5), it can in turn be recovered from just the *set* of these oriented separations.

Recall that a subset σ of a nested separation system $(\tau, \le, {}^*)$ *splits* it if σ is the set of maximal elements of some consistent orientation of τ and $\tau \subseteq \lceil \sigma \rceil$ (which is automatic when τ is finite). These sets σ are proper stars in τ, its *splitting stars*, and they contain no separations that are trivial in τ (Lemma 4.4) or co-trivial (Lemma 3.2). Except for one exceptional case where σ contains a small separation, they are precisely the maximal proper stars in τ (Lemma 4.5).

Let us say that τ is a nested separation system *over* $\mathcal F \subseteq 2^\tau$ if all its splitting stars lie in $\mathcal F$.

Theorem 8.8. *Let $\vec S$ be a finite separation system, and $\mathcal F \subseteq 2^{\vec S}$ a set of stars.*

(i) *If $\vec S$ is nested and over $\mathcal F$, and $\mathcal F$ contains no degenerate singleton, then there exists an essential S-tree (T, α) over $\mathcal F$ whose sets $\{\, \alpha(\vec F_t) \mid t \in T \,\}$ are precisely the stars splitting $\vec S$. The map α is injective, and its image is the tree set which $\vec S$ induces.*

(ii) *If (T, α) is an essential S-tree over $\mathcal F$, then $\alpha(\vec E(T))$ is a tree set over $\mathcal F$ whose splitting stars are precisely the sets $\{\, \alpha(\vec F_t) \mid t \in T \,\}$.*

Proof. (i) If $\vec S = \emptyset$ let T consist of one node. Assume now that $\vec S \neq \emptyset$. Let τ be the set of separations in $\vec S$ that are neither trivial nor co-trivial. By Lemma 2.1 applied to S, we have $\tau \neq \emptyset$. We shall see in a moment that τ contains no degenerate separations either, and is therefore the tree set which $\vec S$ induces.

Let us show that the stars σ splitting τ are precisely those that split $\vec{S}$; in particular, they lie in $\mathcal{F}$. Given σ splitting τ, there is a consistent orientation O of τ whose set of maximal elements is σ. By Lemma 4.1 (i), O extends to a unique consistent orientation O' of $\vec{S}$ obtained by adding all the trivial elements of $\vec{S}$. By Lemma 2.1, each of these lies below an element of O. Hence σ is the set of maximal elements also of O'. It therefore splits $\vec{S}$, and thus lies in $\mathcal{F}$ by assumption. Conversely, the stars splitting $\vec{S}$ are subsets of τ and therefore split it, since maximal elements of a consistent orientation of S can be neither trivial nor cotrivial.

Let us show that τ contains no degenerate $\vec{s} \in \vec{S}$. If it does, then $\tau = \{\vec{s}\}$, since $\vec{S}$ is nested and hence any other $r \in S$ has a trivial orientation $\vec{r} \le \vec{s} = \overleftarrow{s}$, implying $\vec{r}, \overleftarrow{r} \notin \tau$. But then $\tau = \{\vec{s}\}$ splits itself and hence lies in $\mathcal{F}$. This contradicts our assumption that $\mathcal{F}$ contains no degenerate singletons.

We begin the construction of our desired tree T by defining its nodes as the consistent orientations of τ. For each $\vec{s} \in \tau$ there is a unique such orientation $t(\vec{s})$ in which $\vec{s}$ is maximal, by Lemma 4.1 (iii) applied with $P = \{\vec{s}\}$. Let $\vec{T}$ be the directed graph on these nodes with edge set τ, where $\vec{s} \in \tau$ runs from $t(\overleftarrow{s})$ to $t(\vec{s})$. Note that these are distinct, since $\vec{s} \neq \overleftarrow{s}$ as τ has no degenerate elements. Let T be the underlying undirected graph, with pairs $\vec{s}, \overleftarrow{s}$ of directed edges identified into one undirected edge s inheriting its orientations $\vec{s}, \overleftarrow{s}$ from $\vec{T}$.

For each $t \in V(T)$, the set $\vec{F}_t$ of its incoming edges is precisely the set of all $\vec{s} \in \tau$ that are maximal in the orientation t of τ. So these $\vec{F}_t$ are precisely the stars splitting τ, which we have seen lie in $\mathcal{F}$. Thus, once we have checked that T is indeed a tree, we will have shown that it is an S-tree (T, α) over $\mathcal{F}$ with α the identity. It will be essential by definition of τ, our observation that τ contains no degenerate separations, and the fact that different edges of $\vec{T}$ are distinct elements of τ.

We noticed before that $t(\vec{s}) \neq t(\overleftarrow{s})$ for all $\vec{s} \in \tau$, so T has no loops. In fact, T is acyclic: if $\vec{s_0}, \dots, \vec{s_k}$ are the edges of an oriented cycle in $\vec{T}$, then each of these and the inverse of its (cyclic) successor lie in a common oriented star $\vec{F}_t$. Since these $\vec{F}_t$ split τ, they are also stars of separations (Lemma 4.5), which implies that $\vec{s_0} < \dots < \vec{s_k} < \vec{s_0}$ with a contradiction.

To see that T is connected, let t, t' be nodes in different components that agree, as partial orientations of S, on as many $s \in S$ as possible. (Formally: choose t, t' with $|t \cap t'|$ maximum.) Let $\vec{s}$ be maximal in $\tau \smallsetminus (t \cap t')$. As t and t' disagree on s, each contains one of the two orientations of s; we assume that $\vec{s} \in t$. Then $\vec{s}$ is maximal also in t: any $\vec{s'} \in t$ greater than $\vec{s}$ would also lie in t', and hence so would $\vec{s}$ by the consistency of t' (which also orients s). Replacing $\vec{s}$ in t with $\overleftarrow{s}$ therefore changes t into an orientation of τ that is again consistent, by the maximality of $\vec{s}$ in t. In this consistent orientation t'' of τ the separation $\overleftarrow{s}$ is maximal: for any $\overleftarrow{r} > \overleftarrow{s}$ we have $\vec{r} < \vec{s} \in t$, so $\vec{r} \in t$ by the consistency of t and hence $\overleftarrow{r} \notin t$. Hence $s = tt''$, and in particular t'' lies in the same component of T as t. Since it agrees with t' on more separations than t does, we have a contradiction to the choice of t and t'.

(ii) By Lemma 8.7, the map α is injective, and by Lemma 8.2 (i) it preserves the natural ordering of $\vec{E}(T)$. By Lemma 8.2 (ii), also α^{-1} preserves the ordering of every antisymmetric subset of its domain, and hence of any orientation of $\alpha(\vec{E}(T))$. Thus, α and α^{-1} are order isomorphisms on all orientations of their domains.

The consistent orientations of $E(T)$ are clearly those that orient all edges towards some fixed node of T. So the splitting stars of the tree set $\vec{E}(T)$ are the sets $\vec{F}_t$. The splitting stars of $\alpha(\vec{E}(T))$, therefore, are their images $\alpha(\vec{F}_t)$, and these lie in $\mathcal{F}$ by assumption. □

Let us extract some more tangible corollaries from Theorem 8.8. Let $(\vec{S}, \leq, ^*)$ be a separation system. An *isomorphism* between two S-trees (T, α) and (T', α') is an isomorphism φ of the trees T and T' that commutes with α and α', i.e., which satisfies $\alpha(t, t') = \alpha'(\varphi(t), \varphi(t'))$ for all edges $tt' \in \vec{E}(T)$.

Let us call a tree-decomposition $(T, \mathcal{V})$ of a graph *essential* if distinct edges of T are associated with different separations and all these are nontrivial and non-degenerate. For such a tree-decomposition there is a unique bijective map α such that (T, α) is an essential S-tree over stars, where S is the set of separations associated with $(T, \mathcal{V})$.

Our first corollary says that every such S-tree, and hence every such tree-decomposition, can be recovered from its image under α, up to isomorphism. More generally *every*, finite tree set $\vec{S}$ is represented by an essential S-tree that is unique up to isomorphism, and if S consists of separations of a graph then this S-tree comes from a unique essential tree-decomposition:

Corollary 8.9. *Let $\vec{S}$ be a finite tree set.*

(i) *There is an essential S-tree (T, α) with a bijective map $\alpha\colon \vec{E}(T) \to \vec{S}$. This S-tree is unique up to isomorphism.*

(ii) *If S consists of separations of a graph, then this graph has a unique essential tree-decomposition whose associated set of separations is S.*

Proof. (i) Let $\mathcal{F}$ be the set of the splitting stars of $\vec{S}$. Since $\vec{S}$ has no degenerate elements, $\mathcal{F}$ contains no degenerate singletons. By Theorem 8.8 (i), there is an essential S-tree (T, α) with α bijective, since $\vec{S}$ is already a tree set. It remains to show that every other such S-tree (T', α') is isomorphic to (T, α).

By the choice of (T, α), there is a map $f\colon V(T) \to \mathcal{F}$ sending the nodes t of T to the splitting stars $\alpha(\vec{F}_t)$ of $\vec{S}$. By definition of $\mathcal{F}$ this map f is surjective. As distinct t have different sets $\vec{F}_t$, and α is a bijection, f is also injective.

Let $\mathcal{F}' := \{\, \alpha'(\vec{F}_{t'}) \mid t' \in T' \}$. Since α', too, has image S, Theorem 8.8 (ii) implies that S is a tree set also over $\mathcal{F}'$, whose set $\mathcal{F}$ of splitting stars is precisely $\mathcal{F}'$. As before, since α' is a bijection, so is the map $f'\colon V(T') \to \mathcal{F}' = \mathcal{F}$ sending the nodes t' of T' to the splitting stars $\alpha'(\vec{F}_{t'})$ of $\vec{S}$. The composed bijection $f^{-1} \circ f'$ is clearly an isomorphism of S-trees.

(ii) Let (T, α) be the S-tree from (i). Letting $V_t = \bigcap \{\, B \mid (A, B) \in \alpha(\vec{F}_t) \,\}$ for each $t \in T$ we obtain a tree-decomposition $(T, (V_t)_{t \in T})$ associated with S. This tree-decomposition is essential, because (T, α) is. It is unique, even though (T, α) is unique only up to isomorphism, since the V_t are invariant under isomorphisms of S-trees. □

Tree sets, as described in Corollary 8.9, have neither trivial nor degenerate elements. Nested separation systems $\vec{S}$ that do have a degenerate element $\vec{s} = \overleftarrow{s}$, however, are easy to desribe directly. We already observed in Section 3 that such an S has no nontrivial element other than s; in particular, it has no other degenerate element. Hence S has a unique consistent orientation O, the set consisting of $\vec{s}$ and all the trivial elements of $\vec{S}$. Since s is the unique nontrivial witness to their triviality, $\vec{s}$ is the greatest element of O. All the other $\vec{r}, \vec{r'} \in O$ satisfy $\vec{r} < \vec{s} = \overleftarrow{s} > \overleftarrow{r'}$, so O is a star. Unless $S = \{s\}$, this star is improper. The star $O \setminus \{\vec{s}\}$ is also likely to be improper, since trivial separations $\vec{r}, \vec{r'} \in O$ can satisfy $\vec{r} < \vec{r'}$ as well as $\vec{r} < \overleftarrow{r'}$.

It remains to consider finite nested separation systems $\vec{S}$ that have trivial but no degenerate elements. These are also represented by an S-tree; the only difference is that this will not be unique, even up to isomorphism. But for every such S-tree (T, α) the edges of T which α maps to nontrivial separations in S form a (connected) subtree T', where (T', α') with $\alpha' = \alpha \restriction \vec{E}(T')$ is the essentially unique S'-tree from Corollary 8.9 (i) for the tree set $\vec{S'}$ induced by $\vec{S}$:

Corollary 8.10. *Let $\vec{S}$ be a finite nested separation system without degenerate elements.*

(i) *There is an S-tree (T, α) with a bijective map $\alpha \colon \vec{E}(T) \to \vec{S}$. For every such (T, α), the tree set $\vec{S'}$ induced by $\vec{S}$, the edges $e \in T$ with an orientation in $\alpha^{-1}(\vec{S'})$ induce a subtree T' of T.*

(ii) *If S consists of separations of a graph, then this graph has a tree-decomposition whose associated set of separations is S.*

Proof. Let (T', α') be the S'-tree for $\vec{S'}$ as provided by Corollary 8.9. Every $s \in S \setminus S'$ has a trivial orientation $\vec{s}$, for which we pick a non-trivial witness $s' \in S'$ and a node t' of T' incident with an edge e' such that $\alpha'(e') = s'$. Adding for every such s a new node t to T' by an edge $\vec{e} = (t, t')$, and extending α' to a map α mapping every such $\vec{e}$ to its $\vec{s}$, extends (T', α') to the desired S-tree (T, α).

By the uniqueness of (T', α') up to isomorphism, T' will also be connected if we start with any S-tree (T, α) such as in (i) and define T' as stated there.

(ii) The desired tree-decomposition can be obtained from the S-tree (T, α) in (i) as in the proof of Corollary 8.9 (ii). □

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