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Regularity Properties on the Generalized Reals

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Abstract

We investigate regularity properties derived from tree-like forcing notions in the setting of "generalized descriptive set theory", i.e., descriptive set theory on κ^κ and 2^κ, for regular uncountable cardinals κ.

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1. Introduction

Generalized Descriptive Set Theory is an area of research dealing with generalizations of classical descriptive set theory on the Baire space ω^ω and Cantor space 2^ω, to the *generalized Baire space* κ^κ and the *generalized Cantor space* 2^κ, where κ is an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$. Some of the earlier papers dealing with descriptive set theory on $(\omega_1)^{\omega_1}$ were motivated by model-theoretic concerns, see e.g. [1] and [2, Chapter 9.6]. More recently, generalized descriptive set theory became a field of interest in itself, with various aspects being studied for their own sake, as well as for their applications to different fields of set theory.

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This paper is the first systematic study of *regularity properties* for subsets of generalized Baire spaces. We will focus on regularity properties derived from tree-like forcing partial orders, using the framework introduced by Ikegami in [3] (see Definition 3.1) as a generalization of the Baire property, as well as a number of other standard regularity properties (Lebesgue measurability, Ramsey property, Sacks property etc.) In the classical setting, such properties have been studied by many people, see, e.g., [4, 5, 6, 7]. Typically, these properties are satisfied by analytic sets, while the Axiom of Choice can be used to provide counterexamples. On the second projective level one obtains independence results, as witnessed by "Solovay-style" characterization theorems, such as the following:

Theorem 1.1 (Solovay [8])**.** *All $\boldsymbol{\Sigma}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there are co-meager many Cohen reals over $L[r]$.*

Theorem 1.2 (Judah-Shelah [4])**.** *All $\boldsymbol{\Delta}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there is a Cohen real over $L[r]$.*

These types of theorems make it possible to study the relationships between different regularity properties on the second level. Far less is known for higher projective levels, although some results exist in the presence of large cardinals (see [3, Section 5]) and some other results can be found in [9, Chapter 9] and in the recent works [10, 11]. Solovay's model [8] provides a uniform way of establishing regularity properties for all projective sets, starting from ZFC with an inaccessible.

When attempting to generalize descriptive set theory from ω^ω to κ^κ for a regular uncountable κ, at first many basic results remain intact after a straightforward replacement of ω by κ. But, before long, one starts to notice fundamental differences: for example, the generalized $\boldsymbol{\Delta}^1_1$ sets are not the same as the generalized Borel sets; absoluteness theorems, such as Σ^1_1- and Shoenfield absoluteness, are not valid; and in the constructible universe L, there is a $\boldsymbol{\Sigma}^1_1$-good well-order of κ^κ, as opposed to merely a $\boldsymbol{\Sigma}^1_2$-good well-order in the standard setting (see Section 2 for details). Not surprisingly, regularity properties also behave radically different in the generalized context. Halko and Shelah [12] first noticed that on 2^κ, the generalized Baire property provably fails for $\boldsymbol{\Sigma}^1_1$ sets. On the other hand, it holds for the generalized Borel sets, and is independent for generalized $\boldsymbol{\Delta}^1_1$ sets. This suggests that some of the classical theory on the $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ level corresponds to the $\boldsymbol{\Delta}^1_1$ level in the generalized setting.

It should be noted that other kinds of regularity properties have been considered before, sometimes leading to different patterns in terms of consistency of projective regularity. For example, in [13] Schlicht shows that it is consistent relative to an inaccessible that a version of the perfect set property holds for all generalized projective sets. By [14], as well as recent results of Laguzzi and the first author, similar results hold for suitable modifications of the properties studied here.

This paper is structured as follows: Section 2 will be devoted to a brief survey of facts about the "generalized reals". In Section 3 we introduce an abstract notion of regularity and prove that, under certain assumption, the following results hold:

1. Borel sets are "regular".
2. Not all analytic sets are "regular".
3. For $\mathbf{\Delta}^1_1$ sets, the answer is independent of ZFC.

In Section 4 we focus on some concrete examples on the $\mathbf{\Delta}^1_1$-level and generalize some classical results from the $\mathbf{\Delta}^1_2$-level. Section 5 ends with a number of open questions.

2. Generalized Baire spaces

We devote this section to a survey of facts about κ^κ and 2^κ which will be needed in the rest of the paper, as well as specifying some definitions and conventions. None of the results here are new, though some are not widely known or have not been sufficiently documented.

Notation 2.1. $\kappa^{<\kappa}$ denotes the set of all functions from α to κ for some $\alpha < \kappa$, similarly for $2^{<\kappa}$. We use standard notation concerning sequences, e.g., for $s, t \in \kappa^{<\kappa}$ we use $s^\frown t$ to denote the *concatenation* of s and t, $s \subseteq t$ to denote that *s is an initial segment of t* etc. $\kappa^\kappa_\uparrow$ denotes the set of strictly increasing functions from κ to κ, and $\kappa^{<\kappa}_\uparrow$ the set of strictly increasing functions from α to κ for some $\alpha < \kappa$. Also, we will frequently refer to elements of κ^κ or 2^κ as "κ-reals" or "generalized reals".

For finite sequences, it is customary to denote the length by $|s|$. In the generalized context, in order to avoid confusion with cardinality, we denote the *length* of a sequence (i.e., the unique α such that $s \in \kappa^\alpha$ or 2^α) by "len(s)".

2.1. Topology

We always assume that κ is an uncountable, regular cardinal, and that $\kappa^{<\kappa} = \kappa$ holds. The *standard topology* on κ^κ is the one generated by basic open sets of the form $[s] := \{x \in \kappa^\kappa \mid s \subseteq x\}$, for $s \in \kappa^{<\kappa}$; similarly for 2^κ. Many elementary facts from the classical setting have straightforward generalizations to the generalized setting. The concepts *nowhere dense* and *meager* are defined as usual, and a set A has the *Baire property* if and only if $A \triangle O$ is meager for some open O. The following classical results are true regardless of the value of κ:

- *Baire category theorem*: the intersection of κ-many open dense sets is dense.

- *Kuratowski-Ulam theorem* (also called *Fubini for category*): if $A \subseteq \kappa^\kappa \times \kappa^\kappa$ has the Baire property then A is meager if and only if $\{x \mid A_x \text{ is meager}\}$ is comeager, where $A_x := \{y \mid (x, y) \in A\}$.

Definition 2.2. A *tree* is a subset of $\kappa^{<\kappa}$ or $2^{<\kappa}$ closed under initial segments. For a node $t \in T$, we write $\mathrm{Succ}_T(t) := \{s \in T \mid s = t^\frown \langle\alpha\rangle \text{ for some } \alpha\}$. A node $t \in T$ is called

- *terminal* if $\mathrm{Succ}_T(t) = \varnothing$,
- *splitting* if $|\mathrm{Succ}_T(t)| > 1$, and
- *club-splitting* if $\{\alpha \mid t^\frown \langle\alpha\rangle \in T)\}$ is a club in κ.

We use the notation $\mathrm{Split}(T)$ to refer to the set of all splitting nodes of T.

A $t \in T$ is called a *successor node* if $\mathrm{len}(t)$ is a successor ordinal and a *limit node* if $\mathrm{len}(t)$ is a limit ordinal. A tree is *pruned* if it has no terminal nodes, and *$<\kappa$-closed* if for every increasing sequence $\{s_i \mid i < \lambda\}$ of nodes from T, for $\lambda < \kappa$, the limit $\bigcup_{i<\lambda} s_i$ is also a node of T.

Notice that concepts such as *club-splitting*, *successor* and *limit* node, and *$<\kappa$-closed* are inherent to the generalized setting and have no classical counterpart. Most of the trees we consider will be pruned and $<\kappa$-closed.

A *branch through* T is a κ-real $x \in \kappa^\kappa$ or 2^κ such that $\forall\alpha\,(x{\restriction}\alpha \in T)$, and $[T]$ denotes the set of all branches through T. As usual, $[T]$ is topologically closed and every closed set has the form $[T]$ for some tree T.

The Borel and projective hierarchies are defined in analogy to the classical situation: the Borel sets form the smallest collection of subsets of κ^κ or 2^κ containing the basic open sets and closed under complements and κ-unions. A set is $\mathbf{\Sigma}^1_1$ iff it is the projection of a closed (equivalently: Borel) set; it is $\mathbf{\Pi}^1_n$ iff its complement is $\mathbf{\Sigma}^1_n$; and it is $\mathbf{\Sigma}^1_{n+1}$ iff it is the projection of a $\mathbf{\Pi}^1_n$ set, for $n \geq 1$. It is $\mathbf{\Delta}^1_n$ iff it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, and *projective* iff it is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ for some $n \in \omega$.

In spite of the close similarity of the above notions to the classical ones, there are also fundamental differences:

Fact 2.3. $\mathsf{Borel} \neq \mathbf{\Delta}^1_1$.

A proof of this fact can be found in [15, Theorem 18 (1)], and we also refer readers to Sections II and III of the same paper for a more detailed survey of the basic properties of κ^κ and 2^κ.

2.2. The club filter

Sets that will play a crucial role in this paper are those related to the *club filter*. As usual, we may identify 2^κ with $\mathscr{P}(\kappa)$ via characteristic functions.

Fact 2.4. *The set $C := \{a \subseteq \kappa \mid a \text{ contains a club}\}$ is $\mathbf{\Sigma}^1_1$.*

Proof. For every $c \subseteq \kappa$, note that c is closed (in the "club"-sense) if and only if for every $\alpha < \kappa$, $c \cap \alpha$ is closed in α. Therefore, "being closed" is a (topologically) closed property. Being unbounded, on the other hand, is a G_δ property, so "being club" is G_δ. Then for all $a \subseteq \kappa$ we have $a \in C$ iff $\exists c(c \text{ is club and } c \subseteq a)$, which is $\mathbf{\Sigma}^1_1$. $\square$

In [12] it was first noticed that the club filter provides a counterexample to the Baire property.

Theorem 2.5 (Halko-Shelah)**.** *The club filter C does not satisfy the Baire property.*

We will prove a generalization of the above, see Theorem 3.10. An immediate corollary of Theorem 2.5 is that in the generalized setting, analytic sets do not satisfy the Baire property. Although the club filter clearly cannot be Borel (Borel sets *do* satisfy the Baire property, in any topological space satisfying the Baire category theorem), it can consistently be $\mathbf{\Delta}^1_1$ for successors κ.

Theorem 2.6 (Mekler-Shelah; Friedman-Wu-Zdomskyy)**.** *For any successor cardinal κ, it is consistent that the club filter on κ is $\mathbf{\Delta}^1_1$.*

Proof. For $\kappa = \omega_1$, this was first prove in [16]. The argument contained a flaw, which was corrected in [17]. For arbitrary successor cardinals κ, this was proved using different methods in [18]. $\square$

It is also consistent that the club filter is not $\mathbf{\Delta}^1_1$—this will follow from Theorem 3.13.

2.3. *Absoluteness*

Two fundamental results in descriptive set theory are *analytic* (*Mostowski*) *absoluteness* and *Shoenfield absoluteness*. In general, this type of absoluteness does not hold for uncountable κ. For example, let $\kappa = \lambda^+$ for regular λ, pick $S \subseteq \kappa \cap \mathrm{Cof}(\lambda)$ such that both S and $(\kappa \cap \mathrm{Cof}(\lambda)) \setminus S$ are stationary. Let $\mathbb{P}$ be a forcing for adding a club to $S \cup \mathrm{Cof}(<\lambda)$. Then, if Φ is the Σ^1_1 formula defining the club filter $C \subseteq \mathscr{P}(\kappa)$ from Fact 2.4, we have that $V \models \neg\Phi(S \cup \mathrm{Cof}(<\lambda))$ while $V^{\mathbb{P}} \models \Phi(S \cup \mathrm{Cof}(<\lambda))$, so Σ^1_1-absoluteness fails even for κ^+-preserving forcing extensions. On the other hand, Σ^1_1-absoluteness does hold for generic extensions via $<\kappa$-closed forcings.

Lemma 2.7. *Let $\mathbb{P}$ be a $<\kappa$-closed forcing. Then Σ^1_1 formulas are absolute between V and $V^{\mathbb{P}}$.*

Proof. Let $\phi(x)$ be a Σ^1_1 formula with parameters in V. Let $x \in \kappa^\kappa$ and assume $V^{\mathbb{P}} \models \phi(x)$. Let T (in V) be a two-dimensional tree such that $\{x \mid \phi(x)\} = p[T]$, i.e., the projection of T to the first coordinate. Let $h \in \kappa^\kappa \cap V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models (x, h) \in [T]$ and let $\dot{h}$ be a $\mathbb{P}$-name for h.

By induction, build an increasing sequence $\{p_i \mid i < \kappa\}$ of $\mathbb{P}$-conditions, and an increasing sequence $\{t_i \in \kappa^{<\kappa} \mid i < \kappa\}$, such that each $p_i \Vdash t_i \subseteq \dot{h}$. This can be done since at limit stages $\lambda < \kappa$, we can define $t_\lambda := \bigcup_{i<\lambda} t_i$ and pick p_λ below p_i for all $i < \lambda$. Since every p_i forces $(\check{x}, \dot{h}) \in [T]$, it follows that for every i we have $(x\restriction \mathrm{len}(t_i), t_i) \in T$. But then (in V) let $g := \bigcup_{i<\kappa} t_i$, so $(x, g) \in [T]$ and therefore $\phi(x)$ holds. $\square$

2.4. Well-order of the reals

In the classical setting, it is well-known that in L there exists a $\mathbf{\Sigma}^1_2$ well-order of the reals. In fact, the well-order is "$\mathbf{\Sigma}^1_2$-good", meaning that both the relation $<_L$ on the reals, and the binary relation defined by

$$\Psi(x, y) \equiv \text{ ``}x\text{ codes the set of }<_L\text{-predecessors of }y\text{''}$$

is $\mathbf{\Sigma}^1_2$. The proof uses absoluteness of $<_L$ and Ψ between L and initial segments L_δ for countable δ, and the fact that "$E \subseteq \omega \times \omega$ is well-founded" is a $\mathbf{\Pi}^1_1$-predicate on E. In the generalized setting, however, the predicate "$E \subseteq \kappa \times \kappa$ is well-founded" is *closed*, leading to the following result:

Lemma 2.8. *In L, there is a $\mathbf{\Sigma}^1_1$-good well-order of κ^κ.*

Proof. As usual, we have that for $x, y \in \kappa^\kappa$, $x <_L y$ iff $\exists\delta < \kappa^+$ such that $x, y \in L_\delta$ and $L_\delta \models x <_L y$. Using standard tricks, this can be re-written as "$\exists E \subseteq \kappa \times \kappa$ (E is well-founded, $x, y \in \mathrm{ran}(\pi_E)$ and $(\omega, E) \models ZFC^* + V = L + x <_L y$)", where π_E refers to the transitive collapse of (ω, E) onto some $(L_\delta, \in)$ and ZFC^* is a sufficiently large fragment of ZFC. The statement "E is well-founded" is closed because E is well-founded iff $\forall\alpha < \kappa\ E \cap (\alpha \times \alpha)$ is well-founded. Thus we obtain a $\mathbf{\Sigma}^1_1$ statement. A similar argument works with $<_L$ replaced by $\Psi(x, y)$, showing that the well-order is $\mathbf{\Sigma}^1_1$-good. $\square$

2.5. Proper Forcing

A ubiquitous tool in the study of the classical Baire and Cantor spaces is Shelah's theory of *proper forcing*. It is a technical requirement on a forcing notion which is just sufficient to imply preservation of ω_1, while itself being preserved by countable support iterations, and moreover having a multitude of natural examples. Over the years, there have been various attempts at generalizing this theory to higher cardinals (see e.g. [19, 20, 21] for some recent contributions). Of course, we can use the following straightforward generalization:

Definition 2.9. A forcing $\mathbb{P}$ is *κ-proper* if for every sufficiently large θ (e.g. $\theta > 2^{|\mathbb{P}|}$), and for all elementary submodels $M \prec H_\theta$ such that $|M| = \kappa$ and M is closed under $<\kappa$-sequences, for every $p \in \mathbb{P} \cap M$ there exists $q \leq p$ such that for every dense $D \in M$, $D \cap M$ is predense below q.

The above property follows both from the κ^+-c.c. and a κ-version of Axiom A, and implies that κ^+ is preserved, but the property itself is in general not preserved by iterations, see [22, Example 2.4]. Nevertheless, it is a useful formulation that we will need on some occasions.

While a uniform theory for κ-properness is lacking so far, preservation theorems are usually proved either using the κ^+-c.c. or on a case-by-case basis.

Fact 2.10 (Baumgartner)**.** *A forcing $\mathbb{Q}$ is* κ-linked *iff $\mathbb{Q} = \bigcup_{\alpha<\kappa} \mathbb{Q}_\alpha$ where each $\mathbb{Q}_\alpha$ consists of pairwise compatible conditions. A forcing $\mathbb{Q}$ is* well-met *iff for every two compatible conditions $q_1, q_2 \in \mathbb{Q}$ there is a greatest lower bound $q \in \mathbb{Q}$.*

If $\mathbb{P}_\alpha$ is an iteration of length $\alpha > \kappa$ with supports of size $<\kappa$, and every iterand is forced to be κ-linked, $<\kappa$-closed and well-met, then $\mathbb{P}_\alpha$ has the κ^+-c.c.

This was originally proved by Baumgartner in [23], and a modern treatment can be found e.g. in [24, Section V.5] (both expositions deal with $\kappa = \omega_1$ but the proof works for any regular uncountable κ satisfying $\kappa^{<\kappa} = \kappa$).

Fact 2.11.

1. *κ-Sacks forcing $\mathbb{S}_\kappa$ (see Example 3.2) was studied by Kanamori [25], where the following facts were proved:*
   (a) *$\mathbb{S}_\kappa$ satisfies a generalized version of Axiom A (see Definition 3.6 (2)).*
   (b) *Assuming $\diamondsuit_\kappa$, iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-sized supports also satisfy a version of Axiom A.*
   (c) *If κ is inaccessible, then $\mathbb{S}_\kappa$ is* κ^κ-bounding *(meaning that for every $x \in \kappa^\kappa \cap V^{\mathbb{S}_\kappa}$ there exists $y \in \kappa^\kappa \cap V$ such that $x(i) < y(i)$ for sufficiently large $i < \kappa$), and so are arbitrary iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-size supports.*
2. *κ-Miller forcing $\mathbb{M}_\kappa$ (see Example 3.2) was studied by Friedman and Zdomskyy [26], where the following facts were proved:*
   (a) *$\mathbb{M}_\kappa$ satisfies a generalized version of Axiom A.*
   (b) *Assuming κ is inaccessible, iterations of $\mathbb{M}_\kappa$ with $\leq\kappa$-sized supports satisfy a version of Axiom A.*

In particular, $\mathbb{S}_\kappa$, $\mathbb{M}_\kappa$ and their iterations are κ-proper in the sense of Definition 2.9 and thus preserve κ^+.

3. Regularity properties

The regularity properties we will consider in this paper are those derived from definable tree-like forcing notions. In this section we give an abstract treatment following the framework introduced by Ikegami in [3], providing sufficient conditions so that the following facts can be proved uniformly:

1. Regularity for Borel sets is true.
2. Regularity for arbitrary $\mathbf{\Sigma}^1_1$ sets is false.
3. Regularity for arbitrary $\mathbf{\Delta}^1_1$ sets is independent.

3.1. Tree-like forcings on κ^κ

Definition 3.1. A forcing notion $\mathbb{P}$ is called *κ-tree-like* iff

1. the conditions of $\mathbb{P}$ are pruned and $<\kappa$-closed trees on κ^κ or 2^κ ordered by $q \leq p$ iff $q \subseteq p$,
2. the full tree ($\kappa^{<\kappa}$ or $2^{<\kappa}$) is an element of $\mathbb{P}$,
3. for all $T \in \mathbb{P}$ and all $s \in T$ the restriction $T{\uparrow}s := \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}$ is also a member of $\mathbb{P}$,
4. the statement "T is a $\mathbb{P}$-tree" is absolute between models of ZFC, and
5. if $\langle T_\alpha \mid \alpha < \lambda \rangle$ is a decreasing seqeunce of conditions, with $\lambda < \kappa$, then $\bigcap_{\alpha<\lambda} T_\alpha \in \mathbb{P}$.

The first three items are standard, and the fourth one is to make sure that the forcing notion has the same interpretation in all models (in particular in further forcing extensions). Item 5 is a strong form of $<\kappa$-closure of the forcing which is needed for technical reasons. Below are a few examples of κ-tree-like forcings that have either been considered in the literature or are natural generalizations of classical notions.

Example 3.2.

1. *κ-Cohen forcing* $\mathbb{C}_\kappa$. Conditions are the trees corresponding to the basic open sets $[s]$, for $s \in 2^{<\kappa}$ or $\kappa^{<\kappa}$, ordered by inclusion.
2. *κ-Sacks forcing* $\mathbb{S}_\kappa$. A tree T on 2^κ is called a *κ-Sacks tree* if it is pruned, $<\kappa$-closed and
   (a) every node $t \in T$ has a splitting extension in T, and
   (b) for every increasing sequence $\langle s_i \mid i < \lambda \rangle$, $\lambda < \kappa$, of splitting nodes in T, $s := \bigcup_{\alpha<\lambda} s_\alpha$ is a *splitting* node of T.

   $\mathbb{S}_\kappa$ is the partial order of κ-Sacks trees ordered by inclusion.
3. *κ-Miller forcing* $\mathbb{M}_\kappa$. A tree T on $\kappa^{<\kappa}_{\uparrow}$ is called a *κ-Miller tree* if it is pruned, $<\kappa$-closed and
   (a) every node $t \in T$ has a club-splitting extension in T,

(b) for every increasing sequence $\langle s_i \mid i < \lambda \rangle$, $\lambda < \kappa$, of club-splitting nodes in T, $s := \bigcup_{i<\lambda} s_i$ is a club-splitting node of T. Moreover, *continuous club-splitting* is required, which is the following property: for every club-splitting limit node $s \in T$, if $\{s_i \mid i < \lambda\}$ is the set of all club-splitting initial segments of s and $C_i := \{\alpha \mid s_i{}^\frown \langle \alpha \rangle \in T\}$ is the club witnessing club-splitting of s_i for every i, then $C := \{\alpha \mid s^\frown \langle \alpha \rangle \in T\} = \bigcap_{i<\lambda} C_i$ is the club witnessing club-splitting of s.

$\mathbb{M}_\kappa$ is the partial order of κ-Miller trees ordered by inclusion.

4. *κ-Laver forcing* $\mathbb{L}_\kappa$. A tree T on $\kappa_\uparrow^{<\kappa}$ is a *κ-Laver tree* if all nodes $s \in T$ extending the stem of T are club-splitting. $\mathbb{L}_\kappa$ is the partial order of κ-Laver trees ordered by inclusion.

5. *κ-Mathias forcing* $\mathbb{R}_\kappa$. A κ-Mathias condition is a pair (s, C), where $s \subseteq \kappa$, $\mathrm{len}(s) < \kappa$, $C \subseteq \kappa$ is a club, and $\max(s) < \min(C)$. The conditions are ordered by $(t, D) \leq (s, C)$ iff $t \leq s$, $D \subseteq C$ and $t \setminus s \subseteq C$. Formally, this does not follow Definition 3.1, but we can easily identify conditions (s, C) with trees $T_{(s,C)}$ on $\kappa_\uparrow^{<\kappa}$ defined by $t \in T_{(s,C)}$ iff $\mathrm{ran}(t) \subseteq s \cup C$.

6. *κ-Silver forcing* $\mathbb{V}_\kappa$. If κ is inaccessible, let $\mathbb{V}_\kappa$ consist of κ-Sacks-trees T on $2^{<\kappa}$ which are *uniform*, i.e., for $s, t \in T$, if $\mathrm{len}(s) = \mathrm{len}(t)$ then $s^\frown \langle i \rangle \in T$ iff $t^\frown \langle i \rangle \in T$. Alternatively, we can view conditions of $\mathbb{V}_\kappa$ as functions $f : \kappa \to \{0, 1, \{0, 1\}\}$, such that $f(i) = \{0, 1\}$ holds for all $i \in C$ for some club $C \subseteq \kappa$, ordered by $g \leq f$ iff $\forall i\, (f(i) \in \{0, 1\} \to g(i) = f(i))$.

The generalized κ-Sacks forcing was introduced and studied by Kanamori in [25], and the κ-Miller forcing is its natural variant, studied e.g. by Friedman and Zdomskyy in [26]. The requirement on the trees to be "closed under splitting-nodes" (2(b) and 3(b)) ensure that item 5 of Definition 3.1 is satisfied, and thus that the forcings are $<\kappa$-closed. The property called "continuous club-splitting" was introduced in [26] to facilitate the preservation of measurability. We should note that other generalizations of Miller forcing have also been considered, see e.g. [27].

κ-Silver is a natural generalization of Silver forcing, but the standard proof of Axiom A only works for inaccessible κ.

κ-Laver and κ-Mathias are, again, natural generalizations of their classical counterparts; however, since we require the trees to split into club-many successors at all branches above the stem, any two κ-Laver and κ-Mathias conditions with the same stem are compatible, so both $\mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ are κ^+-centered and hence satisfy the κ^+-c.c. Therefore they are perhaps more reminiscent of the classical *Laver-with-filter* and *Mathias-with-filter* forcings on ω^ω, rather than the actual Laver and Mathias forcing posets. Note that if we would drop club-splitting from the definition and only require stationary or κ-sized splitting instead, we would lose $<\kappa$-closure of the forcing.

Remark 3.3. One notion conspicuous by its absence from Example 3.2 is *random forcing*. To date, it is not entirely clear how random forcing should properly be generalized to uncountable κ. Recently Shelah proposed a definition for κ weakly compact, and a different approach was given by the first author and Laguzzi in [28]. However, a consensus on the correct definition for arbitrary κ has not been reached so far, so in this work we choose to avoid random forcing, as well as the concept *null ideal* and *Lebesgue measurability*.

The following definition is based on [3, Definition 2.6 and Definition 2.8]. Let $\mathbb{P}$ be a fixed κ-tree-like forcing.

Definition 3.4. Let A be a subset of κ^κ or 2^κ. Then

1. A is $\mathbb{P}$-*null* iff $\forall T \in \mathbb{P}\, \exists S \leq T$ such that $[S] \cap A = \varnothing$. We denote the ideal of $\mathbb{P}$-null sets by $\mathcal{N}_\mathbb{P}$
2. A is $\mathbb{P}$-*meager* iff it is a κ-union of $\mathbb{P}$-null sets. We denote the κ-ideal of $\mathbb{P}$-meager sets by $\mathcal{I}_\mathbb{P}$.
3. A is $\mathbb{P}$-*measurable* iff $\forall T \in \mathbb{P}\, \exists S \leq T$ such that $[S] \subseteq^* A$ or $[S] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refers to "modulo $\mathcal{I}_\mathbb{P}$".

For a wide class of tree-like forcing notions, the clause "modulo $\mathcal{I}_\mathbb{P}$" can be eliminated from the above definition: see Lemma 3.8 (2).

3.2. Regularity of Borel sets

In ω^ω, it is not hard to prove that if $\mathbb{P}$ is proper then all analytic sets are $\mathbb{P}$-measurable, using forcing-theoretic arguments and absoluteness techniques (see e.g. [7, Proposition 2.2.3]). These methods are generally not available in the generalized setting. However, we would still like to know that, at least, all Borel subsets of κ^κ are $\mathbb{P}$-measurable for all reasonable examples of $\mathbb{P}$.

Remark 3.5. Closed sets are $\mathbb{P}$-measurable for all $\mathbb{P}$. To see this, let $[U]$ be an arbitrary closed set and let $T \in \mathbb{P}$. If $T \subseteq U$ then we are done, otherwise pick $s \in T \setminus U$, then by Definition 3.1 $T{\uparrow}s \in \mathbb{P}$ and $[T{\uparrow}s] \cap [U] = \varnothing$. It is also easy to see that being $\mathbb{P}$-measurable is closed under complements and $<\kappa$-sized unions and intersections.

It remains to verify closure under κ-sized unions and intersections. For that we introduce some definitions that help to simplify the notion of $\mathbb{P}$-measurability, and moreover will play a crucial role for the rest of this paper.

Definition 3.6. Let $\mathbb{P}$ be a κ-tree-like forcing notion on κ^κ or 2^κ. Then we say that:

1. $\mathbb{P}$ is *topological* if $\{[T] \mid T \in \mathbb{P}\}$ forms a topology base for κ^κ (i.e., for all $S, T \in \mathbb{P}$, $[S] \cap [T]$ is either empty or contains $[R]$ for some $R \in \mathbb{P}$).
2. $\mathbb{P}$ satisfies *Axiom A* iff there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$, with $\leq_0 = \leq$, satisfying:

(a) $T \leq_\beta S$ implies $T \leq_\alpha S$, for all $\alpha \leq \beta$.

(b) If $\langle T_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions, with $\lambda \leq \kappa$ (in particular $\lambda = \kappa$) satisfying

$$T_\beta \leq_\alpha T_\alpha \text{ for all } \alpha \leq \beta,$$

then there exists $T \in \mathbb{P}$ such that $T \leq_\alpha T_\alpha$ for all $\alpha < \lambda$.

(c) For all $T \in \mathbb{P}$, D dense below T, and $\alpha < \kappa$, there exists an $E \subseteq D$ and $S \leq_\alpha T$ such that $|E| \leq \kappa$ and E is predense below S.

3. $\mathbb{P}$ satisfies *Axiom A** if 2 above holds, but in 2 (c) we additionally require that "$[S] \subseteq \bigcup\{[T] \mid T \in E\}$".

Example 3.7. In Example 3.2, κ-Cohen, κ-Laver and κ-Mathias are topological. By Fact 2.11, κ-Miller and κ-Sacks satisfy Axiom A, and it is not hard to see that in fact they satisfy Axiom A* as well (a direct consequence of the construction). Assuming κ is inaccessible, a generalization of the classical proof shows that κ-Silver also satisfies Axiom A*.

Lemma 3.8.

1. *If $\mathbb{P}$ is topological then a set A is $\mathbb{P}$-measurable iff it satisfies the property of Baire in the topology generated by $\mathbb{P}$. In particular, all Borel sets are $\mathbb{P}$-measurable.*

2. *If $\mathbb{P}$ satisfies Axiom A* then $\mathcal{N}_\mathbb{P} = \mathcal{I}_\mathbb{P}$, and consequently a set A is $\mathbb{P}$-measurable iff $\forall T \in \mathbb{P}\, \exists S \leq T\, ([S] \subseteq A$ or $[S] \cap A = \varnothing)$ (i.e., we can forget about "modulo $\mathcal{I}_\mathbb{P}$"). Moreover, the collection of $\mathbb{P}$-measurable sets is closed under κ-unions and κ-intersections.*

The proofs are essentially analogous to the classical situation, but let us present them anyway since they are not widely known.

Proof. 1. First of all, notice that if $\mathbb{P}$ is topological then $\mathcal{N}_\mathbb{P}$ is exactly the collection of nowhere dense sets in the $\mathbb{P}$-topology and $\mathcal{I}_\mathbb{P}$ is exactly the ideal of meager sets in the $\mathbb{P}$-topology.

First assume A satisfies the $\mathbb{P}$-Baire property, then let O be an open set in the $\mathbb{P}$-topology such that $A \triangle O$ is $\mathbb{P}$-meager. Given any $T \in \mathbb{P}$, we have two cases: if $[T] \cap O = \varnothing$ then we are done since $[T] \cap A =^* \varnothing$. If $[T] \cap O$ is not empty then there exists a $S \leq T$ such that $[S] \subseteq [T] \cap O$. Then $[S] \subseteq^* A$ holds, so again we are done.

The converse direction is somewhat more involved (cf. [29, Theorem 8.29]). Assume A is $\mathbb{P}$-measurable. Let

- D_1 be a maximal mutually disjoint subfamily of $\{T \in \mathbb{P} \mid [T] \subseteq^* A\}$,
- D_2 be a maximal mutually disjoint subfamily of $\{T \in \mathbb{P} \mid [T] \cap A =^* \varnothing\}$, and

- $D := D_1 \cup D_2$.

Also write $O_1 := \bigcup\{[T] \mid T \in D_1\}$, $O_2 := \bigcup\{[T] \mid T \in D_2\}$ and $O := O_1 \cup O_2$. We will show that $A \triangle O_1$ is $\mathbb{P}$-meager.

Claim 1. *O is $\mathbb{P}$-open dense.*

Proof of Claim. Start with any T. By assumption there exists $S \leq T$ such that $[S] \subseteq^* A$ or $[S] \cap A =^* \varnothing$. In the former case, note that by maximality, there must be some $S' \in D_1$ such that $[S] \cap [S'] \neq \varnothing$. Then find S'' such that $[S''] \subseteq [S] \cap [S']$. Then $[S''] \subseteq O_1$. Likewise, in the case $[S] \cap A =^* \varnothing$ we find a stronger S'' with $[S''] \subseteq O_2$. □ (Claim 1).

Claim 2. *$A \cap O_2$ and $O_1 \setminus A$ are $\mathbb{P}$-meager.*

Proof of Claim. Since the proof of both statements is analogous, we only do the first.

Enumerate $D_2 := \{T_\alpha \mid \alpha < |\kappa^\kappa|\}$. For each α, let $\{X_i^\alpha \mid i < \kappa\}$ be a collection of $\mathbb{P}$-nowhere dense sets, such that $[T_\alpha] \cap A = \bigcup_{i<\kappa} X_i^\alpha$. Now, for every $i < \kappa$, let $Y_i := \bigcup_{\alpha<|\kappa^\kappa|} X_i^\alpha$. We will show that each Y_i is $\mathbb{P}$-nowhere dense. So fix i and pick any $T \in \mathbb{P}$: if $[T]$ is disjoint from all $[T_\alpha]$'s then clearly also $[T] \cap Y_i = \varnothing$. Else, let T_α be such that $[T] \cap [T_\alpha] \neq \varnothing$. Then there exists $S \leq T$ such that $[S] \subseteq [T] \cap [T_\alpha]$. By assumption, $[T_\alpha]$ is disjoint from all $[T_\beta]$'s, and hence from all X_i^β's, for all $\beta \neq \alpha$. Next, since X_i^α is $\mathbb{P}$-nowhere dense, we can find $S' \leq S$ such that $[S'] \cap X_i^\alpha = \varnothing$. But then $[S'] \cap Y_i = \varnothing$, proving that Y_i is indeed $\mathbb{P}$-nowhere dense.

Now clearly $O_2 \cap A$ is completely covered by the collection $\{Y_i \mid i < \kappa\}$, therefore it is meager. □ (Claim 2).

Now it follows from Claim 1 and Claim 2 that $A \triangle O_1 = (O_1 \setminus A) \cup (A \cap O_2) \cup (A \setminus O)$ is a union of three meager sets, hence it is meager.

This proves that the set A has the property of Baire in the topology generated by $\mathbb{P}$.

2. Assume $\mathbb{P}$ satisfies Axiom A*, and let $\{A_i \mid i < \kappa\}$ be a collection of $\mathbb{P}$-null sets. We want to show that $A := \bigcup_{i<\kappa} A_i$ is also $\mathbb{P}$-null. For each i let $D_i := \{T \mid [T] \cap A_i = \varnothing\}$. By assumption, each D_i is dense. Now let $T_0 \in \mathbb{P}$ be given. Using Axiom A* find, inductively, a sequence $\{T_i \mid i < \kappa\}$ as well as a sequence $\{E_i \subseteq D_i \mid i < \kappa\}$ such that

- $T_j \leq_i T_i$ for all $i \leq j$ and
- $[T_i] \subseteq \bigcup\{[T] \mid T \in E_i\}$ for all i.

This can always be done by condition (c) of Axiom A*. Then, by condition (b) there is a T such that $T \leq T_i$ for all i, and hence, $[T] \subseteq \bigcup\{[S] \mid S \in D_i\}$ for all i. In particular, $[T] \cap A_i = \varnothing$ for all $i < \kappa$, proving that $\bigcap A_i$ is $\mathbb{P}$-null.

For the second claim, it suffices to show closure under κ-unions. Consider a collection $\{A_i \mid i < \kappa\}$ of $\mathbb{P}$-measurable sets, and let $T \in \mathbb{P}$. We must find $S \leq T$ such that $[S] \subseteq \bigcup_{i<\kappa} A_i$ or $[S] \cap \bigcup_{i<\kappa} A_i = \varnothing$. If for at least one $i < \kappa$, we can find $S \leq T$ such that $[S] \subseteq A_i$, we are done, so assume that's not the case. Then we have $A_i \cap [T] \in \mathcal{N}_\mathbb{P}$ for all i, because for every $S \in \mathbb{P}$, either $S \not\leq T$ in which case we are done, or $S \leq T$ in which case, by $\mathbb{P}$-measurability of A_i and the fact that $\mathcal{I}_\mathbb{P} = \mathcal{N}_\mathbb{P}$, there exists $S' \leq S$ with $[S'] \subseteq A_i$ or $[S'] \cap A_i = \varnothing$—but by our assumption the former is impossible and so the latter must hold. Therefore each $A_i \cap [T]$ is in $\mathcal{N}_\mathbb{P}$ and again by the above we obtain $\bigcup_{i<\kappa}(A_i \cap [T]) \in \mathcal{N}_\mathbb{P}$, so we can find $S \leq T$ with $[S] \cap \bigcup_{i<\kappa} A_i = \varnothing$. ☐

Note that to prove point 2 above, we do not in fact need the full strength of Axiom A*, but only need that for all $T \in \mathbb{P}$, D dense below T, and $\alpha < \kappa$, there exists $S \leq_\alpha T$ such that $[S] \subseteq \bigcup\{[T] \mid T \in D\}$.

Corollary 3.9. *If $\mathbb{P}$ is either topological or satisfies Axiom A* then all Borel sets are $\mathbb{P}$-measurable.*

3.3. Regularity of $\mathbf{\Sigma}^1_1$ sets

Let us abbreviate "all sets of complexity $\mathbf{\Gamma}$ are $\mathbb{P}$-measurable" by "$\mathbf{\Gamma}(\mathbb{P})$". In the ω^ω case, ZFC proves $\mathbf{\Sigma}^1_1(\mathbb{P})$, and by symmetry $\mathbf{\Pi}^1_1(\mathbb{P})$, but $\mathbf{\Sigma}^1_2(\mathbb{P})$ and $\mathbf{\Delta}^1_2(\mathbb{P})$ are independent of ZFC. But in the case that $\kappa > \omega$ things are dramatically different since by the Halko-Shelah result (Theorem 2.5) $\mathbf{\Sigma}^1_1(\mathbb{C}_\kappa)$ is false, i.e., the Baire property fails for analytic sets. We attempt to find the essential requirements on $\mathbb{P}$ which would allow us to generalize this proof and show, in ZFC, that $\mathbf{\Sigma}^1_1(\mathbb{P})$ fails, i.e., that there is an analytic set which is not $\mathbb{P}$-measurable. It is most convenient to formulate this requirement in terms of the κ-Sacks and κ-Miller forcing notions, see Example 3.2.

Theorem 3.10. *Let $\mathbb{P}$ be a tree-like forcing notion on 2^κ whose conditions are κ-Sacks trees, or a tree-like forcing notion on κ^κ whose conditions are κ-Miller trees. Then $\mathbf{\Sigma}^1_1(\mathbb{P})$ fails.*

Proof. Let us start with the first case. Recall the club-filter C from Fact 2.4, considered as a subset of 2^κ. If C were $\mathbb{P}$-measurable then, in particular, we would have a $T \in \mathbb{P}$ such that $[T] \subseteq^* C$ or $[T] \cap C =^* \varnothing$. First deal with the former case: let $\{X_i \mid i < \kappa\}$ be $\mathbb{P}$-null sets such that $[T] \setminus C = \bigcup_{i<\kappa} X_i$. Inductively, construct a decreasing sequence $\{T_i \mid i < \kappa\}$ of conditions:

- $T_0 = T$.
- Given T_i, first let $T'_i \leq T_i$ be any condition with strictly longer stem, and then let $T_{i+1} \leq T'_i$ be such that $[T_{i+1}] \cap X_i = \varnothing$.
- At limit stages λ, first let $T'_\lambda := \bigcap_{i<\lambda} T_\alpha$, which is a $\mathbb{P}$-condition by item 5 of Definition 3.1. Notice also that $\mathrm{stem}(T'_\lambda) = \bigcup_{i<\lambda} \mathrm{stem}(T_i)$. That is because for every $i < \lambda$, $\bigcup_{i<\lambda} \mathrm{stem}(T_i)$ is in T_i and is the limit of an

increasing sequence $\langle \mathrm{stem}(T_j) \mid i \leq j < \lambda \rangle$ of splitnodes of T_i, hence it is also a splitnode in T_i by condition $2(b)$ of Example 3.2. Therefore it is a splitnode in T'_λ and so is the stem of T'_λ.

Let $T_\lambda \leq T'_\lambda$ be such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown \langle 0 \rangle$.

Now let $x := \bigcup_{i<\kappa} \mathrm{stem}(T_i)$. Then x is a branch through T, $x \notin X_i$ for all i, and moreover, there exists a club $c \subseteq \kappa$ such that $x(i) = 0$ for all $i \in c$. In particular, $x \notin C$—contradiction.

To deal with the second case that $[T] \cap C =^* \varnothing$, proceed analogously except that at limit stages, pick $T_\lambda \leq T'_\lambda$ such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown \langle 1 \rangle$; then it will follow that $x \in C$.

When $\mathbb{P}$ is a tree-like forcing on κ^κ whose conditions are κ-Miller trees, we apply the same argument, but using the following variant of the club-filter: let S be a stationary, co-stationary subset of κ and define

$$C_S := \{a \in \kappa^\kappa \mid \exists c \subseteq \kappa \text{ club such that } \forall i \in c\, (x(i) \in S)\}.$$

Clearly this set is $\mathbf{\Sigma}^1_1$ by the same argument as in Fact 2.4. Proceed exactly as before, choosing members from S or from $\kappa \setminus S$ at limit stages, as desired, which can be achieved using the club-splitting of the trees. $\square$

In the above result, an essential property of the trees T was that $\forall x \in [T]$, the set $\{i < \kappa \mid x{\restriction}i \text{ is a splitting node of } T\}$ formed a club on κ. Recent work of Philipp Schlicht [13] and Giorgio Laguzzi [14] suggests that this property is directly related to the existence of $\mathbf{\Sigma}^1_1$-counterexamples, since for a version of Sacks-, Miller- and Silver-measurability where the trees are *not* required to have this property, it is consistent that all projective sets are measurable.

3.4. Regularity of $\mathbf{\Delta}^1_1$ sets

With $\mathsf{Borel}(\mathbb{P})$ being provable in ZFC and $\mathbf{\Sigma}^1_1(\mathbb{P})$ inconsistent, we are left with the $\mathbf{\Delta}^1_1$-level.

Lemma 3.11 (Folklore)**.** *If $V = L$ then $\mathbf{\Delta}^1_1(\mathbb{P})$ is false for all tree-like $\mathbb{P}$.*

Proof. Use the $\mathbf{\Sigma}^1_1$-good wellorder of the reals of L from Lemma 2.8, and proceed as in the ω^ω-case, obtaining a $\mathbf{\Delta}^1_1$-counterexample as opposed to a $\mathbf{\Delta}^1_2$ one. $\square$

This is not the only method to produce $\mathbf{\Delta}^1_1$-counterexamples to $\mathbb{P}$-measurability. A completely different method, innate to the generalized setting, is to produce models in which the club filter itself is $\mathbf{\Delta}^1_1$, see Lemma 2.6.

It is known that the Baire property on κ^κ holds for $\mathbf{\Delta}^1_1$ sets in κ^+-product/iterations of κ-Cohen forcing, see e.g. [15, Theorem 49 (7)]. We would like to generalize this to other κ-tree-like forcings. First, we need the following technical result, a strengthening of the concept of *κ-proper* (Definition 2.9). This is again similar to the classical case.

Lemma 3.12. *Let $\mathbb{P}$ be κ-tree-like, and assume that $\mathbb{P}$ either has the κ^+-c.c. or satisfies Axiom A*. Then for every elementary submodel $M \prec \mathcal{H}_\theta$ of a sufficiently large $\mathcal{H}_\theta$, with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$, and for every $T \in \mathbb{P} \cap M$, there is $T' \leq T$ such that*

$$[T'] \subseteq^* \{x \in \kappa^\kappa \mid x \text{ is } \mathbb{P}\text{-generic over } M\}.$$

where $\subseteq^*$ means "modulo $\mathcal{I}_\mathbb{P}$" and a κ-real x is $\mathbb{P}$-generic over M if $\{S \in \mathbb{P} \cap M \mid x \in [S]\}$ is a $\mathbb{P}$-generic filter over M.

Proof. First assume that $\mathbb{P}$ has the κ^+-c.c. Let M be an elementary submodel with $|M| = \kappa$.

Claim *A real x is $\mathbb{P}$-generic over M if and only if $x \notin B$ for every Borel $\mathbb{P}$-null set B coded in M.*

Proof. Suppose x is $\mathbb{P}$-generic over M, and let B be a $\mathbb{P}$-null set coded in M. Then by elementarity $M \models$ "B is $\mathbb{P}$-null", and $D := \{S \in \mathbb{P} \cap M \mid [S] \cap B = \varnothing\}$ is in M and $M \models$ "D is dense". Since x is $\mathbb{P}$-generic, there exists $S \in D$ such that $x \in [S]$, and therefore, $x \notin B$.

Conversely, suppose $x \notin B$ for every Borel $\mathbb{P}$-null set coded in M. Let $D \subseteq \mathbb{P}$ be a dense set in M, and let A be a maximal antichain inside D. Let $B := \kappa^\kappa \setminus \bigcup\{[S] \mid S \in (A \cap M)\}$ which is a Borel set since $|A| = \kappa$ and has a code in M. Moreover $B \in \mathcal{N}_\mathbb{P}$ since A is a maximal antichain. Therefore, by assumption, $x \notin B$, and hence $x \in [S]$ for some $S \in A \cap M$, i.e., x is $\mathbb{P}$-generic over M. $\square$ (Claim).

Now it is easy to see that $X := \bigcup\{B \mid B \text{ is a Borel set in } \mathcal{N}_\mathbb{P} \text{ with code in } M\}$ is a κ-union of $\mathbb{P}$-null sets, hence it is itself in $\mathcal{I}_\mathbb{P}$. In particular, there exists $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x \text{ is } \mathbb{P}\text{-generic over } M\} = \kappa^\kappa \setminus X$.

Next, assume instead that $\mathbb{P}$ satisfies Axiom A*. Let $\{D_i \mid i < \kappa\}$ enumerate the dense sets in M, and let $T \in \mathbb{P} \cap M$. As usual, we can apply Axiom A* to inductively find a fusion sequence $\{T_i \mid i < \kappa\}$ and a sequence $\{E_i \subseteq D_i \mid i < \kappa\}$ such that each $E_i \in M$ and $|E_i| \leq \kappa$, and hence $E_i \subseteq M$, and moreover $[T_i] \subseteq \bigcup\{[S] \mid S \in E_i\}$. Let T' be such that $T' \leq T_i$ for all i. Then for every i, $[T'] \subseteq \bigcup\{[S] \mid S \in E_i\}$, so, in particular, every x in $[T']$ is $\mathbb{P}$-generic over M, so we are done. $\square$

Using this strengthening of κ-properness, we are almost in a position to prove that a κ^+-iteration of $\mathbb{P}$ satisfying either the κ^+-c.c. or Axiom A* yields a model of for $\mathbf{\Delta}^1_1(\mathbb{P})$. However, we still have an obstacle, and that is the lack of an abstract preservation theorem for κ-properness, mentioned in Section 2.5. This obstacle makes it impossible to prove the next theorem in an abstract setting including the non-κ^+-c.c. cases. We could formulate it under the assumption that κ-properness is preserved; but in fact we only need several consequences of κ-properness, namely, that κ^+ is preserved and that all new κ-reals appear at some initial stage of the iteration.

Theorem 3.13. *Let $\mathbb{P}$ be a tree-like forcing.*

1. *Suppose $\mathbb{P}$ is κ-linked and well-met (see Fact 2.10), and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $<\kappa$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P})$.*

2. *Suppose $\mathbb{P}$ satisfies Axiom A*, and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $\leq\kappa$. Moreover, assume that $\mathbb{P}_{\kappa^+}$ preserve κ^+ and, moreover, for every $x \in \kappa^\kappa \cap V^{\mathbb{P}_{\kappa^+}}$, there is $\alpha < \kappa^+$ such that $x \in \kappa^\kappa \cap V^{\mathbb{P}_\alpha}$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P})$.*

Proof. The proof works uniformly for both cases. In case 1 we use Fact 2.10 to conclude that $\mathbb{P}_{\kappa^+}$ has the κ^+-c.c., hence preserves κ^+ and has the well-known property that κ-reals in the final extension are caught at an initial stage of the iteration. Note that by Definition 3.1 (5), all tree-like forcings are $<\kappa$-closed.

In $V[G_{\kappa^+}]$, let A be $\mathbf{\Delta}^1_1$, defined by Σ^1_1-formulas ϕ and ψ. Let $S \in \mathbb{P}$ be arbitrary. By the assumption, there exists an $\alpha < \kappa^+$ such that all parameters of ϕ and ψ, as well as S, belong to $V[G_\alpha]$. Moreover, there is a $\beta > \alpha$ such that S belongs to $G(\beta+1)$ (the $(\beta+1)$-st component of the generic filter), since it is dense to force this for some $\beta > \alpha$. Let $x_{\beta+1}$ be the real corresponding to $G(\beta+1)$, i.e., the next $\mathbb{P}$-generic real over $V[G_\beta]$.

We know that in the final model $V[G_{\kappa^+}]$, either $\phi(x_{\beta+1})$ or $\psi(x_{\beta+1})$ holds. As ϕ and ψ are both Σ^1_1 the situation is clearly symmetrical so without loss of generality assume the former. Since $\mathbb{P}$ is $<\kappa$-closed, any iteration of it is also $<\kappa$-closed, so by Lemma 2.7 we have Σ^1_1-absoluteness between $V[G_{\kappa^+}]$ and $V[G_{\beta+1}]$. In particular, $V[G_{\beta+1}] = V[G_\beta][x_{\beta+1}] \models \phi(x_{\beta+1})$. By the forcing theorem, and since we have assumed $S \in G(\beta+1)$, there exists a $T \in V[G_\beta]$ such that $T \leq S$ and $T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\text{gen}})$.

Now, still in $V[G_\beta]$, take an elementary submodel M of a sufficiently large structure, of size κ, containing T. By elementarity, $M \models$ "$T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\text{gen}})$". Going back to $V[G_{\kappa^+}]$, use Lemma 3.12 to find a $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x$ is $\mathbb{P}$-generic over $M\}$. Now note that if x is $\mathbb{P}$-generic over M and $x \in [T]$, then $M[x] \models \phi(x)$. By upwards-Σ^1_1-absoluteness between M and $V[G_{\kappa^+}]$, we conclude that $\phi(x)$ really holds. Since this was true for arbitrary $x \in [T']$, we obtain $[T'] \subseteq^* \{x \mid \phi(x)\} = A$. □

The above theorem can be applied to many forcing partial orders $\mathbb{P}$, in particular those from Example 3.2.

Corollary 3.14. $\mathbf{\Delta}^1_1(\mathbb{P})$ *is consistent for $\mathbb{P} \in \{\mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa\}$, and if κ is inaccessible, also for $\mathbb{P} = \mathbb{V}_\kappa$.*

Proof. The forcings $\mathbb{C}_\kappa, \mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ have the following two properties: any two conditions with the same stem are compatible, and if S, T are two compatible conditions, then $S \cap T$ is a condition. This implies that all three forcings are κ-linked and well-met.

By Fact 2.11 (1), iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-sized supports satisfy κ-properness assuming that $\diamondsuit_\kappa$ holds in the ground model, so $\mathbf{\Delta}^1_1(\mathbb{S}_\kappa)$ holds in $L^{\mathbb{S}_{\kappa^+}}$. By Fact 2.11 (2), iterations of $\mathbb{M}_\kappa$ with $\leq\kappa$-sized supports satisfy κ-properness for inaccessible κ. It seems very plausible that by an analogous argument to [25], the same holds for arbitrary κ assuming $\diamondsuit_\kappa$. However, we will leave out the verification of this (potentially very technical) proof because $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ also follows by a much easier argument, namely Theorem 4.9 (3). Finally, if κ is inaccessible then a straightforward modification of [25, Theorem 6.1] shows that iterations of κ-Silver with $\leq\kappa$-sized supports satisfies κ-properness (the only change in the argument involves the definition of the fusion sequence [25, Definition 1.7] and the amalgamation defined in [25, Page 103]). We leave the details to the reader. □

Remark 3.15. It is clear that in Theorem 3.13 it is enough to add $\mathbb{P}$-generic reals cofinally often, provided that the iteration is $<\kappa$-closed and satisfies the other requirements. For example, we can obtain $\mathbf{\Delta}^1_1(\mathbb{C}_\kappa) + \mathbf{\Delta}^1_1(\mathbb{L}_\kappa) + \mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ simultaneously by employing a κ^+-iteration of $(\mathbb{C}_\kappa * \mathbb{L}_\kappa * \mathbb{R}_\kappa)$ with supports of size $<\kappa$.

Recall that in the classical setting we had Solovay-style characterization theorems for $\mathbf{\Delta}^1_2$ sets, such as Theorem 1.2 and related results (see [5, 3]). In light of Theorem 3.13, one might expect that in the generalized setting, analogous characterization theorems exist for statements concerning $\mathbf{\Delta}^1_1$ sets. However, the following observation shows that this is not the case.

Observation 3.16. Suppose κ is successor. There exists a generic extension of L in which the statement "$\forall r \in 2^\kappa\, \exists x\, (x$ is κ-Cohen over $L[r])$" holds, yet there exists a $\mathbf{\Delta}^1_1$ subset of 2^κ without the Baire property.

Proof. Recall that by Theorem 2.6, it is consistent for the club filter C (Definition 2.4) to be $\mathbf{\Delta}^1_1$-definable. The idea is to adapt the proof of [18, Theorem 1.1] due to Friedman, Wu and Zdomskyy. Since that proof is long and technical, we cannot afford to go into details here, so we only provide a sketch of the argument and leave the details to the reader. In that proof, a model where C is $\mathbf{\Delta}^1_1$ is obtained by a forcing iteration, starting from L, in which cofinally many iterands have the κ^+-c.c. One can then verify that the proof remains correct if, additionally, κ-Cohen reals are added cofinally often to this iteration (in fact, κ-Cohen reals are added naturally in the original proof). Thus we obtain a model in which the club filter is $\mathbf{\Delta}^1_1$ and hence fails to have the Baire property, while clearly the statement "$\forall r \in 2^\kappa\, \exists x\, (x$ is κ-Cohen over $L[r])$" is true. □

A similar argument can be applied to any κ-tree-like forcing $\mathbb{P}$ which satisfies the κ^+-c.c., provided it also satisfies Theorem 3.10 (i.e., whose trees are κ-Sacks or κ-Miller trees).

4. Regularity Properties for $\mathbf{\Delta}^1_1$ sets

In the classical setting, regularity properties related to well-known forcing notions on ω^ω or 2^ω have been investigated, and the exact relationship between statements $\mathbf{\Delta}^1_2(\mathbb{P})$ and $\mathbf{\Sigma}^1_2(\mathbb{P})$ has been studied for various forcing notions $\mathbb{P}$. As we saw in the previous section, for generalized reals the $\mathbf{\Delta}^1_1$-level reflects some of these results. We will focus on the forcing notions from Example 3.2, i.e., κ-Cohen, κ-Sacks, κ-Miller, κ-Laver, κ-Mathias and κ-Silver.

Before proceeding, we make a further comment regarding κ-Laver and κ-Mathias, showing that the ideal $\mathcal{I}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-meager sets and the ideal $\mathcal{I}_{\mathbb{R}_\kappa}$ of $\mathbb{R}_\kappa$-meager sets cannot be neglected when discussing the regularity property generated by them.

Lemma 4.1. *The ideal $\mathcal{N}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-null sets is not equal to the ideal $\mathcal{I}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-meager sets. Also, there is an F_σ set A such that no κ-Laver tree is completely contained or completely disjoint from A. The same holds for $\mathbb{R}_\kappa$.*

Proof. Fix a stationary, co-stationary $S \subseteq \kappa$. For each $i < \kappa$ define $A_i := \{x \in \kappa^\kappa_\uparrow \mid \forall j > i(x(j) \in S)\}$ and $A = \bigcup_{i<\kappa} A_i$. Then each A_i is $\mathbb{L}_\kappa$-null, because any κ-Laver tree T can be extended to some $T' \leq T$ with stem s, such that $\text{len}(s) > i$ and for some $j > i$ we have $s(j) \notin S$, so that clearly $[T'] \cap A_i = \varnothing$. On the other hand, A itself cannot be $\mathbb{L}_\kappa$-null, because every κ-Laver tree T contains a branch $x \in [T]$ such that for all j longer then the stem of T we have $x(j) \in S$, and therefore $x \in A$. It is also clear that the set A is F_σ but every κ-Laver tree T contains a branch x which is in A and another branch y which is not in A. The argument for κ-Mathias is analogous. $\square$

Summarizing, the forcings we have introduced can be neatly divided into two categories as presented in Table 1.

κ-Cohen κ-Laver κ-Mathias	Category 1: topological, κ^+-c.c., ideal $\mathcal{I}_{\mathbb{P}}$ cannot be neglected; $\mathbb{P}$-measurability equivalent to Baire property in $\mathbb{P}$-topology.
κ-Sacks κ-Miller κ-Silver	Category 2: non-topological, Axiom A*, $\mathcal{I}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}}$ can be neglected.

Table 1: Properties of forcings.

4.1. Solovay-style characterizations

By Observation 3.16, we know that a Solovay-style characterization for $\mathbf{\Delta}^1_1(\mathbb{P})$ cannot be achieved in the generalized setting. However, in some cases we can obtain one half of such a characterization.

Lemma 4.2. $\mathbf{\Delta}^1_1(\mathbb{C}_\kappa)$ *implies that for every $r \in \kappa^\kappa$ there exists a κ-Cohen real over $L[r]$.*

Proof. The proof is completely analogous to the classical case, see e.g. [9, Theorem 9.2.1], except that we obtain a $\mathbf{\Delta}^1_1$-counterexample as opposed to a $\mathbf{\Delta}^1_2$ one, using the $\mathbf{\Sigma}^1_1$-good wellorder of L (Lemma 2.8). A central ingredient of the classical proof is the Kuratowski-Ulam (Fubini for Category) theorem, which, as we mentioned, is valid on the generalized Baire space. A detailed argument has also been worked out in the PhD Thesis of Laguzzi, see [30, Theorem 75]. $\square$

Lemma 4.3. *$\mathbf{\Delta}^1_1(\mathbb{S}_\kappa)$ implies that for every $r \in \kappa^\kappa$ there is an $x \in 2^\kappa \setminus L[r]$.*

Proof. This follows directly from Lemma 3.11. $\square$

Let us define, for $x, y \in \kappa^\kappa$, the *eventual domination* relation: $x <^* y$ iff $\exists \alpha \forall \beta > \alpha\ (x(\beta) < y(\beta))$. We will simply say "$y$ dominates x" for $x <^* y$ and if $X \subseteq \kappa^\kappa$ we will say "y dominates X" iff $\forall x \in X\ (x <^* y)$. We will also say "$y$ is unbounded over x" iff $x \not>^* y$ and "y is unbounded over X" iff $\forall x \in X\ (x \not>^* y)$. Note that for the next lemma, it is not relevant whether we talk about domination in the space of all elements of κ^κ or only the strictly increasing ones.

In [5, Theorem 6.1] it is proved that $\mathbf{\Delta}^1_2(\mathbb{M})$ implies the existence of unbounded reals over $L[r]$ for every real r. This generalizes to the κ^κ-context assuming κ is an inaccessible.

Lemma 4.4. *Suppose κ is inaccessible. Then $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ implies that for every $r \in \kappa^\kappa$ there is an $x \in \kappa^\kappa_\uparrow$ which is unbounded over $\kappa^\kappa_\uparrow \cap L[r]$.*

Proof. The proof is based on the proof of [5, Theorem 6.1]. Assuming that there are no unbounded reals over $\kappa^\kappa_\uparrow \cap L[r]$ we will construct a $\mathbf{\Sigma}^1_1$-definable sequence $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ of reals in $L[r]$ which is dominating, well-ordered by $<^*$, and satisfies some additional technical properties. This will yield two non-κ-Miller-measurable sets A and B defined by $A := \{x \in \kappa^\kappa_\uparrow \mid \text{the least } \alpha \text{ such that } x \leq^* f_\alpha \text{ is even}\}$ and $B := \{x \in \kappa^\kappa_\uparrow \mid \text{the least } \alpha \text{ such that } x \leq^* f_\alpha \text{ is odd}\}$, where, by convention, limit ordinals are considered even.

To begin with, we fix an enumeration $\langle \boldsymbol{\sigma}_i \mid i < \kappa \rangle$ of $\kappa^{<\kappa}_\uparrow \setminus \{\varnothing\}$. Let $\ulcorner \sigma \urcorner$ denote i such that $\sigma = \boldsymbol{\sigma}_i$, and also well-order $\kappa^{<\kappa}_\uparrow \setminus \{\varnothing\}$ by $\preceq$, defined by $\sigma \preceq \tau$ iff $\ulcorner \sigma \urcorner \leq \ulcorner \tau \urcorner$. We also use the following notation: for all $\sigma \in \kappa^{<\kappa}_\uparrow$ of successor length, let $\sigma(\text{last})$ denote the last digit of σ, i.e., $\sigma(\text{len}(\sigma) - 1)$.

Next, let $\mathcal{C}$ denote the set $\{\sigma \in \kappa^{<\kappa}_\uparrow \mid \text{len}(\sigma) \text{ is a successor}\}$. Define a fixed function $\varphi_0 : \mathcal{C} \to \kappa$ by letting $\varphi_0(\sigma)$ be the least $i < \kappa$ such that $\boldsymbol{\sigma}_i(0) > \sigma(\text{last})$. The function φ_0 should be understood as a "lower bound" on potential other functions $\varphi : \mathcal{C} \to \kappa$ satisfying $\boldsymbol{\sigma}_{\varphi(\sigma)}(0) > \sigma(\text{last})$.

Let T be a given κ-Miller tree T, and assume, without loss of generality, that every splitting node of T is club-splitting. We recursively define a collection $\left\langle \tilde{\tau}^T_\sigma \mid \sigma \in \kappa^{<\kappa}_\uparrow \right\rangle$ of split-nodes of T, and another collection $\langle \tau^T_\sigma \mid \sigma \in \mathcal{C} \rangle$, as follows:

- $\tilde{\tau}^T_\varnothing = \text{stem}(T)$.

- Assuming $\tilde{\tau}^T_\sigma$ is defined, and given a $\beta < \kappa$, let $\tau^T_{\sigma^\frown\langle\beta\rangle}$ be $\boldsymbol{\sigma}_i$ for the least i such that
  - $\tilde{\tau}^T_\sigma{}^\frown\boldsymbol{\sigma}_i \in \mathrm{Split}(T)$, and
  - $\boldsymbol{\sigma}_i(0) > \beta$.

  Then let $\tilde{\tau}^T_{\sigma^\frown\langle\beta\rangle} := \tilde{\tau}^T_\sigma{}^\frown\tau^T_{\sigma^\frown\langle\beta\rangle}$.
- For σ with $\mathrm{len}(\sigma) = \lambda$ limit, let $\tilde{\tau}^T_\sigma := \bigcup_{\alpha<\lambda} \tilde{\tau}^T_{\sigma\restriction\alpha}$. Note that $\tilde{\tau}^T_\sigma \in \mathrm{Split}(T)$ by the assumption that limits of splitting nodes in T are splitting.

Intuitively, each τ^T_σ, for σ of successor length, gives us a $\preceq$-minimal extension within the tree T, whose first digit is strictly higher then the a-priori-prescribed value $\sigma(\mathrm{last})$. Define a function $\varphi_T : \mathcal{C} \to \kappa$ by $\varphi_T(\sigma) := \ulcorner\tau^T_\sigma\urcorner$. This function will be used as a lower bound later. Notice that for any κ-Miller tree T we have $\varphi_0 \leq \varphi_T$, and in fact $\varphi_0 = \varphi_{(\kappa^{<\kappa}_\uparrow)}$ (i.e., the φ_T for $T = \kappa^{<\kappa}_\uparrow$ = the trivial $\mathbb{M}_\kappa$-condition.)

It is worth noting that since the values of $\varphi_T(\sigma)$ and $\varphi_0(\sigma)$ only depend on $\sigma(\mathrm{last})$, these functions could also be construed as functions from κ to κ. However, for technical reasons, it is necessary to consider them as functions from $\mathcal{C}$ to κ.

Next, for a fixed function $f : \kappa \to \kappa$, another function $\varphi : \kappa^{<\kappa}_\uparrow \to \kappa$ satisfying $\varphi_0 \leq \varphi$, and an ordinal $\beta < \kappa$, we define a special set $S = S(\varphi, f, \beta)$ of κ-reals. This set will be defined by specifying "fronts" S_α, for $\alpha < \kappa$. Each S_α will be a subset of $\kappa^{<\kappa}_\uparrow$, satisfying the following two requirements:

1. $|S_\alpha| < \kappa$, and
2. $\forall\rho \in S_\alpha\ (\mathrm{len}(\rho) \geq \alpha)$.

Moreover, every $\rho \in S_{\alpha+1}$ will be a proper extension of a $\rho' \in S_\alpha$. We construct the S_α recursively as follows:

- $S_0 := \{\boldsymbol{\sigma}_i \mid i \leq \beta\}$.
- $S_1 := \{\rho^\frown\boldsymbol{\sigma}_i \mid \rho \in S_0, i \leq \varphi(\langle\beta\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > \beta\}$.

  Notice that since $\varphi_0(\langle\beta\rangle) \leq \varphi(\langle\beta\rangle)$ there is at least one $\boldsymbol{\sigma}_i$ satisfying the above requirement. In particular, all elements of S_1 have length ≥ 1. It is also clear that $|S_1| < \kappa$.
- Let $\mathrm{height}(S_1) := \sup\{\mathrm{len}(\rho) \mid \rho \in S_1\}$ and let $f^*(1) := \sup(\{\beta\} \cup \{f(\xi) \mid \xi < \mathrm{height}(S_1)\})$. Now let

  $$S_2 := \{\rho^\frown\boldsymbol{\sigma}_i \mid \rho \in S_1, i \leq \varphi(\langle\beta, f^*(1)\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > f^*(1)\}.$$

  Again notice that since $\varphi_0(\langle\beta, f^*(1)\rangle) \leq \varphi(\langle\beta, f^*(1)\rangle)$, there exists at least one $\boldsymbol{\sigma}_i$ as above, so all element of S_2 have length ≥ 2. Also it is clear that $|S_2| < \kappa$.

- Generally, assume S_α is defined as well as $f^*(\xi)$ for all $\xi < \alpha$. Let $\mathrm{height}(S_\alpha) := \sup\{\mathrm{len}(\rho) \mid \rho \in S_\alpha\}$, which is an ordinal $< \kappa$ by the inductive assumption that $|S_\alpha| < \kappa$. Let $f^*(\alpha) := \sup(\{\beta\} \cup \{f(\xi) \mid \xi < \mathrm{height}(S_\alpha)\})$. Then let

  $$S_{\alpha+1} := \{\rho^\frown \boldsymbol{\sigma}_i \mid \rho \in S_\alpha, i \leq \varphi(\langle \beta, f^*(1), \ldots, f^*(\alpha)\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > f^*(\alpha)\}.$$

  As before, $\varphi_0(\langle \beta, f^*(1), \ldots, f^*(\alpha)\rangle) \leq \varphi(\langle \beta, f^*(1), \ldots, f^*(\alpha)\rangle)$ implies that all members of $S_{\alpha+1}$ have length $\geq \alpha + 1$. Also $|S_{\alpha+1}| < \kappa$ is clear.

- Suppose λ is limit. Let S_λ be the collection of $\rho \in \kappa_{\uparrow}^{<\kappa}$ such that $\rho = \bigcup_{\alpha<\lambda} \rho_\alpha$ for some strictly $\subseteq$-increasing sequence $\{\rho_\alpha \mid \alpha < \lambda\}$ with $\rho_\alpha \in S_\alpha$. Clearly all such ρ have length $\geq \lambda$. By the inductive assumption that $|S_\alpha| < \kappa$ for all $\alpha < \lambda$, and the fact that κ is inaccessible, it follows that $|S_\lambda| < \kappa$.

Finally we let $S = S(\varphi, f, \beta)$ to be the set of all κ-reals x such that $x = \bigcup_{\alpha<\kappa} \rho_\alpha$ for some strictly $\subseteq$-increasing sequence $\{\rho_\alpha \mid \alpha < \kappa\}$ with $\rho_\alpha \in S_\alpha$. The essential properties of $S(\varphi, f, \beta)$ are summarized in the next sublemma:

Sublemma 4.5.

1. *For every $S(\varphi, f, \beta)$, there exists a function $g \in \kappa^\kappa$ which bounds $S(\varphi, f, \beta)$ (i.e., $\forall x \in S(\varphi, f, \beta)\ \forall i < \kappa\ ((x(i) < g(i)))$.*
2. *Every $x \in S(\varphi, f, \beta)$ is cofinally often above f (i.e., $x \not<^* f$).*
3. *For every κ-Miller tree T, f and φ satisfying $\varphi_T <^* \varphi$, there exists $\beta < \kappa$ such that $[T] \cap S(\varphi, f, \beta) \neq \varnothing$.*

Proof.

1. By construction, if ρ is any initial segment of any $x \in S(\varphi, f, \beta)$ with $\mathrm{len}(\rho) = \alpha$, then ρ must be an initial segment of some sequence from S_α. We can thus define g by stipulating that $g(\alpha)$ be above $\rho(\alpha)$ for all $\rho \in S_{\alpha+1}$, which can always be done since $|S_{\alpha+1}| < \kappa$. Now it is clear that for every $x \in S(\varphi, f, \beta)$, for every α we have $x(\alpha) < g(\alpha)$ (another way to explain this is: the tree generated by $\bigcup_{\alpha<\kappa} S_\alpha$ is $<\kappa$-branching).
2. By construction, each $S_{\alpha+1}$ contains only those $\rho^\frown \boldsymbol{\sigma}_i$ where $\boldsymbol{\sigma}_i(0) > f^*(\alpha)$. In particular $\boldsymbol{\sigma}_i(0) > f(\mathrm{len}(\rho))$. Therefore $x(\xi) > f(\xi)$ happens cofinally often for every $x \in S(\varphi, f, \beta)$.
3. This is the main point of the proof. First, note that since $\varphi_T <^* \varphi$, there are only $<\kappa$-many σ satisfying $\varphi_T(\sigma) \geq \varphi(\sigma)$. In particular, we can pick $\beta < \kappa$ such that
   (a) $\beta > \ulcorner \mathrm{stem}(T) \urcorner$, and
   (b) $\varphi_T(\langle \beta \rangle^\frown \sigma) < \varphi(\langle \beta \rangle^\frown \sigma)$ holds for *all* σ.

After β has been fixed, the set $S(\varphi, f, \beta)$ is also fixed. In particular, f^* can be computed from f as it was done in the construction of the S_α's. Let

$$\vec{f} := \langle \beta \rangle ^\frown \langle f^*(\alpha) \mid 1 \leq \alpha < \kappa \rangle .$$

and for all $\alpha < \kappa$ use the abbreviation:

$$\rho_\alpha := \tilde{\tau}^T_{\vec{f}\restriction\alpha}.$$

Then $x := \bigcup_{\alpha<\kappa} \rho_\alpha = \bigcup_{\alpha<\kappa} \tilde{\tau}^T_{\vec{f}\restriction\alpha}$ is a branch through $[T]$. On the other hand, we claim that $\rho_\alpha \in S_\alpha$ for all α:

- Since $\ulcorner \mathrm{stem}(T) \urcorner < \beta$ and $\rho_0 = \tilde{\tau}^T_\varnothing = \mathrm{stem}(T)$, by construction $\rho_0 \in S_0$.
- Since $\varphi_T(\langle\beta\rangle) < \varphi(\langle\beta\rangle)$, $\ulcorner \tau^T_{\langle\beta\rangle} \urcorner = \varphi_T(\langle\beta\rangle)$, $\tau^T_{\langle\beta\rangle}(0) > \beta$, and
$$\rho_1 = \tilde{\tau}^T_{\langle\beta\rangle} = \tilde{\tau}^T_\varnothing {}^\frown \tau^T_{\langle\beta\rangle} = \rho_0 {}^\frown \tau^T_{\langle\beta\rangle},$$
by construction $\rho_1 \in S_1$.
- Assume $\rho_\alpha \in S_\alpha$. Since $\varphi_T(\vec{f}\restriction(\alpha+1)) < \varphi(\vec{f}\restriction(\alpha+1))$, $\ulcorner \tau^T_{\vec{f}\restriction(\alpha+1)} \urcorner = \varphi_T(\vec{f}\restriction(\alpha+1))$, $\tau^T_{\vec{f}\restriction(\alpha+1)}(0) > f^*(\alpha)$ and
$$\rho_{\alpha+1} = \tilde{\tau}^T_{\vec{f}\restriction(\alpha+1)} = \tilde{\tau}^T_{\vec{f}\restriction\alpha} {}^\frown \tau^T_{\vec{f}\restriction(\alpha+1)} = \rho_\alpha {}^\frown \tau^T_{\vec{f}\restriction(\alpha+1)},$$
by construction $\rho_{\alpha+1} \in S_{\alpha+1}$.
- For limits λ we have $\rho_\lambda = \tilde{\tau}^T_{\vec{f}\restriction\lambda} = \bigcup_{\alpha<\lambda} \tilde{\tau}^T_{\vec{f}\restriction\alpha} = \bigcup_{\alpha<\lambda} \rho_\alpha$. Since inductively $\rho_\alpha \in S_\alpha$, by definition we have $\rho_\lambda \in S_\lambda$.

Since $\rho_\alpha \in S_\alpha$ for all $\alpha < \kappa$ we obtain $x = \bigcup_{\alpha<\kappa} \rho_\alpha \in S(\varphi, f, \beta)$, as had to be shown. $\square$ (Sublemma)

To complete the proof of the main lemma, assume, towards contradiction, that $\kappa^\kappa_\uparrow \cap L[r]$ is a dominating set, for some r. Construct a sequence $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ of elements of $\kappa^\kappa_\uparrow \cap L[r]$, and an auxiliary sequence $\langle \varphi_\alpha \mid \alpha < \kappa^+ \rangle$ of elements of $\kappa^{\mathcal{C}} \cap L[r]$, in such a way that:

1. $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ and $\langle \varphi_\alpha \mid \alpha < \kappa^+ \rangle$ are well-ordered by $<^*$,
2. $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a dominating subset of $\kappa^\kappa_\uparrow \cap L[r]$ and $\langle \varphi_\alpha \mid \alpha < \kappa^+ \rangle$ is a dominating subset of $\kappa^{\mathcal{C}} \cap L[r]$,
3. all φ_α are pointwise strictly above φ_0,
4. $f_{\alpha+1}$ dominates $S(\varphi_\alpha, f_\alpha, \beta)$ for all β, and
5. both sequences have $\mathbf{\Sigma}^1_1$-definitions.

To see that this can be done, at each step α inductively pick the $<_{L[a]}$-least f_α and φ_α dominating all the previous functions; to satisfy point 4 above, use Sublemma (1) to dominate each $S(\varphi_\alpha, f_\alpha, \beta)$ by a corresponding function g_β, and then dominate $\{g_\beta \mid \beta < \kappa\}$ by another g.

Now, as suggested earlier, define $A := \{x \in \kappa^\kappa_\uparrow \mid$ the least f_α which dominates x is even$\}$ and $B := \{x \in \kappa^\kappa_\uparrow \mid$ the least f_α which dominates x is odd$\}$. Clearly $A \cap B = \varnothing$, and by assumption $A \cup B = \kappa^\kappa_\uparrow$. Since the sequence of f_α's was $\mathbf{\Sigma}^1_1$-definable, the sets A and B are also $\mathbf{\Sigma}^1_1$-definable, hence they are both $\mathbf{\Delta}^1_1$. To reach a contradiction, let T be a κ-Miller tree, and we will show that $[T]$ contains an element in A and an element in B. Since the sequence $\langle \varphi_\alpha \mid \alpha < \kappa^+ \rangle$ is dominating, there exists an α such that for all $\xi \geq \alpha$ we have $\varphi_T <^* \varphi_\xi$. In particular $\varphi_T <^* \varphi_\alpha$ and $\varphi_T <^* \varphi_{\alpha+1}$. By point 3 of the Sublemma, we can find β and β' such that

$$[T] \cap S(\varphi_\alpha, f_\alpha, \beta) \neq \varnothing, \text{ and}$$

$$[T] \cap S(\varphi_{\alpha+1}, f_{\alpha+1}, \beta') \neq \varnothing.$$

Without loss of generality α is even. Let y be an element of the first set. By point 2 of the Sublemma, $y \not<^* f_\alpha$, and by construction, $y <^* f_{\alpha+1}$. Hence $y \in B$. Likewise, let y' be an element of the second set. Then by an analogous argument $y' \not<^* f_{\alpha+1}$ but $y' <^* f_{\alpha+2}$. Hence $y' \in A$. This completes the proof. □

Question 4.6. *Can Lemma 4.4 be proved without assuming that κ is inaccessible?*

So far, these are the only generalizations of classical Solovay-style characterizations known to us. The other result due to Brendle and Löwe linked Laver-measurability with dominating reals. However, that proof does not seem to generalize to the κ^κ-setting because κ-Laver-measurability differs from classical Laver-measurability in the sense that the ideal $\mathcal{I}_{\mathbb{L}}$ cannot be neglected (see Lemma 4.1). Therefore the following is still open:

Question 4.7. *Does $\mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ imply that for every $r \in \kappa^\kappa$, there is an x which is dominating over $L[r]$?*

Likewise, currently we do not have suitable Solovay-style consequences of the assumptions $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$. In the classical setting, there is a connection between these properties and splitting/unsplit reals.

Question 4.8. *Can the hypotheses $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ be linked to the existence of (a suitable generalization of) splitting/unsplit reals?*

4.2. Comparing $\mathbf{\Delta}^1_1(\mathbb{P})$

The next questions we want to ask are: for which $\mathbb{P}$ and $\mathbb{Q}$ does $\mathbf{\Delta}^1_1(\mathbb{P})$ imply $\mathbf{\Delta}^1_1(\mathbb{Q})$, and for which $\mathbb{P}$ and $\mathbb{Q}$ can we construct models where $\mathbf{\Delta}^1_1(\mathbb{P})+\neg\mathbf{\Delta}^1_1(\mathbb{Q})$ holds? We will prove several implications for arbitrary pointclasses $\mathbf{\Gamma}$ in Lemma 4.9. Classical counterparts of such implications are well-known but generally much easier to prove, as the uncountable context provides combinatorial challenges not present when $\kappa = \omega$.

Separating regularity properties is currently very difficult for the following two reasons:

1. We do not have good Solovay-style characterizations, and
2. We do not have good preservation theorems for forcing iterations.

We will finish this section with the only example of such a separation result currently known to us.

Lemma 4.9. *Let $\mathbf{\Gamma}$ be a class of subsets of κ^κ or 2^κ closed under continuous preimages (in particular $\mathbf{\Gamma} = \mathbf{\Delta}^1_1$). Then*

1. $\mathbf{\Gamma}(\mathbb{M}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_\kappa)$.
2. $\mathbf{\Gamma}(\mathbb{V}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_\kappa)$.
3. $\mathbf{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
4. $\mathbf{\Gamma}(\mathbb{L}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
5. $\mathbf{\Gamma}(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
6. *If κ is inaccessible, then $\mathbf{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{V}_\kappa)$.*

Proof.

1. Let $A \subseteq 2^\kappa$ be a set in $\mathbf{\Gamma}$ and let T be a κ-Sacks tree. We must find a κ-Sacks tree below T whose branches are completely contained in or disjoint from A. Let φ be the natural order-preserving bijection identifying $2^{<\kappa}$ with $\mathrm{Split}(T)$, and φ^* the induced homeomorphism between 2^κ and $[T]$. Further, fix a stationary, co-stationary set $S \subseteq \kappa$ and enumerate $S := \{\xi_\alpha \mid \alpha < \kappa\}$ and $\kappa \setminus S := \{\eta_\alpha \mid \alpha < \kappa\}$. Let ψ be a map from $\kappa^{<\kappa}_{\uparrow}$ to $2^{<\kappa}$ defined by:
   - $\psi(\varnothing) = \varnothing$.
   - $\psi(s^\frown \langle\alpha\rangle) := \begin{cases} \psi(s)^\frown \langle 1\rangle {}^\frown 0^{\beta\frown} \langle 1\rangle & \text{if } \alpha \in S \text{ and } \alpha = \xi_\beta \\ \psi(s)^\frown \langle 0\rangle {}^\frown 0^{\beta\frown} \langle 1\rangle & \text{if } \alpha \notin S \text{ and } \alpha = \eta_\beta \end{cases}$

     where 0^β denotes a β-sequence of 0's.
   - $\psi(s) := \bigcup_{\alpha<\lambda} \psi(s{\restriction}\alpha)$, if $\mathrm{len}(s) = \lambda$ for a limit ordinal.

The function ψ is different from a standard encoding of ordinals by binary sequences, but it is clear that ψ is bijective, since there is an obvious algorithm to compute $\psi^{-1}(s)$ for any $s \in 2^{<\kappa}$. The reason for using this specific function is that we want $\psi(s)$ to be a splitting node whenever s is a club-splitting node. Clearly, ψ induces a homeomorphism ψ^* between $\kappa^\kappa_\uparrow$ and $2^\kappa \setminus \mathbb{Q}$, where we use $\mathbb{Q}$ to denote the *generalized rationals*, i.e., $\mathbb{Q} := \{x \in 2^\kappa \mid |\{i \mid x(i) = 1\}| < \kappa\}$.

Let $A' := (\varphi^* \circ \psi^*)^{-1}[A]$, which is in $\mathbf{\Gamma}$ by assumption. By $\mathbf{\Gamma}(\mathbb{M}_\kappa)$ we can find a κ-Miller tree R such that $[R] \subseteq A'$ or $[R] \cap A' = \varnothing$, w.l.o.g. the former. Let $R' := \{\psi(s) \mid s \in R\}$. First, note that R' is a κ-Sacks tree: this follows because for any $s \in \mathrm{Split}(R)$ there are $\alpha \in S$ and $\beta \notin S$ such that both $s^\frown \langle\alpha\rangle$ and $s^\frown \langle\beta\rangle$ are in R, which implies that both $\psi(s)^\frown \langle 1\rangle$ and $\psi(s)^\frown \langle 0\rangle$ are in R', so $\psi(s) \in \mathrm{Split}(R')$. Moreover, since ψ^* is a homeomorphism, we know that $[R'] \setminus \mathbb{Q} = (\psi^*)``[R] \subseteq (\varphi^*)^{-1}[A]$. But since $\mathbb{Q}$ is a set of size $\kappa$ we can easily find a refinement $R'' \subseteq R'$, which is still a $\kappa$-Sacks tree and moreover $[R''] \subseteq (\psi^*)``[R] \subseteq (\varphi^*)^{-1}[A]$. Then $(\varphi^*)``[R'']$ generates a κ-Sacks tree which is completely contained in $[T] \cap A$.

2. Let $A \in \mathbf{\Gamma}$ and $T \in \mathbb{S}_\kappa$ and φ and φ^* be as above. Then $A' := (\varphi^*)^{-1}[A]$ is in $\mathbf{\Gamma}$ so there exists a κ-Silver tree S such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$. As S is a κ-Sacks tree, clearly $\varphi``S$ generates a κ-Sacks tree below T whose branches are completely contained in or completely disjoint from A.

3. Now let $A \subseteq \kappa^\kappa_\uparrow$ be in $\mathbf{\Gamma}$ and let T be a κ-Miller tree. By shrinking if necessary, we may assume T to have the property that all splitting nodes are club-splitting. Let φ be the natural order-preserving bijection between $\kappa^{<\kappa}_\uparrow$ and $\mathrm{Split}(T)$, and φ^* the induced homeomorphism between $\kappa^\kappa_\uparrow$ and $[T]$. Let $A' := (\varphi^*)^{-1}[A]$. As A' has the Baire property by $\mathbf{\Gamma}(\mathbb{C}_\kappa)$, let $[s]$ be a basic open set such that $[s] \subseteq^* A'$ or $[s] \cap A' =^* \varnothing$, and without loss of generality assume the former. Let $\{X_i \mid i < \kappa\}$ be nowhere dense sets such that $[s] \setminus A' = \bigcup_{i<\kappa} X_i$. We will inductively construct a κ-Miller tree S such that $[S] \subseteq A'$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$.

   - Let S_0 be the tree generated by $\{s\}$.
   - Suppose S_i has been defined for $i < \kappa$. Let $\mathrm{Term}(S_i)$ be the collection of terminal branches of S_i (i.e., those $\sigma \in S_i$ such that $\mathrm{Succ}_{S_i}(\sigma) = \varnothing$), and for each $\sigma \in \mathrm{Term}(S_i)$ and $\alpha < \kappa$, let $\tau_{\sigma,\alpha}$ be an extension of $\sigma^\frown \langle\alpha\rangle$ such that $[\tau_{\sigma,\alpha}] \cap X_i = \varnothing$. Now let S_{i+1} be the tree generated by $\{\tau_{\sigma,\alpha} \mid \sigma \in \mathrm{Term}(S_i) \text{ and } \alpha < \kappa\}$.
   - For limits $\lambda < \kappa$, let S_λ be the tree generated by cofinal branches through $\bigcup_{\alpha<\lambda} S_\alpha$.

   By construction, $S := \bigcup_{i<\kappa} S_i$ is a κ-Miller tree (all splitting nodes of S are in fact fully splitting). Moreover $[S] \subseteq [s]$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$. In particular, $[S] \subseteq A'$. But now it follows easily that $\varphi``S$ generates a κ-Miller tree below T, whose branches are completely contained in A.

4. This follows a similar strategy as above, but using the topology generated by $\mathbb{L}_\kappa$ instead of the standard topology. Let $A \in \kappa^\kappa_\uparrow$ be in $\mathbf{\Gamma}$, $T \in \mathbb{M}_\kappa$, φ and φ^* be as above, and let $A' := (\varphi^*)^{-1}[A]$. As A' is $\mathbb{L}_\kappa$-measurable, there is a κ-Laver tree R such that $[R] \subseteq^* A'$ or $[R] \cap A' =^* \varnothing$, where $\subseteq^*$ and $=^*$ means "modulo $\mathcal{I}_{\mathbb{L}_\kappa}$". Without loss of generality assume the former and let $\{X_i \mid i < \kappa\}$ be in $\mathcal{N}_{\mathbb{L}_\kappa}$ such that $[R] \setminus A' = \bigcup_{i<\kappa} X_i$. Again we will construct a κ-Miller tree S such that $[S] \subseteq A'$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$.

   We will need to perform a fusion argument on $\mathbb{M}_\kappa$, so we introduce some terminology. For a κ-Miller tree S, a node $s \in S$ is called an *i-th splitting node* iff $s \in \mathrm{Split}(S)$ and the set $\{j < i \mid s{\restriction}j \in \mathrm{Split}(S)\}$ has order-type i. $\mathrm{Split}_i(S)$ denotes the set of i-th splitting nodes of S. The standard fusion for $\mathbb{M}_\kappa$ (cf. Fact 2.11 (2)) is defined by $S' \leq_i S$ iff $S' \leq S$ and $\mathrm{Split}_i(S') = \mathrm{Split}_i(S)$. We will build a fusion sequence $\{S_i \mid i < \kappa\}$ of κ-Miller trees, but with the following additional property

   $$(*) \qquad \forall i\, \forall s \in \mathrm{Split}_i(S_i)\ (S_i{\uparrow}s \text{ is a } \kappa\text{-Laver tree with stem } s).$$

   Note that if s is as above, then every $t \in S_i$ extending s also has the property that $S_i{\uparrow}t$ is a κ-Laver tree with stem t.

   - Let $S_0 := R$.
   - Suppose S_i has been defined for $i < \kappa$. Pick $\sigma \in \bigcup\{\mathrm{Succ}_{S_i}(\rho) \mid \rho \in \mathrm{Split}_i(S_i)\}$. By $(*)$ we know that $S_i{\uparrow}\rho$, and therefore also $S_i{\uparrow}\sigma$, is a κ-Laver tree. So let $S_\sigma \leq S_i{\uparrow}\sigma$ be a κ-Laver tree such that $[S_\sigma] \cap X_i = \varnothing$. Then let

     $$S_{i+1} := \bigcup\{S_\sigma \mid \sigma \in \bigcup\{\mathrm{Succ}_{S_i}(\rho) \mid \rho \in \mathrm{Split}_i(S_i)\}\}.$$

     By construction S_{i+1} is a κ-Miller tree, $S_{i+1} \leq_i S_i$, and condition $(*)$ is satisfied.
   - For limits $\lambda < \kappa$, let $S_\lambda := \bigcap_{i<\lambda} S_i$. By a standard fusion argument, S_λ is a κ-Miller tree and $S_\lambda \leq_i S_i$ for all $i < \lambda$. Moreover, any $\sigma \in \mathrm{Split}_\lambda(S_\lambda)$ is the extension of a λ-splitting node of S_i for every i, so by condition $(*)$, $S_i{\uparrow}\sigma$ is a κ-Laver tree with stem σ, for every $i < \lambda$. By $<\kappa$-closure of $\mathbb{L}_\kappa$, it follows that $S_\lambda{\uparrow}\sigma = \bigcap_{i<\lambda}(S_i{\uparrow}\sigma)$ is a κ-Laver tree with stem σ, hence S_λ satisfies condition $(*)$.

   By construction, $S := \bigcap_{i<\kappa} S_i$ is a κ-Miller tree, $[S] \subseteq [R]$, and $[S] \cap X_i = \varnothing$ for all $i < \kappa$. In particular, $[S] \subseteq A'$. Now it follows that φ"S generates a κ-Miller tree below T, whose branches are completely contained in A.

5. This part is completely analogous to 4. Note that κ-Mathias conditions are special kinds of κ-Laver trees, and $\mathbb{R}_\kappa$ is also $<\kappa$-closed.

6. Here it is easier to consider $\mathbb{C}_\kappa$ on 2^κ as opposed to κ^κ. It is not hard to see that the two properties are equivalent for $\mathbf{\Gamma}$. Let $A \subseteq 2^\kappa$ be in $\mathbf{\Gamma}$, let

$T \in \mathbb{V}_\kappa$, let φ be the natural order-preserving bijection between 2^κ and the splitnodes of T, and let φ^* be the induced homeomorphism between 2^κ and $[T]$. Let $A' := (\varphi^*)^{-1}[A]$, and using $\mathbf{\Gamma}(\mathbb{C}_\kappa)$ let $s \in 2^{<\kappa}$ be such that $[s] \subseteq^* A'$ or $[s] \cap A' =^* \varnothing$, without loss of generality the former. Let X_i be nowhere dense such that $[s] \setminus A' = \bigcup_{i<\kappa} X_i$. As before, we will inductively construct a κ-Silver tree S such that $[S] \subseteq [s]$ and $[S] \cap X_i = \varnothing$ for all i.

In this construction, it will be easier to view κ-Silver conditions as functions from κ to $\{0, 1, \{0, 1\}\}$. We will use the following notation: for $f : \alpha \to \{0, 1, \{0, 1\}\}$ let

$$[f] := \{x \in 2^\alpha \mid \forall i \, (f(i) \in \{0, 1\} \to x(i) = f(i))\}.$$

Notice that if $f : \kappa \to \{0, 1, \{0, 1\}\}$ and $f(i) = \{0, 1\}$ for club-many i, then the corresponding κ-Silver tree can be defined as $S_f := \{\sigma \in 2^{<\kappa} \mid \sigma \in [f \restriction \mathrm{len}(\sigma)]\}$, and we have $[S_f] = [f]$. We will construct a function f as the limit of f_α's, defined as follows:

- $f_0 := s$.
- Since X_0 is nowhere dense, let τ_1 be such that $[s^\frown \langle 0 \rangle {}^\frown \tau_1] \cap X_0 = \varnothing$. Then let $\tau_2 \supseteq \tau_1$ be such that $[s^\frown \langle 1 \rangle {}^\frown \tau_2] \cap X_0 = \varnothing$. Now set

  $$f_1 := s^\frown \langle \{0, 1\} \rangle {}^\frown \tau_2.$$

  Notice that for any $x \in 2^\kappa$ extending any $\sigma \in [f_1]$ we have $x \notin X_0$.
- Suppose f_i is defined for $i < \kappa$. Let $\{\sigma_\alpha \mid \alpha < 2^i\}$ enumerate all sequences in $[f_i{}^\frown \langle \{0, 1\} \rangle]$ and define $\{\tau_\alpha \mid \alpha < 2^i\}$ by induction as follows:
  - $\tau_0 = \varnothing$.
  - If τ_α is defined let $\tau_{\alpha+1} \supseteq \tau_\alpha$ be such that $[\sigma_\alpha{}^\frown \tau_{\alpha+1}] \cap X_i = \varnothing$.
  - For limits λ let $\tau_\lambda := \bigcup_{\alpha<\lambda} \tau_\alpha$.

  Then define $\tau_{2^i} := \bigcup_{\alpha<2^i} \tau_\alpha$ and notice that $\tau_{2^i} \in 2^\delta$ for $\delta < \kappa$ since κ was inaccessible. Now let

  $$f_{i+1} := f_i{}^\frown \langle \{0, 1\} \rangle {}^\frown \tau_{2^i}.$$

  It is clear that any $x \in 2^\kappa$ extending any $\sigma \in [f_{i+1}]$ is not in X_i.
- For γ limit, let $f_\gamma := \bigcup_{i<\gamma} f_i$.

Finally, we let $f := \bigcup_{i<\kappa} f_i$. By construction $f(i) = \{0, 1\}$ for club-many $i < \kappa$, and clearly every $x \in [f]$ is not in X_i for any $i < \kappa$. Hence $S_f := \{\sigma \in 2^{<\kappa} \mid \sigma \in [f \restriction \mathrm{len}(\sigma)]\}$ is a κ-Silver tree with $[S_f] \subseteq A'$. Then φ“S_f generates a κ-Silver subtree of T which is completely contained in A, as had to be shown. $\square$

Focusing on $\mathbf{\Gamma} = \boldsymbol{\Delta}^1_1$, we can summarize the contents of the above results in Figure 1.[4] Of particular interest are two implications which are present in the classical setting but still seem open in the general setting:

$$\begin{tikzcd}
& \boldsymbol{\Delta}^1_1(\mathbb{V}_\kappa) \arrow[rr, Rightarrow] & & \boldsymbol{\Delta}^1_1(\mathbb{S}_\kappa) \\
\boldsymbol{\Delta}^1_1(\mathbb{R}_\kappa) \arrow[ur, Rightarrow, dashed, "?"] \arrow[r, Rightarrow, dashed, "?"] \arrow[rr, Rightarrow, bend right] & \boldsymbol{\Delta}^1_1(\mathbb{L}_\kappa) \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_1(\mathbb{M}_\kappa) \arrow[ur, Rightarrow] & \\
& & \boldsymbol{\Delta}^1_1(\mathbb{C}_\kappa) \arrow[u, Rightarrow] \arrow[uul, Rightarrow, "\kappa\text{ inacc.}"'] &
\end{tikzcd}$$

Figure 1: Diagram of implications for $\boldsymbol{\Delta}^1_1$.

Question 4.10. *Is $\boldsymbol{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \boldsymbol{\Delta}^1_1(\mathbb{L}_\kappa)$ true? Is $\boldsymbol{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \boldsymbol{\Delta}^1_1(\mathbb{V}_\kappa)$ (at least for κ inaccessible) true?*

As mentioned, currently we can prove only the following separation theorem.

Theorem 4.11. *Suppose κ is inaccessible. Then it is consistent that $\boldsymbol{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\boldsymbol{\Delta}^1_1(\mathbb{S}_\kappa)$ hold whereas $\boldsymbol{\Delta}^1_1(\mathbb{R}_\kappa)$, $\boldsymbol{\Delta}^1_1(\mathbb{L}_\kappa)$, $\boldsymbol{\Delta}^1_1(\mathbb{C}_\kappa)$ and $\boldsymbol{\Delta}^1_1(\mathbb{M}_\kappa)$ fail.*

Proof. It is sufficient to establish $\boldsymbol{\Delta}^1_1(\mathbb{V}_\kappa) + \neg\boldsymbol{\Delta}^1_1(\mathbb{M}_\kappa)$. Perform a κ^+-iteration of κ-Silver forcing, starting in L, with supports of size κ. An argument completely analogous to [25, Theorem 6.1] shows that this iteration of κ-Silver forcing is κ-proper (so the conditions necessary to apply Theorem 3.13 are satisfied, i.e., κ^+ is preserved and κ-reals in the final extension are captured by an initial segment), and moreover, is κ^κ-*bounding*, i.e., every function $f \in \kappa^\kappa$ in the extension is dominated by a $g \in \kappa^\kappa$ in the ground model. By Theorem 3.13 the generic extension satisfies $\boldsymbol{\Delta}^1_1(\mathbb{V}_\kappa)$, while the statement "$\forall r\, \exists x\, (x$ is unbounded over $\kappa^\kappa \cap L[r])$" is false, so by Lemma 4.4 $\boldsymbol{\Delta}^1_1(\mathbb{M}_\kappa)$ fails. □

Notice that by Remark 3.15 and Lemma 4.9 we can obtain $\boldsymbol{\Delta}^1_1(\mathbb{P})$ for all $\mathbb{P} \in \{\mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa\}$, and also for $\mathbb{P} = \mathbb{V}_\kappa$ if κ is inaccessible, simultaneously in one model, namely $L^{(\mathbb{C}_\kappa * \mathbb{L}_\kappa * \mathbb{R}_\kappa)_{\omega_1}}$.

5. Open Questions

We have carried out an initial study of regularity properties related to forcing notions on the generalized reals; but many questions remain open, particularly

[4]We arrange the diagram in this particular way in order to be consistent with previous presentations of similar diagrams, e.g. in [11].

with regard to the specific examples presented in Section 4.

Question 5.1.

1. *Can Lemma 4.4 be proved without assuming that κ is inaccessible?*
2. *Does $\mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ imply that for every $r \in \kappa^\kappa$, there is an x which is dominating over $L[r]$?*
3. *Can the hypotheses $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ be linked to the existence of (a suitable generalization of) splitting/unsplit reals?*

A more long-term goal would be to find a complete diagram of implications for generalized $\mathbf{\Delta}^1_1$ sets.

Question 5.2. *Which additional implications from Figure 1 can be proved in ZFC? Which are consistently false? Specifically, does $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ (at least for κ inaccessible) hold?*

In a more conceptual direction, one should try to better understand the exact role of the club filter, which provides counterexamples for $\mathbf{\Sigma}^1_1$-regularity. For example, perhaps one could prove that the club filter, up to some adequate notion of equivalence, is the *only* $\mathbf{\Sigma}^1_1$-counterexample. Alternatively, one could try to focus on regularity properties such as the ones considered in [13, 14], and try to gain a better understanding why the club filter is a counterexample for some regularity properties but not for others. For example, by recent results of Laguzzi and the first author, projective measurability is consistent for a version of Silver forcing in which the splitting levels occur on a normal measure on κ as opposed to the club filter.

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