HAMBURGER BEITRÄGE
ZUR MATHEMATIK

Heft 564

On the structure of dense graphs with fixed clique number

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Version September 2015

ON THE STRUCTURE OF DENSE GRAPHS WITH FIXED CLIQUE NUMBER

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Abstract. We study structural properties of graphs with fixed clique number and high minimum degree. In particular, we show that there exists a function $L = L(r, \varepsilon)$, such that every K_r-free graph G on n vertices with minimum degree at least $(\frac{2r-5}{2r-3}+\varepsilon)n$ is homomorphic to a K_r-free graph on at most L vertices. It is known that the required minimum degree condition is approximately best possible for this result.

For $r = 3$ this result was obtained by Łuczak [*On the structure of triangle-free graphs of large minimum degree*, Combinatorica **26** (2006), no. 4, 489–493] and, more recently, Goddard and Lyle [*Dense graphs with small clique number*, J. Graph Theory **66** (2011), no. 4, 319-331] deduced the general case from Łuczak's result. Łuczak's proof was based on an application of Szemerédi's regularity lemma and, as a consequence, it only gave rise to a tower-type bound on $L(3, \varepsilon)$. The proof presented here replaces the application of the regularity lemma by a probabilistic argument, which yields a bound for $L(r, \varepsilon)$ that is doubly exponential in poly(ε).

§1. Introduction

1.1. **Notation.** We follow the notation from [4] and briefly review some of it below. The graphs we consider here are undirected, simple, and have no loops and for a graph $G = (V, E)$ we denote by $V = V(G)$ its *vertex set* and by $E = E(G) \subseteq \binom{V}{2}$ its *edge set*. The number of vertices is finite and often denoted by $n = |V|$. By C_r and K_r we denote the *cycle* and the *complete graph/clique* on ℓ vertices. For two adjacent vertices x, y we simply denote its edge by xy. If $x \in X \subseteq V(G)$ and $y \in Y \subseteq V(G)$ then xy is an $X - Y$*-edge*. For disjoint sets X and Y the set of all $X - Y$-edges is a subset of E and is denoted by $E_G(X, Y)$. Moreover, the number of $X - Y$ edges in G is denoted by $e_G(X, Y) = |E_G(X, Y)|$. and for nonempty X and Y the *density* is defined by $d_G(X, Y) = \frac{e_G(X,Y)}{|X||Y|}$. The set of *neighbours* of a vertex v is denoted by $N_G(v)$ and its size $d_G(v) = |N_G(v)|$ is the *degree* of a vertex v, where we sometimes

2010 *Mathematics Subject Classification.* 05C35 (primary), 005C75 (secondary).

Key words and phrases. extremal graph theory, chromatic thresholds.

The second author was supported through the Heisenberg-Programme of the DFG.

suppress the subscript G if there is no danger of confusion. We denote the *minimum degree* of G by $\delta(G)$. For a subset $U \subseteq V$ we define the *common (or joint) neighbourhood* of U as

$$N(U) = \bigcap_{u \in U} N(u)$$

and we emphasise that this differs from the notation in [4]. For later reference we note that the size of $N(U)$ can be easily bounded from below in terms of the minimum degree of $G = (V, E)$ by

$$|N(U)| \geqslant |U| \cdot \delta(G) - (|U| - 1) \cdot |V| \,. \tag{1}$$

Moreover, for a subset $U \subseteq V$ we denote by $G[U]$ the *induced subgraph on* U and we write $G - U$ for $G[V \backslash U]$.

A (vertex) *colouring* of a graph $G = (V, E)$ is a map $c\colon V \to \mathcal{C}$ such that $c(v) \neq c(w)$, whenever v and w are adjacent. The elements of the set $\mathcal{C}$ are called the available *colours*. A graph $G = (V, E)$ is k*-colourable* if there exists colouring $c\colon V \to [k] = \{1, \ldots, k\}$. The *chromatic number* $\chi(G)$ is the smallest integer k such that G has is k-colourable.

A *homomorphism* from a graph G into a graph H is a mapping $\varphi\colon V(G) \to V(H)$ which preserves adjacencies, i.e. for all pairs $uv \in E(G)$ we have $\varphi(u)\varphi(v) \in E(H)$. We write $G \xrightarrow{\text{hom}} H$ to indicate that some homomorphism φ exists. We say G is a *blow-up* of H, if G is obtained from H by replacing every vertex x of H by an independent sets I_x and edges of H correspond to complete bipartite graphs, i.e., I_x, I_y spans a complete bipartite graph in G if $xy \in E(H)$ and otherwise $e_G(I_x, I_y) = 0$. Clearly, if G is a blow-up of H then $G \xrightarrow{\text{hom}} H$ and $K_r \subseteq G$ if and only if $K_r \subseteq H$.

By $F \subseteq G$ we mean that G contains a copy of F, that is there exists an injective homomorphism from F into G. If G contains no such copy of F (i.e., $F \nsubseteq G$), then we say G is F-free. For a graph F

$$\mathrm{Forb}(F) = \{G\colon\ F \nsubseteq G\}$$

denotes the class of F-free graphs, i.e. the collection of those graphs G which do not contain a copy of F. Moreover, we set

$$\mathrm{Forb}_n(F) = \{G\colon\ F \nsubseteq G \text{ and } |V(G)| = n\} \,.$$

An F-free graph $G = (V, E)$ is *maximal* F*-free* if the addition of any edge to G leads to a copy of F in G, i.e., for every $xy \in \binom{V}{2} \backslash E$ we have $F \subseteq (V, E \cup \{xy\})$. Finally, we define the join $G \vee H$ of two graphs G and H as the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$

and edge set

$$E(G \vee F) = E(G) \cup E(F) \cup \{vw\colon\ v \in V(G) \text{ and } w \in V(F)\}\,.$$

1.2. **Chromatic thresholds of graphs.** We are interested in structural properties of graphs $G \in \mathrm{Forb}(F)$, which are forced by an additional minimum degree assumption on G. For example, if F is a clique and $\delta(G)$ is sufficiently high, then Turán's theorem [16] assert that G is $(r-1)$-partite and, in particular, the chromatic number of G is bounded by a constant independent of $|V(G)|$. More generally, Andrásfai [2] raised the following question: *For a given graph F and an integer k, what is smallest condition imposed on the minimum degree $\delta(G)$ such any graph $G \in \mathrm{Forb}(F)$ satisfying this minimum degree condition has chromatic number at most k?* Here we are interested in the case when the minimum degree condition yields an upper bound on $\chi(G)$ independent from the graph G itself. This leads to the so called *chromatic threshold* for a given graph F

$$\delta_\chi(F) = \inf\left\{\alpha \in [0,1]\colon\ \exists k \in \mathbb{N} \text{ s.t. } \chi(G) \leqslant k \ \ \forall G \in \mathrm{Forb}(F) \text{ with } \delta(G) \geqslant \alpha|V(G)|\right\}.$$

If $F' \subseteq F$, then $\mathrm{Forb}(F') \subseteq \mathrm{Forb}(F)$, so obviously $\delta_\chi(F') \leqslant \delta_\chi(F)$. Moreover, it follows from the Erdős–Stone theorem [6] that $\delta_\chi(F) \leqslant \frac{\chi(F)-2}{\chi(F)-1}$ for every graph F with at least one edge.

For $F = K_3$ it was shown in [5] that $\delta_\chi(K_3) \geqslant 1/3$. In the other direction, Thomassen [14] obtained a matching upper bound and, therefore, $\delta_\chi(K_3) = 1/3$. In fact, Erdős and Simonovits [5] asked whether all triangle-free graphs G with $\delta(G) \geqslant (1/3 + o(1))|V(G)|$ are 3-colorable. This was answered negatively by Häggkvist [8], but recently Brandt and Thomassé [3] showed that the chromatic number of such graphs is bounded by 4.

Nikiforov [12] and Goddard and Lyle [7] extended those results from triangles to r-cliques and showed for every $r \geqslant 3$ that

$$\delta_\chi(K_r) = \frac{2r-5}{2r-3} \tag{2}$$

and, in fact, $\chi(G) \leqslant r+1$ for every K_r-free graph G with $\delta(G) > \frac{2r-5}{2r-3}|V(G)|$.

In the case when F is an odd cycle of length at least five it was shown by Thomassen [15] that the chromatic threshold is zero and Łuczak and Thomassé [11] proved that $\delta_\chi(F) \notin (0, 1/3)$ for all graphs F and that $\delta_\chi(F) = 0$ if F is nearly bipartite (a graph is nearly bipartite if it is triangle-free and it admits a vertex partition into two parts such that one part is independent and the other part induces a graph with maximum degree one). Recently, Allen et al. [1] extended of the work of Łuczak and Thomassé and determined the chromatic threshold for every graph F.

1.3. **Homomorphism thresholds of graphs.** Viewing $\chi(G) \leqslant k$ as the property that $G \xrightarrow{\text{hom}} K_k$, one may ask for a graph $G \in \mathrm{Forb}(F)$, whether K_k can be replaced by a graph H of bounded size (independent of G), that is F-free itself. More precisely, in [14] Thomassen posed the following question: *Given a fixed constant c, does there exist a finite family of triangle-free graphs such that every triangle-free graph on n vertices and minimum degree greater than cn is homomorphic to some graph of this family?* To formalise this question we define the *homomorphism threshold* of a graph F

$$\delta_{\text{hom}}(F) = \inf \big\{\alpha \in [0,1] \colon \exists\, k \in \mathbb{N} \text{ s.t. } \forall\, G \in \mathrm{Forb}(F) \text{ with } \delta(G) \geqslant \alpha |V(G)| \\ \exists\, H \in \mathrm{Forb}_k(F) \text{ with } G \xrightarrow{\text{hom}} H \big\}\,.$$

Thomassen then asked to determine $\delta_{\text{hom}}(K_3)$. Since $G \xrightarrow{\text{hom}} H$, implies $\chi(G) \leqslant |V(H)|$, we clearly have

$$\delta_{\text{hom}}(F) \geqslant \delta_{\chi}(F).$$

In [10] Łuczak proved $\delta_{\text{hom}}(K_3) = 1/3$ and, hence, for the triangle K_3 the homomorphic and the chromatic threshold are equal. Recently, Goddard and Lyle [7] extended Łuczak's result showing, that K_r-free graphs with minimum degree bigger than $\frac{2r-5}{2r-3}$ are homomorphic to the join $K_{r-3} \vee H$, where H is a triangle-free graph with $\delta(H) > |V(H)|/3$. Consequently, we have for every $r \geqslant 3$

$$\delta_{\text{hom}}(K_r) = \delta_{\chi}(K_r) = \frac{2r-5}{2r-3}\,. \tag{3}$$

Łuczak's proof in [10] was based on Szemerédi's regularity lemma [13]. We give a different proof of (3), which avoids the regularity lemma and uses a simple probabilistic argument.

Theorem 1.1. *For every integer $r \geqslant 3$ we have*

$$\delta_{\text{hom}}(K_r) = \frac{2r-5}{2r-3}.$$

It seems an interesting open question to determine the homomorphism threshold for other graphs than cliques. In particular, the case of odd cycles of length at least five seems to be a first interesting open case and we put forward the following question.

Question 1. What is $\delta_{\text{hom}}(C_{2\ell+1})$ for $\ell \geqslant 2$?

A somewhat related question concerns the homomorphism threshold for forbidden families of graphs. Note that the definitions of $\mathrm{Forb}(F)$ and $\delta_{\text{hom}}(F)$ straight forwardly extend from one forbidden graph F to forbidden families $\mathcal{F}$ of graphs. In view of Question 1 it is natural to consider the family $\mathcal{C}_{2\ell+1} = \{C_3, \dots, C_{2\ell+1}\}$ of odd cycles of length at most $2\ell + 1$ and we close this introduction with the following open question.

Question 2. What is $\delta_{\mathrm{hom}}(\mathcal{C}_{2\ell+1})$ for $\ell \geqslant 2$?

§2. Simple observations

For an integer $r \geqslant 3$ and $\varepsilon > 0$ the following subclass of $\mathrm{Forb}(K_r)$ will play a prominent rôle

$$\mathcal{F}(r, \varepsilon) = \left\{G = (V, E) \in \mathrm{Forb}(K_r)\colon\ \delta(G) \geqslant \left(\tfrac{2r-5}{2r-3} + \varepsilon\right)|V|\right\},$$

since Theorem 1.1 asserts that there exists some function $L = L(r, \varepsilon)$ and $H \in \mathrm{Forb}(L, K_r)$ such that for every $G \in \mathcal{F}(r, \varepsilon)$ we have $G \xrightarrow{\text{hom}} H$. Note that $\mathcal{F}(r, \varepsilon)$ contains only graphs on at least $r-2$ vertices. We begin with a few observations concerning common neighbourhoods in maximal K_r-free graphs of given minimum degree $\delta(G)$.

Proposition 2.1. *For $r \geqslant 3$ let $G = (V, E)$ be a maximal K_r-free graph. If two distinct vertices u, $v \in V$ are non-adjacent, then $|N(u) \cap N(v)| \geqslant r\delta(G) - (r-2)|V|$.*

Proof. Since $G = (V, E)$ is maximal K_r-free and $uv \notin E$, the joint neighbourhood $N(u) \cap N(v)$ induces a K_{r-2}. Applying (1) to the $r-2$ vertices $w_1, \dots, w_{r-2}$ that span K_{r-2} in the joint neighbourhood of u and v yields $N(\{w_1, \dots, w_{r-2}\}) \geqslant (r-2)\delta(G) - (r-3)|V|$. Moreover, since $N(\{w_1, \dots, w_{r-2}\})$ must be disjoint from $N(u) \cup N(v)$, we obtain

$$\begin{aligned}|V| &\geqslant (r-2)\delta(G) - (r-3)|V| + |N(u) \cup N(v)| \\ &= (r-2)\delta(G) - (r-3)|V| + |N(u)| + |N(v)| - |N(u) \cap N(v)|\end{aligned}$$

and the proposition follows. □

In the proof of the last proposition we used the observation, that the neighbourhood of any two non-adjacent vertices in a maximal K_r-free graph induces a K_{r-2}. Next we note that for maximal K_r-free graphs in $\mathcal{F}(r, \varepsilon)$, we can strengthen this observation and ensure that the clique K_{r-2} is disjoint from an arbitrary given small set of vertices.

Proposition 2.2. *For $r \geqslant 3$ and $\varepsilon > 0$, let $G = (V, E)$ be a maximal K_r-free graph from $\mathcal{F}(r, \varepsilon)$. If two distinct vertices u, $v \in V$ are non-adjacent in G and $U \subseteq V$ satisfies $|U| < \varepsilon|V|$, then $K_{r-2} \subseteq G[(N(u) \cap N(v)) \backslash U]$.*

Proof. Given u, v and U as stated, we first consider any set of $r-3$ vertices $w_1, \dots, w_{r-3} \in V$ and owing to (1) we have

$$N(\{w_1, \dots, w_{r-3}\}) \geqslant (r-3)\delta(G) - (r-4)|V|\,.$$

Moreover, since u and v are non-adjacent Proposition 2.1 tells us that

$$|N(u) \cap N(v)| \geqslant r\delta(G) - (r-2)|V|\,.$$

Consequently, the joint neighbourhood of u, v and $w_1, \dots, w_{r-3}$ satisfies

$$\begin{aligned}|N(\{u, v, w_1, \dots, w_{r-3})| &\geqslant N(\{w_1, \dots, w_{r-3}\}) - (|V| - |N(u) \cap N(v)|)\\ &\geqslant (2r-3)\delta(G) - (2r-5)|V|\end{aligned}$$

and the minimum degree condition from $G \in \mathcal{F}(r, \varepsilon)$ implies that

$$|N(\{u, v, w_1, \dots, w_{r-3})| \geqslant (2r-3)\varepsilon|V| \geqslant 3\varepsilon|V| > |U|\,.$$

Summarising, we have shown that any collection of $r-3$ vertices together with u and v have a joint neighbour outside of U. Selecting w_1 from $(N(u) \cap N(v))\backslash U$ and inductively w_{i+1} from $N(\{u, v, w_1, \dots, w_i)\backslash U$ for $i = 1, \dots, r-3$ yields the desired clique on $w_1, \dots, w_{r-2}$. $\square$

The last observation asserts that any sufficiently large subset of vertices induces a K_{r-2} in a graph G from $\mathcal{F}(r, \varepsilon)$.

Proposition 2.3. *For $r \geqslant 3$ and $\varepsilon > 0$ let $G = (V, E)$ be a graph from $\mathcal{F}(r, \varepsilon)$. If $Z \subseteq V$ satisfies $|Z| \geqslant (\frac{2r-6}{2r-3} + \varepsilon)|V|$, then $K_{r-2} \subseteq G[Z]$.*

Proof. Similarly as in the proof of Proposition 2.2 we consider an arbitrary set of $(r-3)$ vertices $w_1, \dots, w_{r-3} \in V$ and from (1) we infer

$$\begin{aligned}|N(\{w_1, \dots, w_{r-3}\}) \cap Z| &\geqslant (r-3)\delta(G) - (r-4)|V| - (|V| - |Z|)\\ &\geqslant \Big((r-3)\frac{2r-5}{2r-3} + \frac{2r-6}{2r-3} - (r-3)\Big)|V| + (r-2)\varepsilon|V|\\ &= (r-2)\varepsilon|V| > 0\,.\end{aligned}$$

Consequently, any set of $k-3$ vertices has a joint neighbour in Z. Hence, selecting w_1 in Z and inductively w_{i+1} from $N(\{w_1, \dots, w_i\}) \cap Z$ for $i = 1, \dots, r-3$ yields the desired clique on $w_1, \dots, w_{r-2}$. $\square$

§3. Proof of the main result

In the proof of Theorem 1.1 we partition the vertex set of a maximal K_r-free graph $G \in \mathcal{F}(r,\varepsilon)$ into a bounded number of stable sets and show that any two such independent sets are spanning complete or empty bipartite graphs. Consequently, G is a blow-up of a K_r-free graph of bounded size, which is equivalent to the property that G has a K_r-free homomorphic image of bounded size.

We obtain the independent sets in two steps: Roughly speaking, in the first step we consider a random subset $X \subset V(G)$ of bounded size and partition the vertices of $V(G)$ according to their neighbourhood in X. However, since X has only bounded size, a small (but linear sized) set of vertices may have no or only a few neighbours in X and we deal with those vertices in the second step, by considering the neighbourhood into the independent sets from the first step.

Proof of Theorem 1.1. Let $r \geqslant 3$. Owing to (2) we have $\delta_\chi(K_r) = \frac{2r-5}{2r-3}$ and since by definition $\delta_\chi(K_r) \leqslant \delta_{\hom}(K_r)$, we have to prove the matching upper bound on $\delta_{\hom}(K_r)$. Let $\varepsilon > 0$ and set

$$m = \lceil 4\ln(8/\varepsilon)/\varepsilon^2 \rceil + 1\,, \quad T = 2^m\,, \quad \text{and} \quad L = 2^T + T\,. \tag{4}$$

We will show that for any $n > L$ and for every maximal K_r-free graph $G = (V,E)$ from $\mathcal{F}(r,\varepsilon)$ there exists some $H \in \mathrm{Forb}_L(K_r)$ such that $G \xrightarrow{\hom} H$, which clearly suffices to prove the theorem.

In the first part we consider a random subset $X \subseteq V$ of size m chosen uniformly at random from all m-element subsets of V and we consider the random set

$$U_X = \Big\{ v \in V \colon |N(v) \cap X| < \big(\tfrac{2r-5}{2r-3} + \tfrac{\varepsilon}{2}\big)m \Big\}$$

of vertices with "small" degree in X. We show that with positive probability $|U_X| \leqslant \varepsilon n/4$ and $|X \cap U_X| \leqslant \varepsilon m/4$.

It follows from Chernoff's inequality for the hypergeometric distribution (see, e.g., [9, Theorem 2.10, eq. (2.6)] that for a given vertex $v \in V$ we have

$$\mathbb{P}(v \in U_X) \leqslant \exp(-\varepsilon^2 m/4)\,. \tag{5}$$

Consequently,

$$\mathbb{E}[|U_X|] \leqslant \exp(-\varepsilon^2 m/4) \cdot n \overset{(4)}{<} \varepsilon n/8$$

and by Markov's inequality we have

$$\mathbb{P}(|U_X| \leqslant \varepsilon n/4) > 1/2\,. \tag{6}$$

In other words, with probability more than $1/2$ all but at most $\varepsilon n/4$ vertices inherit approximately the minimum degree condition on the randomly chosen set X.

Next we show that with probability at least $1/2$ the intersection of X with U_X is small. This follows from a standard double counting argument. In fact, the same argument giving (5) shows that for every $v \in V$ there are at most $\exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}$ different $(m-1)$-element subsets Y of V for which

$$|N(v) \cap Y| \leqslant \left(\tfrac{2r-5}{2r-3} + \tfrac{\varepsilon}{2}\right) \cdot (m-1)\,. \tag{7}$$

Hence, there are at most $n \exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}$ pairs (v, Y) such that (7) holds. Therefore, there are at most

$$\frac{n \cdot \exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}}{\varepsilon m/4} \overset{(4)}{\leqslant} \frac{1}{2}\binom{n}{m}$$

m-element subsets $X \subseteq V$ that contain at least $\varepsilon m/4$ vertices v such that v and $Y = X\backslash\{v\}$ satisfy (7). Combining this with (6) shows that there exists an m-element set $X \subseteq V$ with the promised properties

$$|U_X| \leqslant \frac{\varepsilon}{4} n \quad \text{and} \quad |X \cap U_X| < \frac{\varepsilon}{4} m\,.$$

Finally, we set

$$Y = X\backslash U_X \quad \text{and} \quad U_Y = \left\{v \in V \colon\; |N(v) \cap Y| < \left(\tfrac{2r-5}{2r-3} + \tfrac{\varepsilon}{4}\right)|Y|\right\}$$

and we note that the induced subgraph on Y satisfies

$$G[Y] \in \mathcal{F}(r, \varepsilon/4)$$

and since $U_Y \subseteq U_X$ we also have

$$|U_Y| \leqslant |U_X| \leqslant \varepsilon n/4\,.$$

Next we define a vertex partition of $V\backslash U_Y$ given by the neighbourhoods in Y. For that we say two vertices v, $w \in V\backslash U_Y$ are equivalent w.r.t. Y, if they have the same neighbours in Y, i.e., $N(v) \cap Y = N(w) \cap Y$. Let $V_1 \dot\cup \ldots \dot\cup V_t = V\backslash U_Y$ be the corresponding partition given by the equivalence classes and let Y_i be the neighbourhood of the vertices from V_i in Y, i.e., for any $v_i \in V_i$ we have

$$N(v_i) \cap Y = Y_i\,.$$

Clearly, $t \leqslant 2^{|Y|} \leqslant 2^{|X|} = 2^m = T$.

We observe that the vertex classes $V_1, \dots, V_t$ are independent sets in G, i.e., for every $i = 1, \dots, t$ we have

$$E_G(V_i) = \varnothing\,. \tag{8}$$

In fact, since every vertex $v \in V \backslash U_Y$ has at least $(\frac{2r-3}{2r-5} + \varepsilon/4)|Y|$ neighbours in Y and since $G[Y] \in \mathcal{F}(r, \varepsilon/4)$ it follows from Proposition 2.3 applied to $G[Y]$ and $Z = Y_i$ that Y_i induces a K_{r-2}. Consequently, the K_r-freeness of G implies that no two vertices v_i, $w_i \in V_i$ can be adjacent in G and (8) follows.

Next we observe that the induced bipartite graphs given by the partition of equivalence classes contain no or all edges, i.e., for every $1 \leqslant i < j \leqslant t$ we have

$$e_G(V_i, V_j) = 0 \quad \text{or} \quad e_G(V_i, V_j) = |V_i||V_j|\,. \tag{9}$$

Suppose for a contradiction that there are (not necessarily distinct) vertices v_i, $w_i \in V_i$ and v_j, $w_j \in V_j$ such that $v_iv_j \in E(V_i, V_j)$ and $w_iw_j \notin E(V_i, V_j)$. Due to the edge v_iv_j the intersection $Y_i \cap Y_j$ must be K_{r-2}-free and, hence, in view of Proposition 2.3 applied to $G[Y]$ and $Z = Y_i \cap Y_j$ we have

$$|Y_i \cap Y_j| < \Big(\frac{2r-6}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|$$

and, therefore,

$$|Y_i \cup Y_j| = |Y_i| + |Y_j| - |Y_i \cap Y_j| > \Big(2\frac{2r-5}{2r-3} - \frac{2r-6}{2r-3} + \frac{\varepsilon}{4}\Big)|Y| = \Big(\frac{2r-4}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|\,. \tag{10}$$

Next we use that $w_i \in V_i$ and $w_j \in V_j$ are non-adjacent. Owing to the maximality of G we can apply Proposition 2.2 to G and U_Y and obtain a clique K_{r-2} outside U_Y in the joint neighbourhood of w_i and w_j. Let R be the vertex set of this K_{r-2}. Since $R \subseteq V \backslash U_Y$ and since the sets V_k are independent for every $k = 1, \dots, t$ the set R intersects $r-2$ classes $V_{k_1}, \dots, V_{k_{r-2}}$ different from V_i and V_j. We consider the joint neighbourhood of R in Y

$$N(R) \cap Y = Y_{k_1} \cap \dots \cap Y_{k_{r-2}}$$

and note that

$$|N(R) \cap Y| \geqslant (r-2)\Big(\frac{2r-5}{2r-3} + \frac{\varepsilon}{4}\Big)|Y| - (r-3)|Y| = \Big(\frac{1}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|\,.$$

However, combined with (10) this implies that either $Y_i \cap N(R) \neq \varnothing$ or $Y_j \cap N(R) \neq \varnothing$. In either case this gives rise to a K_r in G, which yields the desired contradiction and (9) follows.

Note that (8) shows that $G[V \backslash U_Y]$ is homomorphic to a graph H' on $t \leqslant T$ and it follows from (9) that $G[V \backslash U_Y]$ is a blow-up of H'. So in particular H' is K_r-free.

It remains to deal with the vertices in U_Y. For that we first observe that for every vertex $u \in U_Y$ and $i = 1, \dots, t$ we have

$$N(u) \cap V_i = \varnothing \quad \text{or} \quad N(u) \cap V_i = V_i\,. \tag{11}$$

In fact, suppose for a contradiction, that for some v_i, $w_i \in V_i$ we have $uv_i \in E$ while u and w_i are not adjacent. Again the maximality of G and Proposition 2.2 shows that $N(u) \cap N(w_i)$ contains a K_{r-2} avoiding U_Y. However, since by (8) and (9) the vertices v_i and w_i have the same neighbourhood in $V \backslash U_Y$ the same K_{r-2} is also in the neighbourhood of v_i, which together with v_i and u yields a K_r in G. This contradicts $K_r \nsubseteq G$ and (11) follows.

Next we partition U_Y according to the neighbourhoods of its vertices in $V \backslash U_Y$. For every $S \subseteq [t] = \{1, \dots, t\}$ we set

$$V_S = \left\{ u \in U_Y \colon\ N(u) \backslash U_Y = \bigcup_{s \in S} V_s \right\},$$

which yields a partition of U_Y into at most $2^t \leqslant 2^T$ classes. Similar as in (9) and (11) we next observe that for any S, $S' \subseteq [t]$ with $S \neq S'$ we have

$$e_G(V_S, V_{S'}) = 0 \quad \text{or} \quad e_G(V_S, V_{S'}) = |V_S||V_{S'}|\,. \tag{12}$$

The proof is very similar to the proof of (11). Suppose for a contradiction without loss of generalisation there exist vertices v_S, $w_S \in V_S$ and $u \in V_{S'}$ such that $uv_S \in E$ while u and w_S are not adjacent. Then by the maximality of G Proposition 2.2 yields a K_{r-2} in $N(u) \cap N(w_S)$ avoiding U_Y. Owing to (11) the vertices v_S and w_S have the same neighbourhood in $V \backslash U_Y$ and, hence, the same K_{r-2} is also in the neighbourhood of v_S, which together with v_S and u yields a K_r in G. This contradicts $K_r \nsubseteq G$ and (12) follows.

The last thing we have to show is that V_S is independent in G, i.e., for every $S \subseteq [t]$ we have

$$E_G(V_S) = \varnothing\,. \tag{13}$$

This is a direct consequence of (11) and Proposition 2.3. In fact, it follows from (11) that any two vertices u, $v \in V_S$ have the same neighbourhood in $V \backslash U_Y$. Hence, their joint neighbourhood has size at least $(\frac{2r-5}{2r-3} + \frac{3\varepsilon}{4})n$ and Proposition 2.3 yields a K_{r-2} in the joint neighbourhood of u and v. Therefore, u and v cannot be adjacent in G and (13) follows.

Summarising, we have shown that there exists a vertex partition

$$\bigcup_{i=1}^{t} V_i \cup \bigcup_{S \subseteq [t]} V_S = V$$

of V into independent sets (see (8) and (13)) such that all naturally induced bipartite graphs are either complete or empty (see (9), (11), and (12)). Hence, there exists a graph H on $2^T + T \leqslant L$ vertices such that G is a blow-up of H and, therefore, $G \xrightarrow{\text{hom}} H$ and H itself must be K_r-free. This concludes the proof of Theorem 1.1. $\square$

We remark that the size H is doubly exponential in $\mathrm{poly}(1/\varepsilon)$, i.e., there exists some universal constant c such that

$$|V(H)| \leqslant L = 2^{2^{c\ln(\varepsilon)/\varepsilon^2}}$$

holds.

We close by noting that the same approach used in the proof of Theorem 1.1 can be used to show Thomassen's result from [15] that the chromatic threshold of odd cycles of at least five is 0. However, it remains open, if this approach can also be used to address Questions 1 and 2.

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