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Vojtěch Rödl, Atlanta

Andrzej Ruciński, Poznań

Mathias Schacht, Hamburg

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Vojtěch Rödl*
Emory University
Atlanta, USA
rodl@mathcs.emory.edu

Andrzej Ruciński†
A. Mickiewicz University
Poznań, Poland
rucinski@amu.edu.pl

Mathias Schacht‡
Universität Hamburg
Hamburg, Germany
schacht@math.uni-hamburg.de

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Abstract

For two graphs, G and F, and an integer $r \geq 2$ we write $G \rightarrow (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. In 1995, the first two authors established a threshold edge probability for the Ramsey property $G(n,p) \rightarrow (F)_r$, where $G(n,p)$ is a random graph obtained by including each edge of the complete graph on n vertices, independently, with probability p. The original proof was based on the regularity lemma of Szemerédi and this led to tower-type dependencies between the involved parameters. Here, for $r = 2$, we provide a self-contained proof of a quantitative version of the Ramsey threshold theorem with only double exponential dependencies between the constants. As a corollary we obtain a double exponential upper bound on the 2-color Folkman numbers. By a different proof technique, a similar result was obtained independently by Conlon and Gowers.

1 Introduction

For two graphs, G and F, and an integer $r \geq 2$ we write $G \rightarrow (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. By a copy we mean here a subgraph

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of G isomorphic to F. Let $G(n,p)$ be the binomial random graph, where each of $\binom{n}{2}$ possible edges is present, independently, with probability p. In [4] the first two authors established a threshold edge probability for the Ramsey property $G(n,p) \to (F)_r$.

For a graph F, let v_F and e_F stand for, respectively, the number of vertices and edges of F. Assuming $e_F \geq 1$, define

$$d_F = \begin{cases} \frac{e_F - 1}{v_F - 2} & \text{if} \quad e_F > 1 \\ \frac{1}{2} & \text{if} \quad e_F = 1 \end{cases}, \tag{1}$$

and

$$m_F = \max\{d_H \colon H \subseteq F \text{ and } e_H \geq 1\}. \tag{2}$$

Let $\Delta(F)$ be the maximum vertex degree in F. Observe that $m_F = \frac{1}{2}$ for every F with $\Delta(F) = 1$, while for every F with $\Delta(F) \geq 2$ we have $m_F \geq 1$. Moreover, for every k-vertex graph F,

$$m_F \leq m_{K_k} = \frac{k+1}{2}.$$

We now state the main result of [4] in a slightly abridged form.

Theorem 1 ([4])**.** *For every integer $r \geq 2$ and a graph F with $\Delta(F) \geq 2$ there exists a constant $C_{F,r}$ such that if $p = p(n) \geq C_{F,r} n^{-1/m_F}$ then*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)_r) = 1.$$

The original proof of Theorem 1 was based on the regularity lemma of Szemerédi [8] and this led to tower-type dependencies of the involved parameters. In [5] it was noticed that for *two* colors the usage of the regularity lemma could be replaced by a simple Ramsey-type argument. Here we follow that thread and for $r = 2$ prove a quantitative version of Theorem 1 with only double exponential dependencies between the constants.

In order to state the result, we first define inductively four parameters indexed by the number of edges of a k-vertex graph F. For fixed $k \geq 3$ we set

$$a_1 = \frac{1}{2}, \qquad b_1 = \frac{1}{8}, \qquad C_1 = 1, \qquad \text{and} \qquad n_1 = 1 \tag{3}$$

and for each $i = 1, \dots, \binom{k}{2} - 1$, define

$$a_{i+1} = \frac{a_i^{19k^4}}{2^{55k^6}}, \quad b_{i+1} = \frac{a_i^{37k^2}}{2^{118k^4}} b_i^4, \quad C_{i+1} = \frac{2^{122k^4}}{b_i^4 a_i^{37k^2}} C_i, \quad \text{and} \quad n_{i+1} = \frac{2^{14k^3}}{a_i^{4k}} n_i. \tag{4}$$

Note that a_i and b_i decrease with i, while C_i and n_i increase. Finally, for a graph F on k vertices, denote by

$$\mu_F = \binom{n}{k} \frac{k!}{\mathrm{aut}(F)} p^{e_F}$$

the expected number of copies of F in $G(n,p)$ and note that

$$\binom{n}{k} p^{e_F} \leq \mu_F \leq n^k p^{e_F} = n^{v_F} p^{e_F}. \tag{5}$$

For a real number $\lambda > 0$ we write $G \xrightarrow{\lambda} F$ if every 2-coloring of the edges of G produces at least λ monochromatic copies of F. We call a graph F *k-admissible* if $v_F = k$ and either $e_F = 1$ or $\Delta(F) \geq 2$. Now, we are ready to state a quantitative version of Theorem 1.

Theorem 2. *For every $k \geq 3$, every k-admissible graph F, and for all $n \geq n_{e_F}$ and $p \geq C_{e_F} n^{-1/m_F}$,*

$$\mathbb{P}\left(G(n,p) \xrightarrow{a_{e_F}\mu_F} F\right) \geq 1 - \exp(-b_{e_F} p \tbinom{n}{2}).$$

Note that, for $r = 2$, Theorem 1 is an immediate corollary of Theorem 2.

Another consequence of Theorem 2 concerns Folkman numbers. Given an integer $k \geq 3$, the Folkman number $f(k)$ is the smallest integer n for which there exists an n-vertex graph G such that $G \to (K_k)_2$ but $G \not\supset K_{k+1}$. In the special case of $F = K_k$ and $r = 2$, Theorem 2, with $p = C_{\binom{k}{2}} n^{-\frac{2}{k+1}}$, provides a lower bound on $\mathbb{P}(G(n,p) \to (K_k)_2)$. In Section 4, by a standard application of the FKG inequality, we also estimate from below $\mathbb{P}(G(n,p) \not\supset K_{k+1})$, so that the sum of the two probabililities is strictly greater than 1. This, after a careful analysis of the involved constants, provides a self-contained derivation of a double exponential bound for $f(k)$.

Corollary 3. *There exists an absolute constant $c > 0$ such that for every $k \geq 3$*

$$f(k) \leq 2^{k^{ck^2}}.$$

Independently, a similar double exponential bound (with arbitrarily many colors) was obtained by Conlon and Gowers [1]. The method used in [1] is quite different from ours and allows for a further generalization to hypergraphs. After Theorem 2 as well as the result in [1] had been proved, we learned that Nenadov and Steger [7] have found a new proof of Theorem 1 by means of the celebrated containers' method. In [6], we used the ideas from [7] to obtain the bound $f(k) \leq 2^{O(k^4 \log k)}$ which, at least for large k, supersedes Colorary 3. However, the advantage of our approach here is that the proofs of both Theorem 2 and Corollary 3, as opposed to those in [6], are self-contained and, in case of Theorem 2, incorporate the original ideas from [4].

The paper is organized as follows. In Section 3 we prove our main result, Theorem 2. This is preceded by Section 2 collecting preliminary results needed in the main body of the proof. Section 4 is devoted to a proof of Corollary 3

2 Preliminary results

Before we start with the proof of Theorem 2, we need to recall abridged versions of two useful facts from [3, Lemmas 2.52 and 2.51] (see also [4, 5]), which we formulate as Propositions 4 and 5 below.

Given a set Γ and a real number p, $0 \leq p \leq 1$, let Γ_p be the random binomial subset of Γ, that is, a subset obtained by independently including each element of Γ

with probability p. Further, given an increasing family $\mathcal{Q}$ of subsets of a set Γ and an integer h, we denote by $\mathcal{Q}_h$ the subfamily of $\mathcal{Q}$ consisting of the sets $A \in \mathcal{Q}$ having the property that all subsets of A with at least $|A| - h$ elements still belong to $\mathcal{Q}$.

Proposition 4. *Let $0 < c < 1$, $\delta = c^2/9$, $Np \geq 72/\delta^2 = 2^3 3^6/c^4$, and $h = \delta Np/2$. Then for every increasing family $\mathcal{Q}$ of subsets of an N-element set Γ the following holds. If*

$$\mathbb{P}\left(\Gamma_{(1-\delta)p} \notin \mathcal{Q}\right) \leq \exp(-cNp)$$

then

$$\mathbb{P}\left(\Gamma_p \notin \mathcal{Q}_h\right) \leq \exp(-\delta^2 Np/9)\,.$$

Proof. We want to apply [3, Lemma 2.52], which is very similar to Proposition 4. Lemma 2.52 from [3] states that if c and $\delta > 0$ satify

$$\delta(3 + \log(1/\delta)) \leq c \tag{6}$$

and

$$\mathbb{P}(\Gamma_{(1-\delta)p} \notin \mathcal{Q}) \leq \exp(-cNp)$$

then

$$\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) \leq 3\sqrt{Np}\exp(-cNp/2) + \exp(-\delta^2 Np/8)\,. \tag{7}$$

To this end we first note that by assumption of Proposition 4 we have $\delta < 1/9$. Since $\sqrt{x}(\log(1/x)$ is increasing for $x \in (0, 1/\mathrm{e}^2]$ it follows for every $\delta \leq 1/9$ that

$$\sqrt{\delta}\log(1/\delta) \leq \frac{\log(9)}{3} \leq 2\,.$$

Consequently, $\sqrt{\delta}(3 + \log(1/\delta)) \leq 3$ and owing to the assumption $\delta = c^2/9$ this is equivalent to (6). Moreover, since $Np \geq 2^3 3^6/c^4 > (12/c)^2$ we have

$$3\sqrt{Np} \leq \exp(3\sqrt{Np}) \leq \exp(cNp/4)\,.$$

Hence, (7) yields

$$\begin{aligned}\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) &\leq \exp(-cNp/4) + \exp(-\delta^2 Np/8) \leq 2\exp(-\delta^2 Np/8)\\ &\leq \exp(-\delta^2 Np/8 + 1) \leq \exp(-\delta^2 Np/9),\end{aligned}$$

where the last inequality follows by our assumption $Np \geq 72/\delta^2$. $\square$

The following result has appeared in [3] as Lemma 2.51. We state it here for $t = 2$ only.

Proposition 5 ([3])**.** *Let $\mathcal{S} \subseteq \binom{\Gamma}{s}$, $0 \leq p \leq 1$, and $\lambda = |\mathcal{S}|p^s$. Then for every nonnegative integer h, with probability at least $1 - \exp(-\frac{h}{2s})$, there exists a subset $E_0 \subseteq \Gamma_p$ of size h such that $\Gamma_p \setminus E_0$ contains at most 2λ sets from $\mathcal{S}$.*

In the proof of Theorem 2 we will also use an elementary fact about (ϱ, d)-dense graphs. For constants ϱ and d with $0 < d, \varrho \leq 1$ we call an n-vertex graph Γ (ϱ, d)-dense if every induced subgraph on $m \geq \varrho n$ vertices contains at least $d(m^2/2)$ edges. It follows by an easy averaging argument that it suffices to check the above inequality only for $m = \lceil \varrho n \rceil$. Note also that every induced subgraph of a (ϱ, d)-dense n-vertex graph on at least cn vertices is $(\frac{\varrho}{c}, d)$-dense.

It turns out that for a suitable choice of the parameters, (ϱ, d)-dense graphs enjoy a Ramsey-like property. For a two-coloring of (the edges of) Γ we call a sequence of vertices $(v_1, \ldots, v_\ell)$ *canonical* if for each $i = 1, \ldots, \ell - 1$ all the edges $\{v_i, v_j\}$, for $j > i$ are of the same color.

Proposition 6. *For every $\ell \geq 2$ and $d \in (0, 1)$, if $n \geq 2(4/d)^{\ell-2}$ and $0 < \varrho \leq (d/4)^{\ell-2}/2$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains at least*

$$f_n(\ell) := \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} n^\ell$$

canonical sequences of length ℓ.

Proof. First, note that as long as $\varrho \leq 1/2$ every (ϱ, d)-dense graph contains at least $n/2$ vertices with degrees at least $dn/2$. Indeed, otherwise a set of $m = \lceil (n+1)/2 \rceil$ vertices of degrees smaller than $dn/2$ would induce less than $mdn/4 \leq d(m^2/2)$ edges, a contradiction.

We prove Proposition 6 by induction on ℓ. For $\ell = 2$, every ordered pair of adjacent vertices is a canonical sequence and there are at least $2d\binom{n}{2} > f_n(2)$ such pairs if $n \geq 2$. Assume that the proposition is true for some $\ell \geq 2$ and consider an n-vertex (ϱ, d)-dense graph Γ, where $\varrho \leq (d/4)^{\ell-1}/2$ and $n \geq 2(4/d)^{\ell-1}$. As observed above, there is a set U of at least $n/2$ vertices with degrees at least $dn/2$. Fix one vertex $u \in U$ and let M_u be a set of at least $dn/4$ neighbors of u connected to u by edges of the same color. Let $\Gamma_u = \Gamma[M_u]$ be the subgraph of Γ induced by the set M_u. Note that Γ_u has $n_u \geq dn/4 \geq 2(4/d)^{\ell-2}$ vertices and is (ϱ_u, d)-dense with $\varrho_u \leq (d/4)^{\ell-2}/2$. Hence, by the induction assumption, there are at least

$$f_{n_u}(\ell) \geq \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} \left(\frac{dn}{4}\right)^\ell = \left(\frac{1}{4}\right)^{\binom{\ell}{2}+\ell} d^{\binom{\ell+1}{2}} n^\ell$$

canonical sequences of length ℓ in Γ_u. Each of these sequences preceded by the vertex u makes a canonical sequence of length $\ell+1$ in Γ. As there are at least $n/2$ vertices in U, there are at least

$$\frac{n}{2} f_{n_u}(\ell) \geq \left(\frac{1}{4}\right)^{\binom{\ell+2}{2}} d^{\binom{\ell+1}{2}} n^{\ell+1}$$

canonical sequences of length $\ell+1$ in Γ. This completes the inductive proof of Proposition 6. $\square$

Corollary 7. *For every $k \geq 2$, every graph F on k vertices, and every $d \in (0, 1)$, if $n \geq (4/d)^{2k}$ and $0 < \varrho \leq (d/4)^{2k}$, then every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least γn^k monochromatic copies of F, where $\gamma = d^{2k^2} 2^{-5k^2}$.*

Proof. Every canonical sequence $(v_1, \ldots, v_{2k-2})$ contains a monochromatic copy of K_k. Indeed, among the vertices $v_1, \ldots, v_{2k-3}$, some $k-1$ have the same color on all the "forward" edges. Therefore, these vertices together with vertex v_{2k-2} form a monochromatic copy of K_k. On the other hand, every such copy is contained in no more than $k!\binom{2k-2}{k} n^{k-2} = (2k-2)_k n^{k-2}$ canonical sequences of length $2k-2$. Finally, every copy of K_k contains at least one copy of F, and different copies of K_k contain different copies of F. Consequently, by Proposition 6, every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least

$$\frac{f_n(2k-2)}{(2k-2)_k n^{k-2}} = \frac{1}{(2k-2)_k} \left(\frac{1}{4}\right)^{\binom{2k-1}{2}} d^{\binom{2k-2}{2}} n^k > \frac{d^{2k^2}}{2^{5k^2}} n^k$$

monochromatic copies of F. □

3 Proof of Theorem 2

3.1 Preparations and outline

For given $n \in \mathbb{N}$, $p \in (0, 1)$, and a k-vertex graph F we denote by X_F the random variable counting the number of copies of F in $G(n, p)$. We also recall that $\mu_F = \mathbb{E}X_F$.

For fixed $k \geq 3$ we prove Theorem 2 by induction on e_F. We may assume $n \geq k$, as for $n < k$ we have $\mu_F = 0$ and there is nothing to prove.

Base case. Let F_1 be a graph consisting of one edge and $k-2$ isolated vertices. Note that $m_{F_1} = 1/2$ (see (2)) and for every two-coloring of the edges of $G(n, p)$ every copy of F_1 in $G(n, p)$ is monochromatic. Clearly,

$$X_{F_1} = \binom{n-2}{k-2} X_{K_2} \quad \text{and} \quad \mu_{F_1} = \binom{n-2}{k-2} \mu_{K_2} = \binom{n-2}{k-2} \binom{n}{2} p.$$

Thus, by Chernoff's bound (see, e.g., [3, ineq. (2.6)]) we have

$$\mathbb{P}\left(X_{F_1} \leq \frac{1}{2}\mu_{F_1}\right) = \mathbb{P}\left(X_{K_2} \leq \frac{1}{2}\mu_{K_2}\right) \leq \exp\left(-\frac{1}{8}\binom{n}{2}p\right),$$

which holds for any values of p and n. Hence, Theorem 2 follows for $F = F_1$ and with the constants $a_1 = 1/2$, $b_1 = 1/8$, and $C_1 = n_1 = 1$ as given in (3).

Inductive step. Given a graph G, an edge f of G and a nonedge e, that is an edge of the complement of G, we denote by $G - f$ a graph obtained from G by removing f, and by $G + f$ a graph ontained by adding e to G. Let F_{i+1} be a graph with $i + 1 \geq 2$ edges and maximum degree $\Delta(F_{i+1}) \geq 2$. If $i + 1 \geq 3$, then we can remove one edge from F_{i+1} in such a way that the resulting graph F_i still contains at least one vertex of degree at least two, i.e., $\Delta(F_i) \geq 2$. If $i + 1 = 2$, the graph $F_{i+1} = F_2$ consists of a path of length two and $k - 3$ isolated vertices and removing any of the two edges results in the graph $F_i = F_1$. In either case, we may fix an edge $f \in E(F_{i+1})$ such that the graph $F_i = F_{i+1} - f$ is k-admissible. Hence, we can assume that Theorem 2 holds for F_i and for the constants a_i, b_i, C_i, and n_i inductively defined by (3) and (4).

We have to show that Theorem 2 holds for F_{i+1} and constants a_{i+1}, b_{i+1}, C_{i+1}, and n_{i+1} given in (4). To this end, let $n \geq n_{i+1}$ and $p \geq C_{i+1} n^{-1/m_{F_{i+1}}}$. We will expose the random graph $G(n,p)$ in two independent rounds $G(n, p_{\mathrm{I}})$ and $G(n, p_{\mathrm{II}})$ and have $G(n, p) = G(n, p_{\mathrm{I}}) \cup G(n, p_{\mathrm{II}})$. For that, we will fix p_{I} and p_{II} as follows. First we fix auxiliary constants[1]

$$d = \frac{a_i^2}{64^{k^2}}\,, \qquad \varrho = \left(\frac{d}{4}\right)^{2k}, \qquad \gamma = \frac{d^{2k^2}}{2^{5k^2}}\,, \qquad \delta_{\mathrm{II}} = \frac{\gamma^4}{9 \cdot 16^{k^2}}\,, \quad \text{and} \quad \alpha = \frac{\delta_{\mathrm{II}}^2 \gamma}{36}\,. \tag{8}$$

Then p_{I} and $p_{\mathrm{II}} \in (0,1)$ are defined by the equations

$$p = p_{\mathrm{I}} + p_{\mathrm{II}} - p_{\mathrm{I}} p_{\mathrm{II}} \qquad \text{and} \qquad p_{\mathrm{I}} = \alpha p_{\mathrm{II}}\,. \tag{9}$$

Clearly, we have

$$p \geq p_{\mathrm{II}} \geq \frac{p}{2} \geq \alpha p \geq \alpha p_{\mathrm{II}} = p_{\mathrm{I}} \geq \alpha \frac{p}{2}\,. \tag{10}$$

We continue with a short outline of the main ideas of the forthcoming proof.

Outline. First we consider a two-coloring χ, with colors red and blue, of the edges of $G(n, p_{\mathrm{I}})$ (first round). Owing to the induction assumption (Theorem 2 for F_i) we note that with high probability the coloring χ yields many monochromatic copies of F_i. We will say that an unordered pair of vertices $e = \{u, v\}$ is χ*-rich* if $G(n, p_{\mathrm{I}}) + e$ possesses "many" (to be defined later) copies of F_{i+1}, in which e plays the role of the edge f and the rest is a monochromatic copy of F_i. Let Γ_χ be an auxiliary graph of all χ-rich pairs. We will show that with 'high' probability (to be specified later), Γ_χ is, in fact, (ϱ, d)-dense for d and ϱ as in (8) (Claim 8).

To this end, note that if the monochromatic copies of F_i were clustered at relatively few pairs, then we might fall short of proving Claim 8. However, we will show that in the random graph $G(n, p_{\mathrm{I}})$ it is unlikely that many copies of F_i share the same pair of vertices. For that, we will consider the distribution of the graphs T consisting of two copies of F_i which share the vertices of a missing edge f (and possibly other vertices). We will show that the number of those copies is of the same order of magnitude as its

[1]The proof requires several auxiliary constants which at first may appear a bit unmotivated. For example, we now define δ_{II}, while δ_{I} is to be defined only later. Both δ's will be used in applications of Proposition 4.

expectation (Fact 9), and will also require that this holds with high probability. Such a sharp concentration result is known to be false, but Proposition 5 asserts that it can be obtained on the cost of removing a few edges of $G(n, p_{\mathrm{I}})$.

The auxiliary graph Γ_χ is naturally two-colored (by azure and pink), since every χ-rich pair closes either many blue or many red copies of F_i (or both and then we pick the color for that edge, azure or blue, arbitrarily). Consequently, Corollary 7 yields many monochromatic copies of F_{i+1} in Γ_χ and at least half of them are colored, say, pink. That is, there are many copies of F_{i+1} in Γ_χ such that each of their edges closes many red copies of F_i in $G(n, p_{\mathrm{I}})$ under the coloring χ. By Janson's inequality combined with Proposition 4, with high probability, many pink copies will be still present in $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ (second round) even after a fraction of edges is deleted. Thus, we are facing a 'win-win' scenario. Namely, if an extension of χ colors only few pink edges of $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ red then, by the above, many copies of F_{i+1} in $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ have to be colored completely blue. Otherwise, many pink edges of $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ are red, which, by the definition of a pink edge, results in many red copies of F_{i+1} in $G(n, p)$.

Useful estimates. For the verification of several inequalities in the proof, it will be useful to appeal to the following lower bounds for γ, α, and ϱ in terms of powers of a_i and 2. From the definitions in (8), for sufficiently large k, one obtains the following bounds.

$$\begin{aligned}
\gamma =& \frac{a_i^{4k^2}}{2^{12k^4+5k^2}} \geq \frac{a_i^{4k^2}}{2^{13k^4}}\,, \\
\alpha =& \frac{a_i^{36k^2}}{3^6 \cdot 2^{108k^4+53k^2+2}} \geq \frac{a_i^{36k^2}}{2^{109k^4}}\,, \\
\varrho =& \frac{a_i^{4k}}{2^{12k^3+4k}} \geq \frac{a_i^{4k}}{2^{13k^3}}\,.
\end{aligned} \tag{11}$$

We will also make use of the inequalities

$$np \geq C_{i+1}, \tag{12}$$

valid because $m_{F_{i+1}} \geq 1$, and, for every subgraph H of F_{i+1} with $v_H \geq 3$,

$$n^{v_H-2}p^{e_H-1} \geq C_{i+1}^{e_H-1}, \tag{13}$$

valid because

$$m_{F_{i+1}} \geq d_H = \frac{e_H - 1}{v_H - 2}.$$

Of course, (12) follows from (13), by taking H with $d_H = m_{F_{i+1}}$.

3.2 Details.

First round. As outlined above, in the first round we want to show that with high probability the random graph $G(n, p_{\mathrm{I}})$ has the property that for every two-coloring χ

the auxiliary graph Γ_χ (defined below) is (ϱ, d)-dense. For that we set

$$\delta_{\mathrm{I}} = \frac{b_i^2}{36} \tag{14}$$

and for a two-coloring χ call a pair $\{u, v\}$ of vertices *χ-rich* if it closes at least

$$\ell = \frac{a_i}{4^{k^2}} (\varrho n)^{k-2} p_{\mathrm{I}}^i \tag{15}$$

monochromatic copies of F_i in $G(n, p_{\mathrm{I}})$ to a copy of F_{i+1}. Then Γ_χ is an auxiliary n-vertex graph with the edge set being the set of *χ-rich* pairs.

Let $\mathcal{E}$ be the event (defined on $G(n, p_{\mathrm{I}})$) that for every two-coloring χ of $G(n, p_{\mathrm{I}})$ the graph Γ_χ is (ϱ, d)-dense.

Claim 8.

$$\mathbb{P}(\mathcal{E}) \geq 1 - \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}} \binom{\varrho n}{2} p_{\mathrm{I}} + n + 2k^2\right)$$

Before giving the proof of Claim 8 we need one more fact. Let $\{T_1, T_2, \ldots, T_t\}$ be the family of all pairwise non-isomorphic graphs which are unions of two copies of F_i, say $F_i' \cup F_i''$, with the property that adding a single edge completes both, F_i' and F_i'' to a copy of F_{i+1}. We will refer to these graphs as *double creatures* (of F_i). Clearly, with some foresight of future applications,

$$t \leq 2^{\binom{2k-2}{2}} \leq 2^{2k^2-4k} \leq \frac{2^{2k^2-1}}{4\binom{k}{2}}. \tag{16}$$

Let X_j be the number of copies of T_j in $G(U, p_{\mathrm{I}})$, $j = 1, \ldots, t$.

Fact 9. *For every $j = 1, \ldots, t$*

$$\mathbb{E} X_j \leq (\varrho n)^{2k-2} p_{\mathrm{I}}^{2i}.$$

Proof. Let $T := T_j = F_i' \cup F_i''$ be a double creature and set $S = F_i' \cap F_i''$. Then the expected number of copies of T is bounded from above by

$$\mathbb{E} X_T \overset{(5)}{\leq} (\varrho n)^{v_T} p_{\mathrm{I}}^{e_T} = \frac{(\varrho n)^{2k} p_{\mathrm{I}}^{2i}}{(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S}},$$

and it remains to show that

$$(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S} \geq (\varrho n)^2.$$

There is nothing to prove when $v_S = 2$ (and thus $e_S = 0$). Otherwise, pick a pair of vertices f in T such that both, $F_i' + f$ and $F_i'' + f$, are isomorphic to F_{i+1}. Then $J := S + f \subseteq F_{i+1}$. Note that $e_J = e_S + 1$ and $3 \leq v_J = v_S \leq k$. Since $C_{i+1} \geq 2/\alpha$,

$$\begin{aligned}(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S} &\overset{(10)}{\geq} (\varrho n)^{v_J} \left(\frac{\alpha}{2}\right)^{e_S} p^{e_J - 1} \overset{(13)}{\geq} \varrho^{v_S - 2} \left(\frac{\alpha}{2}\right)^{e_S} C_{i+1}^{e_S} (\varrho n)^2 \\ &\geq \varrho^k \frac{\alpha}{2} C_{i+1} (\varrho n)^2 \overset{(11)}{\geq} \frac{1}{2} \frac{a_i^{4k^2}}{2^{13k^4}} \frac{a_i^{36k^2}}{2^{109k^4}} C_{i+1} (\varrho n)^2 \overset{(4)}{\geq} \frac{2^{13k^4-1}}{b_i^4} C_i (\varrho n)^2 \geq (\varrho n)^2. \quad \square\end{aligned} \tag{17}$$

Proof of Claim 8: Let χ be a two-coloring of $G(n, p_{\mathrm{I}})$. Fix a set $U \subseteq [n]$ with $|U| = \varrho n$ (throughout we assume that ϱn is an integer) and consider the random graph $G(n, p_{\mathrm{I}})$ induced on U

$$G(U, p_{\mathrm{I}}) := G(n, p_{\mathrm{I}})[U]\,.$$

By the induction assumption, if $\varrho n \geq n_i$ and $p_i \geq C_i(\varrho n)^{-1/m_{F_i}}$ then, with high probability, there are many monochromatic copies of F_i in $G(U, p_{\mathrm{I}})$. For technical reasons that will become clear only later, we want to strengthen the above Ramsey property so that it is resilient to deletion of a small fraction of edges. For that we apply the induction assumption to the random graph $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$, followed by an application of Propositions 4. We begin by verifying the assumptions of Theorem 2 with respect to F_i and $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. First, note that

$$|U| = \varrho n \geq \varrho n_{i+1} \overset{(11)}{\geq} \frac{a_i^{4k}}{2^{13k^3}} n_{i+1} \overset{(4)}{=} \frac{a_i^{4k}}{2^{13k^3}} \cdot \frac{2^{14k^3}}{a_i^{4k}} n_i = 2^{k^3} n_i \geq n_i\,. \tag{18}$$

It remains to check that

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \geq C_i(\varrho n)^{-1/m_{F_i}}\,. \tag{19}$$

To this end, we simply note that using $\delta_{\mathrm{I}} \leq 1/2$, $\varrho \leq 1$, and $m_{F_{i+1}} \geq \max(1, m_{F_i})$ we have

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \overset{(10)}{\geq} \frac{\alpha p}{4} \geq \frac{\alpha}{4} C_{i+1} \varrho^{1/m_{F_{i+1}}} (\varrho n)^{-1/m_{F_{i+1}}} \geq \frac{\alpha}{4} C_{i+1} \varrho (\varrho n) n^{-1/m_{F_i}}.$$

Furthermore, we have

$$\frac{\alpha\varrho}{4} C_{i+1} \overset{(11)}{\geq} \frac{a_i^{36k^2+4k}}{2^{109k^4+13k^3+2}} \cdot C_{i+1} \overset{(4)}{=} \frac{a_i^{37k^2}}{2^{110k^4}} \cdot \frac{2^{122k^4} C_i}{b_i^4 a_i^{37k^2}} = \frac{2^{12k^2} C_i}{b_i^4} \geq C_i\,. \tag{20}$$

and (19) follows. Thus, we are in position to apply the induction assumption to $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$ and F_i. Let

$$\mu := \mu_{F_i}^{\varrho, \delta_{\mathrm{I}}} := \binom{\varrho n}{k} \frac{k!}{\mathrm{aut}(F_i)} ((1-\delta_{\mathrm{I}})p_{\mathrm{I}})^i \geq \frac{1}{4^{k^2}} (\varrho n)^k p_{\mathrm{I}}^i \tag{21}$$

denote the expected number of copies of F_i in $G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. By Theorem 2 we infer that

$$\begin{aligned} \mathbb{P}\left(G(U, (1-\delta_{\mathrm{I}})p_{\mathrm{I}}) \xrightarrow{a_i\mu} F_i\right) &\geq 1 - \exp\left(-b_i(1-\delta_{\mathrm{I}})p_{\mathrm{I}}\tbinom{\varrho n}{2}\right) \\ &\geq 1 - \exp\left(-\tfrac{b_i}{2} p_{\mathrm{I}} \tbinom{\varrho n}{2}\right)\,. \end{aligned} \tag{22}$$

Next we head for an application of Proposition 4 with $c = b_i/2$, $\delta = \delta_{\mathrm{I}}$, $N = \binom{\varrho n}{2}$, and p_{I}. Note that, indeed, $\delta_{\mathrm{I}} = b_i^2/36 = c^2/9$ (see (14)). Moreover, using $\varrho n \geq 3$ (see (18)) and (12) we see that

$$p_{\mathrm{I}} \binom{\varrho n}{2} \overset{(10)}{\geq} \frac{\alpha p}{2} \cdot \varrho n \geq \frac{\alpha\varrho}{2} \cdot C_{i+1} \overset{(20)}{\geq} \frac{2^{12k^3+1}}{b_i^4} \geq \frac{72}{\delta_{\mathrm{I}}^2}$$

and the assumptions of Proposition 4 are verified. From (22) we infer by Proposition 4 that with probability at least

$$1 - \exp\left(-\frac{\delta_{\mathrm{I}}^2}{9}\binom{\varrho n}{2} p_{\mathrm{I}}\right) \tag{23}$$

$G(U, p_{\mathrm{I}})$ has the property that for every subgraph $G' \subseteq G(U, p_{\mathrm{I}})$ with

$$\left|E(G(U, p_{\mathrm{I}})) \setminus E(G')\right| \leq \frac{\delta_{\mathrm{I}}}{2}\binom{\varrho n}{2} p_{\mathrm{I}}$$

we have

$$G' \xrightarrow{a_i \mu} F_i \,. \tag{24}$$

Our goal is to show that, with high probability, any two-coloring χ of $G(U, p_{\mathrm{I}})$ yields at least $d(|U|^2/2)$ χ-rich edges, and ultimately, by repeating this argument for every set $U \subseteq [n]$ with ϱn vertices, that Γ_χ is (ϱ, d)-dense. The above 'robust' Ramsey property (24) means that after applying Proposition 5 to $G(U, p_{\mathrm{I}})$ the resulting subgraph of $G(U, p_{\mathrm{I}})$ will still have the Ramsey property with high probability.

Let Y be the random variable counting the number of double creatures in $G(U, p_{\mathrm{I}})$. It follows from Fact 9 that

$$\mathbb{E}Y \leq t(\varrho n)^{2k-2} p_{\mathrm{I}}^{2i} \,. \tag{25}$$

Hence, by Proposition 5, applied for every $j = 1, \ldots, t$ to the families $\mathcal{S}_j$ of all copies of T_j in $G(U, p_{\mathrm{I}})$ with

$$h_{\mathrm{I}} = \frac{\delta_{\mathrm{I}}}{2t}\binom{\varrho n}{2} p_{\mathrm{I}} \tag{26}$$

we conclude that with probability at least

$$1 - \sum_{j=1}^{t} \exp\left(-\frac{h_{\mathrm{I}}}{2e(T_j)}\right) \geq 1 - t\exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right) \tag{27}$$

there exists a subgraph $G_0 \subseteq G(U, p_{\mathrm{I}})$ with $|E(G(U, p_{\mathrm{I}}) \setminus E(G_0)| \leq t h_{\mathrm{I}}$ such that G_0 contains at most $2\mathbb{E}Y$ double creatures. Since

$$t h_{\mathrm{I}} \overset{(26)}{=} \frac{\delta_{\mathrm{I}}}{2}\binom{\varrho n}{2} p_{\mathrm{I}},$$

the robust Ramsey property (24) holds with $G' = G_0$.

Recall that a two-coloring χ of $G(n, p_{\mathrm{I}})$ is fixed. For $\{u, v\} \subset U$, let x_{uv} be the number of monochromatic copies of F_i in G_0 which together with the pair $\{u, v\}$ form a copy of F_{i+1}. Owing to (24), we have

$$\sum_{\{u,v\} \in \binom{U}{2}} x_{uv} \geq a_i \mu. \tag{28}$$

By the above application of Proposition 5 we infer that

$$\sum_{\{u,v\}\in\binom{U}{2}} x_{uv}^2 \le 2\cdot\binom{k}{2}\cdot|DC(G_0)| \le 4\binom{k}{2}\mathbb{E}Y \overset{(25),(16)}{\le} 2^{2k^2-1}(\varrho n)^{2k-2}p_{\mathrm{I}}^{2i}\,, \tag{29}$$

where $DC(G_0)$ is the set of all double creatures in G_0. Recall that $\{u,v\}\in E(\Gamma_\chi)$ if it is χ-rich, which is implied by $x_{uv}\ge\ell$, where ℓ is defined in (15). We want to show that $e(\Gamma_\chi[U])\ge d(\varrho n)^2/2$. Since $\ell\le a_i\mu/(\varrho n)^2$ (compare (15) and (21)), it follows from (28) that

$$\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\ge\ell}} x_{uv} \ge \frac{a_i\mu}{2} \overset{(21)}{\ge} \frac{1}{2}\cdot\frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\,.$$

Squaring the last inequality and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left(\frac{1}{2}\cdot\frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\right)^2 \le \Bigg(\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\ge\ell}} x_{uv}\Bigg)^2 &\le e(\Gamma_\chi[U])\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\ge\ell}} x_{uv}^2\\ &\overset{(29)}{\le} e(\Gamma_\chi[U])\cdot 2^{2k^2-1}(\varrho n)^{2k-2}p_{\mathrm{I}}^{2i}\,.\end{aligned}$$

Consequently,

$$e(\Gamma_\chi[U]) \ge \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \ge \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \overset{(8)}{=} d(\varrho n)^2/2\,.$$

Summarizing the above, we have shown that if $G(U,p_{\mathrm{I}})$ has the robust Ramsey property for F_i (24) and if the conclusion of Proposition 5 holds for all $j=1,\dots,t$, then $e(\Gamma_\chi[U])\ge d(\varrho n)^2/2$. The probability that at least one of these events fails is at most (see (23) and (27))

$$\exp\left(-\frac{\delta_{\mathrm{I}}^2}{9}\binom{\varrho n}{2}p_{\mathrm{I}}\right)+t\exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right)\,.$$

Recalling that $t\le 4^{k^2}$ (see (16)) and the definition of h_{I} in (26), Claim 8 now follows by summing up these probabilities over all choices of $U\subseteq[n]$ with $|U|=\varrho n$. More precisely, using the union bound and the estimate $\binom{n}{\varrho n}\le 2^n$, we conclude that the probability that there is a coloring χ for which the graph Γ_χ is not (ϱ,d)-dense is

$$\begin{aligned}\mathbb{P}(\neg\mathcal{E}) &\le 2^n\exp\left(-\frac{1}{9}\delta_{\mathrm{I}}^2\binom{\varrho n}{2}p_{\mathrm{I}}\right)+2^n4^{k^2}\exp\left(-\frac{1}{k^24^{k^2}}\delta_{\mathrm{I}}\binom{\varrho n}{2}p_{\mathrm{I}}\right)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}}+n+2k^2\right)\end{aligned}$$

□

This ends the analysis of the first round.

Second round. Let $\mathcal{B}$ be the conjunction of $\mathcal{E}$ and the event that $|G(n, p_{\mathrm{I}})| \leq n^2 p_{\mathrm{I}}$. In the second round we will condition on the event $\mathcal{B}$ and sum over all two-colorings χ of $G(n, p_{\mathrm{I}})$. Formally, let $\mathcal{A}$ be the (bad) event that there is a two-coloring of the edges of $G(n, p)$ with fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}. (That is, $\neg\mathcal{A}$ is the Ramsey property $G(n, p) \xrightarrow{a_{i+1}\mu_{F_{i+1}}} F_{i+1}$.) Further, given a two-coloring χ of $G(n, p_{\mathrm{I}})$, let $\mathcal{A}_\chi$ be the event that there exists an extension of χ to a coloring $\bar{\chi}$ of $G(n, p)$ yielding altogether fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}.

The following pair of inequalities exhibit the skeleton of our proof of Theorem 2:

$$\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\neg\mathcal{B}) + \sum_{G \in \mathcal{B}} \mathbb{P}\Big(\mathcal{A} | G(n, p_{\mathrm{I}}) = G\Big)\mathbb{P}\Big(G(n, p_{\mathrm{I}}) = G\Big) \tag{30}$$

and

$$\mathbb{P}(\mathcal{A}|G(n, p_{\mathrm{I}}) = G) = \mathbb{P}\left(\bigcup_\chi \mathcal{A}_\chi \middle| G(n, p_{\mathrm{I}}) = G\right) \leq 2^{n^2 p_{\mathrm{I}}} \max_\chi \mathbb{P}(\mathcal{A}_\chi | G(n, p_{\mathrm{I}}) = G). \tag{31}$$

By Claim 8 and Chernoff's inequality (see, e.g., [3, ineq. (2.5)])

$$\begin{aligned}\mathbb{P}(\neg\mathcal{B}) &\leq \mathbb{P}(\neg\mathcal{E}) + \mathbb{P}\Big(|G(n, p_{\mathrm{I}})| > n^2 p_{\mathrm{I}}\Big) \\ &\leq \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2} p_{\mathrm{I}} + n + 2k^2\right) + \exp\left(-\frac{1}{3}\binom{n}{2} p_{\mathrm{I}}\right) \\ &\leq \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2} p_{\mathrm{I}} + n + 2k^2 + 1\right) =: q_{\mathrm{I}}\,. \end{aligned} \tag{32}$$

To complete the proof of Theorem 2 it is thus crucial to find an upper bound on $\mathbb{P}(\mathcal{A}_\chi | G(n, p_{\mathrm{I}}) = G)$ which substantially beats the factor $2^{n^2 p_{\mathrm{I}}}$.

Claim 10. *For every $G \in \mathcal{B}$ and every two-coloring χ of G,*

$$\mathbb{P}(\mathcal{A}_\chi | G(n, p_{\mathrm{I}}) = G) \leq \exp\left(-\frac{\delta_{\mathrm{II}}^2 \gamma}{9} n^2 p_{\mathrm{II}}\right).$$

The edges of Γ_χ are naturally two-colored according to the majority color among the monochromatic copies of F_i attached to them. We color an edge of Γ_χ *pink* if it closes at least $\ell/2$ red copies of F_i and we color it *azure* otherwise. Subsequently, we apply Corollary 7 to Γ_χ for F_{i+1} and d (chosen in (8)). Note that in (8) we chose ϱ to facilitate such an application. Moreover, the required lower bound on n is equivalent to $\varrho n \geq 1$ and this follows from (18). Hence, by Corollary 7 and the choice of γ in (8), we may assume without loss of generality, that there are at least $\gamma n^k/2$ pink copies of F_{i+1} in Γ_χ. In particular, all these copies of F_{i+1} consist entirely of edges closing each at least $\ell/2$ red copies of F_i (from the first round). Let us denote by $\mathcal{F}_\chi$ the family of these copies of F_{i+1}, and let $\Gamma_\chi^{\mathrm{pink}}$ be the subgraph of Γ_χ containing the pink edges. Since every edge may belong to at most n^{k-2} copies of F_{i+1}, we have

$$e(\Gamma_\chi^{\mathrm{pink}}) \geq \frac{(i+1)\cdot|\mathcal{F}_\chi|}{n^{k-2}} \geq \frac{(i+1)\cdot\gamma n^k/2}{n^{k-2}} \geq \gamma n^2\,. \tag{33}$$

In the proof of Claim 10 we intend to use again Proposition 4, this time with $\Gamma = \Gamma_\chi^{\text{pink}}$ and $\mathcal{Q}$ – the property of containing at least

$$\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1} \tag{34}$$

copies of F_{i+1} belonging to $\mathcal{F}_\chi$. For this, however, we need the following fact.

Fact 11. *With δ_{II} chosen in (8),*

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \leq \exp\left(-\frac{\gamma^2}{4^{k^2}} e(\Gamma_\chi^{\text{pink}}) p_{\text{II}}\right).$$

Proof. Consider a random variable Z counting the number of copies F_{i+1} belonging to $\mathcal{F}_\chi$ which are subgraphs of $G(n, (1-\delta_{\text{II}})p_{\text{II}})$. We have

$$\mathbb{E}Z = |\mathcal{F}_\chi|((1-\delta_{\text{II}})p_{\text{II}})^{i+1} \geq \frac{1}{2}\gamma n^k ((1-\delta_{\text{II}})p_{\text{II}})^{i+1} \geq \frac{1}{2} \cdot \frac{1}{2^{\binom{k}{2}}} \gamma n^k p_{\text{II}}^{i+1}, \tag{35}$$

where we used the bound $\delta_{\text{II}} \leq 1/2$.

By Janson's inequality (see, e.g., [3, Theorem 2.14]),

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \leq \mathbb{P}\left(Z \leq \frac{1}{2}\mathbb{E}Z\right) \leq \exp\left(-\frac{(\mathbb{E}Z)^2}{8\bar{\Delta}}\right),$$

where

$$\bar{\Delta} = \sum_{F' \in \mathcal{F}_\chi} \sum_{F'' \in \mathcal{F}_\chi} \mathbb{P}(F' \cup F'' \subseteq G(n, (1-\delta_{\text{II}})p_{\text{II}})),$$

with the double sum ranging over all ordered pairs $(F', F'') \in \mathcal{F}_\chi \times \mathcal{F}_\chi$ with $E(F') \cap E(F'') \neq \emptyset$. The quantity $\bar{\Delta}$ can be bounded from above by

$$\bar{\Delta} \leq \sum_{\widetilde{F} \subseteq F_{i+1}} n^{2k - v(\widetilde{F})} p_{\text{II}}^{2(i+1) - e(\widetilde{F})}, \tag{36}$$

where the sum is taken over all subgraphs $\widetilde{F}$ of F_{i+1} with at least one edge. If $e(\widetilde{F}) = 1$ then

$$n^{v(\widetilde{F})} p_{\text{II}}^{e(\widetilde{F})} = n^{v(\widetilde{F})} p_{\text{II}} \geq n^2 p_{\text{II}}. \tag{37}$$

Otherwise,

$$n^{v(\widetilde{F})} p_{\text{II}}^{e(\widetilde{F})} \geq \frac{n^{v(\widetilde{F})} p^{e(\widetilde{F})}}{2^{e(\widetilde{F})}} \overset{(13)}{\geq} \frac{n^2 p C_{i+1}^{e(\widetilde{F})-1}}{2^{e(\widetilde{F})}} \geq n^2 p \overset{(10)}{\geq} n^2 p_{\text{II}}, \tag{38}$$

where we also used the fact that $C_{i+1} \geq 4$ (see (4)). Combining (36) with the bounds (37) and (38) yields

$$\bar{\Delta} \leq 2^{i+1} n^{2k-2} p_{\text{II}}^{2i+1} \leq 2^{\binom{k}{2}} n^{2k-2} p_{\text{II}}^{2i+1}.$$

Finally, plugging this estimate for $\bar{\Delta}$ and (35) into Janson's inequality we obtain

$$\mathbb{P}\big((\Gamma_\chi^{\text{pink}})_{(1-\delta_{\text{II}})p_{\text{II}}} \notin \mathcal{Q}\big) \leq \exp\left(-\frac{\gamma^2 n^2 p_{\text{II}}}{32 \cdot 2^{2\binom{k}{2}} \cdot 2^{\binom{k}{2}}}\right) \leq \exp\left(-\frac{\gamma^2}{4^{k^2}} e(\Gamma_\chi^{\text{pink}}) p_{\text{II}}\right). \quad \square$$

Proof of Claim 10: We plan to apply Proposition 4 with $c = \gamma^2/4^{k^2}$, $\delta_{\mathrm{II}} = \gamma^4/(9\cdot 16^{k^2})$ (see (8)), $N = e(\Gamma_\chi^{\mathrm{pink}})$, and p_{II}. Therefore, first we have to verify that $e(\Gamma_\chi^{\mathrm{pink}})p_{\mathrm{II}} \geq 72/\delta_{\mathrm{II}}^2$. Indeed,

$$e(\Gamma_\chi^{\mathrm{pink}})\cdot p_{\mathrm{II}} \overset{(10,33)}{\geq} \gamma n^2 \cdot \frac{p}{2} \overset{(12)}{\geq} \frac{\gamma}{2} nC_{i+1} \overset{(4)}{\geq} \frac{\gamma}{2}\cdot\frac{2^{122k^4}}{a_i^{37k^3}} \overset{(11)}{\geq} \frac{72\cdot 81\cdot 16^{2k^2}}{\gamma^8} = \frac{72}{\delta_{\mathrm{II}}^2}\,.$$

Consequently, by Proposition 4, we conclude that with probability at least

$$1-\exp\left(-\frac{\delta_{\mathrm{II}}^2}{9}e(\Gamma_\chi^{\mathrm{pink}})p_{\mathrm{II}}\right) \overset{(33)}{\geq} 1-\exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2 p_{\mathrm{II}}\right)\,, \tag{39}$$

the random graph $(\Gamma_\chi^{\mathrm{pink}})_{p_{\mathrm{II}}}$ has the property that for every subgraph $\Gamma' \subseteq (\Gamma_\chi^{\mathrm{pink}})_{p_{\mathrm{II}}}$ with

$$\left|E((\Gamma_\chi^{\mathrm{pink}})_{p_{\mathrm{II}}})\setminus E(\Gamma')\right| \leq \frac{\delta_{\mathrm{II}}\gamma}{2}n^2 p_{\mathrm{II}} =: h_{\mathrm{II}} \tag{40}$$

we have $\Gamma' \in \mathcal{Q}$, that is, Γ' contains at least $\frac{\gamma}{2^{k^2}}n^k p_{\mathrm{II}}^{i+1}$ copies of F_{i+1} belonging to $\mathcal{F}_\chi$ (see (34)).

Consider now an extension $\bar\chi$ of the coloring χ from $G(n,p_{\mathrm{I}})$ to $G(n,p)$. If in the coloring $\bar\chi$ fewer than h_{II} edges of $(\Gamma_\chi^{\mathrm{pink}})_{p_{\mathrm{II}}}$ are colored red, then, by the above consequence of Proposition 4, the blue part of $(\Gamma_\chi^{\mathrm{pink}})_{p_{\mathrm{II}}}$ contains at least

$$\frac{\gamma}{2^{k^2}}n^k p_{\mathrm{II}}^{i+1} \overset{(10)}{\geq} \frac{\gamma}{4^{k^2}}n^k p^{i+1}$$

copies of F_{i+1}. If, on the other hand, more than h_{II} edges of $(\Gamma_\chi^{pink})_{p_{\mathrm{II}}}$ are colored red, then, by the definition of a pink edge, noting that $i \leq k^2/2$, at least

$$\begin{aligned} h_{\mathrm{II}} \times \frac{\ell}{2} \times \frac{1}{i+1} &\overset{(15,40)}{\geq} \frac{\delta_{\mathrm{II}}\gamma}{2}n^2 p_{\mathrm{II}} \times \frac{a_i}{4^{k^2}k^2}(\varrho n)^{k-2}p_{\mathrm{I}}^i \\ &\overset{(10)}{\geq} \frac{\delta_{\mathrm{II}}\gamma}{4}n^2 p \times \frac{a_i\varrho^k}{4^{k^2}k^2}\left(\frac{\alpha}{2}\right)^i n^{k-2}p^i \\ &\geq \frac{\delta_{\mathrm{II}}\gamma a_i\varrho^k\alpha^{k^2/2}}{16^{k^2}}n^k p^{i+1} \end{aligned}$$

red copies of F_{i+1} arise. Owing to (8), (11), and the choice of a_{i+1} in (4) we have

$$\frac{\gamma}{4^{k^2}} \overset{(11)}{\geq} \frac{a_i^{4k^2}}{2^{13k^4+2k^2}} \overset{(4)}{\geq} a_{i+1}$$

and

$$\frac{\delta_{\mathrm{II}}\gamma a_i\varrho^k\alpha^{k^2/2}}{16^{k^2}} \overset{(8)}{=} \frac{\gamma^5\varrho^k\alpha^{k^2/2}}{9\cdot 2^{8k^2}}a_i \overset{(11)}{\geq} \frac{a_i^{18k^4+24k^2}}{2^{55k^6}} \overset{(4)}{\geq} a_{i+1}\,.$$

Therefore, we have shown that with probability as in (39), indeed any extension $\bar\chi$ of χ yields at least

$$\min\left(\frac{\gamma}{4^{k^2}}\,,\,\frac{\delta_{\mathrm{II}}\gamma a_i\varrho^k\alpha^{k^2/2}}{16^{k^2}}n^k p^{i+1}\right) \geq a_{i+1}n^k p^{i+1} \overset{(5)}{\geq} a_{i+1}\mu_{F_{i+1}}$$

monochromatic copies of F_{i+1}. $\square$

The final touch. To finish the proof of Theorem 2 it is left to verify that indeed $\mathbb{P}(\mathcal{A}) \le \exp(-b_{i+1}\binom{n}{2}p)$. The error probability of the first round is (see (32))

$$\mathbb{P}(\neg\mathcal{B}) \le q_{\mathrm{I}}.$$

Turning to the second round, by Claim 10 and (31), for any $G \in \mathcal{B}$,

$$\mathbb{P}(\mathcal{A}|G(n,p_{\mathrm{I}}) = G) \le 2^{n^2p_{\mathrm{I}}} \cdot \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}}\right) \le \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}} + n^2p_{\mathrm{I}}\right) =: q_{\mathrm{II}}\,, \quad (41)$$

and, consequently, by (30),

$$\mathbb{P}(\mathcal{A}) \le q_{\mathrm{I}} + q_{\mathrm{II}}.$$

Below we show (see Fact 12) that q_{I} and q_{II} are each upper bounded by $\exp(-b_{i+1}n^2p)$. Consequently,

$$\mathbb{P}(\mathcal{A}) \le 2\exp(-b_{i+1}n^2p) \le \exp(1 - b_{i+1}n^2p) \le \exp(-\tfrac{b_{i+1}}{2}n^2p) \le \exp(-b_{i+1}\tbinom{n}{2}p)$$

because

$$\frac{b_{i+1}}{2}n^2p \overset{(12)}{\ge} \frac{b_{i+1}}{2}C_{i+1}n_{i+1} \overset{(4)}{\ge} C_in_{i+1} \ge 1\,.$$

Fact 12.

$$\max(q_{\mathrm{I}}, q_{\mathrm{II}}) \le \exp(-b_{i+1}n^2p)$$

Proof. We first bound q_{I}. Since $\varrho n \ge 3$ (see (18)),

$$\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} \overset{(10)}{\ge} \frac{\delta_{\mathrm{I}}^2\varrho^2\alpha}{16^{k^2}\cdot 6}n^2p \overset{(11,14)}{\ge} \frac{b_i^4a_i^{36k^2+8k}}{6^5\cdot 2^{109k^4+26k^3+4k^2}}np^2 \overset{(4)}{\ge} 2b_{i+1}n^2p$$

while, since $i+1 \ge 2$,

$$n + 2k^2 + 1 \le n + n_{i+1} \le 2n \overset{(4)}{\le} b_{i+1}C_{i+1}n \overset{(12)}{\le} b_{i+1}n^2p\,.$$

Consequently,

$$q_{\mathrm{I}} \le \exp(-2b_{i+1}n^2p + b_{i+1}n^2p) = \exp(-b_{i+1}n^2p)\,.$$

Now we derive the same upper bound for q_{II}. Since

$$p_{\mathrm{I}} \overset{(10)}{\le} \alpha p \overset{(8)}{=} \frac{\delta_{\mathrm{II}}^2\gamma}{36}p$$

while $p_{\mathrm{II}} \overset{(10)}{\ge} p/2$,

$$q_{\mathrm{II}} = \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}} + n^2p_{\mathrm{I}}\right) \le \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{36}n^2p\right)\,.$$

Therefore, the required bound follows from

$$\frac{\delta_{\mathrm{II}}^2\gamma}{36} \overset{(8)}{=} \frac{\gamma^9}{36\cdot 81\cdot 16^{2k^2}} \overset{(11)}{\ge} \frac{a_i^{36k^2}}{2^{118k^4}} \overset{(4)}{\ge} b_{i+1}\,. \quad \square$$

This concludes the proof of the inductive step, i.e., the proof of Theorem 2 for F_{i+1}, given it is true for F_i, $i = 1, \dots, \binom{k}{2} - 1$. The proof of Theorem 2 is thus completed.

4 Proof of Corolary 3

In order to deduce Corollary 3 from Theorem 2, we first need to estimate the parameters a_i, b_i, C_i, n_i, $i = 1, \ldots, \binom{k}{2}$, defined recursively in (4).

Proposition 13. *There exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that for every $k \geq 3$*

$$a_{K_k} \geq 2^{-k^{(c_1 \cdot k^2)}} \qquad b_{K_k} \geq 2^{-k^{(c_2 \cdot k^2)}} \qquad C_{K_k} \leq 2^{k^{(c_3 \cdot k^2)}} \qquad n_{K_k} \leq 2^{k^{(c_4 \cdot k^2)}}.$$

Proof. Throughout the proof we assume that $k \geq k_0$ for some sufficiently large constant k_0. Let $x = 19k^4$, $y = 55k^6$, and set $\alpha_i = \log a_i$, $i = 1, \ldots, \binom{k}{2}$. Recall that $a_1 = \frac{1}{2}$. The recurrence relation (4) becomes now

$$\alpha_i = x\alpha_{i-1} - y,$$

whose solution can be easily found as

$$\alpha_i = -x^{i-1} - y\frac{x^{i-1} - 1}{x - 1}$$

(note that $\alpha_1 = -1$). Hence, for all $i = 1, \ldots, \binom{k}{2}$, and some constant $c_1 > 0$,

$$-\alpha_i = x^{i-1} + y\frac{x^{i-1} - 1}{x - 1} \leq k^{c_1 \cdot i}. \tag{42}$$

In particular,

$$a_{\binom{k}{2}} \geq 2^{-k^{c_1 \cdot \binom{k}{2}}} \geq 2^{-k^{(c_1 \cdot k^2)}}.$$

The recurrence relation for the b_i's is more complex. With $u = 37k^2$ and $v = 118k^4$, it reads as

$$b_i = b_{i-1}^4 a_{i-1}^u 2^{-v}.$$

Thus, recalling that $b_1 = \frac{1}{8}$,

$$b_i 8^{4^{i-1}} = \prod_{j=2}^{i} \left(\frac{b_j}{b_{j-1}^4}\right)^{4^{i-j}} = \prod_{j=2}^{i} \left(a_j^u 2^{-v}\right)^{4^{i-j}}.$$

Setting, $\beta_i = \log b_i$, and taking logarithms of both sides and using (42) we obtain, for some constant $c_2 > 0$,

$$\begin{aligned} -\beta_i = 3 \cdot 4^{i-1} + \sum_{j=2}^{i} 4^{i-j}\left(u(-\alpha_j) + v\right) &\leq 4^i + (i-1)4^{i-2}\left(u(-\alpha_i) + v\right) \\ &\leq 4^i\left[1 + i\left(uk^{(c_1 \cdot i)} + v\right)\right] \leq k^{c_2 \cdot i}, \end{aligned} \tag{43}$$

where in the last step above we used estimates $4^i \leq k^{2i}$ and $i \leq k^2$. In particular,

$$b_{\binom{k}{2}} \geq 2^{-k^{(c_2 \cdot k^2)}}.$$

The recurrence relation for C_i involves not only C_{i-1} and a_{i-1} but also b_{i-1}. Nevertheless, its solution follows the steps of that for b_i. Indeed, we have

$$\frac{C_i}{C_{i-1}} = \frac{2^z}{b_{i-1}^4 a_{i-1}^w},$$

where $z = 122k^4$ and $w = 37k^2$. Recalling that $C_1 = 1$,

$$C_i = \prod_{j=2}^{i} \frac{C_j}{C_{j-1}} = \prod_{j=2}^{i} \frac{2^z}{b_{j-1}^4 a_{j-1}^w}$$

and, consequently, by (42) and (43), for some constant $c_3 > 0$,

$$\begin{aligned}\log C_i \le (i-1)z + \sum_{j=2}^{i} (4(-\beta_j) + w(-\alpha_j)) &\le (i-1)\left(z + 4(-\beta_i) + w(-\alpha_i)\right)\\ &\le k^2 \left(z + 4k^{(c_2\cdot i)} + wk^{(c_1\cdot i)}\right) \le k^{c_3\cdot i}.\end{aligned}$$

In particular,

$$C_{\binom{k}{2}} \le 2^{k^{(c_3\cdot k^2)}}.$$

Similarly, for some constant $c_4 > 0$,

$$n_i = \prod_{j=2}^{i} \frac{n_j}{n_{j-1}} = \prod_{j=2}^{i} \frac{2^{14k^3}}{a_{j-1}^{4k}} \le 2^{k^{(c_4\cdot i)}}$$

and, consequently,

$$n_{\binom{k}{2}} \le 2^{k^{(c_4\cdot k^2)}}. \quad \square$$

We are going to prove Corollary 3 by the probabilistic method. We will show that for some $c > 0$, every $n \ge 2^{k^{c\cdot k^2}}$, and a suitable function $p = p(n)$, with positive probability, $G(n,p)$ has simultaneously two properties: $G(n,p) \to K_k$ and $G(n,p) \not\supset K_{k+1}$. The following simple lower bound on $\mathbb{P}(G(n,p) \not\supset K_{k+1})$ has been already proved in [6] (see lemma 3 therein). For the sake of completeness we reproduce that short proof here.

Lemma 14. *For all $k, n \ge 3$ and $C > 0$, if $p = Cn^{-2/(k+1)} \le \frac{1}{2}$ then*

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp(-C^{\binom{k+1}{2}} n)\,.$$

Proof. By applying the FKG inequality (see, e.g., [3, Theorem 2.12 and Corollary 2.13], we obtain the bound

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) \ge \left(1 - p^{\binom{k+1}{2}}\right)^{\binom{n}{k+1}} \ge \exp\left(-2C^{\binom{k+1}{2}} n^{-k}\binom{n}{k+1}\right) > \exp\left(-C^{\binom{k+1}{2}} n\right),$$

where we used the inequalities $\binom{n}{k+1} < n^{k+1}/2$ and $1 - x \ge e^{-2x}$ for $0 < x < \frac{1}{2}$. $\square$

Now, we are ready to complete the proof of Corollary 3. For convenience, set $\bar{b} = b_{\binom{k}{2}}$, $\bar{C} = C_{\binom{k}{2}}$, and $\bar{n} = n_{\binom{k}{2}}$. Let $n \geq \bar{n}$ and $p = \bar{C}n^{-2/(k+1)}$. By Theorem 2,

$$\mathbb{P}(G(n,p) \to K_k) \geq 1 - \exp\left\{-\bar{b}p\binom{n}{2}\right\}.$$

Let, in addition, $n \geq (2\bar{C})^{(k+1)/2}$. Then, by Lemma 14,

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left\{-\bar{b}p\binom{n}{2}\right\}$$

and, in turn,

$$\mathbb{P}(G(n,p) \to K_k \text{ and } G(n,p) \not\supset K_{k+1}) > 0.$$

Consequently, for every

$$n \geq n_0 := \max(\bar{n}, (2\bar{C})^{(k+1)/2})$$

there exists a graph G with n vertices such that $G \to K_k$ but $G \not\supset K_{k+1}$. Finally, by Proposition 13, there exists $c > 0$ such that $n_0 \leq 2^{k^{c \cdot k^2}}$. This way we have proved that $f(k) \leq n_0 \leq 2^{k^{c \cdot k^2}}$.

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