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An algorithmic hypergraph regularity lemma

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AN ALGORITHMIC HYPERGRAPH REGULARITY LEMMA

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Abstract. Szemerédi's Regularity Lemma is a powerful tools in graph theory. It asserts that all large graphs admit bounded partitions of their edge sets, most classes of which consist of uniformly distributed edges. The original proof of this result was non-constructive and a constructive proof was later given by Alon, Duke, Lefmann, Rödl and Yuster.

Szemerédi's Regularity Lemma was extended to hypergraphs by various authors. Frankl and Rödl gave one such extension in the case of 3-uniform hypergraphs, which was later extended to k-uniform hypergraphs by Rödl and Skokan. W.T. Gowers gave another such extension, using a different concept of regularity than that of Frankl, Rödl and Skokan. In this paper, we give a constructive proof of the Regularity Lemma for hypergraphs.

1. Introduction

Szemerédi's Regularity Lemma [21, 22] is an important tool in combinatorics, with applications ranging across combinatorial number theory, extremal graph theory, and theoretical computer science (see [9, 10] for surveys of applications). The Regularity Lemma hinges on the notion of ε-regularity: a bipartite graph $H = (X \cup Y, E)$ is *ε-regular* if for every $X' \subseteq X$, $|X'| > \varepsilon|X|$, and for every $Y' \subseteq Y$, $|Y'| > \varepsilon|Y|$, we have $|d_H(X', Y') - d_H(X, Y)| < \varepsilon$, where $d_H(X', Y') = |H[X', Y']|/(|X'||Y'|)$ is the *density* of the bipartite graph $H[X', Y']$ induced on the sets X' and Y'. Szemerédi's Regularity Lemma is then stated as follows.

Theorem 1.1 (Szemerédi's Regularity Lemma [21, 22]). *For all $\varepsilon > 0$ and integers $t_0 \geq 1$, there exist integers $T_0 = T_0(\varepsilon, t_0)$ and $N_0 = N_0(\varepsilon, t_0)$ so that every graph G on $N > n_0$ vertices admits a partition of its vertex set $V(G) = V_1 \cup \cdots \cup V_t$, $t_0 \leq t \leq T_0$, satisfying*

(1) *$V(G) = V_1 \cup \cdots \cup V_t$ is equitable: $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$;*
(2) *$V(G) = V_1 \cup \cdots \cup V_t$ is ε-regular: all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j, $1 \leq i < j \leq t$, are ε-regular.*

A constructive proof of Theorem 1.1 was later given by Alon, Duke, Lefmann, Rödl and Yuster. Their result gives that the ε-regular partition $V(G) = V_1 \cup \cdots \cup V_t$ in Theorem 1.1 can be constructed in time $O(M(n))$, where $M(n) = O(n^{2.3727})$ is the time needed to multiply two $n \times n$ matrices with 0, 1-entries over the integers (see [23]). In [8], the running time of $O(M(n))$ was improved to $O(n^2)$.

Szemerédi's Regularity Lemma has been extended to k-uniform hypergraphs, for $k \geq 2$, by various authors. Frankl and Rödl [3] gave one such extension to the case of 3-uniform hypergraphs, using a concept they called *(δ, r)-regularity* (see upcoming Definition 2.12). This regularity lemma was extended to k-uniform hypergraphs, for arbitrary $k \geq 3$, by Rödl and Skokan [17]. Gowers [4, 5] also established a regularity lemma for k-uniform hypergraphs, but used a concept of regularity known as *deviation* (see upcoming Definition 2.6). While the concepts of (δ, r)-regularity and deviation are different, the corresponding Regularity Lemmas have a similar conclusion. Roughly speaking, both lemmas guarantee that every (large) k-uniform hypergraph admits a bounded partition of its edge set (where the edge-partition is defined in a fairly technical way), where most classes of the partition consist of 'regularly distributed' edges. Moreover, both Regularity Lemmas admit a corresponding *Counting Lemma* (see upcoming Theorems 5.1 and 5.2, and see also [12]). The Counting Lemma allows one to estimate the number of fixed subhypergraphs of a given isomorphism type within the 'regular partition' a regularity lemma provides. The combined application of the Regularity and Counting Lemmas is known as the *Regularity Method* for hypergraphs (see [13, 15, 16, 19] for surveys of applications).

The goal of this paper is to establish an algorithmic Hypergraph Regularity Lemma (see upcoming Theorem 3.7). Roughly speaking, we will show that, for every (large) k-uniform hypergraph $\mathcal{H}^{(k)}$, a 'regular partition' of $\mathcal{H}^{(k)}$ can, in fact, be constructed in time polynomial in $|V(\mathcal{H}^{(k)})|$. Thus, combining the work here together with an appropriate Counting Lemma provides an *Algorithmic Regularity Method* for hypergraphs[1].

To prove the algorithmic regularity lemma for hypergraphs, we will proceed along the usual lines. As in the proof of Szemerédi [21, 22] for graphs, we will consider sequences of partitions $\mathscr{P}_i$, $i \geq 1$, of a hypergraph $\mathcal{H}^{(k)}$. For each $\mathscr{P}_i$, $i \geq 1$, we consider the so-called *index* of $\mathscr{P}_i$, denoted $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_i)$, which measures the mean-square density of $\mathcal{H}^{(k)}$ on $\mathscr{P}_i$. When the partition $\mathscr{P}_i$ of $\mathcal{H}^{(k)}$ is irregular, we refine $\mathscr{P}_i$, in the usual way, to produce $\mathscr{P}_{i+1}$. It is well-known that $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{i+1})$ will be non-negligibly larger than $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_i)$, so that this refining process must terminate after constantly many iterations. Now, as in the proof of Alon et al. [1] for graphs, to make the the refinement $\mathscr{P}_{i+1}$ of $\mathscr{P}_i$ constructive, one must be able to *construct* 'witnesses' of the irregularity of $\mathscr{P}_i$. The novel element of our work does precisely this and in Section 2, we state the 'Witness-Construction Theorem' (Theorem 2.16). In Section 3, we state the Algorithmic Regularity Lemma (Theorem 3.7), and in Section 4, we show that Theorem 2.16 implies Theorem 3.7.

The remainder of the paper is devoted to proving Theorem 2.16. For this proof, we will need several technical lemmas. Among these are Gowers' Counting Lemma (see Theorems 5.1 and 5.2), which we present in Section 5. As well, we will need an 'Extension Lemma' (Theorem 5.4), which is a derivative of the Counting Lemma, which we also present in Section 5. Finally, we need an additional lemma, which we call the 'Negative-Extension Lemma' (Theorem 6.2), which we state and prove in Section 6. Using these tools, we prove Theorem 2.16 in Section 7. At the end of the paper, we include an Appendix for the proofs of a few facts we need along the way.

2. Deviation and the Witness-Construction Theorem

In this section, we define the concept of *deviation* (**DEV**) (cf. Definition 2.6), and we present some conditions which are sufficient for implying the property of deviation. We also consider the concept of r-*discrepancy* (r-**DISC**) (cf. Definition 2.12), and present a so-called Witness-Construction theorem (cf. Theorem 2.16). For these purposes, we need some supporting concepts.

2.1. Background concepts: cylinders, complexes and density.

We begin with some basic concepts. For a set X and an integer $j \leq |X|$, let $\binom{X}{j}$ denote the set of all (unordered) j-tuples from X. When $X = [\ell] = \{1, \dots, \ell\}$, we sometimes write $[\ell]^j = \binom{[\ell]}{j}$. Given pairwise disjoint sets $V_1, \dots, V_\ell$, denote by $K^{(j)}(V_1, \dots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph

[1] An algorithmic regularity method for 3-uniform hypergraphs was established by Haxell, Nagle, and Rödl [6, 7] (see also [11]).

with ℓ-partition $V_1 \cup \cdots \cup V_\ell$, which consists of all j-tuples from $V_1 \cup \cdots \cup V_\ell$ meeting each V_a, $1 \leq a \leq j$, at most once. We now define the concept of a 'cylinder'.

Definition 2.1 **(cylinder).** For integers $\ell \geq j \geq 1$, an (ℓ, j)-*cylinder* $\mathcal{H}^{(j)}$ with vertex ℓ-partition $V(\mathcal{H}^{(j)}) = V_1 \cup \cdots \cup V_\ell$ is any subset of $K^{(j)}(V_1, \ldots, V_\ell)$. When $|V_1| = \cdots = |V_\ell| = m$, we say $\mathcal{H}^{(j)}$ is an (m, ℓ, j)-*cylinder.*

In the context of Definition 2.1, fix $j \leq i \leq \ell$ and $\Lambda_i \in [\ell]^i$. We denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda \in \Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (ℓ, j)-cylinder $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda \in \Lambda_i} V_\lambda$. In this setting, $\mathcal{H}^{(j)}[\Lambda_i]$ is an (i, j)-cylinder.

We now prepare to define the concept of a complex. For an integer $i \geq j$, let $\mathcal{K}_i(\mathcal{H}^{(j)})$ denote the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$. Given an $(\ell, j-1)$-cylinder $\mathcal{H}^{(j-1)}$ and an (ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we say $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

Definition 2.2 **(complex).** For integers $1 \leq k \leq \ell$, an (ℓ, k)-*complex* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ is a collection of (ℓ, j)-cylinders, $1 \leq j \leq k$, so that

1. $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_\ell$ is an $(\ell, 1)$-cylinder, i.e., is an ℓ-partition;
2. for each $2 \leq j \leq k$, we have that $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$, i.e., $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

In some cases, we use the notation $\boldsymbol{\mathcal{H}}^{(k)}$ to denote an (ℓ, k)-complex $\{\mathcal{H}^{(j)}\}_{j=1}^k$.

We now define concept of density.

Definition 2.3 **(density).** For integers $2 \leq j \leq \ell$, let $\mathcal{H}^{(j)}$ be an (ℓ, j)-cylinder and let $\mathcal{H}^{(j-1)}$ be an $(\ell, j-1)$-cylinder. If $\mathcal{K}_j(\mathcal{H}^{(j-1)}) \neq \varnothing$, we define the *density of* $\mathcal{H}^{(j)}$ *w.r.t.* $\mathcal{H}^{(j-1)}$ as

$$d(\mathcal{H}^{(j)} | \mathcal{H}^{(j-1)}) = \frac{\big|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})\big|}{\big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big|}.$$

If $\mathcal{K}_j(\mathcal{H}^{(j-1)}) = \varnothing$, we define $d(\mathcal{H}^{(j)} | \mathcal{H}^{(j-1)}) = 0$.

2.2. **Deviation, and sufficient conditions thereof.** In this subsection, we define the concept of deviation (**DEV**), and present some conditions which are sufficient for implying the property of deviation. To that end, we need some supporting concepts.

Definition 2.4 **($(\boldsymbol{\ell}, \boldsymbol{j})$-octohedron).** Let integers $1 \leq j \leq \ell$ be given. The (ℓ, j)-*octohedron* $\mathcal{O}^{(j)} = \mathcal{O}_\ell^{(j)}$ is the complete ℓ-partite j-uniform hypergraph $K^{(j)}(U_1, \ldots, U_\ell)$, where $|U_1| = \cdots = |U_\ell| = 2$, i.e., it is the complete $(2, \ell, j)$-cylinder.

For an (ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we are interested in 'labeled partite-embedded' copies of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$.

Definition 2.5 **(labeled partite-embedding).** Let $\mathcal{H}^{(j)}$ be an (ℓ, j)-cylinder, with ℓ-partition $V(\mathcal{H}^{(j)}) = V_1 \cup \cdots \cup V_\ell$, and let $\mathcal{O}^{(j)} = K^{(j)}(U_1, \ldots, U_\ell)$ be the (ℓ, j)-octohedron. A *labeled, partite-embedding* of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$ is an edge-preserving injection $\psi : U_1 \cup \cdots \cup U_\ell \to V_1 \cup \cdots \cup V_\ell$ so that $\psi(U_i) \subseteq V_i$ for each $1 \leq i \leq \ell$. We write $\mathrm{EMB}(\mathcal{O}^{(j)}, \mathcal{H}^{(j)})$ to denote the family of all labeled partite-embeddings ψ of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$.

We now define the concept of deviation.

Definition 2.6 (deviation (DEV)). Let $\mathcal{H}^{(j)}$ be a (j,j)-cylinder with underlying $(j,j-1)$-cylinder $\mathcal{H}^{(j-1)}$. Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ have common vertex j-partition $V(\mathcal{H}^{(j)}) = V(\mathcal{H}^{(j-1)}) = V_1 \cup \dots \cup V_j$, and let $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. For $\delta > 0$, we say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has (d,δ)-*deviation*, written $\mathbf{DEV}(d,\delta)$, if

$$\sum_{v_1,v_1'\in V_1} \cdots \sum_{v_j,v_j'\in V_j} \prod \Big\{ \omega(J) : J \in K^{(j)}\left(\{v_1,v_1'\},\dots,\{v_j,v_j'\}\right) \Big\} \le \delta \big|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\big|,$$

where for every $v_1, v_1' \in V_1, \dots, v_j, v_j' \in V_j$, and for each $J \in K^{(j)}\Big(\{v_1,v_1'\},\dots,\{v_j,v_j'\}\Big)$,

$$\omega(J) = \begin{cases} 1-d & \text{if } J \in \mathcal{H}^{(j)}, \\ -d & \text{if } J \in \mathcal{K}_j(\mathcal{H}^{(j-1)}) \setminus \mathcal{H}^{(j)}, \\ 0 & \text{if } J \notin \mathcal{K}_j(\mathcal{H}^{(j-1)}). \end{cases}$$

It is easy to extend Definition 2.6 from (j,j)-cylinders to (ℓ,k)-complexes.

Definition 2.7. Let $\boldsymbol{\delta} = (\delta_2,\dots,\delta_k)$ and $\boldsymbol{d} = (d_{\Lambda_j} : \Lambda_j \in [\ell]^j, 2 \le j \le k)$ be sequences of positive reals, and let (ℓ,k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be given. We say the complex $\boldsymbol{\mathcal{H}}$ has $\mathbf{DEV}(\boldsymbol{d},\boldsymbol{\delta})$ if, for each $2 \le j \le h$ and $\Lambda_j \in [\ell]^j$, $(\mathcal{H}^{(j)}[\Lambda_j], \mathcal{H}^{(j-1)}[\Lambda_j])$ has $\mathbf{DEV}(d_{\Lambda_j}, \delta_j)$.

For future reference, we present some easy sufficient conditions for the property of deviation (cf. Definition 2.6). For that, we need the following generalization of Definition 2.5.

Definition 2.8 (labeled partite-embedding). Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.6, and let $\mathcal{S}^{(j)} \subseteq \mathcal{O}^{(j)} = K^{(j)}(U_1,\dots,U_j)$ be an arbitrary $(2,j,j)$-cylinder. We call an injection $\psi : U_1 \cup \dots \cup U_j \to V_1 \cup \dots \cup V_j$ a *labeled partite-embedding* of $\mathcal{S}^{(j)}$ in $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ if it satisfies the following conditions:

(1) ψ is a labeled partite-embedding of $\mathcal{O}^{(j-1)} = K^{(j-1)}(U_1,\dots,U_j)$ in $\mathcal{H}^{(j-1)}$;
(2) for each $J \in \mathcal{O}^{(j)} = K^{(j)}(U_1,\dots,U_j)$, we have
$$J \in \mathcal{S}^{(j)} \quad \Longrightarrow \quad \psi(J) \in \mathcal{H}^{(j)}.$$

We call ψ *labeled, partite-induced embedding* of $\mathcal{S}^{(j)}$ in $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ if it satisfies (1) and (2) above, together with

(2′) for each $J \in \mathcal{O}^{(j)} = K^{(j)}(U_1,\dots,U_\ell)$, we have
$$J \in \mathcal{S}^{(j)} \quad \Longleftrightarrow \quad \psi(J) \in \mathcal{H}^{(j)}.$$

We write $\mathrm{EMB}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))$ to denote the family of all labeled partite-embeddings ψ of $\mathcal{S}^{(j)}$ in $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$. We write $\mathrm{EMB}_{\mathrm{ind}}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))$ to denote the family of all labeled, partite-induced embeddings ψ of $\mathcal{S}^{(j)}$ in $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$.

We now consider the following two properties.

Definition 2.9 ($\mathbf{COUNT}_{\mathrm{emb}}, \mathbf{COUNT}_{\mathrm{ind}}$). Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.8, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. For $\delta > 0$, we say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\mathrm{emb}}(d,\delta)$ if the following condition holds: for every $(2,j,j)$-cylinder $\varnothing \subseteq \mathcal{S}^{(j)} \subseteq \mathcal{O}^{(j)} = K^{(j)}(U_1,\dots,U_j)$, we have

$$\big|\mathrm{EMB}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\big| = (1 \pm \delta) d^{|\mathcal{S}^{(j)}|} \big|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\big|. \tag{1}$$

(Note that when $\mathcal{S}^{(j)} = \varnothing$, it always holds that

$$\big|\mathrm{EMB}(\varnothing, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\big| = (1 \pm \delta) d^{0} \big|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\big|, \tag{2}$$

since every labeled partite-embedding ψ of $\varnothing$ in $\mathcal{H}^{(j)}$ is, equivalently, a labeled partite-embedding of $\mathcal{O}^{(j-1)}$ in $\mathcal{H}^{(j-1)}$.) We say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\text{ind}}(d, \delta)$ if the following condition holds: for every $(2, j, j)$-cylinder $\varnothing \subseteq \mathcal{S}^{(j)} \subseteq \mathcal{O}^{(j)} = K^{(j)}(U_1, \dots, U_j)$,

$$\left|\text{EMB}_{\text{ind}}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right| = (1 \pm \delta) d^{|\mathcal{S}^{(j)}|}(1-d)^{2^j - |\mathcal{S}^{(j)}|}\left|\text{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\right|.$$

The following fact will be useful later in this paper. The proof is easy, and we give it in the Appendix.

Fact 2.10. *Suppose $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ are given as in Definition 2.9, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)}) > 0$, and let $\delta > 0$ be given. Suppose, moreover, that $|\text{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})| = \Omega(n^{2^j})$, where $|V_i| = \Theta(n)$ for all $i \in [j]$.*

1. *$(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\text{emb}}(d, \delta)$ if, and only if, $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\text{ind}}(d, \delta)$;*
2. *If $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\text{emb}}(d, \delta)$, then $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{DEV}(d, \delta)$.*

2.3. **r-discrepancy, and the Witness-Construction Theorem.** In this subsection, we define the concept of r-discrepancy (r-**DISC**), and present the Witness Construction Theorem (cf. Theorem 2.16). We begin with the following extension of the concept of density (cf. Definition 2.3).

Definition 2.11 (***r*-density**). Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.3, and let integer $r \geq 1$ be given. Let $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfy $\bigcup_{i \in [r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)}) \neq \varnothing$. We define the *$r$-density of $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}$* as

$$d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) = \frac{\left|\mathcal{H}^{(j)} \cap \bigcup_{i \in [r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\right|}{\left|\bigcup_{i \in [r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\right|}.$$

We now define the concept of r-discrepancy.

Definition 2.12 (***r*-discrepancy (*r*-DISC)**). Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.3, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. For $\delta > 0$ and an integer $r \geq 1$, we say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has *(d, δ, r)-discrepancy*, written $\mathbf{DISC}(d, \delta, r)$, if for any collection $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$,

$$\left|\bigcup_{i \in [r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\right| > \delta |\mathcal{K}_j(\mathcal{H}^{(j-1)})| \quad \Longrightarrow \quad \left|d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) - d\right| < \delta. \tag{3}$$

For brevity, we sometimes refer to (d, δ, r)-discrepancy as *r-discrepancy*, and sometimes write $\mathbf{DISC}(d, \delta, r)$ as r-**DISC**.

We proceed with the following remark.

Remark 2.13. Note that 1-discrepancy is usually referred to as *discrepancy*, and 1-**DISC** is usually denoted by **DISC** (cf. [11]). □

We will also need the following concept, related to Definition 2.12.

Definition 2.14 (***r*-witness**). Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.12, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. Suppose that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ does not have $\mathbf{DISC}(d, \delta, r)$, for some $\delta > 0$ and integer $r \geq 1$. We call any collection $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ for which

$$\left|\bigcup_{i \in [r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\right| > \delta |\mathcal{K}_j(\mathcal{H}^{(j-1)})| \quad \text{but} \quad \left|d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) - d\right| \geq \delta.$$

an *r-witness* of $\neg\,\mathbf{DISC}(d, \delta, r)$.

We finally present the Witness-Construction Theorem, which concerns a (k,k)-complex $\boldsymbol{\mathcal{H}}$ satisfying the following setup.

Setup 2.15. Let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{(k)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ be a (k,k)-complex, where $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_k$ has $n \le |V_i| \le n+1$ for all $i \in [k]$. Let

$$\boldsymbol{d}_k = \big(d_{\Lambda_j} : \Lambda_j \in [k]^j, 2 \le j \le k\big)$$

satisfy that, for each $2 \le j \le k$ and for each $\Lambda_j \in [k]^j$,

$$d_{\Lambda_j} = d(\mathcal{H}^{(j)}[\Lambda_j] | \mathcal{H}^{(j-1)}[\Lambda_j]).$$

Note, in particular, that $d_{[k]} = d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)})$. We call $\boldsymbol{d}_k$ the *density sequence* for $\boldsymbol{\mathcal{H}}^{(k)}$. Write

$$\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \quad \text{and} \quad \boldsymbol{d}_{k-1} = \big(d_{\Lambda_j} : \Lambda_j \in [k]^j, 2 \le j \le k-1\big),$$

so that $\boldsymbol{d}_{k-1}$ is the density sequence for $\boldsymbol{\mathcal{H}}^{(k-1)}$.

The Witness-Construction Theorem is now given as follows.

Theorem 2.16 (Witness-Construction Theorem). *Let integer $k \ge 2$ be fixed. For all $d_k, \delta_k > 0$, there exists $\delta'_k > 0$ so that for all $d_{k-1} > 0$, there exists $\delta_{k-1} > 0$ so that, ..., for all $d_2 > 0$, there exist $\delta_2 > 0$, positive integer r_0, and positive integer n_0 so that the following holds.*

Set $\boldsymbol{\delta}_{k-1} = (\delta_2, \ldots, \delta_{k-1})$. Let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{(k)}$ be a (k,k)-complex with density sequence $\boldsymbol{d}_k$, as given as in Setup 2.15, where $n \ge n_0$. Suppose $\boldsymbol{d}_k$ satisfies that, for each $2 \le j \le k$ and for each $\Lambda_j \in [k]^j$, $d_{\Lambda_j} \ge d_j$. Assume that

1. *$\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but that*
2. *$(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$.*

Then, there exists an algorithm which constructs, in time $O(n^{3k})$, an r-witness $\mathcal{Q}_1^{(k-1)}, \ldots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$ of $\neg\,\mathbf{DISC}(d_{[k]}, \delta'_k, r)$, for some $r \le r_0$.

3. Algorithmic Hypergraph Regularity Lemma

In this section, we state an Algorithmic Hypergraph Regularity Lemma (see Theorem 3.7, below) for the property of deviation. To state this lemma, we still need some more concepts.

3.1. Families of partitions.

Theorem 3.7 provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, ..., and $(k-1)$-tuples of a given vertex set. We will define the properties of $\mathscr{P}$ in upcoming Definitions 3.1 and 3.2, but we first need to establish some notation and concepts.

We first discuss the structure of these partitions inductively, following the approach of [12]. Let k be a fixed integer and V be a set of vertices. Let $\mathscr{P}^{(1)} = \{V_1, \ldots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \le j \le |\mathscr{P}^{(1)}|$, let $\mathrm{Cross}_j(\mathscr{P}^{(1)}) = K^{(j)}(V_1, \ldots, V_{|\mathscr{P}^{(1)}|})$ be the family of all crossing j-tuples J, i.e., the set of j-tuples which satisfy $|J \cap V_i| \le 1$ for every $1 \le i \le |\mathscr{P}^{(1)}|$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ have been defined for all $1 \le i \le j-1$. Then for every $I \in \mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, there exists a unique class $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. For every $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$, we define the *polyad of J* by $\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup\big\{\mathcal{P}^{(j-1)}(I)\colon I \in [J]^{j-1}\big\}$. Define the family of all polyads $\hat{\mathscr{P}}^{(j-1)} = \big\{\hat{\mathcal{P}}^{(j-1)}(J)\colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\big\}$, which we view as a set (as opposed to a multiset, since $\hat{\mathcal{P}}^{(j-1)}(J) = \hat{\mathcal{P}}^{(j-1)}(J')$

may hold for $J \neq J'$). To simplify notation, we often write the elements of $\hat{\mathscr{P}}^{(j-1)}$ as $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ (dropping the argument J).

Observe that $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}, \tag{4}$$

where '$\prec$' denotes the refinement relation of set partitions. Note that (4) inductively implies that

$$\boldsymbol{\mathcal{P}}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}, \text{ where } \hat{\mathcal{P}}^{(i)}(J) = \bigcup\{\mathcal{P}^{(i)}(I)\colon I \in [J]^i\}, \tag{5}$$

is a $(j, j-1)$-complex (since each $\hat{\mathcal{P}}^{(i)}(J)$ is a (j,i)-cylinder). We may now give Definitions 3.1 and 3.2.

Definition 3.1 (**$\boldsymbol{a}$-family of partitions**). Let V be a set of vertices, and let $k \geq 2$ be a fixed integer. Let $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ be a sequence of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ is an *$\boldsymbol{a}$-family of partitions on V*, if it satisfies the following:

- (a) $\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,
- (b) $\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ refining $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ where, for every $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, $|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}\colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}| = a_j$.

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is *t-bounded*, if $\max\{a_1, \ldots, a_{k-1}\} \leq t$.

3.2. **Properties of families of partitions.** In this subsection, we describe some properties we would like an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ to have.

Definition 3.2 (**$(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family**). Let V be a set vertices, let $\eta > 0$ be fixed, and let $k \geq 2$ be a fixed integer. Let $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$ and $\boldsymbol{D} = (D_2, \ldots, D_{k-1})$ be sequences of positives, and let $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ be a sequence of positive integers.

We say an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V is an *$(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family* if it satisfies the following conditions:

- (a) $\mathscr{P}^{(1)} = \{V_i\colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $\lfloor |V|/a_1 \rfloor \leq |V_i| \leq \lceil |V|/a_1 \rceil$ for $i \in [a_1]$;
- (b) $\big|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\big| \leq \eta |V|^k$;
- (c) all but $\eta |V|^k$ many k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ satisfy that for each $2 \leq j \leq k-1$, and for each $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(d_J, \delta_j)$, where $d_J = d(\mathcal{P}^{(j)}(J) | \hat{\mathcal{P}}^{(j-1)}(J)) \geq D_j$.

Note that in an $(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family of partitions $\mathscr{P}$ on V, properties (b) and (c) above imply that all but $2\eta|V|^k$ many k-tuples $K \in [V]^k$ belong to $\mathrm{Cross}_k(\mathscr{P}^{(1)})$ and satisfy that, for each $2 \leq j \leq k-1$, and for each $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(d_J, \delta_j)$, where $d_J = d(\mathcal{P}^{(j)}(J) | \hat{\mathcal{P}}^{(j-1)}(J)) \geq D_j$.

For future reference, we also define the following concept, related to property (c) in Definition 3.2.

Definition 3.3 (**$(\boldsymbol{\delta}, \geq\boldsymbol{D})$-typical polyad**). Suppose $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is an $(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family of partitions on a vertex set V, where $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$ and $\boldsymbol{D} = (D_2, \ldots, D_{k-1})$. We say a polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ is *$(\boldsymbol{\delta}, \geq\boldsymbol{D})$-typical* if

- (a) $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing$, and fixing any $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, if

(b) the corresponding $(k, k-1)$-complex $\boldsymbol{\mathcal{P}}(K)$ (cf. (5)) satisfies that, for each $2 \le j \le k-1$, and for each $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(d_J, \delta_j)$, where $d_J = d(\mathcal{P}^{(j)}(J) | \hat{\mathcal{P}}^{(j-1)}(J)) \ge D_j$.

Remark 3.4. Note that property (c) of Definition 3.2 can be re-written as

$$\sum \left\{ \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ is not } (\boldsymbol{\delta}, \ge\!\boldsymbol{D})\text{-typical} \right\} \le \eta |V|^k.$$

□

Note that in an $(\eta, \boldsymbol{\delta}, \ge\!\boldsymbol{D}, \boldsymbol{a})$-family $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ (cf. Definition 3.2), the vertices, pairs, ..., and $(k-1)$-tuples of V are under regular control. The following definition describes how the family $\mathscr{P}$ will control the edges of a hypergraph $\mathcal{H}^{(k)}$, where $V = V(\mathcal{H}^{(k)})$.

Definition 3.5 (**$(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$**)**.** Let $\delta_k > 0$ be given. For a k-graph $\mathcal{H}^{(k)}$ and an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V = V(\mathcal{H}^{(k)})$, we say $(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$ if

$$\Big| \bigcup \big\{ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ satisfies that } (\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)}) \text{ does not have } \mathbf{DEV}\big(d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}), \delta_k\big) \big\} \Big| \le \delta_k |V|^k.$$

Before we state the algorithmic hypergraph regularity lemma, we say a word about some notation we use in it.

Remark 3.6. Let $\boldsymbol{D} = (D_2, \dots, D_{k-1}) \in (0,1]^{k-1}$ be a sequence, and for each $2 \le i \le k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function (of $k-i$ many $(0,1]$ variables), where we write $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$. We shall use the notation

$$\boldsymbol{\delta}(\boldsymbol{D}) = (\delta_i(D_i, \dots, D_{k-1}) : 2 \le i \le k-1)$$

to denote the sequence of function values whose i^{th} coordinate, $2 \le i \le k-1$, is $\delta_i(D_i, \dots, D_{k-1})$. We consider this concept since, in most applications of Theorem 3.7, one needs the value δ_i to be sufficiently small not only w.r.t. D_i, but also $D_{i+1}, \dots, D_{k-1}$. □

We now state the algorithmic hypergraph regularity lemma.

Theorem 3.7 (**Algorithmic Hypergraph Regularity Lemma**)**.** *Let $k \ge 2$ be a fixed integer, and let $\eta, \delta_k > 0$ be fixed positives. For each $2 \le i \le k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function, and set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$. Then, there exist $t, n_0 \in \mathbb{N}$ so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \ge n_0$, one may construct, in time $O(n^{3k})$, a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ of $V(\mathcal{H}^{(k)})$ with the following properties:

(i) *$\mathscr{P}$ is a t-bounded $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \ge\!\boldsymbol{D}, \boldsymbol{a}^{\mathscr{P}})$-family on $V(\mathcal{H}^{(k)})$ (cf. Remark 3.6);*

(ii) *$(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$.*

We proceed with the following remark.

Remark 3.8. Similarly as in Szemerédi [21, 22] for graphs, it is well-known that one can prove a hypergraph regularity lemma which 'regularizes' not one, but multiple hypergraphs $\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}$ (on a common vertex set V) simultaneously. More precisely, in the context of Theorem 3.7, the t-bounded $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \ge\!\boldsymbol{D}, \boldsymbol{a}^{\mathscr{P}})$-family above will satisfy that, for each $1 \le i \le s$, the pair $(\mathcal{H}_i^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$, where $t = t(s, k, \eta, \delta_k, \boldsymbol{\delta})$ and $|V| \ge n_0 = n_0(s, k, \eta, \delta_k, \boldsymbol{\delta})$.

We shall prove Theorem 3.7 by induction on $k \ge 2$. To avoid formalism, we shall be proving the case $s = 1$, but our induction hypothesis will assume the general case. □

4. Proof of Theorem 3.7

The proof of Theorem 3.7 is by induction on $k \geq 2$. The induction begins with $k = 2$ as a known base case. Indeed, Alon et al. [1] proved an algorithmic version of the Szemerédi Regularity Lemma, which is Theorem 3.7 ($k = 2$) with **DEV** replaced by **DISC**. Gowers [4, 5] proved that **DEV** and **DISC** are equivalent properties when $k = 2$, and so the base case of Theorem 3.7 holds. We assume Theorem 3.7 holds through $k - 1 \geq 2$, and prove it for $k \geq 3$. To that end, we need a few supporting considerations.

4.1. **Supporting material.** Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n$. Let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be an $\boldsymbol{a}$-family of partitions on V. We define the *index* of $\mathscr{P}$ w.r.t. $\mathcal{H}^{(k)}$ as

$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}) = \frac{1}{n^k} \sum \left\{ d^2(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}) \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \right\}.$$

Clearly,

$$0 \leq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}) \leq 1. \tag{6}$$

The proof of Theorem 3.7 is similar to that of Szemerédi [21, 22], where we will use the following so-called *Index-pumping Lemma* (Lemma 4.1 below). To introduce this lemma, let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n$. Since this proof is by induction on k, suppose we already have a 'regular partition' $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ of V up through $k - 1$. More precisely,

- let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be an arbitrary t-bounded, $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \geq\boldsymbol{D}, \boldsymbol{a})$-family on V.

We now test how $\mathcal{H}^{(k)}$ behaves on $\mathscr{P}$. In particular, we test whether $(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$, which we may do in time $O(n^{2k})$. Indeed,

- for each polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, we test (by using Definition 2.6) whether or not $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ has $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$, where $d_{\hat{\mathcal{P}}^{(k-1)}} = d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)})$.

We arrive at two cases.

Case 1. Suppose we find that most polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ satisfy that $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ has $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$. Then we stop, and $\mathscr{P}$ is the partition we seek in Theorem 3.7.

Case 2. Suppose we find many polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ for which the pair $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ fails to have $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$. Then, for each such $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$,

- Theorem 2.16 builds (in time $O(n^{3k})$) an $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}} = \{\mathcal{Q}^{(k-1)}_1, \ldots, \mathcal{Q}^{(k-1)}_{r_{\hat{\mathcal{P}}^{(k-1)}}}\}$ of $\neg\,\mathbf{DISC}(d_{\hat{\mathcal{P}}^{(k-1)}}, \widetilde{\delta}_k, r_{\hat{\mathcal{P}}^{(k-1)}})$,

where $\widetilde{\delta}_k = \widetilde{\delta}_k(\delta_k) > 0$ depends on δ_k, and where $r_{\hat{\mathcal{P}}^{(k-1)}} \leq r(\boldsymbol{D})$, where $r(\boldsymbol{D})$ depends on $\boldsymbol{D}$. Now,

- Lemma 4.1 (below) constructs, in time $O(n^{k-1})$, a new partition $\mathscr{P}'$ from $\mathscr{P}$ and the witnesses $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$, over those polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ failing to have $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$, where
$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\hat{\mathscr{P}}') \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}) + \frac{\widetilde{\delta}_k^4}{2}.$$

We now state the Index-pumping Lemma precisely.

Lemma 4.1 (Index-pumping Lemma). *Fix an integer $k \geq 2$, and let $\nu, \widetilde{\delta}_k > 0$ be fixed. For each $2 \leq i \leq k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function, where we set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$. Let $r : (0,1]^{k-2} \to \mathbb{N}$ be an arbitrary function. Let $\boldsymbol{D}_{\mathrm{old}} = (D_2^{\mathrm{old}}, \dots, D_{k-1}^{\mathrm{old}}) \in (0,1]^{k-2}$ and $\boldsymbol{a}_{\mathrm{old}} = (a_1^{\mathrm{old}}, \dots, a_{k-1}^{\mathrm{old}}) \in \mathbb{N}^{k-1}$ be fixed. Then, there exist $\boldsymbol{D}_{\mathrm{new}} = (D_2^{\mathrm{new}}, \dots, D_{k-1}^{\mathrm{new}}) \in (0,1]^{k-2}$, $\boldsymbol{a}_{\mathrm{new}} = (a_1^{\mathrm{new}}, \dots, a_{k-1}^{\mathrm{new}}) \in \mathbb{N}^{k-1}$, and $n_0 \in \mathbb{N}$ so that the following holds.*

Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n \geq n_0$. Suppose $\mathscr{P}_{\mathrm{old}} = \mathscr{P}_{\mathrm{old}}(k-1, \boldsymbol{a})$ is a t_{old}-bounded $(\nu, \boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{old}}), \geq\boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}})$-family on V, where $t_{\mathrm{old}} = \max\{a_1^{\mathrm{old}}, \dots, a_{k-1}^{\mathrm{old}}\}$ and where $\boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{old}}) = \big(\delta_i(D_i^{\mathrm{old}}, \dots, D_{k-1}^{\mathrm{old}})\big)_{i=2}^{k-1}$. Suppose that $\hat{\mathscr{P}}_*^{(k-1)} \subseteq \hat{\mathscr{P}}^{(k-1)}$ is a <u>given</u> collection of polyads satisfying the following properties:

(1) *$\forall\, \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_*^{(k-1)}$, one is <u>given</u> an $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$ of $\neg\,$**DISC**$(d_{\hat{\mathcal{P}}^{(k-1)}}, \widetilde{\delta}_k, r_{\hat{\mathcal{P}}^{(k-1)}})$, where $r_{\hat{\mathscr{P}}^{(k-1)}} \leq r(\boldsymbol{D}_{\mathrm{old}}) = r(D_2^{\mathrm{old}}, \dots, D_{k-1}^{\mathrm{old}})$;*

(2)
$$\sum \Big\{ \big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_*^{(k-1)} \Big\} \geq \widetilde{\delta}_k n^k.$$

Then,

(*a*) *there exists a t_{new}-bounded $(\nu, \boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{new}}), \geq\boldsymbol{D}_{\mathrm{new}}, \boldsymbol{a}_{\mathrm{new}})$-family $\mathscr{P}_{\mathrm{new}} = \mathscr{P}_{\mathrm{new}}(k-1, \boldsymbol{a}_{\mathrm{new}})$ on V for which*
$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\mathrm{new}}) \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\mathrm{old}}) + \frac{\widetilde{\delta}_k^4}{2},$$
where $t_{\mathrm{new}} = \max\{a_1^{\mathrm{new}}, \dots, a_{k-1}^{\mathrm{new}}\}$ and where $\boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{new}}) = \big(\delta_i(D_i^{\mathrm{new}}, \dots, D_{k-1}^{\mathrm{new}})\big)_{i=2}^{k-1}$.

(*b*) *Moreover, there exists an algorithm which, in time $O(n^{k-1})$, constructs the partition $\mathscr{P}_{\mathrm{new}}$ above from $\mathscr{P}_{\mathrm{old}}$ and the given collection of witnesses $\{\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}} : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_*^{(k-1)}\}$.*

Lemma 4.1 is essentially given as Lemma 8.3 of [17] and Lemma 6.3 of [5]. The proof of Lemma 4.1 is given in [5, 17], but with no focus to being algorithmic. We shall not give a formal proof of Lemma 4.1, but we will sketch a proof to indicate how its algorithmic part is obtained.

Indeed, the approach in [17] is similar to Szemerédi's [21, 22]. Consider the Venn Diagram of the intersections of the $r_{\hat{\mathcal{P}}^{(k-1)}}$-witnesses $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$, over $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_*^{(k-1)}$. By Statement (1) in the hypothesis of Lemma 4.1, these witnesses are given to us. (In [17], these witnesses are assumed to exist, but here, we will build them with Theorem 2.16.) This Venn diagram has at most
$$2^{|\hat{\mathscr{P}}_*^{(k-1)}| r(\boldsymbol{D}_{\mathrm{old}})}$$
regions (this number is independent of n), where each region is a $(k-1, k-1)$-cylinder. This Venn Diagram defines a refinement $\mathscr{P}'_{\mathrm{old}}$ of $\mathscr{P}_{\mathrm{old}}$, so that $\mathscr{P}'_{\mathrm{old}}$ is itself a partition. The index of $\mathscr{P}'_{\mathrm{old}}$ will be larger than that of $\mathscr{P}_{\mathrm{old}}$ on account of the fact that, in Statement (2), we assumed many k-tuples were lost to polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_*^{(k-1)}$. The $(k-1, k-1)$-cylinders of $\mathscr{P}'_{\mathrm{old}}$ may not have **DEV**(δ_k), so we apply Theorem 3.7 to each (where we assume, by induction on k, that Theorem 3.7 is algorithmic for $k-1$ (cf. Remark 3.8)). This process produces the partition $\mathscr{P}_{\mathrm{new}}$, where it is well-known that, as a refinement of $\mathscr{P}'_{\mathrm{old}}$, we have $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\mathrm{new}}) \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}'_{\mathrm{old}})$. For the formal details of this outline, see [5, 17].

4.2. **Proof of Theorem 3.7.** The proof of Theorem 3.7 was casually revealed when we introduced the Index-pumping Lemma. Here, we proceed with the formal details.

Let $\eta, \delta_k > 0$ be given. For each $2 \leq i \leq k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function, and set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$. We begin our argument by defining some auxiliary parameters.

4.2.1. *Auxiliary parameters for Theorem 3.7.* In all that follows, set

$$d_k = \bar{\delta}_k = \nu = \frac{1}{3}\min\{\delta_k, \eta\} \quad \text{and} \quad t_0 = \lceil 2/\nu \rceil. \tag{7}$$

Let

$$\widetilde{\delta}_k = \delta'_{k,\mathrm{Thm.2.16}}(d_k, \bar{\delta}_k) \tag{8}$$

be the constant guaranteed by the Witness-Construction Theorem (Theorem 2.16). More generally, recall that Theorem 2.16 has the following quantification:

$$\forall\, k, d_k, \delta_k, \exists\, \delta'_k : \forall\, d_{k-1}, \exists\, \delta_{k-1} : \dots \forall\, d_2, \exists\, \delta_2, r_0, n_0 : \dots$$

This means that for each $2 \leq i \leq k-1$, the constant δ_i (which is guaranteed to exist by Theorem 2.16) depends on d_j, for all $i \leq j \leq k-1$ (which were given earlier). In other words, Theorem 2.16 guarantees the existence of the following function

$$\delta_{i,\mathrm{Thm.2.16}}(d_k, x_{k-1}, \dots, x_i) : \{d_k\} \times (0,1]^{k-i-1} \to (0,1) \tag{9}$$

where $x_{k-1} = d_{k-1}, \dots, x_i = d_i \in (0,1]$ are variables. Similarly, with variables $x_{k-1} = d_{k-1}, \dots, x_2 = d_2 \in (0,1]$, let

$$r_0(d_k, x_{k-1}, \dots, x_2) : \{d_k\} \times (0,1]^{k-2} \to \mathbb{N} \tag{10}$$

be the function guaranteed by the Theorem 2.16. We shall assume, w.l.o.g., that for each $2 \leq i \leq k-1$ and for every $x_{k-1}, \dots, x_i \in (0,1]$, we have

$$\delta_i(x_{k-1}, \dots, x_i) \leq \delta_{i,\mathrm{Thm.2.16}}(d_k, x_{k-1}, \dots, x_i). \tag{11}$$

Indeed, for otherwise, we would replace the given function δ_i with the function $\delta_{i,\mathrm{Thm.2.16}}$ and produce a partition $\mathscr{P}$ which is 'more regular' than was sought. In what follows, we set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, and we emphasize that, in what follows,

$$k,\ \nu,\ \widetilde{\delta}_k,\ \boldsymbol{\delta}, \text{ and } r \text{ are } \textit{fixed} \text{ (as a result of (7)–(11))}. \tag{12}$$

It remains to define the promised integer t. Similarly as in the proof of Szemerédi [21, 22], this integer will be determined by an iterative procedure using the Index-pumping Lemma (Lemma 4.1). To that end, recall that Lemma 4.1 has the following quantification:

$$\forall\, k, \nu, \widetilde{\delta}_k, \boldsymbol{\delta}, r, \boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}}, \exists\, \boldsymbol{D}_{\mathrm{new}}, \boldsymbol{a}_{\mathrm{new}}, n_0 : \dots$$

We apply Lemma 4.1 with the fixed choices k, ν, $\widetilde{\delta}_k$, $\boldsymbol{\delta}$, and r from (12) so that Lemma 4.1 defines functions

$$\begin{aligned} \boldsymbol{D}_{\mathrm{new}}(\boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}}) &= \boldsymbol{D}_{\mathrm{new}}(\nu, \widetilde{\delta}_k, \boldsymbol{\delta}, r = r_0, \boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}}) \in (0,1]^{k-2}, \\ \text{and} \quad \boldsymbol{a}_{\mathrm{new}}(\boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}}) &= \boldsymbol{a}_{\mathrm{new}}(\nu, \widetilde{\delta}_k, \boldsymbol{\delta}, r = r_0, \boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}}) \in \mathbb{N}^{k-1}, \end{aligned} \tag{13}$$

where $\boldsymbol{D}_{\mathrm{old}} \in (0,1]^{k-2}$ and $\boldsymbol{a}_{\mathrm{old}} \in \mathbb{N}^{k-1}$ are sequences of variables. (Henceforth, we make the abbreviations $\boldsymbol{D} = \boldsymbol{D}_{\mathrm{new}}$ and $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{new}}$.) Now, we successively define sequences $\boldsymbol{D}^{(i)} \in (0,1]^{k-2}$ and $\boldsymbol{a}^{(i)} \in \mathbb{N}^{k-1}$, as follows. With t_0 given in (7), set

$$\boldsymbol{D}^{(1)} = (d_2 = 1, \dots, d_{k-1} = 1) \quad \text{and} \quad \boldsymbol{a}^{(1)} = (a_1^{(1)} = t_0, a_2^{(1)} = 1, \dots, a_{k-1}^{(1)} = 1). \tag{14}$$

For $i \geq 2$, set (cf. (13))

$$\boldsymbol{D}^{(i)} = \boldsymbol{D}(\boldsymbol{D}^{(i-1)}, \boldsymbol{a}^{(i-1)}) = (d_2^{(i)}, \dots, d_{k-1}^{(i)}),$$
$$\boldsymbol{a}^{(i)} = \boldsymbol{a}(\boldsymbol{D}^{(i-1)}, \boldsymbol{a}^{(i-1)}) = \left(a_1^{(i)}, \dots, a_{k-1}^{(i)}\right),$$
$$\text{and} \quad t_i = \max\left\{a_1^{(i)}, \dots, a_{k-1}^{(i)}\right\} \tag{15}$$

(recall the functions given in (13)). Set (cf. (8))

$$t = \max_{1 \leq i \leq i_{\text{stop}}} t_i, \quad \text{where} \quad i_{\text{stop}} = \left\lfloor \frac{2}{\widetilde{\delta}_k^4} \right\rfloor. \tag{16}$$

This concludes the description of parameters we need to prove Theorem 3.7.

4.2.2. *The argument (algorithm) for Theorem 3.7.* Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where we assume $n = |V|$ is sufficiently large. Our goal is to construct, in time $O(n^{3k})$, a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ of V which is t-bounded (cf. (16)), which is an $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \geq\boldsymbol{D}, \boldsymbol{a})$-family, where $(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$, and where the sequences $\boldsymbol{D}$ and $\boldsymbol{a}$ will be given by $\boldsymbol{D}^{(i)}$ and $\boldsymbol{a}^{(i)}$ (cf. (14) and (15)), resp., for some $1 \leq i \leq i_{\text{stop}}$ (cf. (16)).

To begin, let $V = V_1 \cup \dots \cup V_{t_0}$ (cf. (7)) be a vertex partition satisfying $\lfloor n/t_0 \rfloor \leq |V_i| \leq \lceil n/t_0 \rceil$, for each $1 \leq i \leq t_0$. Let $\mathscr{P}_1 = \{\mathscr{P}_1^{(1)}, \dots, \mathscr{P}_1^{(k-1)}\}$ be an initial family of partitions, where for each $2 \leq j \leq k-1$, the partition $\mathscr{P}_1^{(j)}$ consists of the $\binom{t_0}{j}$ many (j,j)-cylinders $K^{(j)}(V_{i_1}, \dots, V_{i_j})$, where $1 \leq i_1 < \dots < i_j \leq t_0$. Then, $\mathscr{P}_1$ is a t_0-bounded $(\nu, \boldsymbol{\delta}(\boldsymbol{D}^{(1)}), \geq\boldsymbol{D}^{(1)}, \boldsymbol{a}^{(1)})$-family of partitions (cf. (14)). Indeed, all but

$$t_0 \binom{\lceil n/t_0 \rceil}{2} n^{k-2} < \frac{n^k}{t_0} \overset{(7)}{<} \nu n^k$$

many k-tuples $K \in \binom{V}{k}$ belong to $\mathrm{Cross}_k(\mathscr{P}_1^{(1)})$, and every $K \in \mathrm{Cross}_k(\mathscr{P}_1^{(1)})$ satisfies that, for every $2 \leq j \leq k-1$, and for every $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(1,0)$ (cf. Conditions (a)–(c) of Definition 3.2).

For an integer $1 \leq i < i_{\text{stop}}$ (cf. (16)), assume $\mathscr{P}_1, \dots, \mathscr{P}_i$ are constructed families of partitions of V, where

$$\mathscr{P}_i = \mathscr{P}_i(k-1, \boldsymbol{a}_i) \text{ is a } t_i\text{-bounded } (\nu, \boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)}, \boldsymbol{a}^{(i)})\text{-family,} \tag{17}$$

for $\boldsymbol{D}^{(i)}$, $\boldsymbol{a}^{(i)}$ and t_i given in (14)–(15). We proceed with the following Steps 1–4.

Step 1. Identify, in time $O(n^{2k})$, the sets

$$\hat{\mathscr{P}}_{i,\neg\mathbf{DEV}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_i^{(k-1)} : (\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)}) \text{ does not have } \mathbf{DEV}(d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}), \bar{\delta}_k)\right\},$$
$$\hat{\mathscr{P}}_{i,\text{typ}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_i^{(k-1)} : \hat{\mathcal{P}}^{(k-1)} \text{ is } (\boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)})\text{-typical (cf. Definition 3.5)}\right\},$$
$$\hat{\mathscr{P}}_{i,\text{atyp}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_i^{(k-1)} : \hat{\mathcal{P}}^{(k-1)} \text{ is not } (\boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)})\text{-typical}\right\}.$$

Identify, in time $O(n^k)$, the sets (cf. (7))

$$\hat{\mathscr{P}}_{i,\text{dense}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_i^{(k-1)} : d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq d_k\right\}, \quad \hat{\mathscr{P}}_{i,\text{sparse}}^{(k-1)} = \hat{\mathscr{P}}_i^{(k-1)} \setminus \hat{\mathscr{P}}_{i,\text{dense}}^{(k-1)}.$$

Identify, in time $O(1)$, the set

$$\hat{\mathscr{P}}_{i,*}^{(k-1)} = \hat{\mathscr{P}}_{i,\neg\mathbf{DEV}}^{(k-1)} \cap \hat{\mathscr{P}}_{i,\mathrm{typ}}^{(k-1)} \cap \hat{\mathscr{P}}_{i,\mathrm{dense}}^{(k-1)}. \tag{18}$$

(The last identification uses that $|\hat{\mathscr{P}}_{i}^{(k-1)}| = O(1)$.)

Step 2. Compute[2] the sum

$$S_i = \sum \left\{ \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,*}^{(k-1)} \right\}.$$

If $S_i \geq \bar{\delta}_k n^k$ (cf. (7)), we proceed to Step 3. If $S_i < \bar{\delta}_k n^k$, then we stop, and the promised partition is $\mathscr{P} = \mathscr{P}_i$. Indeed, since

$$\hat{\mathscr{P}}_{i,\neg\mathbf{DEV}}^{(k-1)} \subseteq \hat{\mathscr{P}}_{i,*}^{(k-1)} \cup \hat{\mathscr{P}}_{i,\mathrm{atyp}}^{(k-1)} \cup \hat{\mathscr{P}}_{i,\mathrm{sparse}}^{(k-1)},$$

we have (cf. Remark 3.4)

$$\begin{aligned}
&\sum \left\{ \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,\neg\mathbf{DEV}}^{(k-1)} \right\} \\
&\quad \leq S_i + \sum \left\{ \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,\mathrm{atyp}}^{(k-1)} \right\} + \sum \left\{ \left| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \right| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,\mathrm{sparse}}^{(k-1)} \right\} \\
&\qquad\qquad < \bar{\delta}_k n^k + \eta n^k + d_k n^k \overset{(7)}{<} \delta_k n^k,
\end{aligned}$$

so that $\mathscr{P} = \mathscr{P}_i$ has property $\mathbf{DEV}(\delta_k)$ (cf. Definition 3.3). Moreover, since $\mathscr{P}_i$ is a t_i-bounded $(\nu, \boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)}, \boldsymbol{a}^{(i)})$-family, with $\nu < \eta$ (cf. (7)), then it is also an $(\eta, \boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)}, \boldsymbol{a}^{(i)})$-family (cf. (7)), as desired.

Step 3. If $S_i \geq \bar{\delta}_k n^k$, then we will apply Theorem 2.16 to each $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,*}^{(k-1)}$. We first verify that the hypothesis of Theorem 2.16 will be satisfied. To that end, fix $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,*}^{(k-1)}$, and let $\boldsymbol{\mathcal{P}}$ be the corresponding $(k, k-1)$-complex (cf. (5)). In the context of Theorem 2.16, $\boldsymbol{\mathcal{P}}$ plays the role of $\boldsymbol{\mathcal{H}}^{(k-1)}$, and $(\mathcal{H}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})) \cup \boldsymbol{\mathcal{P}}$ plays the role of $\boldsymbol{\mathcal{H}}^{(k)}$. Since

$$\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,*}^{(k-1)} \overset{(18)}{\subseteq} \hat{\mathscr{P}}_{i,\mathrm{typ}}^{(k-1)},$$

we have that $\boldsymbol{\mathcal{P}}$ is $(\boldsymbol{\delta}(\boldsymbol{D}^{(i)}), \geq\boldsymbol{D}^{(i)})$-typical, or in other words (cf. Definition 3.3), $\boldsymbol{\mathcal{P}}$ has $\mathbf{DEV}(\boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}}, \boldsymbol{\delta}(\boldsymbol{D}^{(i)}))$ for some density sequence $\boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}}$ which is coordinate-wise at least $\boldsymbol{D}^{(i)}$. Since

$$\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{i,*}^{(k-1)} \overset{(18)}{\subseteq} \hat{\mathscr{P}}_{i,\neg\mathbf{DEV}}^{(k-1)} \cap \hat{\mathscr{P}}_{i,\mathrm{dense}}^{(k-1)},$$

we have that $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ does not have $\mathbf{DEV}(d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}), \bar{\delta}_k)$, where $d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq d_k$. Moreover, we have chosen the constants d_k, $\bar{\delta}_k$ and $\widetilde{\delta}_k$ (cf. (7) and (8)) and the functions $\boldsymbol{\delta}(\boldsymbol{D}^{(i)})$ and

$$r_0(\boldsymbol{D}^{(i)}) = r_0(d_k, d_{k-1}^{(i)}, \ldots, d_2^{(i)})$$

(cf. (9)–(11)) appropriately for an application of Theorem 2.16. Thus, the hypothesis of Theorem 2.16 is satisfied, and so Theorem 2.16 constructs, in time $O(n^{3k})$, an $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness $\vec{\mathcal{Q}}_{\hat{\mathcal{P}}^{(k-1)}}^{(k-1)}$, given by

$$\mathcal{Q}_{1,\hat{\mathcal{P}}^{(k-1)}}^{(k-1)}, \ldots, \mathcal{Q}_{r_{\hat{\mathcal{P}}^{(k-1)}},\hat{\mathcal{P}}^{(k-1)}}^{(k-1)} \subseteq \hat{\mathcal{P}}^{(k-1)}, \tag{19}$$

[2]Since $S_i = O(n^k)$ has $O(\log n)$ many digits, Step 2 is done in time $O(\log n)$.

of $\neg\,\mathbf{DISC}(d(\mathcal{H}^{(k)},\hat{\mathcal{P}}^{(k-1)}),\widetilde{\delta}_k,r_{\hat{\mathcal{P}}^{(k-1)}})$, where $r_{\hat{\mathcal{P}}^{(k-1)}}\le r_0(\boldsymbol{D}^{(i)})$. Repeat the application of Theorem 2.16 over all $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{i,*}$.

Step 4. If $S_i\ge\bar{\delta}_k n^k$, then we will apply Lemma 4.1 to the family of partitions $\mathscr{P}_i$ and the collection of witnesses $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$, over all $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{i,\#}$. We first verify that the hypothesis of Lemma 4.1 will be satisfied. Indeed, by our induction hypothesis in (17), $\mathscr{P}_i$ is a constructed t_i-bounded $(\nu,\boldsymbol{\delta}(\boldsymbol{D}^{(i)}),\ge\boldsymbol{D}^{(i)},\boldsymbol{a}^{(i)})$ family of partitions. Assumption (1) of Lemma 4.1 is satisfied because the set $\hat{\mathscr{P}}^{(k-1)}_{i,*}$ was constructed in Step 1 (cf. (18)), and for each $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{i,*}$, a corresponding $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$ was constructed in Step 3 (cf. (19)). Assumption (2) of Lemma 4.1 is satisfied because we assume $S_i\ge\bar{\delta}_k n^k$, and so

$$S_i=\sum\left\{\left|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\right|:\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{i,*}\right\}\ge\bar{\delta}_k n^k\overset{(8)}{\ge}\widetilde{\delta}_k n^k.$$

Thus, Lemma 4.1 constructs, in time $O(n^{k-1})$, a t_{i+1}-bounded $(\nu,\boldsymbol{\delta}(\boldsymbol{D}^{(i+1)}),\ge\boldsymbol{D}^{(i+1)},\boldsymbol{a}^{(i+1)})$ family of partitions $\mathscr{P}_{i+1}$, where t_{i+1}, $\boldsymbol{D}^{(i+1)}$, and $\boldsymbol{a}^{(i+1)}$ are given in (15), for which

$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{i+1})\ge\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_i)+\frac{\widetilde{\delta}_k^4}{2}.$$

Return to Step 1 with the newly constructed family $\mathscr{P}_{i+1}$.

From (6), we may repeat Steps 1–4 above at most $i_{\mathrm{stop}}=\lfloor 2/\widetilde{\delta}_k^4\rfloor$ times (cf. (16)), which proves Theorem 3.7.

5. Counting and Extension Lemmas

In this section, we present Counting and Extension Lemmas for regular complexes. All results in this section can be derived, in a standard way, from the following Counting Lemma for cliques due to Gowers [4, 5],

Theorem 5.1 (Clique Counting Lemma, Gowers). *Let integers $\ell\ge k\ge 2$ be fixed. For all $\mu,d_k>0$, there exists $\delta_k>0$ so that for all $d_{k-1}>0$, there exists $\delta_{k-1}>0$ so that, …, for all $d_2>0$, there exists $\delta_2>0$ and positive integer n_0 so that the following holds.*

Set $\boldsymbol{\delta}=(\delta_2,\dots,\delta_k)$, and let $\boldsymbol{d}=(d_{\Lambda_j}:\Lambda_j\in[\ell]^j,2\le j\le\ell)$ be a sequence satisfying, for each $2\le j\le k$, $d_{\Lambda_j}\ge d_j$ for all $\Lambda_j\in[\ell]^j$. Let $\boldsymbol{\mathcal{H}}=\{\mathcal{H}^{(j)}\}_{j=1}^k$ be an (ℓ,k)-complex, where $\mathcal{H}^{(1)}=V_1\cup\dots\cup V_\ell$ has $n_0\le n\le|V_i|\le n+1$ for each $1\le i\le\ell$. If $\boldsymbol{\mathcal{H}}$ has $\mathbf{DEV}(\boldsymbol{d},\boldsymbol{\delta})$, then $\mathcal{H}^{(k)}\in\boldsymbol{\mathcal{H}}$ has

$$\left|\mathcal{K}_\ell(\mathcal{H}^{(k)})\right|=(1\pm\mu)\prod_{j=2}^{k}\prod_{\Lambda_j\in[\ell]^j}d_{\Lambda_j}\times n^\ell$$

many cliques $K^{(k)}_\ell$.

We now present a version of Theorem 5.1 which allows us to count copies of the (ℓ,k)-octohedron $\mathcal{O}^{(k)}=K^{(k)}(U_1,\dots,U_\ell)$, $|U_1|=\dots=|U_\ell|=2$, within an (ℓ,k)-complex $\boldsymbol{\mathcal{H}}$.

Theorem 5.2 (Octohedral Counting Lemma). *Let integers $\ell\ge k\ge 2$ be fixed. For all $\mu,d_k>0$, there exists $\delta_k>0$ so that for all $d_{k-1}>0$, there exists $\delta_{k-1}>0$ so that, …, for all $d_2>0$, there exists $\delta_2>0$ and positive integer n_0 so that the following holds.*

Set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$*, and let* $\boldsymbol{d} = (d_{\Lambda_j} : \Lambda_j \in [\ell]^j, 2 \le j \le k)$ *be a sequence satisfying that for all* $2 \le j \le k$ *and* $\Lambda_j \in [\ell]^j$*,* $d_{\Lambda_j} \ge d_j$*. Let* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ *be an* (ℓ, k)*-complex, where* $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_k$ *has* $n_0 \le n \le |V_i| \le n+1$*,* $1 \le i \le \ell$*. If* $\boldsymbol{\mathcal{H}}$ *has* $\mathbf{DEV}(\boldsymbol{d}, \boldsymbol{\delta})$*, then* $\mathcal{H}^{(k)} \in \boldsymbol{\mathcal{H}}$ *has*

$$\left|\mathrm{EMB}(\mathcal{O}^{(k)}, \mathcal{H}^{(k)})\right| = (1 \pm \mu) \prod_{j=2}^{k} \prod_{\Lambda_j \in [\ell]^j} d_{\Lambda_j} \times n^{2\ell}$$

many labeled partite-isomorphic copies of the (ℓ, k)*-octohedron* $\mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_\ell)$*.*

We next present a type of extension lemma (cf. Lemma 5.4), which we will describe in terms of the following auxiliary graph Γ.

Definition 5.3. For integers $\ell \ge k \ge 2$, let $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be an (ℓ, k)-complex, and let $\mathcal{O}^{(k)}$ be the (ℓ, k)-octohedron. We define the *octohedral-incidence graph* $\vec{\Gamma} = \vec{\Gamma}_{k,\ell}(\boldsymbol{\mathcal{H}})$ of $\boldsymbol{\mathcal{H}}$ as follows. Set $V(\vec{\Gamma}) = \mathcal{K}_\ell(\mathcal{H}^{(k)})$. For $L, L' \in V(\Gamma)$, put $\{L, L'\} \in \vec{\Gamma}$ if, and only if, there exists a labeled partite-embedding ψ of $\mathcal{O}^{(k)}$ in $\boldsymbol{\mathcal{H}}$ with $\operatorname{im}\psi = L \cup L'$, i.e., $L \cup L'$ induces a copy of $\mathcal{O}^{(k)}$ in $\mathcal{H}^{(k)}$.

We now state the Octohedral Extension Lemma.

Theorem 5.4 (Octohedral Extension Lemma). *Fix integers* $\ell \ge k \ge 2$*. For all* $\zeta, d_k > 0$*, there exists* $\delta_k > 0$ *so that for all* $d_{k-1} > 0$*, there exists* $\delta_{k-1} > 0$ *so that, ..., for all* $d_2 > 0$*, there exist* $\delta_2 > 0$ *and positive integer* n_0 *so that the following holds.*

Set $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$*, and let* $\boldsymbol{d} = (d_{\Lambda_j} : \Lambda_j \in [\ell]^j, 2 \le j \le k)$ *be a sequence satisfying that, for all* $2 \le j \le k$ *and for all* $\Lambda_j \in [\ell]^j$*,* $d_{\Lambda_j} \ge d_j$*. Let* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ *be an* (ℓ, k)*-complex, where* $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ *has* $n_0 \le n \le |V_i| \le n+1$ *for each* $i \in [\ell]$*. If* $\boldsymbol{\mathcal{H}}$ *has* $\mathbf{DEV}(\boldsymbol{d}, \boldsymbol{\delta})$ *and if* $\Gamma = \Gamma_{k,\ell}(\boldsymbol{\mathcal{H}})$ *is the octohedral-incidence graph of* $\boldsymbol{\mathcal{H}}$ *(cf. Definition 5.3), then*

(1) *all but* $\zeta|\mathcal{K}_\ell(\mathcal{H}^{(k)})|$ *cliques* $L \in \mathcal{K}_\ell(\mathcal{H}^{(k)})$ *satisfy*

$$\deg_\Gamma(L) = (1 \pm \zeta) \prod_{j=2}^{k} \prod_{\Lambda_j \in [\ell]^j} d_{\Lambda_j}^{2^j - 1} \times n^\ell;$$

(2) *all but* $\zeta|\mathcal{K}_\ell(\mathcal{H}^{(k)})|^2$ *pairs of cliques* $L \ne L' \in \mathcal{K}_\ell(\mathcal{H}^{(k)})$ *satisfy*

$$\deg_\Gamma(L, L') = (1 \pm \zeta) \prod_{j=2}^{k} \prod_{\Lambda_j \in [\ell]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^\ell.$$

6. The Negative-Extension Lemma

In the previous section, we stated Counting and Extension Lemmas corresponding to when a complex $\boldsymbol{\mathcal{H}}$ has the deviation property **DEV**. In this section, we explore what happens when the property of deviation fails to hold. We give our main result as Theorem 6.2, which we call the Negative-Extension Lemma. We first motivate this result.

Suppose $\mathcal{H}^{(k)}$ is a (k, k)-cylinder with underlying $(k, k-1)$-cylinder $\mathcal{H}^{(k-1)}$, where $d = d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)}) > 0$. For $\delta > 0$, suppose that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d, \delta)$. Statement (2) of Fact 2.10 then guarantees that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{COUNT}_{\mathrm{emb}}(d, \delta)$. As such, by Definition 2.9 (recall (1) and (2)), there exists some $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that

$$\left|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\right| \ne (1 \pm \delta) d^{|\mathcal{S}^{(k)}|} \left|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\right|. \tag{20}$$

The Negative-Extension Lemma (Theorem 6.2) will conclude that, as a result of (20), there are ‘many’ k-tuples $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ which ‘belong’ to some unusual number of labeled partite embeddings of $\mathcal{S}^{(k)}$ in $\mathcal{H}^{(k)}$. To make our plan precise, we need some supporting concepts.

6.1. **Supporting concepts, and the Negative-Extension Lemma.** We use the following notation. For a (k,k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$, and for an integer $1 \leq i \leq k$, let $\boldsymbol{\mathcal{H}}^{(i)} \overset{\text{def}}{=} \{\mathcal{H}^{(j)}\}_{j=1}^{i}$. Note that $\boldsymbol{\mathcal{H}}^{(i)}$ is a (k,i)-complex. Now, let

$$\Gamma_i = \Gamma_{i,k}(\boldsymbol{\mathcal{H}}^{(i)}) \tag{21}$$

be the octohedral-incidence graph (cf. Definition 5.3) of $\boldsymbol{\mathcal{H}}^{(i)}$. Clearly,

$$\Gamma_k \subseteq \Gamma_{k-1} \subseteq \cdots \subseteq \Gamma_2. \tag{22}$$

We also use the following variant of the octohedral-incidence graph Γ_k, which accomodates arbitary subhypergraphs $\varnothing \subseteq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \ldots, U_k)$.

Definition 6.1 (incidence digraph, anchor). Fix a $(2,k,k)$-cylinder $\varnothing \subseteq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \ldots, U_k)$. Fix a k-tuple $A = \{a_1, \ldots, a_k\}$, where for each $i \in [k]$, $a_i \in U_i$. Let $\boldsymbol{\mathcal{H}}^{(k)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ be a (k,k)-complex. We define the $(\mathcal{S}^{(k)}, A)$*-incidence digraph* $\vec{\Gamma}_A(\mathcal{S}^{(k)}) = \vec{\Gamma}_A(\mathcal{S}^{(k)}, \boldsymbol{\mathcal{H}}^{(k)})$ of $\boldsymbol{\mathcal{H}}^{(k)}$ as follows. Set $V(\vec{\Gamma}_A(\mathcal{S}^{(k)})) = \mathcal{K}_k(\mathcal{H}^{(k-1)})$. For $K, K' \in V(\vec{\Gamma}_A(\mathcal{S}^{(k)}))$, put $(K, K') \in \vec{\Gamma}_A(\mathcal{S}^{(k)})$ if, and only if, there exists a labeled partite-embedding ψ of $\mathcal{S}^{(k)}$ in $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ (cf. Definition 2.8) so that $\psi(A) = K$ and $\operatorname{im}\psi = K \cup K'$. We will say that A is the *anchor* of $\vec{\Gamma}_A(\mathcal{S}^{(k)})$, and we will write $\bar{A} = (U_1 \cup \cdots \cup U_k) \setminus A$.

When working with the $(\mathcal{S}^{(k)}, A)$-incidence digraph $\vec{\Gamma}_A(\mathcal{S}^{(k)}) = \vec{\Gamma}_A(\mathcal{S}^{(k)}, \boldsymbol{\mathcal{H}})$ of a (k,k)-complex $\boldsymbol{\mathcal{H}}$, we use the following standard notation. For $K, K' \in V(\vec{\Gamma})$, we write

$$N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = \left\{K'' \in V(\vec{\Gamma}_A(\mathcal{S}^{(k)})) : (K, K'') \in \vec{\Gamma}_A(\mathcal{S}^{(k)})\right\},$$
$$N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K, K') = N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \cap N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K'),$$
$$\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = \left|N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K)\right| \quad \text{and} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K, K') = \left|N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K, K')\right|. \tag{23}$$

Note that all neighborhoods and degrees defined above are *out*-neighborhoods and *out*-degrees.

We now consider the following statement **EXT**, which considers a hypergraph $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$, an anchor A for which $\bar{A} \in \mathcal{S}^{(k)}$ (cf. Definition 6.1), and a (k,k)-complex $\boldsymbol{\mathcal{H}}^{(k)}$.

$\mathbf{EXT}_A(\mathcal{S}^{(k)}) = \mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$**.** *Fix $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \ldots, U_k)$, and fix an anchor A for which $\bar{A} \in \mathcal{S}^{(k)}$ (cf. Definition 6.1). Let $\xi > 0$ be given, and let $\boldsymbol{\mathcal{H}}^{(k)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ be a (k,k)-complex with $d_{[k]} = d(\mathcal{H}^{(k)} | \mathcal{H}^{(k-1)}) > 0$. Then, the following condition holds:*

(1) *If $A \in \mathcal{S}^{(k)}$, then all but $\xi|\mathcal{H}^{(k)}|$ edges $H \in \mathcal{H}^{(k)}$ satisfy the following implication:*

$$\textit{If} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(H) > \xi \deg_{\Gamma_{k-1}}(H),$$
$$\textit{then} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(H) = (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(H);$$

(2) *If $A \notin \mathcal{S}^{(k)}$, then all but $\xi|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy the following implication:*

$$\text{If} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K) > \xi \deg_{\Gamma_{k-1}}(K),$$
$$\text{then} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K).$$

For future purposes, it will be convenient to have a compact presentation of the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}) = \mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ (see (26) below). To that end, let

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k)}(\mathcal{S}^{(k)}, A, \boldsymbol{\mathcal{H}}^{(k)}) = \begin{cases} \mathcal{H}^{(k)} & \text{if } A \in \mathcal{S}^{(k)}, \\ \mathcal{K}_k(\mathcal{H}^{(k-1)}) & \text{if } A \notin \mathcal{S}^{(k)}. \end{cases} \tag{24}$$

In the language of $\mathcal{A}^{(k)}$, we will combine Conditions (1) and (2) of $\mathbf{EXT}_A(\mathcal{S}^{(k)})$ into one presentation, as follows. Set

$$\mathcal{A}^{(k)}_{\text{bad}} = \mathcal{A}^{(k)}_{\text{bad}}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \Big\{ K \in \mathcal{A}^{(k)} : \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) > \xi \deg_{\Gamma_{k-1}}(K)$$
$$\text{but} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \neq (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) \Big\}. \tag{25}$$

Then,

$$\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)}) \text{ is true} \quad \Longleftrightarrow \quad |\mathcal{A}^{(k)}_{\text{bad}}| < \xi |\mathcal{A}^{(k)}|. \tag{26}$$

We now state the main result of the section, the Negative-Extension Lemma.

Theorem 6.2 (**The Negative-Extension Lemma).** *Let integer $k \geq 2$ be fixed. For all $d_k, \delta_k > 0$, there exists $\xi > 0$ so that for all $d_{k-1} > 0$, there exists $\delta_{k-1} > 0$ so that, ..., for all $d_2 > 0$, there exist $\delta_2 > 0$ and positive integer n_0 so that the following holds.*

Set $\boldsymbol{\delta}_{k-1} = (\delta_2, \dots, \delta_{k-1})$. Let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{(k)}$ be a (k,k)-complex with density sequence $\boldsymbol{d}_k$, as given in Setup 2.15, where $n \geq n_0$. Suppose $\boldsymbol{d}_k$ satisfies that, for each $2 \leq j \leq k$, $d_{\Lambda_j} \geq d_j$ for all $\Lambda_j \in [k]^j$. Assume that

(1) *$\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but that*
(2) *$(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$.*

Then, there exists a hypergraph $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that, whenever an anchor A satisfies $\bar{A} \in \mathcal{S}^{(k)}$, the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false. In other words, the hypergraphs $\mathcal{A}^{(k)} = (\mathcal{S}^{(k)}, A, \boldsymbol{\mathcal{H}}^{(k)})$ and $\mathcal{A}^{(k)}_{\text{bad}} = \mathcal{A}^{(k)}_{\text{bad}}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ satisfy $|\mathcal{A}^{(k)}_{\text{bad}}| \geq \xi |\mathcal{A}^{(k)}|$.

We proceed to define the constants for Theorem 6.2.

6.2. **The constants of Theorem 6.2.** Let $k \geq 2$ be a fixed integer, and let $d_k, \delta_k > 0$ be given. We define the constant ξ promised by Theorem 6.2 by

$$\xi = \frac{1}{100k2^k} \delta_k d_k^{2^k}. \tag{27}$$

Let $d_{k-1} > 0$ be given. We formally define the constant δ_{k-1} in upcoming (29), but we first motivate how we choose it. To that end, define auxiliary constants (cf. (27))

$$\mu = 1/2 \quad \text{and} \quad \zeta_{k-1} = \xi d_{k-1}^{k2^{k-1}}. \tag{28}$$

Recall from the hypothesis of Theorem 6.2 that we will be working with a $(k, k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=2}^{k-1}$ which has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, where the constants $d_{k-2}, \dots, d_2$ of $\boldsymbol{d}_{k-1}$ and the constants $\delta_{k-1},\dots,\delta_2$ of $\boldsymbol{\delta}_{k-1}$ will be disclosed below. For such a complex $\boldsymbol{\mathcal{H}}^{(k-1)}$, we want $\delta_{k-1}>0$ to be small enough so that the following conditions are satisfied (cf. (28)):

(*a*) we can estimate $|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ within an error of $1\pm\mu$;
(*b*) we can estimate $|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})|$ within an error of $1\pm\mu$;
(*c*) all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ cliques $K\in\mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K)=(1\pm\zeta_{k-1})\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j}d_{\Lambda_j}^{2^j-1}\times n^k.$$

To guarantee that (*a*), (*b*), and (*c*) above are satisfied, we need $\delta_{k-1}>0$ to be small enough to enable applications of Theorems 5.1, 5.2, and 5.4. With d_{k-1} given above, and with $\mu=1/2$ from (28), let

$$\delta_{\mathrm{Thm.5.1},\,k-1}=\delta_{\mathrm{Thm.5.1}}(\ell=k,k-1,\mu=1/2,d_{k-1})>0$$
$$\text{and}\quad\delta_{\mathrm{Thm.5.2},\,k-1}=\delta_{\mathrm{Thm.5.2}}(\ell=k,k-1,\mu=1/2,d_{k-1})>0$$

be the constants guaranteed by Theorems 5.1 and 5.2. With d_{k-1} given above, and with ζ_{k-1} from (28), let

$$\delta_{\mathrm{Thm.5.4},\,k-1}=\delta_{\mathrm{Thm.5.4}}(\ell=k,k-1,\zeta=\zeta_{k-1},d_{k-1})>0$$

be the constant guaranteed by Theorem 5.4. Now, set

$$\delta_{k-1}=\min\{\delta_{\mathrm{Thm.5.1},\,k-1},\delta_{\mathrm{Thm.5.2},\,k-1},\delta_{\mathrm{Thm.5.4},\,k-1}\}\tag{29}$$

which concludes our definition of the promised constant δ_{k-1}.

Inductively, assume $d_{k-1},\delta_{k-1},\dots,d_i,\delta_i,d_{i-1}$ have been disclosed, for a fixed integer i satisfying $3\le i\le k-1$. Moreover, assume that we have defined auxiliary constants (cf. (28))

$$\zeta_{k-1}=\xi d_{k-1}^{k2^{k-1}},\quad \zeta_{k-2}=\xi d_{k-1}^{\binom{k}{k-1}2^{k-1}}d_{k-2}^{\binom{k}{k-2}2^{k-2}},\quad\dots\quad \zeta_{i-1}=\xi\prod_{j=i-1}^{k-1}d_j^{\binom{k}{j}2^j}.\tag{30}$$

We define δ_{i-1} similarly to how we defined δ_{k-1} (cf. (29)). In particular, we want $\delta_{i-1}>0$ to be small enough so that (*a*) and (*b*) above are satisfied with $\mu=1/2$. These tasks are handled by Theorems 5.1 and 5.2, which have the following common quantification of constants:

$$\forall\mu,\ \forall d_{k-1},\ \exists\delta_{k-1}:\dots\forall d_{i-1},\ \exists\delta_{i-1}:\dots$$

With $\mu=1/2$ from (28), and with $d_{k-1},\delta_{k-1},\dots,d_{i-1}$ inductively disclosed above, let

$$\delta_{\mathrm{Thm.5.1},\,i-1}=\delta_{\mathrm{Thm.5.2}}(\ell=k,k-1,\mu=1/2,d_{k-1},\delta_{k-1},\dots,d_i,\delta_i,d_{i-1})>0$$
$$\text{and}\quad\delta_{\mathrm{Thm.5.2},\,i-1}=\delta_{\mathrm{Thm.5.2}}(\ell=k,k-1,\mu=1/2,d_{k-1},\delta_{k-1},\dots,d_i,\delta_i,d_{i-1})>0$$

be the constants guaranteed by Theorems 5.1 and 5.2. We also want $\delta_{i-1}>0$ to be small enough so that (*c*) above is satisfied with ζ_{k-1} from (28). Moreover, we want $\delta_{i-1}>0$ to be small enough so that the following sequence (c') of conditions is satisfied (cf. (30)):

(c') • all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ cliques $K\in\mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K)=(1\pm\zeta_{k-1})\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j}d_{\Lambda_j}^{2^j-1}\times n^k;$$

- all but $\zeta_{k-2}|\mathcal{K}_k(\mathcal{H}^{(k-2)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(k-2)})$ satisfy

$$\deg_{\Gamma_{k-2}}(K) = (1 \pm \zeta_{k-2}) \prod_{j=2}^{k-2} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k;$$

$$\vdots$$

- all but $\zeta_{i-1}|\mathcal{K}_k(\mathcal{H}^{(i-1)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(i-1)})$ satisfy

$$\deg_{\Gamma_{i-1}}(K) = (1 \pm \zeta_{i-1}) \prod_{j=2}^{i-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k.$$

To guarantee that the sequence (c') of conditions above will be satisfied, we fix an integer h satisfying $i-1 \le h \le k-1$, and we appeal to Theorem 5.4, which has the following quantification of constants:

$$\forall \zeta,\ \forall d_h,\ \exists \delta_h : \forall d_{h-1},\ \exists \delta_{h-1} : \ \dots \forall d_{i-1},\ \exists \delta_{i-1} : \dots$$

With $d_h, \delta_h, \dots, d_{i-1}, \delta_{i-1}$ inductively disclosed above, and with $\zeta = \zeta_h$ from (30), let

$$\delta_{\text{Thm.5.4},\ i-1,\ h} = \delta_{\text{Thm.5.4}}(\ell = k, h, \zeta = \zeta_h, d_h, \delta_h, \dots, d_i, \delta_i, d_{i-1})$$

be the constant guaranteed by Theorem 5.4. Set

$$\delta_{\text{Thm.5.4},\ i-1} = \min\left\{\delta_{\text{Thm.5.4},\ i-1,\ h} :\ i-1 \le h \le k-1\right\}.$$

Finally, set

$$\delta_{i-1} = \min\left\{\delta_{\text{Thm.5.1},\ i-1}, \delta_{\text{Thm.5.2},\ i-1}, \delta_{\text{Thm.5.4},\ i-1}\right\}. \tag{31}$$

We continue this way until δ_2 is reached. This concludes our definitions of the constants.

6.3. **The argument for Theorem 6.2.** Set $\boldsymbol{\delta}_{k-1} = (\delta_2, \dots, \delta_{k-1})$, where each δ_j, $2 \le j \le k-1$, was defined in (31). Let $\boldsymbol{\mathcal{H}}^{(k)}$ be a (k,k)-complex with density sequence $\boldsymbol{d}_k$, as given in Setup 2.15, where $n \ge n_0$. Suppose $\boldsymbol{d}_k$ satisfies that, for each $2 \le j \le k$, $d_{\Lambda_j} \ge d_j$ for all $\Lambda_j \in [k]^j$, where d_j was given above. Suppose that $\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$. Theorem 6.2 promises a hypergraph $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that, for any anchor A for which $\bar{A} \in \mathcal{S}^{(k)}$ (cf. Definition 6.1), the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false. We begin our argument by defining the promised hypergraph $\mathcal{S}^{(k)}$.

6.3.1. *Defining the hypergraph $\mathcal{S}^{(k)}$.* First, we appeal to (20), and take any hypergraph $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$ for which

$$\left|\text{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\right| \ne (1 \pm \delta_k) d_{[k]}^{|\mathcal{S}^{(k)}|} \left|\text{EMB}(\mathcal{O}^{(k)}, \mathcal{H}^{(k-1)})\right|. \tag{32}$$

Indeed, Assumption (2) of our hypothesis says that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$. As such, Statement (2) of Fact 2.10 gives that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{COUNT}_{\text{emb}}(d_{[k]}, \delta_k)$. Thus, some $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$ satisfying (32) is guaranteed to exist by Definition 2.9.

Second, take $\varnothing \ne \mathcal{S}^{(k)}_{\min} \subseteq \mathcal{S}^{(k)}$ to be an *edge-minimal* subhypergraph for which

$$\left|\text{EMB}(\mathcal{S}^{(k)}_{\min}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\right| \ne \left(1 \pm \frac{\delta_k}{2^{|\mathcal{S}^{(k)}| - |\mathcal{S}^{(k)}_{\min}|}}\right) d_{[k]}^{|\mathcal{S}^{(k)}_{\min}|} \left|\text{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\right|. \tag{33}$$

(Note that in (33), we require the error $\delta_k/2^{|\mathcal{S}^{(k)}|-|\mathcal{S}^{(k)}_{\min}|}$ to decrease as $|\mathcal{S}^{(k)}_{\min}|$ decreases.) Note that $\mathcal{S}^{(k)}_{\min}$ must exist, because $\mathcal{S}^{(k)}$ itself satisfies (32). Note also that $\mathcal{S}^{(k)}_{\min} \neq \varnothing$, because

$$\big|\mathrm{EMB}(\varnothing, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| = (1 \pm 0) d^0_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|.$$

Since $\mathcal{S}^{(k)}_{\min} \neq \varnothing$ is edge-minimal w.r.t. (33), we have that, for each $e \in \mathcal{S}^{(k)}_{\min}$,

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}_{\min} \setminus \{e\}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| = \left(1 \pm \frac{\delta_k}{2^{|\mathcal{S}^{(k)}|-|\mathcal{S}^{(k)}_{\min}|+1}}\right) d^{|\mathcal{S}^{(k)}_{\min}|-1}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k)}, \mathcal{H}^{(k-1)})\big|. \quad (34)$$

For simplicity of notation, we shall write

$$\delta'_k := \frac{\delta_k}{2^{|\mathcal{S}^{(k)}|-|\mathcal{S}^{(k)}_{\min}|}} \quad \text{and} \quad \mathcal{S}^{(k)} := \mathcal{S}^{(k)}_{\min}. \quad (35)$$

Then, we may rewrite (33) as

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| \neq (1 \pm \delta'_k) d^{|\mathcal{S}^{(k)}|}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|, \quad (36)$$

and we may rewrite (34) as, for each $e \in \mathcal{S}^{(k)}$,

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)} \setminus \{e\}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| = \left(1 \pm \frac{\delta'_k}{2}\right) d^{|\mathcal{S}^{(k)}|-1}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|. \quad (37)$$

This concludes our definition of the promised hypergraph $\mathcal{S}^{(k)}$.

We pause to say a word about the inequality in (36). We have that either

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| < (1 - \delta'_k) d^{|\mathcal{S}^{(k)}|}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|,$$
$$\text{or} \quad \big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| > (1 + \delta'_k) d^{|\mathcal{S}^{(k)}|}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|.$$

In our proof, it will be symmetric to handle either situation above. We therefore assume, w.l.o.g., that the latter holds:

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| > (1 + \delta'_k) d^{|\mathcal{S}^{(k)}|}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|. \quad (38)$$

We proceed to develop a proof by contradiction. Assume the hypergraph $\mathcal{S}^{(k)}$ from (38) doesn't have the desired property of Theorem 6.2. In particular, assume that there exists an anchor A, where $\bar{A} \in \mathcal{S}^{(k)}$, for which the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is true. With this assumption, we will prove the following.

Claim 6.3. *Assuming the statement* $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ *is true for some* $\bar{A} \in \mathcal{S}^{(k)}$*, we have*

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| \leq \left(1 + \frac{3}{4}\delta'_k\right) d^{|\mathcal{S}^{(k)}|}_{[k]} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|.$$

Now, the bound in Claim 6.3 is a direct contradiction with the bound in (38). Thus, it must be the case that for any anchor A, where $\bar{A} \in \mathcal{S}^{(k)}$, the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false, as promised by Theorem 6.2. Thus, to complete the proof of Theorem 6.2, it only remains to prove Claim 6.3.

6.4. **Proof of Claim 6.3.** Assume that the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is true for some anchor A with $\bar{A} \in \mathcal{S}^{(k)}$.

Recall that in (24)–(26), we abbreviated the truth of the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ in terms of the following hypergraphs $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(k)}_{\text{bad}}$:

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k)}(\mathcal{S}^{(k)}, A, \boldsymbol{\mathcal{H}}^{(k)}) = \begin{cases} \mathcal{H}^{(k)} & \text{if } A \in \mathcal{S}^{(k)}, \\ \mathcal{K}_k(\mathcal{H}^{(k-1)}) & \text{if } A \notin \mathcal{S}^{(k)}, \end{cases}$$

$$\mathcal{A}^{(k)}_{\text{bad}} = \mathcal{A}^{(k)}_{\text{bad}}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \Big\{ K \in \mathcal{A}^{(k)} : \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) > \xi \deg_{\Gamma_{k-1}}(K)$$
$$\text{but} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \neq (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) \Big\}.$$

Recall from (26) that our assumption that $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is true is equivalent to

$$|\mathcal{A}^{(k)}_{\text{bad}}| < \xi |\mathcal{A}^{(k)}| \leq \xi |\mathcal{K}_k(\mathcal{H}^{(k-1)})|. \tag{39}$$

Define also the sets

$$\mathcal{A}^{(k)}_{\text{good}} = \mathcal{A}^{(k)}_{\text{good}}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \Big\{ K \in \mathcal{A}^{(k)} : \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) > \xi \deg_{\Gamma_{k-1}}(K)$$
$$\text{and} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) \Big\}, \tag{40}$$

and

$$\mathcal{A}^{(k)}_{0} = \mathcal{A}^{(k)}_{0}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \Big\{ K \in \mathcal{A}^{(k)} : \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) < \xi \deg_{\Gamma_{k-1}}(K) \Big\}. \tag{41}$$

Note that

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k)}_{\text{good}} \cup \mathcal{A}^{(k)}_{\text{bad}} \cup \mathcal{A}^{(k)}_{0} \tag{42}$$

is a partition.

Using the partition $\mathcal{A}^{(k)} = \mathcal{A}^{(k)}_{\text{good}} \cup \mathcal{A}^{(k)}_{\text{bad}} \cup \mathcal{A}^{(k)}_{0}$ from (42), observe that (recall Definition 2.8)

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| = \sum_{K \in \mathcal{A}^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K)$$
$$\stackrel{(42)}{=} \sum_{K \in \mathcal{A}^{(k)}_{\text{good}}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) + \sum_{K \in \mathcal{A}^{(k)}_{\text{bad}}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) + \sum_{K \in \mathcal{A}^{(k)}_{0}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K). \tag{43}$$

We now bound each of the sums above.

First, using the definition of $\mathcal{A}^{(k)}_{\text{good}}$ in (40), we have

$$\sum_{K \in \mathcal{A}^{(k)}_{\text{good}}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \leq (1+\xi) d_{[k]} \sum_{K \in \mathcal{A}^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K)$$
$$= (1+\xi) d_{[k]} \big|\mathrm{EMB}(\mathcal{S}^{(k)} \setminus \{\bar{A}\}, (\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)}))\big| \stackrel{(37)}{\leq} (1+\xi) \left(1 + \frac{\delta'_k}{2}\right) d_{[k]}^{|\mathcal{S}^{(k)}|} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|$$
$$\leq \left(1 + 2\xi + \frac{\delta'_k}{2}\right) d_{[k]}^{|\mathcal{S}^{(k)}|} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|$$
$$\stackrel{(27),(35)}{\leq} \left(1 + \frac{2\delta'_k}{3}\right) d_{[k]}^{|\mathcal{S}^{(k)}|} \big|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\big|. \tag{44}$$

(To see the last inequality, (27) gives $\xi < \delta_k/(2^k \cdot 6)$, and (35) gives $\delta_k/2^k < \delta'_k$.) Second, we take

$$\sum_{K\in\mathcal{A}_0^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \le \sum_{K\in\mathcal{A}_0^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K),$$

since every labeled partite-embedding of $\mathcal{S}^{(k)}$ in $\mathcal{H}^{(k)}$ is also a labeled partite-embedding of $\mathcal{S}^{(k)} \setminus \{\bar{A}\}$ in $\mathcal{H}^{(k)}$. Using the definition of $\mathcal{A}_0^{(k)}$ in (41), we have

$$\begin{aligned}\sum_{K\in\mathcal{A}_0^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) &\le \sum_{K\in\mathcal{A}_0^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K) \\ &\le \xi \sum_{K\in\mathcal{A}^{(k)}} \deg_{\Gamma_{k-1}}(K) = \xi\big|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})\big|. \quad (45)\end{aligned}$$

Third, we take

$$\sum_{K\in\mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \le \sum_{K\in\mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K), \quad (46)$$

since every labeled partite-embedding of $\mathcal{S}^{(k)}$ in $\mathcal{H}^{(k)}$ is also a labeled partite-embedding of $\mathcal{O}^{(k-1)}$ in $\mathcal{H}^{(k-1)}$. More strongly, we have the following bound (which we prove in a moment).

Fact 6.4.

$$\sum_{K\in\mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K) \le 8(k-1)\xi\big|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})\big|.$$

Applying the bounds of (44)–(46) and the bound of Fact 6.4 to (43), we infer

$$\begin{aligned}\big|\mathrm{EMB}(\mathcal{S}^{(k)},(\mathcal{H}^{(k)},\mathcal{H}^{(k-1)})\big| &\le \left(\left(1+\frac{2}{3}\delta'_k\right)d_{[k]}^{|\mathcal{S}^{(k)}|} + \xi + 8(k-1)\xi\right)\big|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})\big| \\ &\le \left(1+\frac{2}{3}\delta'_k + 8k\xi d_k^{-2^k}\right) d_{[k]}^{|\mathcal{S}^{(k)}|}\big|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})\big|, \quad (47)\end{aligned}$$

where we used $|\mathcal{S}^{(k)}| \le 2^k$ and $d_{[k]} \ge d_k$ from the hypothesis of Theorem 6.2. Now, since

$$8k\xi d_k^{-2^k} \overset{(27)}{<} \frac{1}{12\cdot 2^k}\delta_k \overset{(35)}{<} \frac{1}{12}\delta'_k,$$

we have

$$\big|\mathrm{EMB}(\mathcal{S}^{(k)},(\mathcal{H}^{(k)},\mathcal{H}^{(k-1)}))\big| < \left(1+\frac{3}{4}\delta'_k\right) d_{[k]}^{|\mathcal{S}^{(k)}|}\big|\mathrm{EMB}(\mathcal{O}^{(k-1)},\mathcal{H}^{(k-1)})\big|,$$

as promised by Claim 6.3. Thus, it only remains to prove Fact 6.4.

6.5. **Proof of Fact 6.4.** We first outline the main idea of how we bound $\sum_{K\in\mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K)$. To begin, we divide the k-tuples $K \in \mathcal{A}_{\mathrm{bad}}^{(k)}$ into two classes: those for which $\deg_{\Gamma_{k-1}}(K)$ is not 'too large', and those for which it is. More generally, we first partition the set of k-tuples

$V(\Gamma_{k-1}) = \mathcal{K}_k(\mathcal{H}^{(k-1)})$ as follows. With ζ_{k-1} given in (28), define

$$V_{\zeta_{k-1}\text{-good}}(\Gamma_{k-1}) = \Big\{ K \in V(\Gamma_{k-1}) = \mathcal{K}_k(\mathcal{H}^{(k-1)}) : \\ \deg_{\Gamma_{k-1}}(K) < (1+\zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k \Big\}, \\ \text{and} \quad V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) = V(\Gamma_{k-1}) \setminus V_{\zeta_{k-1}\text{-good}}(\Gamma_{k-1}). \quad (48)$$

Then,

$$\mathcal{A}^{(k)}_{\text{bad}} \subseteq \mathcal{A}^{(k)} \overset{(24)}{\subseteq} \mathcal{K}_k(\mathcal{H}^{(k-1)}) = V(\Gamma_{k-1}) \overset{(48)}{=} V_{\zeta_{k-1}\text{-good}}(\Gamma_{k-1}) \cup V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}). \quad (49)$$

As such, with $\zeta_{k-1} \le 1$,

$$\sum_{K \in \mathcal{A}^{(k)}_{\text{bad}}} \deg_{\Gamma_{k-1}}(K) = \sum \Big\{ \deg_{\Gamma_{k-1}}(K) : K \in \mathcal{A}^{(k)}_{\text{bad}} \cap V_{\zeta_{k-1}\text{-good}}(\Gamma_{k-1}) \Big\} \\ + \sum \Big\{ \deg_{\Gamma_{k-1}}(K) : K \in \mathcal{A}^{(k)}_{\text{bad}} \cap V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \Big\} \\ \overset{(48)}{\le} 2|\mathcal{A}^{(k)}_{\text{bad}}| \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k + \sum \Big\{ \deg_{\Gamma_{k-1}}(K) : K \in V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \Big\}. \quad (50)$$

To bound the first term in (50), we have from (26) that $|\mathcal{A}^{(k)}_{\text{bad}}| \le \xi |\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ is 'small'. We will return to this in a moment. To bound the summation in (50), we iterate the approach taken in (48) and (49). Namely, for $2 \le i \le k-1$, we divide the k-tuples $K \in \mathcal{K}_k(\mathcal{H}^{(i)})$ into two classes: those for which $\deg_{\Gamma_i}(K)$ is not 'too large', and those for which it is. More formally, with ζ_i given in (30), define

$$V_{\zeta_i\text{-good}}(\Gamma_i) = \left\{ K \in V(\Gamma_i) = \mathcal{K}_k(\mathcal{H}^{(i)}) : \deg_{\Gamma_i}(K) < (1+\zeta_i) \prod_{j=2}^{i} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k \right\}, \\ \text{and} \quad V_{\zeta_i\text{-bad}}(\Gamma_i) = V(\Gamma_i) \setminus V_{\zeta_i\text{-good}}(\Gamma_i). \quad (51)$$

Now, with $\Gamma_{k-1} \subseteq \Gamma_{k-2}$ from (22), we have

$$V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \subseteq V(\Gamma_{k-1}) \overset{(22)}{\subseteq} V(\Gamma_{k-2}) \overset{(51)}{=} V_{\zeta_{k-2}\text{-good}}(\Gamma_{k-2}) \cup V_{\zeta_{k-2}\text{-bad}}(\Gamma_{k-2}).$$

Thus, with $\zeta_{k-2} \le 1$ and with $\Gamma_{k-1} \subseteq \Gamma_{k-2}$, we may bound the summation of (50) by

$$\sum \Big\{ \deg_{\Gamma_{k-1}}(K) : K \in V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \Big\} \\ \overset{(22)}{\le} \sum \Big\{ \deg_{\Gamma_{k-2}}(K) : K \in V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \cap V_{\zeta_{k-2}\text{-good}}(\Gamma_{k-2}) \Big\} \\ + \sum \Big\{ \deg_{\Gamma_{k-2}}(K) : K \in V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1} \cap V_{\zeta_{k-2}\text{-bad}}(\Gamma_{k-2}) \Big\} \\ \overset{(48)}{\le} 2 \left| V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1}) \right| \prod_{j=2}^{k-2} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k + \sum \Big\{ \deg_{\Gamma_{k-2}}(K) : K \in V_{\zeta_{k-2}\text{-bad}}(\Gamma_{k-2}) \Big\}.$$

Inductively[3], we conclude

$$\sum\Big\{\deg_{\Gamma_{k-1}}(K) : K \in V_{\zeta_{k-1}\text{-bad}}(\Gamma_{k-1})\Big\} \le 2n^k \sum_{i=2}^{k-1}\left(|V_{\zeta_i\text{-bad}}(\Gamma_i)| \prod_{j=2}^{i-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1}\right). \tag{52}$$

Applying (52) to the second term of (50), we have

$$\sum_{K\in\mathcal{A}^{(k)}_{\text{bad}}} \deg_{\Gamma_{k-1}}(K) \le 2n^k\left(|\mathcal{A}^{(k)}_{\text{bad}}|\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1} + \sum_{i=2}^{k-1}\left(|V_{\zeta_i\text{-bad}}(\Gamma_i)|\prod_{j=2}^{i-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1}\right)\right). \tag{53}$$

As we mentioned earlier, $|\mathcal{A}^{(k)}_{\text{bad}}| \le \xi|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ holds from (39). To further bound $|\mathcal{A}^{(k)}_{\text{bad}}|$, we apply Theorem 5.1 with $\mu = 1/2$ to $|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ to get

$$|\mathcal{K}_k(\mathcal{H}^{(k-1)})| \le (1+\mu)\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j} \times n^k < 2\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k,$$

$$\Longrightarrow \quad |\mathcal{A}^{(k)}_{\text{bad}}| \le \xi|\mathcal{K}_k(\mathcal{H}^{(k-1)})| \le 2\xi\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k. \tag{54}$$

Applying (54) to (53) yields

$$\sum_{K\in\mathcal{A}^{(k)}_{\text{bad}}} \deg_{\Gamma_{k-1}}(K) \le 4\xi\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j}\times n^{2k} + 2n^k\sum_{i=2}^{k-1}\left(|V_{\zeta_i\text{-bad}}(\Gamma_i)|\prod_{j=2}^{i-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1}\right). \tag{55}$$

Now, for fixed $2 \le i \le k-1$, we bound $|V_{\zeta_i\text{-bad}}(\Gamma_i)|$. With $\zeta_i > 0$ from (30), Theorem 5.4 gives $|V_{\zeta_i\text{-bad}}(\Gamma_i)| \le \zeta_i|\mathcal{K}_k(\mathcal{H}^{(i)})|$. To further bound $|V_{\zeta_i\text{-bad}}(\Gamma_i)|$, we apply Theorem 5.1 with $\mu = 1/2$ to $|\mathcal{K}_k(\mathcal{H}^{(i)})|$ to get

$$|\mathcal{K}_k(\mathcal{H}^{(i)})| \le (1+\mu)\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k < 2\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k,$$

$$\Longrightarrow \quad |V_{\zeta_i\text{-bad}}(\Gamma_i)| \le \zeta_i|\mathcal{K}_k(\mathcal{H}^{(i)})| \le 2\zeta_i\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k. \tag{56}$$

Applying (56) to (55) gives

$$\begin{aligned}\sum_{K\in\mathcal{A}^{(k)}_{\text{bad}}} \deg_{\Gamma_{k-1}}(K) &\le 2n^{2k}\left(\left(2\xi\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j}\right) + 2\sum_{i=2}^{k-1}\left(\zeta_i\prod_{\Lambda_i\in[k]^i} d_{\Lambda_i}\times\prod_{j=2}^{i-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j}\right)\right)\\ &\le 2n^{2k}\left(\left(2\xi\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j}\right) + 2\sum_{i=2}^{k-1}\left(\zeta_i\prod_{j=2}^{i-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j}\right)\right).\end{aligned} \tag{57}$$

To finish the proof of Fact 6.4, only calculations remain.

[3]Note that the summation in (52) does not include $i = 1$, because every $K \in \mathcal{K}_k(\mathcal{H}^{(1)})$ satisfies $\deg_{\Gamma_1}(K) = n^k$.

Indeed, fix $2 \leq i \leq k-1$. Recall from the hypothesis of Theorem 6.2 that the density sequence $\boldsymbol{d}_k$ satisfies $d_{\Lambda_j} \geq d_j$ for all $\Lambda_j \in [k]^j$ and for all $2 \leq j \leq k$. As such, our definition of ζ_i in (30) gives

$$\zeta_i = \xi \prod_{j=i}^{k-1} d_j^{\binom{k}{j} 2^j} \leq \xi \prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} = \xi \frac{\prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j}}{\prod_{j=2}^{i-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j}}. \tag{58}$$

Applying (58) to (57) gives

$$\sum_{K \in \mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K) \leq 2n^{2k} \left(\left(2\xi \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} \right) + 2\xi(k-2) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} \right)$$
$$= 4\xi(k-1) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} \times n^{2k}. \tag{59}$$

To conclude the proof of Fact 6.4, it only remains to bound $\sum_{K \in \mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K)$ in terms of $|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})|$. To that end, with $\mu = 1/2$, Theorem 5.2 gives

$$\left|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\right| \geq (1-\mu) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} \times n^{2k} \geq \frac{1}{2} \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j} \times n^{2k}. \tag{60}$$

Comparing (59) and (60), we infer

$$\sum_{K \in \mathcal{A}_{\mathrm{bad}}^{(k)}} \deg_{\Gamma_{k-1}}(K) \leq 8\xi(k-1) \left|\mathrm{EMB}(\mathcal{O}^{(k-1)}, \mathcal{H}^{(k-1)})\right|,$$

as promised by Fact 6.4.

7. Proof of Theorem 2.16

The proof of Theorem 2.16 will involve applications of Theorems 5.2, 5.4, and 6.2. In addition to these tools, we will also need the following lemma, a nonconstructive version of which appeared as Lemma 2.6 in [18], where it was called the 'Picking Lemma'. The proof of that version follows by an application of the Markov Inequality, but here, we will need a constructive counterpart, proved in the Appendix.

Lemma 7.1 (**Algorithmic Picking Lemma**). *Let $\sigma_s, \ldots, \sigma_2, c > 0$ be given together with an integer $r \geq 1$. Let X be a set of size m, and let $G_2, \ldots, G_s$ be graphs with vertex set X satisfying $|G_2| \leq \sigma_2 m^2, \ldots, |G_s| \leq \sigma_s m^2$. Then, for every subset $Y \subseteq X$ of size $|Y| \geq cm$, there exists an algorithm which chooses, in time $O(m^3)$, vertices $Z = Z_r = \{z_1, \ldots, z_r\} \subset Y$ so that, for all $2 \leq i \leq s$, $|G_i[Z]| \leq (2(s-1)\sigma_i/c^2)r^2$.*

We proceed to define the constants of Theorem 2.16 (which will be presented in a similar way to how we defined the constants of Theorem 6.2).

7.1. **The Constants of Theorem 2.16.** Let integer $k \geq 2$ be fixed, and let $d_k, \delta_k > 0$ be given. To define the promised constant $\delta'_k > 0$, we appeal to Theorem 6.2, which we recall has the following quantification:

$$\forall d_k,\ \forall \delta_k,\ \exists \xi : \dots$$

With $d_k, \delta_k > 0$ given above, let

$$\xi = \xi_{\text{Thm.6.2}}(k, d_k, \delta_k) > 0 \tag{61}$$

be the constant guaranteed by Theorem 6.2. We define the promised constant δ'_k by

$$\delta'_k = \left(\frac{\xi}{10}\right)^8. \tag{62}$$

Let $d_{k-1} > 0$ be given. We formally define the constant $\delta_{k-1} > 0$ in upcoming (64), but we first motivate how we choose it. To that end, define auxiliary constants

$$\mu = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2} \quad \text{and} \quad \zeta_{k-1} = \frac{d_k^2 \xi^2}{128(k-1)^2} d_{k-1}^{k(2^k-1)}. \tag{63}$$

Recall from the hypothesis of Theorem 2.16 that we will be working with a (k,k)-complex $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=2}^{k-1}$ satisfying that

(1) $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but where
(2) $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$,

where $d_{[k]} \geq d_k$, and where the constants $d_{k-2}, \dots, d_2$ of $\boldsymbol{d}_{k-1}$ and the constants $\delta_{k-1}, \dots, \delta_2$ of $\boldsymbol{\delta}_{k-1}$ will be disclosed below. For such a complex $\boldsymbol{\mathcal{H}}^{(k)}$, we want $\delta_{k-1} > 0$ to be small enough so that the following conditions are satisfied:

(a) there exists a hypergraph $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that, for any anchor A with $\bar{A} \in \mathcal{S}^{(k)}$, the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false (cf. (61));
(b) we can estimate $|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ within an error of $1 \pm \mu$;
(c) all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K) = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j - 1} \times n^k,$$

and all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2$ pairs of cliques $K \neq K' \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K, K') = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^k.$$

To guarantee that (a), (b), and (c) above are satisfied, we need $\delta_{k-1} > 0$ to be small enough to enable applications of Theorems 6.2, 5.1, and 5.4, respectively. With $d_k, \delta_k > 0$ given above, with $\xi > 0$ from (61), and with $d_{k-1} > 0$ given above, let

$$\delta_{\text{Thm.6.2},\ k-1} = \delta_{\text{Thm.6.2},\ k-1}(k, d_k, \delta_k, \xi, d_{k-1}) > 0$$

be the constant guaranteed by Theorem 6.2. With $\mu > 0$ from (63), and with d_{k-1} given above, let

$$\delta_{\text{Thm.5.1},\ k-1} = \delta_{\text{Thm.5.1},\ k-1}(\ell = k, k-1, \mu, d_{k-1}) > 0$$

be the constant guaranteed by Theorem 5.1. With $\zeta_{k-1} > 0$ from (63), and with d_{k-1} given above, let

$$\delta_{\text{Thm.5.4},\ k-1} = \delta_{\text{Thm.5.4},\ k-1}(\ell = k, k-1, \zeta_{k-1}, d_{k-1}) > 0$$

be the constant guaranteed by Theorem 5.4. Now, set

$$\delta_{k-1} = \min\left\{\delta_{\text{Thm.6.2},\,k-1}, \delta_{\text{Thm.5.1},\,k-1}, \delta_{\text{Thm.5.4},\,k-1}\right\}. \tag{64}$$

This concludes our definition of the promised constant $\delta_{k-1} > 0$.

Inductively, assume $d_{k-1}, \delta_{k-1}, \dots, d_i, \delta_i, d_{i-1} > 0$ have been disclosed, for a fixed integer i satisfying $3 \le i \le k-1$. Moreover, assume we have defined auxiliary constants (cf. (63)):

$$\zeta_{k-1} = \frac{d_k^2 \xi^2}{128(k-1)^2} d_{k-1}^{k(2^k-1)}, \quad \zeta_{k-2} = \frac{d_k^2 \xi^2}{128(k-1)^2} d_{k-1}^{\binom{k}{k-1}(2^k-1)} d_{k-2}^{\binom{k}{k-2}(2^{k-1}-1)}, \quad \dots$$

$$\dots \quad \zeta_{i-1} = \frac{d_k^2 \xi^2}{128(k-1)^2} \prod_{j=i-1}^{k-1} d_j^{\binom{k}{j}(2^{j+1}-1)}. \tag{65}$$

We define δ_{i-1} similarly to how we defined δ_{k-1} (cf. (64)). In particular, we want $\delta_{i-1} > 0$ to be small enough so that (a) is satisfied with ξ from (61). This task is handled by Theorem 6.2, which has the following quantification of constants:

$$\forall d_k,\ \forall \delta_k,\ \exists \xi :\ \forall d_{k-1},\ \exists \delta_{k-1} : \dots, \forall d_{i-1},\ \exists \delta_{i-1} : \dots$$

With $d_k, \delta_k > 0$ given above, with ξ given in (61), and with $d_{k-1}, \delta_{k-1}, \dots, d_{i-1}$ inductively disclosed above, let

$$\delta_{\text{Thm.6.2},\,i-1} = \delta_{\text{Thm.6.2},\,i-1}(k, d_k, \delta_k, \xi, d_{k-1}, \delta_{k-1}, \dots, d_{i-1}) > 0$$

be the constant guaranteed by Theorem 6.2. We also want $\delta_{i-1} > 0$ to be small enough so that (b) above is satisfied with $\mu > 0$ from (63). Moreover, we want $\delta_{i-1} > 0$ to be small enough so that the following sequence (b') of conditions is satisfied (cf. (63)):

(b')
- we can estimate $|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ within an error of $1 \pm \mu$;
- we can estimate $|\mathcal{K}_k(\mathcal{H}^{(k-2)})|$ within an error of $1 \pm \mu$;

$$\vdots$$

- we can estimate $|\mathcal{K}_k(\mathcal{H}^{(i-1)})|$ within an error of $1 \pm \mu$.

To guarantee that the sequence (b') of conditions above will be satisfied, we fix an integer h satisfying $i-1 \le h \le k-1$, and appeal to Theorem 5.1, which has the following quantification of constants:

$$\forall \mu,\ \forall d_h,\ \exists \delta_h : \forall d_{h-1},\ \exists \delta_{h-1} :\ \dots \forall d_{i-1},\ \exists \delta_{i-1} :\ \dots$$

With $\mu > 0$ from (63), and with $d_h, \delta_h, \dots, d_{i-1} > 0$ inductively disclosed above, let

$$\delta_{\text{Thm.5.1},\,i-1,\,h} = \delta_{\text{Thm.5.1},\,i-1,\,h}(\ell = k, h, \mu, d_h, \delta_h, \dots, d_{i-1}) > 0$$

be the constant guaranteed by Theorem 5.1. Set

$$\delta_{\text{Thm.5.1},\,i-1} = \min\left\{\delta_{\text{Thm.5.1},\,i-1,\,h} : i-1 \le h \le k-1\right\}.$$

Finally, we also want $\delta_{i-1} > 0$ to be small enough so that (c) above is satisfied with $\zeta_{k-1} > 0$ from (63). Moreover, we want $\delta_{i-1} > 0$ to be small enough so that the following sequence (c') of conditions is satisfied (cf. (65)):

(c')
- all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K) = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1} \times n^k,$$

and all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2$ pairs of cliques $K \neq K' \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K, K') = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^k;$$

- all but $\zeta_{k-2}|\mathcal{K}_k(\mathcal{H}^{(k-2)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(k-2)})$ satisfy

$$\deg_{\Gamma_{k-2}}(K) = (1 \pm \zeta_{k-2}) \prod_{j=2}^{k-2} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j - 1} \times n^k,$$

and all but $\zeta_{k-2}|\mathcal{K}_k(\mathcal{H}^{(k-2)})|^2$ pairs of cliques $K \neq K' \in \mathcal{K}_k(\mathcal{H}^{(k-2)})$ satisfy

$$\deg_{\Gamma_{k-2}}(K, K') = (1 \pm \zeta_{k-2}) \prod_{j=2}^{k-2} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^k;$$

$$\vdots$$

- all but $\zeta_{i-1}|\mathcal{K}_k(\mathcal{H}^{(i-1)})|$ cliques $K \in \mathcal{K}_k(\mathcal{H}^{(i-1)})$ satisfy

$$\deg_{\Gamma_{i-1}}(K) = (1 \pm \zeta_{i-1}) \prod_{j=2}^{i-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j - 1} \times n^k,$$

and all but $\zeta_{i-1}|\mathcal{K}_k(\mathcal{H}^{(i-1)})|^2$ pairs of cliques $K \neq K' \in \mathcal{K}_k(\mathcal{H}^{(i-1)})$ satisfy

$$\deg_{\Gamma_{i-1}}(K, K') = (1 \pm \zeta_{i-1}) \prod_{j=2}^{i-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^k.$$

To guarantee that the sequence (c') of conditions above will be satisfied, we fix an integer h satisfying $i - 1 \leq h \leq k - 1$, and appeal to Theorem 5.4, which has the following quantification of constants:

$$\forall \zeta_h,\ \forall d_h,\ \exists \delta_h : \forall d_{h-1},\ \exists \delta_{h-1} : \dots \forall d_{i-1},\ \exists \delta_{i-1} : \dots$$

With $\zeta_h > 0$ from (65), and with $d_h, \delta_h, \dots, d_{i-1} > 0$ inductively disclosed above, let

$$\delta_{\text{Thm.5.4},\ i-1,h} = \delta_{\text{Thm.5.4},\ i-1,h}(\ell = k, h, \zeta_h, d_h, \delta_h, \dots, d_{i-1}) > 0$$

be the constant guaranteed by Theorem 5.4. Set

$$\delta_{\text{Thm.5.4},\ i-1} = \min \left\{ \delta_{\text{Thm.5.4},\ i-1,h} : i - 1 \leq h \leq k - 1 \right\}.$$

Now, set

$$\delta_{i-1} = \min \left\{ \delta_{\text{Thm.6.2},\ i-1}, \delta_{\text{Thm.5.1},\ i-1}, \delta_{\text{Thm.5.4},\ i-1} \right\}. \tag{66}$$

This concludes our definition of the promised constant $\delta_{i-1} > 0$. We continue this way until $\delta_2 > 0$ is reached.

It remains to define the integer r_0 promised by Theorem 2.16. To that end, set

$$r_0 = 2 \prod_{j=2}^{k-1} d_j^{\binom{k}{j}(2 - 2^j)}, \tag{67}$$

where we omit floors and ceilings for simplicity. Finally, in all that follows, we take the integer n_0 to be sufficiently large whenever needed. This concludes our description of the promised constants.

7.2. **The Algorithm for Theorem 2.16.** Set $\boldsymbol{\delta}_{k-1} = (\delta_2, \dots, \delta_{k-1})$, where each δ_j, $2 \le j \le k-1$, was defined in (66). Let $\boldsymbol{\mathcal{H}}^{(k)}$ be a (k,k)-complex with density sequence $\boldsymbol{d}_k$, as given in Setup 2.15, where $n \ge n_0$. Suppose $\boldsymbol{d}_k$ satisfies that, for each $2 \le j \le k$, $d_{\Lambda_j} \ge d_j$ for all $\Lambda_j \in [k]^j$, where d_j was given above. Suppose $\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{d}_{k-1})$, but that $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$. Our goal is to construct, in time $O(n^{3k})$, a collection of subhypergraphs $\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$, where $r \le r_0$ (cf. (67)), so that

$$\Big| \bigcup_{i \in [r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \Big| > \delta_k' \left| \mathcal{K}_k(\mathcal{H}^{(k-1)}) \right| \quad \text{and} \quad \left| d\left(\mathcal{H}^{(k)} | \mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) - d_{[k]} \right| > \delta_k', \qquad (68)$$

where δ_k' was defined in (62).

Our algorithm will take place in five steps. Before emerging into techical details, we give an overview of the algorithm.

- Assumptions (1) and (2) of Theorem 2.16 allow us to apply the Negative-Extension Lemma to the (k,k)-complex $\boldsymbol{\mathcal{H}}^{(k)}$. In **Step 1**, we will apply Theorem 6.2 to $\boldsymbol{\mathcal{H}}^{(k)}$ to conclude that there exists a hypergraph $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that, for any anchor A for which $\bar{A} \in \mathcal{S}^{(k)}$, the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}})$ is false. In order to find the hypergraph $\mathcal{S}^{(k)}$, we will test, for each $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$, for each fixed choice of anchor A with $\bar{A} \in \mathcal{S}^{(k)}$, and for each k-tuple $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$, whether or not $\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K)$ is 'close' to what is expected. Since $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}})$ is false, our search will find some $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$ so that, for any anchor A with $\bar{A} \in \mathcal{S}^{(k)}$, 'many' $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ will have $\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K)$ being 'far' from what is expected. The running time of Step 1 will be $O(n^{2k})$.

  We will assume, w.l.o.g., that many of the k-tuples K above have $\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K)$ being 'too large', and we will denote the set of such K by $\mathcal{A}_{\mathrm{bad},+}^{(k)}$.

While Step 1 involved the (k,k)-complex $\boldsymbol{\mathcal{H}}^{(k)} = \{\mathcal{H}^{(j)}\}_{j=1}^k$, Steps 2-4 will consider the underlying $(k,k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$.

- Assumption (1) of Theorem 2.16 allows us to apply the Extension Lemma to the $(k,k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)}$. In **Step 2**, we will apply Theorem 5.4 to $\boldsymbol{\mathcal{H}}^{(k-1)}$ to conclude that 'nearly' all $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ have $\deg_{\Gamma_{k-1}}(K)$ being 'close' to what is expected. We will remove, one-by-one, all k-tuples $K \in \mathcal{A}_{\mathrm{bad},+}^{(k)}$ (see Step 1) for which $\deg_{\Gamma_{k-1}}(K)$ is 'far' from what is expected. The application of Theorem 5.4 will guarantee that 'many' k-tuples of $\mathcal{A}_{\mathrm{bad},+}^{(k)}$ remain after this removal, and we will denote this remaining set by $\widetilde{\mathcal{A}}_{\mathrm{bad},+}^{(k)}$. The running time of Step 2 will be $O(n^{2k})$.

- Assumption (1) of Theorem 2.16 allows us to apply the Extension Lemma to each of the complexes $\boldsymbol{\mathcal{H}}^{(k-1)}$, $\boldsymbol{\mathcal{H}}^{(k-2)}$, …, $\boldsymbol{\mathcal{H}}^{(2)}$. In **Step 3**, we will apply Theorem 5.4 to each of the complexes $\boldsymbol{\mathcal{H}}^{(k-1}$, $\boldsymbol{\mathcal{H}}^{(k-2)}$, …, $\boldsymbol{\mathcal{H}}^{(2)}$ to conclude that, for each $2 \le i \le k-1$, 'nearly' all pairs of k-tuples $K \ne K' \in \mathcal{K}_k(\mathcal{H}^{(i)})$ have $\deg_{\Gamma_i}(K, K')$ being 'close' to what is expected. For each $2 \le i \le k-1$, we will consider the auxiliary graph G_i formed by the set of pairs $K \ne K' \in \mathcal{K}_k(\mathcal{H}^{(i)})$ for which $\deg_{\Gamma_i}(K, K')$ is 'far' from what is expected.

The application of Theorem 5.4 will then guarantee that the graphs G_i, $2 \le i \le k-1$, are 'sparse'. The running time of Step 3 will be $O(n^{3k})$.

- In **Step 4**, we will apply the Picking Lemma to the set $Y = \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+} \subseteq \mathcal{K}_{k-1}(\mathcal{H}^{(k-1)}) = X$ (see Step 2) and the graphs G_i (see Step 3), $2 \le i \le k-1$. Lemma 7.1 will choose a set $Z = Z_r \subset Y = \widetilde{\mathcal{A}}^{(k)}_{\text{bad}}$ of size $r \le r_0$ (cf. (67)) so that, for each $2 \le i \le k-1$, the induced subgraph $G_i[Z]$ is still 'sparse'. In other words, for each $2 \le i \le k-1$, most pairs $K \ne K' \in G_i[Z]$ will have $\deg_{\Gamma_i}(K, K')$ being 'close' to what is expected. This property will be a key detail in Step 5. The running time of Step 4 will be $O(n^{3k})$.

- In **Step 5**, we will observe that each $K \in Z$ defines a complex $\boldsymbol{\mathcal{Q}}_K = \{\mathcal{Q}^{(j)}_K\}_{j=2}^{k-1}$ which is a subcomplex of $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=2}^{k-1}$. We will show that the collection $\mathcal{Q}^{(k-1)}_K \in \boldsymbol{\mathcal{Q}}^{(k-1)}$, over all $K \in Z$, is precisely the r-witness we promised in (68). A key ingredient in verifying that $\mathcal{Q}^{(k-1)}_K$, over all $K \in Z$, is the promised r-witness will be that each graph $G_i[Z]$, $2 \le i \le k-1$, is 'sparse'. The running time of Step 5 will be $O(n^k)$.

We now proceed to fill in the details of the outline above, beginning with Step 1.

Step 1: Applying the Negative-Extension Lemma. By Assumptions (1) and (2) of Theorem 2.16, the $(k, k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but $(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$, where $d_{[k]} = d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)})$. As such, with ξ given in (61), Theorem 6.2 guarantees the existence of a subhypergraph $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ so that, for any anchor A with $\bar{A} \in \mathcal{S}^{(k)}$, the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false. Now, with a greedy search, we determine the hypergraph $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$, and we find a 'large' set of k-tuples $K \in \mathcal{K}_k(\mathcal{H}^{(k)})$ witnessing that the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false. Indeed, for each $\varnothing \ne \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$, fix an arbitrary anchor A for which $\bar{A} \in \mathcal{S}^{(k)}$. As we did in (24), define

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k)}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \begin{cases} \mathcal{H}^{(k)} & \text{if } A \in \mathcal{S}^{(k)}, \\ \mathcal{K}_k(\mathcal{H}^{(k-1)}) & \text{if } A \notin \mathcal{S}^{(k)}. \end{cases}$$

Now, for each $K \in \mathcal{A}^{(k)}$,

$$\text{test if } \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K) > \xi \deg_{\Gamma_{k-1}}(K). \tag{69}$$

Since

$$V(\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})) \subseteq V(\Gamma_{k-1}) = \mathcal{K}_k(\mathcal{H}^{(k-1)}),$$

where $|\mathcal{K}_k(\mathcal{H}^{(k-1)})| = O(n^k)$, the test in (69) can be done in time $O(n^k)$. If (69) holds,

$$\text{test if } \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K). \tag{70}$$

The test in (70) can similarly be done in time $O(n^k)$. Thus, over all $K \in \mathcal{A}^{(k)}$, the tests of (69) and (70) can be done in time $O(n^{2k})$.

Now, set (cf. (25))

$$\mathcal{A}^{(k)}_{\text{bad}} = \mathcal{A}^{(k)}_{\text{bad}}(\mathcal{S}^{(k)}, A, \xi, \boldsymbol{\mathcal{H}}^{(k)}) = \Big\{ K \in \mathcal{A}^{(k)} : \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) > \xi \deg_{\Gamma_{k-1}}(K) \\ \text{but} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) \ne (1 \pm \xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \bar{A})}(K) \Big\},$$

which we have identified in time $O(n^{2k})$. Since the statement $\mathbf{EXT}_A(\mathcal{S}^{(k)}, \xi, \boldsymbol{\mathcal{H}}^{(k)})$ is false, there must be some $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$ so that, for any anchor A with $\bar{A} \in \mathcal{S}^{(k)}$, we have

$$\left|\mathcal{A}^{(k)}_{\text{bad}}\right| \geq \xi|\mathcal{A}^{(k)}| \geq \xi|\mathcal{H}^{(k)}| = \xi d_{[k]}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|, \tag{71}$$

where we used that $d_{[k]} = d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)})$. Moreover, the tests of (69) and (70) will (eventually) find the hypergraph $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$ and the corresponding set $\mathcal{A}^{(k)}_{\text{bad}}$, all in time $O(n^{2k})$. (For the remainder of this section, we fix an arbitrary anchor A with $\bar{A} \in \mathcal{S}^{(k)}$.)

We now refine the set $\mathcal{A}^{(k)}_{\text{bad}}$, as follows. Denote by $\mathcal{A}^{(k)}_{\text{bad},+}$ the set of k-tuples $K \in \mathcal{A}^{(k)}_{\text{bad}}$ for which

$$\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) > (1+\xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K), \tag{72}$$

and set $\mathcal{A}^{(k)}_{\text{bad},-} = \mathcal{A}^{(k)}_{\text{bad}} \setminus \mathcal{A}^{(k)}_{\text{bad},+}$. Then, one of $\mathcal{A}^{(k)}_{\text{bad},+}$ or $\mathcal{A}^{(k)}_{\text{bad},-}$ has size at least $\frac{1}{2}|\mathcal{A}^{(k)}_{\text{bad}}|$. In our proof, it will be symmetric to handle these two cases, so we assume, w.l.o.g., that

$$\left|\mathcal{A}^{(k)}_{\text{bad},+}\right| \geq \frac{1}{2}\left|\mathcal{A}^{(k)}_{\text{bad}}\right| \overset{(71)}{\geq} \frac{1}{2}\xi d_{[k]}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|. \tag{73}$$

Clearly, the set $\mathcal{A}^{(k)}_{\text{bad},+}$ can be found in time $O(n^{2k})$, since we will, in fact, identify it as we build $\mathcal{A}^{(k)}_{\text{bad}}$. We now proceed to Step 2.

Step 2: Applying the Extension Lemma to $\boldsymbol{\mathcal{H}}^{(k-1)}$. We apply Theorem 5.4 to the $(k, k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)}$ to further refine the set $\mathcal{A}^{(k)}_{\text{bad},+}$. To that end, by Assumption (1) in the hypothesis of Theorem 2.16, $\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$. With ζ_{k-1} given in (63), Statement (1) of the Extension Lemma guarantees that all but $\zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|$ many elements $K \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$ satisfy

$$\deg_{\Gamma_{k-1}}(K) = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j - 1} \times n^k. \tag{74}$$

Now, let $\widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ denote the set of k-tuples $K \in \mathcal{A}^{(k)}_{\text{bad},+}$ for which (74) holds. Since $\zeta_{k-1} < \frac{1}{4}\xi d_k \leq \frac{1}{4}\xi d_{[k]}$ from (63), we infer from (73) that

$$\left|\widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}\right| \geq \frac{1}{4}\xi d_{[k]}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|. \tag{75}$$

Moreover, we can identify the set $\widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$, arguing similarly as in Step 1.

For future reference, let us now review that every element $K \in \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ has the following properties (on account of (71), (72), and (74)):

$$\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K) > \xi \deg_{\Gamma_{k-1}}(K), \quad \text{where} \quad \deg_{\Gamma_{k-1}}(K) = (1 \pm \zeta_{k-1}) \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j-1},$$
$$\text{and} \quad \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) > (1+\xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K). \tag{76}$$

We now proceed to Step 3.

Step 3: Applying the Extension Lemma to each of $\boldsymbol{\mathcal{H}}^{(k-1)}, \ldots, \boldsymbol{\mathcal{H}}^{(2)}$. We now apply Theorem 5.4 to each of the complexes $\boldsymbol{\mathcal{H}}^{(k-1)}, \ldots, \boldsymbol{\mathcal{H}}^{(2)}$. To that end, for each $2 \leq i \leq k-1$

and with ζ_i given in (65), Statement (2) of the Extension Lemma guarantees that all but $\zeta_i|\mathcal{K}_k(\mathcal{H}^{(i)})|^2$ many pairs $K, K' \in \mathcal{K}_k(\mathcal{H}^{(i)})$ satisfy

$$\deg_{\Gamma_i}(K,K') \le (1+\zeta_i)\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}\times n^k < 2\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}\times n^k. \tag{77}$$

We now record, for each $2\le i\le k-1$, the pairs $K\neq K'\in\mathcal{K}_k(\mathcal{H}^{(i)})$ for which (78) fails. Indeed, for each $2\le i\le k-1$, let G_i be the graph with vertex set $V(G_i)=\mathcal{K}_k(\mathcal{H}^{(k-1)})$ and edge set

$$G_i=\left\{\{K,K'\}\in\binom{\mathcal{K}_k(\mathcal{H}^{(k-1)})}{2} : \deg_{\Gamma_i}(K,K') > 2\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}\times n^k\right\}. \tag{78}$$

Note that the graphs G_i, $2\le i\le k-1$, may be constructed in time $O(n^{3k})$. Indeed, for each $2\le i\le k-1$, the graph G_i has vertex set $\mathcal{K}_k(\mathcal{H}^{(k-1)})$ and the graph Γ_i has vertex set $\mathcal{K}_k(\mathcal{H}^{(i)})\supseteq\mathcal{K}_k(\mathcal{H}^{(k-1)})$, where $|\mathcal{K}_k(\mathcal{H}^{(i)})|=O(n^k)$. As such, we may greedily test the Γ_i-codegree of pairs of vertices of G_i in time $O(n^{3k})$.

Now, for each $2\le i\le k-1$, the application of Theorem 5.4 in (77) gives $|G_i|<\zeta_i|\mathcal{K}_k(\mathcal{H}^{(i)})|^2$. Since

$$V(G_i)=\mathcal{K}_k(\mathcal{H}^{(k-1)})\subseteq\mathcal{K}_k(\mathcal{H}^{(k-2)})\subseteq\cdots\subseteq\mathcal{K}_k(\mathcal{H}^{(i)}),$$

we rewrite $|G_i|\le\zeta_i|\mathcal{K}_k(\mathcal{H}^{(i)})|^2$ in terms of $|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2$. For $i=k-1$, nothing needs to be done. For $2\le i\le k-2$, we employ Theorem 5.1, which says that for each $2\le i\le k-1$,

$$|\mathcal{K}_k(\mathcal{H}^{(i)})| = (1\pm\mu)\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}\times n^k, \tag{79}$$

where μ is given in (63). As such,

$$|\mathcal{K}_k(\mathcal{H}^{(i)})|^2 \le (1+\mu)^2\prod_{j=2}^{i}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^2\times n^{2k} \quad\text{and}\quad |\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2 \ge (1-\mu)^2\prod_{j=2}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^2\times n^{2k},$$

in which case

$$\frac{|G_i|}{|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2}\le\zeta_i\frac{|\mathcal{K}_k(\mathcal{H}^{(i)})|^2}{|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2}\le\zeta_i\left(\frac{1+\mu}{1-\mu}\right)^2\prod_{j=i+1}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{-2}\overset{(63)}{\le}2\zeta_i\prod_{j=i+1}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{-2},$$

or equivalently,

$$|G_i|\le 2\zeta_i\prod_{j=i+1}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{-2}\times|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2.$$

Altogether, we conclude that for each $2\le i\le k-1$,

$$|G_i| < \begin{cases} \zeta_{k-1}|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2 & \text{if } i=k-1,\\ 2\zeta_i\prod_{j=i+1}^{k-1}\prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{-2}\times|\mathcal{K}_k(\mathcal{H}^{(k-1)})|^2 & \text{if } 2\le i\le k-2.\end{cases} \tag{80}$$

We now proceed to Step 4.

Step 4: Applying the Picking Lemma. In the context of the Picking Lemma, set $X = \mathcal{K}_k(\mathcal{H}^{(k-1)})$, where here we write $|X| = m$, and let $G_2, \ldots, G_{k-1}$ be the graphs constructed in (78) of Step 2 on the common vertex set X. Set, for each $2 \le i \le k-1$ and ζ_i given in (65),

$$\sigma_i = \begin{cases} \zeta_{k-1} & \text{if } i = k-1, \\ 2\zeta_i \prod_{j=i+1}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2} & \text{if } 2 \le i \le k-1. \end{cases} \tag{81}$$

Then, (80) gives that, for each $2 \le i \le k-1$, $|G_i| < \sigma_i m^2$. Set

$$c = \frac{1}{4} d_{[k]} \xi, \tag{82}$$

and set $Y = \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ (cf. (75) and (76)). Then,

$$Y = \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+} \subseteq \mathcal{A}^{(k)}_{\text{bad},+} \subseteq \mathcal{A}^{(k)}_{\text{bad}} \subseteq \mathcal{A}^{(k)} \subseteq \mathcal{K}_k(\mathcal{H}^{(k-1)}) = X,$$

and (75) and (82) give $|Y| \ge cm$. Set

$$r = 2\sqrt{\delta'_k} \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2-2^j}, \tag{83}$$

where we omit floors and ceilings for simplicity. Note that, as defined in (83), we have

$$r = 2\sqrt{\delta'_k} \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2-2^j} \le 2 \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2-2^j} \le 2 \prod_{j=2}^{k-1} d_j^{\binom{k}{j}(2-2^j)} \overset{(67)}{=} r_0, \tag{84}$$

where we used that, for each $2 \le j \le k-1$, we have $d_{\Lambda_j} \ge d_j$ for all $\Lambda_j \in [k]^j$.

We apply the (Algorithmic) Picking Lemma (Lemma 7.1) (with $s = k-1$) to select, in time $O(m^3) = O(n^{3k})$, vertices $Z = Z_r = \{K_1, \ldots, K_r\} \subset Y = \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ so that, for each $2 \le i \le k-1$, $|G_i[Z]| < (2(k-1)\sigma_i/c^2)r^2$. The selected vertices $Z = \{K_1, \ldots, K_r\} \subset \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ will play a critical role in our algorithm. One key use we will later have of Z (in Step 4) is summarized in the following claim.

Claim 7.2.

$$\sum_{1 \le a < b \le r} \deg_{\Gamma_{k-1}}(K_a, K_b) \le 2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2 \cdot 2^j - 3} \times n^k.$$

We prefer not to break the flow of the algorithm, and therefore defer this proof until Section 7.3. We continue with Step 5 of our algorithm, which will conclude the proof of Theorem 2.16.

Step 5: Constructing the subhypergraphs $\mathcal{Q}_1^{(k-1)}, \ldots, \mathcal{Q}_r^{(k-1)}$. In Steps 1 and 2, we constructed, in time $O(n^{2k})$ the set $\widetilde{\mathcal{A}}^{(k)}_{\text{bad},+} \subseteq \mathcal{A}^{(k)}$ (cf. (75) and (76)). In Step 3, we constructed, in time $O(n^{3k})$, the graphs G_i, $2 \le i \le k-1$, defined in (78). In Step 4, we used the Picking Lemma to select, in time $O(n^{3k})$, a subset $Z = \{K_1, \ldots, K_r\} \subset \widetilde{\mathcal{A}}^{(k)}_{\text{bad},+}$ for which Claim 7.2 holds. The following claim, which is the last subroutine for proving Theorem 2.16, will now allow us to construct the subhypergraphs $\mathcal{Q}_1^{(k-1)}, \ldots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$ promised in (68).

Claim 7.3. *Recall the hypergraph $\mathcal{S}^{(k)}$ determined in Step 1, whose anchor satisfies $\bar{A} \in \mathcal{S}^{(k)}$. Then, for each $K \in \mathcal{A}^{(k)}$ (cf. (24)), one may construct, in time $O(n^{k-1})$, a $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{k-1}$, where $\mathcal{Q}_K^{(j)} \subseteq \mathcal{H}^{(j)}$ for each $j \in [k-1]$, so that*

(1) $N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K) = \mathcal{K}_k(\mathcal{Q}_K^{(k-1)})$;

Consequently,

(2) $N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = \mathcal{H}^{(k)} \cap N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K) = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{Q}_K^{(k-1)})$.

Remark 7.4. Claim 7.3 holds more generally than we've stated above. In particular, Statement (1) of Claim 7.3 is true for all k-graphs $\varnothing \subseteq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$, and for all choices of anchors A. Statement (2) of Claim 7.3 is a consequence of Statement (1), since

$$N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K) = \mathcal{H}^{(k)} \cap N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K)$$

is a basic identity of the graphs $\vec{\Gamma}_A(\mathcal{S}^{(k)})$ and $\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})$. Thus, Statement (2) of Claim 7.3 is true for all hypergraphs $\varnothing \neq \mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)}$, and for all choices of anchors A for which $\bar{A} \in \mathcal{S}^{(k)}$.

The proof of Claim 7.3 is mechanical, but not difficult. We will first show how Claim 7.3 concludes the proof of Theorem 2.16. We return to the proof of Claim 7.3 in Section 7.4.

To finish the proof of Theorem 2.16, fix $K_i \in Z$, and let $\boldsymbol{\mathcal{Q}}_i^{(k-1)} = \boldsymbol{\mathcal{Q}}_{K_i}^{(k-1)}$ be the $(k, k-1)$-complex constructed in Claim 7.3. For each $1 \leq i \leq r$, we define $\mathcal{Q}_i^{(k-1)} = \mathcal{Q}_{K_i}^{(k-1)} \in \boldsymbol{\mathcal{Q}}_i^{(k-1)}$, and so by Claim 7.3, we have $\mathcal{Q}_i^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$. We prove the hypergraphs $\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$ satisfy the conclusion of Theorem 2.16. Indeed, we already noted in (84) that $r \leq r_0$, as required by Theorem 2.16. As well, it follows from our discussion above that the hypergraphs $\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$ were constructed in time $O(n^{3k})$, as required by Theorem 2.16. It remains to verify the conditions in (68), which we separate into the following two parts.

Fact 7.5.

$$\left| \bigcup_{i \in [r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \right| > \delta'_k \left| \mathcal{K}_k(\mathcal{H}^{(k-1)}) \right|.$$

Fact 7.6.

$$d\left(\mathcal{H}^{(k)} | \mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) > d_{[k]} + \delta'_k.$$

We proceed immediately to the proofs of Facts 7.5 and 7.6.

Proof of Fact 7.5. We use Inclusion-Exclusion to conclude

$$\left| \bigcup_{i \in [r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \right| \geq \sum_{i \in [r]} \left| \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \right| - \sum_{1 \leq i < j \leq r} \left| \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \cap \mathcal{K}_k(\mathcal{Q}_j^{(k-1)}) \right|. \tag{85}$$

To bound the sums above, recall from Claim 7.3 that, for each $i \in [r]$,

$$\left| \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \right| = \left| N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K_i) \right| = \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K_i). \tag{86}$$

Claim 7.3 also gives that, for each $1 \leq i < j \leq r$,

$$\left| \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \cap \mathcal{K}_k(\mathcal{Q}_j^{(k-1)}) \right| = \left| N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K_i) \cap N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K_j) \right| \leq \deg_{\Gamma_{k-1}}(K_i, K_j), \tag{87}$$

where the last inequality holds because $\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\}) \subseteq \Gamma_{k-1}$, which holds because every labeled partite-embedding of $\mathcal{S}^{(k)} \setminus \{\bar{A}\}$ in $\boldsymbol{\mathcal{H}}^{(k)}$ is also a labeled partite-embedding of $\mathcal{O}^{(k-1)}$ in $\boldsymbol{\mathcal{H}}^{(k-1)}$. Applying (86) and (87) to (85) yields

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\Big| \geq \sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i) - \sum_{1\leq i<j\leq r} \deg_{\Gamma_{k-1}}(K_i, K_j).$$

Claim 7.2 immediately bounds the double summation above:

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\Big| \geq \sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i) - 2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k. \tag{88}$$

To bound the single summation in (88), we use that for every $1 \leq i \leq r$, the k-tuple $K_i \in Z \subset Y = \mathcal{A}^{(k)}_{\mathrm{bad},+}$ satisfies the following properties from (76):

$$\begin{aligned}\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i) &\overset{(76)}{>} \xi \deg_{\Gamma_{k-1}}(K_i)\\ &\overset{(76)}{>} \frac{1}{2}\xi \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1} \times n^k \overset{(62)}{=} 5(\delta_k')^{1/8} \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1} \times n^k.\end{aligned} \tag{89}$$

Applying (89) to (88) yields

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\Big| \geq 5r(\delta_k')^{1/8} \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2^j-1} \times n^k - 2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k.$$

Employing the value $r = 2\sqrt{\delta_k'} \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2-2^j}$ from (83) into the inequality above yields

$$\begin{aligned}\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\Big| &\geq 10(\delta_k')^{5/8} \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j} \times n^k - 8\delta_k' \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j} \times n^k\\ &\geq 2\delta_k' \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j} \times n^k \overset{(79)}{\geq} \delta_k' |\mathcal{K}_k(\mathcal{H}^{(k-1)})|.\end{aligned}$$

This proves Fact 7.5. □

Proof of Fact 7.6. By Inclusion-Exclusion, we have

$$\begin{aligned} d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \ldots, \mathcal{Q}_r^{(k-1)}\right) &= \frac{\left|\mathcal{H}^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|}{\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|}\\ &\geq \frac{\sum_{i\in[r]} \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| - \sum_{1\leq i<j\leq r} \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \cap \mathcal{K}_k(\mathcal{Q}_j^{(k-1)})\right|}{\sum_{i\in[r]} \left|\mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|}\\ &\geq \frac{\sum_{i\in[r]} \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| - \sum_{1\leq i<j\leq r} \left|\mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \cap \mathcal{K}_k(\mathcal{Q}_j^{(k-1)})\right|}{\sum_{i\in[r]} \left|\mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|}. \end{aligned} \tag{90}$$

Recall from Claim 7.3 that for each $i \in [r]$, we have

$$\left|\mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| = \left|N_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i)\right| = \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i),$$
$$\text{and} \quad \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| = \left|N_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K_i)\right| = \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K_i).$$

Recall from (87) that, for each $1 \le i < j \le r$, we have that $|\mathcal{K}_k(\mathcal{Q}_i^{(k-1)}) \cap \mathcal{K}_k(\mathcal{Q}_j^{(k-1)})| \le \deg_{\Gamma_{k-1}}(K_i, K_j)$. We may therefore update (90) to say

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge \frac{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K_i) - \sum_{1\le i<j\le r} \deg_{\Gamma_{k-1}}(K_i, K_j)}{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i)}.$$

Claim 7.2 immediately bounds the double summation above:

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge \frac{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K_i) - \left(2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j - 3} \times n^k\right)}{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i)}. \tag{91}$$

To bound the single summation in the numerator, we use that, for every $1 \le i \le r$, the k-tuple $K_i \in Z \subset Y = \mathcal{A}^{(k)}_{\text{bad},+}$ satisfies the following property from (76):

$$\deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)})}(K_i) \overset{(76)}{>} (1+\xi) d_{[k]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i). \tag{92}$$

Applying (92) to (91) yields

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge$$
$$\frac{(1+\xi)d_{[k]} \sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i) - \left(2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j - 3} \times n^k\right)}{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i)}$$
$$= (1+\xi)d_{[k]} - \frac{2r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j - 3} \times n^k}{\sum_{i\in[r]} \deg_{\vec{\Gamma}_A(\mathcal{S}^{(k)}\setminus\{\bar{A}\})}(K_i)}.$$

Employing (89) in the denominator, we have

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge (1+\xi)d_{[k]} - \frac{2r}{5(\delta_k')^{1/8}} \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2^j - 2}.$$

Employing the value $r = 2\sqrt{\delta_k'} \prod_{j=2}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2-2^j}$ from (83) into the inequality above yields

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge (1+\xi)d_{[k]} - \frac{4}{5}(\delta_k')^{3/8} = d_{[k]} + \xi d_{[k]} - \frac{4}{5}(\delta_k')^{3/8}.$$

Now, from the hypothesis of Theorem 2.16, we have $d_{[k]} \ge d_k$, and it follows from the definition of ξ in (61) that $\xi \le d_k \le d_{[k]}$. We therefore have

$$d\left(\mathcal{H}^{(k)}|\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\right) \ge d_{[k]} + \xi^2 - \frac{4}{5}(\delta_k')^{3/8}$$
$$\overset{(62)}{=} d_{[k]} + 100(\delta_k')^{1/4} - \frac{4}{5}(\delta_k')^{3/8} \ge d_{[k]} + 99(\delta_k')^{3/8} > d_{[k]} + \delta_k'.$$

This proves Fact 7.6. □

7.3. **Proof of Claim 7.2.** We shall prove, more generally, that for each $2 \le i \le k-1$,

$$\sum_{\{K_a,K_b\}\in G_i[Z]} \deg_{\Gamma_i}(K_a,K_b) \le \frac{i}{k-1} r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k. \tag{93}$$

We first show that (93) implies Claim 7.2. Indeed, observe that

$$\begin{aligned}\sum_{1\le a<b\le r} \deg_{\Gamma_{k-1}}(K_a,K_b) &= \sum_{\{K_a,K_b\}\in\binom{Z}{2}\setminus G_{k-1}} \deg_{\Gamma_{k-1}}(K_a,K_b) + \sum_{\{K_a,K_b\}\in G_{k-1}} \deg_{\Gamma_{k-1}}(K_a,K_b)\\ &\overset{(78)}{\le} 2\binom{r}{2} \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k + \sum_{\{K_a,K_b\}\in G_{k-1}} \deg_{\Gamma_{k-1}}(K_a,K_b).\end{aligned}$$

Employing (93) with $i = k-1$, we have

$$\sum_{\{a,b\}\in\binom{[r]}{2}} \deg_{\Gamma_{k-1}}(K_a,K_b) \le \left(1 + (k-1)\frac{1}{k-1}\right) r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k,$$

as desired.

To prove (93), we use induction on $2 \le i \le k-1$. Since the base case $i = 2$ will be implicit in the inductive step, we give its discussion in context (see (94) and (95) below). For $3 \le i \le k-1$, we have the recurrence

$$\begin{aligned}\sum_{\{K_a,K_b\}\in G_i[Z]} \deg_{\Gamma_i}(K_a,K_b) &\overset{(22)}{\le} \sum_{\{K_a,K_b\}\in G_i[Z]} \deg_{\Gamma_{i-1}}(K_a,K_b)\\ &= \sum_{\{K_a,K_b\}\in(G_i\setminus G_{i-1})[Z]} \deg_{\Gamma_{i-1}}(K_a,K_b) + \sum_{\{K_a,K_b\}\in(G_i\cap G_{i-1})[Z]} \deg_{\Gamma_{i-1}}(K_a,K_b)\\ &\overset{(78)}{\le} 2|G_i[Z]| \prod_{j=2}^{i-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \times n^k + \sum_{\{K_a,K_b\}\in G_{i-1}[Z]} \deg_{\Gamma_{i-1}}(K_a,K_b).\end{aligned} \tag{94}$$

Note that the last inequality of (94) also holds for $i = 2$. Indeed, when $i = 2$, the summation in (94) is zero, and the first term is $2|G_2[Z]|n^k$. However, when $i = 2$, the following stronger inequality holds:

$$\sum_{\{K_a,K_b\}\in G_2[Z]} \deg_{\Gamma_2}(K_a,K_b) \le |G_2[Z]| \times n^k. \tag{95}$$

Now, for $2 \le i \le k-1$, we claim that

$$2|G_i[Z]| \prod_{j=2}^{i-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3} \le \frac{1}{k-1} r^2 \prod_{j=2}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3},$$

or equivalently,

$$|G_i[Z]| \le \frac{r^2}{2(k-1)} \prod_{j=i}^{k-1} \prod_{\Lambda_j\in[k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}, \tag{96}$$

which, if true, completes our induction step.

To see (96), recall that the Picking Lemma ensures that $|G_i[Z]| \le 2(k-1)(\sigma_i/c^2)r^2$, where

$$\sigma_i \overset{(81)}{=} \begin{cases} \zeta_{k-1} & \text{if } i = k-1, \\ 2\zeta_i \prod_{j=i+1}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2} & \text{if } 2 \le i \le k-2, \end{cases}$$

and $c = \frac{1}{4} d_{[k]} \xi$ was given in (82). To bound $\sigma_{k-1} = \zeta_{k-1}$, we have

$$\sigma_{k-1} = \zeta_{k-1} \overset{(65)}{=} \frac{d_k^2 \xi^2}{128(k-1)^2} d_{k-1}^{k(2\cdot 2^{k-1}-1)} \le \frac{d_{[k]}^2 \xi^2}{128(k-1)^2} \prod_{\Lambda_{k-1} \in [k]^{k-1}} d_{\Lambda_{k-1}}^{2\cdot 2^{k-1}-1}$$
$$\le \frac{d_{[k]}^2 \xi^2}{128(k-1)^2} \prod_{\Lambda_{k-1} \in [k]^{k-1}} d_{\Lambda_{k-1}}^{2\cdot 2^{k-1}k-3},$$

where we used, for $j \in \{k-1, k\}$, $d_{\Lambda_j} \ge d_j$ for all $\Lambda_j \in [k]^j$. Thus, with $|G_{k-1}[Z]| \le 2(k-1)(\sigma_{k-1}/c^2)r^2$ and $c = \frac{1}{4} d_{[k]} \xi$, we have

$$|G_{k-1}[Z]| \le 2(k-1) \frac{\sigma_{k-1}}{c^2} r^2 \le \frac{r^2}{4(k-1)} \prod_{\Lambda_{k-1} \in [k]^{k-1}} d_{\Lambda_{k-1}}^{2\cdot 2^{k-1}-3} < \frac{r^2}{2(k-1)} \prod_{\Lambda_{k-1} \in [k]^{k-1}} d_{\Lambda_{k-1}}^{2\cdot 2^{k-1}-3},$$

which is (96) in the case $i = k-1$.

For $2 \le i \le k-2$, we have from (81) that $\sigma_i = 2\zeta_i \prod_{j=i+1}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2}$. To bound ζ_i in this expression, observe that

$$\zeta_i \overset{(65)}{=} \frac{d_k^2 \xi_1^2}{128(k-1)^2} \prod_{j=i}^{k-1} d_j^{\binom{k}{j}(2\cdot 2^j-1)} \le \frac{d_{[k]}^2 \xi^2}{128(k-1)^2} \prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j-1}$$
$$= \frac{d_{[k]}^2 \xi^2}{128(k-1)^2} \frac{\prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}}{\prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2}} \le \frac{d_{[k]}^2 \xi^2}{128(k-1)^2} \frac{\prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}}{\prod_{j=i+1}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2}},$$

where we used that, for each $2 \le j \le k$, $d_{\Lambda_j} \ge d_j$ for all $\Lambda_j \in [k]^j$. As such, we may bound σ_i by

$$\sigma_i \overset{(81)}{=} 2\zeta_i \prod_{j=i+1}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{-2} \le \frac{d_{[k]}^2 \xi^2}{64(k-1)^2} \prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j-3}.$$

Thus, with $|G_i[Z]| \le 2(k-1)(\sigma_i/c^2)r^2$ and $c = \frac{1}{4} d_{[k]} \xi$, we have

$$|G_i[Z]| \le 2(k-1) \frac{\sigma_i}{c^2} r^2 \le \frac{1}{2(k-1)} r^2 \prod_{j=i}^{k-1} \prod_{\Lambda_j \in [k]^j} d_{\Lambda_j}^{2\cdot 2^j-3},$$

which is (96). This proves Claim 7.2.

7.4. **Proof of Claim 7.3.** In this section, we construct the promised $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{k-1}$. In what follows, we use the following standard notation: for a k-uniform hypergraph $\mathcal{G}$, and for a set $L \subseteq V(\mathcal{G})$, let

$$N_{\mathcal{G}}(L) = \{J \subset V(\mathcal{G}) : L \cup J \in \mathcal{G}\},$$

denote the $\mathcal{G}$-*neighborhood* of L, which is a $(k-|L|)$-uniform hypergraph. Now, to construct the promised $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{k-1}$ is not difficult, but it is a bit formal. We begin with a discussion of the following example, where $k = 4$ and $\mathcal{S}^{(4)} = \mathcal{O}^{(4)} = K^{(4)}(U_1, U_2, U_3, U_4)$, that is, $\mathcal{S}^{(4)}$ is the complete 4-partite 4-uniform hypergraph with $|U_1| = |U_2| = |U_3| = |U_4| = 2$.

7.4.1. *Example: constructing* $\boldsymbol{\mathcal{Q}}^{(3)}$ *when* $\mathcal{S}^{(4)} = \mathcal{O}^{(4)} = K^{(4)}(U_1, U_2, U_3, U_4)$. Since $\mathcal{S}^{(4)} = \mathcal{O}^{(4)}$, we have that the anchor A satisfies $A \in \mathcal{S}^{(4)}$, and therefore, $\mathcal{A}^{(4)} = \mathcal{H}^{(4)}$ (cf. (24)). Now, fix $K \in \mathcal{A}^{(4)} = \mathcal{H}^{(4)}$, and let $K = \{v_1, v_2, v_3, v_4\}$, where $v_i \in V_i$ for all $1 \le i \le 4$.

We construct the promised $(4,3)$-complex $\boldsymbol{\mathcal{Q}}_K^{(3)} = \boldsymbol{\mathcal{Q}}^{(4)}_{\{v_1,v_2,v_3,v_4\}} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^3$ recursively. To begin, set

$$\mathcal{Q}_K^{(1)} = N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_3\}) \cup N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_4\}) \cup N_{\mathcal{H}^{(4)}}(\{v_1, v_3, v_4\}) \cup N_{\mathcal{H}^{(4)}}(\{v_2, v_3, v_4\}),$$

which is a $(4,1)$-cylinder since it is just a partition of vertices into four sets. Next, set

$$\mathcal{Q}_K^{(2)} = \Big(\bigcup_{1 \le i < j \le 4} N_{\mathcal{H}^{(4)}}(\{v_i, v_j\})\Big) \cap \mathcal{K}_2(\mathcal{Q}_K^{(1)}).$$

Then, $\mathcal{Q}_K^{(2)} = \mathcal{Q}^{(2)}_{\{v_1,v_2,v_3,v_4\}}$ consists of six bipartite graphs $\mathcal{Q}^{(2)}_{\{v_i,v_j\}}$, $1 \le i < j \le 4$, where for example,

$$\mathcal{Q}^{(2)}_{\{v_1,v_2\}} = N_{\mathcal{H}^{(4)}}(\{v_1, v_2\})\big[N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_3\}), N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_4\})\big]$$

is the subgraph of $N_{\mathcal{H}^{(4)}}(\{v_1, v_2\})$ induced on $N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_3\}) \cup N_{\mathcal{H}^{(4)}}(\{v_1, v_2, v_4\})$. Finally, set

$$\mathcal{Q}_K^{(3)} = \big(N_{\mathcal{H}^{(4)}}(v_1) \cup N_{\mathcal{H}^{(4)}}(v_2) \cup N_{\mathcal{H}^{(4)}}(v_3) \cup N_{\mathcal{H}^{(4)}}(v_4)\big) \cap \mathcal{K}_3(\mathcal{Q}_K^{(2)}).$$

Then, $\mathcal{Q}_K^{(3)}$ consists of four 3-partite 3-graphs $\mathcal{Q}_{v_1}^{(3)}$, $\mathcal{Q}_{v_2}^{(3)}$, $\mathcal{Q}_{v_3}^{(3)}$, $\mathcal{Q}_{v_4}^{(3)}$, where for example,

$$\mathcal{Q}_{v_1}^{(3)} = N_{\mathcal{H}^{(4)}}(v_1) \cap \mathcal{K}_3\big(\mathcal{Q}^{(2)}_{\{v_1,v_2\}} \cup \mathcal{Q}^{(2)}_{\{v_1,v_3\}} \cup \mathcal{Q}^{(2)}_{\{v_1,v_4\}}\big)$$

is the subhypergraph of $N_{\mathcal{H}^{(4)}}(v_1)$ induced on the triangles of $\mathcal{Q}^{(2)}_{\{v_1,v_2\}} \cup \mathcal{Q}^{(2)}_{\{v_1,v_3\}} \cup \mathcal{Q}^{(2)}_{\{v_1,v_4\}}$. This defines the $(4,3)$-complex $\boldsymbol{\mathcal{Q}}_K^{(3)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^3$, where it is clear that for each $1 \le j \le 3$, $\mathcal{Q}_K^{(j)}$ may be constructed in time $O(n^j)$.

7.4.2. *Defining* $\boldsymbol{\mathcal{Q}}_K^{(k-1)}$ *for general* $\mathcal{S}^{(k)}$. To define the $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{k-1}$ for a general $\mathcal{S}^{(k)} \subseteq \mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$, we proceed similarly to the example above. However, now we must define each of the (k,j)-cylinders $\mathcal{Q}_K^{(j)}$, $1 \le j \le k-1$, inductively. Moreover, we must be mindful of the fact that not all edges of $\mathcal{O}^{(k)} = K^{(k)}(U_1, \dots, U_k)$ may be present in $\mathcal{S}^{(k)}$. (In particular, we are only guaranteed that $\bar{A} \in \mathcal{S}^{(k)}$, by hypothesis.)

We begin by making a few initial preparations. We write the anchor A as $A = \{a_1, \dots, a_k\}$. We then write $\bar{A} = \{b_1, \dots, b_k\}$, which by hypothesis is an element of $\mathcal{S}^{(k)}$. Then, $U_i = \{a_i, b_i\}$ for all $i \in [k]$. Now, fix $K \in \mathcal{A}^{(k)}$, where we recall from (24) that $\mathcal{A}^{(k)} = \mathcal{H}^{(k)}$ if $A \in \mathcal{S}^{(k)}$, and $\mathcal{A}^{(k)} = \mathcal{K}_k(\mathcal{H}^{(k-1)})$ if $A \notin \mathcal{S}^{(k)}$. We write $K = \{v_1, \dots, v_k\}$, where $v_i \in V_i$ for all $i \in [k]$. We will construct the promised complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{k-1}$ recursively.

To construct the promised $(k,1)$-cylinder $\mathcal{Q}_K^{(1)}$, we consider the family $\binom{A}{k-1}$ of all $(k-1)$-tuples from the anchor A. To begin, for $A' = \{a_{h_1}, \dots, a_{h_{k-1}}\} \subset A$, write $A \setminus A' = \{a_{h_k}\}$ and write $K_{A'} = \{v_{h_1}, \dots, v_{h_{k-1}}\}$. Define

$$\mathcal{Q}_{A'}^{(1)} = \begin{cases} N_{\mathcal{H}^{(k)}}(K_{A'}) & \text{if } \{a_{h_1}, \dots, a_{h_{k-1}}, b_{h_k}\} \in \mathcal{S}^{(k)}, \\ N_{\mathcal{K}_k(\mathcal{H}^{(k-1)})}(K_{A'}) & \text{if } \{a_{h_1}, \dots, a_{h_{k-1}}, b_{h_k}\} \notin \mathcal{S}^{(k)}. \end{cases} \tag{97}$$

Define

$$\mathcal{Q}_K^{(1)} = \bigcup_{A' \in \binom{A}{k-1}} \mathcal{Q}_{A'}^{(1)},$$

and note that $\mathcal{Q}_K^{(1)}$ is a $(k,1)$-cylinder. Trivially, $\boldsymbol{\mathcal{Q}}_K^{(1)} = \{\mathcal{Q}_K^{(1)}\}$ is a $(k,1)$-complex whose sole component $\mathcal{Q}_K^{(1)}$ satisfies $\mathcal{Q}_K^{(1)} \subseteq V_1 \cup \cdots \cup V_k = \mathcal{H}^{(1)}$. Moreover, $\boldsymbol{\mathcal{Q}}_K^{(1)}$ may be constructed in time $O(n)$.

For $2 \leq i \leq k-1$, assume we have constructed, in time $O(n^{i-1})$, a $(k, i-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(i-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{i-1}$, where $\mathcal{Q}_K^{(j)} \subseteq \mathcal{H}^{(j)}$ holds for all $j \in [i-1]$. We construct, in time $O(n^i)$, a (k,i)-cylinder $\mathcal{Q}_K^{(i)} \subseteq \mathcal{H}^{(i)} \cap \mathcal{K}_k(\mathcal{Q}_K^{(i-1)})$ by considering the family $\binom{A}{k-i}$ of all $(k-i)$-tuples from $A = \{a_1, \ldots, a_k\}$. For $A' = \{a_{h_1}, \ldots, a_{h_{k-i}}\} \subset A$, write $A \setminus A' = \{a_{h_{k-i+1}}, \ldots, a_{h_k}\}$ and $K_{A'} = \{v_{h_1}, \ldots, v_{h_{k-i}}\}$. Define

$$\mathcal{Q}_{A'}^{(i)} = \begin{cases} N_{\mathcal{H}^{(k)}}(K_{A'}) \cap \mathcal{K}_i(\mathcal{Q}_K^{(i-1)}) & \text{if } \{a_{h_1}, \ldots, a_{h_{k-i}}, b_{h_{k-i+1}}, \ldots, b_{h_k}\} \in \mathcal{S}^{(k)}, \\ N_{\mathcal{K}_k(\mathcal{H}^{(k-1)})}(K_{A'}) \cap \mathcal{K}_i(\mathcal{Q}_K^{(i-1)}) & \text{if } \{a_{h_1}, \ldots, a_{h_{k-i}}, b_{h_{k-i+1}}, \ldots, b_{h_k}\} \notin \mathcal{S}^{(k)}. \end{cases} \tag{98}$$

Define

$$\mathcal{Q}_K^{(i)} = \bigcup_{A' \in \binom{A}{k-i}} \mathcal{Q}_{A'}^{(i)},$$

and note that $\mathcal{Q}_K^{(i)} \subseteq \mathcal{H}^{(i)} \cap \mathcal{K}_i(\mathcal{Q}_K^{(i-1)})$ is a (k,i)-cylinder. As such, together with our induction hypothesis, we may conclude that $\boldsymbol{\mathcal{Q}}_K^{(i)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{i}$ is a (k,i)-complex where $\mathcal{Q}_K^{(j)} \subseteq \mathcal{H}^{(j)}$ holds for each $j \in [i]$. Moreover, $\boldsymbol{\mathcal{Q}}_K^{(i)}$ may be constructed in time $O(n^i)$. Inductively, this defines the promised $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{(k-1)}$.

We claim that, by construction, the $(k, k-1)$-complex $\boldsymbol{\mathcal{Q}}_K^{(k-1)} = \{\mathcal{Q}_K^{(j)}\}_{j=1}^{(k-1)}$ has the properties promised by Claim 7.3. For that, it suffices to prove $\boldsymbol{\mathcal{Q}}_K^{(k-1)}$ satisfies Statement (1) of Claim 7.3 (see Remark 7.4). Indeed, fix $K' \in \mathcal{K}_k(\mathcal{H}^{(k-1)})$. Then, $K' \in N_{\vec{\Gamma}_A(\mathcal{S}^{(k)} \setminus \{\bar{A}\})}(K)$ if, and only if, there exists a labeled partite-embedding ψ of $\mathcal{S}^{(k)} \setminus \{\bar{A}\}$ in $\mathcal{H}^{(k)}$ satisfying $\psi(A) = K$ and $\psi(\bar{A}) = K'$. In other words, $K \cup K'$ induces a copy of $\mathcal{S}^{(k)} \setminus \{\bar{A}\}$ in $\mathcal{H}^{(k)}$, and $K \cup K'$ induces a copy of $\mathcal{O}^{(k-1)}$ in $\mathcal{H}^{(k-1)}$. However, our construction in (98) equivalently places $K' \in \mathcal{K}_{k-1}(\mathcal{Q}_K^{(k-1)})$, and vice-versa.

8. Appendix

8.1. **Proof of Lemma 7.1.** The proof of Lemma 7.1 (which is reduced to Claim 8.1 below) will make a standard appeal to the Method of Conditional Expectations (cf. [14, 20]), which is based on an original idea of Erdős and Selfridge [2]. Before we emerge into these details, we note that it suffices to prove Lemma 7.1 when $c = 1$ (and consequently, $Y = X$). In particular, let $\sigma_s, \ldots, \sigma_2 > 0$ be given, together with an integer $r \geq 1$. Let X be a set of size m, and let $G_2, \ldots, G_s$ be graphs with vertex set X satisfying $|G_2| \leq \sigma_2 m^2, \ldots, |G_s| \leq \sigma_s m^2$.

Suppose there exists an algorithm which chooses, in time $O(|X|^3) = O(m^3)$, vertices $Z = Z_r = \{z_1, \ldots, z_r\} \subset Y = X$ so that, for all $2 \leq i \leq s$,

$$|G_i[Z]| < 2(s-1)\sigma_i r^2. \tag{99}$$

Now, let $Y \subseteq X$ of size $|Y| \geq c|X|$ be given, where $c > 0$ is a constant. Note that the induced subgraphs $G_i[Y]$, $2 \leq i \leq s$, satisfy

$$|G_i[Y]| \leq |G_i| \leq \sigma_i m^2 \quad \Longrightarrow \quad \frac{|G_i[Y]|}{|Y|^2} \leq \sigma_i \frac{m^2}{|Y|^2} \leq \frac{\sigma_i}{c^2} \quad \Longrightarrow \quad |G_i[Y]| \leq \frac{\sigma_i}{c^2}|Y|^2.$$

We apply the algorithm in (99) to the induced subgraphs $G_i[Y]$, $2 \leq i \leq s$. This algorithm chooses, in time $O(|Y|^3) = O(m^3)$, vertices $Z = Z_r = \{z_1, \ldots, z_r\} \subset Y$ so that, for all $2 \leq i \leq s$,

$$|G_i[Z]| = |(G_i[Y])[Z]| \leq 2(s-1)\frac{\sigma_i}{c^2}r^2,$$

as desired.

We now prove (99), i.e., Lemma 7.1 when $c = 1$ and $Y = X$. To that end, let us assume, w.l.o.g., that

$$|G_2| = \sigma_2 m^2, \ \ldots, \ |G_s| = \sigma_s m^2, \quad \text{where} \quad \sigma_s = \max\{\sigma_2, \ldots, \sigma_s\}. \tag{100}$$

Now, for each $2 \leq i \leq s$, define the constant weight function $\omega_i : G_i \to \{\sigma_s/\sigma_i\}$, i.e., for each pair $\{x, x'\} \in G_i$, define

$$\omega_i(\{x, x'\}) = \sigma_s/\sigma_i. \tag{101}$$

Note that G_i has total weight

$$\omega_i(G_i) = \sum_{\{x,x'\} \in G_i} \omega_i(\{x, x'\}) = \frac{\sigma_s}{\sigma_i}|G_i| \stackrel{(100)}{=} \sigma_s m^2.$$

Define $G = G_2 \cup \cdots \cup G_s$. Then, G is a simple weighted graph on vertex set X whose weight function $\omega : G \to \mathbb{R}$ is given by, for each $\{x, x'\} \in G$,

$$\omega(\{x, x'\}) = \sum_{G_i \ni \{x,x'\}} \omega_i(\{x, x'\}). \tag{102}$$

Note that G has total weight

$$\omega(G) = \sum_{\{x,x'\} \in G} \omega(\{x, x'\}) = \sum_{i=2}^{s} \omega_i(G_i) = (s-1)\sigma_s m^2.$$

We make the following claim.

Claim 8.1. *There exists an algorithm which chooses, in time $O(m^3)$, vertices $Z = Z_r = \{z_1, \ldots, z_r\} \subset X$ so that $\omega(G[Z]) \leq 2(s-1)\sigma_s r^2$.*

We defer the proof of Claim 8.1 for a moment in favor of showing how it implies Lemma 7.1.

Let $Z = Z_r = \{z_1, \ldots, z_r\}$ be the set chosen by Claim 8.1. Fix $2 \leq i \leq s$. Then,

$$\omega_i(G_i[Z]) \leq \omega(G[Z]) \leq 2(s-1)\sigma_s r^2. \tag{103}$$

On the other hand, by (101), we have that

$$\omega_i(G_i[Z]) = \frac{\sigma_s}{\sigma_i}|G_i[Z]|. \tag{104}$$

Comparing (103) and (104), we see

$$\frac{\sigma_s}{\sigma_i}|G_i[Z]| = \omega_i(G_i[Z]) \leq \omega(G[Z]) \leq 2(s-1)\sigma_s r^2,$$

from which $|G_i[Z]| \leq 2(s-1)\sigma_i r^2$ follows. Thus, to finish the proof of Lemma 7.1, it only remains to prove Claim 8.1.

8.1.1. *Proof of Claim 8.1.* To select the promised vertices $Z = Z_r = \{z_1, \dots, z_r\} \subset X$, we use the following iterative procedure. For an integer $0 \le p < r$, suppose we have selected vertices $Z_p = \{z_1, \dots, z_p\} \subset X$ (if $p = 0$, then $Z_p = \varnothing$) satisfying the following property:

Let $A_{r-p} \subseteq X \setminus Z_p$ of size $|A_{r-p}| = r - p$ be selected uniformly at random.

$$\text{Then, we have } \mathbb{E}[\omega(G[Z_p \cup A_{r-p}])] \le 2(s-1)\sigma_s r^2. \quad (105)$$

Observe that (105) is true when $p = 0$. Indeed, in this case, $Z_0 = \varnothing$, and $A_r \subseteq X$ is an r-element set selected uniformly at random. Thus, using linearity of expectation, we see that

$$\mathbb{E}[\omega(G[A_r])] \overset{(102)}{=} \sum_{i=2}^{s} \mathbb{E}[\omega(G_i[A_r])] \overset{(101)}{=} \sum_{i=2}^{s} \frac{\sigma_s}{\sigma_i} \mathbb{E}[|G[A_r]|] = \sum_{i=2}^{s} \frac{\sigma_s}{\sigma_i} |G_i| \frac{\binom{r}{2}}{\binom{m}{2}}$$

$$\overset{(100)}{=} (1 + o(1))(s-1)\sigma_s r^2 < 2(s-1)\sigma_s r^2.$$

Thus, (105) is true when $p = 0$. It remains to prove that we may select, in time $O(m^2)$, a vertex $v \in X \setminus Z_p$ so that the set $Z_{p+1} = Z_p \cup \{z\}$ still satisfies the property in (105). Thus, we stop when $p = r$. Indeed, the set Z_r is the desired set, since then $A_{r-p} = \varnothing$, and so we will have, for all $2 \le i \le s$, $\mathbb{E}[\omega(G[Z_r])] = \omega(G[Z_r])$.

To prove the inductive step for (105), we make the following considerations. With the set $Z_p = \{z_1, \dots, z_p\}$ fixed above, define

$$g(Z_p) = g(z_1, \dots, z_p) = \mathbb{E}_{A_{r-p} \in \binom{X \setminus Z_p}{r-p}} [\omega(G[Z_p \cup A_{r-p}])],$$

where the expectation above is taken uniformly over all sets $A_{r-p} \subseteq X \setminus Z_p$ of size $|A_{r-p}| = r - p$. Thus, $g(Z_p)$ is the expected ω-weight of an induced subgraph $G[Z_p \cup A_{r-p}]$ whose vertices contain Z_p, where A_{r-p} runs uniformly over all $(r-p)$-element sets of $X \setminus Z_p$. By our Induction Assumption in (105), we have

$$g(Z_p) = g(z_1, \dots, z_p) \le 2(s-1)\sigma_s r^2. \quad (106)$$

Fix an arbitrary vertex $z \in X \setminus Z_p$, and write $Z_{p+1}^z = Z_p \cup \{z\}$. Define

$$f(z) = g(Z_{p+1}^z) = g(z_1, \dots, z_p, z) = \mathbb{E}_{A_{r-p-1} \in \binom{X \setminus Z_{p+1}^z}{r-p-1}} [\omega(G[Z_{p+1}^z \cup A_{r-p-1}])], \quad (107)$$

where the expectation above is taken uniformly over all sets $A_{r-p-1} \subseteq X \setminus Z_{p+1}^z$ of size $|A_{r-p-1}| = r-p-1$. Thus, $f(z)$ is the average ω-weight of an induced subgraph $G[Z_{p+1}^z \cup A_{r-p-1}]$ whose vertices contain Z_{p+1}^z, where A_{r-p-1} runs uniformly over all $(r-p-1)$-element subsets of $X \setminus Z_{p+1}^z$. As such, the quantity

$$\frac{1}{|X \setminus Z_p|} \sum_{z \in X \setminus Z_p} f(z) = \frac{1}{m-p} \sum_{z \in X \setminus Z_p} f(z)$$

is the average ω-weight of an induced subgraph $G[Z_p \cup A_{r-p}]$ whose vertices contain Z_p, where A_{r-p} runs uniformly over all $(r-p)$-element subsets of $X \setminus Z_p$. Therefore,

$$\frac{1}{m-p} \sum_{z \in X \setminus Z_p} f(z) = g(Z_p) = g(z_1, \dots, z_p) \overset{(106)}{\le} 2(s-1)\sigma_s r^2.$$

Thus, to complete the inductive step for (105), we prove that we may select, in time $O(m^3)$, a vertex $z_0 \in X \setminus Z_p$ so that

$$f(z_0) \le \frac{1}{m-p} \sum_{z \in X \setminus Z_p} f(z). \tag{108}$$

We now proceed to prove (108).

To prove (108), we shall compute, for a fixed vertex $z \in X \setminus Z_p$, the value of $f(z)$ (which is defined in (107)). This computation will take place in (117) below, but to get there, we will need several considerations. To begin, for $z \in X \setminus Z_p$ fixed, we continue to write $Z^z_{p+1} = Z_p \cup \{z\}$. For a vertex $x \in X \setminus Z^z_{p+1}$, let

$$\omega(G[\{x\}, Z^z_{p+1}])$$

denote the total ω-weight of all edges of the form $\{x, y\} \in G$, where $y \in Z^z_{p+1}$. (Note that $G[\{x\}, Z^z_{p+1}]$ is a star centered at x, with pendent vertices consisting of $N_G(x) \cap Z^z_{p+1}$.) Now, define the following equivalence relation $\sim$ on $X \setminus Z^z_{p+1}$ by setting, for each $x, x' \in X \setminus Z^z_{p+1}$,

$$x \sim x' \quad \iff \quad \omega(G[\{x\}, Z^z_{p+1}]) = \omega(G[\{x'\}, Z^z_{p+1}]). \tag{109}$$

Then, we may construct, in time $O(m)$, the partition

$$X \setminus Z^z_{p+1} = X^z_1 \cup \cdots \cup X^z_t \tag{110}$$

induced by $\sim$. For future reference, let us write, for each $1 \le j \le t$,

$$\alpha_j \stackrel{\text{def}}{=} \left\{ \omega(G[\{x\}, Z^z_{p+1}]) : x \in X^z_j \right\}. \tag{111}$$

With the vertex $z \in X \setminus Z_p$ fixed, observe that the partition in (110) satisfies $t = t(z) \le (p+2)^{s-1} = O(1)$. Indeed, for a fixed $x \in X \setminus Z^z_{p+1}$, each of the $(s-1)$ many graphs G_i, $2 \le i \le s$, satisfies $|N_{G_i}(x) \cap Z^z_{p+1}| \in \{0, 1, \ldots, p+1\}$, i.e., $|N_{G_i}(x) \cap Z^z_{p+1}|$ has $(p+2)$ many possible sizes. By (101),

$$\omega_i(G_i[\{x\}, Z^z_{p+1}]) = \frac{\sigma_s}{\sigma_i} |N_{G_i}(x) \cap Z^z_{p+1}|,$$

and so

$$\omega(G[\{x\}, Z^z_{p+1}]) \stackrel{(102)}{=} \sum_{i=2}^{s} \omega_i(G_i[\{x\}, Z^z_{p+1}]) = \sum_{i=2}^{s} \frac{\sigma_s}{\sigma_i} |N_{G_i}(x) \cap Z^z_{p+1}|$$

may assume at most $(p+2)^{s-1}$ possible values, as claimed.

With the vertex $z \in X \setminus Z_p$ still fixed, and with the partition $X \setminus Z^z_{p+1} = X^z_1 \cup \cdots \cup X^z_t$ from (110), we may now compute $f(z)$ (which is defined in (107)). To that end, fix

$$\text{an integer sum } a_1 + \cdots + a_t = r - p - 1, \text{ where } 0 \le a_j \le |X^z_j|,\ 1 \le j \le t. \tag{112}$$

For each $1 \le j \le t$,

$$\text{let } A^z_j \in \binom{X^z_j}{a_j} \text{ be an arbitrary } a_j\text{-subset, and let } A^z = A^z(a_1, \ldots, a_t) = \bigcup_{j=1}^{t} A^z_j. \tag{113}$$

Define

$$\begin{aligned} f(z; a_1, \ldots, a_t) &= \mathbb{E}_{(A^z_1, \ldots, A^z_t) \in \prod_{j=1}^t \binom{X^z_j}{a_j}} [\omega(G[Z^z_{p+1} \cup A^z_1 \cup \cdots \cup A^z_t])] \\ &= \mathbb{E}_{(A^z_1, \ldots, A^z_t) \in \prod_{j=1}^t \binom{X^z_j}{a_j}} [\omega(G[Z^z_{p+1} \cup A^z(a_1, \ldots, a_t)])]. \end{aligned} \tag{114}$$

where the expectation above is taken uniformly over all sequences $(A_1^z, \dots, A_t^z) \in \prod_{j=1}^t \binom{X_j^z}{a_j}$, i.e., the expectation above is taken uniformly over all subsets $A^z = A^z(a_1, \dots, a_t) \subseteq X \setminus Z_{p+1}^z$ of the form in (113). Then, $f(z)$ (which is defined in (107)) is given by

$$f(z) = \sum_{a_1+\cdots+a_t=r-p-1} \frac{\prod_{j=1}^t \binom{|X_j^z|}{a_j}}{\binom{n-p-r}{r-p-1}} f(z; a_1, \dots, a_t), \tag{115}$$

where the sum extends over all indices of the form in (112).

We now expand the expression for $f(z)$ given in (115) by computing each term $f(z; a_1, \dots, a_t)$ (cf. (114)), where $a_1 + \cdots + a_t = r - p - 1$ is of the form in (112). Indeed, by linearity of expectation, we claim that

$$\begin{aligned} f(z; a_1, \dots, a_t) = \omega(G[Z_{p+1}^z]) + \sum_{j=1}^t \alpha_j a_j \\ + \sum_{j=1}^t \omega(G[X_j^z]) \frac{\binom{a_j}{2}}{\binom{|X_j^z|}{2}} + \sum_{1\le j<k\le t} \omega(G[X_j^z, X_k^z]) \frac{a_j a_k}{|X_j^z||X_k^z|}. \end{aligned} \tag{116}$$

Indeed, the first term in (116) is $\omega(G[Z_{p+1}^z])$, which is the total ω-weight of the edges of $G[Z_{p+1}^z]$. The first sum in (116) is the expected ω-weight of $G[A_1^z \cup \cdots \cup A_t^z, Z_{p+1}^z]$ (cf. (111)). The second sum in (116) is the expected ω-weight of $\bigcup_{j=1}^t G[A_j^z]$. Finally, the third sum in (116) is the expected ω-weight of $\bigcup_{1\le j<k\le t} G[A_j^z, A_k^z]$. Thus, applying (116) to (115), we have that

$$\begin{aligned} f(z) = \sum_{a_1+\cdots+a_t=r-p-1} \frac{\prod_{j=1}^t \binom{|X_j^z|}{a_j}}{\binom{n-p-r}{r-p-1}} \Bigg(\omega(G[Z_{p+1}^z]) + \sum_{j=1}^t \alpha_j a_j \\ + \sum_{j=1}^t \omega(G[X_j^z]) \frac{\binom{a_j}{2}}{\binom{|X_j^z|}{2}} + \sum_{1\le j<k\le t} \omega(G[X_j^z, X_k^z]) \frac{a_j a_k}{|X_j^z||X_k^z|} \Bigg), \end{aligned} \tag{117}$$

where the (main) sum extends over all indices of the form in (112).

To prove (108), it remains to choose, in time $O(m^3)$, a vertex $z_0 \in X \setminus Z_p$ so that $f(z_0) \le \frac{1}{m-p} \sum_{z \in X\setminus Z_p} f(z)$. For that, we use the expression in (116) for $f(z)$. Note that, for each $z \in X \setminus Z_p$, the expression for $f(z)$ in (117) depends only on z. Moreover, since all sums above consist of $O(1)$ many terms, we may compute, in time $O(m^2)$, the value of $f(z)$ for a fixed $z \in X \setminus Z_p$. Now, in time $O(m^3)$, we compute all values of $f(z)$ over all $z \in X \setminus Z_p$, and select $z_0 \in X \setminus Z_p$ so that

$$f(z_0) = \min_{z \in X\setminus Z_p} f(z).$$

Then, by our choice of z_0, we have $f(z_0) \le \frac{1}{m-p} \sum_{z\in X\setminus Z_p} f(z)$, which proves (108). This proves Claim 8.1, and hence, concludes the proof of Lemma 7.1.

8.2. **Proof of Fact 2.10.** The equivalence $\mathbf{COUNT}_{\mathrm{emb}} \iff \mathbf{COUNT}_{\mathrm{ind}}$ is trivial to prove. Indeed, let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.9, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)}) > 0$, and fix $\delta > 0$. Note that, for each $\varnothing \subseteq \mathcal{S}^{(j)} \subseteq \mathcal{O}^{(j)} = K^{(j)}(U_1, \dots, U_j)$, we have

$$\left|\mathrm{EMB}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right| = \sum_{\mathcal{S}^{(j)} \subseteq \mathcal{F}^{(j)} \subseteq \mathcal{O}^{(j)}} \left|\mathrm{EMB}_{\mathrm{ind}}(\mathcal{F}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right|, \quad \text{and}$$

$$\left|\mathrm{EMB}_{\mathrm{ind}}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right| = \sum_{\mathcal{S}^{(j)} \subseteq \mathcal{F}^{(j)} \subseteq \mathcal{O}^{(j)}} (-1)^{|\mathcal{F}^{(j)}| - |\mathcal{S}^{(j)}|} \left|\mathrm{EMB}(\mathcal{F}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right|.$$

With these identities, we may apply Definition 2.9 (and the Binomial Theorem) to conclude that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\mathrm{emb}}(d, \delta)$ if, and only if, $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\mathrm{ind}}(d, \delta)$.

Now, suppose $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\mathrm{emb}}(d, \delta)$ and that, for each $i \in [j]$, we have $|V_i| = \Theta(n)$, and that $|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})| = \Omega(n^{2^j})$. Then,

$$\sum_{v_1, v_1' \in V_1} \cdots \sum_{v_j, v_j' \in V_j} \prod \left\{ \omega(J) : J \in K^{(j)}\left(\{v_1, v_1'\}, \dots, \{v_j, v_j'\}\right) \right\}$$
$$= O(n^{2^j - 1}) + \sum_{\varnothing \subseteq \mathcal{S}^{(j)} \subseteq \mathcal{O}^{(j)}} (1-d)^{|\mathcal{S}^{(j)}|} (-d)^{2^j - |\mathcal{S}^{(j)}|} \left|\mathrm{EMB}_{\mathrm{ind}}(\mathcal{S}^{(j)}, (\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)}))\right|.$$

Since $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has $\mathbf{COUNT}_{\mathrm{emb}}(d, \delta)$, it also has $\mathbf{COUNT}_{\mathrm{ind}}(d, \delta)$, and so we conclude

$$\sum_{v_1, v_1' \in V_1} \cdots \sum_{v_j, v_j' \in V_j} \prod \left\{ \omega(J) : J \in K^{(j)}\left(\{v_1, v_1'\}, \dots, \{v_j, v_j'\}\right) \right\}$$
$$\leq O(n^{2^j - 1}) + \left|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\right| d^{2^j} (1-d)^{2^j} \times \delta 2^{2^j}$$
$$\leq O(n^{2^j - 1}) + \delta 2^{-2^j} \left|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\right| \leq \delta \left|\mathrm{EMB}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\right|,$$

where we used $d(1-d) \leq 1/4$.

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