# A braided monoidal category for symplectic fermions 

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#### Abstract

We describe a class of examples of braided monoidal categories which are built from Hopf algebras in symmetric categories. The construction is motivated by a calculation in two-dimensional conformal field theory and is tailored to contain the braided monoidal categories occurring in the study of the Ising model, their generalisation to Tamabara-Yamagami categories, and categories occurring for symplectic fermions.


## 1 Introduction

In this short note we summarise some of the results in [1, 2], where also more extensive references can be found.

We are interested in a particular type of $\mathbb{Z} / 2 \mathbb{Z}$-graded braided monoidal categories. The grade 0 component is the monoidal category $\operatorname{Rep}_{\mathcal{S}}(H)$ of modules over a Hopf algebra $H$ in a symmetric monoidal category $\mathcal{S}$, and the grade 1 component is the category $\mathcal{S}$ itself. We will write $\mathcal{C}=\mathcal{C}_{0}+\mathcal{C}_{1}$ with $\mathcal{C}_{0}=\operatorname{Rep}_{\mathcal{S}}(H)$ and $\mathcal{C}_{1}=\mathcal{S}$. The tensor product functor $*$ on the various components is defined as:

[^0]| $A$ | $B$ | $A * B$ |  | comments |
| :--- | :--- | :--- | :--- | :--- |
| $\mathcal{C}_{0}$ | $\mathcal{C}_{0}$ | $A \otimes B$ | $\in \mathcal{C}_{0}$ | the $H$-action is via the coproduct of $H$ |
| $\mathcal{C}_{0}$ | $\mathcal{C}_{1}$ | $F(A) \otimes B$ | $\in \mathcal{C}_{1}$ | $F: \operatorname{Rep}_{\mathcal{S}}(H) \rightarrow \mathcal{S}$ is the forgetful functor |
| $\mathcal{C}_{1}$ | $\mathcal{C}_{0}$ | $A \otimes F(B)$ | $\in \mathcal{C}_{1}$ |  |
| $\mathcal{C}_{1}$ | $\mathcal{C}_{1}$ | $H \otimes A \otimes B \in \mathcal{C}_{0}$ | the $H$-action is by multiplication |  |

This somewhat ad-hoc looking definition of the tensor product is actually quite natural. The mixed tensor products are the natural left and right action of the monoidal category $\operatorname{Rep}_{\mathcal{S}}(H)$ on $\mathcal{S}$. To obtain the last line in the table, assume that $\mathcal{C}$ can be made rigid. Writing $T$ for the tensor unit of $\mathcal{S}$ considered as an object in $\mathcal{C}_{1}$, we have for all $H$-modules $M$

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}_{0}}\left(T^{*} * T, M\right) \cong \operatorname{Hom}_{\mathcal{C}_{1}}(T, T * M) \cong \operatorname{Hom}_{\mathcal{S}}(1, F(M)) \tag{1}
\end{equation*}
$$

This means that $T^{*} * T$ is a representing object for the functor $M \mapsto$ $\operatorname{Hom}_{\mathcal{S}}(1, F(M))$, and so $T^{*} * T \cong H$ as an $H$-module. If we in addition demand that $T^{*} \cong T$, the last line in the above table follows.

Given the above form of the tensor product functor $*$ on $\mathcal{C}$, one can ask if it is possible to describe associativity and braiding isomorphisms for $*$ in terms of Hopf algebraic data on $H$. Our results for this question are given in Section 4. But before getting there, in Sections 2 and 3 we would like to give the two examples of such $\mathbb{Z} / 2 \mathbb{Z}$-graded braided monoidal categories which were our main motivation when setting up the formalism.

## 2 Tambara-Yamagami categories

For simplicity, we will work over the field $\mathbb{C}$. Consider a fusion category $\mathcal{C}$ over $\mathbb{C}$ whose simple objects are labelled by $G \cup\{m\}$ where $G$ is a finite group and $m$ is an extra label. Suppose that the tensor product $*$ is of the form, for $a, b \in G$,

$$
\begin{equation*}
a * b \cong a b \quad, \quad m * a \cong m \cong a * m \quad, \quad m * m \cong \bigoplus_{g \in G} g . \tag{2}
\end{equation*}
$$

This tensor product is a special case of the one in the above table: the underlying symmetric category $\mathcal{S}$ is vect $(\mathbb{C})$, the category of finite dimensional $\mathbb{C}$ vector spaces. The component $\mathcal{C}_{0}$ is spanned by the simple objects $g \in G$; the component $\mathcal{C}_{1}$ is spanned by $m$ alone, so that $\mathcal{C}_{1} \cong \operatorname{vect}(k)$. The Hopf algebra $H \in \mathcal{S}$ is the function algebra $\operatorname{Fun}(G, \mathbb{C})$.

For any such fusion category $\mathcal{C}$, the group $G$ is necessarily abelian and $\mathcal{C}$ is monoidally equivalent to $\mathcal{C}(\chi, \tau)$, which is defined as follows [3, Thm.3.2]. $\mathcal{C}(\chi, \tau)$ has simple objects and fusion rules as in (2), and its associator is determined by a symmetric non-degenerate bicharacter $\chi: G \times G \rightarrow \mathbb{C}^{\times}$and a choice of $\tau \in \mathbb{C}^{\times}$such that $\tau^{2}=|G|^{-1}$. The associator is a bit lengthy and we refer to [3].

The category $\mathcal{C}(\chi, \tau)$ allows for a braiding if and only if $G$ is an elementary 2-group (i.e. $g g=e$ for all $g \in G$ ) 4]. The braiding isomorphisms are determined by a quadratic form $\sigma$ associated to the bicharacter ${ }^{\dagger} \chi$ and a number $\beta$ such that $\beta \neq 0$ and $\beta^{2}=\tau \sum_{a \in G} \sigma(a)$. Explicitly, under the identifications (1) the braiding is $(a, b \in G)$

$$
\begin{equation*}
c_{a, b}=\chi(a, b) i d_{a b}, c_{a, m}=\sigma(a) i d_{m}=c_{m, a}, c_{m, m}=\bigoplus_{g \in G} \beta \sigma(g)^{-1} i d_{g} . \tag{3}
\end{equation*}
$$

An important example of a braided monoidal category of the above type is provided by the two-dimensional critical Ising model. There, one considers the three irreducible representations $\hat{\mathbf{1}}, \hat{\varepsilon}, \hat{\sigma}$ of the Virasoro algebra which have central charge $c=\frac{1}{2}$ and lowest $L_{0}$-weights $h_{\hat{\mathbf{1}}}=0, h_{\hat{\varepsilon}}=\frac{1}{2}$ and $h_{\hat{\sigma}}=\frac{1}{16}$. The fusion rules are of the form (1) where $\hat{\mathbf{1}}, \hat{\varepsilon}$ generate the group $G=\mathbb{Z} / 2 \mathbb{Z}$ and $m=\hat{\sigma}$ has fusion rule $\hat{\sigma} * \hat{\sigma} \cong \hat{\mathbf{1}} \oplus \hat{\varepsilon}$. The braiding isomorphism $c_{r, s}-$ projected to the simple object $t \in r * s$ - is multiplication by $\exp \left(\pi i\left(h_{r}+\right.\right.$ $\left.h_{s}-h_{t}\right)$ ). Comparing to (3) shows that the braided monoidal structure is determined by $\sigma(\hat{\varepsilon})=\exp (\pi i / 2), \beta=\exp (\pi i / 8)$ and thus $\chi(\hat{\varepsilon}, \hat{\varepsilon})=-1$, $\tau=1 / \sqrt{2}$.

## 3 Symplectic fermions

Continuing with examples from two-dimensional conformal field theory, we now consider symplectic fermions [5]. The mode algebra of $n$ pairs of symplectic fermsions is determined by a $2 n$-dimensional symplectic vector space $\mathfrak{h}$. It is convenient to think of $\mathfrak{h}$ as a purely odd abelian Lie super-algebra with nondegenerate super-symmetric pairing $(-,-)$; we will use this language in the following. The symplectic fermion mode algebra is the affinisation $\mathfrak{h}$ of $\mathfrak{h}$ with central element $K$ and graded bracket $\left[a_{m}, b_{n}\right]=m(a, b) \delta_{m+n, 0} K$, where $m, n \in \mathbb{Z}$ for untwisted (Neveu-Schwarz) representations and $m, n \in \mathbb{Z}+\frac{1}{2}$ for twisted (Ramond) representations.

[^1]Denote by $S(\mathfrak{h})$ the symmetric algebra of $\mathfrak{h}$ in $\operatorname{svect}(\mathbb{C})$, the category of finite-dimensional complex super-vector spaces. Note that as a vector space, $S(\mathfrak{h})$ is simply the exterior algebra of the vector space underlying $\mathfrak{h}$; in particular, $S(\mathfrak{h})$ is finite-dimensional. The categories of untwisted and twisted representations of $\hat{\mathfrak{h}}$ (of a certain type) are equivalent to [2, Thms. 2.4\&2.8]:

$$
\begin{equation*}
\text { (untwisted) } \mathcal{C}_{0}:=\operatorname{Rep}_{\text {svect }} S(\mathfrak{h}) \quad \text { (twisted) } \mathcal{C}_{1}:=\operatorname{svect}(\mathbb{C}) . \tag{4}
\end{equation*}
$$

We would like to stress that for $\operatorname{dim} \mathfrak{h}=2 n>0, \mathcal{C}_{0}$ is not semi-simple.
A conformal field theory calculation endows the category $\mathcal{C}=\mathcal{C}_{0}+\mathcal{C}_{1}$ with a $\mathbb{Z} / 2 \mathbb{Z}$-graded tensor product [2, Thm. 3.13]. This tensor product is of the form stated in Section 1 with symmetric category $\mathcal{S}=\operatorname{svect}(\mathbb{C})$ and Hopf algebra $H=S(\mathfrak{h})$.

The associativity isomorphism is determined by the copairing $C \in \mathfrak{h} \otimes \mathfrak{h}$ dual to the super-symmetric pairing $(-,-)$ on $\mathfrak{h}$, and by a top-form $\lambda$ on $S(\mathfrak{h})$ such that $(\lambda \otimes \lambda)\left(e^{-C}\right)=1$ [2, Thm. 6.2]. To be more specific, pick a basis $\left\{e_{i}\right\}_{i=1, \ldots, 2 n}$ of $\mathfrak{h}$ such that the pairing takes the standard form $\left(e_{2 k-1}, e_{2 k}\right)=$ $1=-\left(e_{2 k}, e_{2 k-1}\right)$ for $k=1, \ldots, n$. Then $C=\sum_{k=1}^{n}\left(e_{2 k} \otimes e_{2 k-1}-e_{2 k-1} \otimes e_{2 k}\right)$ and, if we set $\hat{C}=-2 \sum_{k=1}^{n} e_{2 k} \otimes e_{2 k-1}$, the top-form $\lambda$ is determined by $\lambda\left(\hat{C}^{n}\right)=n!(-2 i)^{n}$. The explicit form of the associativity isomorphisms will be given as a special case of Theorem 1 below.

To describe the braiding, denote by $\omega_{V}$ the parity involution on a supervector space $V$, and by $s_{V, W}: V \otimes W \rightarrow W \otimes V$ the symmetric structure on $\operatorname{svect}(\mathbb{C})$. Then [2, Thm. 6.4]:

$$
\begin{array}{lll}
A & B & c_{A, B}: A * B \rightarrow B * A \\
\hline \mathcal{C}_{0} & \mathcal{C}_{0} & s_{A, B} \circ \exp (-C) \\
\mathcal{C}_{0} & \mathcal{C}_{1} & s_{A, B} \circ\left(\exp \left(\frac{1}{2} \hat{C}\right) \otimes i d_{B}\right)  \tag{5}\\
\mathcal{C}_{1} & \mathcal{C}_{0} & s_{A, B} \circ\left(i d_{A} \otimes \exp \left(\frac{1}{2} \hat{C}\right)\right) \circ\left(i d_{A} \otimes \omega_{B}\right) \\
\mathcal{C}_{1} & \mathcal{C}_{1} & e^{-i \pi \frac{n}{4}} \cdot\left(i d_{S(\mathfrak{h})} \otimes s_{A, B}\right) \circ\left(\exp \left(-\frac{1}{2} \hat{C}\right) \otimes i d_{A} \otimes \omega_{B}\right)
\end{array}
$$

In the last line, note that for $A, B \in \mathcal{C}_{1}$ we have $A * B=S(\mathfrak{h}) \otimes A \otimes B$.

## 4 A unified framework

The braiding isomorphisms (3) and (5) in the two examples just discussed may look quite different at first glance, but - just as was the case for the tensor product $*$ itself - they are actually two instances of the same structure.


Figure 1: Associativity isomorphism $\alpha_{A, B, C}: A *(B * C) \rightarrow(A * B) * C$. The label abc means that $A \in \mathcal{C}_{a}, B \in \mathcal{C}_{b}, C \in \mathcal{C}_{c}$. In the three non-listed cases 000,001 , $100, \alpha_{A, B, C}$ is the identity (or rather the associator of the underlying category $\mathcal{S}$ ). The diagrams are read from bottom to top, the empty and solid dot denote $S$ and $S^{-1}$, respectively, and the three-valent vertices are the product and coproduct. The arrowhead depicts the action of $H$ on a module.

Namely, let $\mathcal{S}$ be a pivotal symmetric monoidal category (i.e. a ribbon category with symmetric braiding $\left.s_{A, B}: A \otimes B \rightarrow B \otimes A\right)$ and let $H$ be a Hopf algebra in $\mathcal{S}$ with invertible antipode. Denote by $\operatorname{Rep}_{\mathcal{S}}(H)$ the monoidal category of left $H$-modules in $\mathcal{S}$ and set

$$
\begin{equation*}
\mathcal{C}=\mathcal{C}_{0}+\mathcal{C}_{1} \quad \text { with } \quad \mathcal{C}_{0}=\operatorname{Rep}_{\mathcal{S}}(H) \quad, \quad \mathcal{C}_{1}=\mathcal{S} \tag{6}
\end{equation*}
$$

On $\mathcal{C}$ we fix the $\mathbb{Z} / 2 \mathbb{Z}$-graded tensor product functor $*$ from Section 1. Denote by $\mu$ the multiplication of $H$, by $\Delta$ the coproduct, and by $S$ the antipode. For a morphisms $x: \mathbf{1} \rightarrow H$ in $\mathcal{S}$, write ${ }_{x} M$ and $M_{x}$ for the endomorphism of $H$ given by left- and right-multiplication with $x$, respectively, and $\mathrm{Ad}_{x}$ for the endomorphism given by $x(-) x^{-1}$. Associativity isomorphisms for $*$ can be obtained from Hopf-algebraic data as follows [1, Cor. 3.17]:

Theorem 1. Let $\gamma: \mathbf{1} \rightarrow H \otimes H$ and $\lambda: H \rightarrow \mathbf{1}$ be two morphisms in $\mathcal{S}$ such that

1. $\gamma$ is a non-degenerate Hopf-copairing,
2. $\lambda$ is a right cointegral for $H$, such that there exits $g: \mathbf{1} \rightarrow H$ with $(i d \otimes \lambda) \circ \Delta=g \circ \lambda$, and such that $(\lambda \otimes \lambda) \circ(i d \otimes S) \circ \gamma=i d_{\mathbf{1}}$,
3. $\gamma$ satisfies the symmetry condition $s_{H, H} \circ \gamma=\left(i d \otimes\left(S^{2} \circ \operatorname{Ad}_{g}^{-1}\right)\right) \circ \gamma$.

Then the natural isomorphisms in Figure 1 define associativity isomorphisms for $*$.


Figure 2: Braiding isomorphisms $c_{A, B}: A * B \rightarrow B * A$. The notation is as in Figure 1

Note that it is not claimed that the above description gives all associativity isomorphisms for $*$; outside of $\mathcal{S}=\operatorname{vect}(k)$ this would require extra assumptions. In [1], the above theorem is actually proved in the more general setting of $\mathcal{S}$ being ribbon but not necessarily symmetric. The relevant Hopf algebra notions are reviewed in [1, Sec. 2].

Example 2. 1. The Tambara-Yamagami categories are recovered for $\mathcal{S}=$ $\operatorname{vect}(\mathbb{C})$ and $H=\operatorname{Fun}(G, \mathbb{C})$. The cointegral and copairing are $\lambda=\tau \sum_{a \in G} \delta_{a}$ and $\gamma=\sum_{a, b \in G} \chi(a, b) \delta_{a} \otimes \delta_{b}$.
2. For symplectic fermions take $\mathcal{S}=\operatorname{svect}(\mathbb{C})$ and $H=S(\mathfrak{h})$. The copairing is $\gamma=e^{C}$ and the cointegral $\lambda$ is as given in Section 3,
3. Another example for $\mathcal{S}=\operatorname{vect}(\mathbb{C})$ is provided by Sweedler's four-dimensional Hopf algebra, which is not semi-simple [1, Sec.3.8.3]. This illustrates that Theorem 1 is more general than Tambara-Yamagami categories even in the vector space case.

For the braiding isomorphisms we need to fix an involutive monoidal automorphism $\omega$ of the identity functor on $\mathcal{S}$. For $\mathcal{S}=\operatorname{vect}(\mathbb{C}), \omega$ is necessarily the identity, but for $\mathcal{S}=\operatorname{svect}(\mathbb{C})$ there are already two choices: the identity and parity involution. We have [1, Thm. 1.2 \& Rem. 4.11]:

Theorem 3. Let $H, \gamma, \lambda$ and $g$ be as in Theorem 1 and let $\sigma: \mathbf{1} \rightarrow H$ and $\beta: \mathbf{1} \rightarrow \mathbf{1}$ be invertible. Suppose that

1. $\gamma$ is determined through $\sigma$ by $\gamma=\left({ }_{\sigma^{-1}} M \otimes M_{\sigma^{-1}}\right) \circ \Delta \circ \sigma$.
2. $\lambda$ satisfies $\lambda \circ S=\lambda \circ \operatorname{Ad}_{\sigma}$ and $\lambda \circ \sigma=\beta \circ \beta$.
3. $\mathrm{Ad}_{\sigma}$ is a Hopf-algebra isomorphism $H \rightarrow H_{\text {cop }}$ (the opposite coalgebra).
4. $S \circ \sigma={ }_{g} M \circ \sigma=M_{g^{-1}} \circ \sigma$.
5. $\omega$ evaluated on $H$ satisfies $\left(i d \otimes \omega_{H}\right) \circ \gamma=\left(\operatorname{Ad}_{\sigma} \otimes\left(\operatorname{Ad}_{\sigma}^{-1} \circ S\right)\right) \circ \gamma$.

Then the natural isomorphisms in Figure define a braiding on $\mathcal{C}$.
If $\sigma^{2}$ is central in $H$, then $\mathcal{C}$ can be made into a ribbon category with twist isomorphisms $\theta_{A}=\sigma^{-2}$.(-) for $A \in \mathcal{C}_{0}$ (i.e. the left action of $\sigma^{-2}$ on the $H$-module $A$ ), and $\theta_{A}=\beta^{-1} \omega_{A}$ for $A \in \mathcal{C}_{1}$, see [1, Prop. 4.18].

Example 4. 1. In the Tambara-Yamagami case, $\sigma$ and $\beta$ are as in Section 2. Comparing (3) and Figure 2 shows that the braiding isomorphisms match ( $S=i d_{H}$ for an elementary 2-group, and $\mathrm{Ad}_{\sigma}=i d_{H}$ since $H$ is commutative). 2. For symplectic fermions choose $\sigma=\exp \left(\frac{1}{2} \hat{C}\right)$ and $\beta=e^{-\pi i n / 4}$. Then (5) agrees with Figure 2,
3. Sweedler's Hopf algebra is quasi-triangular, but the resulting braiding on $\mathcal{C}_{0}$ does not extend to all of $\mathcal{C}$ (at least via the above construction). However, one can find a 16 -dimensional semi-simple Hopf algebra in vect $(\mathbb{C})$ which is neither commutative nor co-commutative for which Theorems 1 and 3 apply [1, Sec.4.7.4]. This is another instance where our setting is more general than the Tambara-Yamagami case even for $\mathcal{S}=\operatorname{vect}(\mathbb{C})$.

## References

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[^1]:    ${ }^{\dagger}$ This means that $\sigma: G \rightarrow \mathbb{C}^{\times}$satisfies $\sigma(a)=\sigma\left(a^{-1}\right), \sigma(e)=1$, and that $\chi(a, b) \sigma(a) \sigma(b)=\sigma(a b)$ for all $a, b \in G$.

