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New Large-Rank Nichols Algebras Over Nonabelian Groups With Commutator Subgroup $\mathbb{Z}_2$

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Abstract. In this article, we explicitly construct new finite-dimensional, indecomposable Nichols algebras with Dynkin diagrams of type $A_n, C_n, D_n, E_{6,7,8}, F_4$ over any group G with commutator subgroup isomorphic to $\mathbb{Z}_2$. The construction is generic in the sense that the type just depends on the rank and center of G, and thus positively answers for all groups of this class a question raised by Susan Montgomery in 1995 [Mon95][AS02].

Our construction uses the new notion of a covering Nichols algebra as a special case of a covering Hopf algebra [Len12] and produces non-faithful Nichols algebras. We give faithful examples of Doi twists for type A_3, C_3, D_4, C_4, F_4 over several nonabelian groups of order 16 and 32. These are hence the first known examples of nondiagonal, finite-dimensional, indecomposable Nichols algebras of rank > 2 over non-abelian groups.

Contents

1. Introduction

The Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel'd module M over a group Γ is a quotient of the tensor algebra $T(M)$. It has a natural structure of a Hopf algebra in a braided category satisfying a certain universal property. Finite-dimensional Nichols algebras arise naturally e.g. as quantum Borel part in the classification of finite-dimensional pointed Hopf algebras [AS10], such as the small quantum groups $u_q(\mathfrak{g})$. Heckenberger classified all finite-dimensional Nichols algebras for Γ abelian [Hec09]. For Nichols algebras over arbitrary semisimple Yetter-Drinfeld modules, Andruskiewitsch, Heckenberger and Schneider defined a Weyl groupoid and a generalized root system in [AHS10], and further developed it in [HS10].

However, the existence of a finite-dimensional Nichols algebra over a non-abelian group still seems to be a rather rare and difficult phenomenon. The first examples were discovered by Milinski and Schneider in [MS00] over Coxeter groups and are of rank 1 except for one of rank 2 over $\mathbb{D}_4$. Here *rank* refers to the number of irreducible summands in the underlying Yetter-Drinfel'd module, and equivalently to the rank of the Weyl groupoid and the root system. Andruskiewitsch, Graña, Heckenberger, Lochmann and Vendramin have in [AG03][GHV11][HLV12] constructed several large finite-dimensional Nichols algebras of rank 1. On the other hand, strong conditions have been developed to rule out the existence of finite-dimensional Nichols-algebras over many groups such as higher alternating groups and most sporadic groups, see e.g. [AFGV10].
Moreover, during the work on this article, Schneider, Heckenberger and Vendramin have in series of papers completed the classification of Nichols algebras of rank 2, see [HV13], by narrowing down the possibilities using the root system theory and constructing the remaining by hand. So far, no examples of higher rank over nonabelian groups have been constructed.[1]

In this article, we explicitly construct finite-dimensional indecomposable Nichols algebras with root systems of type $A_n,\ C_n,\ D_n,\ E_{6,7,8},\ F_4$ and hence arbitrary rank over nonabelian groups G that are central stem-extensions of

[1]Added 04/2015: Recently Heckenberger and Vendramin have in [HV14] completed the classification of all Nichols algebras of rank > 1. In rank > 3 and characteristic 0 the Nichols algebras constructed in the present article turn out to be all; this is very furtunate, because our construction gives very much control over the Nichols algebra (generators and relations, cohomology etc.). There is a single exceptional example in rank 3 and in characteristic 3 there is an additional series B_n which seemingly can be constructed with the approach in the present article from an additional B_n series of diagonal Nichols algebras with braiding ± 1 in characteristic 3.

an abelian group Γ, i.e.:

$$\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma \qquad \Sigma^* \subset [G,G] \cap Z(G)$$

As a side remark, we mention that the construction is a special case of our new notion of a covering Hopf algebra [Len12], applied to the bosonization of known finite-dimensional Nichols algebra over the abelian group Γ. The covering construction itself does not depend on Γ being abelian: For example, in [Len12] we have constructed a covering Nichols algebra of dimension 24^2 over $G = GL_2(\mathbb{F}_3)$, which is a $\mathbb{Z}_2$-stem-extension of $\Gamma = \mathbb{S}_4$ and an open case called $\mathbf{C_4}$ in the list [FGV07].

More concretely, we proceed in Section 3.1 as follows: Suppose $M = \bigoplus_{i\in I} M_i$ is a semisimple Yetter-Drinfel'd module over the abelian finite group Γ with simple 1-dimensional summands $M_i,\ i \in I$, diagonal braiding matrix q_{ij} and known finite-dimensional Nichols algebra $\mathcal{B}(M)$. Furthermore, suppose that a finite abelian group Σ acts on the vector space M, such that the Γ-graduation as well as the the self-braiding operators $c_{M_iM_i} = q_{ii}$ and the monodromy operators $c_{M_iM_j}c_{M_iM_j} = q_{ij}q_{ji}$ are preserved. We usually assume the Σ-action induced from a permutation action on I. In this case the assumed compatibility with the braiding lets Σ act on the q-diagram and Dynkin-diagram of M (having nodes $i \in I$) by graph automorphisms.

However, Σ does not act on M by Yetter-Drinfel'd module-automorphisms: The braiding matrix q_{ij} itself is generally not preserved, but is supposed to be modified under the action of each $p \in \Sigma$ as prescribed by a bimultiplicative form $\langle \bar{g}_i, \bar{g}_j \rangle_p$ with respect to the Γ-graduation $\bar{g}_i, \bar{g}_j \in \Gamma$ of M_i, M_j.

The bimultiplicative forms $\langle\rangle_p$ will usually be induced from a group-2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$ via $\langle \bar{g}, \bar{h} \rangle_p := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$. We call such an action a twisted symmetry action of Σ on M with respect to the the 2-cocycle σ.

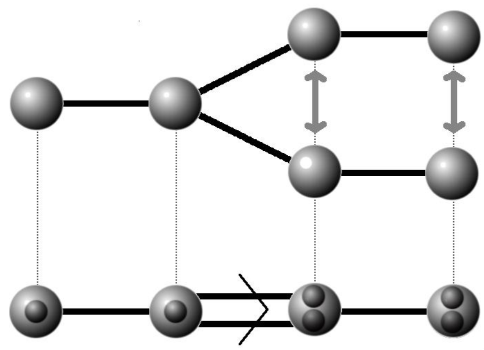
With these notions we construct in Section 3.2 a covering Yetter-Drinfel'd module $\tilde{M}$ over a stem-extension $\Sigma^* \to G \to \Gamma$ with Γ abelian as follows: We start with a Γ-Yetter-Drinfel'd module M and a twisted permutation symmetry action of Σ on M (i.e. induced from Σ permuting I) with respect to a group-2-cocycle σ representing the given stem-extension. We decompose M into simultaneous eigenspaces $M^{[\lambda]}$ for eigenvalues $\lambda \in \Sigma^*$ of the twisted symmetry action of Σ and use this Σ^*-graduation to refine the Γ-graduation on M to a G-graduation. Note that formerly Γ-homogeneous elements in M are usually not G-homogeneous. By pulling back also the Γ-action on M to a G-action, we obtain a covering Yetter-Drinfel'd module $\tilde{M}$ over the nonabelian group G. As a braided vector space, $\tilde{M}$ is isomorphic to M. The Nichols algebra $\mathcal{B}(\tilde{M})$ of the covering Yetter-Drinfel'd module $\tilde{M}$ is called covering Nichols algebra and is isomorphic to $\mathcal{B}(M)$ as an algebra.

When we apply this construction in case Γ abelian to a semisimple Yetter-Drinfel'd module $M = \bigoplus_{i\in I} M_i$ with Σ acting by twisted permutation symmetries, then the different irreducible Γ-Yetter-Drinfel'd modules M_i laying on an orbit of the twisted symmetry Σ become a single irreducible G-Yetter-Drinfel'd module with increased dimension. This changes the Cartan matrix and hence the Dynkin diagram (see [HS10] Definition 6.4) of $\tilde{M}$ compared with M as described in Theorem 3.15: Nodes of the Dynkin diagram of M in a Σ-orbit give rise to a single nodes of the Dynkin diagram of $\tilde{M}$. The root system is reduced to the subsystem fixed by Σ acting on the Dynkin diagram of M by graph automorphisms. This behaviour is classically known as diagram folding of a Lie algebra by an outer automorphism (for a purely root system approach to folding see e.g. [Gin06] p. 47).

Example. *(Section 5.3) There exists a* 6*-dimensional Yetter-Drinfel'd module* M *over* $\Gamma = \mathbb{Z}_2^4$ *with* Γ*-homogeneous components of dimension* $1, 1, 2, 2$ *which is the sum of* 6 *simple 1-dimensional Yetter-Drinfel'd modules* M_i*. The Cartan matrix of* M *is as the semisimple Lie algebra* E_6 *and the Nichols algebra* $\mathcal{B}(M)$ *has dimension* 2^{36}*. Moreover,* M *admits an action of* $\Sigma = \mathbb{Z}_2$ *by twisted symmetries, corresponding to a diagram automorphism of the* E_6 *root system.*

The covering Yetter-Drinfel'd module $\tilde{M}$ *over the stem-extension* $G = \mathbb{Z}_2^2 \times \mathbb{D}_4$ *is the sum of* 4 *simple Yetter-Drinfel'd modules* $\tilde{M}_i$ *of dimensions* $1, 1, 2, 2$*. The covering Nichols algebra* $\mathcal{B}(\tilde{M})$ *is indecomposable, has also dimension* 2^{36} *and its Dynkin diagram is* F_4*. This corresponds to the inclusion of the semisimple Lie algebras* $F_4 \subset E_6$ *as fixed points of the outer Lie algebra automorphism of* E_6*.*

This example is visualized as follows:



For twisted permutation symmetries of prime order $\Sigma = \mathbb{Z}_p$ we use the suggestive terms inert/split for G-nodes $\tilde{M}_i$ of dimension $1/p$. They correspond to Σ-orbits of length $1/p$ of Γ-nodes M_i and by Lemma 3.14 to central/noncentral G-graduation (conjugacy class!). We also use the terms inert / ramified / split for G-edges between G-nodes $\tilde{M}_i, \tilde{M}_j$ that are inert,inert / inert,split / split,split. They correspond to edges between between Γ-nodes in different Σ-orbits of length $1,1$ / $1,p$ / p,p and by Theorem 3.15 to G-edges of type A_2 / B_2 / A_2. All these cases are shown in the preceding example for $p = 2$.

In order to summarize combinatorial considerations, we introduce in Section 4.2 the notion of a symplectic root system. A symplectic root system for a given Cartan matrix resp. Dynkin diagram is a decoration of the diagram nodes by values in a finite symplectic vector space, such that the Weyl group acts as symplectic isometries. Note that such structures and especially the group of isometries have already been studied under the name "vanishing lattices" in [Chm82], [Jan83] in the context of singularity theory. The existence of a symplectic root system encodes nontrivial necessary conditions on the existence of a covering Nichols algebra: By [HS10] Prop. 8.1., nodes in the diagram have to be connected if the G-decorations are noncommuting. Meanwhile we have in [Len15] completely classified symplectic root systems for arbitrary given graphs over the field $\mathbb{F}_2$, which is the case relevant to this article. This technical result determines for a given Dynkin diagram the possible size of rank and center of a nonabelian group G which realizes the diagram as a finite-dimensional covering Nichols algebra. For example, to realize the diagram D_n, the group G needs to have a larger center than for other diagrams, while groups G with even larger center can only support disconnected diagrams.

Note that in [Len12] Theorem 6.1 we also checked for possible covering Nichols algebras for other primes $\Sigma = \mathbb{Z}_p$ and found that the ones given in the present paper of Cartan type and $\Sigma = \mathbb{Z}_2$ are indeed the only admissible choices. However, there are additional diagrams not corresponding to semisimple Lie algebras that lead to new **de**-composable covering Nichols algebras with $\Sigma = \mathbb{Z}_2$ and even one $D_4 \to G_2$ for $\Sigma = \mathbb{Z}_3$.

The application of the Construction Theorem 3.6 proceeds case-by-case, depending on the assumed symmetric Dynkin diagram of $\mathcal{B}(M)$ and a respective symplectic root system, whose existence imposes restrictions on rank and center of G. For the constructions we asumme G fulfills an additional technical condition (2-saturated) to simplify statements involving even-order generating sets. For groups where this is not the case, we may construct Nichols algebras that are disconnected and/or contain a connected component solely over an abelian group. This is discussed in Section 5.5, especially Lemma 5.9 and Corollary 5.13. For each connected Dynkin diagram we compactly describe dimension, root system and Hilbert series of the newly constructed covering Nichols algebra $\mathcal{B}(\tilde{M})$.

- In Section 5.1 we treat the generic, unramified case: Given a simply-laced Dynkin diagram X_n of type ADE, we use the symplectic root system to define a Γ-Yetter-Drinfel'd module $M = N \oplus N_\sigma$ with finite-dimensional Nichols algebra $\mathcal{B}(M)$, where $N \ncong N_\sigma$ each have the given Dynkin diagram X_n and the braiding matrices only contain entries ± 1. The crucial aspect of the symplectic root system is that it ensures the overall Dynkin diagram of M to be a disconnected union $X_n \times X_n$, which corresponds to $c_{NN_\sigma} c_{N_\sigma N} = id$, and hence again a finite-dimensional Nichols algebra.

  Then, interchanging N, N_σ gives by construction an action of $\Sigma = \mathbb{Z}_2$ by twisted symmetries on M. Note that the well-known example of an indecomposable Nichols algebra over $G = \mathbb{D}_4$ (see [MS00] resp. example Section 3.4 in this article) is our model for this case and corresponds to the diagram $X_n = A_2$. We also give an example $X_n = A_4$ in Section 5.2.
- In Section 5.3 we construct the exceptional example visualized above, where the twisted symmetry acts on a single Γ-Yetter-Drinfel'd module M of type E_6. The covering Nichols algebra $\mathcal{B}(\tilde{M})$ over G has Dynkin diagram F_4. Thereby ramified edges appear, which connect simple G-Yetter-Drinfel'd modules of different dimension. We use a symplectic root system to construct the split part of M with diagram $A_2 \times A_2$ and use an ad-hoc continuation by two inert nodes to E_6.
- In Section 5.4 we construct an infinite family of ramified covering Nichols algebras of type $A_{2n-1} \to C_n$ similar to the previous case $E_6 \mapsto F_4$. Again we use a symplectic root system for the split part of the diagram $A_{n-1} \times A_{n-1}$ and an explicit continuation by one inert node to A_{2n-1}.
- In Section 5.5 we describe how to construct Nichols algebras $\mathcal{B}(\tilde{M})$ with disconnected Dynkin diagrams and prove in particular, that any group G with $[G, G] \cong \mathbb{Z}_2$ (regardless of the order) admits at least one finite-dimensional indecomposable Nichols algebra.

We summarize the properties of the constructed Nichols algebras with connected Dynkin diagram in the following table, where the first and second column give necessary and sufficient conditions on the group G. For presenting the main result, we also need an additional technical assumption on G (2-saturated, Definition 5.1) that gives us control over the size of even-order generating systems in a group. We use the conventions $(n \bmod 2) \in \{0,1\}$ and denote by $\Phi^+(X_n)$ a fixed set of positive roots in a root system X_n (the Nichols algebra dimensions are 2-powers, because all self-braidings are $q_{\alpha\alpha} = -1$).

$dim_{\mathbb{F}_2}(G/G^2)$	$dim_{\mathbb{F}_2}(Z(G)/G^2)$	Dynkin-D. of $\tilde{M}$	$\dim(\mathcal{B}(M))$	$= \dim\left(\mathcal{B}(\tilde{M})\right)$
n	$n \bmod 2$	$A_{n\geq 2}$	$2^{\lvert\Phi^+(A_n\times A_n)\rvert}$	$= 2^{n(n+1)}$
$n = 6,7,8$	$n \bmod 2$	$E_{6,7,8}$	$2^{\lvert\Phi^+(E_n\times E_n)\rvert}$	$= 2^{72}, 2^{126}, 2^{240}$
n	$2-(n \bmod 2)$	$D_{n\geq 4}$	$2^{\lvert\Phi^+(D_n\times D_n)\rvert}$	$= 2^{2n(n-1)}$
$n = 4$	2	F_4	$2^{\lvert\Phi^+(E_6)\rvert}$	$= 2^{36}$
n	$2-(n \bmod 2)$	$C_{n\geq 3}$	$2^{\lvert\Phi^+(A_{2n-1})\rvert}$	$= 2^{n(2n-1)}$

Especially, in Corollary 5.13 we find indecomposable Nichols algebras (possibly with disconnected Dynkin diagram) over all groups G, that are $\mathbb{Z}_2$-stem-extensions of an abelian group Γ and thus positively answer for such groups a respective question raised by Susan Montgomery in [Mon95] for pointed Hopf algebras by providing the bosonizations $H = \Bbbk[G]\#\mathcal{B}(\tilde{M})$. See [AS02] Question 3.17 for the Nichols algebra formulation.

By construction, a Nichols algebra $\mathcal{B}(\tilde{M})$ obtained this way is non-faithful, diagonal and as an algebra isomorphic to the corresponding Nichols algebra $\mathcal{B}(M)$ over the abelian Γ. However, the knowledge of $H^2(G,\Bbbk^\times)$ together with Matsumoto's spectral sequence often allows to obtain Doi twists that are truly new faithful, non-diagonal, finite-dimensional, indecomposable Nichols Algebras. We give explicit examples of type $A_2,\ A_3,\ C_3,\ D_4,\ C_4,\ F_4$ over nonabelian groups of order 16 and 32 in Section 6.

2. Preliminaries

Throughout this article we suppose $\Bbbk = \mathbb{C}$, all groups are finite and all vector spaces finite-dimensional. The field with p elements is denoted by $\mathbb{F}_p$. The dihedral, quaternion, symmetric and alternating groups are denoted by $\mathbb{D}_4, \mathbb{Q}_8, \mathbb{S}_n, \mathbb{A}_n$. The Dynkin diagrams of the semisimple Lie algebras of rank n are denote by $A_n, B_n, C_n, D_n, E_n, F_4, G_2$. The multiplicative group of the field $\Bbbk$ is denoted by $\Bbbk^\times$, while the dual group is denoted $\Gamma^* = \mathrm{Hom}(\Gamma, \Bbbk^\times)$. We frequently call the generator of the multiplicatively denoted group $\mathbb{Z}_2 = \langle\theta\rangle$.

The following notions are standard. We summarize them to fix notation and refer to [Hec08] for a detailed account.

Definition 2.1. *A* Yetter-Drinfel'd module M *over a group* Γ *is a* Γ*-graded vector space,* $M = \bigoplus_{g\in\Gamma} M_g$ *with a* Γ*-action on* M *such that* $g.M_h = M_{ghg^{-1}}$ *Note that over* $\Bbbk = \mathbb{C}$ *any Yetter-Drinfel'd module* M *is semisimple, i.e. the direct sum of* n *simple Yetter-Drinfel'd modules, and we call* n rank*. Call* M

- (link-) indecomposable*, iff the support* $\{g \mid M_g \neq 0\}$ *generates all* Γ*.*
- minimally indecomposable*, iff* M *is indecomposable and no proper sub-Yetter-Drinfel'd module is indecomposable. Every indecomposable Yetter-Drinfel'd module contains a minimally indecomposable one.*
- faithful*, iff the* Γ*-action on* M *is faithful.*

Lemma 2.2. *The map* $c_{MM} : M \otimes M \to M \otimes M$ *defined by*

$$M_g \otimes M_h \ni v \otimes w \overset{c_{MM}}{\longmapsto} g.w \otimes v \in M_{ghg^{-1}} \otimes M_g$$

fulfills the Yang-Baxter-equation

$$(id \otimes c_{MM})(c_{MM} \otimes id)(id \otimes c_{MM}) = (c_{MM} \otimes id)(id \otimes c_{MM})(c_{MM} \otimes id)$$

turning M *into a* braided vector space *with* braiding c_{MM}*.*

Example 2.3. *For* abelian *groups* Γ*, the compatibility condition implies the stability of the homogeneous components* M_g*. For* $\Bbbk = \mathbb{C}$ *all simple Yetter-Drinfel'd modules* M_i *are 1-dimensional and isomorphic to some* $\mathcal{O}_{g_i}^{\chi_i} := x_i\Bbbk$ *with* Γ*-graduation* g_i *and* Γ*-action defined by a 1-dimensional character* $\chi_i : \Gamma \to \Bbbk^\times$ *via* $g.x_i := \chi_i(g)x_i$*. The braiding* c_{MM} *is hence* diagonal *with* braiding matrix $q_{ij} := \chi_j(g_i)$*.*

$$x_i \otimes x_j \overset{c_{MM}}{\longmapsto} q_{ij}(x_j \otimes x_i)$$

Definition 2.4. *Let* M *be a Yetter-Drinfel'd module over an arbitrary group* Γ *and let* $e_k \in M_{g_k}$ *be a fixed homogeneous basis; for* Γ *abelian one may choose* $e_k := x_k$*. Consider the tensor algebra* $T(M)$*, which can be identified with the algebra of words in the letters* $\{e_k\}_{k=1\ldots\dim(M)}$ *and is again a* Γ*-Yetter-Drinfel'd module. We can uniquely obtain* skew derivations $\partial_i : T(M) \to T(M)$ *by*

$$\partial_k(1) = 0 \qquad \partial_k(e_l) = \delta_{kl}1 \qquad \partial_k(x \cdot y) = \partial_k(x) \cdot (g_k.y) + x \cdot \partial_k(y)$$

The Nichols algebra $\mathcal{B}(M)$ *is the quotient of* $T(M)$ *by the largest homogeneous ideal* $\mathfrak{I}$ *in degree* ≥ 2*, invariant under all* ∂_k*. It is a* Γ*-Yetter-Drinfel'd module.*

Following [Hec08] we draw a q-diagram for Yetter-Drinfel'd module M over an abelian group by drawing a node for each basis element x_i spanning a corresponding 1-dimensional simple summand $M_i = \mathcal{O}_{g_i}^{\chi_i} = x_i\mathbb{k}$ of M. We draw an edge between x_i, x_j whenever $q_{ij}q_{ji} \neq 1$ (i.e. $c^2_{M_iM_j} \neq id$) and decorate each node i by the complex numbers q_{ii} and each edge ij by $q_{ij}q_{ji}$. It turns out that this data is all that is needed to determine the Nichols algebra $\mathcal{B}(M) = \mathcal{B}(\bigoplus_{i\in I} M_i)$.

Definition 2.5. *The* adjoint action $\mathcal{B}(M) \otimes \mathcal{B}(M) \to \mathcal{B}(M)$ *is given by*

$$x \otimes y \longmapsto \mathrm{ad}\,(x)\,(y) := x^{(1)}yS(x^{(2)})$$

For Yetter-Drinfel'd modules $N, L \subset \mathcal{B}(M)$ we define the ad-space *by*

$$\mathrm{ad}\,(N)\,(L) := \{\mathrm{ad}\,(x)\,(y) \;\mid x \in N,\; y \in L\}$$

The root system theory in [AHS10] describes $\mathcal{B}(M)$ in terms of Nichols algebras over iterated ad-spaces between the simple summands M_i (simple roots). Especially if M is over an abelian group, all $M_i = x_i\mathbb{k}$ are 1-dimensional and we obtain a (slightly different) PBW-basis of iterated braided commutators, as already observed by Kharchenko [Kha08]. Compare the lecture notes [Hec08].

Definition 2.6. *For a finite-dimensional Nichols algebra $\mathcal{B}(M)$ with semisimple Yetter-Drinfel'd module $M = \bigoplus_{i\in I} M_i$ consider the following structure constants for $i \neq j \in I$:*

$$C_{i,j} := \max_m \left(\mathrm{ad}^m\,(M_i)\,(M_j) := \mathrm{ad}\,(M_i)\,(\mathrm{ad}\,(M_i)\,(\cdots M_j)) \neq \{0\}\right)$$

Together with $C_{i,i} := 2$ they define a Cartan matrix [2] $(C_{i,j})_{i,j\in I}$ *of M. One may draw a* Dynkin diagram *with node set I and edges decorated by $(C_{i,j}, C_{j,i})$.*
For some cases pictorial representations are custom, e.g. double line and arrow for $(-1,-2)$ as in the root system B_2.

Theorem 2.7. *For a finite-dimensional Nichols algebra $\mathcal{B}(M)$ over an abelian group with diagonal braiding matrix q_{ij} the Cartan matrix can be equivalently obtained by $C_{i,j} := \min_m \left(q_{ii}^{-m} = q_{ij}q_{ji} \text{ or } q_{ii}^{m+1} = 1\right)$, see e.g.* [Hec08] *Prop. 5.5.*

If especially the Cartan matrix $C_{i,j}$ is the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$, then the Nichols algebra has a PBW-basis consisting of monomials in iterated braided commutators according to the positive roots in the classical root system of $\mathfrak{g}$. However, several additional exotic examples of

[2] Be warned, that this corresponds to choosing for Lie algebras $C_{i,j} = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$ according to Kac, Jantzen, Carter etc. and is transpose to the convention e.g. in Humphreys book.

finite-dimensional Nichols algebras exist, that possess unfamiliar Dynkin diagrams, such as a multiply-laced triangle, and where Weyl reflections may connect different Γ-Yetter-Drinfel'd modules (yielding a Weyl groupoid). Heckenberger completely classified all Nichols algebras over abelian Γ in [Hec09].

3. Covering Nichols Algebras

From now on, we always suppose a central extension of an abelian group Γ:

$$1 \to \Sigma^* \to G \xrightarrow{\pi} \Gamma \to 1 \qquad \Sigma \subset Z(\Gamma)$$

where $\Sigma^* = \mathrm{Hom}(\Sigma, \Bbbk^\times)$. We denote elements in G by letters such as g, whereas elements in the abelian quotient Γ are denoted by $\bar{g}$. A Γ-Yetter-Drinfel'd module is denoted by M, whereas the covering G-Yetter-Drinfel'd module will be denoted by $\tilde{M}$.

3.1. Twisted Symmetries.

Definition 3.1. *Let M be a Yetter-Drinfel'd module over an abelian group Γ*

$$M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} \mathcal{O}_{\bar{g}_i}^{\chi_i} = \bigoplus_{i\in I} x_i\Bbbk \qquad q_{ij} = \chi_j(\bar{g}_i)$$

written as a sum of simple 1-dimensional Yetter-Drinfel'd modules M_i, $i \in I$.
We call a linear bijection $f_0:\ M \to M$ a twisted symmetry*, iff*

- *f_0 preserves the Γ-grading (resp. is colinear)*
- *f_0 preserves self-braiding and monodromy:*

$$(f_0 \otimes f_0)c_{M_iM_i} = c_{M_iM_i}(f_0 \otimes f_0)$$

$$(f_0 \otimes f_0)c_{M_iM_j}c_{M_jM_i} = c_{M_iM_j}c_{M_jM_i}(f_0 \otimes f_0)$$

The braiding itself needs not to be preserved. However, we wish to control the modification by the following notions, regardless of Γ being abelian:

Definition 3.2. *Let $M = \bigoplus_{i\in I} M_i$ be a semisimple, indecomposable Γ-Yetter-Drinfel'd module decomposed into 1-dimensional summands M_i.*

- *We call f_0 a twisted symmetry with respect to a given bimultiplicative form $\langle,\rangle:\ \Gamma \times \Gamma \to \Bbbk^\times$, iff f_0 is a twisted symmetry and*

  $$c_{M_iM_j}(f_0 \otimes f_0) = \langle \bar{g}_i, \bar{g}_j \rangle (f_0 \otimes f_0)c_{M_iM_j}$$

  Note that since f_0 preserves the monodromy $c_{M_iM_j}c_{M_jM_i}$, the form is always skew-symmetric $\langle \bar{g}, \bar{h} \rangle = \langle \bar{h}, \bar{g} \rangle^{-1}$ and since f_0 preserves the self-braiding $c_{M_iM_i}$, the form is isotropic $\langle \bar{g}, \bar{g} \rangle = 1$. It is hence symplectic.

- *We call* f_0 *a twisted symmetry with respect to a group-2-cocycle* $\sigma_0 \in Z^2(\Gamma, \mathbb{k}^\times)$*, iff it is a twisted symmetry with respect to the form*

$$\langle \bar{g}, \bar{h} \rangle := \sigma_0(\bar{g}, \bar{h})\sigma_0^{-1}(\bar{h}, \bar{g})$$

Corollary 3.3. *Any twisted symmetry* $f_0 : M \to M$ *of an indecomposable Yetter-Drinfel'd module* M *with respect to a given form* $\langle , \rangle : \Gamma \times \Gamma \to \mathbb{k}$ *is an isomorphism of Yetter-Drinfel'd modules* $M \to M_{\langle , \rangle}$*, where* $M_{\langle , \rangle}$ *is* M *as* Γ*-graduated vector space and the* Γ*-action on a homogeneous element* $v \in M_{\bar{h}}$ *is modified to*

$$\bar{g}._{\langle , \rangle} v := \langle \bar{g}, \bar{h} \rangle (\bar{g}.v)$$

Remark 3.4. *In particular, any twisted symmetry* f_0 *of an indecomposable Yetter-Drinfel'd module* M *with respect to a given group-2-cocycle* $\sigma_0 \in Z^2(\Gamma, \mathbb{k}^\times)$ *is an isomorphism of Yetter-Drinfel'd modules* $M \to M_{\sigma_0}$*, where* M_{σ_0} *has accordingly modified* Γ*-action*

$$\bar{g}._{\sigma_0} v := \sigma_0(\bar{g}, \bar{h})\sigma_0^{-1}(\bar{h}, \bar{g})(\bar{g}.v)$$

or equivalently twisted characters $\chi_i^{\sigma_0}(\bar{g}) := \sigma_0(\bar{g}, \bar{h})\sigma_0^{-1}(\bar{h}, \bar{g})\chi_i^{\sigma_0}(\bar{g})$*. By e.g.* [Mas08] *Prop 5.2 this condition precisely means that* f_0 *can be extended to an isomorphism of Nichols algebras* $\mathcal{B}(M) \to \mathcal{B}(M_{\sigma_0})$ *to the Doi twist* $\mathcal{B}(M_{\sigma_0}) \cong \mathcal{B}(M)_{\sigma_0}$*.*

Next, we consider a family of twisted symmetries that respect a group law:

Definition 3.5. *Let* M *be a Yetter-Drinfel'd module over an abelian group* Γ *and* Σ *a finite abelian group. We say* Σ acts as twisted symmetries *on* M *iff*

- Σ *acts on the vector space* M*, i.e. denoting the action of an* $p \in \Sigma$ *by* $f_p : M \to M$ *we demand* $f_1 = id_M$ *and* $f_p f_q = f_{pq}$ *for all* $p, q \in \Sigma$*.*
- *For each* $p \in \Sigma$ *the action* f_p *is a twisted symmetry.*

For a given group-2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$*, we say that* Σ *acts on* M *as twisted symmetry with respect to* σ*, iff for all* $p \in \Sigma$ *the twisted symmetry* f_p *of* M *is a twisted symmetry with respect to the group-2-cocycle*

$$\sigma_p(\bar{g}, \bar{h}) := \sigma(\bar{g}, \bar{h})(p)$$

3.2. **Main Construction Theorem.** Suppose a given central extension of an abelian group Γ:

$$1 \to \Sigma^* \to G \xrightarrow{\pi} \Gamma \to 1 \qquad \Sigma \subset Z(\Gamma)$$

It can be described in terms of a cohomology class of 2-cocycles $[\sigma] \in H^2(\Gamma, \Sigma^*)$. We fix a set-theoretic section $s : \Gamma \to G$ of π which is normalized, i.e. $s(1) = 1$. This corresponds to a choice of a specific representing 2-cocycle

$\sigma \in Z^2(\Gamma, \Sigma^*)$ with $s(\bar{g})s(\bar{h}) = \sigma(\bar{g}, \bar{h})s(\bar{g}\bar{h})$. Different choices of s, σ will in what follows produce identical forms $\langle \bar{g}, \bar{h} \rangle_p := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$ and hence identical notions of twisted symmetry.

Theorem 3.6 (Covering Construction)**.** *Suppose now M to be a Yetter-Drinfel'd module over Γ and an action of Σ on M as twisted symmetries with respect to the $\sigma \in Z^2(\Gamma, \Sigma^*)$ fixed above. Because Σ is abelian, we may simultaneously diagonalize the action and decompose M into eigenspaces $M^{[\lambda]}$ with simultaneously eigenvalues $\lambda \in \Sigma^*$. Then the following structures define a G-Yetter-Drinfel'd module $\tilde{M}$, which we call the* covering Yetter-Drinfel'd module *of M:*

- *$\tilde{M} := M$ as vector space*
- *The G-action is the pullback of the Γ-action via π.*
  Especially $\Sigma^* \subset G$ acts trivially and thus $\tilde{M}$ is not faithful.
- *The eigenspaces $M^{[\lambda]}$ give rise to the G-homogeneous layers via $\tilde{M}_h := M_{\bar{h}}^{[hs(\bar{h})^{-1}]}$. Note that $\pi(hs(\bar{h})^{-1}) = 1$ so, $\lambda := hs(\bar{h})^{-1}$ is indeed an element of* $\mathrm{Ker}(\pi) = \Sigma^*$.

Proof. To prove M to be a well-defined Yetter-Drinfel'd module, we have to check that $\tilde{M}$ fulfills the (nonabelian) Yetter-Drinfel'd condition $g.\tilde{M}_h = \tilde{M}_{ghg^{-1}}$:

Claim 1: The Γ-action permutes simultaneous eigenspaces $M^{[\lambda]}$ as follows:

$$\bar{g}.M_{\bar{h}}^{[\lambda]} = M_{\bar{h}}^{[\sigma(\bar{g},\bar{h})\sigma^{-1}(\bar{h},\bar{g})\cdot\lambda]}$$

This can just be calculated: Let $v \in M_{\bar{h}}^{[\lambda]}$, i.e. the twisted symmetry action of any $p \in \Sigma$ is $f_p(v) = \lambda(p)v$, then by the defining property of a twisted symmetry

$$\begin{aligned} f_p(\bar{g}.v) &= \sigma_p(\bar{g}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{g})\bar{g}.f_p(v) \\ &= \sigma_p(\bar{g}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{g}) \cdot \lambda(p)v \\ &= \left(\sigma(\bar{g}, \bar{h})\sigma^{-1}(\bar{h}, \bar{g}) \cdot \lambda\right)(p) \cdot v \end{aligned}$$

and thus $\bar{g}.v$ is a simultaneous eigenvector of the Σ-action with eigenvalues $\sigma_p(\bar{g}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{g}) \cdot \lambda$ as claimed.

Claim 2: Γ abelian implies the commutator in G can be expressed as

$$[G, G] \ni [g, h] = \sigma(\bar{g}, \bar{h})\sigma^{-1}(\bar{h}, \bar{g}) \in \Sigma^*$$

This is by definition of σ true for elements $s(\bar{g}), s(\bar{h})$ in $\mathrm{Im}(s)$:

$$\begin{aligned}
[s(\bar{g}), s(\bar{h})] &= s(\bar{g})s(\bar{h}) \cdot s(\bar{g})^{-1} \cdot s(\bar{h})^{-1} \\
&= s(\bar{g})s(\bar{h}) \cdot s(\bar{g}\bar{h})^{-1} s(\bar{g}\bar{h})^{-1} \cdot s(\bar{g})^{-1} \cdot s(\bar{h})^{-1} \\
(\Gamma \text{ abelian}) \quad &= s(\bar{g})s(\bar{h})s(\bar{g}\bar{h})^{-1} \cdot s(\bar{h}\bar{g})^{-1} s(\bar{g})^{-1} \cdot s(\bar{h})^{-1} \\
&= \sigma(\bar{g}, \bar{h}) \cdot \sigma^{-1}(\bar{h}, \bar{g})
\end{aligned}$$

General elements $g, h \in G$ differ from such elements in $\mathrm{Im}(s)$ by a factor in $\mathrm{Ker}(\pi) = \Sigma^* \subset G$. Because Σ^* was supposed central in G, this does not change the commutator $[g, h]$, while the right hand side of the claim anyway only depends on the images $\bar{g}, \bar{h} \in \Gamma$. Thus the claim holds in for general $g, h \in G$ as well.

Claims 1 and 2 imply the asserted Yetter-Drinfel'd condition $g.\tilde{M}_h = \tilde{M}_{ghg^{-1}}$

$$\begin{aligned}
g.\tilde{M}_h &= \bar{g}.M_{\bar{h}}^{[hs(\bar{h})^{-1}]} \\
(\text{claim 1}) \quad &= M_{\bar{h}}^{[\sigma(g,h)\cdot\sigma^{-1}(h,g)\cdot hs(\bar{h})^{-1}]} \\
(\text{claim 2}) \quad &= M_{\bar{h}}^{[ghg^{-1}h^{-1}\cdot hs(\bar{h})^{-1}]} \\
(\Gamma \text{ abelian}) \quad &= M_{\bar{h}}^{[ghg^{-1}s(\bar{g}\bar{h}\bar{g}^{-1})^{-1}]} = \tilde{M}_{ghg^{-1}}
\end{aligned}$$

□

If a central extension $\Sigma^* \to G \to \Gamma$ is a stem extension $\Sigma^* \subset [G, G]$, then it is an easy group theoretic fact that any preimage of any generating system of Γ generates G. Hence:

Corollary 3.7. *For a stem extension, the covering $\tilde{M}$ of an indecomposable Γ-Yetter-Drinfel'd module M is an indecomposable G-Yetter-Drinfel'd module.*

By construction the G-action on $\tilde{M}$ factorizes to the Γ-action on M, thus:

Corollary 3.8. *$M, \tilde{M}$ are isomorphic as braided vector spaces. Especially $\mathcal{B}(M), \mathcal{B}(\tilde{M})$ are isomorphic as $\mathbb{Z}$-graded algebras (see e.g.* [AS02] *Section 5.1).*

Remark 3.9. *Note that in* [Len12] *we gave a much more general construction:*

- *A Hopf algebra H and a group Σ of Bigalois objects H_p yield a Hopf algebra structure on the direct sum, the covering Hopf algebra (see* [Len12] *Thm. 1.6):*

$$\Omega := \bigoplus_{p \in \Sigma} H_p$$

fitting into an exact sequence of Hopf algebras (see [Len12] *Thm. 1.13)*

$$0 \to \Bbbk^{\Sigma} \to \Omega \to H \to 0$$

The covering Hopf algebra Ω thereby can only be pointed, if among others H is pointed and Σ is an abelian group.

- *If we specialize this to the bosonization $H = \Bbbk[\Gamma]\#\mathcal{B}(M)$ of a Nichols algebra of a Yetter-Drinfel'd module M over an arbitrary group Γ, we yield a covering Nichols algebra $\mathcal{B}(\tilde{M})$ over a central extension.*

$$1 \to \Sigma^* \to G \to \Gamma \to 1$$

The construction in [Len12] *Thm. 4.3 uses a newly defined coaction. The direct formulation in the Construction Theorem 3.6 follows after diagonalizing the twisted symmetries. As an example for Γ non-abelian, we have also constructed e.g. a Nichols algebra of dimension 24^2 over $GL_2(\mathbb{F}_3) \to \mathbb{S}_4$.*

3.3. **Impact On The Dynkin Diagram: Folding.** First we observe, that if a twisted symmetry of a Γ-Yetter-Drinfel'd module M directly permutes the simple summands M_i (and thus the index set I), then it is already an automorphism of the q-diagram of M:

Definition 3.10. *Let $M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} x_i$ be a vector space decomposed into 1-dimensional sub-vector spaces according to some index set I, e.g. the decomposition of a Yetter-Drinfel'd module over an abelian group Γ decomposed into simple summands. Then an action $(f_p)_{p\in\Sigma}$ of a group Σ on M is called* permutation action*, iff it is induced by a permutation representation ρ on I, i.e.*

$$\rho : \ \Sigma \to \mathrm{Aut}(I) = \mathbb{S}_{|I|} \qquad \forall_{i\in I}\ f_p(x_i) = x_{p.i}$$

If M is moreover a diagonally braided vector space with braiding matrix q_{ij} with respect to the basis $\{x_i\}_{i\in I}$, then we denote the permuted braiding matrix by $(q^{(p)})_{ij} := q_{p.i,p.j}$.

Remark 3.11. *Note without proof that the technical assumption in the previous definition is an implicit necessity to obtain minimally indecomposable Nichols algebras. Certain non-minimally indecomposable Nichols algebras may however require the consideration of non-permutation actions (e.g in Remark 5.5).*

We may now for abelian Γ express the twisted symmetry condition of a given permutation action in terms of the permuted braiding matrix. We especially recognize Σ to consist necessarily of automorphism of the q-diagram:

Corollary 3.12. *Let $M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} x_i$ be a Yetter-Drinfel'd module over an abelian group Γ and Σ a group with permutation action on the decomposed M. Then the action is a action by twisted symmetries according to Definition 3.5 iff*

- *The Σ-action on I only permutes elements i,j with M_i, M_j in the same homogeneous component of M.*
- *The Σ-action on I preserves the self-braiding*

$$(q^{(p)})_{ii} = q_{ii}$$

- *The Σ-action on I preserves the monodromy*

$$(q^{(p)})_{ij}(q^{(p)})_{ji} = q_{ij}q_{ji}$$

Moreover, the Σ-action is a twisted symmetry with respect to a 2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$, iff the braiding matrix transforms under the action of all $p \in \Sigma$ according to the prescribed bimultiplicative form

$$\langle \bar{g}, \bar{h} \rangle_p := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$$

$$q_{ij}^{(p)} = \langle \bar{g}_i, \bar{g}_j \rangle_p q_{ij}$$

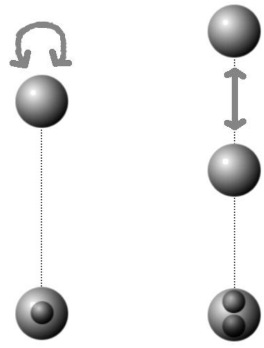
Hence especially the permutation action is an automorphism of the q-diagram of M, that has by definition node set I and is decorated with q_{ii} and $q_{ij}q_{ji}$.

Next we calculate the Dynkin diagram of the covering Nichols algebra $\mathcal{B}(\tilde{M})$ of a Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel'd module M over an abelian group Γ. We restrict to specific scenarios appearing in the present article (especially $p = 2$), but similar calculations can be carried out for other situations as well.

Definition 3.13. *Let $M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} \mathcal{O}_{\bar{g}_i}^{\chi_i}$ be some Γ-Yetter-Drinfel'd module and let $\Sigma = \mathbb{Z}_2 = \langle\theta\rangle$ act on M as twisted permutation symmetries. Then I decomposes into orbits of length 1 resp. 2. We call such nodes $i \in I$* inert *resp.* split*.*

In this situation the splitting behaviour of a node is determined by whether it's decoration is central in G:

Lemma 3.14. *Let Σ acts as twisted permutation symmetries: If a node $i \in I$ is inert, then necessarily its decoration $\bar{g}_i \in \Gamma$ has central image $s(\bar{g}_i) \in G$. If moreover all 1-dimensional simple summands M_i of M are mutually non-isomorphic, then the converse holds also: A node $i \in I$ is inert resp. split iff $s(\bar{g}_i) \in G$ is central resp. noncentral in G.*



Hence $\tilde{M}$ decomposes into simple G-Yetter-Drinfel'd modules $\tilde{M}_{\tilde{k}}$, where the new nodes $\tilde{k} \subset I$ are Σ-orbits of cardinality 1 resp. 2 and are called inert resp. split as well. These subsets of I hence form the new nodes set $\tilde{I}$.

Proof. Because we consider an action of $\Sigma = \mathbb{Z}_2$ on a set I, the orbits have length $1, 2$.

For the **first claim**, we assume some $i_1 \in I$ with $s(\bar{g}_{i_1}) \notin Z(G)$ and prove i_1 to be a split node: By assumption of noncentrality there exist some $g \in G$ with commutator

$$[g, s(\bar{g}_{i_1})] = \theta^* \in \Sigma^* \subset G$$

This is, because Γ is abelian and hence every commutator lays in the kernel Σ^* of the central extension; if the commutator is nontrivial, it has to coincide with the generator θ^* of $\Sigma^* \cong \mathbb{Z}_2$, i.e. the element with $\theta^*(\theta) = -1_{\mathbb{k}}$. By claim 2 in the proof of Theorem 3.6 we then have

$$\sigma(\bar{g}, \bar{g}_{i_1})\sigma^{-1}(\bar{g}_{i_1}, \bar{g}) = [g, s(\bar{g}_{i_1})] = \theta^*$$

We assumed Σ to act by twisted permutation symmetries f_p, hence $f_\theta(M_{i_1}) =: M_{i_2}$ is another summand of M and we wish to prove $i_1 \neq i_2$. This finally follows from Remark 3.4, as the twisted Γ-action on M_{i_2} is

$$\bar{g}._{\sigma_\theta} v := \sigma_\theta(\bar{g}, \bar{h})\sigma_\theta^{-1}(\bar{h}, \bar{g}) \cdot \bar{g}.v = \sigma(\bar{g}, \bar{h})\sigma^{-1}(\bar{h}, \bar{g})(\theta) \cdot \bar{g}.v = \theta^*(\theta) \cdot \bar{g}.v = -\bar{g}.v$$

This is a different Γ-action, hence $M_{i_1} \not\cong M_{i_2}$ and $i_1 \neq i_2$ and thus the node is split.

For the **second claim**, assume now moreover that all 1-dimensional simple summands M_i of M are mutually non-isomorphic, then we also prove the converse: Suppose some $i_1 \in I$ with $s(\bar{g}_{i_1}) \in Z(G)$, then we prove i_1 to be inert: By assumption of centrality the commutator $[g, s(\bar{g}_i)] = 1$ for all

$g \in G$. By claim 2 in the proof of Theorem 3.6 we have

$$\sigma(\bar{g}, \bar{g}_{i_1})\sigma^{-1}(\bar{g}_{i_1}, \bar{g}) = [g, s(\bar{g}_{i_1})] = 1_{\Sigma^*}$$

We assumed Σ acts as twisted permutation symmetries, hence $f_\theta(M_{i_1})$ is also a summand M_{i_2} of M; we wish to prove $i_1 = i_2$. By Remark 3.4, the twisted Γ-action on M_{i_2} is:

$$\bar{g}.{}_{\sigma_p} v := \sigma_p(\bar{g}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{g})(\bar{g}.v) = \bar{g}.v$$

Thus $M_{i_1} \cong M_{i_2}$ and by the additional assumption hence $i_1 = i_2$ and i_1 is inert. $\square$

Theorem 3.15. *Let $M = \bigoplus_i M_i$ be a Yetter-Drinfel'd module over the abelian group Γ and $\mathbb{Z}_2 \to G \to \Gamma$ a stem extension as above. Suppose again the simple summands M_i to be mutually nonisomorphic and consider the covering $\tilde{M} = \bigoplus_{\tilde{i}\in\tilde{I}} \tilde{M}_{\tilde{i}}$ constructed in Theorem 3.6. The set $\tilde{I}$ of simple summands $\tilde{M}_{\tilde{i}}$ has been described in the previous Lemma 3.14.*

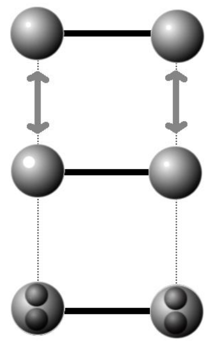
We now assume several situations (relevant to this article) for the Cartan matrix resp. Dynkin diagram of M and calculate in these situations Cartan matrix $\tilde{C}_{\tilde{k},\tilde{l}}$ and hence Dynkin diagram of the covering $\tilde{M}$ ("folded diagrams"):

(1) **Disconnected:** *Let $\tilde{k}, \tilde{l} \in \tilde{I}$ be arbitrary nodes (split or inert) and suppose all elements $k \in \tilde{k} \subset I$ are disconnected to all elements in $l \in \tilde{l} \subset I$ i.e. $q_{kl}q_{lk} = 1$ and $\mathrm{ad}\,(x_k)\,(x_l) = 0$. Equivalently we may assume the q-subdiagram of M to consist of mutually disconnected components $\tilde{k} \times \tilde{l}$. Then the Cartan matrix of $\mathcal{B}(\tilde{M})$ is also diagonal $\tilde{C}_{\tilde{k}\tilde{l}} = 0$, i.e. the covering nodes $\tilde{k}, \tilde{l}$ of $\tilde{M}$ are disconnected as well.*

(2) **Inert Edge:** *Let $\tilde{k} = \{k\}, \tilde{l} = \{l\} \subset I$ be inert nodes. Then the Cartan matrix entry in the covering Nichols algebra is of identical type $\tilde{C}_{\tilde{k},\tilde{l}} = C_{k,l}$.*

(3) **Split Edge:** *Let $\tilde{k} = \{k_1, k_2\}$ and $\tilde{k} = \{k_1, k_2\}$ both be split and (after possible renumbering) let the Dynkin diagram of $\mathcal{B}(M)$ restricted to the 4 simple Γ-Yetter-Drinfel'd modules $M_{k_1}, M_{k_2}, M_{l_1}, M_{l_2}$ be of type $A_2 \times A_2$. This means by definition:*

$$\mathrm{ad}\,(M_{k_i})\,(M_{l_i}) =: N_i \neq \{0\} \qquad i = 1, 2$$



where $N_i \subset \mathcal{B}(M)$ and all other ad-spaces are trivial. Then the Cartan matrix entry in the covering Nichols algebra over G is of type A_2 with $\mathrm{ad}\left(\tilde{M}_k\right)\left(\tilde{M}_l\right) = N_1 \oplus N_2 \subset \mathcal{B}(\tilde{M})$

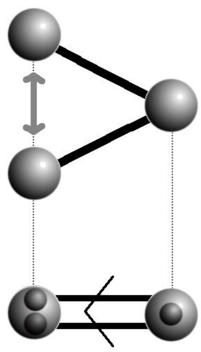
$$\begin{pmatrix} C_{\tilde{k}\tilde{k}} & C_{\tilde{k}\tilde{l}} \\ C_{\tilde{l}\tilde{k}} & C_{\tilde{l}\tilde{l}} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

See the example over $G \cong \mathbb{D}_4 \to \mathbb{Z}_2^2 \cong \Gamma$ in Section 3.4.

(4) **Ramified Edge:** *Let $\tilde{k} = \{k_1, k_2\} \in \tilde{I}$ be split and $\tilde{l} = \{l\}$ be inert. Let the Dynkin diagram of $\mathcal{B}(M)$ restricted to the 3 simple Γ-Yetter-Drinfel'd modules M_{k_1}, M_{k_2}, M_l be of type A_3 with l the middle node, i.e.*

$$\mathrm{ad}\,(M_{k_i})\,(M_l) =: N_i \neq \{0\} \qquad i = 1, 2$$
$$\mathrm{ad}\,(M_{k_2})\,(N_1) = \mathrm{ad}\,(M_{k_1})\,(N_2) =: N_{12} \neq \{0\}$$

and all other ad-spaces are trivial. Then the Cartan matrix entry in the covering Nichols algebra over G is of type B_2 with the split node $\tilde{k}$ the shorter root:



$$\begin{pmatrix} C_{\tilde{k}\tilde{k}} & C_{\tilde{k}\tilde{l}} \\ C_{\tilde{l}\tilde{k}} & C_{\tilde{l}\tilde{l}} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

This case can never occur isolated in indecomposable coverings, but the reader my check for example the ramified edge in $E_6 \mapsto F_4$ in Section 5.3.

Proof. The assumed standard-Lie-type of M in each case above translates into the knowledge of the resp. ad-spaces corresponding to the positive roots (see Definition 2.6). Because $\mathcal{B}(M) \cong \mathcal{B}(\tilde{M})$ as algebra and braided vector space (see Corollary 3.8), we can hence directly calculate case-by-case the respective ad-spaces of $\tilde{M}$ and hence the nondiagonal Cartan matrix

$$\tilde{C}_{\tilde{k}\tilde{l}} := -\max_m \left(\mathrm{ad}^m \left(\tilde{M}_i\right)\left(\tilde{M}_j\right) \neq \{0\}\right)$$

Here the new simple summands $\tilde{M}_i$ of dimension 1 or 2 determined in Lemma 3.14 and the calculations take place in $\mathcal{B}(M)$.

Note that in contrast to the proceeding e.g. in [HV13] we do not determine a Nichols algebra over ad-spaces as Yetter-Drinfel'd modules in $\mathcal{B}(\tilde{M})$, but merely piece together the assumed ad-spaces in $\mathcal{B}(M)$.

(1) In this case all ad-spaces are trivial:

$$\begin{aligned} \mathrm{ad}\left(\tilde{M}_i\right)\left(\tilde{M}_j\right) &= \mathrm{ad}\left(\bigoplus_{k\in\tilde{k}} M_k\right)\left(\bigoplus_{l\in\tilde{l}} M_l\right) \\ &= \sum_{k,l} \mathrm{ad}\,(M_k)\,(M_l) \\ &= \{0\} \end{aligned}$$

(2) If both $\tilde{k} = \{k\}$ and $\tilde{l} = \{l\}$ are inert, then $M_{\tilde{l}}, M_{\tilde{k}}$ are equal to M_l, M_k and, using again that $\mathcal{B}(M) = \mathcal{B}(\tilde{M})$, all ad-spaces coincide.

(3) We calculate the ad-spaces of $\tilde{M}$ over G from the assumed type $A_2 \times A_2$ of M over Γ, i.e. both N_i have trivial $\mathrm{ad}(N_i)$:

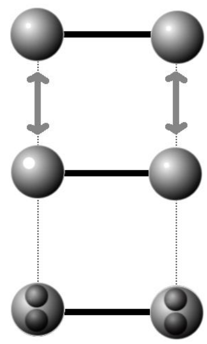
$$\begin{aligned}
\mathrm{ad}\left(\tilde{M}_{\tilde{k}}\right)\left(\tilde{M}_{\tilde{l}}\right) &= \mathrm{ad}\left(M_{k_1} \oplus M_{k_2}\right)\left(M_{l_1} \oplus M_{l_2}\right) \\
&= \mathrm{ad}\left(M_{k_1}\right)\left(M_{l_1}\right) + \mathrm{ad}\left(M_{k_1}\right)\left(M_{l_2}\right) \\
&\quad + \mathrm{ad}\left(M_{k_2}\right)\left(M_{l_1}\right) + \mathrm{ad}\left(M_{k_2}\right)\left(M_{l_2}\right) \\
&= N_1 \oplus N_2 \\
\mathrm{ad}^2\left(\tilde{M}_{\tilde{k}}\right)\left(\tilde{M}_{\tilde{l}}\right) &= \mathrm{ad}\left(M_{k_1} \oplus M_{k_2}\right)\left(N_1 \oplus N_2\right) \\
&= \{0\}
\end{aligned}$$

(4) We calculate the ad-spaces of $\tilde{M}$ over G from the assumed type A_3 of M over Γ, i.e. the N_{12} below has trivial $\mathrm{ad}(N_{12})$:

$$\begin{aligned}
\mathrm{ad}\left(\tilde{M}_{\tilde{k}}\right)\left(\tilde{M}_{\tilde{l}}\right) &= \mathrm{ad}\left(M_{k_1} \oplus M_{k_2}\right)\left(M_l\right) \\
&= N_1 \oplus N_2 \\
\mathrm{ad}^2\left(\tilde{M}_{\tilde{k}}\right)\left(\tilde{M}_{\tilde{l}}\right) &= \mathrm{ad}\left(M_{k_1} \oplus M_{k_2}\right)\left(N_1 \oplus N_2\right) \\
&= N_{12} \\
\mathrm{ad}^3\left(\tilde{M}_{\tilde{k}}\right)\left(\tilde{M}_{\tilde{l}}\right) &= \mathrm{ad}\left(M_{k_1} \oplus M_{k_2}\right)\left(N_{12}\right) \\
&= \{0\} \\
\mathrm{ad}^2\left(\tilde{M}_{\tilde{l}}\right)\left(\tilde{M}_{\tilde{k}}\right) &= \mathrm{ad}\left(M_l\right)\left(N_1 \oplus N_2\right) \\
&= \{0\}
\end{aligned}$$

□

3.4. **Example: Folding $A_2 \times A_2$ to A_2 over the group $\mathbb{D}_4$.**
In [MS00] Milinski and Schneider gave examples of indecomposable Nichols algebras over the non-abelian Coxeter groups $G = \mathbb{D}_4, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5$. We want to show how the first case $G = \mathbb{D}_4$ may be constructed as a covering Nichols algebra $\mathcal{B}(\tilde{M})$ of a certain diagonal $\mathcal{B}(M)$ with q-diagram $A_2 \times A_2$ over $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.



The example shall demonstrate the systematic approach in this article and exhibits already several crucial points of interest:

- The construction of an M with a twisted symmetry of order 2 is a model for the so-called "unramified case" of the covering construction in Section 5.1: The diagram consists of disconnected copies of a q-diagram interchanged by the twisted symmetry.
- As described in [MS00] p. 21, Graña had remarked, that the Nichols algebra possesses a "strange" alternative basis that is non-homogeneous in the G-grading, but of precise type $A_2 \times A_2$. This allowed Schneider and Milinski to more easily write down the relations. The covering construction precisely reproduces this basis as the formerly homogeneous basis of M that is no longer homogeneous in $\tilde{M}$. This existence of a finer diagonal PBW-basis continues throughout this article (only in the unramified cases this finer PBW-basis type $X_n \times X_n$ as here, ramified cases are more involved).
- The Nichols algebra itself is non-faithful and even diagonal. On the other hand, there is a Doi twist of $\mathcal{B}(M)$, which is a faithful indecomposable Nichols algebra over $\mathbb{D}_4$. We will produce faithful Doi twists as well in Section 6.

Example 3.16. *We start with a specific* 4*-dimensional diagonal Yetter-Drinfel'd module over* $\Gamma := \mathbb{Z}_2^2 = \langle v, w \rangle$ *(for a systematic construction see Section 5.1)*

$$M = \bigoplus_{i=0}^{4} \mathcal{O}_{\bar{g}_i}^{\chi_i} = \bigoplus_{i=0}^{4} y_i \mathbb{k}$$

$$\bar{g}_1 = \bar{g}_3 = v \qquad \bar{g}_2 = \bar{g}_4 = w$$

$$\chi_1 = \chi_4 = (-1, -1) \qquad \chi_3 = (-1, +1) \qquad \chi_2 = (+1, -1)$$

where the tuples denote the character value on the generators: $\chi = (\chi(v), \chi(w))$. *According to* [Hec09]*, this* M *is of type* $A_2 \times A_2$ *and hence the Hilbert series* *of the Nichols algebra $\mathcal{B}(M)$ is as follows (denoting $[n]_t := \frac{t^n-1}{t-1}$ and esp. $[2]_t = 1+t$)*

$$\mathcal{H}(t) = \left((1+t)(1+t)(1+t^2)\right)^2 = [2]_t^4[2]_{t^2}^2 \qquad \dim\left(\mathcal{B}(M)\right) = \mathcal{H}(1) = 2^6 = 64$$

The covering Nichols algebra $\mathcal{B}(\tilde{M})$ over the $\mathbb{Z}_2$-stem-extension $G = \mathbb{D}_4$ of Γ is of type A_2 with nodes of dimension 2. But since $\mathcal{B}(\tilde{M}) \cong \mathcal{B}(M)$, it is still a diagonal, with now non-homogeneous diagonal $\tilde{M}$-basis y_1, y_2, y_3, y_4, has the same Hilbert series and dimension and a finer PBW-basis of type $A_2 \times A_2$.

More precisely, the construction proceeds step-by step as follows:

Let $a^4 = b^2 = 1$ be the usual generators of $\mathbb{D}_4$. We choose a splitting $s : \mathbb{D}_4 \to \Gamma$ by sending the elements $1, v, w, vw$ to $1, b, ab, a^3 = bab$. The group-2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$ and especially the evaluation on the generator $\theta \in \mathbb{Z}_2 \cong \Sigma$ can hence be calculated explicitly (rows, columns are labeled $1, v, w, vw$):

$$\sigma_\theta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

To apply the Construction Theorem 3.6 we need an action of Σ on M by twisted symmetries with respect to σ. Hence we first calculate from σ the nontrivial bimultiplicative form $\langle\rangle_\theta$ in Definition 3.2:

$$\langle -, v\rangle_\theta = \sigma_\theta(-, v)\sigma_\theta^{-1}(v, -) = (+1, -1)$$
$$\langle -, w\rangle_\theta = \sigma_\theta(-, w)\sigma_\theta^{-1}(w, -) = (-1, +1)$$

This form immediately determines the twisted characters by Corollary 3.3:

$$\chi_1^{\sigma_\theta}(-) = (+1, -1)\chi_1(-) = \chi_3(-)$$
$$\chi_3^{\sigma_\theta}(-) = (+1, -1)\chi_3(-) = \chi_1(-)$$
$$\chi_2^{\sigma_\theta}(-) = (-1, +1)\chi_2(-) = \chi_4(-)$$
$$\chi_4^{\sigma_\theta}(-) = (-1, +1)\chi_4(-) = \chi_2(-)$$

Hence switching y_1, y_3 respectively y_2, y_4 is a twisted symmetry with respect to the bimultiplicative form $\langle -, -\rangle_\theta$ (Definition 3.2). Moreover, taking this map as f_θ (and $f_1 := id$) defines an action of Σ on M by twisted symmetries with respect to the cocycle σ (Definition 3.5). The covering construction (Theorem 3.6) hence yields an indecomposable Nichols algebra of dimension $\dim \mathcal{B}(\tilde{M}) = \dim \mathcal{B}(M) = 64$ over $G = \mathbb{D}_4$.

To connect to the notation in [MS00] we now also calculate the G-homogeneous components, as they follow from the construction theorem as f_p-eigenvectors to the trivial eigenvalue $1^* \in \Sigma^*$ with $1^*(\theta) = 1$ or the unique nontrivial eigenvalue $\theta^* \in \Sigma^*$ with $\theta^*(\theta) = -1$:

$$x_1 := y_1 + y_3 \in M_v^{[1^*]} = \tilde{M}_b$$
$$x_2 := y_2 + y_4 \in M_w^{[1^*]} = \tilde{M}_{ab}$$
$$x_3 := y_1 - y_3 \in M_v^{[\theta^*]} = \tilde{M}_{1,a^2b}$$
$$x_4 := y_2 - y_4 \in M_w^{[\theta^*]} = \tilde{M}_{1,a^3b}$$

4. Symplectic Root Systems

4.1. Symplectic $\mathbb{F}_p$-Vector Spaces And Stem Extensions.

Suppose we are given a finite group G with commutator subgroup $[G,G] = \mathbb{Z}_p$. Such a group is clearly always a stem-extension of its abelianization $\Gamma = G/[G,G]$. As usual e.g. for p-groups (see e.g. [Hup83]) we consider the *commutator map* $[,]$, which is skew-symmetric and isotropic:

$$G \times G \xrightarrow{[,]} [G,G] = \mathbb{Z}_p$$
$$g, h \mapsto [g,h] = ghg^{-1}h^{-1}$$
$$[h,g] = [g,h]^{-1} \quad [g,g] = 1$$

Because $[G,G]$ is central, the map is multiplicative in both arguments:

$$\begin{aligned}[g,h][g',h] &= (ghg^{-1}h^{-1})(g'hg'^{-1}h^{-1}) \\ &= g(g'hg'^{-1}h^{-1})hg^{-1}h^{-1} \\ &= gg'hg'^{-1}g^{-1}h^{-1} \\ &= [gg',h]\end{aligned}$$

and factors to $\langle , \rangle : \ \Gamma \times \Gamma \to \mathbb{Z}_p$. Because of bimultiplicativity, $[g^p, h] = [g,h]^p = 1$ holds and thus the commutator map even factorizes one step further to $V := \Gamma/\Gamma^p \cong \mathbb{F}_p^n$

$$V \times V \xrightarrow{\langle , \rangle} \mathbb{F}_p \qquad \textbf{denoted additively}$$

Remark 4.1. *Note that by claim 2 in the proof of Theorem 3.6, this bimultiplicative form coincides with the form*

$$\langle \bar{g}, \bar{h} \rangle_\theta := \sigma(\bar{g}, \bar{h})(\theta)\sigma^{-1}(\bar{h}, \bar{g})(\theta)$$

associated by Definition 3.5 to any 2-cocycle σ representing the present stem-extension

$$\langle \theta \rangle \cong \mathbb{Z}_p = \Sigma \to G \to \Gamma$$

Thus, the form $\langle , \rangle$ determines directly the notion of twisted symmetry in this situation.

Theorem 4.2 (Burnside Basis Theorem, [Hup83] Thm. 3.15 p. 273f)**.** *For $|G| = p^N$ every minimal generating set of G corresponds to a $\mathbb{F}_p$-basis in the quotient $V := G/(G^p[G,G])$, where $G^p[G,G]$ is the Frattini subgroup of G. Especially every minimal generating set consists precisely of $n = \dim_{\mathbb{F}_p}(V)$ elements. Note that for $p = 2$ we have $V = G/(G^2[G,G]) = G/G^2$.*

In what follows, we shall consider $V = G/([G,G]G^p)$ as a *symplectic vector space* $\mathbb{F}_p^n$ with (possibly degenerate!) *symplectic form* $\langle v, w\rangle$. For a sub-vector space $W \subset V$ we define the *orthogonal complement*:

$$W^\perp := \{v \in V \mid \forall_{w\in W}\ \langle v, w\rangle = 0\}$$

Especially $V^\perp = Z(G)/([G,G]G^p)$ is the *nullspace* of vectors orthogonal on all of V (note that always $\langle v, v\rangle = 0$). For $V^\perp = \{0\}$ we call V *nondegenerate*.

It is well known (see e.g. [Hup83]) that there is always a *symplectic basis* $\{x_i, y_i, z_j\}_{i,j}$ consisting of mutually orthogonal nullvectors $z_j \in V^\perp$ and *symplectic base pairs* $\langle x_i, y_i\rangle = 1$ generating a maximal nondegenerate subspace. Note especially, that nondegenerate symplectic vector spaces hence always have even dimension! They lead for example to extraspecial groups $G = p_\pm^{\dim(V)+1}$, especially for $p = 2$ and $\dim(V) = 2$ to $G = \mathbb{D}_4, \mathbb{Q}_8$.

4.2. **Symplectic Root Systems Of Type ADE Over $\mathbb{F}_2$.**

Definition 4.3. *Given a symplectic vector space V over $\mathbb{F}_2$ and a graph $\mathcal{D}$, we define a* symplectic root system *for this graph as a decoration ϕ : $Nodes(\mathcal{D}) \to V$, such that $\mathrm{Im}(\phi)$ generates V and nodes $i \neq j$ are connected iff $\langle\phi(i), \phi(j)\rangle = 1_{\mathbb{F}_2}$ (note that always $\langle v, v\rangle = 0$). If $\mathrm{Im}(\phi)$ is even a $\mathbb{F}_2$-basis of V, we call the symplectic root system* minimal*.*

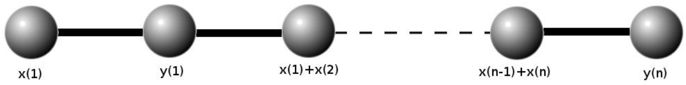
Remark 4.4. *We will use the notion for one directly on simply-laced Dynkin diagrams $\mathcal{D}$, but also as tool for the ramified case $\mathcal{D}'$, where only a part of the diagram $\mathcal{D} \subset \mathcal{D}'$ is split (such as $\mathcal{D} = A_2, A_{n-1}$ for ramified $E_6 \mapsto F_4$ and $A_{2n-1} \mapsto C_n$). Minimal symplectic root systems thereby correspond to minimally indecomposable covering Nichols algebras.*

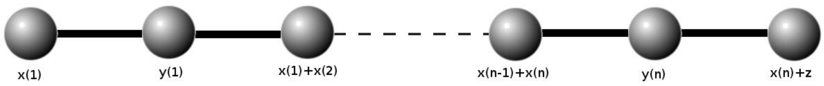
Note that in [Len15] we completely classified symplectic root systems for arbitrary given graphs over the field $\mathbb{F}_2$. In [Len15] Cor. 5.9 we have proven that every graph admits a unique minimal symplectic root systems. We have also given explicit descriptions, given below, of the symplectic root systems for Cartan type Dynkin diagrams. However, that the reader may directly verify that the following decorations indeed do form symplectic root systems for all simply-laced Dynkin diagrams.

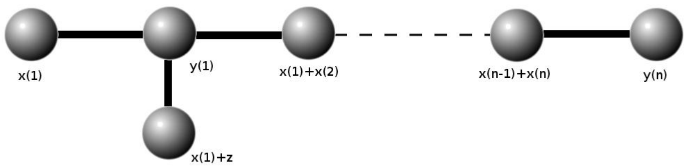
Theorem 4.5. *The graph of a simply laced Dynkin diagram of rank n admits a minimal symplectic root system over the symplectic vector space V of dimension n and typically minimal nullspace dimension $k = \dim(V^\perp)$:*

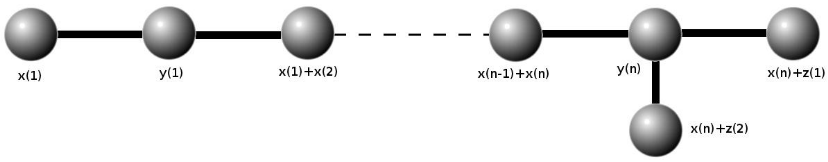
- $k = 0$ *for n even, i.e.* $V = \langle \{x_i, y_i\}_i \rangle_{\mathbb{k}}$
- $k = 1$ *for n odd, i.e.* $V = \langle \{x_i, y_i\}_i, z \rangle_{\mathbb{k}}$
- $k = 2$ *for type D_n and n even, i.e.* $V = \langle \{x_i, y_i\}_i, z_1, z_2 \rangle_{\mathbb{k}}$.

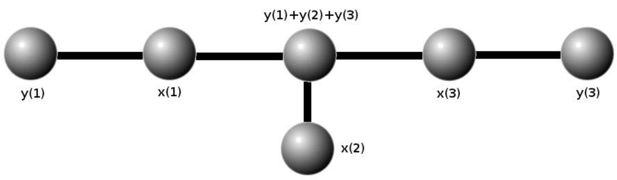
Proof. Explicit symplectic root systems are given by the following decorations ϕ. One checks easily in every instance, that indeed $\langle \phi(i), \phi(j) \rangle = 1$ iff i, j are connected:

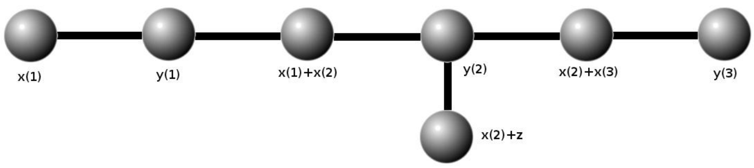


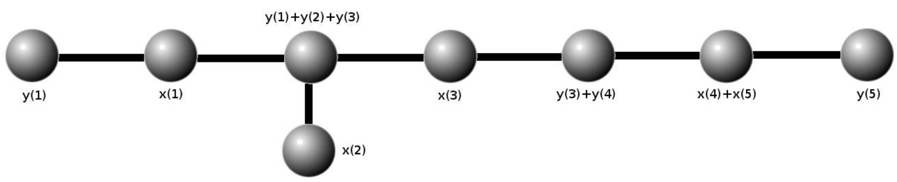












□

5. Main Constructions For Commutator Subgroup $\mathbb{Z}_2$

Suppose a nonabelian group G with commutator subgroup $[G, G] = \mathbb{Z}_2$, which is hence a stem-extension $\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma$ of an abelian group Γ. Using Heckenberger's classification [Hec09] of finite-dimensional Nichols algebras $\mathcal{B}(M)$ over abelian groups Γ and symplectic root systems, we now construct finite-dimensional minimally indecomposable covering Nichols algebras $\mathcal{B}(\tilde{M})$ with connected Dynkin diagram, depending on 2-rank and 2-center of G.

We need the following additional technical assumption to present the main result, that gives us control over the size of even-order generating systems in a group. Groups that fail this assumption can easily be treated if explicit generating systems are at hand, see e.g. Corollary 5.10. Usually such a group still admits **dis**connected covering Nichols algebras.

Definition 5.1. *A nilpotent group G is* 2-saturated*, if for any prime p*

$$\dim_{\mathbb{F}_p}(G/[G,G]G^p) \leq \dim_{\mathbb{F}_2}(G/[G,G]G^2) \qquad \big(= \dim_{\mathbb{F}_2}(G/G^2)\big)$$

Especially every 2-group is 2-saturated.

Lemma 5.2. *If a nilpotent group G is 2-saturated, then any $\mathbb{F}_2$-basis of the elementary abelian quotient $V = G/G^2$ can be lifted to a minimally generating set of G. Especially any minimal generating set of G contains only elements of even order and has size $\dim_{\mathbb{F}_2}(G/G^2)$.*

Proof. By [Hup83] Thm 2.3 (p. 260f) every nilpotent group G is a direct product of p-groups G_p, i.e. groups of prime power. By the Burnside Basis Theorem 4.2 a minimal generating set of any G_p corresponds to a $\mathbb{F}_p$-basis of the elementary abelian quotient $G_{/p}[G_p, G_p]G_p^p \cong \mathbb{F}_p^{N_p}$ with $N_p := \dim_{\mathbb{F}_p}(G/[G,G]G^p)$. The group G is 2-saturated, iff N_2 is a maximal value of all N_p. Hence under this assumption any $\mathbb{F}_2$-basis of

$$V = G_2/([G_2, G_2]G_2^2) = G_2/G_2^2 \cong G/G^2$$

hence of rank N_2, allows for the choice of generating set in the other G_p as well. The respective direct products form a generating set in G. Omitting one element would omitt an element in the $\mathbb{F}_2$-basis, which can hence not be generating any more. Thus the constructed generating set is minimal. □

In Section 6 we will furthermore give nondiagonal (and especially some faithful!) Doi twists of the diagonal Nichols algebras constructed below. These examples of rank ≤ 4 over various groups of order 16 and 32 are then the first indecomposable faithful Nichols algebras over nonabelian groups of rank > 2.

Theorem 5.3. *For any group G with $[G,G] \cong \mathbb{Z}_2$ consider the invariants*

$$\begin{array}{llll} V & := G/G^2 \cong \Gamma/\Gamma^2 & \dim_{\mathbb{F}_2}(V) =: \text{2-rank} \\ V^\perp & = Z(G)/G^2 & \dim_{\mathbb{F}_2}(V^\perp) =: \text{2-center} \end{array}$$

and denote $(n \bmod 2) \in \{0,1\}$. Assume G to be 2-saturated, then G admits a finite-dimensional minimally indecomposable Nichols algebra $\mathcal{B}(\tilde{M})$ with the following connected Dynkin diagram, depending on 2-rank and 2-center of G. They are covering Nichols algebras of some $\mathcal{B}(M)$ over Γ constructed below:

- **Unramified** *(generic) simply-laced components from a disconnected double with a symplectic root system for V:*

$dim_{\mathbb{F}_2}(V)$	$dim_{\mathbb{F}_2}(V^\perp)$	M	$\tilde{M}$
n	$n \bmod 2$	$A_n \times A_n$	$A_{n\geq 2}$
n	$n \bmod 2$	$E_n \times E_n$	$E_{n=6,7,8}$
n	$2-(n \bmod 2)$	$D_n \times D_n$	$D_{n\geq 4}$

- **Ramified** *components from a single diagram with an order 2 automorphism and a symplectic root system decomposed as $V := V_{inert} \oplus^\perp V_{split}$:*

$dim_{\mathbb{F}_2}(V)$		$dim_{\mathbb{F}_2}(V^\perp)$	M	$\tilde{M}$
$2+2$	$=4$	2	E_6	F_4
$1+(n-1)$	$=n$	$2-(n \bmod 2)$	A_{2n-1}	$C_{n\geq 3}$

Proof. The theorem states a list of constructions, provided that G is of the assumed form with specific invariants $dim_{\mathbb{F}_2}(V)$ and $dim_{\mathbb{F}_2}(V^\perp)$. These construction are then carried out in the subsequent sections:

- The unramified cases in Theorem 5.6.
- The ramified case $E_6 \mapsto F_4$ in Theorem 5.7.
- The ramified case $A_{2n-1} \mapsto C_n$ in Theorem 5.8.

In each of the quoted theorems, the assumed invariants of G characterize the type (dimension, nullspace) of the symplectic vector space G/G^2 as described in Section 4.1. We then invoke Theorem 4.5, which returns a minimal symplectic root system precisely for the distinct choice of invariants assumed in the statement. This basis of $V = G/G^2$ is then lifted using Lemma 5.2 using the additional assumption of G to be saturated. This yields a set of conjugacy classes, which we complete to a Yetter-Drinfeld modules M with twisted symmetry as prescribed by an ad-hoc choice of realizing characters. This finally stages the application of the covering construction Theorem 3.6 to yield $\mathcal{B}(\tilde{M})$. □

Remark 5.4. *For further link-***de***composable Nichols algebras see* [Len12] *Sec. 6.6:*

- *unramified $A_1 \times A_1 \mapsto A_1$*

- *ramified* $A_3 \mapsto B_2$, $D_{n+1} \mapsto B_n$ *and several alike non-Cartan diagrams.*
- *an isolated loop diagrams* $A_2 \mapsto A_1$, $q \in \Bbbk_3$.
- *ramified* $D_4 \to G_2$ *which is the only covering with* $\Sigma \cong \mathbb{Z}_3$.

Before we proceed to the proof, we comment on possible outlooks to groups of higher nilpotency class:

Remark 5.5. *Note that non-minimally indecomposable covering Nichols algebras over* G *might be interesting as well, especially because in this case there might exist a secondary covering Nichols algebra* $\mathcal{B}(\tilde{\tilde{M}})$ *over a group* $\tilde{G}$ *of nilpotency class* 3*, that use a twisted symmetry of the primary covering Nichols algebra* $\mathcal{B}(\tilde{M})$ *over* G*.*

In [Len15] *we have found non-minimal symplectic root systems as well, especially of type* D_{n+1}*. As a conjecture, this might give rise to covering Nichols algebras*

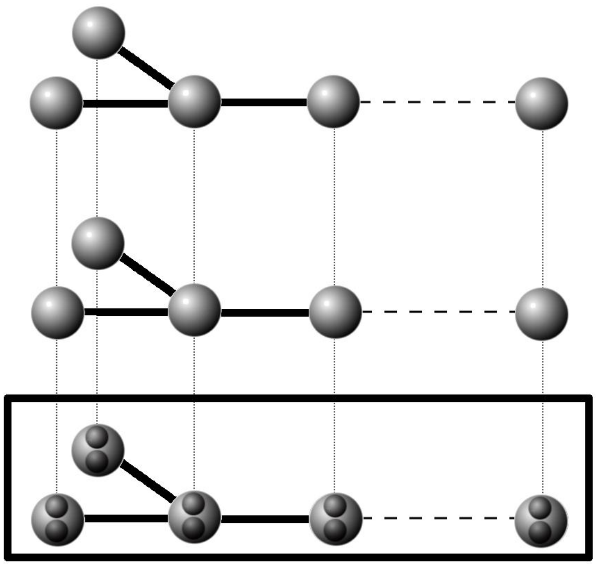
$$D_{n+1} \times D_{n+1} \mapsto D_{n+1} \mapsto B_n$$

over a group $\tilde{G}$ *of nilpotency class* 3*. In this case,* $\tilde{\tilde{M}}$ *would be the sum of a unique simple Yetter-Drinfel'd module of dimension* 4 *and* $n-1$ *simple Yetter-Drinfel'd modules of dimension* 2*. Such a Nichols algebra* $\mathcal{B}(\tilde{\tilde{M}})$ *would exhibit a* B_n*-Dynkin diagram, while the Hilbert series were the square of a Hilbert series of a Nichols algebra with diagram* D_{n+1} *over an abelian group* Γ*.*

During the review of this article, such a Nichols algebra for $n=2$ *has indeed be discovered in the classification of rank 2 in* [HV13]*. However, the assumed Hilbert series* $\mathcal{H}(t) = \left([2]_t^3[2]_{t^2}^2[2]_{t^3}\right)^2$ *turns out to be just a large divisor of the found Hilbert series, which indicates an additional extension. We hope nevertheless that the present approach can help to further understand the structure of the newly discovered Nichols algebra, such as providing a PBW-basis.*

5.1. **Unramified Cases** $ADE \times ADE \mapsto ADE$**.** The most natural and generic way to construct a Yetter-Drinfel'd module with twisted symmetry $\mathbb{Z}_2$ has already been demonstrated on the case $\mathbb{D}_4$ in Section 3.4; we take Γ-Yetter-Drinfel'd modules N as well as N_σ with modified Γ-action (Remark 3.4) and force twisted symmetry by considering $M := N \oplus N_\sigma$. The symplectic root system will assures that the diagram of M consists indeed of disconnected identical subdiagrams for N, N_σ (while $N \not\cong N_\sigma$). We will subsequently calculate an explicit example for $A_4 \times A_4 \mapsto A_4$ in Section 5.2.

The following image illustrates the covering of type $D_n \times D_n \mapsto D_n$:



Theorem 5.6. *Suppose a simply-laced Dynkin diagram X_n of rank n and G an arbitrary 2-saturated group (Def 5.1) with $[G,G] = \mathbb{Z}_2$ and $\Gamma := G/[G,G]$, such that*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(G/G^2) \overset{!}{=} n \geq 2$
- $dim_{\mathbb{F}_2}(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G^2) \overset{!}{=} \begin{cases} 2, & \textit{for } X_n = D_{2m} \\ n \bmod 2, & \textit{else} \end{cases}$

Then there exists a Γ-Yetter-Drinfel'd module $M = N \oplus N_\sigma$ with N, N_σ disconnected in M and an twisted permutation action of $\Sigma \cong \mathbb{Z}_2$ interchanging $N \leftrightarrow N_\sigma$. The covering Yetter-Drinfel'd module over G is hence $\tilde{M} = \bigoplus_{i=1}^n \tilde{M}_i$ of dimension $2n$ with:

- *$[G,G]$ acts trivially on $\tilde{M}$, which is diagonal, but V acts faithfully.*
- *$\tilde{M}$ is minimally indecomposable.*
- *$\mathcal{B}(\tilde{M})$ is finite-dimensional, with Hilbert series the square of the Hilbert series single diagram in the diagonal case over $\mathbb{Z}_2^n$, especially*

$$\dim\left(\mathcal{B}(\tilde{M})\right) = \mathcal{H}(1) = 2^{|\Phi^+(X_n \times X_n)|}$$

- *$\tilde{M}$ has the prescribed Cartan matrix and Dynkin diagram X_n with all nodes $\tilde{M}_i$ of dimension 2 (i.e. underlying conjugacy class of length 2).*

An explicit example of type A_4 will be discussed in the subsequent subsection 5.2. Several faithful Doi twist and hence nondiagonal Nichols algebras for small rank D_4, A_2, A_3 over nonabelian G are given in Section 6.

Proof. The strategy has been outlined above:

Step 1: We first construct a Γ-Yetter-Drinfel'd module $N := \bigoplus_{i=1}^n \mathcal{O}_{\bar{g}_i}^{\chi_i}$, such that

- N is minimally indecomposable
- The braiding matrix only contains ± 1
- The quotient V acts faithfully
- Nodes i, j are connected iff $\langle \bar{g}_i, \bar{g}_j \rangle \neq 0$ (i.e. any lifts $g_i, g_j \in G$ discommute)
- The Nichols algebra is finite-dimensional and has the prescribed Dynkin diagram

This is done by using precisely the symplectic root systems constructed in Sections 4.1-4.2: $V := G/G^2$ is a symplectic vector space as described in the cited Section with dimension $dim_{\mathbb{F}_2}(G/G^2)$ and nullspace dimension $dim_{\mathbb{F}_2}(Z(G)/G^2)$. Hence the assumptions of the present theorem exactly match those of *cit. loc.* and we get a symplectic root system basis $\phi(i)$ $(1 \leq i \leq n)$ of V, i.e. $\langle \phi(i), \phi(j) \rangle \neq 0$ iff i, j are connected. Because G was assumed 2-saturated, this basis of V can be lifted to a minimally generating set g_i of G.

We define an indecomposable Yetter-Drinfel'd module N by using the images of the minimally generating set $\bar{g}_i$ of $\Gamma = G/[G, G]$ (=coaction). Then we construct suitable characters $\chi_i : \Gamma \to \mathbb{k}^\times$ (=action) that realize the given diagram with braiding matrix ± 1. Because the $\phi(i)$ were a basis of Γ/Γ^2, there is exactly one χ_i such that $\chi_i(\bar{g}_j) = -1$ if $i = j$ or $i < j$ are connected and $+1$ otherwise. Then $N := \bigoplus_i \mathcal{O}_{\bar{g}_i}^{\chi_i}$ has by construction a braiding matrix with monodromy $q_{ij} q_{ji} \neq 1$ precisely iff lifts g_i, g_j discommute in G. Note by construction, as $\mathbb{F}_2$-matrix $\chi_1, \ldots \chi_n$ is triangular, hence V acts faithful, which also proves this part of the statement.

Step 2: The central extension in question is $\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma$. Take a Section s and $\sigma \in Z^2(\Gamma, \Sigma^*)$; we have seen during the proof of Theorem 3.6 in claim 2 that

$$\sigma(\bar{a}, \bar{b}) \sigma^{-1}(\bar{b}, \bar{a}) = [a, b]$$

Because the chosen generators $g_i \in G$ map to the symplectic root system $\phi(i)$, we know the commutators $[g_i, g_j] \in [G, G] = \Sigma^*$: Take θ the generator of $\Sigma = \mathbb{Z}_2$, then the twisted Γ-action after applying f_θ on an element $v_{\bar{b}} \in M_{\bar{b}}$

reads as:

$$\begin{aligned}\bar{a}.f_\theta(v_{\bar{b}}) &\overset{!}{=} \sigma_\theta(\bar{a},\bar{b})\sigma_\theta^{-1}(\bar{b},\bar{a})f_\theta(\bar{a}.v_{\bar{b}})\\ &= \left(u(\bar{a},\bar{b})u^{-1}(\bar{b},\bar{a})\right)(\theta)f_\theta(\bar{a}.v_{\bar{b}})\\ &= \left(\langle\bar{a},\bar{b}\rangle\right)(p)f_\theta(\bar{a}.v_{\bar{b}})\end{aligned}$$

Hence any decorating character on some decorating group element $\chi_k(g_l)$ picks up an additional -1 iff $[g_k, g_l] \neq 1$ iff $\langle\bar{g_k},\bar{g_l}\rangle \neq 0$.

Step 3: We now construct a Γ-Yetter-Drinfel'd module with an action $\Sigma = \mathbb{Z}_2$ by twisted permutation symmetries as in the example $\mathbb{D}_4$ in Section 3.4. We start with the indecomposable $N = \bigoplus_{i=1}^n N_i$ constructed in step 1. Then we add the necessary twisted image $f_\theta(N)$ for θ the generator of $\Sigma = \mathbb{Z}_2$ (see Remark 3.4):

$$N_\sigma := N_{\sigma_\theta} = N_{\sigma(\theta)}$$

N_σ is hence the sum of simple Yetter-Drinfel'd modules $N_{\sigma i}$ given by the same group elements $\phi(i)$ but with twisted Γ-action:

$$\chi_i^{\sigma_\theta}(\bar{b}) := \left(\langle\phi(i),\bar{b}\rangle\right)(\theta)\chi_i(\bar{b})$$

$$\bar{a}._{\sigma_\theta}v_{\bar{b}} = \left(\langle\phi(i),\bar{b}\rangle\right)(\theta)$$

By construction $M := N \oplus N_\sigma$ admits a twisted symmetry f_θ interchanging $N_i \leftrightarrow N_{\sigma i}$.

Step 4: We yet have to check that M still has a finite Nichols algebra, so we determine its full Dynkin diagram – as intended, we prove now, that it really consists of two disconnected copies of the given one. First be reminded on Corollary 3.12 that twisted symmetries leave Dynkin diagrams and q-diagram invariant, so the diagrams of N, N_σ coincide.

Hence the tricky part is, that there are no additional mixed edges between any $N_i \leftrightarrow N_{\sigma j}$, i.e. $c_{N_i,N_{\sigma j}}c_{N_{\sigma j},N_i} = id$. This is precisely where we need the specific base choice $\phi(i)$ to be a symplectic root system (Definition 4.3) together with the fact that all $q_{ij} = \pm 1$. We have to calculate the mixed braiding factors:

$$\begin{aligned}q &:= q_{N_i,N_{\sigma j}}q_{N_{\sigma j},N_i}\\ &= \chi_i(\phi(j))\chi_j^{\sigma_\theta}(\phi(i))\\ &= \chi_i(\phi(j))\cdot\sigma_\theta(\phi(j),\phi(i))\sigma_\theta^{-1}(\phi(i),\phi(j))\chi_j(\phi(i))\\ &= \langle\phi(i),\phi(j)\rangle(\theta)\chi_i(\phi(j))\chi_j(\phi(i))\\ &= \langle\phi(i),\phi(j)\rangle(\theta)q_{ij}q_{ji}\end{aligned}$$

We have to distinguish two cases that yield $q = 1$ in different ways:

- Suppose i,j disconnected in the original diagram. Then $q_{ij}q_{ji} = 1$ and at the same time by construction $\langle\phi(i),\phi(j)\rangle = 0$, hence $q = 1$.
- Suppose i,j connected by a single edge. Then $q_{ij}q_{ji} = -1$ and at the same time by construction $\langle\phi(i),\phi(j)\rangle \neq 0$, hence $= \theta^*$ for the generator of $\Sigma^* \cong \mathbb{Z}_2$ with $\theta^*(\theta) = -1$. Hence we again get $q = 1$.

Step 5: Thus we are done: We constructed a twist-symmetric indecomposable Yetter-Drinfel'd module M over Γ with finite-dimensional Nichols algebra and Hilbert series $\mathcal{H}_M(t) = \mathcal{H}_N(t)\mathcal{H}_{N_\sigma}(t) = \mathcal{H}_N(t)^2$. Hence the covering G-Yetter-Drinfel'd module $\tilde{M}$ is indecomposable and the Nichols algebra $\mathcal{B}(\tilde{M})$ over G has Hilbert series $\mathcal{H}_{\tilde{M}}(t) = \mathcal{H}_M(t)$, especially it is finite-dimensional. $\square$

5.2. **Example $A_4 \times A_4 \mapsto A_4$.** We realize A_4 as prescribed over a group G with 2-rank $dim_{\mathbb{F}_2}(G/G^2) = 4$ and no 2-center $dim_{\mathbb{F}_2}(Z(G)/G^2) = 0$, such as the extraspecial group $G = 2^{4+1}_{+} = \mathbb{D}_4 * \mathbb{D}_4$ (the central product identifies the two dihedral centers), which is generated by mutually discommuting involutions x, y and x', y', corresponding to a symplectic basis of the nondegenerate symplectic vector space $V = \Gamma = \mathbb{F}_2^4$. We need a Γ-Yetter-Drinfel'd module of type $A_4 \times A_4$ admitting an involutory twisted symmetry

$$M = N \oplus N_\sigma =: (M_1 \oplus M_2 \oplus M_3 \oplus M_4) \oplus (M_5 \oplus M_6 \oplus M_7 \oplus M_8)$$

where each $M_k = \mathcal{O}_{\bar{g}_k}^{\chi_k}$ is 1-dimensional. The group elements are determined by the respective symplectic root system in Theorem 4.5:

$$\bar{g}_1 = \bar{g}_5 = x \qquad \bar{g}_2 = \bar{g}_6 = y \qquad \bar{g}_3 = \bar{g}_7 = xx' \qquad \bar{g}_4 = \bar{g}_8 = y'$$

Then the characters χ_k for $k \leq 4$ were defined in such a way that $\chi_k(\bar{g}_k) = -1$, and $\chi_k(\bar{g}_l) = -1$ for edges $k < l$ and $+1$ else. This has to be basis-transformed to be expressed as row vector showing the values in the original basis $(\chi(x), \chi(y), \chi(x'), \chi(y'))$:

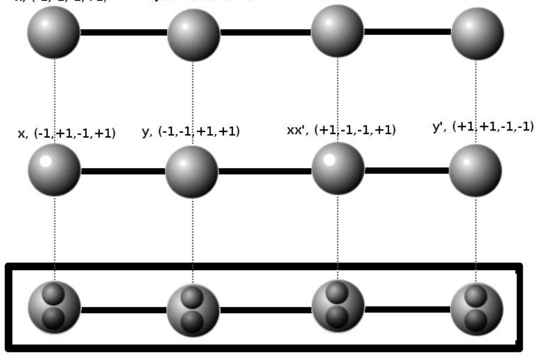
$$\begin{array}{llll} \chi_1 & = (-1,-1,-1,+1) & \chi_2 & (+1,-1,-1,+1) \\ \chi_3 & = (+1,+1,-1,-1) & \chi_4 & (+1,+1,+1,-1) \end{array}$$

As calculated in general, the twisted characters $\chi_{4+k} = \chi_k^\sigma$ catch an additional -1 on every element G-discommuting with g_k resp. $\not\perp \bar{g}_k$ in V:

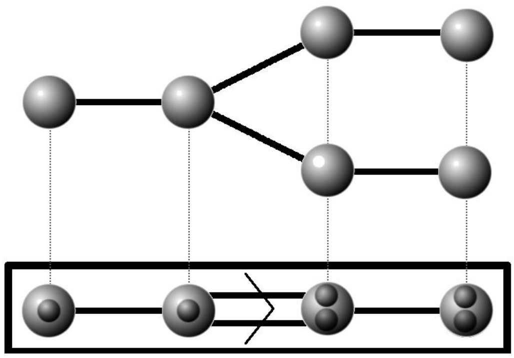
$$\begin{array}{llll} \chi_1 & = (-1,+1,-1,+1) & \chi_2 & (-1,-1,+1,+1) \\ \chi_3 & = (+1,-1,-1,+1) & \chi_4 & (+1,+1,-1,-1) \end{array}$$

Altogether we find the following covering Nichols algebra $\mathcal{B}(\tilde{M})$ with Hilbert series $\mathcal{H}(t)$ and dimension $\dim(\mathcal{B}(\tilde{M}))$ as for M and hence the square of the Hilbert series of the prescribed diagram A_4 of M for $q = -1$ [Hec09]:

$$\mathcal{H}(t) = \left([2]_t^4 [2]_{t^2}^3 [2]_{t^3}^2 [2]_{t^4}\right)^2 \qquad \dim(\mathcal{B}(\tilde{M})) = \mathcal{H}(1) = 2^{20} = 2^{|\Phi^+(A_4 \times A_4)|}$$



5.3. **Ramified Case** $E_6 \mapsto F_4$**.** The examples of the last two Sections are "generically" exploit a disconnected doubling of a rather arbitrary Dynkin diagram and yield simply laced coverings. Every (nonabelian) edge corresponds to the $\mathbb{D}_4$ example above and all conjugacy classes have same lengths. It turns out, that the *ramified case* involving different conjugacy class lengths is far more restrictive! We shall now give an example of this type, with $\Sigma = \mathbb{Z}_2$ the diagram automorphism of a single E_6-diagram and the covering Nichols algeb



Theorem 5.7. *Suppose G a 2-saturated group with $[G,G] = \mathbb{Z}_2$ and $\Gamma := G/[G,G]$ s.t.*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(G/G^2) = 4$
- $dim_{\mathbb{F}_2}(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G^2) = 2$

Then there exists a suitable Γ-Yetter-Drinfel'd module M of type E_6 with an involutory diagram automorphisms. The covering G-Yetter-Drinfel'd module $\tilde{M}$ decomposes into 4 simple Yetter-Drinfel'd module like $\tilde{M} = \bigoplus_{\tilde{k}=1}^4 \tilde{M}_{\tilde{k}}$, has dimension 6 and moreover:

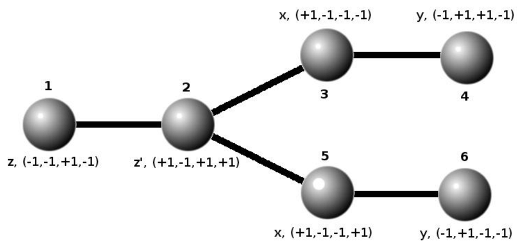
- *$[G,G]$ acts trivially on $\tilde{M}$, which is hence diagonal, but V acts faithfully.*
- *$\tilde{M}$ is minimally indecomposable.*
- *$\mathcal{B}(\tilde{M})$ has Hilbert series*
$$\mathcal{H}(t) = [2]_t^6[2]_{t^2}^5[2]_{t^3}^5[2]_{t^4}^5[2]_{t^5}^4[2]_{t^6}^3[2]_{t^7}^3[2]_{t^8}^2[2]_{t^9}[2]_{t^{10}}[2]_{t^{11}}$$
  and is thus especially of dimension
$$\dim\left(\mathcal{B}(\tilde{M})\right) = \mathcal{H}(1) = 2^{36} = 2^{|\Phi^+(E_6)|}$$
- *$\tilde{M}$ has the Dynkin diagram F_4, where short roots correspond to conjugacy classes of length 2 and long roots to a central elements.*

In section 6 we give an example of a nondiagonal Doi twist of $\mathcal{B}(\tilde{M})$ over $\mathbb{Z}_2^2 \times \mathbb{D}_4$.

Proof. Denote by $\bar{z}, \bar{z}', \bar{x}, \bar{y} \in \Gamma$ some lifts of a basis of the 4-dimensional symplectic vector space $V = G/G^2 = \Gamma/\Gamma^2$ with 2-dimensional nullspace, such that $\bar{z}, \bar{z}'$ were nullvectors and $\bar{x}, \bar{y}$ was a symplectic base pair in V. As G was assumed to be 2-saturated (Def. 5.1), we may choose these lifts to be a minimally generating system of Γ as well. Because $[G,G] \cong \mathbb{Z}_2$, any further lifts to $z, z', x, y \in G$ will obey by Section 4.1:

$$z, z' \in Z(G) \qquad [x,y] \neq 1$$

We directly construct the Γ-Yetter-Drinfel'd module $\bigoplus_{k=1}^{6} \mathcal{O}_{\bar{g}_k}^{\chi_k}$ of type E_6, but otherwise proceed as in the unramified case. Note that the following could also be derived systematically using the (rather trivial) symplectic root system $\bar{x}, \bar{y}$ for the aspired split part of V and character via some ordering of the nodes, as it is done for the remaining ramified case below; but here we want to keep everything explicit! Further denote any character $\chi \in \Gamma^*$ as row-vectors containing the basis images $(\chi(\bar{z}), \chi(\bar{z}'), \chi(\bar{x}), \chi(\bar{y}))$, then M shall be (we've introduced additional signs for the faithfulness-statement):



One can check directly, that $q_{ii} = -1$ and the $q_{ij}q_{ji} = \pm 1$ exactly match the given diagram. Furthermore, already $\chi_1, \chi_2, \chi_3, \chi_4$ is $\mathbb{F}_2$-linearly independent and z, z' have been constructed to act as -1 on x resp. y, hence the faithfulness assertions hold. This defined a proper Nichols algebra $\mathcal{B}(M)$ of dimension 2^{36} and the prescribed Hilbert series by [HS10] Theorem 4.5.

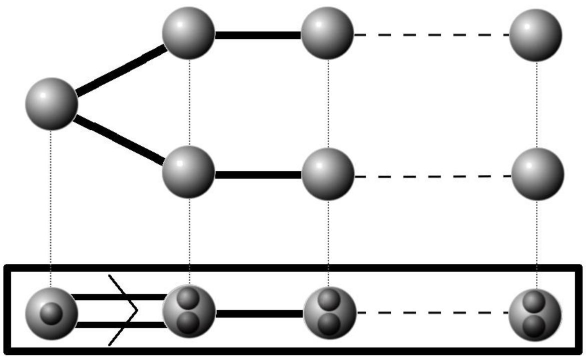
We calculate now, that the E_6 diagram automorphisms f_θ is here a twisted symmetry:

$$\chi_1^\sigma(g_k) = \sigma_\theta(g_k, g_1)\sigma_\theta^{-1}(g_1, g_k)\chi_1(g_k) = \langle \bar{g}_k, z\rangle \chi_1(g_k) = \chi_1(g_k)$$

$$\chi_3^\sigma(z) = \langle z, x\rangle \chi_3(z) = \chi_3(z) = +1 = \chi_5(z)$$
$$\chi_3^\sigma(z') = \langle z', x\rangle \chi_3(z') = \chi_3(z') = -1 = \chi_5(z')$$
$$\chi_3^\sigma(x) = \langle x, x\rangle \chi_3(x) = \chi_3(x) = -1 = \chi_5(x)$$
$$\chi_3^\sigma(y) = \langle y, x\rangle \chi_3(z') = -\chi_3(y) = +1 = \chi_5(y)$$

This shows $\chi_1^\sigma = \chi_1$ and $\chi_3^\sigma = \chi_5$. The same calculations prove $\chi_2^\sigma = \chi_2$ and $\chi_4^\sigma = \chi_6$, hence the generator $f_\theta : M \to M$ defines a twisted symmetry action of $\Sigma = \mathbb{Z}_2$ on M. The covering Nichols algebra $\mathcal{B}(\tilde{M})$ over G then has the asserted properties. $\square$

5.4. **Ramified Cases** $A_{2n-1} \mapsto C_n$**.** The second ramification will be treated more systematically, by completely reducing it to the unramified case $A_{n-1} \times A_{n-1} \mapsto A_{n-1}$ and an additional inert node causing an additionally ramified edge.



Theorem 5.8. *Suppose G a 2-saturated group with $[G,G] = \mathbb{Z}_2$ and $\Gamma := G/[G,G]$, s.t.*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(G/G^2) = n \geq 3$
- $dim_{\mathbb{F}_2}(V^{\perp}) = dim_{\mathbb{F}_2}(Z(G)/G^2) = 1 + (n-1 \bmod 2)$

Then there exists a suitable Γ-Yetter-Drinfel'd module of type A_{2n-1} with an involutory diagram automorphisms. The covering Yetter-Drinfel'd module $\tilde{M}$ over G has rank n, dimension $2n-1$ and moreover:

- *$[G,G]$ acts trivially on $\tilde{M}$, which is diagonal, but V acts faithfully.*
- *$\tilde{M}$ is minimally indecomposable.*
- *$\mathcal{B}(\tilde{M})$ has Hilbert series*

$$\mathcal{H}(t) = [2]_{t^1}^{2n-1}[2]_{t^2}^{2n-2}[2]_{t^3}^{2n-3}\cdots[2]_{t^{2n-1}}^{1}$$

  and is thus especially of dimension

$$\dim\left(\mathcal{B}(\tilde{M})\right) = \mathcal{H}(1) = 2^{n(2n-1)} = 2^{|\Phi^+(A_{2n-1})|}$$

- *$\tilde{M}$ has the nonabelian Dynkin diagram C_n where short roots correspond to conjugacy classes of length 2 and the long root to a central element.*

Exemplary nondiagonal and even faithful Doi twists of type C_3, C_4 over $\mathbb{D}_4 \times \mathbb{Z}_2$ resp. $\mathbb{D}_4 \times \mathbb{Z}_2^2$ a given in Section 6.

Proof. As in the ramified case $E_6 \mapsto F_4$ above, we use the prescribed dimension $1 + (n-1 \bmod 2)$ nullspace of $V = G/G^2$ to decompose $V = \bar{z}\mathbb{F}_2 \oplus W$ with $\dim(W^{\perp}) = n - 1 \bmod 2$ for the split nodes and $z \in Z(\Gamma)$ for the inert node.

Our main goal is to construct a Γ-Yetter-Drinfel'd module M of dimension $1+2(n-1)$ and Dynkin diagram A_{2n-1} with the involutory diagram automorphism a twisted symmetry. The starting point is the Yetter-Drinfel'd module constructed in the proof of Section 5.1 of dimension $2(n-1)$ and Dynkin diagram $A_{n-1}\times A_{n-1}$, numbered $2\ldots 1+2(n-1)$, with an involutory twisted symmetry over the subgroup $\Gamma'\subset\Gamma$ generated by any lifts of W. Denote the leftmost nodes $2,3$ of both copies by $\mathcal{O}_{\bar{g}}^{\chi'},\mathcal{O}_{\bar{g}}^{\chi''}$. We extend all used characters trivially to Γ except

$$\chi(\bar{z})=-1 \qquad \chi(\bar{g})=-1 \qquad \chi(\bar{g}_k)=+1$$

for all other $\bar{g}_k$, which is possible because the images of $\bar{g}=\bar{g}_1,\ldots\bar{g}_n$ form a W-basis. Note that the former Yetter-Drinfel'd module had already been proven to be faithful over the Γ-quotient W, with $\bar{z}$ now acting trivial on all but the new node M_1, hence faithfulness of V again holds.

First we have to check that M indeed has decorated diagram $A_{1+2(n-1)}$ and hence the asserted Hilbert series by [HS10] Theorem 4.5. We've shown that already for the subdiagram $A_{n-1}\times A_{n-1}$, and the additional node M_1 obeys for $k\geq 4$:

$$\begin{aligned} q_{11} &= \chi(\bar{z})=-1 \\ q_{12}q_{21} &= \chi(\bar{g})\chi'(\bar{z})=(-1)(+1)=-1 \\ q_{13}q_{31} &= \chi(\bar{g})\chi''(\bar{z})=(-1)(+1)=-1 \\ q_{1k}q_{k1} &= \chi(\bar{g}_k)\chi_k(\bar{z})=(+1)(+1)=+1 \end{aligned}$$

Secondly we have to extend the established twisted symmetry f_p of $A_{n-1}\times A_{n-1}$ by $f_p(x_1):=x_1$, which is possible by z's centrality in G:

$$\begin{aligned} \chi^\sigma(\bar{h}) &= \sigma(\bar{z},\bar{h})\sigma_p^{-1}(\bar{h},\bar{z})\chi(\bar{h}) \\ &= \langle\bar{h},\bar{z}\rangle\chi(\bar{h})=\chi(\bar{h}) \end{aligned}$$

The covering Yetter-Drinfel'd module over G then has the asserted properties. □

5.5. **Disconnected Diagrams.** So far we have constructed covering Nichols algebras $\mathcal{B}(\tilde{M})$ with connected Dynkin diagram over 2-saturated groups G. We will now show, how disconnected diagrams can be realized, regardless of the technical assumption of G to be 2-saturated. As a corollary, we will note that every group G with $[G,G]=\mathbb{Z}_2$ and no restrictions on 2-rank and 2-center admits (possibly disconnected) finite-dimensional indecomposable Nichols algebras $\mathcal{B}(\tilde{M})$. Note that in the next Lemma we actually construct a whole family of covering Nichols algebras for arbitrary 2-rank and 2-center (without claiming these are all), but for the existence corollary, very simple choices suffice.

Lemma 5.9. *Let G be a 2-saturated group with $[G,G]\cong\mathbb{Z}_2$ and 2-rank and 2-center*

$$n := \dim(V) = dim_{\mathbb{F}_2}(G/G^2)$$
$$k := \dim(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G^2)$$

For every numerical decomposition $(n,k)=\sum_i(n_i,k_i)+(0,k_0)$, where all (n_i,k_i) appear as 2-rank and 2-center in the list of Theorem 5.3, we can construct a minimally indecomposable covering Nichols algebra over G with connected ramified or unramified components as prescribed by Theorem 5.3 for (n_i,k_i), as well as an additional inert part $(0,k_0)$, that is an arbitrary finite dimensional indecomposable Nichols algebra $\mathcal{B}(M^{(0)})$ over $\mathbb{Z}_2^{k_0}$ from [Hec09].

Proof. Consider again $V=G/G^2$ as a symplectic $\mathbb{F}_2$-vector space. The assumed decomposition

$$(\dim(V),\dim(V^\perp))=\sum_i(n_i,k_i)+(0,k_0)$$

implies an orthogonal decomposition of V as symplectic vector space into

$$V\cong\bigoplus_i^{\perp}V_i\oplus^{\perp}V_0$$

where $(n_i,k_i)=(\dim(V_i),\dim(V_i^{\perp}))$ and $k_0=\dim(V_0)$. Apply the constructions of Sections 5.1-5.4 that yield Yetter-Drinfel'd modules $M^{(i)}$ over $\Gamma=G/[G,G]$ that factorize over $\Gamma\to V$ and where $\mathcal{B}(M^{(i)})$ having a connected Dynkin diagram of the respective type. Then consider

$$M:=\bigoplus_i M^{(i)}\oplus M^{(0)}$$

where the action of V_i on $M^{(j)}$ for $i\neq j$ is trivial and $M^{(0)}$ is the assumed Yetter-Drinfel'd module over the abelain group $V_0=\mathbb{Z}_2^{k_0}$. Hence $c_{M^{(i)}M^{(j)}}c_{M^{(j)}M^{(i)}}=id$, thus we have trivial adjoint action $\mathrm{ad}\left(\mathcal{B}(M^{(i)})\right)\left(\mathcal{B}(M^{(j)})\right)=0$ and the multiplication in $\mathcal{B}(M)$ yields an isomorphism of vector spaces

$$\mathcal{B}(M)\cong\bigotimes_i\mathcal{B}(M^{(i)})\otimes\mathcal{B}(M^{(0)})$$

and the Dynkin diagram of $\mathcal{B}(M)$ consists of mutually disconnected components, each of the respective type of $\mathcal{B}(M^{(i)})$ and $\mathcal{B}(M^{(0)})$.

Take f_θ the sum of the twisted symmetries employed in the construction of each $M^{(i)}$ and trivial on $M^{(0)}$. Then the covering Nichols algebra $\tilde{M}$ has as Dynkin diagram the mutually disconnected Dynkin diagrams of each

$\mathcal{B}(\tilde{M}^{(i)})$ as they follow from the respective construction and a disconnected inert part with the given $\mathcal{B}(M^{(0)})$. □

Corollary 5.10. *Let the group beof the form $G = G_{ab} \times G_{sat}$ with G_{ab} abelian and G_{sat} a 2-saturated group with $[G_{sat}, G_{sat}] \cong \mathbb{Z}_2$. Note that such a decomposition may not be unique. Suppose M_{ab} a minimally indecomposable G_{ab}-Yetter-Drinfel'd module with finite-dimensional Nichols algebra $\mathcal{B}(M_{ab})$. Suppose further over the 2-saturated group G_{sat} an indecomposable finite-dimensional Nichols algebra $\mathcal{B}(M_{sat})$ from the list in this article, i.e. Theorem 5.3 for connected resp. Lemma 5.9 for disconnected diagrams. Then, the indecomposable G-Nichols algebra $\mathcal{B}(M_{ab} \oplus M_{sat})$ is as a vector space isomorphic to $\mathcal{B}(M_{ab}) \otimes \mathcal{B}(M_{sat})$, hence finite-dimensional, and has as Dynkin diagram a disjoint union of the diagrams of $\mathcal{B}(M_{ab}), \mathcal{B}(M_{sat})$.*

Example 5.11. *Every group G with $[G, G] \cong \mathbb{Z}_2$ is nilpotent and can thus be written as a product $G = G_{odd} \times G_2$, where G_{odd} has odd order and is hence abelian, while G_2 is a 2-group with $[G_2, G_2] \cong \mathbb{Z}_2$ and especially 2-saturated. By the previous lemma we may obtain a finite-dimensional indecomposable Nichols algebra $B(M)$ over G by joining a finite-dimensional indecomposable Nichols algebra over the abelian group $G_{ab} := G_{odd}$ with a finite-dimensional indecomposable Nichols algebra constructed by Lemma 5.9 over $G_{sat} := G_2$.*

Let us come to an explicit easy and generic decomposition:

Example 5.12. *Let thus be (n, k) the type of the symplectic $\mathbb{F}_2$-vector space $V = G_2/G_2^2$ as in Lemma 5.9. Applying the lemma, to the particular decomposition $(n, 0) + (0, k)$ yields a finite-dimensional indecomposable covering Nichols algebra $\mathcal{B}(M_2)$, which has as Dynkin diagram a disjoint union of an unramified A_{2n} and an arbitrary inert finite-dimensional indecomposable Nichols algebra over the abelian group $\mathbb{Z}_2^k$, such as a disjoint union of k diagrams of type A_1*

Corollary 5.13. *Every group G with $[G, G] \cong \mathbb{Z}_2$ admits a finite-dimensional indecomposable Nichols algebra $\mathcal{B}(\tilde{M})$. This answers for groups of this class positively a question raised by Susan Montgomery in 1995* [Mon95][AS02]*: There exist a finite-dimensional indecomposable pointed Hopf algebra with coradical $\Bbbk[G]$, namely the bosonization $\Bbbk[G]\#\mathcal{B}(M)$ with $\mathcal{B}(M)$ constructed in the previous example, or others constructed by Lemma 5.9.*

6. Explicit Examples Of Nondiagonal Nichols Algebras

The covering Nichols algebras $\mathcal{B}(\tilde{M})$ over nonabelian groups G constructed in this article are by construction non-faithful, because the G-action is the pullback of the action of the quotient Γ. Especially the commutators $[G, G]$

act trivially, so the braiding of $\tilde{M}$ is still diagonal.

However, over G there may exist Doi twists $\mathcal{B}(\tilde{M}_\eta)$ by a G-group-2-cocycle $\eta \in Z^2(G, \Bbbk^\times)$, such that the action of the subgroup $\Sigma^* = [G,G] \cong \mathbb{Z}_2$ on $\tilde{M}_\eta$ is nontrivial. Then $\tilde{M}$ has a nondiagonal braiding, and it even may be faithful, depending on the precise action of the other central elements.

In the following we shall derive a criterion for the existence of such non-diagonal twistings and give a list of examples for covering Nichols algebras for Rank $2, 3, 4$.

6.1. **Doi Twists And Matsuomots Spectral Sequence.** We already noted in Remark 3.4, that a Doi twist of the Nichols algebra produces the following twisted action on the twisted Yetter-Drinfel'd module $\tilde{M}_\eta$:

$$a._\eta v_h = \eta(aga^{-1}, a)\eta^{-1}(a,g)a.v_h$$

Hence the central subgroup $\Sigma^* \subset G$ with trivial action on $\tilde{M}$ acts on $\tilde{M}_\eta$ by multiplication with the scalar

$$\gamma(\eta)(a,g) := \eta(g,a)\eta^{-1}(a,g) \qquad \gamma(\eta) \in \Sigma^* \otimes G$$

This expression appears already in literature on group cohomology, namely in Matsumoto's extension [NI64] for central group extensions of the general Lyndon-Hochschild-Serre spectral sequence:

$$1 \to \Gamma^* \to G^* \to \Sigma \to H^2(\Gamma, \Bbbk^\times) \to H^2(G, \Bbbk^\times)_\Sigma \xrightarrow{\gamma} \Sigma^* \otimes G$$

Here, $H^2(G, \Bbbk^\times)_\Sigma$ denotes the kernel of the restriction map to Σ^* and the map γ yields as expected a bimultiplicative pairing that exactly matches the expression above!

Theorem 6.1. *Let $\Sigma^* = \mathbb{Z}_p \to G \to \Gamma$ be a stem-extension, M a Γ-Yetter-Drinfel'd module with finite-dimensional Nichols algebra $\mathcal{B}(M)$ and $\tilde{M}$ be the covering G-Yetter-Drinfel'd module. If we assume that the following holds*

$$p\frac{|H^2(G, \Bbbk^\times)|}{|H^2(\Gamma, \Bbbk^\times)|} > 1$$

then there exists a group-2-cocycle $\eta \in Z^2(G, \Bbbk^\times)$, such that the Doi twist $\tilde{M}_\eta$ has nontrivial action of Σ^* and is hence nondiagonal.

Proof. We use Matsumoto's sequence to enumerate the number of different Σ^*-actions $|\mathrm{Im}(\gamma)|$, that can be achieved by Doi twisting. Note that

- For stem-extensions we have $G^* = \Gamma^*$, so the first terms disappear.

- In our case $\Sigma^* = \mathbb{Z}_p$ we have $H^2(\Sigma^*, \Bbbk^\times) = 1$, so the restriction kernel is all $H^2(G, \Bbbk^\times)_\Sigma = H^2(G, \Bbbk^\times)$

so Matsumoto's sequence takes the following form

$$1 \to \Sigma \to H^2(\Gamma, \Bbbk^\times) \to H^2(G, \Bbbk^\times) \xrightarrow{\gamma} \Sigma^* \otimes G$$

In particular, counting elements shows the claim:

$$|\mathrm{Im}(\gamma)| = |H^2(G, \Bbbk^\times)| \cdot |H^2(\Gamma, \Bbbk^\times)|^{-1} \cdot |\mathbb{Z}_p| > 1$$

□

Remark 6.2. *This approach has also classificatory value in special cases: In* [Len12] *sections 7.2 – 7.4 we prove for several exemplary groups G of order* 16 *and* 32*, that these Doi twists already exhaust all Σ^*-actions, that are possible on a G-Yetter-Drinfel'd module with finite dimensional Nichols algebra by* [HS10]*. Thereby,* **all** *finite-dimensional Nichols algebras over G are Doi twists of covering Nichols algebras.*
Especially for the last cases in [Len12] *section 7.4, having $G = \mathbb{Z}_2^2$ and a certain commutator structure, there is no possible covering Nichols algebra and this disproves existence of finite-dimensional link-indecomposable Nichols algebras over these groups at all.*

6.2. **Examples Of Rank 2.** A symplectic vector space of rank $2 = 2n + k$ can be of the following two types:

Type $(n, k) = (0, 2)$ means that $\bar{G}, G$ are abelian.

Type $(n, k) = (1, 0)$ induces Nichols algebras over the following type of group:

$$\mathbb{Z}_2 \to G \to \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad [G, G] = \mathbb{Z}_2 \qquad Z(G) = G^2$$

Take as example $G = \mathbb{D}_4$, then $H^2(G, \Bbbk^\times) = \mathbb{Z}_2 = H^2(\Gamma, \Bbbk^\times)$ and hence by Theorem 6.1 this 2-cocyle causes a nondiagonal Doi twist (the other stem extension $\mathbb{Q}_8$ has not enough cohomology for nondiagonal twists).
Recall the generators $g, h, \epsilon \in \mathbb{D}_4$ with $gh = \epsilon hg, g^2 = \epsilon, h^2 = \epsilon^2 = 1$.

A symplectic vector space of type $(1, 0)$ admits by Theorem 4.5 a symplectic root system of type A_2, hence we obtained in Section 5.1 an unramified covering Nichols algebra $A_2 \times A_2 \mapsto A_2$. This yields the well known example, discussed in Section 3.4:

Example 6.3 (Type A_2, see [MS00] Example 6.5)**.**

$$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi}$$

Consider the diagonalizable Yetter-Drinfel'd module M with

$$\chi(h) = \phi(gh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = 1$$

as well the nondiagonal (even faithful) Doi twist M_η with

$$\chi(h) = \phi(gh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = -1$$

Both Nichols algebras $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type A_2, but possess a finer PBW-basis of type $A_2 \times A_2$. Hilbert series and dimension are hence:

$$\mathcal{H}(t) = \left([2]_t^2[2]_{t^2}\right)^2 \qquad \dim = 2^6 = 64$$

Remark 6.4. *Here as well as in the following examples, a reflection R_i (see* [AHS10]*) turns the Yetter-Drinfel'd module M into e.g.*

$$R_1 M = \mathcal{O}_{[h]}^{\chi'} \oplus \mathcal{O}_{[g]}^{\phi'}$$

where now the generator g has order 4 (see [Len12] *Sec. 5.3 and 8). This corresponds to the choice of a different 2-cocycle $\sigma' \in Z^2(\mathbb{Z}^2, \Bbbk^\times)$ in the same cohomology class then σ, especially the induced symplectic forms $\sigma(g,h)\sigma^{-1}(h,g)$ coincides. Equivalently, for a given 2-cocycle it corresponds to the choice of a different (but isometric) symplectic root system $(\overline{h}, \overline{g})$ instead of $(\overline{h}, \overline{gh})$.*

Note that we can uniformly write the diagonal and nondiagonal examples and all reflections by giving the following character relations and the symplectic root system directly, yielding a class of Yetter-Drinfel'd modules as in [HV13]*:*

$$M = \mathcal{O}_{[v]}^{\chi} \oplus \mathcal{O}_{[w]}^{\phi} \qquad \bar{v} \not\perp \bar{w} \qquad \chi(v) = \phi(w) = -1 \qquad \chi(\epsilon)\phi(\epsilon) = 1$$

6.3. **Examples Of Rank 3.** A symplectic vector space of rank $3 = 2n + k$ can be of the following two types:

Type $(n,k) = (0,3)$ means that $\bar{G}, G$ are abelian.

Type $(n,k) = (1,1)$ induces Nichols algebras over the following type of group:

$$\mathbb{Z}_2 \to G \to \mathbb{Z}_2^3 \qquad [G,G] = \mathbb{Z}_2 \qquad Z(G)/G^2 = \mathbb{Z}_2$$

Take as example for this type $G = \mathbb{D}_4 \times \mathbb{Z}_2$, then by Küneth's formula $H^2(G, \Bbbk^\times) = \mathbb{Z}_2^3 = H^2(\Gamma, \Bbbk^\times)$. Hence by Theorem 6.1 there is a 2-cocyle that causes a nondiagonal Doi twist. Recall the $\mathbb{D}_4$-generators g, h, ϵ and add a central generator z.

A symplectic vector space of type $(1,1)$ admits by Theorem 4.5 symplectic root systems of type A_3 or $A_2 \times A_1$, hence we get the following Nichols algebras:

- From the symplectic root system of type A_3 we obtained in Section 5.1 an unramified covering Nichols algebra $A_3 \times A_3 \mapsto A_3$ as follows:

Example 6.5 (Type A_3)**.**

$$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi} \oplus \mathcal{O}_{[zh]}^{\psi}$$

Consider the diagonalizable Yetter-Drinfel'd modules M with

$$\chi(h) = \phi(gh) = \psi(zh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = 1$$

$$\chi(zh)\psi(h) = 1$$

as well as nondiagonal (for some choices faithful) Doi twists M_η with

$$\chi(h) = \phi(gh) = \psi(zh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = -1$$

$$\chi(zh)\psi(h) = 1$$

All Nichols algebras $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type A_3, but possess a finer PBW-basis of type $A_3 \times A_3$. Hilbert series and dimension are hence:

$$\mathcal{H}(t) = \left([2]_t^3 [2]_{t^2}^2 [2]_{t^3}\right)^2 \qquad \dim = 2^{12} = 4,096$$

- From the symplectic root system of type $A_2 \times A_1$ we obtained in Section 5.4 a ramified covering Nichols algebra $A_5 \mapsto C_3$ as follows:

Example 6.6 (Type C_3)**.**

$$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi} \oplus \mathcal{O}_{\{z\}}^{\psi}$$

Consider the diagonalizable Yetter-Drinfel'd modules M with

$$\chi(h) = \phi(gh) = \psi(z) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = 1$$

$$\chi(z)\psi(h) = 1 \qquad \phi(z)\psi(gh) = -1$$

as well as nondiagonal (for some choices faithful) Doi twists M_η with

$$\chi(h) = \phi(gh) = \psi(z) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = -1 \qquad \psi(\epsilon) = 1$$

$$\chi(z)\psi(h) = 1 \qquad \phi(z)\chi(gh) = -1$$

All Nichols algebras $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type C_3, but possess a finer PBW-basis of type A_5. Hilbert series and dimension are hence:

$$\mathcal{H}(t) = [2]_t^5 [2]_{t^2}^4 [2]_{t^3}^3 [2]_{t^4}^2 [2]_{t^5} \qquad \dim = 2^{15} = 32,768$$

- Finally we have Nichols algebras with disconnected Dynkin diagram $A_1 \times A_2$ containing an abelian support and an A_2-Nichols algebra over $\mathbb{D}_4$.

6.4. **Examples Of Rank 4.** A symplectic vector space of rank $4 = 2n + k$ can be of the following three types:

Type $(n,k) = (0,4)$ means that $\bar{G}, G$ are abelian.

Type $(n,k) = (2,0)$ admits a symplectic root system of type A_4. The respective Nichols algebra over $G = \mathbb{D}_4 * \mathbb{D}_4$ has Hilbert series

$$H(t) = \left([2]_t^4 [2]_{t^2}^3 [2]_{t^3}^2 [2]_{t^4}\right)^2 \qquad \dim = 2^{20} = 1,048,576$$

It is thoroughly discussed in Example 5.2, but over groups of order 16 and 32 we were not able to find a nondiagonal Doi twist (is there none at all?).

Type $(n,k) = (1,2)$ induces Nichols algebras over the following type of group:

$$\mathbb{Z}_2 \to G \to \mathbb{Z}_2^4 \qquad [G,G] = \mathbb{Z}_2 \qquad Z(G)/G^2 = \mathbb{Z}_2^2$$

Take as example for this type $G = \mathbb{D}_4 \times \mathbb{Z}_2^2$, then by Küneth's formula $H^2(G, \Bbbk^\times) = \mathbb{Z}_2^6 = H^2(\Gamma, \Bbbk^\times)$. Hence by Theorem 6.1 there is a 2-cocyle that causes a nondiagonal Doi twist. Recall the $\mathbb{D}_4$-generators g, h, ϵ and add central generators z, w.

A symplectic vector space of type $(n,k) = (1,2)$ admits by Theorem 4.5 symplectic root systems of type D_4 or $A_1 \times A_3$ or $A_1 \times A_1 \times A_2$, hence we get the following Nichols algebras:

- From the symplectic root system of type D_4 we obtained in Section 5.1 an unramified covering Nichols algebra $D_4 \times D_4 \mapsto D_4$ as follows:

  Example 6.7 (Type D_4)**.**

  $$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi} \oplus \mathcal{O}_{[zh]}^{\psi} \oplus \mathcal{O}_{[wh]}^{\rho}$$

  Consider the diagonalizable Yetter-Drinfel'd modules M with

  $$\chi(h) = \phi(gh) = \psi(zh) = \rho(wh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = \rho(\epsilon) = 1$$
  $$\chi(zh)\psi(h) = 1 \qquad \chi(wh)\rho(h) = 1 \qquad \psi(wh)\rho(zh) = 1$$

  as well as nondiagonal (for some choices faithful) Doi twists M_η with

  $$\chi(h) = \phi(gh) = \psi(zh) = \rho(wh) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = \rho(\epsilon) = -1$$
  $$\chi(zh)\psi(h) = 1 \qquad \chi(wh)\rho(h) = 1 \qquad \psi(wh)\rho(zh) = 1$$

  All Nichols algebras $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type D_4, but possess a finer PBW-basis of type $D_4 \times D_4$. Hilbert series and dimension are hence:

  $$H(t) = \left([2]_t^4 [2]_{t^2}^3 [2]_{t^3}^3 [2]_{t^4} [2]_{t^5}\right)^2 \qquad \dim = 2^{24} = 16,777,216$$

- From the symplectic root system of type $A_1 \times A_3$ we obtained in Section 5.4 a ramified covering Nichols algebra $A_7 \mapsto C_4$ as follows:

Example 6.8 (Type C_4).

$$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi} \oplus \mathcal{O}_{[zh]}^{\psi} \oplus \mathcal{O}_{\{w\}}^{\rho}$$

Consider the diagonalizable Yetter-Drinfel'd modules M with

$$\chi(h) = \phi(gh) = \psi(zh) = \rho(w) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = \rho(\epsilon) = 1$$

$$\chi(zh)\psi(h) = 1 \qquad \chi(w)\rho(h) = 1 \qquad \phi(w)\rho(gh) = 1 \qquad \psi(w)\rho(zh) = -1$$

as well as nondiagonal (for some choices faithful) Doi twists M_η with

$$\chi(h) = \phi(gh) = \psi(zh) = \rho(w) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = -1 \qquad \rho(\epsilon) = 1$$

$$\chi(zh)\psi(h) = 1 \qquad \chi(w)\rho(h) = 1 \qquad \phi(w)\rho(gh) = 1 \qquad \psi(w)\rho(zh) = -1$$

All Nichols algebra $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type C_4, but possess a finer PBW-basis of type A_7. Hilbert series and dimension are hence:

$$H(t) = [2]_t^7 [2]_{t^2}^6 [2]_{t^3}^5 [2]_{t^4}^4 [2]_{t^5}^3 [2]_{t^6}^2 [2]_{t^7}$$
$$\dim = 2^{28} = 268,435,456$$

- From the symplectic root system of type $A_1 \times A_1 \times A_2$ we obtained in Section 5.3 a ramified covering Nichols algebra $E_6 \mapsto F_4$ as follows:

Example 6.9 (Type F_4).

$$M = \mathcal{O}_{[h]}^{\chi} \oplus \mathcal{O}_{[gh]}^{\phi} \oplus \mathcal{O}_{\{z\}}^{\psi} \oplus \mathcal{O}_{\{w\}}^{\rho}$$

Consider the diagonalizable Yetter-Drinfel'd modules M with

$$\chi(h) = \phi(gh) = \psi(z) = \rho(w) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = \psi(\epsilon) = \rho(\epsilon) = 1$$

$$\chi(z)\psi(h) = 1 \qquad \chi(w)\rho(h) = 1 \qquad \phi(w)\rho(gh) = 1$$

$$\phi(z)\psi(gh) = -1 \qquad \psi(w)\rho(z) = -1$$

as well as nondiagonal (for some choices faithful) Doi twists M_η with

$$\chi(h) = \phi(hg) = \psi(z) = \rho(w) = -1 \qquad \chi(\epsilon) = \phi(\epsilon) = -1 \qquad \psi(\epsilon) = \rho(\epsilon) = 1$$

$$\chi(z)\psi(h) = 1 \qquad \chi(w)\rho(h) = 1 \qquad \phi(w)\rho(gh) = 1$$

$$\phi(z)\psi(gh) = -1 \qquad \psi(w)\rho(z) = -1$$

All Nichols algebras $\mathcal{B}(M), \mathcal{B}(M_\eta)$ are standard of type F_4, but possess a finer PBW-basis of type E_6. Hilbert series and dimension are hence:

$$\mathcal{H}(t) = [2]_t^6 [2]_{t^2}^5 [2]_{t^3}^5 [2]_{t^4}^5 [2]_{t^5}^4 [2]_{t^6}^3 [2]_{t^7}^3 [2]_{t^8}^2 [2]_{t^9} [2]_{t^{10}} [2]_{t^{11}}$$

$$\dim = 2^{36} = 68,719,476,736$$

- We have again several Nichols algebras $X_2^{ab} \cup A_2$ with disconnected Dynkin diagram and partly abelian support from the symplectic root system $A_1 \times A_1 \times A_2$. Here X_2^{ab} may be any diagonal Nichols algebra of rank 2. Moreover, we get a ramified $A_1^{ab} \times C_3$. From the other symplectic root system $A_1 \times A_3$ we get $A_1^{ab} \times A_3$.

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