Topological group cohomology of Lie groups and Chern-Weil theory for compact symmetric spaces

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Abstract

In this paper we analyse the topological group cohomology of finite-dimensional Lie groups. We introduce a technique for computing it (as abelian groups) for torus coefficients by the naturally associated long exact sequence. The upshot in there is that certain morphisms in this long exact coefficient sequence can be accessed (at least for semi-simple Lie groups) very conveniently by the Chern-Weil homomorphism of the naturally associated compact dual symmetric space. Since the latter is very well-known, this gives the possibility to compute the topological group cohomology of the classical simple Lie groups. In addition, we establish a relation to characteristic classes of flat bundles.

Keywords: Topological group, group cohomology, classifying space, symmetric space, compact dual, subalgebra non-cohomologous to zero, Chern-Weil homomorphism, flat characteristic class, bounded continuous cohomology

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Introduction

Topological group cohomology is the cohomology theory for topological groups that incorporates both, the algebraic and the topological structure of a topological group G with coefficients in some topological G-module A. There are two obvious guesses for this, which already capture parts of the theory in special cases.

The first one is the (singular) cohomology of the classifying space BG of G. This leads to well-defined cohomology groups $H^n_{\pi_1(BG)}(BG; A)$ for discrete coefficient groups A (where $H^n_{\pi_1(BG)}$ denotes the cohomology of the corresponding local coefficient system on BG). However, if A is non-discrete, then $H^n_{\pi_1(BG)}(BG; A)$ is not even well-defined, since BG is only defined up to homotopy equivalence. Moreover, BG is trivial if G is contractible, so no homotopy invariant construction on BG could capture for instance the Heisenberg group as a central extension of $\mathbb{R} \times \mathbb{R}$ by U(1).

The second obvious guess would be the cohomology of the cochain complex of continuos A-valued functions (see Section 1). We call this the van Est cohomology $H^n_{vE}(G; A)$ of G, since it has first been exhaustively analysed (in the case of Lie groups) by van Est in the 50's and 60's. However, this has a reasonable interpretation as a relative derived functor only in the case that A is a topological vector space [HM62], and captures in this respect the case that is contrary to the case of discrete coefficients.

The topological group cohomology interpolates between these two extreme case. It has first been defined by Segal and Mitchison in [Seg70] (see also [Del74, Moo76, Cat77, Bry00, Fla08]) and recently been put into a unifying framework in [WW13]. We denote the corresponding cohomology groups by $H^n(G; A)$ with no additional subscript. If $A = \mathfrak{a}/\Gamma$ for some contractible *G*-module \mathfrak{a} and some submodule Γ , then the topological group cohomology interpolates between the classifying space cohomology and the van Est cohomology in the sense that there is a long exact sequence

$$\cdots \to H^{n-1}(G;A) \to H^n_{\pi_1(BG)}(BG;\Gamma) \to H^n_{vE}(G;\mathfrak{a}) \to H^n(G;A) \to H^{n+1}_{\pi_1(BG)}(BG;\Gamma) \to \cdots$$
(1)

(see Section 2).

The bulk of this paper is devoted to analyse this exact sequence in the case that G is a Lie groups and that the coefficients are smooth. In particular, we establish a connection to Lie algebra cohomology that we then exploit in the sequel to calculate certain important morphisms of the above sequence in explicit terms. This then permits to calculate $H^n(G; U(1))$ for some (in principle all) semi-simple Lie groups in terms of the (well-known) Chern-Weil homomorphism of compact symmetric spaces. Moreover, we establish a connection to characteristic classes of flat bundles.

We now shortly list the results of the individual sections. Section 1 recalls the basic facts about topological group cohomology. In Section 2 we introduce the long exact sequence (1) and reinterpret it in terms of relative group cohomology. In particular, we will motivate why it is natural to think of the morphisms

$$\varepsilon^n \colon H^n_{\pi_1(BG)}(BG; \Gamma) \to H^n_{vE}(G; \mathfrak{a}) \tag{2}$$

as connecting morphisms (instead of $H^{n-1}(G; A) \to H^n_{\pi_1(BG)}(BG; \Gamma)$). Since these morphisms play a distinguished rôle for the whole theory we call them *characteristic morphisms*¹. Note that both sides of (2) are well-known in many cases, so the question arises whether ε^n also has a known interpretation.

In Section 3 we establish the relation to Lie algebra cohomology. Those cohomology classes which have trivial Lie algebra cohomology classes have a natural interpretation as flat bundles (or higher bundles, such as bundle gerbes). This gives in particular rise to the interpretation of the image of ε^n as flat characteristic classes.

Section 4 then treats the case in which all characteristic morphisms vanish. This condition can be checked very conveniently for semi-simple Lie groups, since there it can be read off the associated compact dual G_u/K of the non-compact symmetric space G/K naturally associated to G (where $K \leq G$ is a maximal compact subgroup). In the case that all characteristic morphisms vanish the cohomology groups $H^n(G; U(1))$ may be computed as

$$H^n(G; \mathfrak{a}/\Gamma) \cong H^n_{\operatorname{Lie}}((\mathfrak{g}, K), \mathfrak{a}) \oplus H^{n+1}_{\pi_1(BG)}(BG; \Gamma).$$

In Section 5 we then show that the characteristic homomorphisms ε^n can be computed in terms of the compact dual G_u/K . More precisely, let $f: G_u/K \to BK$ be a classifying map for the principal K-bundle $G_u \to G_u/K$ and let $j: \Gamma \to \mathfrak{a}$ denote the inclusion. The main result of Section 5 is then the following

Theorem. Suppose G is a semi-simple Lie group that acts trivially on a and suppose $\Gamma \leq a$ is discrete. Then

¹See Remark 3.10 for the interpretation in terms of flat characteristic classes and the relation to the characteristic morphisms $H^n_{\text{Lie}}((\mathfrak{g}, K); \mathbb{R}) \to H^n_{\text{gp}}(G, \mathbb{R})$ from [Mor01].

there exist isomorphisms $H^n(G;\Gamma) \xrightarrow{\cong} H^n_{top}(BK;\Gamma)$ and $H^n(G;\mathfrak{a}) \xrightarrow{\cong} H^n_{top}(G_u/K;\mathfrak{a})$ such that the diagram

$$\begin{array}{c} H^n(G;\Gamma) & \xrightarrow{\varepsilon^n} & H^n(G;\mathfrak{a}) \\ \downarrow \cong & \downarrow \cong \\ H^n_{\mathrm{top}}(BK;\Gamma) & \xrightarrow{j_*} & H^n_{\mathrm{top}}(BK;\mathfrak{a}) & \xrightarrow{f^*} & H^n_{\mathrm{top}}(G_u/K;\mathfrak{a}) \end{array}$$

commutes.

Since j_* and f^* can be computed explicitly by the Chern-Weil homomorphism of $G_u \to G_u/K$, the preceding theorem gives a very good control on the long exact sequence (1). In particular, it gives a good control on which classes in $H^n(G; \mathfrak{a})$ are flat.

In Section 6 we then treat some examples. One of the perhaps most interesting consequences is that ε^{2q} does not vanish on the Euler class $E_q \in H^{2q}(\mathrm{SL}_{2q}(\mathbb{R});\mathbb{Z}) \cong H^{2q}_{\mathrm{top}}(B \operatorname{SO}_{2q};\mathbb{Z})$, and thus $\varepsilon^{2q}(E_q)$ yields a flat characteristic class.

This brings us to an analysis of the results obtained in this paper. At first, the computational results obtained in Section 6 and the connection to the Chern-Weil homomorphism of compact symmetric spaces is new. The flatness of the Euler class is of course not new (cf. [Mil58, Dup78]), but what is new is the perspective on this phenomena that topological group cohomology yields. Moreover, the perspective to the Chern-Weil homomorphism as a push-forward in coefficients within the same theory seems to be new. In particular, this gives a conceptual interpretation of many seemingly ad-hoc relations between the cohomology of classifying spaces and the van Est cohomology (see [BIMW08, Kar04, Dup79, Dup76]).

Since the aforementioned flatness of the Euler class and the relation between the cohomology of classifying spaces and the van Est cohomology all occur naturally in the context of bounded continuous cohomology (see also [Mon06, Mon01]), this suggest that there is a close relation between topological group cohomology and bounded continuous cohomology. We expect that a further analysis of the techniques presented in this paper might also lead to new applications and insights there.

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1 A recap of topological group cohomology

The purpose of this section is to fix notation and to introduce concepts. More detailed expositions can be found in [WW13, Gui80, Seg70, HM62]. Throughout this section, G is an arbitrary topological group² and Aa topological G-module. By a topological G-module we mean a locally contractible topological abelian group A that is a G-module such that the action map $G \times A \to A$ is continuous. A short exact sequence $A \to B \to C$ of topological G-modules is defined to be a sequence of topological G-modules such that B is a principal A-bundle over C. The sequence $A \to B \xrightarrow{q} C$ is said to be topologically trivial if the principal bundle is trivial, i.e., if there exists $\sigma: C \to B$ continuous such that $q \circ \sigma = \mathrm{id}_C$. Moreover, let G act continuously from the left on a space X (in case X = G we will always consider the action by left multiplication). Then we endow Map(X, A) (arbitrary set maps for the moment) with the left action of G given by $(g.f)(x) := g.(f(g^{-1}.x))$.

 $^{^{2}}$ With this we mean a group object in the category of compactly generated Hausdorff spaces, i.e., we endow products with the compactly generated product topology.

Now there are several cohomology groups associated to this setting:

a) The van Est cohomology

$$H^n_{\rm vE}(X;A) := H^n(C^0_{\rm vE}(X,A)^G \xrightarrow{d} C^1_{\rm vE}(X,A)^G \xrightarrow{d} \cdots)$$

with

$$C_{\rm vE}^n(X,A) := C(X^{n+1},A) \quad \text{and} \quad df(g_0,...,g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i f(g_0,...,\widehat{g_i},...,g_{n+1}). \tag{3}$$

If X = G, then we obtain the van Est cohomology $H^n_{vE}(G; A)$ (which is called $H^n_{glob,c}(G; A)$ in $[WW13]^3$). If, moreover, G is a Lie group, A is a smooth G-module, X is a manifold and the action is smooth, then we also have the corresponding smooth version

$$H^n_{\mathrm{vE},s}(X;A) := H^n(C^0_{\mathrm{vE},s}(X;A)^G \xrightarrow{d} C^1_{\mathrm{vE},s}(X,A))^G \xrightarrow{d} \cdots)$$

with $C_{\mathrm{vE},s}^n(X,A) := C^{\infty}(X^{n+1},A)^4$. Note that if G is a finite-dimensional Lie group and $A = \mathfrak{a}$ is a smooth and quasi-complete locally convex G-module, then by [HM62, Theorem 5.1] the inclusion $C_{\mathrm{vE},s}^n(G,\mathfrak{a}) \hookrightarrow C_{\mathrm{vE}}^n(G,\mathfrak{a})$ induces an isomorphism $H_{\mathrm{vE},s}^n(G;\mathfrak{a}) \cong H_{\mathrm{vE}}^n(G;\mathfrak{a})$.

b) The Segal-Mitchison cohomology (for simplicity we only consider the case X = G)

$$H^n_{\rm SM}(G;A) := H^n(C(G,EA)^G \xrightarrow{d} C(G,B_GA)^G \xrightarrow{d} \cdots)$$

where $B_G A := C(G, EA)/A$ and EA is a chosen model for the universal bundle of the topological abelian group A such that $EA \to BA$ admits a local section [Seg70, Appendix A]. If A is contractible, then we may assume that EA = A and thus one sees that $H^n_{\rm SM}(G; A) \cong H^n_{\rm vE}(G; A)$ in this case [Seg70, Proposition 3.1]. On the other hand, if $A = A^{\delta}$ is discrete, then [Seg70, Proposition 3.3] shows that $H^n_{\rm SM}(G; A) \cong H^n_{\pi_1(BG)}(BG; A)$ (where $BG := |BG_{\bullet}|$ is the classifying space of G and $H^n_{\pi_1(BG)}$ denotes the sheaf cohomology of the local system of the $\pi_1(BG) \cong \pi_0(G)$ -action on the discrete group A).

c) The locally continuous cohomology

$$H^n_{\rm loc}(X;A) := H^n(C^0_{\rm loc}(X,A)^G \xrightarrow{d} C^1_{\rm loc}(X,A)^G \xrightarrow{d} \cdots),$$

where

 $C^n_{\text{loc}}(X,A) := \{ f \colon X^{n+1} \to A \mid f \text{ is continuous on some neighbourhood of the diagonal } \Delta^{n+1}X \}.$

By abuse of notation we sometimes refer to the elements of $C_{loc}^n(X, A)$ as locally continuous maps or cochains. Again, if G = X, then we obtain the locally continuous cohomology $H_{loc}^n(G; A)$. We have a natural morphism $H_{vE}^n(X; A) \to H_{loc}^n(X; A)$ induced from the inclusion $C_{vE}^n(X, A) \hookrightarrow C_{loc}^n(X, A)$. Note that this is an isomorphism if either X is contractible [Fuc11, Theorem 5.16] or if X = G is metrisable and A is contractible by group homomorphisms [FW12, Proposition 3.6].

If, moreover, G is a Lie group, A is a smooth G-module, X is a manifold and the action is smooth, then we also have the corresponding smooth version

$$H^n_{\mathrm{loc},s}(X;A) := H^n(C^0_{\mathrm{loc},s}(X,A)^G \xrightarrow{d} C^1_{\mathrm{loc},s}(X,A)^G \xrightarrow{d} \cdots),$$

³To match up with [WW13] one has to pass from the homogeneous cochain complex to the inhomogeneous one, i.e., identify $\operatorname{Map}(G^{n+1}, A)^G$ with $\operatorname{Map}(G^n, A)$ via $f \mapsto F$ with $F(g_1, ..., g_n) := F(1, g_1, g_1g_2, ..., g_1 \cdots g_n)$ (see [Bro94, Section I.5], [Gui80, n^o I.3.1] or [Nee04, Appendix B]).

⁴In the smooth category we endow products with the usual product smooth structure

where

 $C^n_{\text{loc s}}(X, A) := \{ f \colon X^{n+1} \to A \mid f \text{ is smooth on some neighbourhood of the diagonal } \Delta^{n+1}X \}.$

By abuse of notation we sometimes refer to the elements of $C^n_{\text{loc},s}(X, A)$ as locally smooth maps or cochains. Again, if G = X, then we obtain locally smooth cohomology $H^n_{\text{loc},s}(G; A)$ of G considered in [WW13]. If we assume, furthermore, that G is finite-dimensional and \mathfrak{a} is quasi-complete, then the inclusion $C^n_{\text{loc},s}(G; A) \hookrightarrow C^n_{\text{loc}}(G, A)$ is a quasi-isomorphism, i.e., induces an isomorphism in cohomology $H^n_{\text{loc},s}(G; A) \cong H^n_{\text{loc}}(G; A)$ [WW13, Proposition I.7]. We will often identify $H^n_{\text{loc},s}(G; A)$ with $H^n_{\text{loc}}(G; A)$ via this identification.

If $A \to B \to C$ is a short exact sequence of topological G-modules, then we have long exact sequences

$$\cdots \to H^{n-1}_{\mathrm{SM}}(G;C) \xrightarrow{\delta^{n-1}} H^n_{\mathrm{SM}}(G;A) \to H^n_{\mathrm{SM}}(G;B) \to H^n_{\mathrm{SM}}(G;C) \xrightarrow{\delta^n} H^n_{\mathrm{SM}}(G;A) \to \cdots$$

and

$$\cdots \to H^{n-1}_{\text{loc}}(G;C) \xrightarrow{\delta^{n-1}} H^n_{\text{loc}}(G;A) \to H^n_{\text{loc}}(G;B) \to H^n_{\text{loc}}(G;C) \xrightarrow{\delta^n} H^n_{\text{loc}}(G;A) \to \cdots$$

(cf. [Seg70, Proposition 2.3] and [WW13, Remark I.2]). These long exact sequences are natural with respect to morphisms of short exact sequences (cf. [WW13, Section VI]). Since $H^n_{SM}(G; A)$ and $H^n_{loc}(G; A)$ coincide for (loop) contractible A with $H^n_{VE}(G; A)$ (cf. [Seg70, Proposition 3.1] and [FW12]), this implies that we have isomorphisms of δ -functors (cf. [WW13, Section VI])

$$H^n_{\rm SM}(G;A) \cong H^n_{\rm loc}(G;A) \tag{4}$$

(under the additional assumption that the product topology on G^n is compactly generated, see [WW13, Corollary IV.8]). The same argument shows that the Segal-Mitchison and the locally continuous cohomology coincides (under some mild additional assumptions) with many other cohomology theories for topological groups, as for instance the simplicial group cohomology from [Del74, Bry00] (see [WW13, Corollary IV.7]), the measurable group cohomology from [Moo76] (see [WW13, Remark IV.13]) and the cohomology groups from [Fla08] (see [WW13, Remark IV.12]). We believe that this is the "correct" notion of a cohomology theory for topological groups and thus call it the *topological group cohomology*. In case that we do not refer to a specific cocycle model we will just denote it by $H^n(G; A)$.

Note that the argument leading to the isomorphism $H^n_{SM}(G; A) \cong H^n_{loc}(G; A)$ does not show that the topological group cohomology is isomorphic to the van Est cohomology, since for the van Est cohomology we only have a long exact sequence

$$\cdots \to H^{n-1}_{vE}(G;C) \xrightarrow{\delta^{n-1}} H^n_{vE}(G;A) \to H^n_{vE}(G;B) \to H^n_{vE}(G;C) \xrightarrow{\delta^n} H^n_{vE}(G;A) \to \cdots$$

if the short exact sequence $A \to B \to C$ is topologically trivial.

Remark 1.1. The functors H_{vE}^n , H_{SM}^n and H_{loc}^n are also natural in the first argument in the sense that a continuous morphism $\varphi \colon H \to G$ induces morphisms $\varphi^* \colon H^n(G; A) \to H^n(H; \varphi^*A)$, where φ^*A denotes the pull-back module. Indeed, φ induces morphisms of cochain complexes

$$\varphi^* \colon C(G^{n+1}, A)^G \to C(H^{n+1}, \varphi^* A)^H \quad \text{ and } \quad \varphi^* \colon C^n_{\text{loc}}(G, A)^G \to C(H, \varphi^* A)^H, \quad f \mapsto f \circ \varphi,$$

which induce morphisms $\varphi^* \colon H^n_{\rm vE}(G; A) \to H^n_{\rm vE}(H; \varphi^* A)$ and $\varphi^* \colon H^n_{\rm loc}(G; A) \to H^n_{\rm loc}(H; \varphi^* A)$ in cohomology. The morphisms for $H^n_{\rm SM}$ are induced as follows. Recall from [Seg70, Appendix A] that the module structure

The morphisms for H^n_{SM} are induced as follows. Recall from [Seg70, Appendix A] that the module structure on EA is induced from the action of the simplicial topological group G_{\bullet} (i.e., $G_n := G$) on the abelian simplicial topological group EA_{\bullet} (i.e., $EA_n := A^{n+1}$) via the diagonal action of G on A^{n+1} . In the geometric realisation we thus obtain a map

$$G \times EA = |G_{\bullet}| \times |EA_{\bullet}| \cong |G_{\bullet} \times EA_{\bullet}| \to |EA_{\bullet}| = EA$$

defining the module structure. From this one sees that we have $\varphi^* EA = E\varphi^* A$. We now observe that a morphism $\psi \colon \varphi^* A \to C$ induces a morphism

$$E_{\varphi,\psi} \colon \varphi^* C(G, EA) \to C(H, EC), \quad f \mapsto E(\psi) \circ f \circ \varphi.$$

Since $E_{\varphi,\psi}$ preserves the constant maps it induces a morphism

$$B_{\varphi,\psi} \colon \varphi^* B_G A \to B_H C$$

Inductively we obtain morphisms $B^n_{\varphi,\psi}$: $\varphi^* B^n_G A \to B^n_H C$ and thus morphisms

$$E_{\varphi,B^n_{\varphi,\psi}}:\varphi^*C(G,EB^n_GA)\to C(H,EB^n_HC), \quad f\mapsto E(B^n_{\varphi,\psi})\circ f\circ\varphi$$

In particular, if we set $C = \varphi^* A$, then we have morphisms of cochain complexes

that induce morphisms $\varphi^* \colon H^n_{\mathrm{SM}}(G; A) \to H^n_{\mathrm{SM}}(H; \varphi^* A)$ in cohomology.

Obviously, if $\alpha \colon A \to D$ is a morphism of topological *G*-modules, then we get a morphism $\varphi^* \alpha \colon \varphi^* A \to \varphi^* D$ and the diagram

$$\begin{array}{c} H^n(G;A) & \xrightarrow{\alpha_*} & H^n(G;D) \\ \downarrow^{\varphi^*} & \downarrow^{\varphi^*} \\ H^n(H;\varphi^*A) & \xrightarrow{\alpha_*} & H^n(H;\varphi^*D) \end{array}$$

commutes.

Proposition 1.2. The isomorphisms $H^n_{SM}(G; A) \cong H^n_{loc}(G; A)$ from (4) are natural with respect to φ^* , i.e., if $\varphi: H \to G$ is a morphisms of topological groups, then the diagram

$$\begin{aligned} H^{n}_{\mathrm{SM}}(G; A) & \xrightarrow{\cong} H^{n}_{\mathrm{loc}}(G; A) \\ & \downarrow \varphi^{*} & \downarrow \varphi^{*} \\ H^{n}_{\mathrm{SM}}(H; \varphi^{*}A) & \xrightarrow{\cong} H^{n}_{\mathrm{loc}}(H; \varphi^{*}A) \end{aligned}$$
 (5)

commutes for each $n \in \mathbb{N}_0$.

Proof. We have the following δ -functors (cf. [WW13, Section VI])

 $\mathbf{G}\text{-}\mathbf{Mod} \to \mathbf{Ab}, \quad A \mapsto H^n_{\mathrm{SM}}(G;A), \quad A \mapsto H^n_{\mathrm{SM}}(H;\varphi^*A), \quad A \mapsto H^n_{\mathrm{loc}}(G;A), \quad A \mapsto H^n_{\mathrm{loc}}(H;\varphi^*A).$

Observe that φ^* constitute morphisms of δ -functors. Since

$$H^n_{\rm SM}(G; E_G B^n_G A) = H^n_{\rm loc}(G; E_G B^n_G A) = 0$$

it suffices by [WW13, Theorem VI.2] to observe that (5) commutes for n = 0. The latter is trivial.

Remark 1.3. One relation that we obtain from the above is in the case that the *G*-action is also continuous for the discrete topology A^{δ} on *A* (this happens for instance if *G* is locally contractible and G_0 acts trivially). Then we have isomorphisms

$$\zeta^n \colon H^n_{\pi_1(BG)}(BG; A^{\delta}) \xrightarrow{=} H^n_{SM}(G; A^{\delta})$$

(cf. Remark in §3 of [Seg70]). If we identify $H^n_{SM}(G; A^{\delta})$ with $H^n_{\text{loc}}(G; A^{\delta})$, then the morphism $A^{\delta} \to A$ induces a morphism

$$\flat^n \colon H^n_{\pi_1(BG)}(BG; A) \to H^n_{\mathrm{loc}}(G; A),$$

which is of course an isomorphism if A already is discrete. On the other hand, if G is discrete, then \flat^n is the well-known isomorphism $H^n_{\rm gp}(G; A) \cong H^n_{\pi_1(BG)}(BG; A)$ [Bro94]. Here $H^n_{\rm gp}(G; A)$ is the group cohomology of the abstract group G with coefficients in A, which coincides (literally at the cochain level) with $H^n_{\rm loc}(G^{\delta}; A^{\delta})$. From the explicit description of ζ^n it follows that ζ^n and \flat^n are natural with respect to morphisms of groups and of coefficients, i.e., if $\varphi: H \to G$ is a morphism of topological groups and $\alpha: A \to D$ is a morphism of topological G-modules, then the diagrams

$$\begin{array}{cccc} H^n_{\pi_1(BG)}(BG;A) & \stackrel{\flat^n}{\longrightarrow} H^n_{\text{loc}}(G;A) & H^n_{\pi_1(BG)}(BG;A) & \stackrel{\flat^n}{\longrightarrow} H^n_{\text{loc}}(G;A) \\ & & \downarrow^{\alpha_*} & \downarrow^{\alpha_*} & \text{and} & \downarrow^{\varphi^*} & \downarrow^{\varphi^*} \\ H^n_{\pi_1(BG)}(BG;D) & \stackrel{\flat^n}{\longrightarrow} H^n_{\text{loc}}(G;D) & H^n_{\pi_1(BH)}(BH;\varphi^*A) & \stackrel{\flat^n}{\longrightarrow} H^n_{\text{loc}}(H;A) \end{array}$$

commute (and likewise for ζ^n). If G is a finite-dimensional Lie group and $A = \mathfrak{a}/\Gamma$ for \mathfrak{a} locally convex and quasi-complete, then we may interpret \flat^n as a natural morphism to $H^n_{\text{loc},s}(G;A)$ via the identification $H^n_{\text{loc},s}(G;A) \cong H^n_{\text{loc},s}(G;A)$.

We will follow the convention that we denote morphism in cohomology that are induced by morphisms of groups, spaces or coefficient modules by upper and lower stars. Morphisms that are induced by manipulations of cochains will be denoted by the corresponding cohomology index. If we use the upper star as the index of cohomology groups when referring to the whole cohomology algebra, instead to a single abelian group in one specific degree (and a plus there refers to the cohomology in positive degree). For the convenience of the reader, we collect here the definitions of the cohomology groups and some of their chain complexes that we will use throughout (in the order in which they appear in the text):

$$\begin{array}{c} H^n_{\rm vE}(X,A) \\ H^n_{\rm vE,s}(X,A) \\ H^n_{\rm SM}(G,A) \\ H^n_{\rm I_{0C}}(BG;A) \\ H^n_{\rm loc,s}(X,A) \\ H^n_{\rm loc,s}(G;A) \\ H^n_{\rm loc,s}(G;A) \\ H^n_{\rm loc,s}(G,K);A) \end{array} \begin{array}{c} C^n_{\rm vE}(X,A)^G \\ C^\infty_{\rm vE,s}(X,A)^G \\ C^\infty_{\rm vE,s}(X,A)^G \\ C^\infty_{\rm vE,s}(X,A)^G \\ C^\infty_{\rm vE,s}(X,A)^G \\ C^\infty_{\rm loc}(X,A)^G \\ \{f: X^{n+1} \rightarrow A \mid f \text{ is cont. on some neighbh. of } \Delta^{n+1}X\}^G \\ \{f: X^{n+1} \rightarrow A \mid f \text{ is smooth on some neighbh. of } \Delta^{n+1}X\}^G \\ \{f: X^{n+1} \rightarrow A \mid f \text{ is smooth on some neighbh. of } \Delta^{n+1}X\}^G \\ \{f: X^{n+1} \rightarrow A \mid f \text{ is smooth on some neighbh. of } \Delta^{n+1}X\}^G \\ any of H^n_{\rm SM}(G;A), H^n_{\rm loc}(G;A) (or H^n_{\rm loc,s}(G;A)) \\ cohomology of the (underlying) topological space X with coeff. \\ in the (abstract) abelian group A \\ H^n_{\rm loc,s}(G;A) \\ H^n_{\rm loc,s}(G;A) \\ H^n_{\rm loc,s}(G;A) \\ C^{0,n}_{\rm loc,s}(G,A) \\ C^{0,n}$$

2 The long exact sequence and the characteristic morphisms

In this section we analyse the long exact sequence in topological group cohomology for torus (or more generally $K(\Gamma, 1)$) coefficients. We will try to motivate why it is a good idea to look at this sequence. The general assumptions in this section are as in the previous one. We only assume, in addition, that the coefficient module is $A = \mathfrak{a}/\Gamma$ for some contractible *G*-module \mathfrak{a} and a discrete submodule $\Gamma \leq \mathfrak{a}$. Recall that $H^n(G; A)$

always refers to the topological group cohomology of G with coefficients in A, which can be realised by the models $H^n_{\rm SM}(G; A)$ or $H^n_{\rm loc}(G; A)$.

Remark 2.1. The exact coefficient sequence $\Gamma \to \mathfrak{a} \to A$ induces a long exact sequence

$$\cdots \to H^{n-1}(G; A) \to H^n(G; \Gamma) \to H^n(G; \mathfrak{a}) \to H^n(G; A) \to H^{n+1}(G; \Gamma) \to \cdots .$$
(6)

As described in the previous section we have isomorphisms

$$H^n(G;\mathfrak{a}) \cong H^n_{\mathrm{vE}}(G;\mathfrak{a}) \quad \text{and} \quad H^n(G;\Gamma) \cong H^n_{\pi_1(BG)}(BG;\Gamma).$$

This leads to a long exact sequence

$$\dots \to H^{n-1}(G;A) \to H^n_{\pi_1(BG)}(BG;\Gamma) \to H^n_{vE}(G;\mathfrak{a}) \to H^n(G;A) \to H^{n+1}_{\pi_1(BG)}(BG;\Gamma) \to \dots$$
(7)

From the long exact sequence above, certain morphisms will turn out to be particularly important. We give them a distinguished name.

Definition 2.2. The morphisms $\varepsilon^n \colon H^n(G; \Gamma) \to H^n(G; \mathfrak{a})$, induced by the inclusion $\Gamma \hookrightarrow \mathfrak{a}$ will be called *characteristic morphisms* in the sequel.

Note that in the theory of flat characteristic classes there is also the notion of *characteristic morphism* (see [Mor01, Section 2.3]). Our characteristic morphism is a refinement of the one occurring there that is more sensitive to the underlying topological information (see Remark 3.10). In particular, the characteristic morphisms in [Mor01, Section 2.3] are likely to be injective [Mor01, Theorem 2.22]. In contrast to this we note that from the vanishing of $H^n_{vE}(G; \mathfrak{a}) \cong H^n(G; \mathfrak{a})$ for compact Lie groups and quasi-complete \mathfrak{a} [HM62] and the contractibility of BG for contractible G we have the following vanishing result for the characteristic morphism in the topologically trivial situations:

Lemma 2.3. The characteristic morphisms vanish if either G is contractible or if G is a compact Lie group and \mathfrak{a} is quasi-complete and locally convex.

This suggest that the characteristic morphisms are likely to vanish. We will show in the sequel that this is often the case and, if not, is the source of interesting geometric structure in terms of flat characteristic classes (cf. Remark 3.10 and Section 6). What we will show in the remainder of this section is that it is appropriate to think of the characteristic morphisms as some kind of connecting morphisms.

Proposition 2.4. Suppose G is 1-connected. If $q: \mathfrak{a} \to A = \mathfrak{a}/\Gamma$ denotes the quotient morphism, then

$$q_* \colon H^n_{\mathsf{vE}}(G;\mathfrak{a}) \to H^n_{\mathsf{vE}}(G;A) \tag{8}$$

is an isomorphism for $n \geq 1$.

Proof. We first show surjectivity. Let $f: G^n \to A$ be continuous and satisfy df = 0. By the dual Dold-Kan correspondence we may assume without loss of generality that f(1, ..., 1) = 0. Since G is 1-connected, there exists a unique continuous $\tilde{f}: G^n \to \mathfrak{a}$ such that $\tilde{f}(1, ..., 1) = 0$ and $q \circ \tilde{f} = f$. Thus $q \circ d\tilde{f} = d(q \circ \tilde{f}) = df = 0$ and since $d\tilde{f}$ is uniquely determined by $q \circ d\tilde{f} = 0$ and $d\tilde{f}(1, ..., 1) = 0$ is follows that $d\tilde{f} = 0$. Thus (8) is surjective.

Injectivity is argued similarly. If $q \circ \tilde{f} = db$ for some continuous $b: G^{n-1} \to A$, then we can lift b to some continuous $\tilde{b}: G^{n-1} \to \mathfrak{a}$. Making the appropriate assumptions on the values in (1, ..., 1), one can adjust things so that $d\tilde{b} = \tilde{f}$ and conclude that (8) is injective.

Corollary 2.5. If G is 1-connected, then the natural morphism $H^n_{vE}(G; A) \to H^n_{loc}(G; A)$ fits into the long exact sequence

$$\cdots \to H^{n-1}_{\text{loc}}(G; A) \to H^n_{\pi_1(BG)}(BG; \Gamma) \to H^n_{\text{vE}}(G; A) \to H^n_{\text{loc}}(G; A) \to H^{n+1}_{\pi_1(BG)}(BG; \Gamma) \to \cdots$$

In particular, $H^n_{vE}(G; A) \cong H^n_{loc}(G; A)$ if G is contractible (the latter is a specialisation of [Fuc11, Theorem 5.16]).

We now work towards an interpretation of the long exact sequence for finite-dimensional Lie groups.

Remark 2.6. Suppose that $K \leq G$ is a closed subgroup. Then the inclusion $i: K \to G$ induces a restriction morphism

$$i^* \colon H^n(G; A) \to H^n(K; A)$$

(given on H_{loc}^n by restricting cochains to K, whence the name). On the other hand, G acts on the quotient G/K by left multiplication and we set

$$H^n_{vE}((G,K);A) := H^n_{vE}(G/K,A)$$
 and $H^n_{loc}((G,K);A) := H^n_{loc}(G/K;A).$

Note that $H^n_{vE}((G, K); A)$ and $H^n_{loc}((G, K); A)$ are the relative versions of the van Est and the locally continuous cohomology (compare to the relative Lie algebra cohomology in Section 3). Since the quotient map $p: G \to G/K$ is G-equivariant it induces morphisms in cohomology

$$p^* \colon H^n_{\operatorname{loc}}((G,K);A) \to H^n_{\operatorname{loc}}(G;A),$$

given on the cochain level by $f \mapsto f \circ (p \times \cdots \times p)$.

Proposition 2.7. Suppose $K \leq G$ is a closed subgroup such that G/K is 1-connected. If $q: \mathfrak{a} \to A = \mathfrak{a}/\Gamma$ denotes the quotient morphism, then

$$q_* \colon H^n_{\mathrm{vE}}((G, K); \mathfrak{a}) \to H^n_{\mathrm{vE}}((G, K); A)$$

is an isomorphism for $n \geq 1$.

Proof. Replacing G by G/K, the prof of Proposition 2.4 carries over verbatim.

Proposition 2.8. Suppose that G is a finite-dimensional Lie group with finitely many components, that $K \leq G$ is a maximal compact subgroup and that \mathfrak{a} is a quasi-complete locally convex space. Then $H^n_{\text{loc}}(G;\mathfrak{a}) \cong H^n_{\text{loc}}((G,K);A)$ for $n \geq 1$.

Proof. We consider the following diagram of morphisms of cochain complexes, in which the maps are precompositions with $p: G \to G/K$, post-compositions with $q: \mathfrak{a} \to A$ and inclusions of locally continuous maps into continuous ones:



Now α_1 is a quasi-isomorphism, i.e., it induces an isomorphism in cohomology, by [Gui80, Corollaire III.2.2]. Moreover, α_2 is a quasi-isomorphism by [FW12, Proposition III.6]. The contractibility of G/K also implies that α_3 and α_5 are quasi-isomorphisms by [Fuc11, Theorem 3.16]. Consequently, β_1 is a quasi-isomorphism. In addition, α_4 induces an isomorphism in cohomology if $n \ge 1$ by Proposition 2.7 since G/K is contractible. This induces the desired isomorphisms $H^n_{loc}(G; \mathfrak{a}) \cong H^n_{loc}((G, K); A)$ for $n \ge 1$.

Remark 2.9. We will denote the isomorphism from the preceding proposition by

$$\psi^n \colon H^n_{\text{loc}}(G; \mathfrak{a}) \xrightarrow{\cong} H^n_{\text{loc}}((G, K); A).$$

The same argument also shows that there is an isomorphism in the locally smooth cohomology, which we also denote by

$$\psi^n \colon H^n_{\mathrm{loc},s}(G;\mathfrak{a}) \xrightarrow{\cong} H^n_{\mathrm{loc},s}((G,K);A).$$

Note that ψ^n is not implemented by a canonical morphism on the cochain level. However, we have the morphism $\beta_1 = p^* \colon C^n_{\text{loc}}(G/K, \mathfrak{a})^G \to C^n_{\text{loc}}(G, \mathfrak{a})^G$. The preceding proof shows that this also induces an isomorphism

$$p^* \colon H^n_{\mathrm{loc}}(G; \mathfrak{a}) \xrightarrow{\cong} H^n_{\mathrm{loc}}((G, K); \mathfrak{a})$$

in cohomology for all $n \in \mathbb{N}_0$.

If $i: K \to G$ denotes the inclusion, then this is a homotopy equivalence, and same is true for the induced map of classifying spaces $Bi: BK \to BG$. Thus the induced map in cohomology Bi^* is an isomorphism and the commuting diagram

$$\begin{array}{c} H^{n+1}_{\mathrm{loc}}(G;\Gamma) \xrightarrow{\cong} H^{n+1}_{\pi_1(BG)}(BG;\Gamma) \\ \downarrow^{i^*} \qquad \qquad \downarrow^{Bi^*} \\ H^{n+1}_{\mathrm{loc}}(K;\Gamma) \xrightarrow{\cong} H^{n+1}_{\pi_1(BK)}(BK;\Gamma) \end{array}$$

shows hat $i^* \colon H^n_{\text{loc}}(G; \Gamma) \to H^n_{\text{loc}}(K, \Gamma)$ is an isomorphism.

With respect to these identifications, the characteristic morphisms $\varepsilon^n \colon H^n_{\text{loc}}(G; \Gamma) \to H^n_{\text{loc}}(G; \mathfrak{a})$ induce morphisms $\tilde{\varepsilon}^n \colon H^n_{\text{loc}}(K; A) \to H^{n+1}_{\text{loc}}((G, K); A)$ that make

$$\begin{array}{c} H_{\mathrm{loc}}^{n}(K;A) & \xrightarrow{\tilde{\varepsilon}^{n}} & H_{\mathrm{loc}}^{n+1}((G,K);A) \\ & \downarrow^{\delta^{n}} & \uparrow^{\psi^{n+1}} \\ H_{\mathrm{loc}}^{n+1}(K;\Gamma) & \xrightarrow{(i^{*})^{-1}} & H_{\mathrm{loc}}^{n+1}(G;\Gamma) & \xrightarrow{\varepsilon^{n+1}} & H_{\mathrm{loc}}^{n+1}(G;\mathfrak{a}) \end{array}$$

commute. We shall call the morphisms $\tilde{\varepsilon}^n$ also *characteristic morphisms*.

The following proposition illustrates that one should think of the characteristic morphism as some kind of connecting homomorphism.

Proposition 2.10. Suppose that G is a finite-dimensional Lie group with finitely many components, that $K \leq G$ is a maximal compact subgroup, that \mathfrak{a} is a quasi-complete locally convex G-module and that $\Gamma \leq \mathfrak{a}$ is a discrete submodule. Then the sequence

$$H^{1}_{\text{loc}}((G,K);A) \xrightarrow{p^{*}} \cdots \xrightarrow{\tilde{\varepsilon}^{n-1}} H^{n}_{\text{loc}}((G,K);A) \xrightarrow{p^{*}} H^{n}_{\text{loc}}(G;A) \xrightarrow{i^{*}} H^{n}_{\text{loc}}(K;A) \xrightarrow{\tilde{\varepsilon}^{n}} H^{n+1}_{\text{loc}}((G,K);A) \to \cdots$$
(9)

 $is \ exact.$

Proof. We first observe that

$$\begin{split} H^n_{\mathrm{loc}}(G;\mathfrak{a}) & \longrightarrow H^n_{\mathrm{loc}}(G;A) \xrightarrow{\delta^n} H^{n+1}_{\mathrm{loc}}(G;\Gamma) & \longrightarrow H^{n+1}_{\mathrm{loc}}(G;\mathfrak{a}) \\ & \downarrow^{i^*} & \downarrow^{i^*} \\ H^n_{\mathrm{loc}}(K;\mathfrak{a}) & \longrightarrow H^n_{\mathrm{loc}}(K;A) \xrightarrow{\delta^n} H^{n+1}_{\mathrm{loc}}(K;\Gamma) & \longrightarrow H^{n+1}_{\mathrm{loc}}(K;\mathfrak{a}) \end{split}$$

commutes and has exact rows. If $n \ge 1$, then we have $H^n_{\text{loc}}(K;\mathfrak{a}) \cong H^n_{\text{vE}}(K;\mathfrak{a}) = 0$ by [FW12, Corollary II.8] and [BW00, Lemma IX.1.10] and thus $\delta^n \colon H^n_{\text{loc}}(K;A) \to H^{n+1}_{\text{loc}}(K;\Gamma)$ is an isomorphism. Thus the exactness of (9) follows from the definition of $\tilde{\varepsilon}^n$.

3 The relation to relative Lie algebra cohomology

We now discuss the relation of topological group cohomology (in the guise of locally smooth group cohomology) to Lie algebra cohomology. Whilst the previous sections are results on the topological group cohomology that were derived more or less from their definitions, the perspective of Lie algebra cohomology will really bring new facets into the game. To this end, we will have to use the locally smooth model $H^n_{\text{loc},s}(G; A)$ quite intensively.

Unless mentioned otherwise, G will throughout this section be a finite-dimensional Lie group with finitely many components and K will be a maximal compact subgroup. The coefficient module is always of the form $A = \mathfrak{a}/\Gamma$, where \mathfrak{a} is a smooth, locally convex and quasi-complete G-module and Γ is a discrete submodule. We will denote the corresponding quotient morphisms by $p: G \to G/K$ and $q: \mathfrak{a} \to A$ and injections by $i: K \to G$ and $j: \Gamma \to \mathfrak{a}$. Moreover, \mathfrak{g} denotes the Lie algebra of G and \mathfrak{k} the Lie algebra of K. Note that A is then also a module for K and that \mathfrak{a} is also a module for \mathfrak{g} and for \mathfrak{k} .

We first recall some basic notions. Let $\mathfrak{h} \leq \mathfrak{g}$ be an arbitrary subalgebra. The relative Lie algebra cohomology $H^n_{\text{Lie}}((\mathfrak{g},\mathfrak{h});\mathfrak{a})$ is the cohomology of the based and invariant cochains in the Chevalley-Eilenberg complex $C^n_{\text{CE}}(\mathfrak{g},\mathfrak{a}) := \text{Hom}_{\mathbb{R}}(\Lambda^n \mathfrak{g},\mathfrak{a})$, i.e.,

$$C^n_{\rm CE}((\mathfrak{g},\mathfrak{h}),\mathfrak{a}):=\operatorname{Hom}_{\mathbb{R}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}),\mathfrak{a})^{\mathfrak{h}}\cong\{\omega\colon\Lambda^n\mathfrak{g}\to\mathfrak{a}\mid i_y(\omega)=0\text{ and }\theta_y(\omega)=0\text{ for all }y\in\mathfrak{h}\},$$

where $i_y(\omega)(x_1, ..., x_{n-1}) := \omega(y, x_1, ..., x_{n-1})$ and

$$\theta_y(\omega)(x_1, ..., x_n) := \sum_{i=1}^n \omega(x_1, ..., [x_i, y], ..., x_n) + y . \omega(x_1, ..., x_n)$$

with respect to the Chevalley-Eilenberg differential

$$d_{\rm CE}\omega(x_0,...,x_n) := \sum_{1 \le i \le n} (-1)^i x_i ...(x_0,...,\hat{x_i},...,x_n) + \sum_{1 \le i < j \le n} (-1)^{i+j} \omega([x_i,x_j],x_0,...,\hat{x_i},...,\hat{x_j},...,x_n).$$

(cf. [BW00, Section I.1], [Gui80, n^o II.3] or [GHV76, Chapter X]). If $i: \mathfrak{h} \to \mathfrak{g}$ denotes the inclusion, then we have a sequence of cochain complexes

$$C^n_{\rm CE}((\mathfrak{g},\mathfrak{h}),\mathfrak{a}) \hookrightarrow C^n_{\rm CE}(\mathfrak{g},\mathfrak{a}) \xrightarrow{i^*} C^n_{\rm CE}(\mathfrak{h},\mathfrak{a}).$$

This gives rise to a sequence in cohomology

$$H^n_{\text{Lie}}((\mathfrak{g},\mathfrak{h});\mathfrak{a}) \xrightarrow{\kappa^n} H^n_{\text{Lie}}(\mathfrak{g};\mathfrak{a}) \xrightarrow{i^*} H^n_{\text{Lie}}(\mathfrak{h};\mathfrak{a})$$
(10)

which is of order two, but which is in general far from being exact. For instance, $H_{\text{Lie}}^n((\mathfrak{g},\mathfrak{h});\mathfrak{a}) \to H_{\text{Lie}}^n(\mathfrak{g};\mathfrak{a})$ vanishes frequently (see Section (4)). The sequence (10) is much more a part of the spectral sequence

$$E_2^{p,q} = H^p_{\text{Lie}}(\mathfrak{h};\mathbb{R}) \otimes H^q_{\text{Lie}}((\mathfrak{g},\mathfrak{h});\mathfrak{a}) \Rightarrow H^{p+q}_{\text{Lie}}(\mathfrak{g};\mathfrak{a})$$

[Kos50, Chapitre VI], where the morphisms from (10) occur as edge homomorphisms.

In case that $\mathfrak{h} = \mathfrak{k}$ and K is not connected there is a subcomplex

$$C_{\mathrm{CE}}^n((\mathfrak{g},K),\mathfrak{a}) := \mathrm{Hom}(\Lambda^n\mathfrak{g}/\mathfrak{k},\mathfrak{a})^K$$

of $C_{CE}^{n}((\mathfrak{g},\mathfrak{k}),\mathfrak{a})$, whose cohomology we denote by $H_{Lie}^{n}((\mathfrak{g},K);\mathfrak{a})$. We clearly have an induced sequence in cohomology

$$H^n_{\operatorname{Lie}}((\mathfrak{g},K);\mathfrak{a}) \xrightarrow{\kappa^n} H^n_{\operatorname{Lie}}(\mathfrak{g};\mathfrak{a}) \xrightarrow{i^*} H^n_{\operatorname{Lie}}(\mathfrak{k};\mathfrak{a})$$

and $C_{\text{CE}}^n((\mathfrak{g}, K), \mathfrak{a}) = C_{\text{CE}}^n((\mathfrak{g}, \mathfrak{k}), \mathfrak{a})$ if K is connected.

We now introduce the differentiation homomorphism from locally smooth to Lie algebra cohomology.

Remark 3.1. (cf. [Nee06, Section V.2], [Gui80, n^o III.7.3]) We want to differentiate in the identity, so we first identify $C_{\text{loc},s}^n(G, A)^G$ via $f \mapsto F$ with $F(g_1, ..., g_n) := F(1, g_1, g_1g_2, ..., g_1 \cdots g_n)$ with

 $\widetilde{C}^n_{\text{loc.s}}(G, A) := \{ f \colon G^n \to A \mid f \text{ is smooth on some identity neighbourhood} \}.$

(cf. [Nee04, Appendix B]). It will be convenient to work with normalised cochains, so we set

$$\widetilde{C}^{0,n}_{\text{loc},s}(G,A) := \{ f \in \widetilde{C}^n_{\text{loc},s}(G,A) \mid f(g_1,...,g_n) = 0 \text{ if } g_i = 1 \text{ for some } i \}$$

and observe that the inclusion $\widetilde{C}^{0,n}_{\mathrm{loc},s}(G,A) \hookrightarrow \widetilde{C}^n_{\mathrm{loc},s}(G,A)$ is a quasi-isomorphism by the dual Dold-Kan correspondence.

Now suppose that M is a manifold and that $f: M^n \to A$ is smooth. For $v_n \in T_{m_n}M_n$ we then set

$$\partial_n(v_n)f(m_n): M^{n-1} \to \mathfrak{a}, \quad (m_1, ..., m_{n-1}) \mapsto \delta(f)(0_{m_1}, ..., 0_{m_{n-1}}, v_k),$$

where $\delta(f) := f^*(\omega_{MC})$ is the left logarithmic derivative of f (equivalently the pull-back of the Maurer-Cartan form ω_{MC} of A). With this we now set

$$D^n \colon \widetilde{C}^n_{\mathrm{loc},s}(G,A) \to C^n_{\mathrm{CE}}(\mathfrak{g},\mathfrak{a}), \quad D^n(f)(v_1,...,v_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \partial_1(v_{\sigma(1)}) \cdots \partial_n(v_{\sigma(n)}) f(1,...,1),$$

where M = U for $U \subseteq G$ an identity neighbourhood such that $f|_{U \times \cdots U}$ is smooth and $v_1, \dots, v_n \in \mathfrak{g} = T_1 U$. This induces for $n \ge 1$ (and if $A = \mathfrak{a}$ also for n = 0) a morphism in cohomology

$$D^n \colon H^n_{\mathrm{loc},s}(G;A) \to H^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a})$$

(see [Nee06, Theorem V.2.6], [Nee04, Appendix B], [Gui80, n^o III.7.3] or [EK64, Appendix I]). The kernel of D^n are precisely those cohomology classes which possess representatives by cochains in $\widetilde{C}^n_{\text{loc},s}(G, A)$ that are constant on some identity neighbourhood. These are the *flat* classes in the locally smooth cohomology. They comprise precisely the image of the morphism

$$(A^{\delta} \to A)_* \colon H^n_{\mathrm{loc},s}(G; A^{\delta}) \to H^n_{\mathrm{loc},s}(G; A)$$

[WW13, Remark V.14]. We thus obtain an exact sequence

$$H^n_{\operatorname{top}}(BG; A^{\delta}) \xrightarrow{\flat^n} H^n_{\operatorname{loc},s}(G; A) \xrightarrow{D^n} H^n_{\operatorname{Lie}}(\mathfrak{g}; \mathfrak{a}).$$

Lemma 3.2. If we set

$$\widetilde{C}_{\text{loc},s}((G,K),A) := \{ f \in \widetilde{C}^n_{\text{loc},s}(G,A) \mid f(g_1,...,g_n) = k_0 \cdot f(k_0^{-1}g_1k_1,...,k_{n-1}^{-1}g_nk_n)$$
 for all $g_1,...,g_n \in G, k_0,...,k_n \in K \},$

then $\alpha : C^n_{\text{loc},s}(G/K, A)^G \to \widetilde{C}^n_{\text{loc},s}((G, K), A), f \mapsto F \text{ with } F(g_1, ..., g_n) := f(p(1), p(g_1), p(g_1g_2), ..., p(g_1 \cdots g_n))$ is an isomorphism of cochain complexes. Moreover, D^n maps the subcomplex

$$\widetilde{C}^{0,n}_{\mathrm{loc},s}((G,K),A) := \widetilde{C}^n_{\mathrm{loc},s}((G,K),A) \cap \widetilde{C}^{0,n}_{\mathrm{loc},s}(G,A)$$

to $C_{CE}^{n}((\mathfrak{g}, K); \mathfrak{a})$ and induces a morphism

$$D^n \colon H^n_{\mathrm{loc},s}((G,K);A) \to H^n_{\mathrm{Lie}}((\mathfrak{g},K);\mathfrak{a})$$

in cohomology.

Proof. Noting that we have a K-equivariant diffeomorphism $G \cong G/K \times K$ it is clear that

$$Map((G/K)^{n+1}, A) \to \{ f \in Map(G^{n+1}, A) \mid f(g_0, ..., g_n) = f(g_0k_0, ..., g_nk_n)$$

for all $g_0, ..., g_n \in G, k_0, ..., k_n \in K \}, \quad f \mapsto f \circ (p \times \cdots \times p)$

is an isomorphism. A straight-forward calculation then shows that the composition with the isomorphism $f \mapsto F$ is α and that $\operatorname{im}(\alpha^n) = \widetilde{C}^n_{\operatorname{loc},s}((G,K),A)$.

To verify $D^n(\widetilde{C}^{0,n}_{\text{loc},s}((G,K);A)) \subseteq C^n_{\text{CE}}((\mathfrak{g},K),\mathfrak{a})$ we observe that

$$f(kg_1k^{-1}, ..., kg_nk^{-1}) = k.f(g_1, ..., g_n)$$
 for all $k \in K, g_1, ..., g_n \in G$

implies $k.(D^n f) = D^n f$ for all $k \in K$. Moreover,

$$f(k, g_1, ..., g_{n-1}) = k^{-1} f(1, g_1, ..., g_{n-1}) = 0$$
 for all $k \in K, g_1, ..., g_{n-1} \in G$

implies that $D^n f(y, x_1, ..., x_{n-1})$ vanishes if $y \in \mathfrak{k}$, $x_1, ..., x_{n-1} \in \mathfrak{g}$. To finish the proof we notice that the inclusion $\widetilde{C}^{0,n}_{\text{loc},s}((G,K);A) \hookrightarrow \widetilde{C}^n_{\text{loc},s}((G,K);A)$ induces an isomorphism in cohomology, so D^n is uniquely determined by its values on the subcomplex $\widetilde{C}^{0,n}_{\text{loc},s}((G,K);A)$.

The following isomorphism is sometimes also called the *van Est* isomorphism. Note that $H^n_{\text{Lie}}((\mathfrak{g}, K); \mathfrak{a}) = H^n_{\text{Lie}}((\mathfrak{g}, \mathfrak{k}); \mathfrak{a})$ if K is connected.

Theorem 3.3. Under the hypothesis from the beginning of this section the morphism

$$D^n: H^n_{\operatorname{loc},s}((G,K);A) \to H^n_{\operatorname{Lie}}((\mathfrak{g},K);\mathfrak{a})$$

is an isomorphism.

Proof. Consider the diagram of morphisms of cochain complexes

$$C_{\mathrm{vE},s}^{n}(G/K,\mathfrak{a})^{G} \xrightarrow{(f \mapsto q \circ f)} C_{\mathrm{vE},s}^{n}(G/K,A)^{G} \xrightarrow{D^{n}} C_{\mathrm{loc},s}^{n}((G,K),A)^{G} \xrightarrow{D^{n}} C_{\mathrm{CE}}^{n}((\mathfrak{g},K),\mathfrak{a})$$

which obviously commutes. Now $D^n: H^n_{\mathrm{vE},s}(G/K;\mathfrak{a}) \to H^n_{\mathrm{Lie}}((\mathfrak{g},\mathfrak{k});\mathfrak{a})$ is an isomorphism by [Gui80, Corollaire III.7.2 and n^o III.7.3] and the morphisms in the top row are quasi-isomorphisms by Proposition 2.7 and [Fuc11, Section 7]. This shows the claim.

It is this isomorphism that will enable us to access the cohomology groups $H^n(G; A)$ (mostly in the model $H^n_{loc,s}(G; A)$). This is mostly because it connects to the well-understood algebraic picture by the following, obvious fact.

Proposition 3.4. The diagram

$$\begin{split} H^n_{\mathrm{loc},s}((G,K);A) & \xrightarrow{p^*} H^n_{\mathrm{loc},s}(G;A) \xrightarrow{i^*} H^n_{\mathrm{loc},s}(K;A) \\ & \downarrow_{D^n} & \downarrow_{D^n} & \downarrow_{D^n} \\ & H^n_{\mathrm{Lie}}((\mathfrak{g},K);\mathfrak{a}) \xrightarrow{\kappa^n} H^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a}) \xrightarrow{i^*} H^n_{\mathrm{Lie}}(\mathfrak{k};\mathfrak{a}) \end{split}$$

commutes. In particular, the composed morphisms



vanish.

Corollary 3.5. If κ^n is injective, then the characteristic morphisms ε^n and $\tilde{\varepsilon}^n$ vanish.

Remark 3.6. The fact that $D^n \circ p^* \circ \tilde{\varepsilon}^n$ vanishes can also be understood by considering the commuting diagram

$$H^{n}_{\mathrm{loc},s}(G;\Gamma) \xrightarrow{j_{*}=\varepsilon^{n}} H^{n}_{\mathrm{loc},s}(G;\mathfrak{a}) \xrightarrow{q_{*}} H^{n}_{\mathrm{loc},s}(G;A)$$

$$\downarrow^{D^{n}} \qquad \downarrow^{D^{n}}$$

$$H^{n}_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a}) \xrightarrow{H^{n}_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a})}$$

The identifications from Remark 2.9 turn $D^n \circ p^* \circ \tilde{\varepsilon}^n$ into $D^n \circ \varepsilon^n = D^n \circ j_*$. But the image of j_* consists, on the cochain level, of maps that vanish on an identity neighbourhood, so that all derivatives of these maps vanish in the identity. Consequently, $D^n \circ j_*$ vanishes. However, the fact that $D^n \circ p^* \circ \tilde{\varepsilon}^n$ vanishes is not that important (let alone obvious), it is the conjunction with the fact that it factors as $\kappa^n \circ D^n \circ \tilde{\varepsilon}^n$ that will be important.

We end this section with the following very convenient relation between the locally smooth Lie group cohomology, the abstract group cohomology and the Lie algebra cohomology.

Theorem 3.7. Suppose G is a Lie group with finitely many components and \mathfrak{a} is a quasi-complete G-module on which G_0 acts trivially. Let $\xi^n \colon H^n_{\mathrm{loc},s}(G;\mathfrak{a}) \to H^n_{\mathrm{loc},s}(G^{\delta};\mathfrak{a}^{\delta}) = H^n_{\mathrm{gp}}(G;\mathfrak{a})$ be induced by mapping locally smooth cochains to abstract cochains in the bar complex. Then the diagram



commutes and the sequence

$$H^{n}_{\mathrm{loc},s}(G;\mathfrak{a}^{\delta}) \xrightarrow{(\mathfrak{a}^{\delta} \to \mathfrak{a})_{*}} H^{n}_{\mathrm{loc},s}(G;\mathfrak{a}) \xrightarrow{D^{n}} H^{n}_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a})$$
(12)

is exact. In particular, $D^n \colon H^n_{\text{loc},s}(G;\mathfrak{a}) \to H^n_{\text{Lie}}(\mathfrak{g};\mathfrak{a})$ factors through $\kappa^n \colon H^n_{\text{Lie}}((\mathfrak{g},K);\mathfrak{a}) \to H^n_{\text{Lie}}(\mathfrak{g};\mathfrak{a})$ and isomorphisms.

Proof. The left rectangle commutes by Proposition 1.2 and the triangles by the definition of the morphisms on the cochain level. As already observed, the sequence (12) is exact by [WW13, Remark V.14].

Corollary 3.8. Suppose the hypothesis of Theorem 3.7 hold. If, in addition, $\kappa^n \colon H^n_{\text{Lie}}((\mathfrak{g}, K); \mathfrak{a}) \to H^n_{\text{Lie}}(\mathfrak{g}; \mathfrak{a})$ is injective, then $H^n_{\text{top}}(BG; \mathfrak{a}) \to H^n_{\text{top}}(BG^{\delta}; \mathfrak{a})$ vanishes.

Corollary 3.9. Suppose the hypothesis of Theorem 3.7 hold. If, in addition, G admits a cocompact lattice, then

$$\operatorname{im}\left(H^{n}_{\operatorname{top}}(BG;\mathfrak{a})\to H^{n}_{\operatorname{top}}(BG^{o};\mathfrak{a})\right)\cong \operatorname{ker}(H^{n}_{\operatorname{Lie}}((\mathfrak{g},K);\mathfrak{a})\to H^{n}_{\operatorname{Lie}}(\mathfrak{g};\mathfrak{a})).$$

This happens for instance if G is semi-simple.

Proof. If G admits a cocompact lattice, then the ξ^n is injective by [Bla85, 16° Théorème] (if we use the isomorphism $H^n_{vE,s}(G;\mathfrak{a}) \to H^n_{loc,s}(G;\mathfrak{a})$ induced by $C^n_{vE,s}(G,\mathfrak{a}) \hookrightarrow C^n_{loc,s}(G,\mathfrak{a})$ to pull back ξ^n to $H^n_{vE}(G;\mathfrak{a})$). Thus ξ^n maps ker $(D^n) = \operatorname{im}((\mathfrak{a}^{\delta} \to \mathfrak{a})_*)$ isomorphically onto $\operatorname{im}(\xi^n \circ (\mathfrak{a}^{\delta} \to \mathfrak{a})_*) = \operatorname{im}((G^{\delta} \to G)^*)$. That semi-simple Lie groups admit cocompact lattices is [Bor63, Theorem C].

Remark 3.10. We now interpret diagram (11) of Theorem 3.7 in terms of flat characteristic classes. Recall that a flat characteristic class is an element in $H^n_{\text{top}}(BG^{\delta};\mathbb{R})$ (or also $H^n_{\text{top}}(BG^{\delta};\mathbb{Z})$) [Dup78, Chapter 9]. Note also that ξ^n is called characteristic morphism in the theory of flat characteristic classes (if one identifies $H^n_{\text{loc},s}(G,\mathbb{R})$ with $H^n_{\text{Lie}}((\mathfrak{g},K);\mathbb{R})$ via the van Est isomorphism) [Mor01, Section 2.3].

Then the image of $b^n \colon H^n_{\text{top}}(BG; \mathbb{R}^{\delta}) \to H^n_{\text{loc},s}(G; \mathbb{R})$ consists of those cohomology classes that are represented by locally smooth cochains that vanish on some identity neighbourhood. These are precisely the *flat* cohomology classes in $H^n_{\text{loc},s}(G; \mathbb{R})$ in the sense that the associated Lie algebra cohomology class vanishes (if n = 2, then the flat classes in $H^2_{\text{loc},s}(G; \mathbb{R})$ are precisely those classes that are represented by a flat principal bundle $\mathbb{R} \to \widehat{G} \to G$ [Nee02]). From (11) it thus follows that the flat classes in $H^n_{\text{loc},s}(G; \mathbb{R})$ map under ξ^n to flat characteristic classes.

The relation to our characteristic morphism ε^n is given by the diagram

$$H^{n}_{\operatorname{top}}(BG;\mathbb{Z}) \xrightarrow{j_{*}} H^{n}_{\operatorname{top}}(BG;\mathbb{R}) \xrightarrow{j_{*}} H^{n}_{\operatorname{top}}(BG;\mathbb{R})$$

$$(G^{\delta} \to G)^{*} \xrightarrow{(G^{\delta} \to G)^{*}} (G^{\delta} \to G)^{*} \xrightarrow{(G^{\delta} \to G)^{*}} H^{n}_{\operatorname{loc},s}(G^{\delta};\mathbb{R}) \xrightarrow{j_{*}} H^{n}_{\operatorname{loc},s}(G^{\delta};\mathbb{R}) \xrightarrow{j_{*}} H^{n}_{\operatorname{loc},s}(G^{\delta};\mathbb{R})$$

$$H^{n}_{\operatorname{top}}(BG^{\delta};\mathbb{Z}) \xrightarrow{j_{*}} H^{n}_{\operatorname{loc},s}(G^{\delta};\mathbb{R}) \xrightarrow{j_{*}} H^{n}_{\operatorname{top}}(BG^{\delta};\mathbb{R})$$

which commutes by the naturality of the involved morphisms. Note that

$$(G^{\delta} \to G)^* \colon H^n_{\mathrm{loc},s}(G; \mathbb{Z}) \cong H^n_{\mathrm{top}}(BG; \mathbb{Z}) \to H^n_{\mathrm{loc},s}(G^{\delta}; \mathbb{Z}) \cong H^n_{\mathrm{top}}(BG^{\delta}; \mathbb{Z})$$

is injective by [Mil83, Corollary 1]. If we assume, moreover, that G is semi-simple, then

$$(G^{\delta} \to G)^* \colon H^n_{\mathrm{loc},s}(G, \mathbb{R}) \cong H^n_{\mathrm{vE}}(G, \mathbb{R}) \to H^n_{\mathrm{loc},s}(G^{\delta}, \mathbb{R}) = H^n_{\mathrm{gp}}(G; \mathbb{R})$$

is also injective (cf. Corollary 3.9). Thus our characteristic morphism *coincides* (on the image of $(G^{\delta} \to G)^*$) with

$$j^* \colon H^n(BG^{\delta}; \mathbb{Z}) \to H^n(BG^{\delta}; \mathbb{R}).$$

4 Subalgebras non-cohomologous to zero

In this section we will analyse under which conditions all characteristic morphisms vanish. The setting is the same as in Section 3.

Definition 4.1. (cf. [GHV76, Section X.5]) Let $\mathfrak{h} \leq \mathfrak{g}$ be a subalgebra. We say that $\mathfrak{h} \leq \mathfrak{g}$ is non-cohomologous to zero (shortly n.c.z.) for \mathfrak{a} if

$$\kappa^n \colon H^n_{\operatorname{Lie}}((\mathfrak{g},\mathfrak{h});\mathfrak{a}) \to H^n_{\operatorname{Lie}}(\mathfrak{g};\mathfrak{a})$$

is injective for all $n \in \mathbb{N}_0$. If $\mathfrak{h} \leq \mathfrak{g}$ is n.c.z. for $\mathfrak{a} = \mathbb{R}$, then we simply say that $\mathfrak{h} \leq \mathfrak{g}$ is n.c.z. More generally, we say that the maximal compact subgroup $K \leq G$ s n.c.z. for \mathfrak{a} if

$$\kappa^n \colon H^n_{\operatorname{Lie}}((\mathfrak{g}, K); \mathfrak{a}) \to H^n_{\operatorname{Lie}}(\mathfrak{g}; \mathfrak{a})$$

is injective for all $n \in \mathbb{N}_0$ and shortly that $K \leq G$ is n.c.z. if it is so for $\mathfrak{a} = \mathbb{R}$.

Note that for G connected we have that $\mathfrak{k} \leq \mathfrak{g}$ is n.c.z. for \mathfrak{a} if and only if $K \leq G$ is n.c.z.

Proposition 4.2. If $K \leq G$ is n.c.z., then all characteristic morphisms $\tilde{\varepsilon}^n$ vanish and the long exact sequence from Proposition 2.10 splits for each $n \geq 1$ into short exact sequences

$$0 \to H^n_{\text{loc}}((G,K);A) \xrightarrow{p^*} H^n_{\text{loc}}(G;A) \xrightarrow{i^*} H^n_{\text{loc}}(K;A) \to 0$$
(13)

In particular, $i^* \colon H^n_{\text{loc}}(G; A) \to H^n_{\text{loc}}(K; A)$ is then surjective and $p^* \colon H^n_{\text{loc}}((G, K); A) \to H^n_{\text{loc}}(G; A)$ is then injective for each $n \ge 1$. Moreover, we have in this case

$$H^{n}(G,A) \cong H^{n}_{\text{Lie}}((\mathfrak{g},K),\mathfrak{a}) \oplus H^{n+1}_{\pi_{1}(BG)}(BG;\Gamma)$$

$$\tag{14}$$

as abelian groups.

Proof. From Corollary 3.5 we immediately deduce the splitting of the long exact sequence. By Theorem 3.3 we have $H_{loc}^n((G, K); A) \cong H_{Lie}^n((\mathfrak{g}, K), \mathfrak{a})$ and as in Remark 2.9 we see that $H_{loc}^n(K; A) \cong H_{\pi_1(BG)}^{n+1}(BG; \Gamma)$. Since $H_{Lie}^n((\mathfrak{g}, K), \mathfrak{a})$ is a divisible abelian group the short exact sequence splits.

Determining whether $\mathfrak{k} \leq \mathfrak{g}$ is n.c.z. is particularly convenient for semi-simple \mathfrak{g} by the Cartan decomposition.

Remark 4.3. Suppose \mathfrak{g} is semi-simple and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} . Then we have $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, and we denote by $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$ the Lie algebra with the same underlying vector space and bracket defined for $x, y \in \mathfrak{k}$ and $v, w \in \mathfrak{p}$ by

$$[x, y]_u := [x, y], \quad [x, v]_u := [x, v], \quad [v, w]_u := -[v, w].$$

Then \mathfrak{g}_u is a compact real form of the complexification $\mathfrak{g}_{\mathbb{C}}$ and \mathfrak{k} is a subalgebra of \mathfrak{g}_u . More precisely, \mathfrak{g}_u is isomorphic to the subalgebra which is the direct sum of \mathfrak{k} and $i\mathfrak{p}$ as subspaces of $\mathfrak{g}_{\mathbb{C}}$ (see [HN12, Section 13.1+2] or [Hel78, Section III.7] for details). Moreover, we have the identity $\mathfrak{g} = \mathfrak{g}_u$ as \mathfrak{k} -modules. If \mathfrak{a} is the trivial \mathfrak{g} -module (also considered as trivial \mathfrak{g}_u -module), then this identity induces an isomorphism of \mathfrak{k} -modules

$$\{\omega \colon \Lambda^n \mathfrak{g} \to \mathfrak{a} \mid i_y(\omega) = 0 \text{ for all } y \in \mathfrak{k}\} = \{\omega \colon \Lambda^n \mathfrak{g}_u \to \mathfrak{a} \mid i_y(\omega) = 0 \text{ for all } y \in \mathfrak{k}\}$$
(15)

and thus an isomorphism $\mu^n \colon H^n_{\text{Lie}}((\mathfrak{g}, \mathfrak{k}); \mathfrak{a}) \xrightarrow{\cong} H^n_{\text{Lie}}((\mathfrak{g}_u, \mathfrak{k}); \mathfrak{a}).$

Lemma 4.4. Suppose \mathfrak{g} is real semi-simple, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} and set $\mathfrak{g}_u := \mathfrak{k} \oplus \mathfrak{i}\mathfrak{p}$. If \mathfrak{a} is the trivial \mathfrak{g} - and \mathfrak{g}_u -module, then $\kappa^n \colon H^n_{\mathrm{Lie}}((\mathfrak{g}, \mathfrak{k}); \mathfrak{a}) \to H^n_{\mathrm{Lie}}(\mathfrak{g}; \mathfrak{a})$ is injective if and only if $\kappa^n \colon H^n_{\mathrm{Lie}}((\mathfrak{g}_u, \mathfrak{k}); \mathfrak{a}) \to H^n_{\mathrm{Lie}}(\mathfrak{g}; \mathfrak{a}) \to H^n_{\mathrm{Lie}}(\mathfrak{g}; \mathfrak{a})$ is injective. In particular, $\mathfrak{k} \leq \mathfrak{g}$ is n.c.z. for \mathfrak{a} if and only if $\mathfrak{k} \leq \mathfrak{g}_u$ is n.c.z. for \mathfrak{a} .

Proof. The diagram

$$\begin{aligned} H^n_{\mathrm{Lie}}((\mathfrak{g},\mathfrak{k});\mathfrak{a}) &\xrightarrow{\kappa^n} H^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a}) &\longrightarrow H^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{a}) \otimes \mathbb{C} \xrightarrow{\cong} H^n_{\mathrm{Lie}}(\mathfrak{g}_{\mathbb{C}},\mathfrak{a}_{\mathbb{C}}) \\ & \downarrow \cong \\ H^n_{\mathrm{Lie}}((\mathfrak{g}_u,\mathfrak{k});\mathfrak{a}) \xrightarrow{\kappa^n} H^n_{\mathrm{Lie}}(\mathfrak{g}_u;\mathfrak{a}) &\longleftrightarrow H^n_{\mathrm{Lie}}(\mathfrak{g}_u;\mathfrak{a}) \otimes \mathbb{C} \xrightarrow{\mu} H^n_{\mathrm{Lie}}((\mathfrak{g}_u)_{\mathbb{C}},\mathfrak{a}_{\mathbb{C}}) \end{aligned}$$

commutes. This shows the claim.

Remark 4.5. The big advantage of $H^n_{\text{Lie}}((\mathfrak{g}_u, \mathfrak{k}); \mathfrak{a})$ over $H^n_{\text{Lie}}((\mathfrak{g}, \mathfrak{k}); \mathfrak{a})$ is that \mathfrak{g}_u is a compact Lie algebra, and thus $H^n_{\text{Lie}}((\mathfrak{g}_u, \mathfrak{k}); \mathfrak{a})$ can be accessed as the de Rham cohomology of a *compact* symmetric space. Let \widetilde{G}_u be the simply connected Lie group with Lie algebra \mathfrak{g}_u . Then the embedding $\mathfrak{k} \to \mathfrak{g}_u$ induces an embedding $\widetilde{K} \to \widetilde{G}_u$. In particular, $\pi_1(K)$ embeds into \widetilde{G}_u and we set $G_u := \widetilde{G}_u/\pi_1(K)$. From this it is clear that $K = \widetilde{K}/\pi_1(K)$ embeds into G_u and we will identify K via this embedding with a subgroup of G_u . We call the pair (G_u, K) the *dual pair* of (G, K). Note that the property that K embeds into G_u determines uniquely the quotient of \widetilde{G}_u that we have to take. If G is linear, then another method for obtaining G_u is to take a maximal compact subgroup of the complexification $G_{\mathbb{C}}$ that contains K. However, the complexification exists in the semi-simple case if and only if G is linear (cf. [HN12, Proposition 16.1.3]).

Now there is the canonical morphism $\nu: H^n_{\text{Lie}}((\mathfrak{g}_u, \mathfrak{k}); \mathfrak{a}) \to H^n_{\text{dR}}(G_u/K; \mathfrak{a}) \cong H^n_{\text{top}}(G_u/K; \mathfrak{a})$ that maps ω to the left invariant differential form on G_u/K with value ω in $T_e(G_u/K) \cong \mathfrak{g}_u/\mathfrak{k}$. This is an isomorphism, for instance by [GHV76, Proposition XI.1.I] or [FOT08, Theorem 1.28].

Example 4.6. a) If G is compact, then $\mathfrak{k} \leq \mathfrak{g}$ is clearly n.c.z. for each \mathfrak{a} .

b) Suppose G is complex simple, considered as a simple real Lie group and \mathfrak{a} is the trivial module. Then \mathfrak{k} is a compact real form of \mathfrak{g} and $\mathfrak{k} \oplus i\mathfrak{k}$ is a Cartan decomposition of \mathfrak{g} . Consequently, $\mathfrak{g}_u = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_i := \mathfrak{k}$ (we introduced the indices to distinguish the different copies of \mathfrak{k}). Then \mathfrak{k} embeds as \mathfrak{k}_1 into \mathfrak{g}_u and we have

$$H^n_{\text{Lie}}((\mathfrak{k}_1 \oplus \mathfrak{k}_2, \mathfrak{k}_1); \mathfrak{a}) \cong H^n(\text{Hom}(\Lambda^{\bullet} \mathfrak{k}_2, \mathfrak{a})^{\mathfrak{k}_1}) \cong H^n(\text{Hom}(\Lambda^{\bullet} \mathfrak{k}_2, \mathfrak{a})) \cong H^n_{\text{Lie}}(\mathfrak{k}_2, \mathfrak{a}),$$

which embeds into $H^n_{\text{Lie}}(\mathfrak{k}_1 \oplus \mathfrak{k}_2, \mathfrak{a})$ by the Künneth Theorem.

c) If \mathfrak{a} is the trivial \mathfrak{g} -module, then $\mathfrak{h} \leq \mathfrak{g}$ is n.c.z. for \mathfrak{a} if and only if $H^*_{\text{Lie}}((\mathfrak{g}, \mathfrak{h}); \mathfrak{a})$ is generated by 1 and elements of odd degree [GHV76, Theorem X.10.19]. Moreover, this is the case if and only if $H^*_{\text{Lie}}((\mathfrak{g}_u, \mathfrak{h}); \mathfrak{a})$ is n.c.z., which in turn is equivalent to $H^n_{\text{top}}(G_u/K; \mathfrak{a})$ being generated by 1 and elements of odd degree [GHV76, Theorem 11.5.VI].

Example 4.7 $(G = SL_{2q+1})$. Let $G = SL_p$ with $p = 2q + 1 \ge 3$ odd. Then $K = SO_p$ and $G_u = SU_p$. Then we have by [MT91, Theorem III.6.7] that $H^*_{top}(SU_p / SO_p; \mathbb{R})$ is generated by 1 and elements of odd degree. Thus $\mathfrak{k} \le \mathfrak{g}$ is n.c.z. by Example 4.6 c) and we have the description of $H^n(SL_{2q+1}(\mathbb{R}); U(1))$ from (14).

From Corollary 3.8 we also obtain immediately

Corollary 4.8. Suppose \mathfrak{a} is the trivial \mathfrak{g} -module. If $\mathfrak{k} \leq \mathfrak{g}$ is n.c.z. for \mathfrak{a} , then $H^n_{top}(BG; \mathfrak{a}^{\delta}) \to H^n_{top}(BG^{\delta}; \mathfrak{a}^{\delta})$ vanishes.

Note that Corollary 4.8, together with Example 4.6 a) and b) give the well-known vanishing of $H^n_{top}(BG; \mathfrak{a}^{\delta}) \to H^n_{top}(BG^{\delta};$ in case that G is either compact or G is complex and semi-simple with finitely many components. The latter is usually proved directly via Chern-Weil theory, cf. [Knu01, Section 5.1] or [Mil83] and the relation of the Chern-Weil homomorphism to the van Est cohomology [Bot73]. This is also implicit in here, as the next section shows.

5 Semi-simple Lie groups

In this section we will compute the characteristic homomorphism in terms of the Chern-Weil homomorphism of the compact dual of the symmetric space naturally associated to the non-compact symmetric space G/K. In particular, this will enable us to analyse cases in which not all characteristic homomorphisms vanish.

The Setting is the same as in Section 3, except that we assume, in addition, that \mathfrak{g} is semi-simple and the induced \mathfrak{g} module structure on \mathfrak{a} is trivial⁵. Moreover, we choose and fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of \mathfrak{g} . We will use the notation from Remark 4.3 and Remark 4.5.

Theorem 5.1. Suppose G is a connected finite-dimensional Lie group, that $\mathfrak{g} = L(G)$ is semi-simple and that G acts trivially on the quasi-complete locally convex space \mathfrak{a} . If $f: G_u/K \to BK$ is a classifying map for the principal K-bundle $G_u \to G_u/K$, then the diagram

$$H^{n}_{\operatorname{loc},s}(G;\Gamma) \xrightarrow{(\flat^{n})^{-1}} H^{n}_{\operatorname{top}}(BG;\Gamma) \xrightarrow{Bi^{*}} H^{n}_{\operatorname{top}}(BK;\Gamma)$$

$$\downarrow^{j_{*}}$$

$$\downarrow^{j_{*}}$$

$$H^{n}_{\operatorname{top},s}(BK;\mathfrak{a}) \xrightarrow{j_{*}}$$

$$\downarrow^{j_{*}}$$

$$H^{n}_{\operatorname{top},s}(G;\mathfrak{a}) \xrightarrow{(p^{*})^{-1}} H^{n}_{\operatorname{loc},s}((G,K);\mathfrak{a}) \xrightarrow{D^{n}} H^{n}_{\operatorname{Lie}}((\mathfrak{g},\mathfrak{k});\mathfrak{a}) \xrightarrow{\mu^{n}} H^{n}_{\operatorname{Lie}}((\mathfrak{g}_{u},\mathfrak{k});\mathfrak{a}) \xrightarrow{\nu^{n}} H^{n}_{\operatorname{top}}(G_{u}/K;\mathfrak{a})$$

$$(16)$$

commutes and all horizontal morphisms are in fact isomorphisms.

Note that the cohomology of G_u/K and the morphisms $f^* \colon H^n_{top}(BK; \mathfrak{a}) \to H^n_{top}(G_u/K; \mathfrak{a})$ are well understood, for instance for $\mathfrak{a} = \mathbb{R}$ and simple G (see for instance [GHV76, Mim95, FOT08]). We will list some examples and applications of the theorem in the next section.

Proof. That all horizontal morphisms are isomorphisms has been argued in the previous sections. We will deduce the commutativity of the diagram by establishing a sequence of commuting diagrams that will give (16) in the end. We first consider

which commutes by the naturality of Bi^* and b^n . From this it follows that $j_*: H^n_{\text{loc},s}(G; \Gamma) \to H^n_{\text{loc},s}(G; \mathfrak{a})$ factors through $H^n_{\text{top}}(BG, \mathfrak{a})$ and thus vanishes (by Hopf's Theorem) if n is odd. Since $j_*: H^n_{\text{top}}(BK, \Gamma) \to H^n_{\text{top}}(BK, \mathfrak{a})$ vanishes for n odd for the same reason it suffices to show the commutativity of (16) if n = 2m is even.

We now consider the algebraic Chern-Weil homomorphism

$$\operatorname{CW}^m_{(\mathfrak{q},\mathfrak{k})}\colon \operatorname{Hom}_{\mathbb{R}}(S^m\mathfrak{k};\mathfrak{a})^{\mathfrak{k}} \to H^{2m}_{\operatorname{Lie}}((\mathfrak{g},\mathfrak{k});\mathfrak{a}),$$

which is defined as follows (cf. [GHV76]). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} and let $\pi_{\mathfrak{p}} \colon \mathfrak{g} \to \mathfrak{p}$ and $\pi_{\mathfrak{k}} \colon \mathfrak{g} \to \mathfrak{k}$ be the corresponding projections. Note that both are morphisms of \mathfrak{k} -modules. Then we set

$$CW^{1}_{(\mathfrak{g},\mathfrak{k})} := \frac{1}{2}\pi^{*}_{\mathfrak{p}} \circ d_{CE} \circ \pi^{*}_{\mathfrak{k}}, \text{ i.e.}, \qquad CW^{1}_{(\mathfrak{g},\mathfrak{k})}(\lambda)(x,y) := \frac{1}{2}\lambda([\pi_{\mathfrak{p}}(x),\pi_{\mathfrak{p}}(y)])$$

⁵It would be desirable to have the results of this and the preceding section also for non-trivial coefficients. However, the techniques presented in this paper do not simply generalise to non-trivial coefficients for the following reasons: A^{δ} might not be a *G*-module any more; \mathfrak{a} is not a \mathfrak{g}_u -module in a natural way; the Weil homomorphism is not well-defined, since the identity $W^1_{(\mathfrak{a},\mathfrak{p})} = d_{CE} \circ \pi_{\mathfrak{p}}^* - \pi_{\mathfrak{p}}^* \circ d_{CE}$ does not hold any more.

Then $\mathrm{CW}^1_{(\mathfrak{g},\mathfrak{k})}(\lambda)$ is \mathfrak{k} -invariant and clearly satisfies $i_y(\mathrm{CW}^1_{(\mathfrak{g},\mathfrak{k})}(\lambda)) = 0$ for all $y \in \mathfrak{k}$. Thus we have

$$d_{\rm CE} \circ {\rm CW}^{1}_{(\mathfrak{g},\mathfrak{k})} = \pi^*_{\mathfrak{p}} \circ d_{\rm CE} \circ {\rm CW}^{1}_{(\mathfrak{g},\mathfrak{k})} = \pi^*_{\mathfrak{p}} \circ d_{\rm CE} \circ (d_{\rm CE} \circ \pi^*_{\mathfrak{k}} - \pi^*_{\mathfrak{k}} \circ d_{\rm CE}) = -\pi^*_{\mathfrak{p}} \circ d_{\rm CE} \circ \pi^*_{\mathfrak{k}} \circ d_{\rm CE} = 0$$
(18)

where $CW^{1}_{(\mathfrak{g},\mathfrak{k})} = d_{CE} \circ \pi^{*}_{\mathfrak{k}} - \pi^{*}_{\mathfrak{k}} \circ d_{CE}$ follows from

$$\lambda(\pi_{\mathfrak{k}}([\pi_{\mathfrak{p}}(x),\pi_{\mathfrak{p}}(y)])) = \lambda(\pi_{\mathfrak{k}}([x-\pi_{\mathfrak{k}}(x),y-\pi_{\mathfrak{k}}(y)])) = \lambda(\pi_{\mathfrak{k}}([x,y])) - \lambda([\pi_{\mathfrak{k}}(x),\pi_{\mathfrak{k}}(y)])$$

and the last identity in (18) follows from the fact that $d_{\rm CE}$ preserves $\operatorname{im}(\pi_{\mathfrak{k}}^*)$. Thus $\operatorname{CW}^1_{(\mathfrak{g},\mathfrak{k})}(\lambda)$ is closed and hence represents a class in $H^2_{\rm Lie}((\mathfrak{g},\mathfrak{k});\mathfrak{a})$. Since $H^{\rm even}_{\rm Lie}((\mathfrak{g},\mathfrak{k});\mathfrak{a})$ is commutative the case m = 1 determines a unique morphism of algebras $\operatorname{CW}^*_{(\mathfrak{g},\mathfrak{k})}$: $\operatorname{Hom}_{\mathbb{R}}(S^*\mathfrak{k},\mathfrak{a})^{\mathfrak{k}} \to H^{2*}_{\rm Lie}((\mathfrak{g},\mathfrak{k});\mathfrak{a})$.

The algebraic Chern-Weil homomorphism, together with the universal Chern-Weil isomorphism

$$\widetilde{\mathrm{CW}}^m \colon \operatorname{Hom}(S^m \mathfrak{k}; \mathfrak{a})^{\mathfrak{k}} \xrightarrow{\cong} H^{2m}_{\operatorname{top}}(BK; \mathfrak{a})$$

now give rise to a diagram

We claim that this diagram commutes as well. To this end, let

$$I^{2m} \colon H^{2m}_{\operatorname{Lie}}((\mathfrak{g}, \mathfrak{k}); \mathfrak{a}) \to H^{2m}_{\operatorname{vE}, s}(G/K; \mathfrak{a})$$

be the inverse of the van Est isomorphism, as described explicitly in [Gui80, n^o III.7.3] or in [Dup76, Proposition 1.5]. This has the property that

$$H^{2m}_{\text{Lie}}((\mathfrak{g},\mathfrak{k});\mathfrak{g}) \xrightarrow{I^{2m}} H^{2m}_{\text{vE},s}(G/K;\mathfrak{g}) \xrightarrow{\cong} H^{2m}_{\text{loc},s}((G,K);\mathfrak{g})$$
(20)

commutes, where the unlabelled isomorphism is induced by the inclusion $C_{vE,s}^n(G/K, \mathfrak{a}) \hookrightarrow C_{loc,s}^n((G, K), \mathfrak{a})$ of chain complexes (cf. [Fuc11, Section 7]). Now let $D \leq G$ be a cocompact lattice in G (which exists by [Bor63, Theorem C]) and let $\iota: D \to G$ denote the inclusion. Then the restriction $B\iota^*: H^n_{vE,s}(G; \mathfrak{a}) \to H^n_{vE}(D; \mathfrak{a}) = H^n_{gp}(D, \mathfrak{a})$ is injective by [Bla85, 15° Théorème]. Thus we have the commuting diagram

Since $b^{2m} \colon H^{2m}_{top}(BD; \mathfrak{a}) \to H^{2m}_{gp}(D; \mathfrak{a})$ is just the isomorphism between the cohomology of the classifying space and the bar resolution for the discrete group D, the inner diagram commutes by [Dup76, Corollary 1.3, Proposition 1.5 and Lemma 4.6]⁶. Since $B\iota^* \colon H^{2m}_{loc,s}(G; \mathfrak{a}) \to H^{2m}_{gp}(D; \mathfrak{a})$ is injective we thus conclude that the outer

⁶One can also argue without using a cocompact lattice by [Dup78, Theorem 9.12], but then one needs to assume that ξ^n is injective, which follows in [Bla85, 16° Théorème] from the existence of a cocompact lattice.

diagram, and thus diagram (19), commutes.

We now consider the algebraic Chern-Weil homomorphism $\operatorname{CW}_{(\mathfrak{g}_u,\mathfrak{k})}^m$ for \mathfrak{g}_u with respect to the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$. Note that the underlying vector spaces of \mathfrak{g} and \mathfrak{g}_u are the *same*, as well as the subspaces \mathfrak{p} and $i\mathfrak{p}$. Since the cochains representing $\operatorname{CW}_{(\mathfrak{g},\mathfrak{k})}^m(\lambda)$ and $\operatorname{CW}_{(\mathfrak{g}_u,\mathfrak{k})}^m(\lambda)$ only depend on the projections onto \mathfrak{k} and $i\mathfrak{p}$ (respectively \mathfrak{k} and \mathfrak{p}) we conclude that the diagram

commutes.

Since the Chern-Weil homomorphism CW_{π} for the bundle $\pi: G_u \to G_u/K$ factors through the universal Chern-Weil isomorphism and f^* we obtain by [GHV76, Section 8.26] the commuting diagram

$$\operatorname{Hom}_{\mathbb{R}}(S^{n}\mathfrak{k},\mathfrak{a})^{\mathfrak{k}} \xrightarrow{\operatorname{CW}^{n}_{(\mathfrak{g}_{u},\mathfrak{k})}} H^{2n}_{\operatorname{Lie}}((\mathfrak{g}_{u},\mathfrak{k});\mathfrak{a})$$

$$\cong \bigvee_{\widetilde{\operatorname{CW}}^{n}} \xrightarrow{\operatorname{CW}^{n}_{\pi}} \cong \bigvee_{\mu} \cdot \cdot \cdot$$

$$H^{2n}_{\operatorname{top}}(BK;\mathfrak{a}) \xrightarrow{f^{*}} H^{2n}_{\operatorname{top}}(G_{u}/K;\mathfrak{a})$$

$$(22)$$

If we paste the above diagrams (17), (19), (21) and (22), then the outer diagram yields precisely (16). This finishes the proof.

Remark 5.2. Note that $j_*: H^n_{top}(BK;\Gamma) \to H^n_{top}(BK;\mathfrak{a})$ is very well-behaved in this particular case. If $H_{odd}(BK;\Gamma)$ is finitely generated and torsion free, then we get

from the Universal Coefficient Theorem. Thus j_* is injective in this case. If, moreover, \mathfrak{a} is separable and Γ is countable, then it is free [Nee02, Remark 9.5 (c)], and thus $\operatorname{Hom}(H^n(BK);\Gamma)$ injects into $\operatorname{Hom}(H^n(BK);\mathfrak{a})$. If $\mathfrak{a} = \mathbb{R}^n$ and Γ is a lattice in \mathbb{R}^n , then so is $\operatorname{Hom}(H^n(BK);\Gamma)$ in $\operatorname{Hom}(H^n(BK);\mathbb{R}^n)$ and a basis for Γ then gives a basis for $\operatorname{Hom}(H^n(BK);\Gamma)$. All the above assumptions are in particular satisfied for $\mathfrak{a} = \mathbb{R}, \Gamma = \mathbb{Z}$ and $K = \operatorname{U}_q, \operatorname{SU}_q, \operatorname{Sp}_q, \operatorname{SO}_q$ (see [Spa66, Theorem 5.5.10], [Swi75, Theorem 16.17], [MT91, Corollary III.3.11] and [Bro82]).

6 Examples

We will stick in this section to examples of simple linear Lie groups and the trivial coefficient modules \mathbb{Z} , \mathbb{R} and $U(1) = \mathbb{R}/\mathbb{Z}$. We will calculate the characteristic homomorphisms $\varepsilon^n \colon H^n_{\text{loc}}(G;\mathbb{Z}) \to H^n_{\text{loc}}(G;\mathbb{R})$ via the commuting diagram

from Theorem 5.1 (and identify ε^n with $f^* \circ j_*$ by this). This will then give complete information on $H^n_{\text{loc}}(G; U(1))$ (as abelian group) via the long exact sequence from Section 2.

To this end we first recall the following facts from [GHV76, Section 11.5] on the cohomology of a homogeneous space G/H of a general compact connected Lie group G with closed and connected subgroup H. Let $\pi: G \to G/H$ denote the corresponding principal H-bundle, and let $f: G/H \to BH$ be a classifying map for π . Then there is an isomorphism

$$\Phi \colon H^*_{\operatorname{top}}(G/H; \mathbb{R}) \xrightarrow{\cong} A_{\pi} \otimes \Lambda \widehat{P}_{\pi}$$

of graded algebras, where $A_{\pi} := \mathrm{CW}_{\pi}^*(\mathrm{Hom}_{\mathbb{R}}(S^*\mathfrak{h},\mathbb{R})^{\mathfrak{h}}) = \mathrm{im}(f^* \colon H^*_{\mathrm{top}}(BH;\mathbb{R}) \to H^*_{\mathrm{top}}(G/H;\mathbb{R}))$ is the image of the Chern-Weil homomorphism of π and $\hat{P}_{\pi} := P_G \cap \mathrm{im}(\pi^* \colon H^*_{\mathrm{top}}(G/H;\mathbb{R}) \to H^*_{\mathrm{top}}(G;\mathbb{R}))$ for $P_G \leq H^*_{\mathrm{top}}(G;\mathbb{R})$ the graded subspace of primitive elements. Moreover, Φ makes the diagram

$$\begin{array}{c} A_{\pi} & \longrightarrow A_{\pi} \otimes \Lambda \widehat{P}_{\pi} & \longrightarrow H^{2*}_{\mathrm{top}}(G; \mathbb{R}) \\ \uparrow^{CW_{\pi}} & \uparrow^{\Phi} & \| \\ \mathrm{Hom}_{\mathbb{R}}(S^{*}\mathfrak{h}, \mathbb{R})^{\mathfrak{h}} \xrightarrow{\mathrm{CW}_{\pi}^{*}} H^{2*}_{\mathrm{top}}(G/H; \mathbb{R}) \xrightarrow{\pi^{*}} H^{2*}_{\mathrm{top}}(G; \mathbb{R}) \\ & \downarrow^{\widetilde{\mathrm{CW}}^{*}} & \| \\ H^{2*}_{\mathrm{top}}(BH; \mathbb{R}) \xrightarrow{f^{*}} H^{2*}_{\mathrm{top}}(G/H; \mathbb{R}) \end{array}$$

commute.

Example 6.1 ($G = SL_p(\mathbb{C})$). Then $K = SU_p$ and $G_u = SU_p \times SU_p$. By Example 4.6 b) we have that $\mathfrak{k} \leq \mathfrak{g}$ is n.c.z. for all \mathfrak{a} and thus Proposition 4.2 yields

$$H^{n}(\mathrm{SL}_{p}(\mathbb{C}), U(1)) \cong H^{n}_{\mathrm{Lie}}((\mathfrak{sl}_{p}(\mathbb{C}), \mathfrak{su}_{p}), \mathbb{R}) \oplus H^{n}_{\mathrm{top}}(B \operatorname{SU}_{p}; \mathbb{Z}) \cong H^{n}_{\mathrm{top}}(\operatorname{SU}_{p}; \mathbb{R}) \oplus H^{n}_{\mathrm{top}}(B \operatorname{SU}_{p}; \mathbb{Z})$$

for each $n \ge 1$ (cf. Remarks 4.3 and 4.5), and the groups on the right are well-known (see for instance [FOT08, Corollary 1.86] and [Hat02, Corollary 4D.3]).

We now run through some interesting and illustrative cases in which G is a connected non-compact real form of a simple complex Lie group, K is the maximal compact of G and G_u the maximal compact of the complexification $G_{\mathbb{C}}$ (see [Hel78, Chapter X] for notation and details).

Example 6.2 $(G = SL_p(\mathbb{R}))$. Then $K = SO_p$ and $G_u = SU_p$. The case $p = 2q + 1 \ge 3$ has been treated in Example 4.7. If $p = 2q \ge 4$ is even, then

$$H^*_{\operatorname{top}}(BK;\mathbb{R}) \cong \operatorname{Hom}_{\mathbb{R}}(S^*\mathfrak{so}_{2q};\mathbb{R})^{\mathfrak{so}_{2q}} = \Lambda(P_1,...,P_{q-1},E_q)$$

is generated by the Pontryagin classes $P_1, ..., P_{q-1}$, where P_i is of degree 4i, and the Euler class E_q of degree 2q. Moreover, $E_q^2 = P_q$ [MS74, Theorem 15.9]. We now consider the kernel of CW_{π}^* . By [GHV76, Proposition 10.6.III] it is generated (as an algebra without unit) by the image of i^* : $Hom_{\mathbb{R}}(S^*\mathfrak{su}_{2q}, \mathbb{R}) \to Hom_{\mathbb{R}}(S^*\mathfrak{so}_{2q}, \mathbb{R})$. From [GHV76, Example 11.11.4] we get

$$i^*(C_i) = \begin{cases} 0 & \text{if } i \text{ odd} \\ (-1)^{i/2} P_{i/2} & \text{if } i \text{ even} \end{cases}$$

where $C_2, ..., C_{2q}$ denote the Chern classes (C_1 is missing since we consider SU_p , not U_p). Thus we have $A_{\pi} = \mathbb{R}[E_q]/(E_q^2)$ and by [GHV76, Proposition 10.26.VII] we have that ker $(i^*) \cong \widehat{P}_{\pi}$ is generated by the suspensions of the odd Chern classes $\widetilde{C}_3, \widetilde{C}_5, ..., \widetilde{C}_{2q-1}$ with $\widetilde{C}_i \in H^{2i-1}_{\mathrm{top}}(\mathrm{SU}_p; \mathbb{R})$. Consequently,

$$H^*_{\text{top}}(\mathrm{SU}_p / \mathrm{SO}_p; \mathbb{R}) \cong \left(\mathbb{R}[E_q] / (E_q^2) \right) \otimes \Lambda(\widetilde{C}_3, \widetilde{C}_5, ..., \widetilde{C}_{2q-1}).$$

In particular, $f^*: H^{2q}_{top}(B \operatorname{SO}_{2q}; \mathbb{R}) \to H^{2q}_{top}(\operatorname{SU}_{2q} / \operatorname{SO}_{2q}; \mathbb{R})$ does not vanish on the Euler class. By Remark 5.2, $j_*: H^n_{top}(B \operatorname{SO}_{2q}; \mathbb{Z}) \to H^n_{top}(B \operatorname{SO}_{2q}; \mathbb{R})$ is injective. Thus the characteristic homomorphism

$$\varepsilon^{2q} \colon H^{2q}(\mathrm{SL}_{2q}(\mathbb{R});\mathbb{Z}) \to H^{2q}(\mathrm{SL}_{2q}(\mathbb{R});\mathbb{R})$$

does not vanish on the integral Euler class and $0 \neq \varepsilon^{2q}(E_q) \in H^{2q}(\mathrm{SL}_{2q}(\mathbb{R});\mathbb{R})$ is flat (cf. Remark 3.10).

Example 6.3 $(G = SU_{2p}^*)$. Then $K = Sp_p$ and $G_u = SU_{2p}$. From [MT91, Theorem III.6.7] one sees that $H^*_{top}(G_u/K;\mathbb{R})$ is generated by 1 and elements of odd degree. Thus \mathfrak{sp}_p is n.c.z. in \mathfrak{su}_{2p} by Example 4.6 c). Consequently, all characteristic homomorphisms

$$\varepsilon^n \colon H^n(\mathrm{SU}_{2p}^*;\mathbb{Z}) \to H^n(\mathrm{SU}_{2p}^*;\mathbb{R})$$

vanish and we have the description of $H^n_{top}(SU^*_{2p}; U(1))$ from (14).

Example 6.4 ($G = \operatorname{Sp}_p(\mathbb{R})$). Then $K = U_p$ and $G_u = \operatorname{Sp}_p$. From [MT91, Theorem III.6.9 (1)] one sees that $H^*_{\operatorname{top}}(G_u/K;\mathbb{R})$ is evenly graded and that

$$f^* \colon H^*_{\operatorname{top}}(B\operatorname{U}_p; \mathbb{R}) \to H^*_{\operatorname{top}}(\operatorname{Sp}_p/\operatorname{U}_p; \mathbb{R})$$

is surjective with kernel generated by the alternating products $\{\sum_{i+j=2k}(-1)^i C_i C_j \mid k \geq 1\}$ of the Chern classes. By Remark 5.2, $j_*: H^n_{top}(B \operatorname{U}_p; \mathbb{Z}) \to H^n_{top}(B \operatorname{U}_p; \mathbb{R})$ is injective. Thus the characteristic homomorphism

$$\varepsilon^n \colon H^n(\mathrm{Sp}_p(\mathbb{R});\mathbb{Z}) \to H^n(\mathrm{Sp}_p(\mathbb{R});\mathbb{R})$$

has as kernel precisely the integral linear combinations of the alternating products $\sum_{i+j=2k} (-1)^i C_i C_j$ of the *integral* Chern classes (for $k \ge 1$). In particular, $0 \ne \varepsilon^{2n}(C_n) \in H^{2n}(\operatorname{Sp}_p(\mathbb{R});\mathbb{R})$ is flat (cf. Remark 3.10).

With [MT91, Theorem III.6.9 (2+3)], a similar argument also applies to $G = SU_{(p,q)}$ and $G = Sp_{(p,q)}$.

Remark 6.5. The results on the non-vanishing of the characteristic morphisms on the Euler class or the Chern classes are a refinement of some well-known identities in the abstract group cohomology $H^n_{\rm gp}(G;\mathbb{R})$ for $G = \operatorname{SL}_p(\mathbb{R})$ and $G = \operatorname{Sp}_n(\mathbb{R})$ (see [Mil58] and [Dup78, Section 9]). What follows from re results of this paper is that these classes do not only live in $H^n_{\rm gp}(G;\mathbb{R})$, but that they lift to the topological group cohomology $H^n(G;\mathbb{R})$.

We end this section with establishing the following stability result.

Proposition 6.6. The natural homomorphisms $SL_p(\mathbb{C}) \to SL_{p+1}(\mathbb{C})$ and $SL_p(\mathbb{R}) \to SL_{p+1}(\mathbb{R})$ induce isomorphisms

$$H^{n}(\mathrm{SL}_{p+1}(\mathbb{C}); U(1)) \xrightarrow{\cong} H^{n}(\mathrm{SL}_{p}(\mathbb{C}); U(1)) \quad and \quad H^{n}(\mathrm{SL}_{p+1}(\mathbb{R}); U(1)) \xrightarrow{\cong} H^{n}(\mathrm{SL}_{p}(\mathbb{R}); U(1))$$

for sufficiently large p.

Proof. From the descriptions of

$$H^n_{\text{top}}(B\operatorname{SO}_p;\mathbb{Z}) \cong H^n_{\text{top}}(B\operatorname{SL}_p(\mathbb{R});\mathbb{Z})$$
 and $H^n_{\text{top}}(\operatorname{SU}_p/\operatorname{SO}_p;\mathbb{R}) \cong H^n_{\text{vE}}(\operatorname{SL}_p(\mathbb{R});\mathbb{R})$

in [Bro82] and [MT91, Theorem III.6.7 (2)] on sees that $\mathrm{SL}_p(\mathbb{R}) \to \mathrm{SL}_{p+1}(\mathbb{R})$ induces an isomorphism for sufficiently big p. Thus the same holds for $H^n(\mathrm{SL}_p(\mathbb{R}); U(1))$ by the long exact sequence (7) from Remark 2.1 and the Five Lemma. The argument for $\mathrm{SL}_p(\mathbb{C})$ is exactly the same (cf. [MT91, Corollary III.3.11 and Theorem III.5.5]).

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