ON THE HIGHER TOPOLOGICAL HOCHSCHILD HOMOLOGY OF $\mathbb{F}_p$ AND COMMUTATIVE $\mathbb{F}_p$-GROUP ALGEBRAS

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Abstract. We extend Torleif Veen's calculation of higher topological Hochschild homology $\mathsf{THH}_*^{[n]}(\mathbb{F}_p)$ from $n \leqslant 2p$ to $n \leqslant 2p+2$ for p odd, and from $n = 2$ to $n \leqslant 3$ for $p = 2$. We calculate higher Hochschild homology $\mathsf{HH}_*^{[n]}(k[x])$ over k for any integral domain k, and $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x]/x^{p^\ell})$ for all $n > 0$. We use this and étale descent to calculate $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[G])$ for all $n > 0$ for any cyclic group G, and therefore also for any finitely generated abelian group G. We show a splitting result for higher THH of commutative $\mathbb{F}_p$-group algebras and use this technique to calculate higher topological Hochschild homology of such group algebras for as large an n as $\mathsf{THH}_*^{[n]}(\mathbb{F}_p)$ is known for.

1. Introduction

Given a commutative ring R and an R-module M, Jean-Louis Loday introduced a functor $\mathcal{L}(R, M)$ which takes a based simplicial set $X.$ to the simplicial R-module which consists in degree n of M tensored with one copy of R for each element in $X_n \setminus \{*\}$. The homotopy groups of the image of the Loday functor turn out to be independent of the simplicial structure used for $X.$; they depend only on its homotopy type.

Applying this functor to the usual simplicial model of $\mathbb{S}^1$ with one non-degenerate 0-cell and one non-degenerate 1-cell, we get the classical Hochschild complex whose homology is $\mathsf{HH}_*(R; M)$. Extending this, the higher topological Hochschild homology groups $\mathsf{HH}_*^{[n]}(R; M)$ were defined by Teimuraz Pirashvili [P] as the homotopy groups of $\mathcal{L}(R, M)$ evaluated on $\mathbb{S}^n$.

Morten Brun, Gunnar Carlsson, and Bjørn Dundas introduced a topological version of $\mathcal{L}(R, M)$ for a ring spectrum R and an R-module spectrum M [BCD]. When evaluated on $\mathbb{S}^n$, it yields the spectrum $\mathsf{THH}^{[n]}(R; M)$, the higher topological Hochschild homology of R with coefficients in M. For $M = R$ with the obvious action by multiplication M is omitted from the notation.

Higher (topological) Hochschild homology features in several different contexts. There are stabilization maps in the algebraic context

$$\mathsf{HH}_*^{[1]}(R) \to \mathsf{HH}_{*+1}^{[2]}(R) \to \ldots \to H\Gamma_{*-1}(R)$$

starting with Hochschild homology and ending with Gamma homology in the sense of Alan Robinson and Sarah Whitehouse [RW]. In the topological setting they start with $\mathsf{THH}(R)$ and end in topological André-Quillen homology, $\mathsf{TAQ}(R)$,

$$\mathsf{THH}_*^{[1]}(R) \to \mathsf{THH}_{*+1}^{[2]}(R) \to \ldots \to \mathsf{TAQ}_{*-1}(R).$$

The k-invariants of commutative ring spectra live in topological André-Quillen cohomology [Ba] and obstructions for E_∞-ring structures on spectra live in Gamma cohomology [R, GH], so these two cohomology theories are of great interest.

The evaluation of the Loday functor on higher dimensional tori is the same as iterated topological Hochschild homology and this features in the program for detecting red-shift in

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algebraic K-theory. Calculations of iterated topological Hochschild homology use higher THH as an important ingredient.

Work of Benoit Fresse [F] identifies Hochschild homology of order n (in the disguise of E_n-homology) with the homology groups of an algebraic n-fold bar construction, thus $\mathsf{HH}_*^{[n]}(R)$ can be viewed as the homology of an n-fold algebraic delooping.

In his thesis Torleif Veen [V1, V2] used a decomposition result for $\mathcal{L}(R, M)$ to calculate $\mathsf{THH}_*^{[n]}(\mathbb{F}_p) = \pi_*(\mathsf{THH}^{[n]}(\mathbb{F}_p))$ for all $n \leqslant 2p$ and any odd prime p. For small n such calculations were earlier done by John Rognes. Veen inductively sets up a spectral sequence of Hopf algebras calculating $\mathsf{THH}_*^{[n]}(\mathbb{F}_p)$ from $\mathsf{THH}_*^{[n-1]}(\mathbb{F}_p)$ with the base case $\mathsf{THH}_*^{[1]}(\mathbb{F}_p)$ being known by work of Marcel Bökstedt [B]. Veen explains why the spectral sequence has to collapse for $n \leqslant 2p$. By a careful analysis of the structure of the spectral sequence, motivated by computer calculations, we show that it actually collapses for $n \leqslant 2p + 2$ (Proposition 4.4), thus getting a calculation of $\mathsf{THH}_*^{[n]}(\mathbb{F}_p)$ for those n. The computer analysis also found potential nontrivial differentials in the spectral sequence when $n = 2p + 3$. We actually believe that the differential will end up vanishing for all n. We intend to return to this question in a future paper with Maria Basterra and Michael Mandell. At $p = 2$ Veen calculates $\mathsf{THH}_*^{[n]}(\mathbb{F}_2)$ up to $n = 2$. We include the $n = 3$ case and also show that the generator in $\mathsf{THH}_2(\mathbb{F}_2)$ stabilizes to a non-trivial element in the first topological André-Quillen homology group of $\mathbb{F}_2$ (Proposition 5.4).

We prove that for an $\mathbb{F}_p$-algebra A and an abelian group G,

$$\mathsf{THH}_*^{[n]}(A[G]) \cong \mathsf{THH}_*^{[n]}(A) \otimes \mathsf{HH}_*^{[n]}(\mathbb{F}_p[G]).$$

Using this, we calculate $\mathsf{THH}_*^{[n]}(\mathbb{F}_p[G])$ for any finitely generated abelian group G for $n \leqslant 2p+2$. To extend this to general abelian groups, observe that higher Hochschild homology commutes with direct limits.

The actual calculations of higher Hochschild homology that we do are of $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x])$ and of $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x]/x^m)$ for any m.

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2. Comparing the bar construction and its homology for some basic algebras

We consider the two-sided bar construction $\mathsf{B}(k, A, k)$ where k is a commutative ring and $A = k[x]$ or $A = k[x]/x^m$. The generator x will be allowed to be of any even degree; if $A = k[x]/x^2$ or $2 = 0$ in k, x can be of any degree. Note that since k is commutative, A is also a graded commutative ring, and so $\mathsf{B}(k, A, k)$ is a differential graded augmented commutative k-algebra, with multiplication given by the shuffle product.

Our goal in this section is to establish quasi-isomorphisms between $\mathsf{B}(k, A, k)$ and its homology ring $\mathrm{Tor}_*^A(k, k)$ which are maps of differential graded augmented k-algebras. (We use the zero differential on the homology ring.) The quasi-isomorphisms are adapted from [LL], where similar maps are studied on the Hochschild complex for variables x which have to be of degree zero, but may satisfy other monic polynomial equations. The reason that we need these quasi-isomorphisms is that in Section 8 we will be looking at iterated bar constructions of the form $\mathsf{B}(k, \mathsf{B}(k, A, k), k)$. If we know that there is some differential graded algebra C with quasi-isomorphisms that are algebra maps between $\mathsf{B}(k, A, k)$ and C, we then get quasi-isomorphisms that are algebra maps between $\mathsf{B}(k, \mathsf{B}(k, A, k), k)$ and $\mathsf{B}(k, C, k)$. In the cases we study, the rings $C = \mathrm{Tor}_*^A(k, k)$ are very simple, and in fact involve rings of the form of the A's we deal with in this section, or tensor products of them. Thus the $\mathsf{B}(k, C, k)$ can again be compared to simpler graded algebras, and the process can continue.

The following propositions also re-prove what $\mathrm{Tor}_*^A(k,k)$ is for the A's we are interested in, but those are old and familiar results; our motivation is understanding the bar complex $\mathsf{B}(k,A,k)$ as a differential graded algebra, not just its homology ring.

We will assume that our ground ring k is an integral domain to simplify the proofs – in this paper we will only use the calculations for $k=\mathbb{F}_p$.

We will use the notation $\Lambda(y)=k[y]/y^2$ for the exterior algebra on y over k, and $\Gamma(y)$ for the divided power algebra on y over k, spanned over k by elements $\gamma_i(y)$, $i\geqslant 0$, with $\gamma_i(y)\cdot\gamma_j(y)=\binom{i+j}{i}\gamma_{i+j}(y)$.

Proposition 2.1. *Let k be an integral domain, and let x be of even degree. Then there exist quasi-isomorphisms*

$$\pi\colon \mathsf{B}(k,k[x],k)\to\Lambda(\epsilon x)$$

and

$$\mathrm{inc}\colon \Lambda(\epsilon x)\to\mathsf{B}(k,k[x],k)$$

which are maps of differential graded augmented commutative k-algebras, with $|\epsilon x|=|x|+1$.

Proof. We define the quasi-isomorphisms as follows: Let $\pi:\mathsf{B}(k,k[x],k)\to\Lambda(\epsilon x)$ be given by $\pi(1\otimes 1)=1$,

$$\pi(1\otimes x^i\otimes 1)=\begin{cases}\epsilon x & \text{if } i=1,\\ 0 & \text{otherwise.}\end{cases}$$

and $\pi=0$ on $\mathsf{B}_n(k,k[x],k)$ for $n>1$. Let $\mathrm{inc}:\Lambda(\epsilon x)\to\mathsf{B}(k,k[x],k)$ be given by $\mathrm{inc}(1)=1\otimes 1$ and $\mathrm{inc}(\epsilon x)=1\otimes x\otimes 1$. Then π and inc are chain maps, and $\pi\circ\mathrm{inc}=\mathrm{id}_{\Lambda(\epsilon x)}$. Therefore inc_* induces an isomorphism from $\Lambda(\epsilon x)$ to a direct summand of $H_*(\mathsf{B}(k,k[x],k))=\mathrm{Tor}_*^{k[x]}(k,k)$, and π_* projects back onto that summand. But the resolution

$$0\to\Sigma^{|x|}k[x]\xrightarrow{\cdot x}k[x]$$

of k shows that the rank of $\mathrm{Tor}_*^{k[x]}(k,k)$ over k in each degree is equal to that of $\Lambda(\epsilon x)$, and since k is an integral domain, the direct summand must then be equal to all of $H_*(\mathsf{B}(k,k[x],k))$. Thus π and inc are quasi-isomorphisms. In this case, both maps preserve the multiplication because both $\mathsf{B}(k,k[x],k)$ and $\Lambda(\epsilon x)$ are graded commutative, so the square of anything in odd degree must be zero. □

Proposition 2.2. *Let k be an integral domain, let $m\geqslant 2$ be an integer, and let x be of even degree. Then there exist quasi-isomorphisms*

$$\pi\colon \mathsf{B}(k,k[x]/x^m,k)\to\Lambda(\epsilon x)\otimes\Gamma(\varphi^0x)$$

and

$$\mathrm{inc}\colon \Lambda(\epsilon x)\otimes\Gamma(\varphi^0x)\to\mathsf{B}(k,k[x]/x^m,k)$$

which are maps of differential graded augmented commutative k-algebras, with $|\epsilon x|=|x|+1$ and $|\varphi^0x|=2+m|x|$.

Proof. Let $\pi:\mathsf{B}(k,k[x]/x^m,k)\to\Lambda(\epsilon x)\otimes\Gamma(\varphi^0x)$ be given by

$$\pi(1\otimes x^{a_1}\otimes\cdots\otimes x^{a_n}\otimes 1)=\begin{cases}x^{a_1+a_2-m}\cdots x^{a_{n-1}+a_n-m}\,\gamma_{(\frac{n}{2})}(\varphi^0x) & n\text{ even},\\ x^{a_1-1}x^{a_2+a_3-m}\cdots x^{a_{n-1}+a_n-m}\epsilon x\cdot\gamma_{(\frac{n-1}{2})}(\varphi^0x) & n\text{ odd},\end{cases}$$

where $0\leqslant a_i<m$ and where we interpret $x^s=0$ for $s\neq 0$: for $s<0$, this is because we define it to be so; for $s>0$, this is because $k[x]/x^m$ acts by first applying the augmentation. Therefore, if n is even, we get $\gamma_{(\frac{n}{2})}(\varphi^0x)$ if and only if $a_1+a_2=m,\ a_3+a_4=m,\ \dots,\ a_{n-1}+a_n=m$ and otherwise we get zero. For odd n we get $\epsilon x\cdot\gamma_{(\frac{n-1}{2})}(\varphi^0x)$ if and only if $a_1=1,\ a_2+a_3=m,\ \dots,\ a_{n-1}+a_n=m$ and zero otherwise. To see that π is a chain map, we only need to show that it sends boundaries to zero, which can be checked directly using the stringent conditions under which a monomial is sent to a nonzero element.

Let $\mathrm{inc}\colon \Lambda(\epsilon x)\otimes\Gamma(\varphi^0 x)\to \mathsf{B}(k,k[x]/x^m,k)$ be given by

$$\mathrm{inc}(\gamma_i(\varphi^0 x)) = 1\otimes (x^{m-1}\otimes x)^{\otimes i}\otimes 1 \in \mathsf{B}_{2i}(k,k[x]/x^m,k)$$

and

$$\mathrm{inc}(\epsilon x\cdot\gamma_i(\varphi^0 x)) = 1\otimes x\otimes (x^{m-1}\otimes x)^{\otimes i}\otimes 1 \in \mathsf{B}_{2i+1}(k,k[x]/x^m,k).$$

Since $x^m=0$ and since the augmentation sends x to zero, every face map d_j vanishes on the image of inc, so clearly the boundary vanishes too and inc is a chain map.

As before, we get that $\pi\circ\mathrm{inc} = \mathrm{id}_{\Lambda(\epsilon x)\otimes\Gamma(\varphi^0 x)}$, and since the periodic resolution

$$\ldots\to\Sigma^{(m+1)|x|}k[x]/x^m \xrightarrow{\cdot x} \Sigma^{m|x|}k[x]/x^m \xrightarrow{\cdot x^{m-1}} \Sigma^{|x|}k[x]/x^m \xrightarrow{\cdot x} k[x]/x^m$$

shows that $\Lambda(\epsilon x)\otimes\Gamma(\varphi^0 x)$ has the same rank over k in each dimension as $H_*(\mathsf{B}(k,k[x]/x^m,k)) = \mathrm{Tor}_*^{k[x]/x^m}(k,k)$, by the same argument as in Proposition 2.1, π and inc are quasi-isomorphisms.

To show that π is multiplicative, consider $\pi((1\otimes x^{a_1}\otimes\cdots\otimes x^{a_\ell}\otimes 1)\cdot(1\otimes x^{a_{\ell+1}}\otimes\cdots\otimes x^{a_{\ell+n}}\otimes 1))$ which is the sum over all (ℓ,n)-shuffles σ of

$$\mathrm{sgn}(\sigma)\pi(1\otimes x^{a_{\sigma(1)}}\otimes\cdots\otimes x^{a_{\sigma(\ell+n)}}\otimes 1).$$

In the case where ℓ and n are both even, observe that this term is equal to $\mathrm{sgn}(\sigma)\gamma_{(\frac{\ell+n}{2})}(\varphi^0 x)$ if and only if $a_{\sigma(1)}+a_{\sigma(2)}=m,\ldots,\ a_{\sigma(\ell+n-1)}+a_{\sigma(\ell+n)}=m$. If there is some pair $2i-1$, $2i$ for which $\sigma(2i-1)$ is in one of the sets $\{1,\ldots,\ell\}$, $\{\ell+1,\ldots,\ell+n\}$ and $\sigma(2i)$ is in the other, the term associated to σ will cancel with the term associated to the permutation which is exactly like σ except for switching $\sigma(2i-1)$ and $\sigma(2i)$. Thus we will be left with terms associated with shuffles σ which shuffle pairs of coordinates, and for these it is clear that $\pi(1\otimes x^{a_{\sigma(1)}}\otimes\cdots\otimes x^{a_{\sigma(\ell+n)}}\otimes 1)\neq 0$ if and only if both $\pi(1\otimes x^{a_1}\otimes\cdots\otimes x^{a_\ell}\otimes 1)\neq 0$ and $\pi(1\otimes x^{a_{\ell+1}}\otimes\cdots\otimes x^{a_{\ell+n}}\otimes 1)\neq 0$. And there will be exactly $\binom{\frac{\ell+n}{2}}{\frac{\ell}{2}}$ (ℓ,n)-shuffles σ with $\sigma(2i)=\sigma(2i-1)+1$ for all i.

A similar argument works if ℓ is odd and n is even. Then the terms corresponding to shuffles σ which do not satisfy $\sigma(1)=1$ and $\sigma(2i+1)=\sigma(2i)+1$ for all $1\leqslant i<(\ell+n)/2$ will cancel in pairs, and the terms corresponding to the $\binom{\frac{\ell+n-1}{2}}{\frac{\ell-1}{2}}$ shuffles which do will be nonzero if and only if the images of both factors will be nonzero. Commutativity then establishes multiplicativity for the case ℓ even, n odd. If both ℓ and n are odd then all (ℓ,n)-shuffles σ will have a mixed pair $2i-1$, $2i$ for which $\sigma(2i-1)$ is in one of the sets $\{1,\ldots,\ell\}$, $\{\ell+1,\ldots,\ell+n\}$ and $\sigma(2i)$ is in the other, so all the terms will cancel and so the product will map to zero, which is also the product of the images of the factors.

To show that inc is multiplicative, it suffices to show that $\mathrm{inc}(\epsilon x)\cdot\mathrm{inc}(\gamma_i(\varphi^0 x)) = \mathrm{inc}(\epsilon x\cdot\gamma_i(\varphi^0 x))$ and that $\mathrm{inc}(\gamma_i(\varphi^0 x))\cdot\mathrm{inc}(\gamma_j(\varphi^0 x)) = \mathrm{inc}(\gamma_i(\varphi^0 x)\cdot\gamma_j(\varphi^0 x)) = \binom{i+j}{i}\mathrm{inc}(\gamma_{i+j}(\varphi^0 x))$. The first claim follows from the fact that shuffles which allow two adjacent x's from different factors cancel in pairs, leaving only the unique $(1,2i)$-shuffle σ with $\sigma(1)=1$. The second claim follows from the fact that shuffles which do not preserve the pairs $x^{m-1}\otimes x$ cancel in pairs, and there are $\binom{i+j}{i}$ shuffles which preserve the pairs. Thus both quasi-isomorphisms respect the multiplication.

□

Proposition 2.3. *Let k be an integral domain, let x be of odd degree and let $\rho^0 x$ be an element with $|\rho^0 x| = |x|+1$. Then there exist quasi-isomorphisms*

$$\pi\colon \mathsf{B}(k,\Lambda(x),k)\to\Gamma(\rho^0 x)$$

and

$$\mathrm{inc}\colon \Gamma(\rho^0 x)\to\mathsf{B}(k,\Lambda(x),k)$$

which are maps of differential graded augmented commutative k-algebras.

If $k=\mathbb{F}_2$, this proposition and its proof also work if x has even degree, and the result agrees with the result of Proposition 2.2 for $m=2$.

Proof. We use the same quasi-isomorphisms as in Proposition 2.2, and the argument showing that they are quasi-isomorphisms is the same as well, but the multiplicative structure is different and much easier to analyze. The maps from Proposition 2.2 give, in the case of $m = 2$,

$$\pi(1 \otimes x^{a_1} \otimes \cdots \otimes x^{a_n} \otimes 1) = \begin{cases} \gamma_n(\rho^0 x) & \text{if } \; a_i = 1 \text{ for all } 1 \leqslant i \leqslant n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{inc}(\gamma_n(\rho^0 x)) = 1 \otimes x^{\otimes n} \otimes 1.$$

Since x is of odd degree,

$$(1 \otimes x^{\otimes i} \otimes 1) \cdot (1 \otimes x^{\otimes j} \otimes 1) = \binom{i+j}{i}(1 \otimes x^{\otimes(i+j)} \otimes 1)$$

for all $i, j \geqslant 0$ and so both π and inc respect multiplication. □

Notation 2.4.

(a) If $k = \mathbb{F}_p$, we can decompose the divided power algebra as

$$\Gamma(\rho^0 x) \cong \bigotimes_{i \geqslant 0} \mathbb{F}_p[\gamma_{p^i}(\rho^0 x)]/(\gamma_{p^i}(\rho^0 x))^p$$

and we will denote the generators $\gamma_{p^i}(\rho^0 x)$ by $\rho^i x$.

(b) Similarly, if $k = \mathbb{F}_p$

$$\Gamma(\varphi^0 x) \cong \bigotimes_{i \geqslant 0} \mathbb{F}_p[\gamma_{p^i}(\varphi^0 x)]/(\gamma_{p^i}(\varphi^0 x))^p$$

and $\varphi^i x$ is short for the generator $\gamma_{p^i}(\varphi^0 x)$ of the ith truncated polynomial algebra.

3. Veen's spectral sequence and iterated tors

Our main computational tool is the bar spectral sequence, set up in [V2], which is closely related to the bar constructions we use in Section 8 and calculate the homology of in Section 2. Let $H\mathbb{F}_p$ denote the Eilenberg-MacLane spectrum of $\mathbb{F}_p$. Veen uses the Brun-Carlsson-Dundas [BCD] model $\Lambda_{\mathbb{S}^n} H\mathbb{F}_p$ for topological Hochschild homology of order n of $H\mathbb{F}_p$, $\mathsf{THH}^{[n]}(\mathbb{F}_p) = H\mathbb{F}_p \otimes \mathbb{S}^n$.

Theorem 3.1. [V2, §7] *There exists a strongly convergent spectral sequence of* $\mathbb{F}_p$*-Hopf algebras*

$$E^2_{r,s} = \mathrm{Tor}^{\pi_*(\Lambda_{\mathbb{S}^{n-1}} H\mathbb{F}_p)}_{r,s}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{r+s}(\Lambda_{\mathbb{S}^n} H\mathbb{F}_p).$$

Thus this spectral sequence uses $\mathsf{THH}^{[n-1]}_*(\mathbb{F}_p)$ as an input in order to calculate $\mathsf{THH}^{[n]}_*(\mathbb{F}_p)$. As long as it keeps collapsing at E^2, calculating $\mathsf{THH}^{[n]}_*(\mathbb{F}_p)$ is simply a process of starting with $\mathsf{THH}_*(\mathbb{F}_p) = \mathsf{THH}^{[1]}_*(\mathbb{F}_p) \cong \mathbb{F}_p[\mu]$ with $|\mu| = 2$ (as calculated by Bökstedt in [B]) and applying $\mathrm{Tor}^{-}_*(\mathbb{F}_p, \mathbb{F}_p)$ iteratively $n - 1$ times.

By [V2, Theorem 7.6], this is what happens for $n \leqslant 2p$, and so $\mathsf{THH}^{[n]}(\mathbb{F}_p) \cong B_n$ for $n \leqslant 2p$, where $B_n = \mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p)$ is the iterated Tor ring as explained above and defined in Definition 3.2 below. We will actually show in Section 4 that $\mathsf{THH}^{[n]}_*(\mathbb{F}_p) \cong B_n$ up to $n \leqslant 2p + 2$. We believe that it should be possible to use spectrum analogs of the methods of Section 2 in order to understand the homotopy type of the iterated Tor spectra rather than just their homotopy rings, and prove that $\mathsf{THH}^{[n]}_*(\mathbb{F}_p) \cong B_n$ for all $n > 0$, and are working on showing that with Maria Basterra and Michael Mandell.

It is well-known and follows from the calculations of Section 2 that $\mathrm{Tor}^{\mathbb{F}_p[x]}_*(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda(\epsilon x)$ with $|\epsilon x| = 1 + |x|$, which would be odd if $|x|$ were even; that $\mathrm{Tor}^{\Lambda[y]}_*(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma(\rho^0 y)$ if $|y|$ is odd, with $|\rho^0 y| = |y| + 1$, and that $\mathrm{Tor}^{\mathbb{F}_p[z]/z^m}(\mathbb{F}_p, \mathbb{F}_p) = \Lambda(\epsilon z) \otimes \Gamma(\varphi^0 z)$ when $|z|$ is even, with $|\epsilon z| = |z| + 1$ and $|\varphi^0 z| = 2 + m|z|$. The latter includes the case $\mathrm{Tor}^{\Lambda[y]}_*(\mathbb{F}_p, \mathbb{F}_p)$ if $|y|$

$$
\begin{tikzcd}
 & \bigotimes_{k\geqslant 0}\Lambda(\epsilon\rho^k\epsilon\omega) \\
\mathbb{F}_p[\omega] \to \Lambda(\epsilon\omega) \to \Gamma(\rho^0\epsilon\omega) \cong \bigotimes_{k\geqslant 0}\mathbb{F}_p[\rho^k\epsilon\omega]/(\rho^k\epsilon\omega)^p \arrow[ur] \arrow[dr] & \cdots \\
 & \bigotimes_{k\geqslant 0}\Gamma(\varphi^0\rho^k\epsilon\omega) \cong \bigotimes_{k,i\geqslant 0}\mathbb{F}_p[\varphi^i\rho^k\epsilon\omega]/(\varphi^i\rho^k\epsilon\omega)^p
\end{tikzcd}
$$

Figure 1. Evolution of elements.

$$
\begin{tikzcd}
 & & & & & \cdots \\
 & & & \bigotimes_{k\geqslant 0}\Lambda \arrow[r] & \bigotimes_{k\geqslant 0}\Gamma \arrow[ur] \arrow[dr] & \\
 & & & & & \cdots \\
\mathbb{F}_p[\omega] \arrow[r] & \Lambda \arrow[r] & \Gamma \arrow[uur] \arrow[dr] & & \bigotimes_{k,i}\Lambda \arrow[r] & \cdots \\
 & & & \bigotimes_{k\geqslant 0}\Gamma \arrow[ur] \arrow[dr] & & \cdots \\
 & & & & \bigotimes_{k,i}\Gamma \arrow[ur] \arrow[dr] & \\
 & & & & & \cdots
\end{tikzcd}
$$

Figure 2. Schematic overview of iterated Tor-terms.

is even, as well as the case of $\mathrm{Tor}_*^{\Gamma(y)}(\mathbb{F}_p, \mathbb{F}_p)$ for $|y|$ even, since the ground ring is $\mathbb{F}_p$ and so $\Gamma(y) \cong \bigotimes_{k\geqslant 0}\mathbb{F}_p[\varphi^k(y)]/(\varphi^k(y))^p$.

One can prove that the Tor over a finite tensor product is the tensor product of the Tor's directly, using projective resolutions of the single factors and the fact that $\mathbb{F}_p$ is $\mathbb{F}_p$-flat. Calculating Tor with the two-sided bar resolution shows that Tor respects direct limits also in the ring variable as well as in the module variables.

So we can encode the result of taking iterated $\mathrm{Tor}_*^{-}(\mathbb{F}_p, \mathbb{F}_p)$ in a flowchart as in Figure 1. or more schematically as in Figure 2. This notation for elements in iterated Tor-terms goes back to Cartan (compare [C, §1]).

Definition 3.2. Let B_n be the algebra generated by all words of length n of the following form (as illustrated in the flowchart), modulo the relations implied in the description of the algebras above (free for μ, exterior for $\epsilon\omega$, polynomial truncated at the pth power for $\rho^k\omega$ or $\varphi^k\omega$ for $k \geqslant 0$ and any word ω of length $n-1$):

- The rightmost letter must be μ.
- If there is something to the left of μ, it must be ϵ.
- If there is something to the left of an ϵ, it must be a ρ^k for some $k \geqslant 0$.
- If there is something to the left of a ρ^k for any $k \geqslant 0$, it must be either an ϵ or a φ^j for some $j \geqslant 0$.
- Similarly, if there is something to the left of a φ^k for any $k \geqslant 0$, it must be either an ϵ or a φ^j for some $j \geqslant 0$.

Observe, by the discussion above, that B_n is the algebra we get if we apply the functor $\mathrm{Tor}_*^-(\mathbb{F}_p, \mathbb{F}_p)$ iteratively $n-1$ times, starting with the algebra $\mathbb{F}_p[\mu]$.

Definition 3.3. Let B'_n be defined as the algebra generated by all words of length n defined as above, except that the rightmost letter must be x rather than μ; the letter directly to its left, if there is one, should be an ϵ. This follows the rules of the flowchart t with $\omega = x$, and will be useful in calculating $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x])$.

Definition 3.4. Let $B''_n = B''_n(m)$ be defined as the algebra generated by all words of length n ending with $\omega = x$ modulo the same relations as before and also the relation $x^m = 0$. In this case, if there is a letter immediately to the left of x, it has to be either ϵ or φ^k for some $k \geqslant 0$. The other rules are unchanged. This will be used in calculating $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x]/x^m)$. As the m should usually be clear from the context, we will omit it from the notation.

When we write such a word in an iterated Tor-term, the leftmost letter in the word carries the information about what kind of algebra the element corresponding to that word generates, the one before the last letter remembers what kind of algebra the generator came from, and so on; exponents remember what component of a divided power algebra the word came from at a particular stage.

The bidegrees of the words are computed using the following recursive formulas:

- $|\mu| = 2$ for the THH calculation, and $|x| = 0$ for the HH calculation, as explained above,
- $||\epsilon w|| = (1, |w|)$,
- $||\rho^i w|| = p^i(1, |w|)$, and
- $||\varphi^\ell w|| = p^\ell(2, p|w|)$.

The bidegrees will be important in the THH calculation. Note that when we write $|w|$ on the right hand side of the formulas, we mean the total degree of w. For the HH calculations, we will only care about total degrees.

4. Pushing Veen's bounds

In this section, we will work over $\mathbb{F}_p$ and assume that $p > 2$. In a Hopf algebra, ψ will denote the comultiplication. The following is a trivial generalization of [V2, Proposition 4.1], adapted to the needs of our calculation. It provides a little bit more information about the first nontrivial differential one could have in Veen's spectral sequence.

Lemma 4.1. *Suppose that Veen's spectral sequence of Theorem 3.1 collapses at E^2 and has no nontrivial multiplicative extensions for all $i < n$, so that $\pi_*(\Lambda_{\mathbb{S}^{n-1}} H\mathbb{F}_p) \cong B_{n-1}$. Suppose also that in Veen's spectral sequence for $\pi_*(\Lambda_{\mathbb{S}^n} H\mathbb{F}_p)$, $d^j \equiv 0$ for all $2 \leqslant j < i$. If $d^i \not\equiv 0$, then there exists a generator $\gamma_{p^k}(x)$ in the $E^2 = E^i$ term such that $d^i(\gamma_{p^k}(x))$ is a nonzero linear combination of generators of exterior algebras.*

Proof. If $d^i \not\equiv 0$, there exists an $a \in E^i_{*,*}$ such that $d^i(a) \neq 0$. Choose such an a of lowest degree. Recall that $E^i_{*,*}$ is a tensor product of graded exterior algebras and graded divided power algebras. Writing a as a linear combination of pure tensors, we see that there must be a pure tensor b such that $d^i(b) \neq 0$. If we can write $b = b'b''$ (with b', b'' of strictly lower degree), then by the Leibniz rule, $d^i(b) = d^i(b')b'' \pm b'd^i(b'')$; by our assumption on the minimality of

b's degree, this sum must be zero, contradicting the fact that $d^i(b) \neq 0$. Thus b must be indecomposable, that is: it must be a constant multiple of a generator. If the bidegree of b is (k, ℓ), then the bidegree of $d^i(b)$ must be $(k-i, \ell+i-1)$, and for $d^i(b)$ to be nontrivial, we must have $k \geqslant i \geqslant 2$. Since all generators of an exterior algebra have bidegree $(1, \ell)$ for some ℓ, we see that b must be of the form $\gamma_{p^k}(x)$ for some x, and of even degree.

Now consider $d^i(b)$. It must be primitive: writing $\psi(b) = 1 \otimes b + b \otimes 1 + \sum_j b'_j \otimes b''_j$, with b'_j and b''_j of lower degree, we obtain that

$$\psi(d^i(b)) = 1 \otimes d^i(b) + d^i(b) \otimes 1 + \sum_j (d^i(b'_j) \otimes b''_j \pm b'_j \otimes d^i(b''_j)) = 1 \otimes d^i(b) + d^i(b) \otimes 1.$$

The only primitive elements of odd degree in $E^i_{*,*}$ are generators of exterior algebras. □

Our goal is to show that Veen's bound of $n = 2p$ can be pushed to $n = 2p + 2$ by a further analysis of bi-degrees and the Hopf algebra structure, but no further: at $n = 2p + 3$ there will always be a differential candidate, which we believe will in fact vanish, but that needs to be established by other methods.

Definition 4.2.

- Let $\#w$ denote the length of a word w, that is: the number of letters used to write w.
- For a word w we write $w^{[n]}$ for the word consisting of w concatenated n times.

Lemma 4.3. *The only word w with $\#w \leqslant 2p + 1$ and $|w| = 4p^k$ for $k \geqslant 0$ is equal to $\rho^k \epsilon \mu$.*

Proof. Since the total degree $|w|$ is even, w must start with a ρ^ℓ or a φ^ℓ. Suppose first that $w = \rho^\ell \epsilon w'$. If $\ell < k$ then $|w'| = 4p^{k-\ell} - 2$, so by [V2, Lemma 7.2 part 5] we know that w' equals $(\rho^0\epsilon)^{[p-2]}\mu$ or starts with $(\rho^0\epsilon)^{[p-2]}\varphi^0$ or $(\rho^0\epsilon)^{[p-1]}$. In the first case $|w'| = 2p - 2$, which is not of the form $4p^{k-\ell} - 2$. In the second case, the beginning of w' is of length $2p - 3$, but it requires a tail of length 3 or more, and thus $\#w' \geqslant 2p$, which is not possible. In the third case, the beginning is of length $2p - 2$, and so the only way we could get $\#w' = 2p - 1$ is by having $w' = (\rho^0\epsilon)^{[p-1]}\mu$, but then $|w'| = 2p \neq 4p^{k-1}$ and this case is also impossible.

Thus $\ell = k$, so that $w' = \mu$ and $w = \rho^k \epsilon \mu$.

Now suppose that $w = \varphi^\ell w'$. Then $p|w'| = 4p^{k-\ell} - 2$. However, this can only happen when $p = 2$, a contradiction. So there are no such possible words w, and we are done. □

We have the following extension of Veen's Theorem 7.6:

Proposition 4.4. *When $n \leqslant 2p+2$ there are no non-trivial differentials in the spectral sequence of Theorem 3.1, and there is an $\mathbb{F}_p$-Hopf algebra isomorphism $\pi_*(\Lambda_{\mathbb{S}^n} H\mathbb{F}_p) \cong B_n$.*

Proof. For $n \leqslant 2p$, [V2, Theorem 7.6] gives us exactly the desired result. Thus we simply need to analyze two cases: $n = 2p + 1$ and $n = 2p + 2$. In order to extend Veen's argument to these cases, we will need to show that

(a) there are no possible non-trivial differentials in the spectral sequence, and
(b) there are no possible multiplicative extensions.

(a) Suppose that there exists a possible nonzero differential. This means that there exists an indecomposable element α and a primitive element β with $|\alpha| = |\beta| + 1$; as discussed in Lemma 4.1 we can assume that α is of the form $\gamma_{p^k}(x)$, or in other words that it is of the form $\rho^k w$ or $\varphi^k w$ for some admissible word w of length $2p$ or $2p + 1$, respectively. In order for there to be a differential which might not be trivial on α, we must have $k \geqslant 1$, so $|\alpha| \equiv 0 \pmod{2p}$.

Then $|\beta| \equiv -1 \pmod{2p}$. As β is primitive it is a linear combination of words that start with ϵ. From [V2, Lemma 7.2] we know that a word with such a degree is either equal to $\epsilon(\rho^0\epsilon)^{[p-2]}\mu$ or starts with $\epsilon(\rho^0\epsilon)^{[p-2]}\varphi^0$ or $\epsilon(\rho^0\epsilon)^{[p-1]}\rho^k$ or $\epsilon(\rho^0\epsilon)^{[p-1]}\varphi^k$ for some $k \geqslant 1$. The first of these has length $2p - 2$ so is not under consideration. The second must end with a suffix which has length at least 3, so we'll need to consider it in both cases. The third and fourth possibilities must end with a suffix of length at least 2, so we'll only need to consider them in the $2p+2$ case.

CASE 1: $n = 2p+1$. All words that can be the target of differentials must be of the form

$$\beta = \epsilon(\rho^0\epsilon)^{[p-2]}\varphi^0\rho^k\epsilon\mu \qquad k \geqslant 0.$$

This word has degree $4p^{k+1}+2p-1$. Thus any possible differential comes from a word of degree $4p^{k+1}+2p$. As α must start with a ρ^k or a φ^k, we know that α must equal $\varphi^1 w$, where $\#w = 2p$ and $|w| = 4p^k$ or $\rho^1\epsilon w$, where $\#w = 2p-1$ and $|w| = 4p^k$. However, both of these cases are impossible by Lemma 4.3, so there are no possible differentials.

CASE 2: $n = 2p+2$. We have two possible words that might be targets of differentials:

$$\beta_1 = \epsilon(\rho^0\epsilon)^{[p-2]}\varphi^0\varphi^k\rho^\ell\epsilon\mu,$$
$$\beta_2 = \epsilon(\rho^0\epsilon)^{[p-1]}\rho^{k+1}\epsilon\mu.$$

In both cases, $k, \ell \geqslant 0$. We have

$$|\beta_1| = 4p^{k+\ell+2} + 2p^{k+1} + 2p - 1 \qquad |\beta_2| = 4p^{k+1} + 2p - 1.$$

Thus we have two possibilities for α, with $|\alpha_1| = 4p^{k+\ell+2} + 2p^{k+1} + 2p$ and $|\alpha_2| = 4p^{k+1} + 2p$. As α_2 must start with a ρ^k or a φ^k, $k \geqslant 1$, it must be of the form $\rho^1\epsilon w$ or $\varphi^1 w$ for some w of length $2p$ or $2p+1$, respectively, with $|w| = 4p^k$ or $|w| = 4p^{k-1}$. But we know (by Lemma 4.3) that this is impossible, so it remains to consider the first case, where α_1 must equal either $\rho^1\epsilon w$ with $\#w = 2p$ and $|w| = 4p^{k+\ell+1} + 2p^k$ or $\varphi^1\rho^m\epsilon w$ with $\#w = 2p-1$ and $|w| = 4p^{k+\ell-m} + 2p^{k-m-1} - 2$.

CASE 2A: $\alpha_1 = \rho^1\epsilon w$. First, note that $w \neq \rho^a\epsilon w'$, because in this case $|w'| = 4p^{k+\ell-a+1} + 2p^{k-a} - 2$ and $\#w' = 2p-2$, and $|w'|$ is either equal to $4p^{k+\ell-a+1}$ (which is a contradiction by Lemma 4.3 because $\#w' = 2p-2 > p \geqslant 3$) or equivalent to $-2 \bmod 2p$, which demands a word longer than $2p-2$. Thus $w = \varphi^a w'$. Then $p|w'| = 4p^{k+\ell-a+1} + 2p^{k-a} - 2$, which means that $a = k$ and $|w'| = 4p^{\ell+1}$. But $\#w' = 2p-1 > 3$, a contradiction by Lemma 4.3, and so w does not exist.

CASE 2B: $\alpha_1 = \varphi^1\rho^m\epsilon w$. We know that $|w| = 4p^{k+\ell-m} + 2p^{k-m-1} - 2$. If $k = m+1$ then this is equal to $4p^{k+\ell-m} \geqslant 4p$, and by Lemma 4.3 we know that no such w exists. If $k > m+1$ then $|w| \equiv -2 \pmod{2p}$ and we know by [V2, Lemma 3.3.2 part 5] that w must start with $(\rho^0\epsilon)^{[p-2]}\varphi^0$ or $(\rho^0\epsilon)^{[p-1]}\rho^k$ or $(\rho^0\epsilon)^{[p-1]}\varphi^k$ for some $k \geqslant 1$. However, there are no words of length $2p-1$ that start with any of these prefixes, so w cannot exist.

(b) To solve the multiplicative extension problem we need to determine what the pth powers of elements can be. Let z be a generator of lowest degree with $z^p \neq 0$. Then we have

$$\psi(z^p) = \psi(z)^p = 1 \otimes z^p + z^p \otimes 1 + \sum (z')^p \otimes (z'')^p = 1 \otimes z^p + z^p \otimes 1,$$

so z^p must be primitive. However, in addition we know that $|z^p| = p|z|$, so $|z^p| \equiv 0 \pmod{2p}$. By the proof of [V2, Lemma 7.5] the shortest primitive word with degree equivalent to 0 modulo $2p$ of degree larger than $2p$ is equal to $w = (\rho^0\epsilon)^{[p-1]}\varphi^0\rho^k\epsilon\mu$ for $k \geqslant 1$. Thus it has length $2p+2$, so we do not need to worry about multiplicative extensions in the $n = 2p+1$ case.

In the $n = 2p+2$ case, we need some extra care. The degree of w is $|w| = 4p^{k+1} + 2p$, so we see that $|z| = 4p^k + 2$. Therefore $z = \rho^0\epsilon w$ or $z = \varphi^0\rho^\ell w$. In the first case we have $\#w = 2p$ and $|w| = 4p^k$, so by Lemma 4.3 this cannot happen. In the second case, we can deduce $\#w = 2p$ and $|w| = 4p^{k-\ell-1} - 1$. Note that we must have $k - \ell - 1 > 0$, as otherwise this clearly cannot happen. But then we know that $|w| \equiv -1 \pmod{2p}$, and by [V2, Lemma 7.5] it must have length at least $2p+1$. Thus such a word does not exist, and we see that there are no multiplicative extensions when $n = 2p+2$, either.

□

As we mentioned above, it is not possible to continue pushing the bound using this type of analysis, and while the spectral sequence may continue to collapse for $n > 2p+2$ (as we believe it will) we cannot deduce this purely from degree considerations:

Proposition 4.5. *For $n = 2p+3$ there is a potential non-trivial differential.*

Proof. Let

$$w = \varphi^1(\rho^0\epsilon)^{[p-1]}\varphi^0\rho^0\epsilon\mu \qquad \text{and} \qquad v = \epsilon(\rho^0\epsilon)^{[p-2]}\varphi^0\rho^2\epsilon\rho^0\epsilon\mu.$$

We have

$$||w|| = (2p, 6p^3) \qquad \text{and} \qquad ||v|| = (1, 6p^3+2p-2).$$

Thus we have a differential d^{2p-1} in the spectral sequence that is potentially non-trivial. $\square$

Remark 4.6. We do not claim that this is the shortest possible differential. It may be that for more complicated words there exist shorter possible differentials; indeed, at $n = 2p+4$ it is easy to find potential differentials of length $p-1$.

We found the above potential differential using a computer program written in Haskell; we include the code in Appendix A.

5. $\mathsf{THH}^{[n]}(\mathbb{F}_2)$, up to $n = 3$ and a stable element

Marcel Bökstedt showed [B] that THH of $\mathbb{F}_2$ is isomorphic to a polynomial algebra on a generator in degree 2, $\mathbb{F}_2[\mu]$. Using Torleif Veen's [V2] spectral sequence

$$E^2_{r,s} = \mathrm{Tor}^{\mathsf{THH}^{[n]}_*(\mathbb{F}_2)}_{r,s}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \mathsf{THH}^{[n+1]}_{r+s}(\mathbb{F}_2)$$

we obtain

$$\mathsf{THH}^{[2]}_*(\mathbb{F}_2) \cong \mathbb{F}_2[\beta]/\beta^2$$

where β is a generator in degree three (see also [V1, Proposition 2.3.1]).

Using Proposition 2.3 we get a spectral sequence calculating $\mathsf{THH}^{[3]}_*(\mathbb{F}_2)$ with E^2-term

$$\mathrm{Tor}^{\mathsf{THH}^{[2]}_*(\mathbb{F}_2)}_{*,*}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(x)]/\gamma_{2^i}(x)^2, \text{ with } |x| = 4.$$

The generators are concentrated in bidegrees of the form $(k, 3k)$ so there are no non-trivial differentials and the spectral sequence collapses. Also, since the only possible products are those which are detected by the E^∞ term, there are no multiplicative extension issues, so we get:

Proposition 5.1. *Let x denote a generator in degree 4, then*

$$\mathsf{THH}^{[3]}_*(\mathbb{F}_2) \cong \bigotimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(x)]/\gamma_{2^i}(x)^2.$$

As

$$\mathrm{Tor}^{\bigotimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(x)]/\gamma_{2^i}(x)^2}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{i\geqslant 0}\mathrm{Tor}^{\mathbb{F}_2[\gamma_{2^i}(x)]/\gamma_{2^i}(x)^2}(\mathbb{F}_2,\mathbb{F}_2)$$

we have to understand the single factors first. For each factor of the tensor product, by Proposition 2.2

$$\mathrm{Tor}^{\mathbb{F}_2[\gamma_{2^i}(x)]/\gamma_{2^i}(x)^2}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{j\geqslant 0}\mathbb{F}_2[\gamma_{2^j}(y_i)]/\gamma_{2^j}(y_i)^2 \cong \Gamma_{\mathbb{F}_2}(y_i)$$

with the y_i's being elements of bidegree $(1, 2^{i+2})$. But the E^2-term is now a tensor product of these building blocks

$$E^2_{*,*} \cong \bigotimes_{i\geqslant 0}\Gamma_{\mathbb{F}_2}(y_i) \cong \bigotimes_{i\geqslant 0}\bigotimes_{j\geqslant 0}\mathbb{F}_2[\gamma_{2^j}(y_i)]/\gamma_{2^j}(y_i)^2$$

thus excluding non-trivial differentials is harder.

Lemma 5.2. *The elements in the first column of the spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{\mathsf{THH}^{[3]}_*}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \mathsf{THH}^{[4]}_*(\mathbb{F}_2, \mathbb{F}_2)$$

are not in the image of d^r for any r.

Proof. The spectral sequence is a bar spectral sequence and the filtration that gives rise to it is compatible with the multiplication in the bar construction. Therefore the spectral sequence is (at least) one of algebras. It therefore suffices to show that none of the indecomposable elements can hit anything in the first column. Note that the only elements on the first column are the y_i's.

The bidegree of an element $\gamma_{2^j}(y_i)$ is $(2^j, 2^j \cdot 2^{i+2})$ and if a $d^r(\gamma_{2^j}(y_i))$ is in the first column for $r \geqslant 2$ then $r = 2^j - 1$ and the relation in the internal degree forces $2^j(2^{i+1}+1) - 2$ to be of the form 2^{k+2}. Since $r \geqslant 2$, we must have $j \geqslant 2$, but then $2^j(2^{i+1}+1) - 2 = 2(2^{j-1}(2^{i+1}+1) - 1)$ is not of the form 2^{k+2}.

So no indecomposable element hits anything in the first column. Products of such elements cannot hit a y_i either, because this would decompose y_i (the spot $(0,0)$ cannot be hit by a differential for degree reasons), so all the y_i must survive to the E^∞ term. □

Remark 5.3. Veen [V2, Proposition 3.5] describes the stabilization map

$$\sigma\colon \mathsf{THH}^{[n]}_*(R) \to \mathsf{THH}^{[n+1]}_{*+1}(R)$$

for every commutative ring spectrum R. It sends a class $[z] \in \mathsf{THH}^{[n]}_q(R)$ to the element in $\mathsf{THH}^{[n+1]}_{q+1}(R)$ that corresponds to $1 \otimes [z] \otimes 1 \in B_1(\pi_0(R), \mathsf{THH}^{[n]}_q(R), \pi_0(R))$. From the first cases we can read off that σ sends $\mu \in \mathsf{THH}^{[1]}_2(\mathbb{F}_2)$ to $\beta \in \mathsf{THH}^{[2]}_3(\mathbb{F}_2)$ and β to $x \in \mathsf{THH}^{[3]}_4(\mathbb{F}_2)$. We know that the y_i's give rise to non-trivial elements in $\mathsf{THH}^{[4]}_{1+2^{i+2}}(\mathbb{F}_2)$ and that $\sigma(x) = y_0 \in \mathsf{THH}^{[4]}_5(\mathbb{F}_2)$.

Proposition 5.4. *The iterative classes $\sigma^i(y_0)$ are all non-trivial and therefore give rise to a non-trivial class in topological André-Quillen homology,* TAQ*,*

$$\mathsf{TAQ}_1(\mathbb{F}_2) := \varinjlim_n \mathsf{THH}^{[n]}_{1+n}(\mathbb{F}_2).$$

Proof. We know that the classes $\sigma^i(y_0)$ are always cycles in the corresponding spectral sequences, so we have to show that they cannot be hit by any differential. We do not know whether the $\gamma_{2^j}(y_i)$'s survive but we know that the E^∞-term is a subquotient of the E^2-term and hence we get at most elements in $\mathsf{THH}^{[4]}_*(\mathbb{F}_2)$ that have a total degree corresponding to products of the $\gamma_{2^j}(y_i)$'s. By an iteration of this argument we can calculate possible bidegrees of elements that would arise if there were no non-trivial differentials. Let ℓ be bigger or equal to two and consider elements $\gamma_{2^{i_{\ell+1}}}(y_{i_1,\ldots,i_\ell})$ of bidegree

$$(2^{i_{\ell+1}}, 2^{i_{\ell+1}}(2^{i_\ell} + 2^{i_\ell+i_{\ell-1}} + \ldots + 2^{i_\ell+i_{\ell-1}+\ldots+i_2} + 2^{i_\ell+i_{\ell-1}+\ldots+i_2+i_1+2})).$$

A product of elements $\gamma_{2^{i_{1,\ell+1}}}(y_{i_{1,1},\ldots,i_{1,\ell}})$ up to $\gamma_{2^{i_{m,\ell+1}}}(y_{i_{m,1},\ldots,i_{m,\ell}})$ then has homological degree $\sum_{j=1}^r 2^{i_{j,\ell+1}}$ and internal degree

$$\sum_{j=1}^r 2^{i_{j,\ell+1}+i_{j,\ell}} + \ldots + \sum_{j=1}^r 2^{i_{j,\ell+1}+i_{j,\ell}+\ldots+i_{j,2}} + \sum_{j=1}^r 2^{i_{j,\ell+1}+i_{j,\ell}+\ldots+i_{j,2}+i_{j,1}+2}.$$

We know that $y_{0,\ldots,0} = \gamma_{2^0}(y_{0,\ldots,0})$ has bidegree $(1, \ell-1+4) = (1, \ell+3)$. If a differential d^s hits this element, then it has to start in something of bidegree $(1+s, \ell+3-s+1) = (s+1, \ell+4-s)$. For $s \geqslant 2$ the only possible bidegrees are $(3, \ell+2)$ up to $(\ell+5, 0)$.

The element $\gamma_2(y_{0,\ldots,0})$ has bidegree $(2, (\ell-1)2+8) = (2, \ell+(\ell+6))$ and as ℓ is at least 2 the internal degree is already larger than $\ell+2$, so this element cannot be a suitable source for a nontrivial differential. All other potential bidgrees have larger internal degree, thus there are no non-trivial differentials.

□

Maria Basterra and Michael Mandell calculated $\mathsf{TAQ}_*(H\mathbb{F}_p)$ for every prime p (see [La, §6] for a written account) and there is precisely one generator in $\mathsf{TAQ}_1(H\mathbb{F}_p)$.

Remark 5.5. For odd primes p it is easy to see that the generator $\mu \in \mathsf{THH}_2(\mathbb{F}_p)$ stabilizes to a non-trivial class in $\mathsf{TAQ}_1(H\mathbb{F}_p)$. The stabilizations of μ are represented by the words $((\rho^0\epsilon)^\ell\mu)$ and $(\epsilon(\rho^0\epsilon)^\ell\mu)$ in the spectral sequences (for some ℓ), so we have to show that these elements cannot be hit by any differential. Both types of elements are of bidegree $(1, m)$ for some m. If $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ should hit an element in such a spot, then we get $s = r + 1$ and $t = m - r + 1$. As r is greater or equal to 2, the differential can only start from bidegrees of the form $(3, m-1), \dots, (m+2, 0)$. If a term arises in the same spectral sequence as a stabilization of μ with bidegree $(1, m)$, then it is generated by words of length m, which means that it has internal degree at least m. But such terms cannot hit a term with bidegree $(1, m)$, so the stabilizations of μ survive.

6. A splitting of $\mathsf{THH}^{[n]}(A[G])$ for abelian groups G

If G is an abelian group, then the suspension spectrum of G_+ is an E_∞ ring spectrum, so it can be made into a commutative S-algebra $S^0[G]$ for instance by the methods of [EKMM]. If R is another commutative S-algebra, so is $R \wedge S^0[G]$. Applying the formula for the product of two simplicial objects, we get that for any n and any commutative S-algebras A and B,

$$\mathsf{THH}^{[n]}(A \wedge B) \simeq \mathsf{THH}^{[n]}(A) \wedge \mathsf{THH}^{[n]}(B),$$

which in our case yields

$$\mathsf{THH}^{[n]}(R \wedge S^0[G]) \simeq \mathsf{THH}^{[n]}(R) \wedge \mathsf{THH}^{[n]}(S^0[G]).$$

If R is a general S-algebra, we could take $R \wedge S^0[G]$ with coordinate-wise product to be the definition of $R[G]$. If $R = HA$ is the Eilenberg Mac Lane spectrum of a commutative ring, this is a model of the Eilenberg Mac Lane spectrum $H(A[G])$. This is because $HA \wedge S^0[G]$ has only one nontrivial stable homotopy group; $HA \wedge S^0[G]$ is the coproduct in the category of commutative S-algebras so the obvious inclusions induce a map of commutative S-algebras $HA \wedge S^0[G] \to H(A[G])$ which induces a multiplicative isomorphism on that unique nontrivial homotopy group. The product on an Eilenberg Mac Lane spectrum is determined by what it does on the unique nontrivial homotopy group, so we get

$$\mathsf{THH}^{[n]}(A[G]) \simeq \mathsf{THH}^{[n]}(A) \wedge \mathsf{THH}^{[n]}(S^0[G]). \tag{1}$$

As usual, when we talk of the topological Hochschild homology of a ring, we mean the topological Hochschild homology of its Eilenberg Mac Lane spectrum.

Proposition 6.1. *If A is a commutative $\mathbb{F}_p$-algebra, then for any $n \geqslant 1$ and any abelian group G,*

$$\mathsf{THH}^{[n]}_*(A[G]) \cong \mathsf{THH}^{[n]}_*(A) \otimes \mathsf{HH}^{[n]}_*(\mathbb{F}_p[G]).$$

Proof. We can rewrite the splitting in (1) above as

$$\mathsf{THH}^{[n]}(A[G]) \simeq \mathsf{THH}^{[n]}(A) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge \mathsf{THH}^{[n]}(S^0[G]),$$

which yields a spectral sequence with E^2-term

$$\mathrm{Tor}^{\mathbb{F}_p}_{*,*}(\mathsf{THH}^{[n]}_*(A), \pi_*(H\mathbb{F}_p \wedge \mathsf{THH}^{[n]}(S^0[G]))) \cong \mathsf{THH}^{[n]}_*(A) \otimes H_*(\mathsf{THH}^{[n]}(S^0[G]); \mathbb{F}_p)$$

converging to $\mathsf{THH}^{[n]}_*(A[G])$. (Recall that for a commutative $\mathbb{F}_p$-algebra A, $\mathsf{THH}^{[n]}(A)$ is an HA-module, and so its homotopy groups are $\mathbb{F}_p$-vector spaces.) Since the spectral sequence is concentrated in the 0th column, it collapses, yielding

$$\mathsf{THH}^{[n]}_*(A[G]) \cong \mathsf{THH}^{[n]}_*(A) \otimes H_*(\mathsf{THH}^{[n]}(S^0[G]); \mathbb{F}_p) \cong \mathsf{THH}^{[n]}_*(A) \otimes \mathsf{HH}^{[n]}_*(\mathbb{F}_p[G]),$$

where the fact that $H_*(\mathsf{THH}^{[n]}(S^0[G]);\mathbb{F}_p) \cong \mathsf{HH}_*^{[n]}(\mathbb{F}_p[G])$ follows from the fact that $H_*(S^0[G];\mathbb{F}_p)$ consists only of $\mathbb{F}_p[G]$ in dimension zero and the Künneth formula. □

Note that this proof goes through if we replace G by any commutative monoid M.

7. The higher Bökstedt spectral sequence

The aim of this section is to provide a Bökstedt spectral sequence for $\mathsf{THH}_*^{[n]}$.

Notation 7.1. For the remainder of the paper $\mathbb{S}^1$ will always denote the standard model of the 1-sphere with two non-degenerate simplices, one in dimension zero and one in dimension one. For $n \geqslant 1$ we take the n-fold smash product of this model as a simplicial model of $\mathbb{S}^n$.

Assume that R is a cofibrant commutative S-algebra (in the setting of [EKMM]). Then the simplicial spectrum $\mathsf{THH}^{[n]}(R)_\bullet$ has k-simplices

$$\mathsf{THH}^{[n]}(R)_k = \bigwedge_{\mathbb{S}^n_k} R.$$

The inclusion from the 'subspectrum' of degenerate simplices into the simplicial spectrum (which is actually a map of co-ends, as in [EKMM, p.182]) is a cofibration, because the degeneracies are induced by the unit of the algebra and the fact that R is cofibrant as a commutative S-algebra [EKMM, VII Theorem 6.7] guarantees that the smash product has the correct homotopy type. Therefore the simplicial spectrum $\mathsf{THH}^{[n]}(R)_\bullet$ is proper.

By [EKMM, X 2.9] properness implies that there is a spectral sequence for any homology theory E with

$$E^2_{r,s} = H_r(E_s(\mathsf{THH}^{[n]}(R)_\bullet))$$

converging to $E_{r+s}\mathsf{THH}^{[n]}(R)$. Note that for every s, $E_s(\mathsf{THH}^{[n]}(R)_\bullet)$ is a simplicial abelian group; $H_r(E_s(\mathsf{THH}^{[n]}(R)_\bullet))$ denotes its r'th homology group.

In the following we identify the E^2-term in good cases.

If $E_*(R)$ is flat over E_*, then we get that $E_s(\mathsf{THH}^{[n]}(R)_r)$ is

$$\pi_s(E \wedge_S \mathsf{THH}^{[n]}(R)_r) \cong \pi_s(E \wedge \bigwedge_{\mathbb{S}^n_r} R) \cong \pi_s(\bigwedge_{\mathbb{S}^n_r}^{E} E \wedge R) \cong (\bigotimes_{\mathbb{S}^n_r}^{E_*} E_*(R))_s$$

where $\bigwedge^E$ indicates that the smash product is taken over E. Taking the rth homology of the corresponding chain complex gives precisely

$$E^2_{r,s} \cong \mathsf{HH}^{[n]}_{r,s}(E_*(R))$$

where r is the homological degree and s the internal one. Therefore the Bökstedt spectral sequence for higher THH is of the following form.

Proposition 7.2. *Let R be a cofibrant commutative S-algebra and let E be a homology theory such that $E_*(R)$ is flat over E_*. Then there is a spectral sequence*

$$E^2_{r,s} \cong \mathsf{HH}^{[n]}_{r,s}(E_*(R)) \Rightarrow E_{r+s}(\mathsf{THH}^{[n]}(R)).$$

For $E = H\mathbb{F}_p$ we get $\mathsf{HH}^{[n]}_{r,s}((H\mathbb{F}_p)_*(R))$ for instance. If we then set $R = H\mathbb{F}_p$ as well, we obtain

$$E^2_{r,s} \cong \mathsf{HH}^{[n]}_{r,s}((H\mathbb{F}_p)_*(H\mathbb{F}_p))$$

thus we have to calculate Hochschild homology of order n of the dual of the mod-p Steenrod algebra, $\mathcal{A}_*(p)$.

For $p = 2$ this is a polynomial algebra in classes ξ_i of degree $2^i - 1$ and for $i \geqslant 1$. We can write $\mathcal{A}_*(2)$ as

$$\mathcal{A}_*(2) \cong \bigotimes_{i \geqslant 1} \mathbb{F}_2[\xi_i].$$

Recall that Pirashvili defines Hochschild homology of order n of a commutative k-algebra A the homotopy groups of the Loday functor $\mathcal{L}(A, A)$ evaluated on a simplicial model of $\mathbb{S}^n$ [P, 5.1]. For a finite pointed set of the form $\{0, \dots, m\}$ with 0 as basepoint $\mathcal{L}(A, A)\{0, \dots, m\}$ is $A^{\otimes m+1}$ and a map of finite pointed sets $f\colon \{0, \dots, m\} \to \{0, \dots, M\}$ induces a map of tensor powers by

$$f_*(a_0 \otimes \ldots \otimes a_m) = b_0 \otimes \ldots \otimes b_M,\ b_i = \prod_{f(j)=i} a_j$$

where the product over the empty set spits out the unit of the algebra A. For a finite pointed simplicial set $X.$ the Loday functor on $X.$ is then defined to be the simplicial k-module with m-simplices

$$\mathcal{L}(A, A)(X.)_m = \mathcal{L}(A, A)(X_m).$$

Therefore, for any two commutative algebras A, B we have

$$\mathcal{L}(A \otimes B, A \otimes B) \cong \mathcal{L}(A, A) \otimes \mathcal{L}(B, B)$$

as functors and so

$$\pi_*\mathcal{L}(A \otimes B, A \otimes B)(\mathbb{S}^n) \cong \pi_*(\mathcal{L}(A, A)(\mathbb{S}^n) \otimes \mathcal{L}(B, B)(\mathbb{S}^n)).$$

If all the algebras involved are flat as k-modules, we can identify this with

$$\pi_*(\mathcal{L}(A, A)(\mathbb{S}^n)) \otimes \pi_*(\mathcal{L}(B, B)(\mathbb{S}^n)).$$

In our case, where we are working over $\mathbb{F}_p$, we can therefore break down Bökstedt's spectral sequence $\mathsf{HH}^{[n]}_{r,s}(\mathcal{A}_*(p))$ into a tensor product of the higher Hochschild homology of the different tensored factors of $\mathcal{A}_*(p)$.

We know that

$$\mathsf{HH}^{[n]}_*(k[x]; k) \cong H_*(K(\mathbb{Z}, n); k)$$

(see for instance [LR, p. 207]). Here $\mathsf{HH}^{[n]}_*(k[x]; k)$ denotes Hochschild homology of order n of $k[x]$ with coefficients in k. So we have to understand what difference an internal grading makes and what changes if we take coefficients in $k[x]$ and not just in k.

8. Higher Hochschild homology of (truncated) polynomial algebras

In this section we will explain how to compute the higher Hochschild homology of the rings $k[x]$ over any integral domain k, and $\mathbb{F}_p[x]/x^{p^\ell}$ over $\mathbb{F}_p$. By varying the ground ring over which the tensor products in the Loday construction are taken, we can exhibit higher Hochschild homology as iterated Hochschild homology. Because we will be varying the ground rings, we introduce the notation $\mathcal{L}^k(R, M)$ to indicate the ground ring k in the Loday construction.

These methods were suggested to us by Michael Mandell based on his work with Maria Basterra on TAQ computations. Note that most of this section involves formal constructions that could be applied to augmented commutative $H\mathbb{F}_p$-algebra spectra as well.

Lemma 8.1. *Let k be a commutative ring, and let R be a commutative k-algebra. Then there is an isomorphism of functors from pointed simplicial sets to simplicial augmented commutative R-algebras*

$$\mathcal{L}^k(R, R) \cong \mathcal{L}^R(R \otimes_k R, R),$$

where R acts on $R \otimes_k R$ by multiplying the first coordinate, and the augmentation map is the multiplication $R \otimes_k R \to R$.

Proof. We can define a natural transformation $\mathcal{L}^k(R, R) \to \mathcal{L}^R(R \otimes_k R, R)$ by mapping $R \hookrightarrow R \otimes_k R$ via $r \mapsto 1 \otimes r$ over each simplex other than the base point, and using the identity over the base point. This map is simplicial, and is an isomorphism in each simplicial degree. $\square$

Remark 8.2. For any commutative ring R and augmented commutative R-algebra C, there is an isomorphism of simplicial augmented commutative R-algebras

$$\mathsf{B}^R(R,C,R) \cong \mathcal{L}^R(C,R)(\mathbb{S}^1),$$

where B^R denotes the two-sided bar construction with tensors taken over R and $\mathbb{S}^1$ is the model of the 1-sphere as in 7.1. This is simply because we can map the two R's on the sides of the bar complex to the 0th (coefficient) coordinate in the Hochschild homology complex.

Lemma 8.3. *Let R be a commutative ring, and let C be an augmented commutative R-algebra. Let $X.$ and $Y.$ be pointed simplicial sets. Then there is an isomorphism between the diagonals of the bisimplicial augmented commutative R-algebras*

$$\mathcal{L}^R(\mathcal{L}^R(C,R)(X.),R)(Y.) \cong \mathcal{L}^R(C,R)(X. \wedge Y.)$$

If $X.$ is a pointed simplicial set, then we denote by $\tilde{X}_k$ the k-simplices of X that are not the basepoint.

Proof. In degree k we can identify the diagonal of the bisimplicial sets as

$$\bigotimes_{\tilde{Y}_k}((\bigotimes_{\tilde{X}_k} C)\otimes R)\otimes R \cong \bigotimes_{\tilde{X}_k\times\tilde{Y}_k} C\otimes R.$$

Here, tensor products are all taken over R. The non-basepoint k-simplices in $X.\wedge Y.$ are exactly $\tilde{X}_k \times \tilde{Y}_k$, and the simplicial face maps in both cases are induced from those of $X.$ and $Y.$ in the same way. □

Corollary 8.4. *For any commutative ground ring k and commutative k-algebra R, the nth higher Hochschild homology complex of R over k, $\mathsf{HH}^{[n]}(R)$, can be written as*

$$\mathsf{HH}^{[n]}(R) \cong \mathsf{B}^R(R,\mathsf{HH}^{[n-1]}(R),R).$$

Proof. By Lemmata 8.1 and 8.3 and Remark 8.2,

$$\begin{aligned}\mathsf{HH}^{[n]}(R) = \mathcal{L}^k(R,R)(\mathbb{S}^n) &\cong \mathcal{L}^R(R\otimes_k R,R)(\mathbb{S}^n) \cong \mathcal{L}^R(\mathcal{L}^R(R\otimes R,R)(\mathbb{S}^{n-1}),R)(\mathbb{S}^1)\\ &\cong \mathcal{L}^R(\mathsf{HH}^{[n-1]}(R),R)(\mathbb{S}^1) \cong \mathsf{B}^R(R,\mathsf{HH}^{[n-1]}(R),R).\end{aligned}$$

□

Remark 8.5. Our results in Corollary 8.4 are not new. They can be found in the literature for slightly different settings: For instance, Veen [V2] establishes such an identification for ring spectra and the [BCD]-model in order to construct his spectral sequence and Ginot-Tradler-Zeinalian prove in an $(\infty,1)$-category setting that the Hochschild functor sends homotopy pushouts on space level to derived tensor products [GTZ, 3.27 c)].

Now we can calculate $\mathsf{HH}^{[n]}(R)$ inductively. To work with the bar construction, observe first that if we calculate $\mathsf{B}^R(R,C,R)$ for an augmented commutative R-algebra C and if there is an augmented commutative k-algebra C' so that $C \cong R\otimes C'$ as an augmented commutative $R\otimes k$-algebra (that is, the augmentation $C \to R$ is the tensor product of the identity of R with an augmentation $C' \to k$), then by grouping the R's together we get

$$\mathsf{B}^R(R,C,R) \cong \mathsf{B}^R(R,R,R)\otimes \mathsf{B}^k(k,C',k) \cong R\otimes \mathsf{B}^k(k,C',k)$$

as simplicial augmented commutative $R \cong R\otimes k$-algebras. Also, if we have a tensor product of augmented commutative k-algebras C and D,

$$\mathsf{B}^k(k,C\otimes D,k) \cong \mathsf{B}^k(k,C,k)\otimes \mathsf{B}^k(k,D,k)$$

as simplicial augmented commutative k-algebras.

In [B], Bökstedt used such decompositions to calculate the Hochschild homology of the dual of the Steenrod algebra. He observed that for any commutative ring k,

$$k[x]\otimes k[x] \cong k[x]\otimes C',$$

as augmented commutative algebras, where $k[x]$ is embedded as $k[x]\otimes k \subset k[x]\otimes k[x]$, and $C' \subset k[x]\otimes k[x]$ is the sub-algebra generated over k by the element $x' = x\otimes 1 - 1\otimes x$. Note that $C' = k[x'] \cong k[x]$.

Theorem 8.6. *Let k be an integral domain. There is an isomorphism of simplicial augmented commutative k-algebras*

$$\mathsf{HH}^{[n]}(k[x]) \cong k[x] \otimes \underbrace{\mathsf{B}(k, \mathsf{B}(k, \cdots \mathsf{B}(k}_{n \text{ times}}, k[x], k) \cdots, k), k)$$

where we take the diagonal of the multisimplicial set on the right. This induces an isomorphism of the associated chain complexes.

Moreover, there is a map of augmented differential graded k-algebras which is a quasi-isomorphism on the associated chain complexes

$$\mathsf{HH}^{[n]}(k[x]) \cong k[x] \otimes \underbrace{\mathrm{Tor}^{\mathrm{Tor}^{\cdots \mathrm{Tor}^{k[x]}(k,k)\cdots}(k,k)}(k,k)}_{n \text{ times}} \cong k[x]\otimes B'_{n+1},$$

for B'_{n+1} from Definition 3.3.

Here the Tor-expressions and B'_{n+1} are viewed as differential graded k-algebras with respect to the trivial differential; thus it follows automatically that the higher Hochschild homology groups of $k[x]$ are, respectively, isomorphic to the part of them which has the appropriate degree.

Proof. The first part of the claim is proved inductively. From Bökstedt's decomposition we get

$$\mathsf{HH}^{[1]}(k[x]) \cong \mathsf{B}^{k[x]}(k[x], k[x]\otimes C', k[x]) \cong k[x] \otimes \mathsf{B}(k, C', k) \cong k[x]\otimes \mathsf{B}(k, k[x], k)$$

as simplicial augmented commutative k-algebras. From this decomposition and the same kind of splitting, we then get by Corollary 8.4 that $\mathsf{HH}^{[2]}(k[x]) \cong k[x]\otimes \mathsf{B}(k, \mathsf{B}(k, k[x], k), k)$, and the general statement follows by an iteration of this argument.

The second part uses the quasi-isomorphisms of differential graded algebras from Section 2. The point is that we have a multiplicative quasi-isomorphism $\mathsf{B}(k, k[x], k) \simeq \Lambda(\epsilon x)$, which means that we have multiplicative quasi-isomorphisms $\mathsf{B}(k, \mathsf{B}(k, k[x], k), k) \simeq \mathsf{B}(k, \Lambda(\epsilon x), k) \simeq \Gamma(\rho^0\epsilon x)$, and so on. Thus instead of having a Veen-type spectral sequence, which one can easily get for Hochschild homology following the method that Veen used for topological Hochschild homology, we have a complex of algebras. □

Remark 8.7. As mentioned before, we believe that an argument along the lines of the above proof can show that Veen's spectral sequence collapses at E^2 for certain commutative ring spectra. To this end one has to establish that the higher topological Hochschild homology bar constructions of these ring spectra are weakly equivalent via multiplicative maps to the homotopy rings of the bar construction (taken over the Eilenberg Mac Lane spectrum of $\mathbb{F}_p$ rather than over $\mathbb{F}_p$). Such an argument would be analogous to our proof that there are multiplicative quasi-isomorphisms between the bar constructions $\mathsf{B}(k, A, k)$ (for certain algebras A) and their homology algebras as in Section 2.

In low dimensions we can identify $\mathsf{HH}^{[n]}(\mathbb{F}_p[x])$ as follows: We know that Hochschild homology of $\mathbb{F}_p[x]$, $\mathsf{HH}_*(\mathbb{F}_p[x])$ is isomorphic to $\Lambda_{\mathbb{F}_p[x]}(\epsilon x)$ with $|\epsilon x| = 1$. For Hochschild homology of order two we obtain

$$\mathsf{HH}^{[2]}_*(\mathbb{F}_p[x]) \cong \Gamma_{\mathbb{F}_p[x]}(\rho^0\epsilon x),\ |\rho^0\epsilon x| = 2.$$

In the next step we get

$$\mathsf{HH}^{[3]}_*(\mathbb{F}_p[x]) \cong \mathrm{Tor}_{*,*}^{\Gamma_{\mathbb{F}_p[x]}(\rho^0\epsilon x)}(\mathbb{F}_p[x], \mathbb{F}_p[x]) \cong \left(\bigotimes_{k\geqslant 0} \Lambda_{\mathbb{F}_p[x]}(\epsilon\rho^k\epsilon x)\right) \otimes \left(\bigotimes_{k\geqslant 0} \Gamma_{\mathbb{F}_p[x]}(\varphi^0\rho^k\epsilon x)\right).$$

Using the flowcharts in Figure 1 and Figure 2 one can explicitly calculate Hochschild homology of higher order.

Specifying $k=\mathbb{F}_p$ and using Bökstedt's method again, if we consider the ring $\mathbb{F}_p[x]/x^{p^\ell}$ we obtain

$$\mathbb{F}_p[x]/x^{p^\ell} \otimes \mathbb{F}_p[x]/x^{p^\ell} \cong \mathbb{F}_p[x]/x^{p^\ell} \otimes C'',$$

as augmented commutative algebras, where $\mathbb{F}_p[x]/x^{p^\ell}$ is embedded as $\mathbb{F}_p[x]/x^{p^\ell} \otimes k \subset \mathbb{F}_p[x]/x^{p^\ell} \otimes \mathbb{F}_p[x]/x^{p^\ell}$, and $C'' \subset \mathbb{F}_p[x]/x^{p^\ell} \otimes \mathbb{F}_p[x]/x^{p^\ell}$ is the $\mathbb{F}_p$-sub-algebra generated by the element $x' = x\otimes 1 - 1 \otimes x$, with the relation $(x')^{p^\ell}=0$ so that again $C'' = \mathbb{F}_p[x']/(x')^{p^\ell} \cong \mathbb{F}_p[x]/x^{p^\ell}$.

We use this to get a calculation of the higher Hochschild homology groups of $\mathbb{F}_p[x]/x^{p^\ell}$. In [P], Pirashvili calculated the nth higher Hochschild homology groups of $k[x]/x^a$ for any a when n is odd and k is a field of characteristic zero using Hodge decomposition techniques.

Theorem 8.8. *There is an isomorphism of simplicial augmented commutative $\mathbb{F}_p$-algebras*

$$\mathsf{HH}^{[n]}(\mathbb{F}_p[x]/x^{p^\ell}) \cong \mathbb{F}_p[x]/x^{p^\ell} \otimes \underbrace{\mathsf{B}(\mathbb{F}_p, \mathsf{B}(\mathbb{F}_p, \cdots \mathsf{B}(\mathbb{F}_p,}_{n \text{ times}} \mathbb{F}_p[x]/x^{p^\ell}, \mathbb{F}_p)\cdots, \mathbb{F}_p), \mathbb{F}_p)$$

where we take the diagonal of the multisimplicial set on the right. This induces an isomorphism of the associated chain complexes.

Moreover, there is a map of augmented differential graded $\mathbb{F}_p$-algebras which is a quasi-isomorphism on the associated chain complexes

$$\mathsf{HH}^{[n]}(\mathbb{F}_p[x]/x^{p^\ell}) \cong \mathbb{F}_p[x]/x^{p^\ell} \otimes \underbrace{\mathrm{Tor}^{\mathrm{Tor}^{\cdots \mathrm{Tor}^{\mathbb{F}_p[x]/x^{p^\ell}}(\mathbb{F}_p,\mathbb{F}_p)\cdots}(\mathbb{F}_p,\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)}_{n \text{ times}} \cong \mathbb{F}_p[x]/x^{p^\ell} \otimes B''_{n+1},$$

for B''_{n+1} from Definition 3.3. The Tor*-expressions and B''_{n+1} are again viewed as differential graded $\mathbb{F}_p$-algebras with a trivial differential.*

9. Étale and Galois descent

Ordinary Hochschild homology satisfies étale and Galois descent: Weibel and Geller [WG] showed that for an étale extension $A \to B$ of commutative k-algebras one has

$$\mathsf{HH}_*(B) \cong \mathsf{HH}_*(A) \otimes_A B$$

and if $A\to B$ is a Galois extension of commutative k-algebras in the sense of Auslander-Goldman [AG] with finite Galois group G, then

$$\mathsf{HH}_*(A) \cong \mathsf{HH}_*(B)^G.$$

We will show that these properties translate to higher order Hochschild homology. In the following let k be again an arbitrary commutative unital ring and let n be greater or equal to one.

Theorem 9.1.
(a) *If A is a commutative étale k-algebra, then $\mathsf{HH}^{[n]}_*(A) \cong A$.*
(b) *If $A \to B$ is an étale extension of commutative k-algebras, then*

$$\mathsf{HH}^{[n]}_*(B) \cong \mathsf{HH}^{[n]}_*(A) \otimes_A B.$$

(c) *If $A \to B$ is a G-Galois extension with G a finite group, then*

$$\mathsf{HH}^{[n]}_*(A) \cong \mathsf{HH}^{[n]}_*(B)^G.$$

Proof. The first claim follows from the second, but we also give a direct proof: Étale k-algebras have Hochschild homology concentrated in degree zero. Therefore Veen's spectral sequence yields

$$\mathrm{Tor}^{\mathsf{HH}_*(A)}_{p,q}(A,A) \cong \mathrm{Tor}^A_{p,q}(A,A) = A$$

in the $p=q=0$-spot and thus we get $\mathsf{HH}^{[2]}_*(A) = A$ concentrated in degree zero. An iteration of this argument shows the claim for arbitrary n.

For étale descent we deduce from Corollary 8.4 that

$$\mathsf{HH}_*^{[2]}(B) \cong \mathrm{Tor}_*^{\mathsf{HH}_*^{[1]}(B)}(B,B) \cong \mathrm{Tor}_*^{\mathsf{HH}_*^{[1]}(A)\otimes_A B}(A\otimes_A B, A\otimes_A B) \cong \mathrm{Tor}_*^{\mathsf{HH}_*^{[1]}(A)}(A,A)\otimes_A B$$

and the latter is exactly $\mathsf{HH}_*^{[2]}(A)\otimes_A B$. Note that the maps $B = \mathsf{HH}_0(B) \to \mathsf{HH}_*(B)$ and $\mathsf{HH}_*(A) \to \mathsf{HH}_*(B)$ used for the Weibel-Geller isomorphism induce a map of graded commutative rings $\mathsf{HH}_*(A)\otimes_A B \to \mathsf{HH}_*(B)$, and the argument above shows that our formulas for higher Hochschild homology are ring maps as well.

Iterating this argument, we get that $\mathsf{HH}_*^{[n]}(B) \cong \mathsf{HH}_*^{[n]}(A)\otimes_A B$ for all n as graded commutative rings.

Any G-Galois extension as above is in particular an étale extension, so we get

$$\mathsf{HH}_*^{[n]}(B) \cong \mathsf{HH}_*^{[n]}(A)\otimes_A B.$$

The G-action on the left hand side corresponds to the G-action on the B-factor on the right hand side and thus taking G-fixed points yields

$$\mathsf{HH}_*^{[n]}(B)^G \cong \mathsf{HH}_*^{[n]}(A)\otimes_A (B^G) \cong \mathsf{HH}_*^{[n]}(A)\otimes_A A \cong \mathsf{HH}_*^{[n]}(A).$$

□

10. Group algebras of finitely generated abelian groups

The results of the preceding sections allow us to compute $\mathsf{THH}_*^{[n]}$ of group algebras of finitely generated abelian groups over $\mathbb{F}_p$. If G is a finitely generated abelian group, then we know from Section 6 that we need to determine $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[G])$ because $\mathsf{THH}_*^{[n]}(\mathbb{F}_p[G])$ is isomorphic to the tensor product of $\mathsf{THH}_*^{[n]}(\mathbb{F}_p)$ and $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[G])$. In addition we know that $\mathbb{F}_p[G]$ can be written as a tensor product

$$\mathbb{F}_p[G] \cong \mathbb{F}_p[\mathbb{Z}]^{\otimes r}\otimes \mathbb{F}_p[C_{q_1^{\ell_1}}]\otimes\ldots\otimes \mathbb{F}_p[C_{q_s^{\ell_s}}]$$

where r is the rank of G and the $C_{q_i^{\ell_i}}$'s are the torsion factors of G for some primes q_i. As $\mathsf{HH}_*^{[n]}$ sends tensor products to tensor products, we only have to determine the tensor factors $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[\mathbb{Z}])$ and $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[C_{q_i^{\ell_i}}])$.

Proposition 10.1.

- *For the group algebra $\mathbb{F}_p[\mathbb{Z}] \cong \mathbb{F}_p[x^{\pm 1}]$ we get*
$$\mathsf{HH}_*^{[n]}(\mathbb{F}_p[\mathbb{Z}]) \cong \mathbb{F}_p[x^{\pm 1}]\otimes B'_{n+1}.$$
- *If q is a prime not equal to p, then $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[C_{q^\ell}]) \cong \mathbb{F}_p[C_{q^\ell}]$ where the latter is concentrated in homological degree zero.*
- *For $q = p$,*
$$\mathsf{HH}_*^{[n]}(\mathbb{F}_p[C_{p^\ell}]) \cong \mathbb{F}_p[x]/x^{p^\ell}\otimes B''_{n+1}.$$

Proof. The group algebra $\mathbb{F}_p[\mathbb{Z}] \cong \mathbb{F}_p[x^{\pm 1}]$ is étale over $\mathbb{F}_p[x]$ and therefore by Theorem 9.1 we obtain

$$\mathsf{HH}_*^{[n]}(\mathbb{F}_p[\mathbb{Z}]) \cong \mathsf{HH}_*^{[n]}(\mathbb{F}_p[x])\otimes_{\mathbb{F}_p[x]}\mathbb{F}_p[x^{\pm 1}]$$

and hence the first statement follows from Theorem 8.6.

The group algebra $\mathbb{F}_p[C_{q^\ell}]$ is an étale algebra over $\mathbb{F}_p$ for q not equal to p, so Theorem 9.1 also implies the second claim.

We know that $\mathbb{F}_p[C_{p^\ell}] \cong \mathbb{F}_p[x]/x^{p^\ell}$ because $\mathbb{F}_p[x]/x^{p^\ell}-1 = \mathbb{F}_p[x]/(x-1)^{p^\ell}$. Thus $\mathsf{HH}_*^{[n]}(\mathbb{F}_p[C_{p^\ell}])$ is determined by Theorem 8.8. □

Thus if we express G as

$$G = \mathbb{Z}^r\times C_{p^{i_1}}\times\ldots\times C_{p^{i_a}}\times C_{q_1^{j_1}}\times\ldots\times C_{q_b^{j_b}}$$

with $r, a, b \geqslant 0$, $i_s, j_t \geqslant 1$ and primes $q_i \neq p$, then we obtain

$$\begin{aligned}\mathsf{THH}_*^{[n]}(\mathbb{F}_p[G]) &\cong \mathsf{THH}_*^{[n]}(\mathbb{F}_p) \otimes \mathsf{HH}_*^{[n]}(\mathbb{F}_p[\mathbb{Z}]^{\otimes r} \otimes \bigotimes_{s=1}^{a} \mathbb{F}_p[x]/x^{p^{i_s}} \otimes \bigotimes_{t=1}^{b} \mathbb{F}_p[C_{q_b^{j_b}}]) \\ &\cong \mathsf{THH}_*^{[n]}(\mathbb{F}_p) \otimes \left(\mathsf{HH}_*^{[n]}(\mathbb{F}_p[x]) \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x^{\pm 1}]\right)^{\otimes r} \otimes \bigotimes_{s=1}^{a} \mathsf{HH}_*^{[n]}(\mathbb{F}_p[x]/x^{p^{i_s}}) \otimes \bigotimes_{t=1}^{b} \mathbb{F}_p[C_{q_b^{j_b}}].\end{aligned}$$

For instance, unravelling the definitions gives

$$\begin{aligned}\mathsf{THH}_*^{[2]}(\mathbb{F}_3[\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}]) &\cong \mathsf{THH}_*^{[2]}(\mathbb{F}_3) \otimes \mathsf{HH}_*^{[2]}(\mathbb{F}_3[x]) \otimes_{\mathbb{F}_3[x]} \mathbb{F}_3[x^{\pm 1}] \otimes \mathbb{F}_3[C_2] \otimes \mathsf{HH}_*^{[2]}(\mathbb{F}_3[x]/x^3) \\ &\cong \Lambda_{\mathbb{F}_3}(\epsilon y) \otimes (\mathbb{F}_3[x] \otimes B_3') \otimes_{\mathbb{F}_3[x]} \mathbb{F}_3[x^{\pm 1}] \otimes \mathbb{F}_3[C_2] \otimes \mathbb{F}_3[x]/x^3 \otimes B_3'' \\ &\cong \Lambda_{\mathbb{F}_3}(\epsilon y) \otimes \mathbb{F}_3[x^{\pm 1}] \otimes B_3' \otimes \mathbb{F}_3[C_2] \otimes \mathbb{F}_3[x]/x^3 \otimes B_3''\end{aligned}$$

with B_3' and B_3'' as explained in Definitions 3.3 and 3.4 and where ϵy is a generator of degree three.

Appendix A. Code

Below is the Haskell code for generating possible differentials. The code finds all admissible words of a given length n that fit into a particular portion of the E^2 page and then looks for words that have consecutive degrees. As the shortest differential must go from an indecomposable to a primitive, we do not generate any powers or products of words, as none of these can support a shortest nonzero differential.

```
import System.Environment
import Data.List
import qualified Data.Set as S

main = do
  (prime:n:limit:_) <- getArgs
  putStrLn $ concat $ map pairToString
                          (possibleD (read n :: Integer)
                                     (read limit :: Integer)
                                     (read prime :: Integer))
-----------------

data VeenWord = M | E VeenWord | Rk VeenWord | Pk VeenWord
type Ppoly = [(Integer, (Integer,Integer))]

-- takes a sum and a list length and makes all lists of the length that
-- add up to at most m; this is the maximum degree of any particular
-- generator
varValueLists 0 m = [[]]
varValueLists 1 m = map (\a -> [a]) [0..m]
varValueLists n m = foldr (\l ls ->
                           let s = sum l
                           in (map (\a -> a:l) [0..m-s]) ++ ls)
                          [] (varValueLists (n-1) m)

makeKey M _ = "u"
makeKey (E w) l = "e" ++ (makeKey w l)
makeKey (Rk w) (a:as) = "r^" ++ (show a) ++ (makeKey w as)
makeKey (Pk w) (a:as) = "l^" ++ (show a) ++ (makeKey w as)
makeKey _ _ = error "Incorrect number of variables"
```

```
constantPoly n = [(n,(0,0))]
numVars = foldr (\(a,(_,c)) m -> if a == 0 || c == 0 then m
                                   else if c >= m then c else m) 0
compress p =
  let addup x [] = [x]
      addup x@(a,pair) ys@((a',pair'):l) =
        if pair == pair' then (a+a',pair):l else x:ys
  in foldr addup [] p

-- plugs in for variable number 1, shifts other variables down;
-- keep in mind that variable 3 is really the sum of three variables,v1,v2,v3
plugInV1 p v = compress $ map (\(a,(b,c)) -> if c >= 1
                                             then (a,(b+v,c-1))
                                             else (a,(b,c)))
                              p

plugInP :: Integer -> Ppoly -> Integer
plugInP prime p =
  let a ^^ n
        | n < 0 = error "Exponent must be positive"
        | n == 0 = 1
        | otherwise = a * (a ^^ (n-1))
  in if any (\(_,(_,c)) -> c /= 0) p
     then error "To plug in p you need to have no variables"
     else sum $ map (\(a,(b,_)) -> a * (prime ^^ b)) p

plugInAllVars :: Integer -> Ppoly -> [Integer] -> Integer
plugInAllVars prime p l = plugInP prime (foldl plugInV1 p l)

polyToString :: Ppoly -> String
polyToString =
  let monoToString (a,(b,c)) =
        (show a) ++ (if (b,c) == (0,0) then ""
                     else " P^{" ++ (if b /= 0 then (show b) ++ "+" else "")
                          ++ (if c /= 0
                              then "v_" ++ (show c)
                              else "") ++ "}")
  in (intercalate " + ") . (map monoToString)

addN n ((m,(0,0)):l) = (m+n,(0,0)):l
addN n l = (n,(0,0)):l
shiftBy1 = map (\(a,(b,c)) -> (a,(b+1,c)))
shiftByVar = map (\(a,(b,c)) -> (a,(b,c+1)))

degree :: VeenWord -> Ppoly
degree M = constantPoly 2
degree (E x) = addN 1 (degree x)
degree (Rk x) = shiftByVar $ addN 1 $ degree x
degree (Pk x) = shiftByVar $ addN 2 $ shiftBy1 $ degree x

bidegree :: VeenWord -> (Ppoly, Ppoly)
```

```
bidegree M = (constantPoly 0, constantPoly 2)
bidegree (E x) = (constantPoly 1, degree x)
bidegree (Rk x) = (shiftByVar $ constantPoly 1, shiftByVar $ degree x)
bidegree (Pk x) = (shiftByVar $ constantPoly 2,
                   shiftByVar $ shiftBy1 $ degree x)


makeAdmissibleWords n
  | n < 1 = error "makeAdmissibleWords needs positive integer"
  | n == 1 = [M]
  | otherwise =
      let words :: VeenWord -> [VeenWord] -> [VeenWord]
          words M l = (E M):l
          words w@(E _)  l = (Rk w):l
          words w@(Rk _) l = (E w):(Pk w):l
          words w@(Pk _) l = (E w):(Pk w):l
      in foldr words [] (makeAdmissibleWords (n-1))

--this takes a word and a pair of limits (which must be positive integers)
--and a prime p
--and generates all versions of the word and all powers of each version that
--will fit inside those limits
makeVersions :: VeenWord -> Integer -> Integer -> [(String,(Integer,Integer))]
makeVersions w maxdeg prime =
  let maxpow = (log (fromIntegral maxdeg))/(log (fromIntegral prime))
      estimate_bounds = floor(maxpow) :: Integer

      -- note that hom has at most one variable, which must have the same
      -- value as the first variable in inter
      (hom, inter) = bidegree w
      possibleVarValues = varValueLists (numVars inter) estimate_bounds
  in map (\l -> (makeKey w l, plugInAllVars prime hom   l,
                              plugInAllVars prime inter l))
         possibleVarValues

generateAllElts n maxdeg prime = concat $
                                  map (\w -> makeVersions w maxdeg prime)
                                      (makeAdmissibleWords n)

consecutivePairs l =
  [ (a,b,x-x') | a@(_,(x,y)) <- l, b@(_,(x',y')) <- l, x+y == x'+y'+1, x-x'>1]

possibleD n x prime = consecutivePairs $ generateAllElts n x prime

pairToString (a,b,deg) =
  let showThis (k,(x,y)) = k ++ (show (x,y))
  in (showThis a) ++ " ---> " ++ (showThis b) ++ ": " ++ (show deg) ++ "\n"
```

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